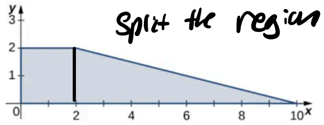
18.1 Section 5.6, Exercise 301

computation to final answer left to you all :-) In the following exercises, the region R is the trapezoidal region determined by the lines $y = \frac{1}{4}x + \frac{5}{2}$, $y = 0$, $y = 2$, and $x = 0$. Find the mass of R with the density function $\rho(x, y) = 3xy$.



Split the region

$$\text{mass} = \iint_R \rho \, dA$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} 3xy \, dy \, dx + \int_{x=2}^{10} \int_{y=0}^{\frac{1}{4}x+\frac{5}{2}} 3xy \, dy \, dx$$

$$= 3\int_{x=0}^{2} x \, dx \int_{y=0}^{2} y \, dy + \int_{x=}^{10} \left[\frac{3xy^2}{2}\right]_{y=0}^{\frac{1}{4}x+\frac{5}{2}} dx$$

$$= 3 \cdot 2 \cdot 2 + \int_{x=2}^{10} \frac{3}{2}x\left(\frac{1}{4}x + \frac{5}{2}\right) dx$$

$$= 12 + \int_{x=2}^{10} \frac{3}{8}x^2 + \frac{15}{4}x \, dx$$

$$= 12 + \left[\frac{x^3}{8} + \frac{15}{8}x^2\right]_{x=2}^{10}$$

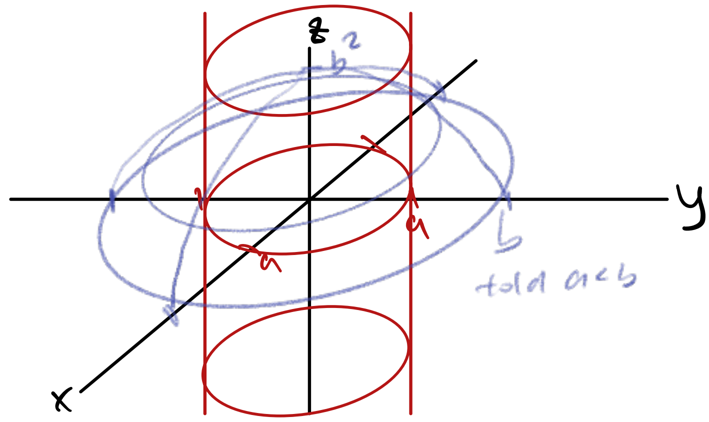
$$= 12 + \frac{1}{8}\left[(1000-8) + 15 \cdot (100-4)\right]$$

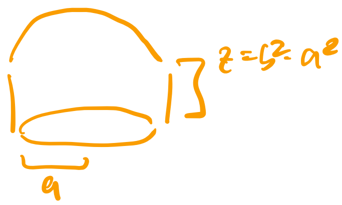
$$= 316$$

18.2 Section 5.6, Exercise 346

computation to final answer left to you all :-) The solid Q is bounded by the cylinder $x^2+y^2=a^2$, the paraboloid $b^2-z=x^2+y^2$, and the $\underbrace{xy \text{ plane}}_{z=0}$, where $0<a<b$. Find the mass of the solid if its density is given by $\rho(x,y,z)=\sqrt{x^2+y^2}$.

$\rho = r$

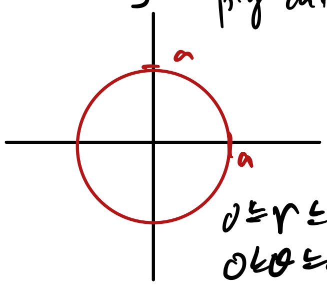




$x^2+y^2=a^2 \qquad b^2-z=x^2+y^2$

$a^2=b^2-z \;\rightarrow\; z=b^2-a^2$

proj onto xy-plane



$0\le r\le a$

$0\le\theta\le 2\pi$

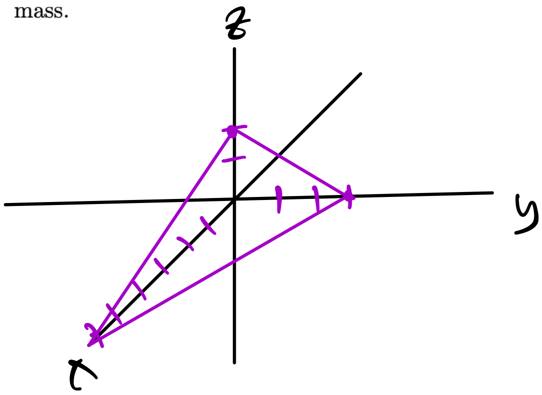
$$\int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{b^2-r^2}[\,r\,]\,r\,dz\,dr\,d\theta$$

$$= 2\pi\int_{r=0}^{a}(b^2-r^2)r^2\,dr \qquad = 2\pi\int_{r=0}^{a} b^2r^2 \;\; r^4\,dr$$

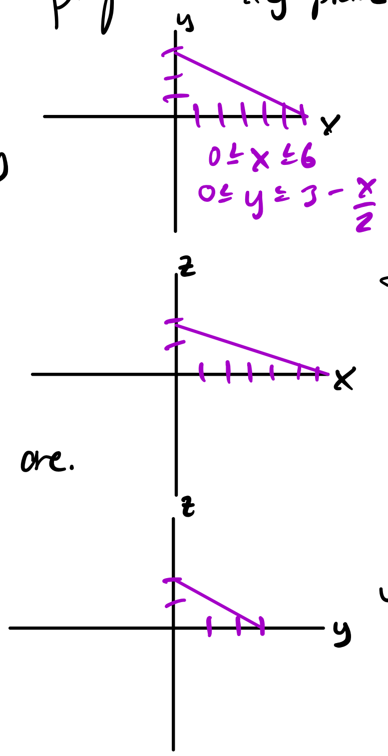
$$= 2\pi\left[\frac{b^2r^3}{3}+\frac{r^5}{5}\right]_{r=0}^{a} = \boxed{2\pi\left[\frac{a^3b^2}{3}+\frac{a^5}{5}\right]}$$

18.3 Section 5.6, Example 5.62

try different orders of integration - you should get same answer regardless Suppose that Q is a solid region bounded by $x+2y+3z=6$ and the coordinate planes and has density $\rho(x,y,z)=x^2yz$. Find the total mass.



proj onto xy-plane



$0 \le x \le 6$

$0 \le y \le 3 - \frac{x}{2}$

Sket these up if you want

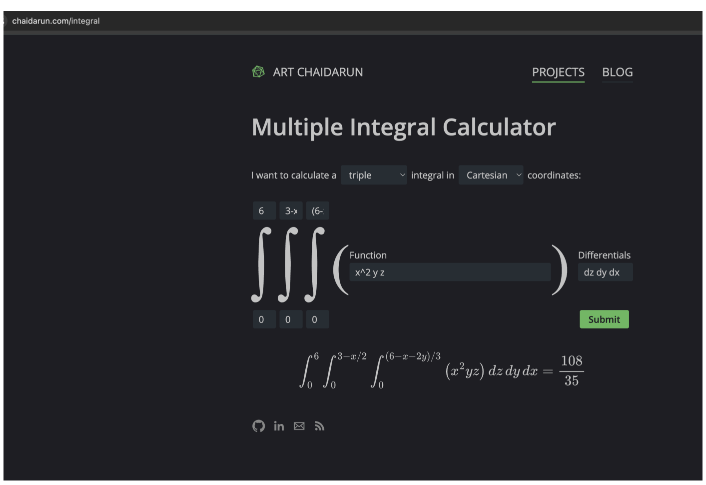
there are $3! = 6$ orders of integration. I'll just do one.

$$\int_{x=0}^{6}\int_{y=0}^{3-\frac{x}{2}}\int_{z=0}^{\frac{6-x-2y}{3}} x^2yz\,dz\,dy\,dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{3-\frac{x}{2}} x^2y\,\frac{(6-x-2y)^2}{2\cdot 9}\,dy\,dx$$

$\vdots$

plugged into Wolfram Alpha $= \frac{108}{35}$



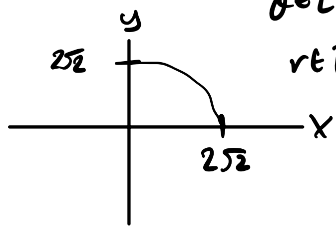
18.4 Challenge: Section 5.6, Exercise 343

The mass of a solid Q is given by

$$\int_0^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \int_{\underset{r}{\sqrt{x^2+y^2}}}^{\overset{\sqrt{16-r^2}}{\overbrace{\sqrt{16-x^2-y^2}}}} \left(x^2+y^2+z^2\right)^n \, dz \, dy \, dx$$

where n is an integer. Determine n such that the mass of the solid is $(2-\sqrt{2})\cdot\pi$.

convert to spherical



$\theta \in [0, \frac{\pi}{2}]$

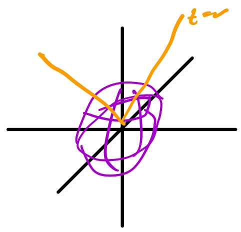
$r \in [0, 2\sqrt{2}]$ - $\rho \sin\varphi$

$z = \rho\cos\varphi \in [r, \underbrace{\sqrt{16-r^2}}_{\text{sphere of radius } 4}]$

$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{\varphi=0}^{\frac{\pi}{4}} \int_{\rho=0}^{4} (\rho^2)^n \, \rho^2 \sin\varphi \; d\rho \, d\varphi \, d\theta$$

$\rho\sin\varphi = \rho\cos\varphi$

$\varphi = \frac{\pi}{4}$



$\sqrt{16-r^2} = r$

$r^2 = 16 - r^2$

$2r^2 = 16$

$r = \sqrt{8}$

$$\to \frac{\pi}{2} \int_{\varphi=0}^{\frac{\pi}{4}} \sin\varphi \, d\varphi \int_{\rho=0}^{4} (\rho^2)^{n+1} \, d\rho$$

$$= \frac{\pi}{2} \underbrace{\left[-\cos\varphi\right]_0^{\frac{\pi}{4}}}_{\left(-\frac{\sqrt{2}}{2}+1\right)} \left[\frac{\rho^{2n+3}}{2n+3}\right]_0^4 = (2-\sqrt{2})\pi$$

$$\cancel{\pi} \underbrace{\cancel{\left[\frac{1}{2} - \frac{\sqrt{2}}{4}\right]}}_{\frac{2-\sqrt{2}}{4}} \cdot \frac{4^{2n+3}}{2n+3} = (2-\sqrt{2})\cancel{\pi}$$

$4^{2n+2} = 2n+3$

let $n = -1$.

$4^{-2+2} = 4^0 = 1 = 2\cdot(-1)+3 = 1$

$n = -1$