21-259 Recitations

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Contents

1 08-26-2025

1.1 Basics

This class is **Calculus in Three Dimensions**, thus the class will require you to think in 3-D.

- Get used to drawing in 3-D.
  My preference is to draw the usual vertical and horizontal axes.
  Label the horizontal axis y and the vertical axis z.
  Then draw a diagonal from the bottom left to the top right, passing through the origin. Label this axis x. Note that you can interchange any of the axis labels as necessary (which will come in handy when we start doing 3-D integration).
- In $\mathbb{R}^3$, the (Euclidean) distance (also referred to as the L_2 norm) between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1.2 Planes and Spheres

In this course, you will often deal with planes and spheres (e.g., spherical coordinates).

- You will learn the canonical formula for a plane later (finding it involves cross products). For now, we will find equations for planes parallel to another plane.

Ex. Write an equation of the plane passing through point $(21, 2, 59)$ that is parallel to the xz-plane.

When a plane is parallel to the xz-plane, it means only the x and z coordinates may vary.

Thus taking the y-value, we get the equation $y = 2$.

Ex. Write an equation of the plane passing through points $(2, 125, 9), (21, 25, 9), (5, 7, 9)$ that is parallel to the xy-plane.

When a plane is parallel to the xy-plane, it means only the x and y coordinates may vary.

Conveniently, the z-value in all 3 coordinates are equal; we get the equation $z = 9$.

- A sphere is the set of all points in space equidistant from a fixed point, the center of the sphere. For center (a, b, c) and radius r, we represent the sphere by the ("canonical") equation:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Ex. Find the equation of the sphere with diameter $\overline{PQ}$ where $P = (2, -1, -3)$ and $Q = (-2, 5, -1)$.

First, find the center, which is at the midpoint of the diameter $\overline{PQ}$:

$$C = \left(\frac{2 + (-2)}{2}, \frac{-1 + 5}{2}, \frac{-3 + (-1)}{2}\right) = (0, 2, -2)$$

Next, find the radius using the distance formula (half the length of the diameter)

$$r = \frac{1}{2}\left\|\overline{PQ}\right\| = \frac{1}{2}\sqrt{(-2 - 2)^2 + (5 - (-1))^2 + (-1 - (-3))^2} = \frac{1}{2}\sqrt{56} = \sqrt{\frac{56}{4}} = \sqrt{14}$$

Thus the sphere is given by $x^2 + (y - 2)^2 + (z + 2)^2 = 14$

1.3 Vector Notation

Vectors are quantities with magnitude and direction. In $\mathbb{R}^3$, the standard unit vectors are

$$\hat{i} = \langle 1, 0, 0\rangle \quad \hat{j} = \langle 0, 1, 0\rangle \quad \hat{k} = \langle 0, 0, 1\rangle$$

There are several notations. Fix points in $\mathbb{R}^3$ $P = (0, 2, 1)$ and $Q = (2, 5, 9)$
Let's represent the vector $\overrightarrow{PQ} = \langle x_Q - x_P, y_Q - y_P, z_Q - z_P\rangle$.

- Component Form: $\overrightarrow{PQ} = \langle 2-0, 5-2, 9-1\rangle = \langle 2, 3, 8\rangle$
- Using the unit vectors: $\overrightarrow{PQ} = 2\hat{i} + 3\hat{j} + 8\hat{k}$

Ex: Consider the vectors $v = \langle 0, 2, 1\rangle$ and $u = \langle 2, 5, 9\rangle$. Find a unit vector in the direction of $3v + u$.

You will need the following rules (taken from your textbook)

RULE: PROPERTIES OF VECTORS IN SPACE

Let $\mathbf{v} = \langle x_1, y_1, z_1\rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2\rangle$ be vectors, and let k be a scalar.

Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1, kz_1\rangle$

Vector addition: $\mathbf{v} + \mathbf{w} = \langle x_1, y_1, z_1\rangle + \langle x_2, y_2, z_2\rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2\rangle$

Vector subtraction: $\mathbf{v} - \mathbf{w} = \langle x_1, y_1, z_1\rangle - \langle x_2, y_2, z_2\rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2\rangle$

Vector magnitude: $\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$

Unit vector in the direction of v: $\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\langle x_1, y_1, z_1\rangle = \left\langle \frac{x_1}{\|\mathbf{v}\|}, \frac{y_1}{\|\mathbf{v}\|}, \frac{z_1}{\|\mathbf{v}\|}\right\rangle$, if $\mathbf{v} \neq \mathbf{0}$

Solution: first find $3v + u$, then find unit vector in that direction.

$$\begin{aligned} 3v + u &= \langle 3(0), 3(2), 3(1)\rangle + \langle 2, 5, 9\rangle && \text{(scalar multiplication)} \\ &= \langle 0, 6, 3\rangle + \langle 2, 5, 9\rangle = \langle 2, 11, 12\rangle \\ \frac{1}{\|3v + u\|}(3v + u) &= \frac{\langle 2, 11, 12\rangle}{\sqrt{2^2 + 11^2 + 12^2}} = \langle \frac{2}{\sqrt{269}}, \frac{11}{\sqrt{269}}, \frac{12}{269}\rangle && \text{(unit vector in direction)} \end{aligned}$$

Observe that 269 is prime, so you can't simplify the denominator. Thus we're done. Not all numbers will be pretty, but as a tip: make sure your answers are feasible. If you are taking the length of a vector with 1-digit coordinates and get a 5-digit number, that's probably wrong.

DON'T FORGET THE SQUARE ROOT $\sqrt{}$ when taking norms!!!!

1.4 Computing 3x3 Determinants

Useful for computing cross products. You are free to use whatever trick you want so long as you show your work. (note vertical bars means determinant, while square brackets mean matrix).

1.4.1 Cofactor Method

Key notes: $+, -, +$ (**DON'T FORGET THE MIDDLE IS NEGATIVE**) and the 2x2 sub-determinants are the elements not in the same row or column.

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= aei - afg - bdi + bfg + cdh - ceg \end{aligned}$$

For cross products between two vectors $u = \langle u_x, u_y, u_z \rangle$ and $v = \langle v_x, v_y, v_z \rangle$, you just compute the 3x3 determinant $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$

1.4.2 Diagonal Method

Recall that the 2x2 determinant is computed using the diagonals: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, where the diagonals going from right to left are positive and diagonals going from left to right are negative. We extend this idea to 3x3 determinants:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \xRightarrow{expand} \begin{matrix} & & a & b & c & & \\ & f & d & e & f & d & \\ h & i & g & h & i & g & h \end{matrix} \xRightarrow{compute} aei + bfg + cdh - afh - bdi - ceg$$

which is the same result as derived from the cofactor method.

2 08-28-2025

2.1 Section 2.3, Checkpoint 2.23: Finding the Angle between Two Vectors

Find the measure of the angle, in radians, formed by vectors $a = \langle 1, 2, 0 \rangle$ and $b = \langle 2, 4, 1 \rangle$. Round to the nearest hundredth.

2.2 Section 2.3, Checkpoint 2.24: Identifying Orthogonal Vectors

For which value of x is $p = \langle 2, 8, -1 \rangle$ orthogonal to $q = \langle x, -1, 2 \rangle$?

2.3 Section 2.3, Checkpoint 2.27: Resolving Vectors into Components

Express $v = 5i - j$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $u = 4i + 2j$.

Properties of the Cross Product

Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be vectors in space, and let c be a scalar.

i.	$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$	Anticommutative property
ii.	$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$	Distributive property
iii.	$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$	Multiplication by a constant
iv.	$\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$	Cross product of the zero vector
v.	$\mathbf{v} \times \mathbf{v} = \mathbf{0}$	Cross product of a vector with itself
vi.	$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$	Scalar triple product

2.4 Section 2.4, Checkpoint 2.33 (quick)

Use the properties of the cross product to calculate $(i \times k) \times (k \times j)$.

2.5 Section 2.4, Checkpoint 2.38

Find the area of the parallelogram $PQRS$ with vertices $P(1, 1, 0), Q(7, 1, 0), R(9, 4, 2)$, and $S(3, 4, 2)$.

2.6 Section 2.4, Example 2.44: Evaluating Torque

A bolt is tightened by applying a force of 6 N to a 0.15-m wrench (Figure 2.62). The angle between the wrench and the force vector is 40°. Find the magnitude of the torque about the center of the bolt. Round the answer to two decimal places.

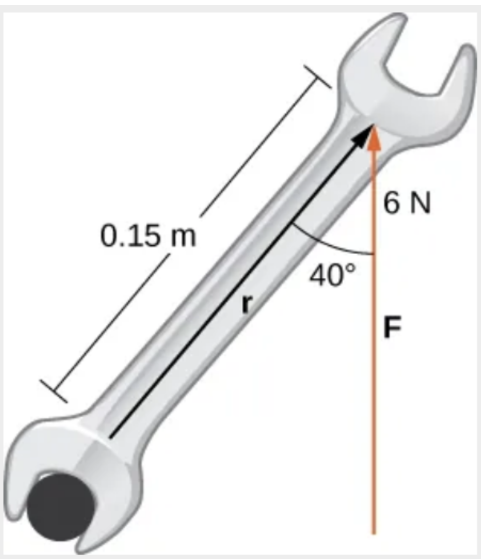


Figure 2.62 Torque describes the twisting action of the wrench.

3 09-02-2025

3.1 Section 2.5, Checkpoint 2.45

Find the distance between point $(0, 3, 6)$ and the line with parametric equations

$$x = 1 - t, y = 1 + 2t, z = 5 + 3t$$

3.2 Section 2.5, Checkpoint 2.46

Describe the relationship between the lines with the following parametric equations:

$$x = 1 - 4t, y = 3 + t, z = 8 - 6t$$

$$x = 2 + 3s, y = 2s, z = -1 - 3s$$

3.3 Section 2.5, Checkpoint 2.47

Find an equation of the plane containing the lines L1 and L2:

$$L1 : x = -y = z$$

$$L2 : \frac{x-3}{2} = y = z - 2$$

On the original handout, I mistyped and had $L_2 = x - 32$. This has been corrected, as the error makes the question unsolvable since the lines would be skew.

Worked Solution

Let $L_1 : x = -y = z = t$. The parameterized form of L_1 is thus $\langle t, -t, t\rangle = (0, 0, 0) + \langle 1, -1, 1\rangle t$

Let $L_2 : \frac{x-3}{2} = y = z - 2 = t$. The parameterized form of L_2 is thus $\langle 2t + 3, t, t + 2\rangle = (3, 0, 2) + \langle 2, 1, 1\rangle t$.

A normal vector to the plane that contains both lines is the cross product of the two lines' direction vectors, which was $\vec{n} = \langle -2, 1, 3\rangle$ *(work done in recitation)*.

Recall the equation of a plane is $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$, where $\vec{r} = (x, y, z)$, $\vec{n}$ is a normal vector of the plane, and $\vec{r_0}$ is some arbitrary point in the plane. *Note I said **a** normal vector, since we could scale $\vec{n}$ arbitrarily.*

We have $\vec{n}$, so we just need to get $\vec{r_0}$. *In recitation, I mentioned that we can pick any arbitrary point on either line to create the equation of the plane that contains L_1 and L_2. This holds true, and I will arbitrarily pick some points below and prove that the resulting plane equation is the same.*

- In recitation, we picked the easy point $(0, 0, 0)$, which we know to lie in L_1.

$$\begin{aligned}\vec{n} \cdot (\vec{r} - \vec{r_0}) &= \langle -2, 1, 3\rangle \cdot (x - 0, y - 0, z - 0)\\ &= -2x + y + 3z = 0\end{aligned}$$

- But what if we had picked $(3, 0, 2)$, which was our anchor point for L_2?

$$\begin{aligned}\vec{n} \cdot (\vec{r} - \vec{r_0}) &= \langle -2, 1, 3\rangle \cdot (x - 3, y - 0, z - 2)\\ &= -2(x - 3) + y + 3(z - 2) = 0\\ &= -2x + 6 + y + 3z - 6 = 0\\ &= -2x + y + 3z = 0 && \text{(the 6's cancel!)}\end{aligned}$$

- As an extension, derive the plane equation you get if you choose the intersection point of L_1 and L_2! You'll see that you still reach the same plane equation.

3.4 Section 2.5, Checkpoint 2.48

Find the distance between point $P = (5, -1, 0)$ and the plane given by $4x + 2y - z = 3$.

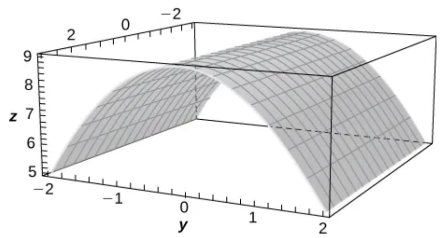
4 09-04-2025

Note: We do NOT cover the foci of paraboloids or ellipsoids.

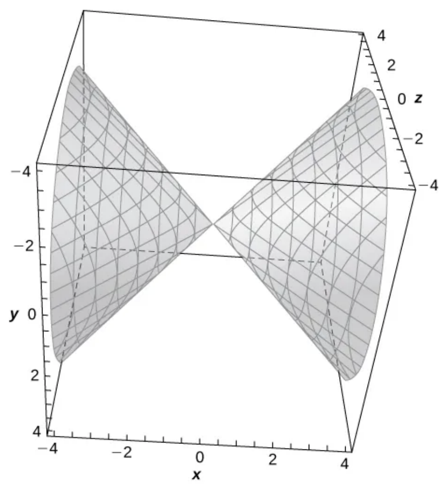
4.1 Section 2.6, Exercises 309-312

For the following exercises, the graph of a quadric surface is given.

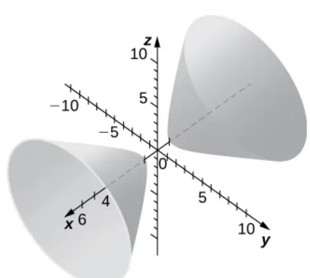
a. Specify the name of the quadric surface.
b. Determine the axis of the quadric surface.



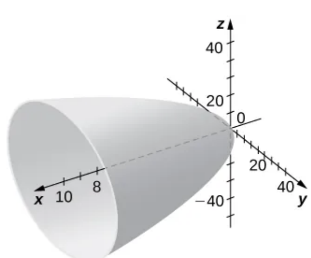
309.



310.



311.



312.

4.2 Section 2.6, Example 2.59 Identifying Equations of Quadric Surfaces

Identify the surfaces represented by the given equations.

(a) $16x^2 + 9y^2 + 16z^2 = 144$ **Solution** Observe that all the terms are squared, and all the coefficients are positive (but not equal) so this is an ellipsoid.

(b) $9x^2 - 18x + 4y^2 + 16y - 36z + 25 = 0$ **Solution** We have two squared terms, and z term of degree one. We could rearrange this to get $36z = 9x^2 - 18x + 4y^2 + 16y + 25$, which appears to be a elliptic paraboloid.

4.3 Section 2.6, Checkpoint 2.54

Identify the surface represented by the equation $9x^2 + y^2 - z^2 + 2z - 10 = 0$.

Solution We have three squared terms, two positive coefficients and one negative coefficient. So this is a double cone.

4.4 Section 2.6, Exercise 350

Find the equation of the quadric surface with points $P(x, y, z)$ that are equidistant from point $Q(0, 2, 0)$ and plane of equation $y = -2$. Identify the surface.

Solution First, the distance from a point (x, y, z) to the plane $y = -2$ is just $|y|$. *This is because the plane $y = -2$ has normal vector $\hat{j}$; projecting $\overrightarrow{PQ}$ onto $\hat{j}$ just yields the y component.*

To do this, we want $\langle x, y, z \rangle$ that satisfies

$$\begin{aligned}
\sqrt{(x-0)^2 + (y-2)^2 + (z-0)^2} &= y - (-2) \\
\sqrt{x^2 + (y-2)^2 + z^2} &= y + 2 \\
x^2 + (y-2)^2 + z^2 &= (y+2)^2 \\
x^2 + y^2 - 4y + 4 + z^2 &= y^2 + 4y + 4 \\
x^2 + z^2 &= 8y
\end{aligned}$$

This is the form of an elliptic paraboloid.

5 09-09-2025

5.1 Section 3.1, Exercise 8

Find the limit of the following vector valued function at the indicated value of t.

$$\lim_{t\to 4}\left\langle \sqrt{t-3}, \frac{\sqrt{t}-2}{t-4}, \tan\left(\frac{\pi}{t}\right)\right\rangle$$

Solution We can just substitute for the x and z components. The middle one requires more thought; direct substitution gives $\frac{0}{0}$, so we apply L'Hopital's rule.

$$\lim_{t\to 4}\frac{\sqrt{t}-2}{t-4} = \lim_{t\to 4}\frac{\frac{1}{2\sqrt{t}}}{1} = \frac{1}{4}$$

So the limit is

$$\left\langle 1, \frac{1}{4}, 1\right\rangle$$

5.2 Section 3.1, Exercise 22

Eliminate the parameter t, write the equation in Cartesian coordinates, then sketch the graphs of the vector-valued functions. *Hint: solve first equation for x in terms of t and substitute this result into the second equation.*

$$\mathbf{r}(t) = 2t\hat{i} + t^2\hat{j} \qquad \text{let } x = 2t, y = t^2$$

THEOREM 3.3

Properties of the Derivative of Vector-Valued Functions

Let **r** and **u** be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

i.	$\frac{d}{dt}[c\mathbf{r}(t)]$	$=$	$c\mathbf{r}'(t)$	Scalar multiple
ii.	$\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)]$	$=$	$\mathbf{r}'(t) \pm \mathbf{u}'(t)$	Sum and difference
iii.	$\frac{d}{dt}[f(t)\mathbf{u}(t)]$	$=$	$f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$	Scalar product
iv.	$\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)]$	$=$	$\mathbf{r}'(t)\cdot\mathbf{u}(t) + \mathbf{r}(t)\cdot\mathbf{u}'(t)$	Dot product
v.	$\frac{d}{dt}[\mathbf{r}(t)\times\mathbf{u}(t)]$	$=$	$\mathbf{r}'(t)\times\mathbf{u}(t) + \mathbf{r}(t)\times\mathbf{u}'(t)$	Cross product
vi.	$\frac{d}{dt}[\mathbf{r}(f(t))]$	$=$	$\mathbf{r}'(f(t))\cdot f'(t)$	Chain rule
vii.	If $\mathbf{r}(t)\cdot\mathbf{r}(t)$	$=$	c, then $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$.	

5.3 Section 3.2, Checkpoint 3.5

Calculate the derivative of the function

$$\mathbf{r}(t) = (t \ln t)\hat{\imath} + (5e^{t})\hat{\jmath} + (\cos t - \sin t)\hat{k}$$

5.4 Section 3.2, Checkpoint 3.7

Find the unit tangent vector for the vector-valued functions

$$\mathbf{r}(t) = (t^2 - 3)\hat{\imath} + (2t + 1)\hat{\jmath} + (t - 2)\hat{k}$$

5.5 Section 3.2, Checkpoint 3.8

Calculate the following integral:

$$\int_1^3 \left[(2t + 4)\hat{\imath} + (3t^2 - 4t)\hat{\jmath}\right]\, dt$$

6 09-11-2025

6.1 Section 3.3, Checkpoint 3.9

Calculate the arc length of the parameterized curve

$$\mathbf{r}(t) = \langle 2t^2 + 1, 2t^2 - 1, t^3 \rangle, \quad 0 \leq t \leq 3$$

Extension: write the arc length parametrization by solving for t in terms of s

6.2 Section 3.3, Checkpoint 3.13

Find the equation of the osculating circle of the curve defined by the vector-valued function

$$y = 2x^2 - 4x + 5 \text{ at } x = 1$$

6.3 Section 3.3, Exercise 3.143

Find the equation for the osculating plane at point $t = \dfrac{\pi}{4}$ on the curve

$$\mathbf{r}(t) = \cos(2t)\hat{\imath} + \sin(2t)\hat{\jmath} + t\hat{k}$$

7 09-16-2025 - Midterm 1 tomorrow

Logistics: see @35 on Piazza.

- 50 mins
- no electronics at all
- single-page, typical 8.5x11 or A4 size paper, front and back, can typeset or handwrite
- write your answer in the provided answer box so we can grade faster

Points, vectors, lines, planes, quadrics. See the handout for quadrics that I gave out, and consult the wikipedia if you need help making your formula sheet.

Dot and Cross products and their meanings

- how does the dot product relate to projections/components?
- how does the cross product relate to areas/volumes?
- how do dot and cross products help when calculating work and torque?

Calculus on vector-valued functions (limits, derivatives, integrals)

- Remember examples from previous recitations - sometimes we had you do L'Hopital's or do questions that required knowledge of limits from 2-D calculus.

Arc length + arc length parameterization

- What is the formula?

Curvature + unit tangent, normal, binormal

- Also includes osculating planes/circles
- know the formulas (or write them down on your formula sheet)

motion in space

- apply what you know about projections/components to find out the "component" of the tangential/normal components of acceleration
- plus other questions regarding position, velocity, speed, acceleration.

What can I do to study? **DO A PRACTICE TEST** and check your answers afterwards. Read up on the textbook (see @35 on Piazza for which sections to ignore) and get a good night's sleep. I will be proctoring your exam, so I'll see you all tomorrow :-)

8 09-18-2025 - Post Midterm 1, No Content

9 09-23-2025

9.1 Section 4.1, Checkpoint 4.4

Find the domain and range of the function $f(x, y) = \sqrt{36 - 9x^2 - 9y^2}$

9.2 Section 4.1, Checkpoint 4.5

Find the equation of the level surface of the function

$$g(x, y, z) = x^2 + y^2 + z^2 - 2x + 4y - 6z$$

corresponding to $c = 2$, and describe the surface, if possible.

9.3 Section 4.2, Exercise 60

Evaluate $\lim_{(x,y)\to(1,2)} x$ or explain why the limit does not exist.

9.4 Section 4.2, Exercise 77

$$\text{Evaluate } \lim_{(x,y)\to(0,0)} \ln\left(x^2 + y^2\right) \text{ or explain why the limit does not exist.}$$

*Does the limit exist on **any** path?*

9.5 Section 4.2, Exercise 81

$$\text{Evaluate } \lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^4}{x + 2y^2} \text{ or explain why the limit does not exist.}$$

Do you see any algebraic manipulations?

9.6 Section 4.2, Exercise 86

$$\text{Evaluate} \quad \lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2} \quad \text{or explain why the limit does not exist.}$$

(a) Along the x-axis ($y = 0$)

(b) Along the y-axis $x = 0$

(c) Along the path $y = 2x$

9.7 Section 4.2, Exercise 88

$$\text{Evaluate} \quad \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} \quad \text{or explain why the limit does not exist.}$$

(a) Along the x-axis ($y = 0$)

(b) Along the y-axis $x = 0$

(c) Along the path $y = x^2$

9.8 Section 4.2, Exercise 95

Determine the region in which the function is continuous. Explain your answer

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Hint: maybe this coordinate system isn't helpful. Also remember the Squeeze Theorem?
We do NOT expect you to use Squeeze Theorem with completely general functions

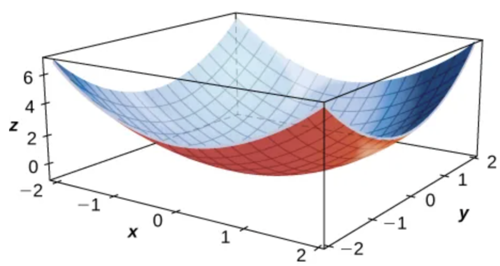
10 09-25-2025

10.1 Section 4.3, Checkpoint 4.13

Calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for the function $f(x, y) = \tan\left(x^3 - 3x^2y^2 + 2y^4\right)$ by holding the opposite variable constant, then differentiating.

10.2 Section 4.3, Exercises 114-117

Calculate the sign of the partial derivative using the graph of the surface f below.



114. $f_x(1, 1)$
115. $f_x(-1, 1)$
116. $f_y(1, 1)$
117. $f_x(0, 0)$

10.3 Section 4.4, Checkpoint 4.19

Find the equation of the tangent plane to the surface defined by the function

$$f(x, y) = x^3 - x^2y + y^2 - 2x + 3y - 2 \text{ at the point } (-1, 3)$$

10.4 Section 4.4, Checkpoint 4.22

Find the differential dz of the function

$$f(x, y) = 4y^2 + x^2 y - 2xy$$

and use it to approximate Δz at point $(1, -1)$. Use $\Delta x = 0.03$ and $\Delta y = -0.02$. What is the exact value of Δz?

11 09-30-2025

11.1 Section 4.5, Checkpoint 4.23

Calculate $\frac{dz}{dt}$ given the following functions. Express your final answer in terms of t.

$$z = f(x, y) = x^2 - 3xy + 2y^2$$

$$x = x(t) = 3\sin 2t$$

$$y = y(t) = 4\cos 2t$$

11.2 Section 4.5, Exercise 253

The radius of a right circular cone is increasing at 3 cm/min whereas the height of the cone is decreasing at 2 cm/min. Find the rate of change of the volume of the cone when the radius is 13 cm and the height is 18 cm.

11.3 Section 4.5, Example 4.30

(a) Calculate $\frac{dy}{dx}$ if y is defined implicitly as a function of x via the equation

$$3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$$

What is the equation of the tangent line to the graph of this curve at point $(2, 1)$?

(b) Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, given $x^2 e^y - yze^x = 0$.

11.4 Section 4.6, Example 4.31

Let $\theta = \arccos\left(\frac{3}{5}\right)$. Find the directional derivative $D_u f(x, y)$ of $f(x, y) = x^2 - xy + 3y^2$ in the direction of $u = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$. What is $D_u f(-1, 2)$?

Use the definition of directional derivative (x-h defn)

11.5 Section 4.6, Example 4.32

Let $\theta = \arccos\left(\frac{3}{5}\right)$. Find the directional derivative $D_u f(x, y)$ of $f(x, y) = x^2 - xy + 3y^2$ in the direction of $u = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$. What is $D_u f(-1, 2)$?

Use the partial derivatives

11.6 Section 4.6, Checkpoint 4.30

Find the direction for which the directional derivative of $g(x, y) = 4x - xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

12 10-02-2025

12.1 Section 4.7, Checkpoint 4.35

Use the second derivative to find the local extrema of the function

$$f(x, y) = x^3 + 2xy - 6x - 4y^2$$

12.2 Section 4.7, Checkpoint 4.36

Given $z = f(x, y)$ is continuous and differentiable on a closed, bounded set D, the strategy to find absolute extrema of f on D is to

1. Determine the critical points of f in D
2. Calculate f at each of these critical points
3. Determine the maximum and minimum values of f on the boundary of its domain
4. Find the absolute maximum and minimum values of f by comparing the values from steps 2 and 3.

Use the above strategy to find the absolute extrema of

$$f(x, y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3 \text{ on } \{(x, y) : 0 \leq x \leq 2 \text{ and } -1 \leq y \leq 3\}$$

13 10-07-2025 - Midterm 2 tomorrow

Use the method of Lagrange Multipliers to find the maximum and minimum values of the function subject to the given constraints.

13.1 Section 4.8, Exercise 369

Minimize $f(x, y) = x^2 + y^2$ on the hyperbola $xy = 1$.

13.2 Section 4.8, Exercise 373

The curve $x^3 - y^3 = 1$ is asymptotic to the line $y = x$. Find the point(s) on the curve $x^3 - y^3 = 1$ farthest from the line $y = x$.

What is the function we seek to minimize?

13.3 Section 4.8, Exercise 374

Maximize $U(x, y) = 8x^{\frac{4}{5}}y^{\frac{1}{5}}$ under constraint $4x + 2y = 12$.

13.4 Section 4.8, Exercise 379

Maximize $f(x, y, z) = x^2 + y^2 + z^2$ under constraints $x + .y + z = 9$ and $x + 2y + 3z = 20$.

13.5 Section 4.8, Exercise 388

A large container in the shape of a rectangular solid must have a volume of $480m^3$. The bottom of the container costs $\$5/m^2$ to construct, whereas the top and sides cost $\$3/m^2$ to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has the minimum cost.

14 10-09-2025 - Post Midterm 2, No Content

15 10-21-2025

Properties of Double Integrals

Let $f(x,y)$ and $g(x,y)$ are integrable over the rectangular region R, and let S and T be subregions of R. I've added one-word ways to remember the each property:

- (Addition) The sum $f(x,y)+g(x,y)$ is integrable and

$$\iint_R \left[f(x,y)+g(x,y)\right]\ dA = \iint_R f(x,y)\ dA + \iint_R g(x,y)\ dA$$

- (Multiplication) If c is a constant, then $c \cdot f(x,y)$ is integrable and

$$\iint_R c \cdot f(x,y)\ dA = c \iint_R f(x,y)\ dA$$

- (Partitioning the Region) If $R = S \cup T$ and $S \cap T = \varnothing$, except some overlap on the boundaries, then

$$\iint_R f(x,y)\ dA = \iint_S f(x,y)\ dA + \iint_T f(x,y)\ dA$$

- (Dominance) If $f(x,y) \geq g(x,y)$ for $(x,y) \in R$, then

$$\iint_R f(x,y)\ dA \geq \iint_R g(x,y)\ dA$$

- (Extrema) If for $m \in \mathbb{R}$ and $M \in \mathbb{R}$, then

$$m \cdot \text{Area}(R) \leq \iint_R f(x,y)\ dA \leq M \cdot \text{Area}(R)$$

  Think of it as this, since R is a closed and bounded region, f has absolute minimum m and absolute maximum M on R. Then we can do

$$m \leq f(x,y) \text{ on } R \leq M \implies \iint_R m\ dA \leq \iint_R f(x,y)\ dA \leq \iint_R M\ dA$$

- In the case where $f(x,y)$ can be factored as a product of a function $g(x)$ only dependent on x and a function $h(y)$ only dependent on y, then over the region $R = \{(x,y) : x \in [a,b], y \in [c,d]\}$, the double integral can be written as

$$\iint_R f(x,y)\ dA = \left(\int_a^b g(x)\ dx\right) \cdot \left(\int_c^d h(y)\ dy\right)$$

Fubini's Theorem

Suppose that $f(x,y)$ is a function of two variables that is continuous over a rectangular region $r = \{(x,y) \in \mathbb{R} : x \in [a,b], y \in [c,d]\}$. Then we can switch the order of integration, i.e.,

$$\iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dx\ dy$$

Generally, Fubini's holds if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

15.1 Section 5.1, Checkpoint 5.2

(a) Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3 - x + 4y)\ dy\ dx$$

(b) Show that

$$0 \leq \iint_R \sin \pi x \cos \pi y\ dA \leq \frac{1}{32}$$

where $R = \left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, \frac{1}{2}\right]$.

15.2 Section 5.1, Checkpoint 5.5

Find the volume of the solid bounded above by the graph of $f(x, y) = xy \sin\left(x^2 y\right)$ and below by the xy-plane on the rectangular region $R = [0, 1] \times [0, \pi]$

15.3 Section 5.2, Checkpoint 5.12

Find the volume of the solid bounded above by $f(x, y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$, where $x \in [0, 1]$.

15.4 Section 5.2, Checkpoint 5.15

Evaluate the improper integral

$$\iint_D \frac{y}{\sqrt{1-x^2-y^2}}\, dA$$

where $D = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$

15.5 Section 5.1, Exercise 36

Find the average value of the function over the given rectangles.

$$f(x, y) = x^4 + 2y^3, R = [1, 2] \times [2, 3]$$

15.6 Section 5.2, Exercise 68, 69

See the textbook, Type I vs Type II region problem.

15.7 Section 5.2, Exercise 98

$$\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2 \; dx \; dy$$

16 10-23-2025

Remember that when we convert to polar, our Jacobian (coming soon) is r. So

$$\iint_R f(x,y)\ dx\ dy = \iint_R f(r,\theta) r\ dr\ d\theta$$

16.1 Section 5.3, heckpoint 5.17

Sketch the region $R = \{(r,\theta) : r \in [1,2], \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\}$

16.2 Section 5.3, Checkpoint 5.20

Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

16.3 Section 5.3, Checkpoint 5.21

Find the area enclosed inside the cardioid $r = 3 - 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$

16.4 Section 5.3, Checkpoint 5.22

Evaluate the integral

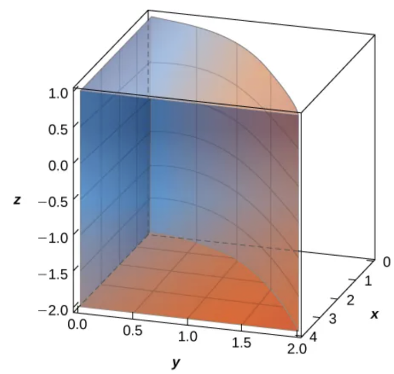
$$\iint_{\mathbb{R}^2} e^{-4\cdot(x^2+y^2)}\ dx\ dy$$

17 10-28-2025

17.1 Section 5.4, Exercise 212

Evaluate the triple integral over the bounded region $E = \{(x, y, z) : (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$, where D is the projection of E onto the xy-plane.

E is the solid bounded by $y = \sqrt{x}$, $x = 4$, $y = 0$, $z = -2$, and $z = 1$ (see figure below).



Evaluate $\iiint_E xyz \, dV$ (use $dx\ dy\ dz$).

17.2 Section 5.5, Checkpoints 5.28

Consider the region E inside the right circular cylinder with equation $r = 2\sin\theta$, bounded below by the $r\theta$-plane and bounded above by $z = 4 - y$. Set up a triple integral with a function $f(r, \theta, z)$ in cylindrical coordinates.

17.3 Section 5.5, Checkpoints 5.31

Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho = 2$ and bounded below by the cone $\phi = \frac{\pi}{3}$.

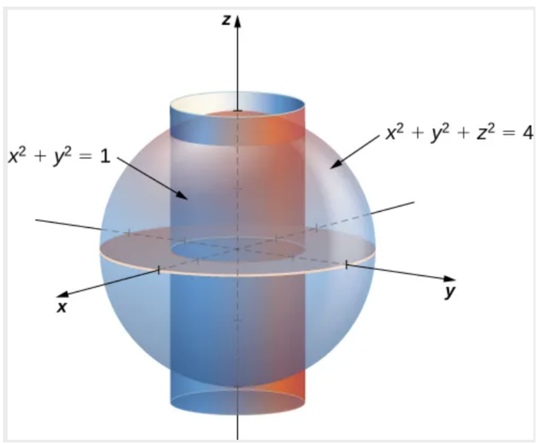
17.4 Section 5.5, Example 5.50

the book does not draw a picture, but you should Convert the following rectangle into cylindrical coordinates:

$$\int_{y=-1}^{1} \int_{x=0}^{\sqrt{1-y^2}} \int_{z=x^2+y^2}^{\sqrt{x^2+y^2}} xyz\ dz\ dx\ dy$$

17.5 Section 5.5, Checkpoint 5.32

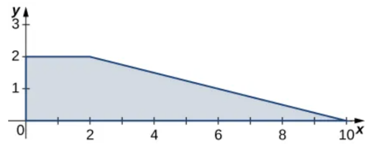
cyl. and sph. only Use cylindrical and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ but outside the cylinder $x^2 + y^2 = 1$.



18 10-30-2025

18.1 Section 5.6, Exercise 301

computation to final answer left to you all :-) In the following exercises, the region R is the trapezoidal region determined by the lines $y = \frac{1}{4}x + \frac{5}{2}$, $y = 0$, $y = 2$, and $x = 0$. Find the mass of R with the density function $\rho(x, y) = 3xy$.



18.2 Section 5.6, Exercise 346

computation to final answer left to you all :-) The solid Q is bounded by the cylinder $x^2 + y^2 = a^2$, the paraboloid $b^2 - z = x^2 + y^2$, and the xy plane, where $0 < a < b$. Find the mass of the solid if its density is given by $\rho(x, y, z) = \sqrt{x^2 + y^2}$.

18.3 Section 5.6, Example 5.62

try different orders of integration - you should get same answer regardless Suppose that Q is a solid region bounded by $x + 2y + 3z = 6$ and the coordinate planes and has density $\rho(x, y, z) = x^2yz$. Find the total mass.

18.4 Challenge: Section 5.6, Exercise 343

The mass of a solid Q is given by

$$\int_0^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} \left(x^2 + y^2 + z^2\right)^n \, dz \, dy \, dx$$

where n is an integer. Determine n such that the mass of the solid is $(2 - \sqrt{2}) \cdot \pi$.

19 11-04-2025

PROBLEM-SOLVING STRATEGY

Change of Variables

1. Sketch the region given by the problem in the xy-plane and then write the equations of the curves that form the boundary.
2. Depending on the region or the integrand, choose the transformations $x = g(u, v)$ and $y = h(u, v)$.
3. Determine the new limits of integration in the uv-plane.
4. Find the Jacobian $J(u, v)$.
5. In the integrand, replace the variables to obtain the new integrand.
6. Replace $dy\,dx$ or $dx\,dy$, whichever occurs, by $J(u, v)\,du\,dv$.

19.1 Section 5.7, Exercise 359, 360

The given function $T : S \to R$, $T(u, v) = (x, y)$ on the region $S = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$ bounded by the unique square is given. $R \subset \mathbb{R}^2$ is the image of S under T.

(a) Justify that T is a C^1 transformation

(b) Find the images of the vertices of the unit square S through the function T

(c) Determine the image R of the unit square S and graph it

359. $= 2u - v$, $y = u + 2v$

360. $x = u^2$, $y = v^2$

19.2 Section 5.7, Checkpoint 5.46

Make appropriate changes of variables in the integral

$$\iint_R \frac{4}{(x-y)^2}\ dy\ dx$$

where R is the trapezoid bounded by the lines $x - y = 2$, $x - y = 4$, $x = 0$, and $y = 0$. Write the resulting integral.

19.3 Section 5.7, Checkpoint 5.47

Using the substitutions $x = v$, $y = \sqrt{u + v}$, evaluate the integral

$$\iint_R y \sin\left(y^2 - x\right) \, dA$$

where R is the region bounded by the lines $\sqrt{x}$, $x = 2$, and $y = 0$.

19.4 Section 5.7, Checkpoint 5.48

Let D be the region in xyz-space defined by $1 \leq x \leq 2$, $0 \leq xy \leq 2$, and $0 \leq z \leq 1$. Evaluate

$$\iiint_D (x^2y + 3xyz) \, dx \, dy \, dz$$

by using the transformation $u = x$, $v = xy$, and $w = 3z$.

20 11-06-2025

21 11-11-2025 - Midterm 3 tomorrow

22 11-13-2025 - Post Midterm 3, No Content

23 11-18-2025

25 11-25-2025

26 12-02-2025 - Final Review

27 12-04-2025 - Final Review