1 08-26-2025

1.1 Basics

This class is **Calculus in Three Dimensions**, thus the class will require you to think in 3-D.

- Get used to drawing in 3-D.

  My preference is to draw the usual vertical and horizontal axes.

  Label the horizontal axis y and the vertical axis z.

  Then draw a diagonal from the bottom left to the top right, passing through the origin. Label this axis x. Note that you can interchange any of the axis labels as necessary (which will come in handy when we start doing 3-D integration).

- In $\mathbb{R}^3$, the (Euclidean) distance (also referred to as the L_2 norm) between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1.2 Planes and Spheres

In this course, you will often deal with planes and spheres (e.g., spherical coordinates).

- You will learn the canonical formula for a plane later (finding it involves cross products). For now, we will find equations for planes parallel to another plane.

Ex. Write an equation of the plane passing through point $(21, 2, 59)$ that is parallel to the xz-plane.

When a plane is parallel to the xz-plane, it means only the x and z coordinates may vary.

Thus taking the y-value, we get the equation $y = 2$.

Ex. Write an equation of the plane passing through points $(2, 125, 9), (21, 25, 9), (5, 7, 9)$ that is parallel to the xy-plane.

When a plane is parallel to the xy-plane, it means only the x and y coordinates may vary.

Conveniently, the z-value in all 3 coordinates are equal; we get the equation $z = 9$.

- A sphere is the set of all points in space equidistant from a fixed point, the center of the sphere. For center (a, b, c) and radius r, we represent the sphere by the ("canonical") equation:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Ex. Find the equation of the sphere with diameter $\overline{PQ}$ where $P = (2, -1, -3)$ and $Q = (-2, 5, -1)$.

First, find the center, which is at the midpoint of the diameter $\overline{PQ}$:

$$C = \left(\frac{2 + (-2)}{2}, \frac{-1 + 5}{2}, \frac{-3 + (-1)}{2} \right) = (0, 2, -2)$$

Next, find the radius using the distance formula (half the length of the diameter)

$$r = \frac{1}{2} \left\| \overline{PQ} \right\| = \frac{1}{2} \sqrt{(-2 - 2)^2 + (5 - (-1))^2 + (-1 - (-3))^2} = \frac{1}{2} \sqrt{56} = \sqrt{\frac{56}{4}} = \sqrt{14}$$

Thus the sphere is given by $x^2 + (y - 2)^2 + (z + 2)^2 = 14$

1.3 Vector Notation

Vectors are quantities with magnitude and direction. In $\mathbb{R}^3$, the standard unit vectors are

$$\hat{i} = \langle 1,0,0 \rangle \quad \hat{j} = \langle 0,1,0 \rangle \quad \hat{k} = \langle 0,0,1 \rangle$$

There are several notations. Fix points in $\mathbb{R}^3$ $P = (0,2,1)$ and $Q = (2,5,9)$
Let's represent the vector $\overrightarrow{PQ} = \langle x_Q - x_P, y_Q - y_P, z_Q - z_P \rangle$.

- Component Form: $\overrightarrow{PQ} = \langle 2 - 0, 5 - 2, 9 - 1 \rangle = \langle 2,3,8 \rangle$
- Using the unit vectors: $\overrightarrow{PQ} = 2\hat{i} + 3\hat{j} + 8\hat{k}$

Ex: Consider the vectors $v = \langle 0,2,1 \rangle$ and $u = \langle 2,5,9 \rangle$. Find a unit vector in the direction of $3v + u$.

You will need the following rules (taken from your textbook)

RULE: PROPERTIES OF VECTORS IN SPACE

Let $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle$

Vector addition: $\mathbf{v} + \mathbf{w} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$

Vector subtraction: $\mathbf{v} - \mathbf{w} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$

Vector magnitude: $\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$

Unit vector in the direction of $\mathbf{v}$: $\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\langle x_1, y_1, z_1 \rangle = \left\langle \frac{x_1}{\|\mathbf{v}\|}, \frac{y_1}{\|\mathbf{v}\|}, \frac{z_1}{\|\mathbf{v}\|} \right\rangle$, if $\mathbf{v} \neq \mathbf{0}$

Solution: first find $3v + u$, then find unit vector in that direction.

$$3v + u = \langle 3(0), 3(2), 3(1) \rangle + \langle 2,5,9 \rangle \qquad \text{(scalar multiplication)}$$
$$= \langle 0,6,3 \rangle + \langle 2,5,9 \rangle = \langle 2,11,12 \rangle$$
$$\frac{1}{\|3v+u\|}(3v+u) = \frac{\langle 2,11,12 \rangle}{\sqrt{2^2 + 11^2 + 12^2}} = \langle \frac{2}{\sqrt{269}}, \frac{11}{\sqrt{269}}, \frac{12}{269} \rangle \qquad \text{(unit vector in direction)}$$

Observe that 269 is prime, so you can't simplify the denominator. Thus we're done. Not all numbers will be pretty, but as a tip: make sure your answers are feasible. If you are taking the length of a vector with 1-digit coordinates and get a 5-digit number, that's probably wrong.
DON'T FORGET THE SQUARE ROOT $\sqrt{}$ when taking norms!!!!

1.4 Computing 3x3 Determinants

Useful for computing cross products. You are free to use whatever trick you want so long as you show your work. (note vertical bars means determinant, while square brackets mean matrix).

1.4.1 Cofactor Method

Key notes: $+, -, +$ (**DON'T FORGET THE MIDDLE IS NEGATIVE**) and the 2x2 sub-determinants are the elements not in the same row or column.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei - afg - bdi + bfg + cdh - ceg$$

For cross products between two vectors $u = \langle u_x, u_y, u_z \rangle$ and $v = \langle v_x, v_y, v_z \rangle$, you just compute the 3x3 determinant $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$

1.4.2 Diagonal Method

Recall that the 2x2 determinant is computed using the diagonals: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, where the diagonals going from right to left are positive and diagonals going from left to right are negative. We extend this idea to 3x3 determinants:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \xLongrightarrow{expand} \begin{matrix} a & b & c \\ f & d & e & f & d \\ h & i & g & h & i & g & h \end{matrix} \xLongrightarrow{compute} aei + bfg + cdh - afh - bdi - ceg$$

which is the same result as derived from the cofactor method.