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On the Doppler Bias of Hyperbolic Frequency Modulation Matched Filter Time of Arrival Estimates

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***Abstract*—The purpose of this paper is to introduce an extended matched filter for hyperbolic frequency modulation (HFM) waveforms in active sonar systems, along with an exact closed-form solution for the Doppler bias in time of arrival estimates when using this filter. Conventional HFM matched filtering employs a single undilated replica of the transmitted signal and estimates the time of an echo's arrival using the peak squared magnitude of the processed data. It is well known that when the echo is dilated (the term dilate is used in a generic sense to mean either stretching or compressing) due to the Doppler effect, such an estimate of the time of arrival is biased by an amount depending on the range rate v, however, a closed-form solution for this Doppler bias does not exist. Using the extended HFM matched filter in place of a conventional one allows for an exact closed-form solution for the Doppler bias. The closed-form solution for the Doppler bias of time of arrival estimates using the extended HFM matched filter is presented. This solution applies to both broadband and narrowband HFM signals, and suggests that a previously published approximation of the bias for a conventional HFM matched filter is valid only for narrowband HFM signals.**

***Index Terms*—Acoustic signal processing, frequency modulation, matched filters, sonar detection.**

I. Introduction

THE hyperbolic frequency modulation (HFM) chirp signal is commonly used in radar and sonar systems for accurate ranging of received echoes, in part due to its insensitivity [1] to dilation from the Doppler effect. The wideband ambiguity function of the HFM pulse [2] is a narrow ridge in the delay-Doppler plane that results in a biased time of arrival (TOA) estimate for dilated echoes when using the peak magnitude of the HFM matched filter response. Recently, an approximate expression for this Doppler bias has been derived [3]. In this paper, the conventional zero-Doppler (undilated) HFM matched filter is extended (in both time and frequency). When such an extended HFM filter is used, the exact Doppler bias can be written in closed-form. The subsequent development reviews conventional HFM matched filtering, defines the extended HFM matched filter, derives the exact bias in TOA due to echo Doppler, and shows that for the exact bias to apply the chirp replica must be extended (in time and frequency) to ensure that the frequency range of all expected echoes—based on knowledge of practical target speeds—is encompassed by the extension.

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II. Development

A. Conventional HFM Matched Filter

An HFM transmission of duration T, starting at frequency f_0 and ending at frequency f_1 is given by

$$s(t) = \begin{cases} e^{-j\phi_0 \ln\left(\frac{t_0 - t}{t_0}\right)}, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

with the parameters t_0 and ϕ_0 defined as

$$t_0 = \frac{f_1}{f_1 - f_0} T \text{ and } \phi_0 = 2\pi f_0 t_0. \tag{2}$$

The instantaneous frequency of the HFM pulse as a function of time is

$$f_s(t) = \frac{f_0 f_1}{f_1 - (f_1 - f_0)\frac{t}{T}} \;. \tag{3}$$

Equation (3) confirms that the HFM sweep begins at time $t = 0$ at frequency f_0 and ends at time $t = T$ at frequency f_1.

An echo from a target presenting range rate v is dilated in time. To accommodate bistatic (as well as monostatic) geometries, v is defined as the average range rate presented to the source and receiver. Ignoring propagation loss and noise, such an echo arriving at the receiver at time τ can be written as

$$x(t) = \begin{cases} \sqrt{\eta} e^{-j\phi_0 \ln\left(\frac{t_0 - \eta(t-\tau)}{t_0}\right)}, & \tau \le t \le \tau + T/\eta \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

where the dilation factor η is defined as

$$\eta = \frac{c + v}{c - v} \tag{5}$$

and c is the speed of sound in water. Here, the naval convention that positive v corresponds to a closing target speed is used. The instantaneous frequency of the echo—valid over the time interval $\tau \le t \le \tau + T/\eta$ – is

$$f_e(t) = \frac{f_0 f_1}{\frac{f_1}{\eta} - (f_1 - f_0)\frac{t - \tau}{T}} \;. \tag{6}$$

At time $t = \tau$, the instantaneous frequency of the echo is ηf_0, and at time $t = \tau + T/\eta$ the instantaneous frequency is ηf_1.

A zero Doppler replica for matched filtering is constructed by taking the conjugate of the HFM pulse reversed in time

$$h(t) = s^*(-t) = \begin{cases} e^{j\phi_0 \ln\left(\frac{t_0 + t}{t_0}\right)}, & -T \le t \le 0 \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

The matched filter processing of the received echo produces

$$y(t) = \int h(\sigma)\, x(t-\sigma)\, d\sigma = \sqrt{\eta} \int_{a(t)}^{b(t)} e^{-j\phi_0 \ln\left(\frac{t_0 - \eta(t-\sigma-\tau)}{t_0+\sigma}\right)} d\sigma \tag{8}$$

where the limits on integration $a(t) \le b(t)$ are chosen to ensure that both $h(\sigma)$ and $x(t-\sigma)$ are within the ranges of their respective HFM sweeps

$$a(t) = \max(t-\tau-T/\eta, -T) \tag{9}$$

and

$$b(t) = \min(t-\tau, 0). \tag{10}$$

Since the convolution integral is identically zero when $a(t) > b(t)$, (8) through (10) apply for the time interval $-T \le t-\tau \le T/\eta$, and the maximum of the squared envelope must lie on this interval.

B. Extended HFM Matched Filter

An extended HFM matched filter is defined as

$$h_e(t) = \begin{cases} e^{j\phi_0 \ln\left(\frac{t_0+t}{t_0}\right)}, & -T_1 \le t \le T_2 \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

where $T_1 \ge T$ and $T_2 \ge 0$. The filter is *matched* in that the sweep from f_1 at time $t = -T$ to f_0 at time $t = 0$ (recall that the filter is reversed in time relative to the transmitted signal) is exactly retained, but the trajectory is extended in time, in the positive and negative directions so that frequencies higher than f_1 and lower than f_0 are included in the sweep.

C. Maximum Squared Magnitude of the Extended HFM Filter Response

Using the extended HFM matched filter to process the received echo produces

$$y_e(t) = \int h_e(\sigma)\, x(t-\sigma)\, d\sigma = \sqrt{\eta} \int_{a_e(t)}^{b_e(t)} e^{-j\phi_0 \ln\left(\frac{t_0 - \eta(t-\sigma-\tau)}{t_0+\sigma}\right)} d\sigma \tag{12}$$

where the limits of integration $a_e(t) \le b_e(t)$ are given by

$$a_e(t) = \max(t-\tau-T/\eta, -T_1) \tag{13}$$

and

$$b_e(t) = \min(t-\tau, T_2). \tag{14}$$

Equation (12) applies for $-T_1 \le t-\tau \le T/\eta + T_2$ (outside of which the integral is identically zero as before), and the maximum of the squared envelope must lie on this interval. The bounds of the integral in (12) are portrayed graphically in Fig. 1.

The derivation for the closed-form TOA bias of the extended HFM matched filtering will proceed by first assuming that T_1 and T_2 can be made large enough to be neglected in (13) and (14). Under this assumption, the maximum squared envelope of the response can be found by employing the Cauchy–Schwarz inequality. Once the maximum is found, conditions on how large T_1 and T_2 need to be to ensure that they can be neglected as assumed are presented.

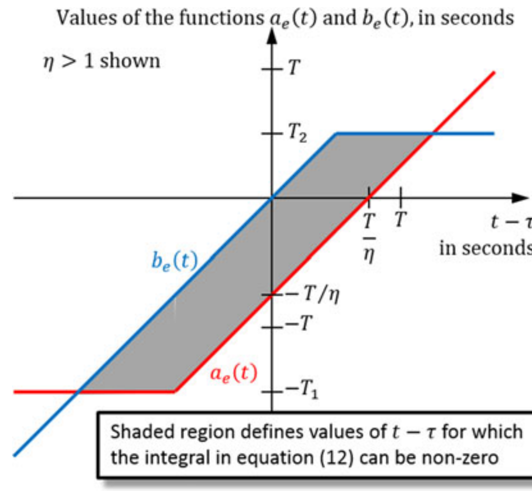

Fig. 1. Graphical depiction of the bounds of integration for (12).

For large T_1 and T_2, the response is

$$y_e(t) = \sqrt{\eta} \int_{t-\tau-T/\eta}^{t-\tau} e^{-j\phi_0 \ln\left(\frac{t_0 - \eta(t-\sigma-\tau)}{t_0+\sigma}\right)} d\sigma\ . \tag{15}$$

The TOA estimate for the echo corresponds to the peak of the squared magnitude of the response

$$\hat{\tau} = \underset{t}{\operatorname{argmax}} \left| \sqrt{\eta} \int_{t-\tau-T/\eta}^{t-\tau} e^{-j\phi_0 \ln\left(\frac{t_0 - \eta(t-\sigma-\tau)}{t_0+\sigma}\right)} d\sigma \right|^2 = \eta \underset{t}{\operatorname{argmax}} \left| \int_{t-\tau-T/\eta}^{t-\tau} e^{-j\phi_0 \ln\left(\frac{t_0 - \eta(t-\sigma-\tau)}{t_0+\sigma}\right)} d\sigma \right|^2. \tag{16}$$

Using the Cauchy–Schwarz inequality

$$\left| \int_a^b f(\sigma)\, g^*(\sigma)\, d\sigma \right|^2 \le \left(\int_a^b |f(\sigma)|^2 d\sigma \right) \left(\int_a^b |g(\sigma)|^2 d\sigma \right) \tag{17}$$

with the definitions

$$f(\sigma) = e^{-j\phi_0 \ln\left(\frac{t_0 - \eta(t-\sigma-\tau)}{t_0+\sigma}\right)} \tag{18}$$

and

$$g(\sigma) = 1 \tag{19}$$

it is easy to show that

$$\left| \int_{t-\tau-T/\eta}^{t-\tau} e^{-j\phi_0 \ln\left(\frac{t_0 - \eta(t-\sigma-\tau)}{t_0+\sigma}\right)} d\sigma \right|^2 \le \left(\frac{T}{\eta}\right)^2 \tag{20}$$

since both $f(\sigma)$ and $g(\sigma)$ have unit magnitude.

Thus, if there is a value of t for which the argument of the natural logarithm

$$\frac{t_0 - \eta(t-\sigma-\tau)}{t_0+\sigma} \tag{21}$$

is constant with respect to σ, then at that value of t the exponential in (16) can be brought outside the integral. Since the exponential is unit magnitude, the squared magnitude of the integral in (16) at this value of t is then $(T/\eta)^2$ and is, by the Cauchy–Schwarz argument above, a maximum.

Returning to the issue of deriving the TOA bias, the argument of the natural logarithm in (21) can, after some minor algebraic manipulation, be written as

$$\eta + \frac{(1-\eta)\,t_0 - \eta\,(t-\tau)}{t_0+\sigma} \tag{22}$$

which is constant (and equal to η) when the numerator of the second term is identically zero. This occurs at time

$$t_{\max} = \frac{1-\eta}{\eta}t_0 + \tau. \tag{23}$$

Thus, the estimated TOA is

$$\hat{\tau} = \frac{1-\eta}{\eta}t_0 + \tau \tag{24}$$

and the bias β in the estimate is

$$\beta = \hat{\tau} - \tau = \frac{1-\eta}{\eta}t_0 = \left(\frac{1-\eta}{\eta}\right)\left(\frac{f_1}{f_1-f_0}\right)T. \tag{25}$$

This closed-form solution for the TOA bias β using the extended HFM matched filter is the main result of this paper.

It is worth asking if the methodology leading to (25) can be applied more generally. Although the Cauchy–Schwarz inequality can be invoked for any waveform, it appears that the special form of the HFM signal allows one to find a way to make the integrand independent of the variable of integration. The author has been unable to employ a similar strategy using a linear frequency modulation (LFM) signal. The decomposition in (17) raises the possibility that amplitude modulation (or simple shading) of an HFM signal might be considered. Since the length of the matched filter (extended) does not match the length of the transmitted signal, it would be difficult (perhaps impossible) to apply an amplitude modulation to both signals and have them remain matched after the echo is dilated. Even for the special case when the transmitted signal is shaded nonuniformly (amplitude modulation that is nonnegative for all t), it appears that the left-hand side of (20) does not, in general, reach the value of the right-hand side.

The exact extended HFM bias β differs slightly from an approximation of the bias for conventional HFM matched filters that was derived using an instantaneous frequency-based model [3]. Accounting for differences in notation, which include an opposite sign for v, the previously published approximate bias is

$$\tilde{\beta} \approx \left(\frac{1-\eta}{\eta}\right)\left(\frac{f_c}{f_1-f_0}\right)T \tag{26}$$

where

$$f_c = \frac{f_0+f_1}{2} \tag{27}$$

is the center frequency of the HFM sweep. To compare (25) and (26), it is useful to introduce the fractional bandwidth $\gamma = (f_1-f_0)/(2f_c) = (f_1-f_0)/(f_1+f_0)$, and rewrite (25) and (26) as

$$\beta = \left(\frac{1-\eta}{\eta}\right)\left(\frac{1}{2\gamma}+\frac{1}{2}\right)T \tag{28}$$

and

$$\tilde{\beta} \approx \left(\frac{1-\eta}{\eta}\right)\left(\frac{1}{2\gamma}\right)T. \tag{29}$$

The ratio $\beta/\tilde{\beta} \approx 1+\gamma$ is nearly unity when the fractional bandwidth is small (γ near zero) indicating the two expressions are in close agreement for narrowband cases, but approaches a value of two for broadband cases (γ near unity). This suggests that the approximation $\tilde{\beta}$ is valid only for narrowband HFM signals and is consistent with the fact that the broadband model used in its derivation [3] ignores the time dilation of the echo. The bias β in (25) is exact (for an extended HFM matched filter) and applies to both broadband and narrowband signals. Fig. 2 compares the exact extended HFM bias β with the approximation $\tilde{\beta}$ for a 1-s long sweep from 500 to 1000 Hz. This waveform has a relatively high fractional bandwidth ($\gamma = 1/3$), resulting in noticeable differences between the exact solution and the approximation.

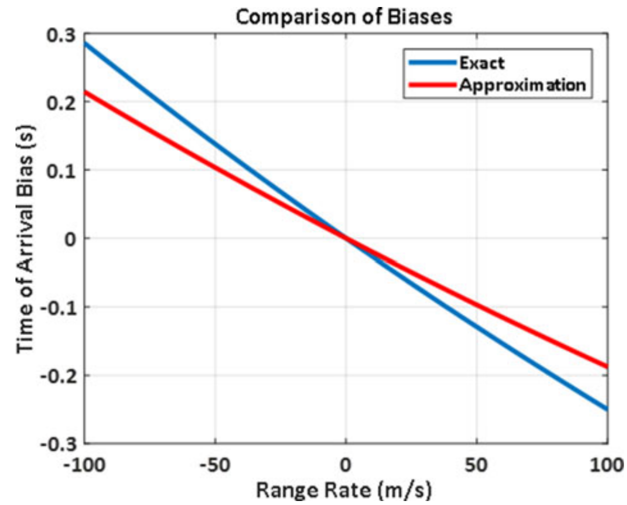

Fig. 2. Comparison of the exact extended HFM bias with the approximation [3].

D. Design of Extended HFM Matched Filter

What remains is to verify the assumption that T_1 and T_2 can be made large enough to be neglected in the limits of integration in (12). When T_1 is selected to ensure that $t_{\max} - \tau - T/\eta \geq -T_1$ the lower limit matches the analysis of the extended HFM matched filter response in (15). Plugging $t_{\max}$ into this constraint yields

$$\frac{1-\eta}{\eta}t_0 - \frac{T}{\eta} \geq -T_1 \tag{30}$$

which after substituting for t_0 and performing some minor algebraic manipulations becomes

$$T_1 \geq \left(\frac{f_1 - f_0/\eta}{f_1-f_0}\right)T. \tag{31}$$

Similar analysis of the upper limit produces the constraint on T_2 that ensures $t_{\max} - \tau \leq T_2$

$$T_2 \geq \left(\frac{f_1/\eta - f_1}{f_1-f_0}\right)T. \tag{32}$$

Consequently, any T_1 and T_2 that are large enough to satisfy the above inequalities will ensure that at the peak magnitude response, the limits of integration match the analysis for the extended HFM matched filter in (15), and the closed-form solution for the HFM bias in (25) applies.

The minimum duration required to ensure that the maximum found in (16) through (23) is preserved occurs when T_1 and T_2 exactly meet their respective conditions in (31) and (32). Using (3) to evaluate the instantaneous frequency of the extended HFM matched filter—which is reversed in time compared to the transmitted signal $s(t)$—at these points shows that

$$f_s\left(T_1\right) = \eta f_1 \tag{33}$$

and

$$f_s\left(-T_2\right) = \eta f_0. \tag{34}$$

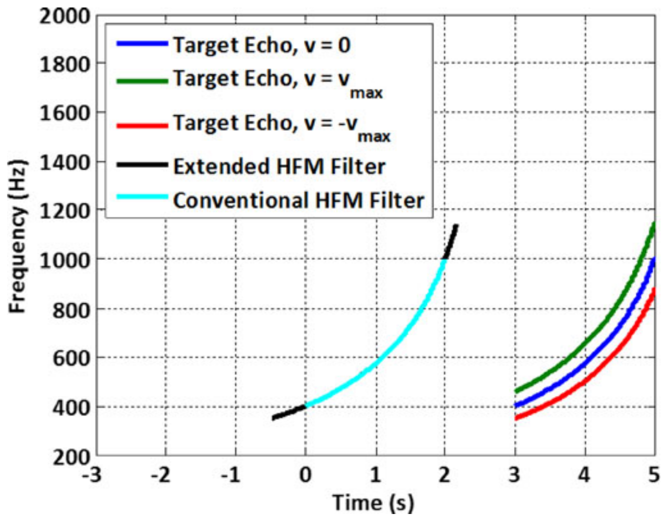

Fig. 3. Designing the extended HFM matched filter to encompass plausible target speeds.

Thus, the minimum duration limits on the extended HFM matched filter to preserve the maximum magnitude response (and the closed-form solution for the bias) exactly match the starting and ending frequencies of the Doppler dilated echo—that is, it is an exact matched filter for the dilated echo.

Consequently, if a range of plausible target speeds, say $|v| \leq v_{\max}$, is known in advance, then the extended HFM matched filter in (11) can be designed to encompass all possible echoes in this range by choosing T_1 and T_2 so that

$$f_s\left(T_1\right) = \eta_{\max} f_1 = \left(\frac{c + v_{\max}}{c - v_{\max}}\right) f_1 \tag{35}$$

and

$$f_s\left(-T_2\right) = \eta_{\min} f_0 = \left(\frac{c - v_{\max}}{c + v_{\max}}\right) f_0. \tag{36}$$

Equations (35) and (36) assume that $f_1 > f_0$, but similar conditions, which are easily derived, apply when $f_1 < f_0$.

Fig. 3 illustrates the instantaneous frequency of the various signals of importance: A zero Doppler echo, an echo from a target closing at speed $v_{\max}$, an echo from a target opening at speed $v_{\max}$, a conventional (zero Doppler) HFM matched filter, and an extended matched filter designed to encompass all plausible echo frequencies. Each of the echoes shown arrives at $t = 3$ s.

The intuitively pleasing result is that when the limits of the extended HFM matched filter are chosen so that the instantaneous frequency sweeps from the minimum plausible echo frequency $\eta_{\min} f_0$ to the maximum plausible echo frequency $\eta_{\max} f_1$ (based on the known range of target speeds and the transmitted HFM pulse), the bias in the magnitude squared extended HFM matched filter response is exactly

$$\left(\frac{1 - \eta}{\eta}\right)\left(\frac{f_1}{f_1 - f_0}\right) T \tag{37}$$

for any dilation factor η within the range $\eta_{\min} \leq \eta \leq \eta_{\max}$.

The TOA bias in (25) can be used as an engineering approximation to the bias that is observed for a conventional HFM matched filter. As an example, the upper panel of Fig. 4 shows the TOA bias for the extended HFM matched filter as a function of echo range rate v. The bias is evaluated numerically for the same broadband HFM sweep as in Fig. 2 (sweeping from 500 to 1000 Hz in 1 s) and lies exactly on the curve predicted by (25)—which appears in Fig. 2. The TOA bias for a conventional HFM matched filter, also evaluated numerically, is indistinguishable from the extended HFM curve given the scale of the plot. In both cases, a high-resolution (10 ns) matched filter is applied to an ideal echo of known arrival time and known dilation. The measured arrival time corresponds to the peak of the squared envelope. The lower panel of Fig. 4 shows the error between the bias predicted by (25) and the bias numerically computed for the conventional HFM matched filter. The error in approximating the conventional HFM bias with the extended HFM analytical result is several orders of magnitude smaller than value of the bias. Near zero, the error function is very steep and very nearly linear in the range rate.

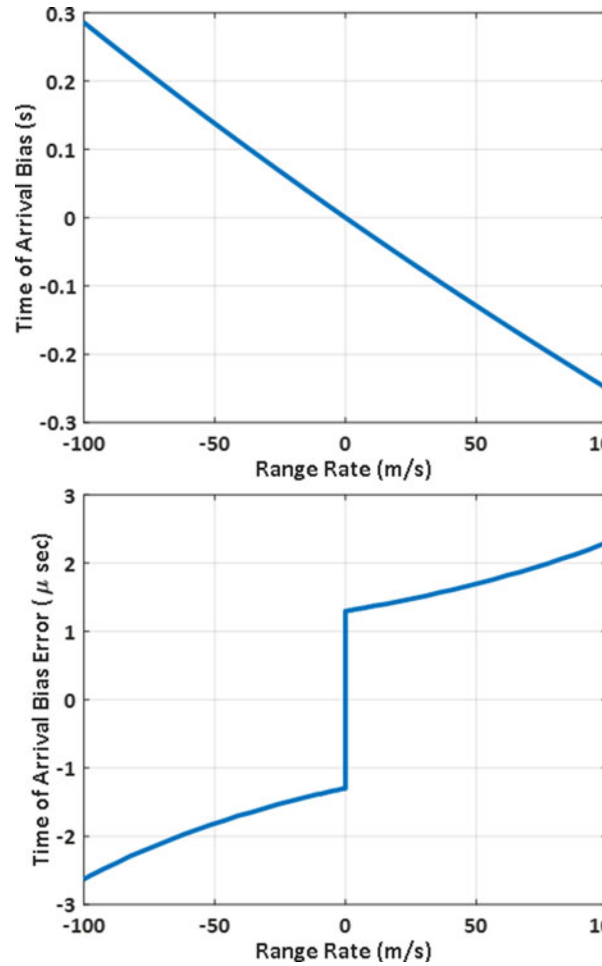

Fig. 4. Numerically evaluated TOA bias for the extended HFM matched filter (top), and the error between the exact analytical expression for the extended HFM TOA bias and the observed TOA bias for a conventional HFM matched filter (bottom).

III. Conclusion

A closed-form solution for the TOA bias of HFM echoes due to Doppler effects when using the extended HFM matched filter has been presented. This solution enables exact compensation of the bias when the target range rate is known, and describes exactly the peak value of the extended HFM waveform ambiguity function for any range rate v, and is applicable to both broadband and narrowband HFM signals. The author has been unsuccessful in applying the same technique to develop closed-form expressions for the bias in LFM signals. The solution does not strictly apply for nonuniformly shaded HFM waveforms, though for reasonable shadings it seems unlikely that the bias will change significantly. The solution does not strictly apply for conventional HFM matched filtering (where the matched filter is not extended), but is a very good approximation over a wide range of chirp parameters.

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