NASA TECHNICAL NOTE

NASA TN D-6218

NASA TN D-6218

A METHOD FOR REDUCING THE SENSITIVITY OF OPTIMAL NONLINEAR SYSTEMS TO PARAMETER UNCERTAINTY

by

Jarrell R. Elliott
Langley Research Center

and

William F. Teague
University of Kansas

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JUNE 1971

1. Report No. NASA TN D-6218	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle A METHOD FOR REDUCING THE SENSITIVITY OF OPTIMAL NONLINEAR SYSTEMS TO PARAMETER UNCERTAINTY		5. Report Date June 1971
		6. Performing Organization Code
7. Author(s) Jarrell R. Elliott and William F. Teague (University of Kansas)		8. Performing Organization Report No. L-7485
9. Performing Organization Name and Address NASA Langley Research Center Hampton, Va. 23365		10. Work Unit No. 125-17-06-03
		11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Technical Note
		14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

Mathematical relationships are derived and used to establish a procedure for reshaping the optimal solution so as to reduce the statistical uncertainty in the terminal conditions of the system due to system parameter uncertainties of known statistical properties. The procedure introduces the use of an augmented performance index which contains a scalar measure of the system sensitivity partial derivatives. A nonlinear multiparameter optimal-rocket-trajectory problem was solved by using an algorithm based on the method of steepest descent to illustrate the procedure.

17. Key Words (Suggested by Author(s)) Sensitivity Optimal control Nonlinear systems	18. Distribution Statement Unclassified — Unlimited		
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 40	22. Price* $3.00

A METHOD FOR REDUCING THE SENSITIVITY
OF OPTIMAL NONLINEAR SYSTEMS
TO PARAMETER UNCERTAINTY

By Jarrell R. Elliott
Langley Research Center
and
William F. Teague
University of Kansas

SUMMARY

The parameters of a nonlinear dynamical system which is to be controlled optimally are not always accurately known. It may therefore be desirable to accept a reduction in the predicted nominal performance of the system in exchange for the ability to better predict the outcome of the system, or plant, operation.

In this paper relationships to predict mathematically the sensitivity of the system to parameter errors are derived and used to establish a procedure for reshaping the optimal solution to reduce the statistical uncertainty in the terminal conditions of the system due to known statistical characteristics of the system parameters. The procedure requires the introduction of an augmented performance index which is a linear combination of the original performance index and a positive scalar measure of the system sensitivity composed of the weighted sum of the variances of the performance index and terminal constraints. The augmented performance index, the sensitivity partial derivatives, and the original state variables are used in the formulation of a new, higher dimensioned optimization problem of the same form as the original problem. The new problem introduces certain weighting factors which permit different relative importance to be attached to different types of sensitivity, such as position relative to velocity, and which allow for adjustment of performance degradation and of sensitivity reduction.

The procedure developed was illustrated by solving a nonlinear multiparameter rocket-trajectory problem. An algorithm based on the method of steepest descent was used to solve the problem because of the widespread use and proven versatility of this numerical technique. The example solutions serve to show the tradeoffs made possible by changing the weighting factors and to illustrate the radically different solutions one can obtain when sensitivity considerations are included in the problem formulation.

INTRODUCTION

In calculating the open-loop control time history of a physical process or plant, it is common to assume that the plant behavior or outcome can be predicted with the aid of a mathematical model of the plant with known or deterministic values of the plant parameters. More often than not, however, the plant parameters are stochastic, not deterministic, and serious discrepancies between predicted outcome and actual outcome may occur as a result of plant-parameter variations of one of two types: (1) parameter variations during plant operation, or (2) errors in the estimates of fixed plant parameters. These effects (that is, the sensitivity of a dynamical process to parameter variations) should be a consideration in the design of any control system.

While many studies related to plant sensitivity have been reported in the literature (see refs. 1 to 14 for a nonexhaustive list), most of them deal with linear problems. It also appears that little effort has been devoted to computational aspects (a notable exception is ref. 14). The present paper deals with the stochastic nature of the parameters in nonlinear problems and shows how a well-known computational algorithm in optimization theory may be applied to obtain numerical results.

The problem considered is a multivariable, multiparameter optimal control problem (with terminal constraints) whose mathematical model consists of a set of ordinary first-order nonlinear differential state equations. The analysis is limited to parameter variations of type (2). The sensitivity of the performance index and the sensitivity of terminal constraints to parameter variations, that is, the partial derivatives of these quantities with respect to the parameters, are used to construct a scalar measure of plant sensitivity. This sensitivity measure, which is the expected value of the weighted sum of the squares of the sensitivity partials, is multiplied by a weighting factor and added to the original performance index to form an augmented performance index. Minimization of this augmented performance index, with appropriate weighting factors, results in solutions which are less sensitive to parameter variations.

A simple nonlinear example problem, representative of rocket flight in a uniform gravitational field, is worked out in detail to show the steps required in setting up and solving plant sensitivity problems and to illustrate how reducing the sensitivity of a plant to parameter variations can modify the open-loop control time history and state-variable time history of the plant. The example also serves the purpose of demonstrating the use of the cumputational algorithm in solving a typical problem.

SYMBOLS

A $n \times q$ sensitivity coefficient matrix, $\partial \underline{f} / \partial \underline{a}$

A_i partition of matrix A $(i=1,2,\ldots,n)$

$\underline{a}$ system parameter vector with components $a_1,\ldots,a_q$

$\bar{\underline{a}}$ mean (nominal) value of $\underline{a}$

C_D aerodynamic drag coefficient

D aerodynamic drag

dP "length" of control step

$(dP)^2$ control step-size measure

$E\{\}$ expectation of random variable $\{\}$

F $n \times n$ state coefficient matrix

F_i partition of matrix F $(i=1,2,\ldots,n)$

f_1,f_2,f_3 components of $\underline{f}$

$\underline{f}$ governing equations of state

$\hat{\underline{f}}$ governing equations of augmented state

g gravitational acceleration

$\underline{g}_i$ vector elements of augmented-state differential equations $(i=1,2,\ldots,n)$

$I_{\phi\phi}, I_{\psi\phi}, I_{\psi\psi}$ constant matrices used in steepest-descent algorithm and defined by equations (A16)

J_A augmented performance index

3

K_D	modified drag coefficient
$\underline{L}$	vector set of terminal constraint values (in the example problem, L_1 and L_2 are the values for altitude and vertical velocity component, respectively)
l	number of terminal constraints
m	dimension of control vector $\underline{u}$; vehicle mass
m_0	initial vehicle mass
$\dot{m}$	time rate of change of mass
n	dimension of state vector $\underline{x}$
$\hat{n}$	number of augmented-state variables
P	covariance matrix
q	dimension of system parameter vector
S	sensitivity matrix, $\partial \underline{x} / \partial \underline{a}$; frontal surface area
S_i	partition of matrix S $(i=1,2,...,n)$
T	vehicle thrust
t	time (independent variable)
t_f	final time
t_0	initial time
u	horizontal velocity component, $\dot{x}_3$
$\underline{u}$	m-dimensional control vector
V	total velocity magnitude
v	vertical velocity component, $\dot{x}_4$
W	$m \times m$ (symmetric) control weighting matrix
w_i	relative weighting factors $(i=1,2,...,l+1)$

4

w_s	sensitivity weighting factor
$\underline{x}$	n-dimensional state vector
$\underline{\hat{x}}$	ñ-dimensional augmented-state vector
$\hat{x}_2, \hat{x}_3$	components of $\underline{\hat{x}}$
x	range (with numerical subscript denoting a specific state variable)
y	altitude
γ	flight-path angle
θ	thrust-attitude control angle
λ_ϕ	n-dimensional adjoint vector associated with payoff
Λ	$(l + 1) \times n$ variable matrix integrating factor
Λ_ψ	$l \times n$ matrix of adjoint variables associated with terminal constraints
μ	constant Lagrange multiplier
$\underline{\nu}$	l-dimensional constant Lagrange multiplier vector
ρ	atmospheric density
ϕ	sensitivity performance index
ϕ_p	performance index (not including sensitivity)
ϕ_s	sensitivity measure
$\underline{\psi}$	l-dimensional terminal constraint vector
ψ_i	components of $\underline{\psi}$ (i=1,2,...,l)
$(\ )^T$	transpose of matrix ( )

A dot over a symbol indicates a derivative with respect to time. The symbol δ denotes a variation in a quantity, and Δ denotes a first-order perturbation.

PROBLEM STATEMENT

Consider the following fixed-time open-loop optimal control problem:
Minimize

$$\phi_p = \phi_p\left(\underline{x}(t_f)\right)$$

for the system

$$\underline{\dot{x}} = \underline{f}\left(\underline{x},\underline{u},\underline{a},t\right) ; \quad \underline{x}\left(t_o\right) = \underline{x}_o \quad \text{(given)}$$

with terminal constraints

$$\underline{\psi} - \underline{L} = 0$$

where

ϕ_p is a scalar performance index,

$\underline{x}$ is an n-dimensional state vector,

$\underline{f}$ is an n-dimensional state derivative vector,

$\underline{u}$ is an m-dimensional control vector,

$\underline{a}$ is a q-dimensional parameter vector, whose mean and covariance matrix are known $\left(E\{\underline{a}\} = \underline{\bar{a}}; \quad E\{\delta a \; \delta a^T\} = P\right)$, and

$\underline{\psi}$ is an l-dimensional terminal constraint vector.

$\left(\text{All vectors are column vectors unless superscripted by } T \text{ which indicates vector or matrix transpose, except } \dfrac{\partial(\text{scalar})}{\partial(\text{vector})}, \text{ which is a row vector.}\right)$

It is desired to combine some measure of sensitivity ϕ_s to ϕ_p such that when the combination is minimized, the performance index ϕ_p and the terminal constraints $\underline{\psi}$ will be less sensitive to perturbations in the parameters $\underline{a}$. It should be realized that this reduction in sensitivity will degrade the performance index ϕ_p. However, since a system may have more performance capability than is needed, a system user may be willing to sacrifice some performance in order to reduce terminal perturbations or

6

errors in ϕ_p and $\underline{\psi}$. In order to provide different degrees of performance loss and sensitivity reduction, a weighting constant is put on the sensitivity measure ϕ_s, and the sensitivity performance index to be minimized is then defined as

$$\phi = \phi_p + w_s \phi_s$$

The sensitivity measure should reflect the fact that the system is stochastic. The sensitivity measure will therefore be taken to be an expected value of the weighted sum of the squares of the perturbations in ϕ_p and each ψ_i due to parameter variations. (It is necessary to square the perturbation in ϕ_p and in each ψ_i $(i=1,2,\ldots,l)$ to insure that the measure is positive). Now since the parameter variations are of type (2) described in the introduction, the perturbations to first order are

$$\Delta \phi_p = \frac{\partial \phi_p}{\partial \underline{a}} \, \delta \underline{a}$$

and

$$\Delta \psi_i = \frac{\partial \psi_i}{\partial \underline{a}} \, \delta \underline{a} \qquad\qquad (i=1,2,\ldots,l)$$

The squares of these perturbations may be expressed as

$$\left(\Delta \phi_p\right)^2 = \frac{\partial \phi_p}{\partial \underline{a}} \, \delta \underline{a} \, \delta \underline{a}^T \left(\frac{\partial \phi_p}{\partial \underline{a}}\right)^T$$

and

$$\left(\Delta \psi_i\right)^2 = \frac{\partial \psi_i}{\partial \underline{a}} \, \delta \underline{a} \, \delta \underline{a}^T \left(\frac{\partial \psi_i}{\partial \underline{a}}\right)^T \qquad\qquad (i=1,2,\ldots,l)$$

The sensitivity measure contains certain weighting factors, to be called relative weighting factors, which are required because one Δ^2 quantity may be considered more, or less, important than another (for example, one may be more concerned with velocity errors than position errors in some application although both are used as terminal constraints). Finally, the sensitivity measure is

$$\phi_s = E\left\{ w_1 \left(\Delta \phi_p\right)^2 + w_2 \left(\Delta \psi_1\right)^2 + \cdots + w_{l+1} \left(\Delta \psi_l\right)^2 \right\}$$

$$= E\left\{ w_1 \frac{\partial \phi_p}{\partial \underline{a}} \, \delta \underline{a} \, \delta \underline{a}^T \left(\frac{\partial \phi_p}{\partial \underline{a}}\right)^T + w_2 \frac{\partial \psi_1}{\partial \underline{a}} \, \delta \underline{a} \, \delta \underline{a}^T \left(\frac{\partial \psi_1}{\partial \underline{a}}\right)^T + \cdots + w_{l+1} \frac{\partial \psi_l}{\partial \underline{a}} \, \delta \underline{a} \, \delta \underline{a}^T \left(\frac{\partial \psi_l}{\partial \underline{a}}\right)^T \right\}$$

$$(1)$$

On a trajectory defined by some open-loop control time history, the partial derivatives inside the expected value are constant. By well-known identities involving the expected value (see ref. 15), equation (1) may be rewritten as

$$\phi_s = w_1 \frac{\partial \phi_p}{\partial \underline{a}} E\left\{\delta \underline{a}\ \delta a^T\right\}\left(\frac{\partial \phi_p}{\partial \underline{a}}\right)^T + w_2 \frac{\partial \psi_1}{\partial \underline{a}} E\left\{\delta \underline{a}\ \delta \underline{a}^T\right\}\left(\frac{\partial \psi_1}{\partial \underline{a}}\right)^T + \ldots$$

$$+ w_{l+1} \frac{\partial \psi_l}{\partial \underline{a}} E\left\{\delta \underline{a}\ \delta a^T\right\}\left(\frac{\partial \psi_l}{\partial \underline{a}}\right)^T$$

Since $E\left\{\delta a\ \delta a^T\right\} \equiv P$, this equation becomes

$$\phi_s = w_1 \frac{\partial \phi_p}{\partial \underline{a}} P\left(\frac{\partial \phi_p}{\partial \underline{a}}\right)^T + w_2 \frac{\partial \psi_1}{\partial \underline{a}} P\left(\frac{\partial \psi_1}{\partial \underline{a}}\right)^T + \ldots + w_{l+1} \frac{\partial \psi_l}{\partial \underline{a}} P\left(\frac{\partial \psi_l}{\partial \underline{a}}\right)^T \tag{2}$$

where P is the covariance matrix of the random parameters which is assumed to be known along with the mean value, that is

$$E\left\{\underline{a}\right\} = \overline{\underline{a}}$$

This expression for ϕ_s (eq. (2)) may be recognized as the weighted sum of the variances of the perturbations in performance and constraints. Finally the sensitivity performance index for minimization is

$$\phi = \phi_p\left(\underline{x}(t_f)\right) + w_s \phi_s\left(\frac{\partial \phi_p}{\partial \underline{a}}, \frac{\partial \psi}{\partial \underline{a}}\right) \tag{3}$$

Now return to the problem of computing $\Delta \phi_p$ and $\Delta \psi_i$. Since these perturbations are explicitly functions of the terminal state, they may be written as

$$\left.\begin{aligned} \Delta \phi_p &= \frac{\partial \phi_p}{\partial \underline{a}} \delta \underline{a} = \frac{\partial \phi_p}{\partial \underline{x}} \frac{\partial \underline{x}}{\partial \underline{a}} \delta \underline{a} \\[2em] \Delta \psi_i &= \frac{\partial \psi_i}{\partial \underline{a}} \delta \underline{a} = \frac{\partial \psi_i}{\partial \underline{x}} \frac{\partial \underline{x}}{\partial \underline{a}} \delta \underline{a} \qquad\qquad (i=1,2,\ldots,l) \end{aligned}\right\} \tag{4}$$

The remaining problem is that of determining $\frac{\partial \underline{x}}{\partial \underline{a}}(t_f)$. In reference 16 it is shown that for parameter variations of type (2) and for the system differential equations

$$\underline{\dot{x}} = \underline{f}\left(\underline{x}, \underline{u}(t), \underline{a}, t\right) = \underline{f}\left(\underline{x}, \underline{a}, t\right), \quad \frac{\partial \underline{x}}{\partial \underline{a}}(t) \text{ is the solution of}$$

8

$$\frac{d}{dt}\left(\frac{\partial \underline{x}}{\partial \underline{a}}\right) = \frac{\partial \underline{f}}{\partial \underline{a}}\frac{\partial \underline{x}}{\partial \underline{a}} + \frac{\partial \underline{f}}{\partial \underline{a}}; \quad \frac{\partial \underline{x}}{\partial \underline{a}}(t_0) = 0 \tag{5}$$

By defining

$$S \equiv \frac{\partial \underline{x}}{\partial \underline{a}}$$

$$F \equiv \frac{\partial \underline{f}}{\partial \underline{x}}$$

and

$$A \equiv \frac{\partial \underline{f}}{\partial \underline{a}}$$

equation (5) may be written as

$$\frac{d}{dt}S = FS + A; \quad S(t_0) = 0 \tag{6}$$

Then ϕ_S becomes

$$\phi_S = w_1 \frac{\partial \phi_p}{\partial \underline{x}}S(t_f)PS^T(t_f)\left(\frac{\partial \phi_p}{\partial \underline{x}}\right)^T + w_2 \frac{\partial \psi_1}{\partial \underline{x}}S(t_f)PS^T(t_f)\left(\frac{\partial \psi_1}{\partial \underline{x}}\right)^T + \ldots$$

$$+ w_{l+1} \frac{\partial \psi_l}{\partial x}S(t_f)PS^T(t_f)\left(\frac{\partial \psi_l}{\partial \underline{x}}\right)^T \tag{7}$$

A simple, easy-to-work-with form for ϕ_S may be determined by reordering and possibly introducing some new variables such that

$$\phi_p\left(\underline{x}(t_f)\right) = x_1$$

and

$$\psi_i\left(\underline{x}(t_f)\right) = x_{i+1} \qquad (i=1,2,\ldots,l)$$

This procedure may, if new variables are required, introduce a new system of differential equations. The notation used will remain unchanged, however, and the system of differential equations will continue to be called $\underline{\dot{x}} = \underline{f}$.

Now partition S, where S_i^T (i=1,2,...,n) denotes the rows of S, as follows:

$$S = \begin{bmatrix} S_1{}^T \\ \hline S_2{}^T \\ \hline \vdots \\ \hline S_n{}^T \end{bmatrix}$$

Then

$$\phi = \left(x_1 + w_1 S_1{}^T P S_1 + w_2 S_2{}^T P S_2 + \ldots + w_{l+1} S_{l+1}{}^T P S_{l+1}\right)\Big|_{t_f}$$

Note that $S_i{}^T = F_i{}^T S + A_i{}^T$; F and A have been partitioned in the same manner as S, that is,

$$F = \begin{bmatrix} F_1{}^T \\ \hline F_2{}^T \\ \hline \vdots \\ \hline F_n{}^T \end{bmatrix} \quad ; \quad A = \begin{bmatrix} A_1{}^T \\ \hline A_2{}^T \\ \hline \vdots \\ \hline A_n{}^T \end{bmatrix}$$

Observe that this optimization problem involving the sensitivity index has the same form as the original problem involving only the performance index (but with more dimensions) and may be solved by any of several algorithms. Restated, the problem is to minimize

$$\phi = \phi\left(x_1, S_1, S_2, \ldots, S_{l+1}\right) \tag{8}$$

subject to the differential constraints

$$\left. \begin{aligned} \dot{\underline{x}} &= \underline{f} = \underline{f}\left(\underline{x}, \underline{u}, \underline{\bar{a}}, t\right) \; ; \quad \underline{x}\left(t_o\right) = \underline{x}_o \\ \dot{S}_1 &= S^T F_1 + A_1 = \underline{g}_1\left(\underline{x}, S_i, \underline{u}, \underline{\bar{a}}, t\right) \; ; \quad S_1\left(t_o\right) = 0 \\ & \vdots \\ \dot{S}_n &= S^T F_n + A_n = g_n\left(\underline{x}, S_i, \underline{u}, \underline{\bar{a}}, t\right) \; ; \quad S_n\left(t_o\right) = 0 \end{aligned} \right\} \tag{9}$$

10

and the terminal conditions

$$\underline{\psi}\left(\underline{x}(t_f)\right) = \underline{L} \tag{10}$$

The augmented-state terms are defined as follows:

$$\left.\begin{aligned}
\underline{\hat{x}}^T &= \left[\underline{x}^T \;\middle|\; S_1^{\;T} \;\middle|\; S_2^{\;T} \;\middle|\; \ldots \;\middle|\; S_n^{\;T}\right] \\
\underline{\hat{f}}^T &= \left[\underline{f}^T \;\middle|\; \underline{g}_1^{\;T} \;\middle|\; \underline{g}_2^{\;T} \;\middle|\; \ldots \;\middle|\; \underline{g}_n^{\;T}\right]
\end{aligned}\right\} \tag{11}$$

Then the differential constraints are $\dfrac{dx}{dt} = \underline{\hat{f}}\left(\underline{\hat{x}},\underline{u},\underline{\bar{a}},t\right)$. Thus it may be seen that the only difference between this problem and the original problem is that it is now possible to control, at the expense of performance index ϕ_p, the sensitivity of the trajectory, and the problem is now an $n(1 + q)$-dimensional state-variable problem rather than an n-dimensional one. Also several weighting factors $\left(w_s \text{ and } w_1, w_2, \ldots, w_{l+1}\right)$ have been introduced into the problem. These weighting factors are to be used in applications to control the loss in the performance index ϕ_p and to properly emphasize the importance of one type of terminal error relative to another type. The contribution and effect of these weighting factors will be illustrated in an example problem.

It should be pointed out that solving the differential equations for the sensitivity partials gives all the information needed to perform a first-order error analysis on any of the state variables of the original system. In the example problem, a comparison will be made between first-order error analysis results obtained by using the sensitivity partials and first-order error analysis results for which one-sigma errors were introduced one at a time into the equations of motion.

The discussion herein deals with the fixed-time problem. However, fixed time was used for convenience and clarity only and is not a limitation of the formulation.

While almost any of the algorithms for solving optimization problems would be applicable here, it was decided to use the steepest-descent algorithm as outlined in reference 16 because of its widespread use and proven versatility. The procedure used in this algorithm is to linearize about a solution $\underline{\hat{x}}^*(t)$ provided by some reasonably chosen control time history $\underline{u}^*(t)$ (which in general neither minimizes ϕ nor satisfies $\underline{\psi} - \underline{L} = 0$) and to solve for $\delta\underline{u}(t)$ $\left(\text{which improves } \phi(t_f) \text{ and } \underline{\psi}(t_f)\right)$. A new solution provided by $\underline{u}^*(t) + \delta\underline{u}(t)$ is obtained and the procedure is repeated successively until $\underline{\psi} - \underline{L}$ is sufficiently close to zero and $\phi(t_f)$ can no longer be decreased. This final solution is said to be optimal although no necessary conditions for optimality have been

satisfied. A brief derivation of the necessary relationships with notation common to many steepest-descent programs is provided in appendix A. The algorithm was applied to the following example problem.

A NUMERICAL EXAMPLE

Problem Statement

The example problem is a fixed-time problem in which it is required to determine the thrust-attitude program of a single-stage rocket vehicle starting from rest and going to specified terminal conditions of altitude and vertical velocity which will maximize the final horizontal velocity. The idealizing assumptions made are the following:

(1) A point-mass vehicle

(2) A flat, nonrotating earth

(3) A constant-gravity field, $g = 9.8$ m/sec^2 (32.2 ft/sec^2)

(4) Constant thrust and mass-loss rate

(5) A nonlifting body in a nonvarying atmosphere with a constant drag parameter $K_D = \frac{1}{2}\rho C_D S$, where S is the frontal surface area.

The coordinate system and pertinent geometric relations and terms are shown in figure 1. The differential equations of motion needed in the algorithm setup are

$$\left.\begin{array}{l} \dfrac{du}{dt} = \dfrac{1}{m}\left(T \cos \theta - K_D uV\right) = \dot{x}_1 = f_1 \\[2mm] \dfrac{dy}{dt} = v = \dot{x}_2 = f_2 \\[2mm] \dfrac{dv}{dt} = \dfrac{1}{m}\left(T \sin \theta - K_D uV\right) - g = \dot{x}_3 = f_3 \end{array}\right\} \tag{12}$$

where m is the vehicle mass and where

$$V = \sqrt{u^2 + v^2}$$

and

$$m(t) = m_0 + \dot{m}t$$

An equation for dx/dt is not included in the equations of motion becuase x does not enter into the problem. However the equation $\dot{x} = u$ was integrated separately to obtain range. The equation for dm/dt is analytically integrable, since dm/dt is a constant, and is therefore not included in the set of differential equations.

The parameters of the problem, which will be considered fixed constants whose precise values are unknown, are thrust level, mass-loss rate, initial vehicle mass, and modified drag coefficient. The parameter vector is defined as

$$\underline{a}^T = \left[T, \dot{m}, m_0, K_D\right]$$

These parameters are assumed to be statistically independent and to have a normal distribution function with mean and one-sigma values as shown in table I.

TABLE I.- SYSTEM PARAMETER VALUES

Parameter	Mean value	One-sigma value, fraction of mean value
Thrust, T, kN (lbf)	17.8 (4000)	0.0067
Mass-loss rate, $\dot{m}$, kg/sec (slugs/sec)	-9.1 (-0.62)	0.0167
Initial mass, m_0, kg (slugs)	1433.6 (98.259)	0.010
Modified drag coefficient, K_D, kg/m (slugs/ft)	0.048 (0.001), 0.479 (0.01)	0.0167

These values are considered representative of the current state of knowledge in solid rocketry. (The mean values used were chosen to conform to a flight of 100 seconds for a rocket with specific impulse of 200 seconds, an average acceleration $\frac{1}{100}\int_{t=0}^{100} \frac{T}{m(t)}\,dt$ of 2g, and a ratio of initial to final mass $m(t_0)/m(t_f)$ equal to e, the base of the natural logarithm.)

The boundary conditions at $t = 0$ and $t = 100$ seconds are shown in table II.

TABLE II.- BOUNDARY CONDITIONS

Variable	Initial conditions (t = 0)	Terminal conditions (t = 100 seconds)
u, m/sec (ft/sec)	0	Maximum
y, m (ft)	0	15 240 (50 000)
v, m/sec (ft/sec)	0	0

In the notation previously introduced

13

$$\phi_p(t_f) = -u(t_f)$$

$$\psi_1 = y(t_f) - L_1 = 0 \; ; \quad L_1 = 15\,240 \text{ m } (50\,000 \text{ ft})$$

$$\psi_2 = v(t_f) - L_2 = 0 \; ; \quad L_2 = 0 \text{ m/sec } (0 \text{ ft/sec})$$

$$\underline{x}^T = \begin{bmatrix} -u,y,v \end{bmatrix}$$

and the vector control variable $\underline{u}$ in the general formulation is the thrust-attitude angle θ. This completes the problem statement (without trajectory-sensitivity considerations).

Sensitivity Relations

The sensitivity-matrix differential equation is

$$\frac{dS}{dt} = FS + A$$

where

$$S = \begin{bmatrix} -\dfrac{\partial u}{\partial T} & -\dfrac{\partial u}{\partial \dot{m}} & -\dfrac{\partial u}{\partial m_o} & -\dfrac{\partial u}{\partial K_D} \\[2ex] \dfrac{\partial y}{\partial T} & \dfrac{\partial y}{\partial \dot{m}} & \dfrac{\partial y}{\partial m_o} & \dfrac{\partial y}{\partial K_D} \\[2ex] \dfrac{\partial v}{\partial T} & \dfrac{\partial v}{\partial \dot{m}} & \dfrac{\partial v}{\partial m_o} & \dfrac{\partial v}{\partial K_D} \end{bmatrix} = \begin{bmatrix} S_1^{\;T} \\ \hline S_2^{\;T} \\ \hline S_3^{\;T} \end{bmatrix} \tag{13}$$

$$F = -\frac{K_D}{m} \begin{bmatrix} V + \dfrac{u^2}{V} & 0 & -\dfrac{uv}{V} \\[2ex] 0 & 0 & -\dfrac{m}{K_D} \\[2ex] -\dfrac{uv}{V} & 0 & V + \dfrac{v^2}{V} \end{bmatrix} = \begin{bmatrix} F_1^{\;T} \\ \hline F_2^{\;T} \\ \hline F_3^{\;T} \end{bmatrix} \tag{14}$$

and

$$A = -\frac{1}{m} \begin{bmatrix} \cos\theta & tf_1 & f_1 & -uV \\[2ex] 0 & 0 & 0 & 0 \\[2ex] -\sin\theta & t(f_3 + g) & (f_3 + g) & vV \end{bmatrix} = \begin{bmatrix} A_1^{\;T} \\ \hline A_2^{\;T} \\ \hline A_3^{\;T} \end{bmatrix} \tag{15}$$

14

By using the partition notation indicated, differential equations for the row vectors may be written as

$$\frac{dS_i{}^T}{dt} = F_i{}^T S + A_i{}^T \qquad (i=1,2,3)$$

or in column vector form

$$\frac{dS_i}{dt} = S^T F_i + A_i \qquad (i=1,2,3) \qquad (16)$$

The sensitivity measure is

$$\phi_S = E\left\{ w_1\, \Delta u^2(t_f) + w_2\, \Delta y^2(t_f) + w_3\, \Delta v^2(t_f) \right\} \qquad (17)$$

which may be written as

$$\phi_S = \sum_{i=1}^{3} w_i S_i{}^T P S_i(t_f)$$

where

$$P = \begin{bmatrix} \left[\left(\frac{2}{3}\right)\left(\frac{T}{100}\right)\right]^2 & 0 & 0 & 0 \\[2ex] 0 & \left[\left(\frac{5}{3}\right)\left(\frac{\dot{m}}{100}\right)\right]^2 & 0 & 0 \\[2ex] 0 & 0 & \left[\frac{m_o}{100}\right]^2 & 0 \\[2ex] 0 & 0 & 0 & \left[\left(\frac{5}{3}\right)\left(\frac{K_D}{100}\right)\right]^2 \end{bmatrix} \qquad (18)$$

is a diagonal covariance matrix due to the assumption of statistical independence in the parameters.

Problem Statement Including Sensitivity

The augmented-state vector is

$$\hat{\underline{x}}^T = \left[x^T \,\vdots\, S_1{}^T \,\vdots\, S_2{}^T \,\vdots\, S_3{}^T \right]$$

$$= \left[-u, y, v, \frac{\partial u}{\partial T}, \frac{\partial u}{\partial \dot{m}}, \frac{\partial u}{\partial m_o}, \frac{\partial u}{\partial K_D}, \frac{\partial y}{\partial T}, \cdots, \frac{\partial y}{\partial K_D}, \frac{\partial v}{\partial T}, \cdots, \frac{\partial v}{\partial K_D} \right]$$

15

with the initial conditions

$$\hat{\underline{x}}^T(t_o) = \begin{bmatrix} 0,0,. & . .,0 \end{bmatrix}$$

The problem restatement with sensitivity considerations is to minimize

$$\phi = \phi_p + w_s \phi_s$$

subject to

$$\frac{d\hat{\underline{x}}^T}{dt} = \hat{\underline{f}}^T(\hat{\underline{x}}, \theta, \overline{\underline{a}}, t) = \begin{bmatrix} \dfrac{dx^T}{dt} & \dfrac{dS_1^T}{dt} & \dfrac{dS_2^T}{dt} & \dfrac{dS_3^T}{dt} \end{bmatrix} ; \quad \hat{\underline{x}}(t_o) = 0 \tag{19}$$

and

$$\left. \begin{aligned} \psi_1 &= \hat{x}_2(t_f) - L_1 = 0 ; \quad L_1 = 15\,240 \text{ m} \quad (50\,000 \text{ ft}) \\ \psi_2 &= \hat{x}_3(t_f) - L_2 = 0 ; \quad L_2 = 0 \text{ m/sec} \quad (0 \text{ ft/sec}) \end{aligned} \right\} \tag{20}$$

Numerical Results

As previously mentioned, the steepest-descent algorithm of reference 17 was implemented to obtain numerical results. These results were computed with the use of a fourth-order integration subroutine with a fixed step size of 0.5 second. For this example problem each iteration in the solution required integration of 15 augmented state differential equations plus the forward integration of the range equation and backwards integration of 45 adjoint variable differential equations. Each iteration required about 22 seconds on the Control Data 6600 computer system used with no special effort having been made to keep computer run time down. Several check solutions were computed with the use of a 0.125-second fixed step size. These check solutions always agreed to at least five, and usually to at least seven, significant figures with those computed with the use of the 0.5-second step size.

Numerical results for a variety of cases were obtained by changing the values of K_D (the modified drag coefficient), w_s (the sensitivity weighting factor), and w_1, w_2, w_3 (the set of relative weighting factors). The different combinations for which results were obtained are shown in table III.

TABLE III.- CATALOG OF CASES

Relative-weighting-factor sets	K_D	w_s
Set (1): $w_1 = w_3 = 1$; $w_2 = 10^{-2}$	0	0, 0.1, 0.396, 1.0, 10.0
Set (2): $w_1 = w_3 = 1$; $w_2 = 10^{-4}$	0	0, 0.1, 0.3, 1.0, 10.0
	0.001	0, 0.1, 1.0, 10.0

The relative-weighting-factor sets were chosen on the basis of the relative importance attached to velocity (w_1 applies to horizontal velocity and w_2 applies to vertical velocity) and position (w_3 applies to altitude errors). Set (1) gives equal weight to a 0.305-m/sec (1-ft/sec) velocity error and a 3.05-meter (10-foot) altitude error, while set (2) gives equal weight to a 0.305-m/sec (1-ft/sec) velocity error and a 30.5-meter (100-foot) altitude error.

The modified drag coefficient K_D was set at zero, simulating vacuum flight, and at 0.001, a representative value for small rockets in the earth's atmosphere. Setting K_D at zero reduced the number of parameters to three. It also made possible, for $w_s = 0$, an analytical calculus-of-variations solution for the optimal thrust-attitude time history; namely, the well known linear tangency law (ref. 18).

$$\tan \theta(t) = \alpha + \beta t$$

where α and β are constants determined by the boundary conditions of the problem. This calculus-of-variations solution was used to validate the steepest-descent-algorithm programing and solution. The comparison of results showed negligible differences.

Also, for $K_D = 0$, it can be shown that

$$\frac{\partial y}{\partial T} = \frac{1}{T}\left[y(t_f) + \frac{1}{2}(t_f)^2 g\right]$$

and

$$\frac{\partial v}{\partial T} = \frac{1}{T} v(t_f) + t_f g$$

Therefore, these partials will not change with w_s for $K_D = 0$.

As an independent check on the sensitivity partials, perturbations in u, v, and y at t_f due to plus and minus one-sigma errors in each parameter, taken one at a time, were computed. For example, the one-sigma (1σ) perturbation in u due to parameter errors was computed by using the relation

$$\Delta u_{1\sigma}^{(2)} = \frac{1}{2} \left\{ \left[u\left(t_f\right)_{+1\sigma_T} - u\left(t_f\right)_{-1\sigma_T} \right]^2 + \left[u\left(t_f\right)_{+1\sigma_{\dot{m}}} - u\left(t_f\right)_{-1\sigma_{\dot{m}}} \right]^2 \right.$$

$$\left. + \left[u\left(t_f\right)_{+1\sigma_{\dot{m}_O}} - u\left(t_f\right)_{-1\sigma_{\dot{m}_O}} \right]^2 + \left[u\left(t_f\right)_{+1\sigma_{K_D}} - u\left(t_f\right)_{-1\sigma_{K_D}} \right]^2 \right\}^{1/2} \tag{21}$$

where $u\left(t_f\right)_{+1\sigma_T}$ means u at t_f on a trajectory with T increased by its one-sigma value, where $u\left(t_f\right)_{-1\sigma_{\dot{m}}}$ means u at t_f on a trajectory with $\dot{m}$ decreased by its one-sigma value, and so forth. Similar computations were made for $\Delta v_{1\sigma}^{(2)}$ and $\Delta y_{1\sigma}^{(2)}$. One-sigma perturbation values computed by this method were compared with one-sigma values computed by using the sensitivity partial derivatives. For example, the one-sigma (1σ) perturbation in u was computed as follows by using the sensitivity partials:

$$\Delta u_{1\sigma}^{(1)} = \left[\left(\frac{\partial u}{\partial T} \delta T_{1\delta} \right)^2 + \left(\frac{\partial u}{\partial \dot{m}} \delta m_{1\sigma} \right)^2 + \left(\frac{\partial u}{\partial \dot{m}_O} \delta m_{O,1\sigma} \right)^2 + \left(\frac{\partial u}{\partial K_D} \delta K_D \right)^2 \right]^{1/2} \tag{22}$$

where $\delta T_{1\sigma}$, $\delta \dot{m}_{1\sigma}$ and so forth are one-sigma perturbations in the parameters. Similar computations were made for $\Delta v_{1\sigma}^{(1)}$ and $\Delta y_{1\sigma}^{(1)}$. Superscript (1) refers to 1σ perturbations obtained by using sensitivity partials, and superscript (2) indicates that the method of computation was to obtain an average perturbation value by assuming first plus and then minus 1σ variations in each parameter.

Discussion of Numerical Results

Numerical results for the converged trajectories are summarized in figures 2 to 11, which give control time histories and trajectory plots, and in tables that give sensitivity partials, one-sigma perturbation values, and ϕ_s and ϕ_p.

Figure 2 shows the control time history used initially and the converged control time history after 10 iterations for the conditions indicated $\left(K_D = 0; \ w_s = 0\right)$ to illustrate how a reasonably chosen control time history is modified by the algorithm to obtain an "optimal" solution. Optimal here means that a gradient function has been reduced several orders of magnitude or to some reasonable value. A plot of $\tan \theta(t)$ as a function of time was made for this converged trajectory, and the intercept and slope of a straight line

fairing to that plot are compared in table IV with the values of the constants α and β required in the calculus-of-variations (C.O.V.) solution.

TABLE IV.- LINEAR-TANGENCY-LAW CONSTANTS

Constants	Faired values	C.O.V. values
α	3.49	3.49
β	-0.0401	-0.0404

The agreement is considered to be good, in view of the crude fairing method and other factors relating to how the C.O.V. values of α and β were obtained. Also the trajectories showed good agreement.

Table V(a) and figures 3, 4, and 5 summarize converged numerical results for $K_D = 0$ and set (1) relative weighting factors for several values of the sensitivity weighting factors. The sensitivity partials of table V(a) show consistent trends, with altitude and horizontal-velocity sensitivity partials decreasing in magnitude while vertical-velocity sensitivity partials are increasing in magnitude with increases in w_S. These trends are also reflected in the root-sum-square perturbations; one-sigma (1σ) perturbation values of altitude and horizontal velocity decrease while those of vertical velocity increase. The percentage of the original value (value with $w_S = 0$) of $\Delta u_{1\sigma}$, $\Delta y_{1\sigma}$, $\Delta v_{1\sigma}$, ϕ_S, and ϕ_p is plotted in figure 3 against sensitivity weighting factor. For values of w_S larger than about 0.3, little change takes place in $\Delta v_{1\sigma}$, $\Delta y_{1\sigma}$, and ϕ_S, but the performance $\left(\phi_p\right)$ continues to degrade along with a decrease in $\Delta u_{1\sigma}$. The control time histories for these cases, shown in figure 4, exhibit an interesting charac-teristic. As w_S becomes larger, the control tends toward a bang-bang type of control where the thrust is either directed straight up (vertical) or straight down. While it appears that the steepest-descent program indicates the existence of a bang-bang optimal control law, attempts to predict this behavior analytically have been unsuccessful. Altitude is plotted against range in figure 5 where it may be observed that the trajectory becomes steeper as w_S increases.

Table V(b) and figures 6 to 9 summarize the results for $K_D = 0$ and set (2) rela-tive weighting factors. Set (2) puts less emphasis on the altitude sensitivity partials than set (1). Thus altitude sensitivity partials increase for set (2) rather than decrease as they did for set (1), and both horizontal and vertical-velocity partials decrease. Figure 6 clearly illustrates these results in the plots of $\Delta y_{1\sigma}$, $\Delta v_{1\sigma}$, and $\Delta u_{1\sigma}$. By comparing this figure with figure 3, it may be seen that $\Delta y_{1\sigma}$ and $\Delta v_{1\sigma}$ have switched positions on the plots. Also it appears that ϕ_S begins to level off and remain essentially con-stant at $w_S \approx 10.0$ in figure 6 whereas it leveled off and became essentially constant at

TABLE V.- SENSITIVITY PARTIALS AND RELATED INFORMATION

(a) $K_D = 0$; $w_1 = w_3 = 1$, $w_2 = 10^{-2}$ (Set (1))

Sensitivity partials and related terms (*)	Values for sensitivity weighting factor w_S, of —				
	0	0.1	0.396	1.0	10.0
$\partial u/\partial T$	1.106	0.973	0.372	0.163	0.020
$\partial u/\partial \dot{m}$	-6 322	-5 970	-2 044	-838	-101
$\partial u/\partial m_o$	-84.98	-77.34	-28.07	-11.93	-1.44
$\Delta u_{1\sigma}^{(1)}$	110.1	101.3	36.1	15.2	1.8
$\Delta u_{1\sigma}^{(2)}$	110.2	101.4	36.2	15.3	1.8
$\partial y/\partial T$	52.75	52.75	52.75	52.75	52.75
$\partial y/\partial \dot{m}$	-96 140	-82 950	-73 640	-72 790	-72 610
$\partial y/\partial m_o$	-2 755	-2 672	-2 613	-2 608	-2 606
$\Delta y_{1\sigma}^{(1)}$	3 209	3 100	3 025	3 019	3 017
$\Delta y_{1\sigma}^{(2)}$	3 210	3 100	3 025	3 019	3 017
$\partial v/\partial T$	0.805	0.805	0.805	0.805	0.805
$\partial v/\partial \dot{m}$	-1 419	-1 646	-1 805	-1 820	-1 823
$\partial v/\partial m_o$	-41.74	-43.17	-44.18	-44.27	-44.29
$\Delta v_{1\sigma}^{(1)}$	48.57	50.50	51.91	52.04	52.07
$\Delta v_{1\sigma}^{(2)}$	48.57	50.51	51.93	52.12	52.18
ϕ_p	4 423	3 892	1 489	651	79
ϕ_S	117 500	108 900	95 500	94 100	93 700

*Superscripts (1) and (2) refer to one-sigma (1σ) perturbations obtained by using the sensitivity-partial-derivative method and the parameter method, respectively.

TABLE V.- SENSITIVITY PARTIALS AND RELATED INFORMATION – Continued

(b) $K_D = 0$; $w_1 = w_3 = 1$, $w_2 = 10^{-4}$ (Set (2))

Sensitivity partials and related terms (*)	Values for sensitivity weighting factor, w_s, of –				
	0	0.1	0.3	1.0	10.0
$\partial u/\partial T$	1.106	1.084	0.723	0.448	0.104
$\partial u/\partial \dot{m}$	-6 322	-5 869	-2 267	-103	174
$\partial u/\partial m_O$	-84.98	-81.22	-43.76	-18.88	-3.40
$\Delta u_{1\sigma}^{(1)}$	110.1	104.4	52.6	22.1	4.5
$\Delta u_{1\sigma}^{(2)}$	110.2	104.4	52.6	22.1	4.5
$\partial y/\partial T$	52.75	52.75	52.75	52.75	52.75
$\partial y/\partial \dot{m}$	-96 100	-102 100	-108 500	-112 100	-126 400
$\partial y/\partial m_O$	-2 755	-2 793	-2 833	-2 856	-2 946
$\Delta y_{1\sigma}^{(1)}$	3 209	3 260	3 315	3 347	3 474
$\Delta y_{1\sigma}^{(2)}$	3 210	3 260	3 316	3 347	3 475
$\partial v/\partial T$	0.805	0.805	0.805	0.805	0.805
$\partial v/\partial \dot{m}$	-1 419	-1 316	-1 206	-1 146	-899
$\partial v/\partial m_O$	-41.74	-41.09	-40.40	-40.02	-38.46
$\Delta v_{1\sigma}^{(1)}$	48.57	47.71	46.82	46.35	44.45
$\Delta v_{1\sigma}^{(2)}$	48.57	47.72	46.82	46.33	44.40
ϕ_p	4 423	4 336	2 891	1 791	417
ϕ_s	15 510	14 240	6 060	3 760	3 200

*Superscripts (1) and (2) refer to one-sigma (1σ) perturbations obtained by using the sensitivity-partial-derivative method and the parameter method, respectively.

$w_S \approx 1.0$ in figure 3. Part of the reason is the change in the magnitude of ϕ_S due to changing only the relative values of the relative weighting factors without regard to their magnitude. This is clearly illustrated by the near-order-of-magnitude difference in ϕ_S for set (1) and set (2) at $w_S = 0$. (Compare tables V(a) and V(b).) The trajectories for the two sets are the same; ϕ_S changes with the change from set (1) to set (2). Reducing the values of ϕ_S in table V(a) by dividing by the constant $\frac{117\,500}{15\,510} \approx 7.57$ so that both cases have the same ϕ_S at $w_S = 0$, and incorporating this constant into the weighting factor by multiplying each w_S of table V(a) by 7.57, allows a more reasonable comparison. This comparison is shown in figure 7 where the ordinate is called w_S (adjusted). This figure shows the differences which come about due to different relative-weighting-factor sets.

The control time histories for relative-weighting-factor set (2) are shown in figure 8. Again there is a tendency toward a bang-bang type of control, as w_S increases, which may be noted by observing that the angle difference between the nearly constant attitude portion of the control time history at the beginning and near the end of flight is about 180° for both $w_S = 1.0$ and $w_S = 10.0$. Figure 9 shows that in comparison with the set (2) trajectory for $w_S = 0$, the other set (2) trajectories are generally less steep. The opposite result is shown for set (1) in figure 5; in comparison with the set (1) trajectory for $w_S = 0$, the other set (1) trajectories are more steep. These results indicate the importance of the relative weighting factors in shaping the trajectories.

Data for a nonzero value of K_D, $K_D = 0.001$, and relative-weighting-factor set (2) are shown in table V(c) and figures 10 and 11. Data for $w_S = 0.1$ are shown in table V(c) but not in figures 10 and 11 because the control time histories and trajectories essentially coincide with those for $w_S = 0$. In table V(c) it may be observed that while consistent data trends are shown in the first three w_S columns, the data in the column for $w_S = 10.0$ are not consistent. The consistent data trends in the first three columns are very similar to those in table V(b). The inconsistency of the last column may be explained by examination of figures 10 and 11 where the radically different character of the control time history and trajectory for $w_S = 10.0$ may be seen. This result for $w_S = 10.0$ was so unusual that it was believed necessary to verify the answer. Accordingly, verification was obtained by iterating to the same result (essentially) from an additional two different nominal trajectories. These results therefore appear correct. It is believed that the data inconsistency results from the different character of the trajectory — that is, the bending back of the trajectory as seen in figure 11. With the exception of this $w_S = 10.0$ case, the trajectories are less steep with increasing w_S just as they were in figure 9 for a similar case with zero drag.

For all of the cases discussed herein, good agreement was observed (see tables V(a), V(b), and V(c)) between the one-sigma perturbations computed by using sensitivity partials and those computed by using the parameter-perturbation method.

TABLE V.- SENSITIVITY PARTIALS AND RELATED INFORMATION – Concluded

(c) $K_D = 0.001$; $w_1 = w_3 = 1$, $w_2 = 10^{-4}$ (Set (2))

Sensitivity partials and related terms (*)	Values for sensitivity weighting factor, w_S, of –			
	0	0.1	1.0	10.0
$\partial u/\partial T$	25.35	25.24	24.35	12.07
$\partial u/\partial \dot{m}$	-756.9	-747.1	-668.1	-002.8
$\partial u/\partial m_O$	-8.11	-7.99	-7.04	-1.27
$\partial u/\partial K_D$	-687 000	-688 900	-697 500	-359 900
$\Delta u_{1\sigma}^{(1)}$	17.37	17.28	16.52	6.61
$\Delta u_{1\sigma}^{(2)}$	17.37	17.28	16.52	6.61
$\partial y/\partial T$	31.94	31.96	32.11	31.81
$\partial y/\partial \dot{m}$	-44 640	-44 840	-46 370	-44 800
$\partial y/\partial m_O$	-1 438	-1 440	-1 456	-1 428
$\partial y/\partial K_D$	-14 190 000	-14 200 000	-14 220 000	-14 350 000
$\Delta y_{1\sigma}^{(1)}$	1 730	1 732	1 751	1 712
$\Delta y_{1\sigma}^{(2)}$	1 729	1 732	1 751	1 712
$\partial v/\partial T$	16.18	16.13	15.81	17.01
$\partial v/\partial \dot{m}$	-7.8	4.6	101.8	34.5
$\partial v/\partial m_O$	-3.80	-3.68	-2.81	-3.52
$\partial v/\partial K_D$	-278 700	-280 400	-293 500	-313 700
$\Delta v_{1\sigma}^{(1)}$	7.358	7.309	7.102	7.766
$\Delta v_{1\sigma}^{(2)}$	7.357	7.311	7.100	7.766
ϕ_p	1 824	1 824	1 811	911
ϕ_S	655	652	630	403

*Superscripts (1) and (2) refer to one-sigma (1σ) perturbations obtained by using the sensitivity-partial-derivative method and the parameter method, respectively.

CONCLUDING REMARKS

It has been shown how a sensitivity measure, composed of the weighted sum of the variances of the performance index and terminal constraints, may be added to the performance index of a stochastic optimal control problem to achieve a reduction in performance and constraint sensitivity due to parameter perturbations.

It was necessary to assume that the parameters of the system remained fixed during system operation and that the stochastic nature of the problem came about because of inexact knowledge of these fixed values.

It was shown how this technique increased the dimensionability of the optimization problem and introduced weighting factors or constants for use as design parameters. These weighting factors permit different relative importance to be attached to different types of sensitivity, such as position relative to velocity, and allow for adjustment of performance degradation and of sensitivity reduction.

The feasibility of solving nonlinear multiparameter problems by using this technique was illustrated by solving an example rocket-trajectory problem through application of a steepest-descent algorithm. The example solutions also served to illustrate the tradeoffs made possible by changes in the weighting factors.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., April 14, 1971.

APPENDIX A

STEEPEST-DESCENT DERIVATION

The problem is to find the control time history $u(t)$, $t_o \leqq t \leqq t_f$, which minimizes

$$\phi = \phi\left(\underline{x}\left(t_f\right)\right) \tag{A1}$$

subject to the differential constraints

$$\underline{\dot{x}} = \underline{f}\left(\underline{x},\underline{u},t\right) \; ; \; \underline{x}\left(t_o\right) = \underline{x}_o \quad \text{(given)} \tag{A2}$$

and the terminal constraints

$$\underline{\psi} = \underline{\psi}\left(\underline{x}\left(t_f\right)\right) = \underline{L} \tag{A3}$$

The solution is obtained iteratively. Choose a reasonable time history $\underline{u}^*(t)$ and obtain $\underline{x}^*(t)$, a solution to equation (A2). This solution, in general, is such that neither ϕ is minimum nor $\underline{\psi} = \underline{L}$.

Linearizing about this solution gives

$$\underline{\dot{x}} - \underline{\dot{x}}^* = \frac{\partial \underline{f}}{\partial \underline{x}}\left(\underline{x}^*,\underline{u}^*,t\right)\left(\underline{x} - \underline{x}^*\right) + \frac{\partial \underline{f}}{\partial \underline{u}}\left(\underline{x}^*,\underline{u}^*,t\right)\left(\underline{u} - \underline{u}^*\right)$$

or

$$\delta\underline{\dot{x}} = \frac{\partial \underline{f}}{\partial \underline{x}}(t)\delta\underline{x} + \frac{\partial \underline{f}}{\partial \underline{u}}(t)\delta\underline{u} \tag{A4}$$

Multiplying equation (A4) by $\Lambda(t)$, a variable matrix integrating factor, and integrating by parts yields

$$\Lambda\delta\underline{x}\bigg|_{t_o}^{t_f} = \int_{t_o}^{t_f}\left(\frac{d\Lambda}{dt} + \Lambda\frac{\partial \underline{f}}{\partial \underline{x}}\right)\delta\underline{x}\,dt + \int_{t_o}^{t_f}\Lambda\frac{\partial \underline{f}}{\partial \underline{u}}\,\delta\underline{u}\,dt \tag{A5}$$

Now, for convenience, let $\Lambda(t)$ be the solution of

$$\frac{d\Lambda}{dt} + \Lambda\frac{\partial \underline{f}}{\partial \underline{x}} = 0 \tag{A6}$$

with

$$\Lambda\left(t_f\right) = \frac{\partial \tilde{\underline{\psi}}}{\partial \underline{x}}\left(t_f\right) \tag{A7}$$

where

$$\tilde{\underline{\psi}} = \left[\frac{\phi}{\underline{\psi}}\right] \tag{A8}$$

Then equation (A5) becomes

$$\frac{\partial \tilde{\underline{\psi}}}{\partial \underline{x}} \, \delta \underline{x}\left(t_f\right) - \Lambda\left(t_o\right) \, \delta \underline{x}\left(t_o\right) = \int_{t_o}^{t_f} \Lambda \, \frac{\partial f}{\partial \underline{u}} \, \delta \underline{u} \, dt$$

or, letting $\delta \tilde{\underline{\psi}} \equiv \frac{\partial \tilde{\underline{\psi}}}{\partial \underline{x}} \, \delta \underline{x}\left(t_f\right)$,

$$\delta \tilde{\underline{\psi}} = \left[\delta \phi, \delta \psi_1, \ldots, \delta \psi_l\right]^T = \Lambda\left(t_o\right) \, \delta \underline{x}\left(t_o\right) + \int_{t_o}^{t_f} \Lambda \, \frac{\partial f}{\partial \underline{u}} \, \delta \underline{u} \, dt \tag{A9}$$

Partition $\Lambda(t)$ as follows:

$$\Lambda(t) = \left[\frac{\lambda_\phi^{\,T}(t)}{\Lambda_\psi(t)}\right]$$

Then, since $\underline{x}\left(t_o\right)$ is given and $\delta \underline{x}\left(t_o\right) = 0$,

$$\delta \phi = \int_{t_o}^{t_f} \lambda_\phi^{\,T} \, \frac{\partial f}{\partial \underline{u}} \, \delta \underline{u} \, dt \tag{A10}$$

and

$$\delta \underline{\psi} = \int_{t_o}^{t_f} \Lambda_\psi \, \frac{\partial f}{\partial \underline{u}} \, \delta \underline{u} \, dt \tag{A11}$$

Now for some measure of allowable control perturbation

$$(dP)^2 = \int_{t_o}^{t_f} \delta \underline{u}^T W \, \delta \underline{u} \, dt \; ; \quad W = W^T > 0 \tag{A12}$$

minimize $\delta \phi$ and choose $\delta \underline{\psi}$ such that $\underline{\psi}\left(\underline{x}\left(t_f\right)\right) = L$ will be satisfied (or more nearly so). Form an augmented function to be minimized

$$J_A = \delta \phi + \mu \left[(dP)^2 - \int_{t_o}^{t_f} \delta \underline{u}^T W \, \delta \underline{u} \, dt\right] + \underline{\nu}^T \left(\delta \underline{\psi} - \int_{t_o}^{t_f} \Lambda_\psi \frac{\partial f}{\partial \underline{u}} \, \delta \underline{u} \, dt\right) \tag{A13}$$

26

where μ and $\underline{\nu}$ are constant Lagrange multipliers. For J_A to be an extremum, its variation with respect to $\delta\underline{u}$ must be zero — that is,

$$\delta J_A = \left(\int_{t_o}^{t_f} \lambda_\phi{}^T \frac{\partial f}{\partial \underline{u}} - 2\mu\, \delta\underline{u}^T W - \underline{\nu}^T \Lambda_\psi \frac{\partial f}{\partial \underline{u}}\right)\delta(\delta\underline{u})\, dt = 0$$

This requirement implies that

$$\lambda_\phi{}^T \frac{\partial f}{\partial \underline{u}} - 2\mu\, \delta\underline{u}^T W - \underline{\nu}^T \Lambda_\psi \frac{\partial f}{\partial \underline{u}} = 0$$

for all t. Solving for $\delta\underline{u}$,

$$\delta\underline{u} = \frac{1}{2\mu} W^{-1}\left(\frac{\partial f}{\partial \underline{u}}\right)^T \left(\lambda_\phi - \Lambda_\psi{}^T \underline{\nu}\right) \tag{A14}$$

In order to solve for μ and $\underline{\nu}$ this expression for $\delta\underline{u}$ is substituted into equations (A10) and (A11) to obtain

$$\left.\begin{aligned}
\underline{\nu} &= -2\mu I_{\psi\psi}{}^{-1}\, d\underline{\psi} + I_{\psi\psi}{}^{-1} I_{\psi\phi} \\
2\mu &= -\left\{\left(I_{\phi\phi} - I_{\psi\phi}{}^T I_{\psi\psi}{}^{-1} I_{\psi\phi}\right)\Big/\left[(dP)^2 - d\underline{\psi}^T I_{\psi\psi}{}^{-1}\, d\underline{\psi}\right]\right\}^{1/2}
\end{aligned}\right\} \tag{A15}$$

where the sign on the radical is minus because ϕ is being minimized and where

$$\left.\begin{aligned}
I_{\phi\phi} &\equiv \int_{t_o}^{t_f} \lambda_\phi{}^T \frac{\partial f}{\partial \underline{u}} W^{-1}\left(\frac{\partial f}{\partial \underline{u}}\right)^T \lambda_\phi\, dt \\[2ex]
I_{\psi\phi} &\equiv \int_{t_o}^{t_f} \Lambda_\psi \frac{\partial f}{\partial \underline{u}} W^{-1}\left(\frac{\partial f}{\partial \underline{u}}\right)^T \lambda_\phi\, dt \\[2ex]
I_{\psi\psi} &\equiv \int_{t_o}^{t_f} \Lambda_\psi \frac{\partial f}{\partial \underline{u}} W^{-1}\left(\frac{\partial f}{\partial \underline{u}}\right)^T \Lambda_\psi{}^T\, dt
\end{aligned}\right\} \tag{A16}$$

Now let

$$\underline{u}(t) = \underline{u}^*(t) + \delta\underline{u}(t)$$

be the new control time history used to obtain a solution to equation (A2) and repeat the same procedure until $\underline{\psi} - \underline{L}$ and the gradient of the payoff-constraint surface

$$\frac{d\phi}{dP} = -\sqrt{I_{\phi\phi} - I_{\psi\phi}{}^T I_{\psi\psi}{}^{-1} I_{\psi\phi}} \tag{A17}$$

are sufficiently close to zero.

27

REFERENCES

1. Dorato, P.: On Sensitivity in Optimal Control Systems. IEEE Trans. Automat. Contr., vol. AC-8, no. 3, July 1963, pp. 256-257.

2. Pagurek, Bernard: Sensitivity of the Performance of Optimal Control Systems to Plant Parameter Variations. IEEE Trans. Automat. Contr., vol. AC-10, no. 2, Apr. 1965, pp. 178-180.

3. Witsenhausen, H. S.: On the Sensitivity of Optimal Control Systems. IEEE Trans. Automat. Contr., vol. AC-10, no. 4, Oct. 1965, pp. 495-496.

4. Sinha, N. K.; and Atluri, Satya Ratnam: Sensitivity of Optimal Control Systems. Proceedings Fourth Annual Allerton Conference on Circuit and System Theory, William R. Perkins and J. P. Cruz, Jr., eds., Univ. of Illinois and IEEE, 1966, pp. 508-516.

5. Breakwell, John V.; Speyer, Jason L.; and Bryson, Arthur E.: Optimization and Control of Nonlinear Systems Using the Second Variation. J. Soc. Ind. Appl. Math. Contr., ser. A, vol. 1, no. 2, 1963, pp. 193-223.

6. Kreindler, Eliezer: Synthesis of Flight Control Systems Subject to Vehicle Parameter Variations. AFFDL-TR-66-209, U.S. Air Force, Apr. 1967. (Available from DDC as AD 653 600.)

7. Higginbotham, Ronald E.: Design of Sensitivity-Constrained Optimal Linear State-Regulator Control Systems. Conference Record of Second Asilomar Conference on Circuits & Systems, 68 C 64-ASIL, IEEE, 1968, pp. 330-335.

8. Haas, V. B.; and Steinberg, A. M.: Minimum Sensitivity Optimal Control for Nonlinear Systems. TR-EE67-10 (NGR 15-005-021), School Elec. Eng., Purdue Univ., Aug. 1967. (Available as NASA CR-88863.)

9. Rohrer, R. A.; and Sobral, M., Jr.: Sensitivity Considerations in Optimal System Design. IEEE Trans. Automat. Contr. vol. AC-10, no. 1, Jan. 1965, pp. 43-48.

10. Chan, S. Y.; and Chuang, K.: A Study of Effects on Optimal Control Systems Due to Variations in Plant Parameters. Int. J. Contr., First ser., vol. 7, no. 3, Mar. 1968, pp. 281-298.

11. Patrick, L. Benjamin; and D'Angelo, Henry: Optimal Control of Plants With Random Slowly-Varying Parameters. Proceedings Fifth Annual Allerton Conference on Circuit and System Theory, J. B. Cruz, Jr., and T. N. Trick, eds., Univ. of Illinois and IEEE, 1967, pp. 711-720.

12. Holtzman, J. M; and Horing, S.: The Sensitivity of Terminal Conditions of Optimal Control Systems to Parameter Variations. IEEE Trans. Automat. Contr., vol. AC-10, no. 4, Oct. 1965, pp. 420-426.

13. Kokotovic, P. V.; and Rutman, R. S.: Sensitivity of Automatic Control Systems (Survey). Automat. Remote Contr., vol. 26, no. 4, Apr. 1965, pp. 727-749.

14. Rosenbaum, Richard C.; and Willwerth, Robert E.: Launch Vehicle Error Sensitivity Study. NASA CR-1512, 1970.

15. Lindgren, B. W.; and McElrath, G. W.: Introduction to Probability and Statistics. Second ed., Macmillan Co., 1966.

16. Coddington, Earl A.; and Levinson, Norman: Theory of Ordinary Differential Equations. McGraw-Hill Book Co., Inc., 1955.

17. Bryson, A. E.; and Denham, W. F.: A Steepest-Ascent Method for Solving Optimum Programming Problems. Trans. ASME, ser. E: J. Appl. Mech., vol. 29, no. 2, June 1962, pp. 247-257.

18. Bryson, Arthur E., Jr.; and Ho, Yu-Chi: Applied Optimal Control. Blaisdell Pub. Co., c.1969.
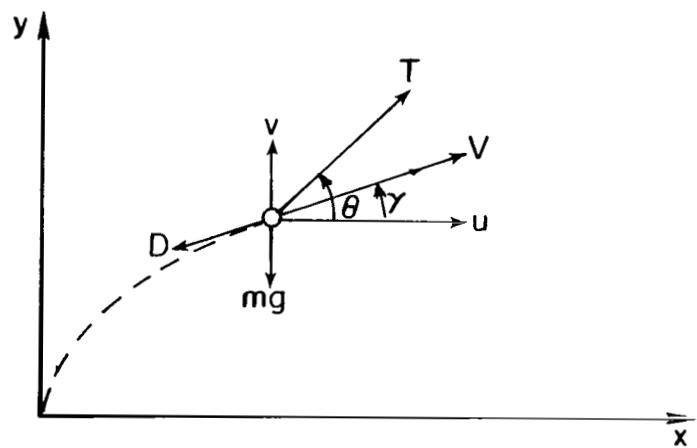
Figure 1.- Coordinate system and geometric relations.
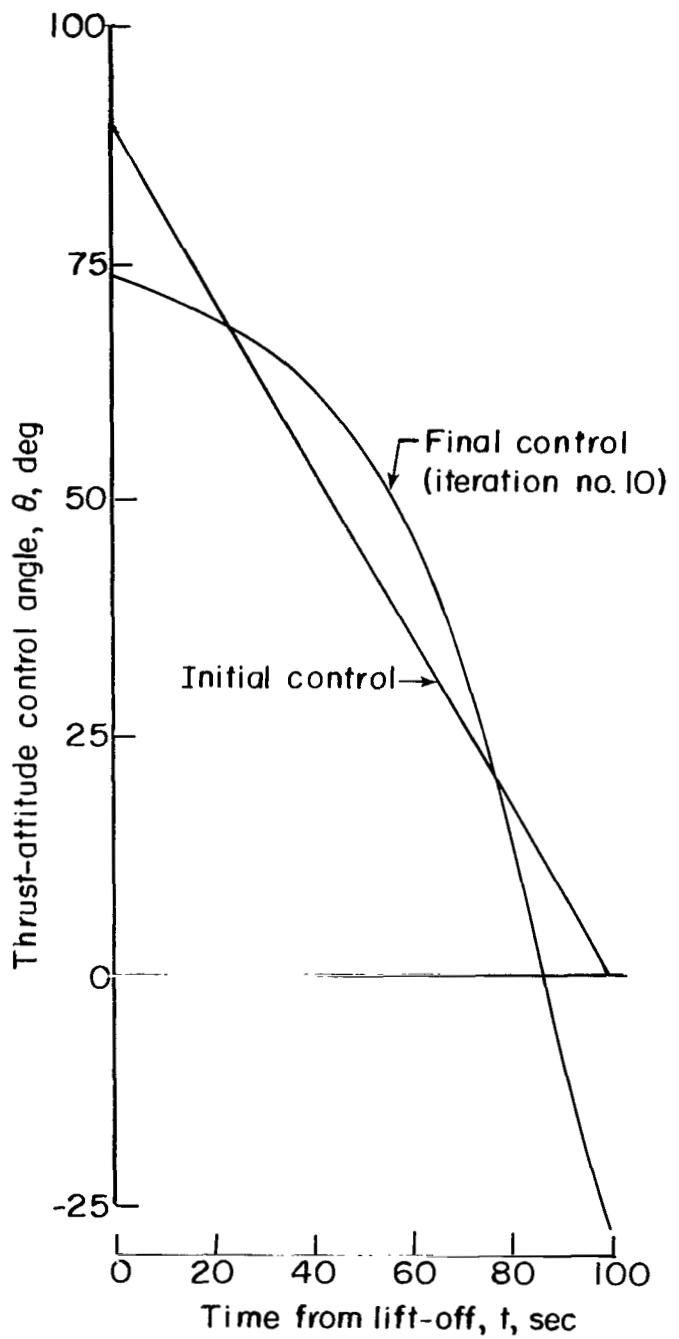
Figure 2.- A comparison of the initial guessed thrust-attitude
control time history with the steepest-descent generated
optimal thrust-attitude control time history. $K_D = 0$; $w_s = 0$.
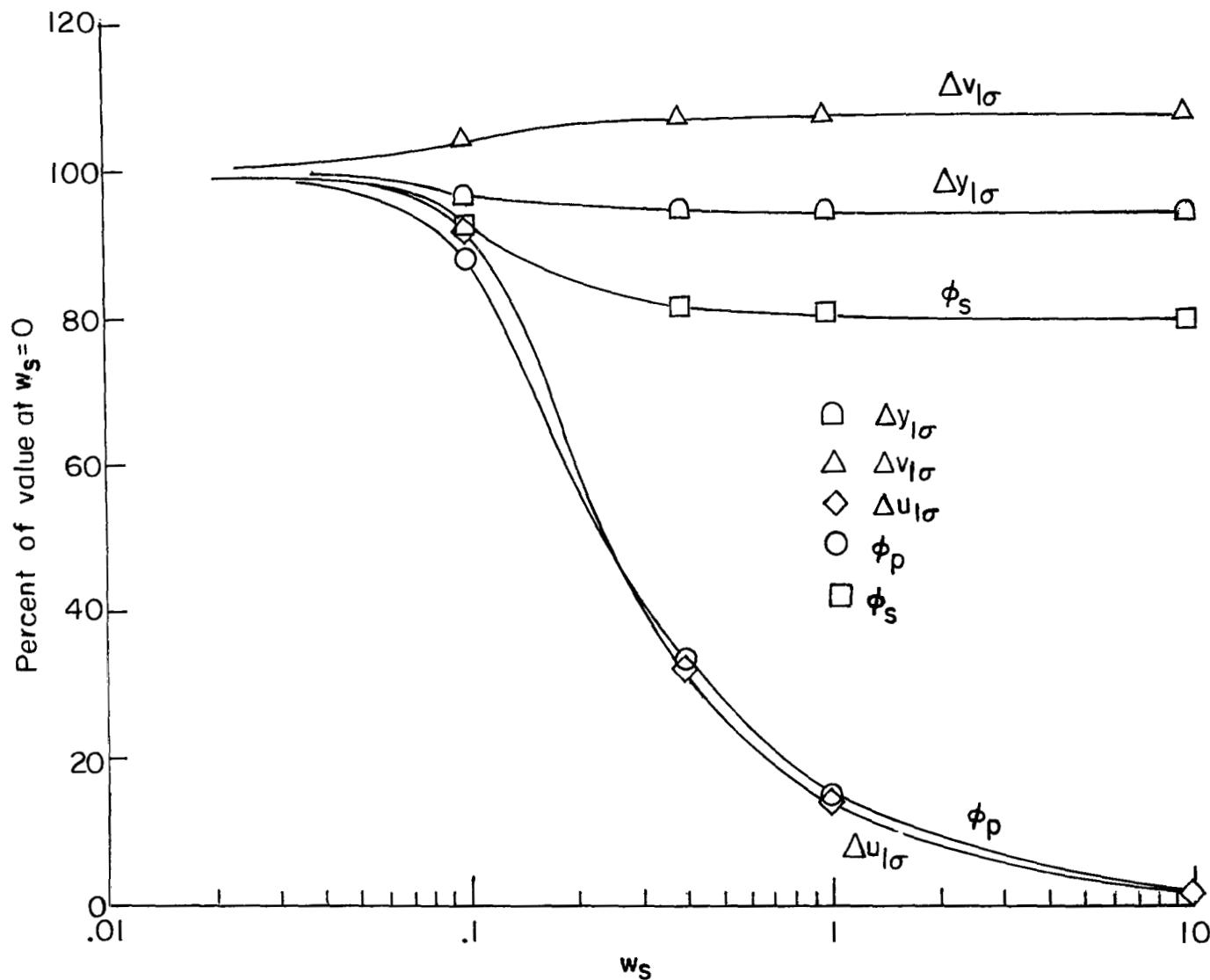
Figure 3.- One-sigma (1σ) error components and sensitivity and performance indices plotted against sensitivity weighting factor. $K_D = 0$; $w_1 = w_3 = 1$, $w_2 = 10^{-2}$.
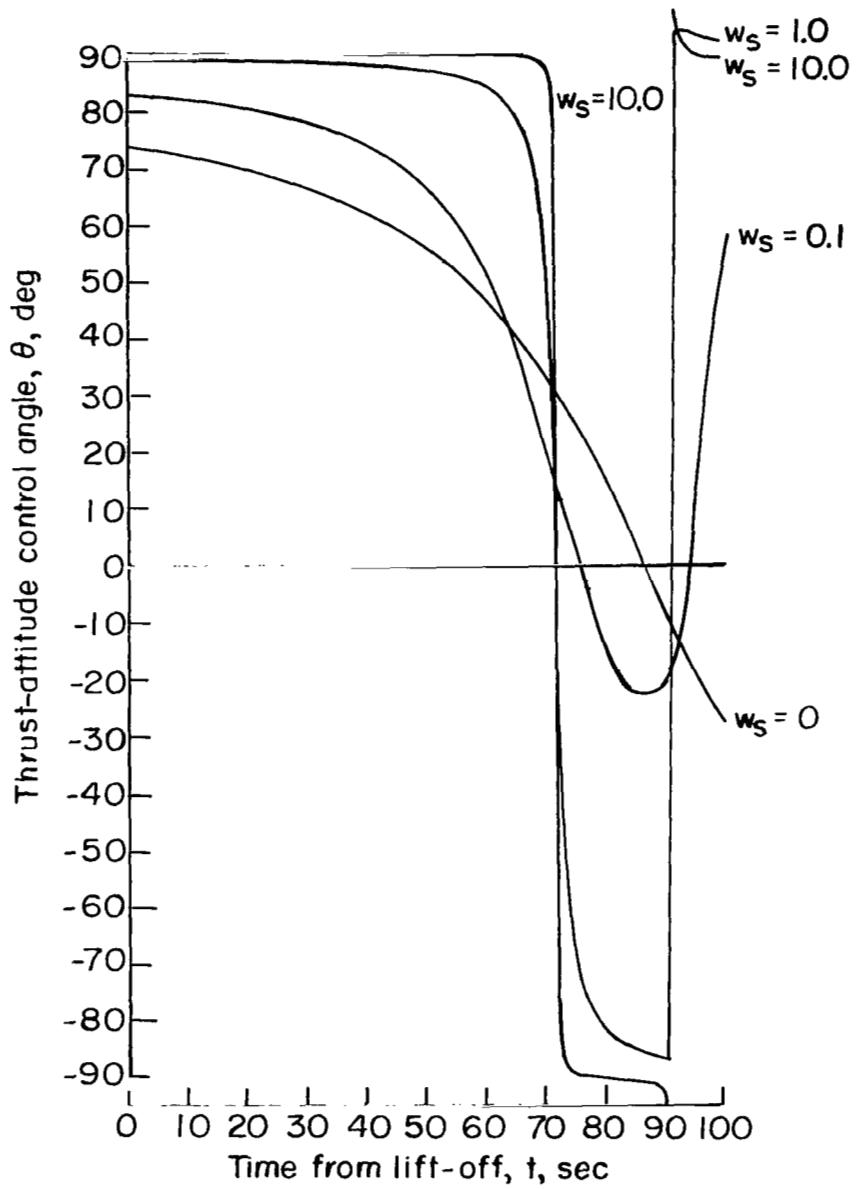
Figure 4.- Comparison of optimal control time histories for different sensitivity weighting factors $K_D = 0$; $w_1 = w_3 = 1$, $w_2 = 10^{-2}$.
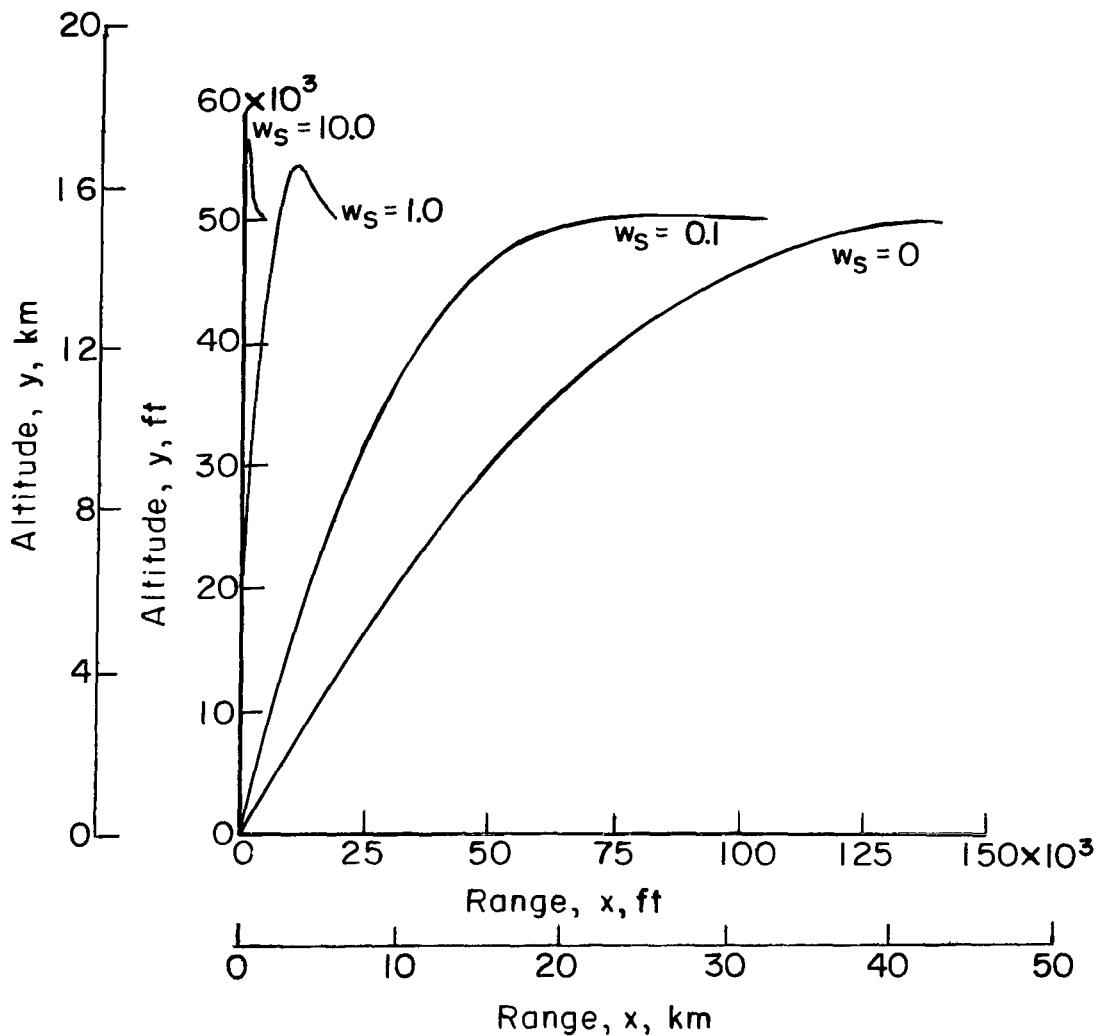
Figure 5.- Comparison of optimal trajectories for different sensitivity weighting factors. $K_D = 0$; $w_1 = w_3 = 1$, $w_2 = 10^{-2}$.
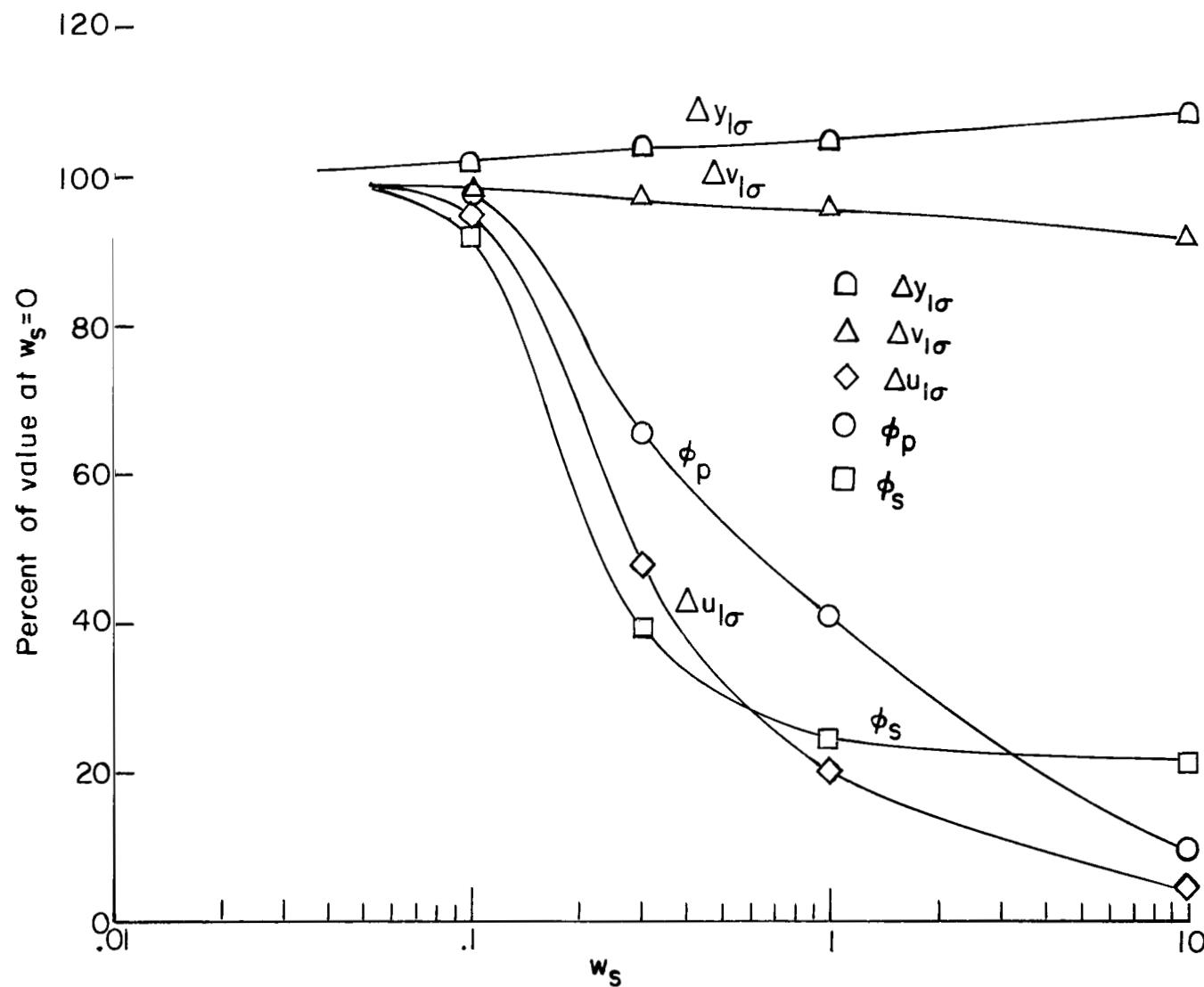
Figure 6.- One-sigma (1σ) error components and sensitivity and performance indices plotted against sensitivity weighting factor. $K_D = 0$; $w_1 = w_3 = 1$, $w_2 = 10^{-4}$.
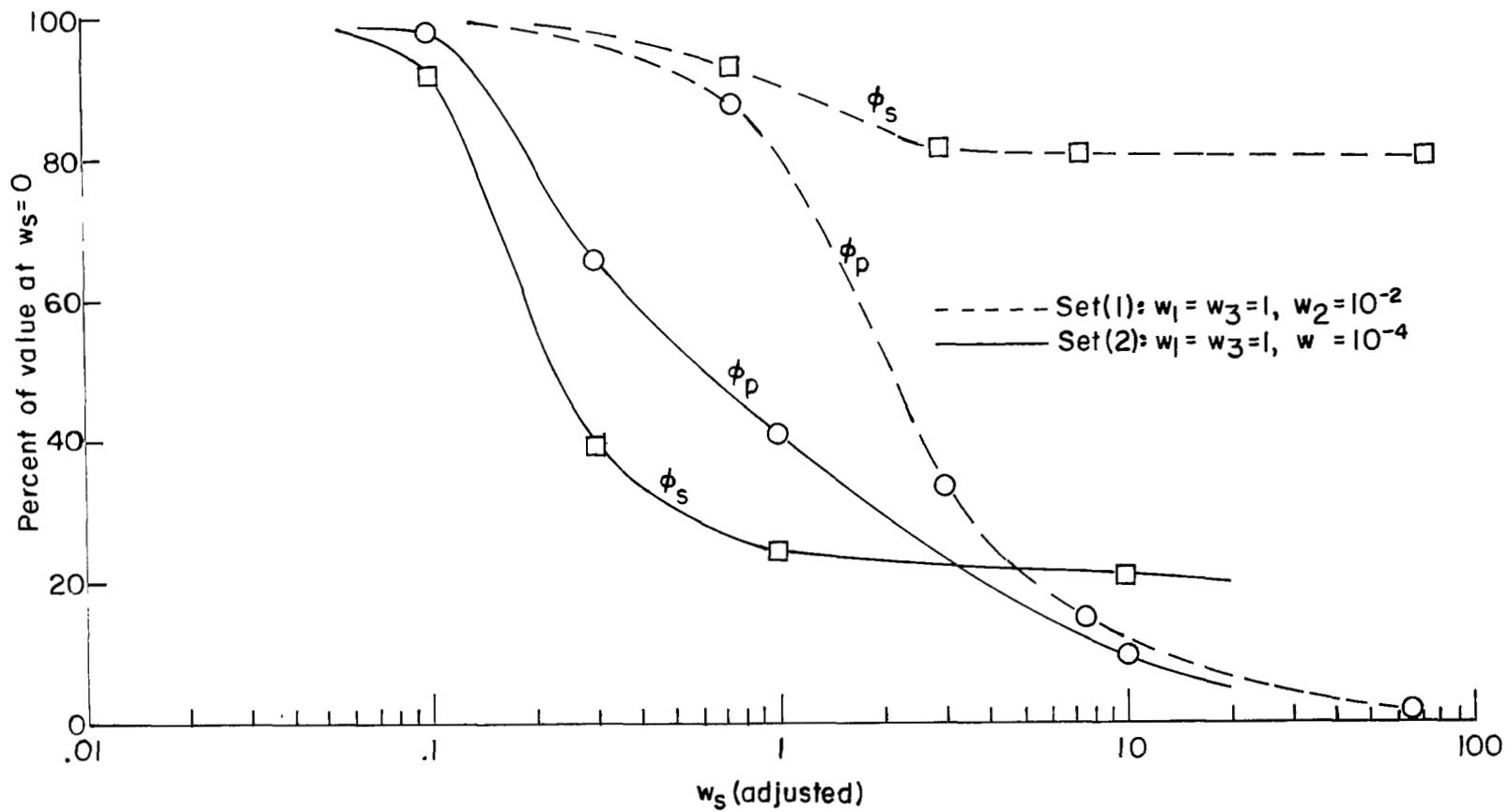
Figure 7.- Set (1) and set (2) sensitivity and performance indices plotted against a sensitivity weighting factor adjusted for equal sensitivity at $w_2 = 0$. $K_D = 0$.
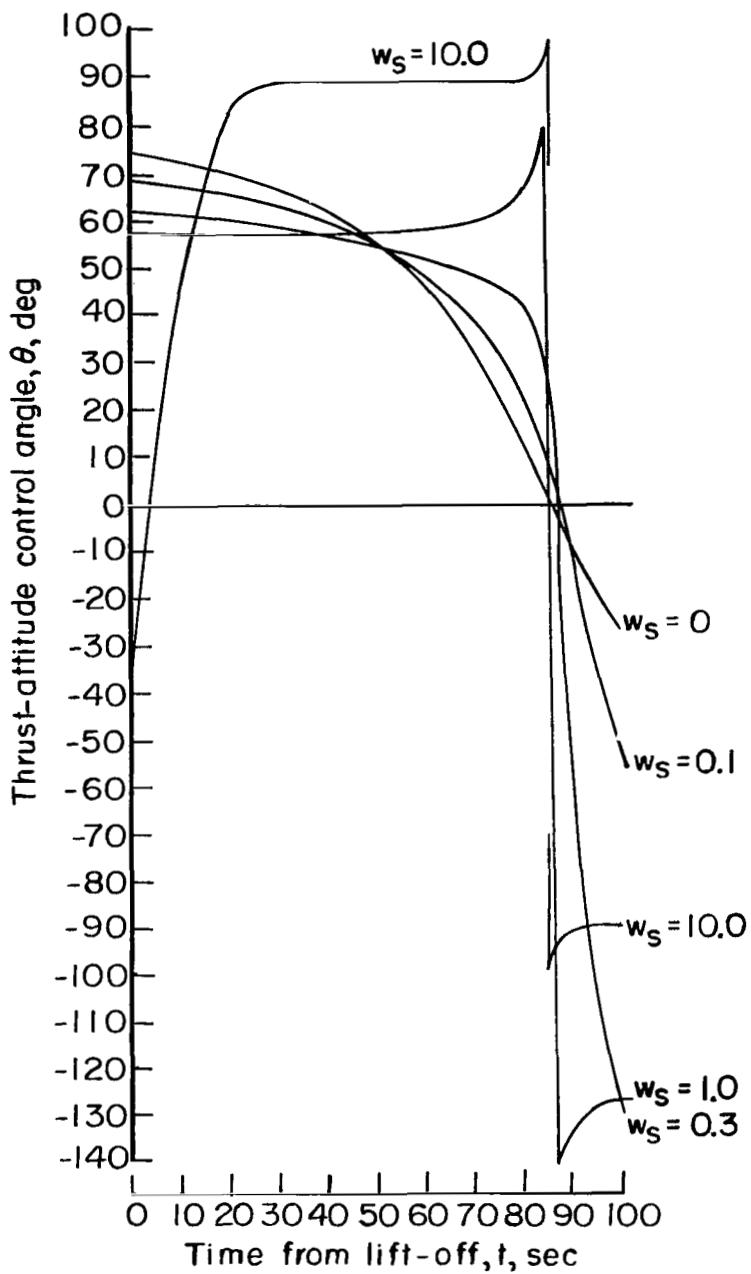
Figure 8.- Comparison of optimal control time histories for different sensitivity weighting factors. $K_D = 0$; $w_1 = w_3 = 1$, $w_2 = 10^{-4}$.
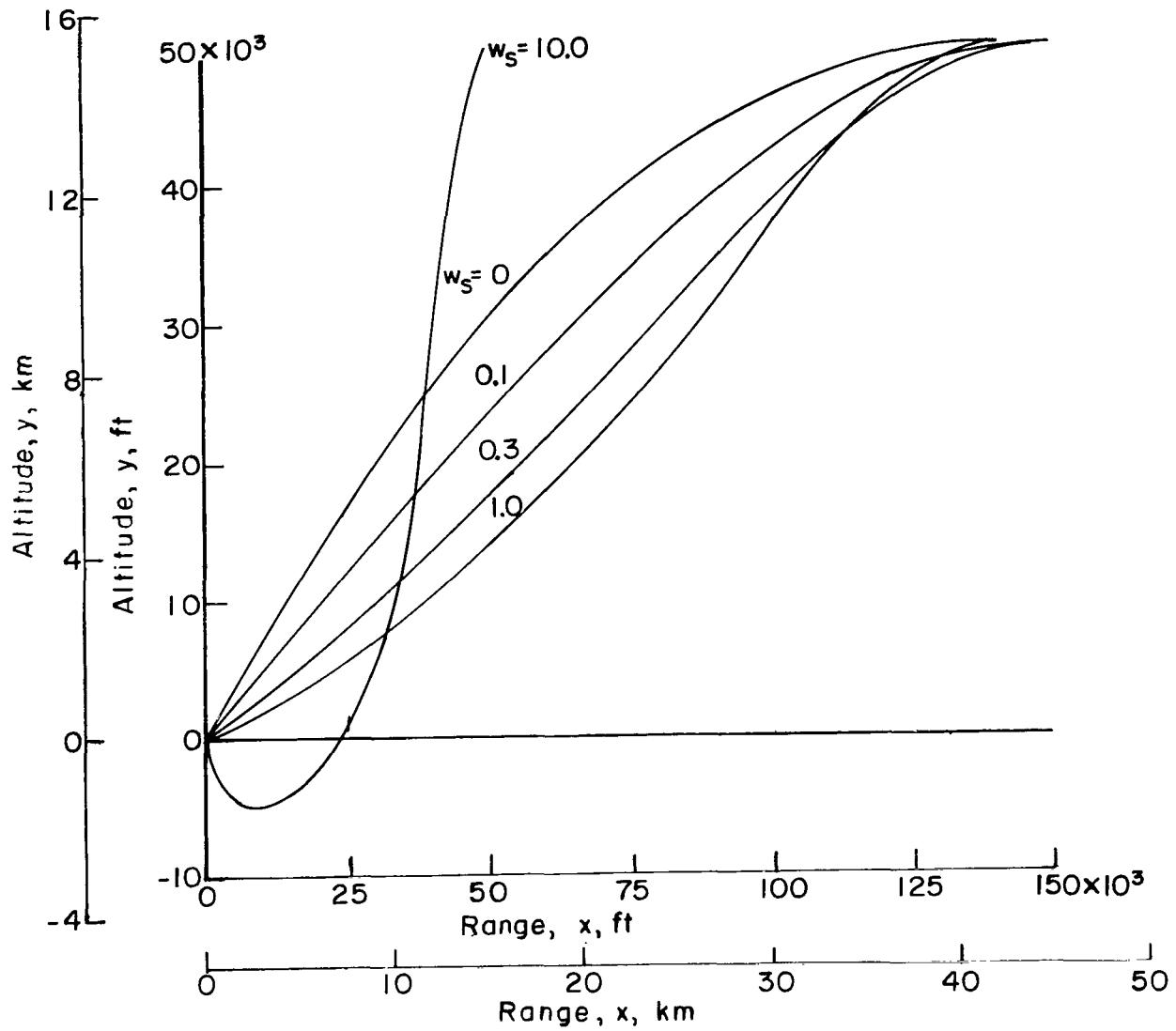
Figure 9.- Comparison of optimal trajectories for different sensitivity weighting factors. $K_D = 0$; $w_1 = w_3 = 1$, $w_2 = 10^{-4}$.
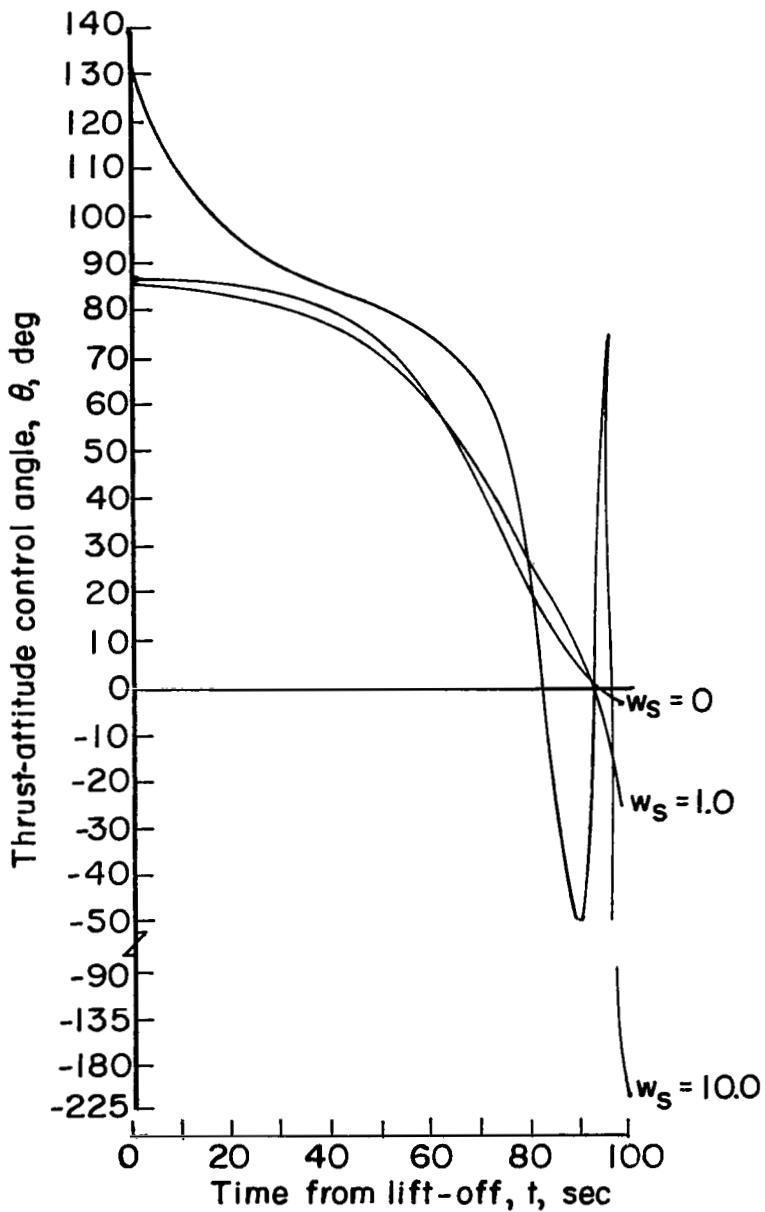
Figure 10.- Comparison of optimal control time histories for different sensitivity weighting factors. $K_D = 0.001$; $w_1 = w_3 = 1$, $w_2 = 10^{-4}$.
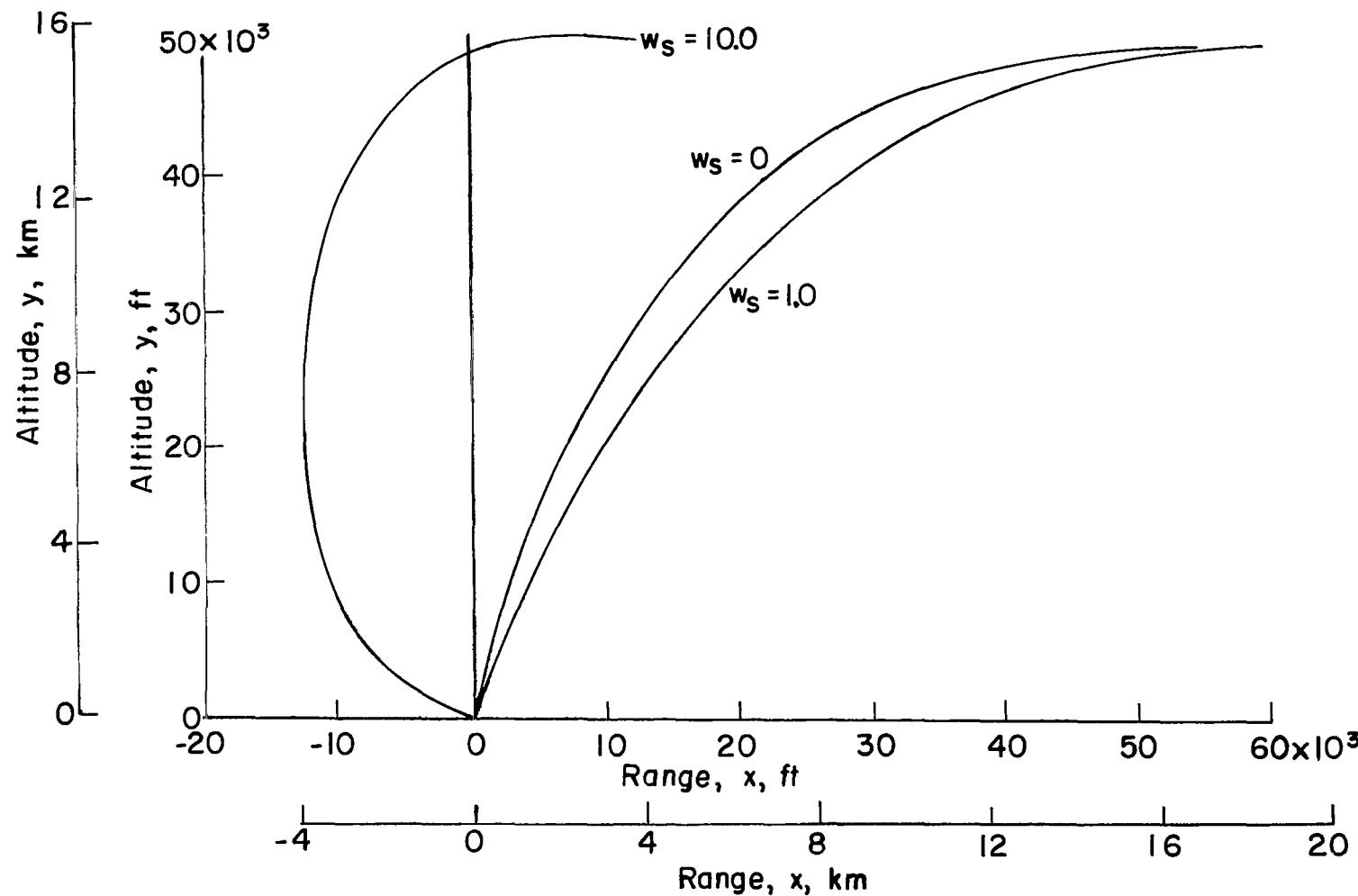
Figure 11.- Comparison of optimal trajectories for different sensitivity weighting factors. $K_D = 0.001$; $w_1 = w_3 = 1$, $w_2 = 10^{-4}$.

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

— NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION OFFICE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Washington, D.C. 20546