15 10-21-2025

Properties of Double Integrals

Let $f(x,y)$ and $g(x,y)$ are integrable over the rectangular region R, and let S and T be subregions of R. I've added one-word ways to remember the each property:

- (Addition) The sum $f(x,y)+g(x,y)$ is integrable and

$$\iint_R \left[f(x,y)+g(x,y)\right]\ dA = \iint_R f(x,y)\ dA + \iint_R g(x,y)\ dA$$

- (Multiplication) If c is a constant, then $c \cdot f(x,y)$ is integrable and

$$\iint_R c \cdot f(x,y)\ dA = c \iint_R f(x,y)\ dA$$

- (Partitioning the Region) If $R = S \cup T$ and $S \cap T = \varnothing$, except some overlap on the boundaries, then

$$\iint_R f(x,y)\ dA = \iint_S f(x,y)\ dA + \iint_T f(x,y)\ dA$$

- (Dominance) If $f(x,y) \geq g(x,y)$ for $(x,y) \in R$, then

$$\iint_R f(x,y)\ dA \geq \iint_R g(x,y)\ dA$$

- (Extrema) If for $m \in \mathbb{R}$ and $M \in \mathbb{R}$, then

$$m \cdot \text{Area}(R) \leq \iint_R f(x,y)\ dA \leq M \cdot \text{Area}(R)$$

  Think of it as this, since R is a closed and bounded region, f has absolute minimum m and absolute maximum M on R. Then we can do

$$m \leq f(x,y) \text{ on } R \leq M \implies \iint_R m\ dA \leq \iint_R f(x,y)\ dA \leq \iint_R M\ dA$$

- In the case where $f(x,y)$ can be factored as a product of a function $g(x)$ only dependent on x and a function $h(y)$ only dependent on y, then over the region $R = \{(x,y) : x \in [a,b], y \in [c,d]\}$, the double integral can be written as

$$\iint_R f(x,y)\ dA = \left(\int_a^b g(x)\ dx\right) \cdot \left(\int_c^d h(y)\ dy\right)$$

Fubini's Theorem

Suppose that $f(x,y)$ is a function of two variables that is continuous over a rectangular region $r = \{(x,y) \in \mathbb{R} : x \in [a,b], y \in [c,d]\}$. Then we can switch the order of integration, i.e.,

$$\iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dx\ dy$$

Generally, Fubini's holds if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

15.1 Checkpoint 5.1

Use the function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$.

Divide R into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square $(0, 1), (1, 1), (0, 2), (1, 2)$ to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

15.2 Checkpoint 5.2

(a) Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3 - x + 4y)\ dy\ dx$$

(b) Show that

$$0 \leq \iint_R \sin \pi x \cos \pi y\ dA \leq \frac{1}{32}$$

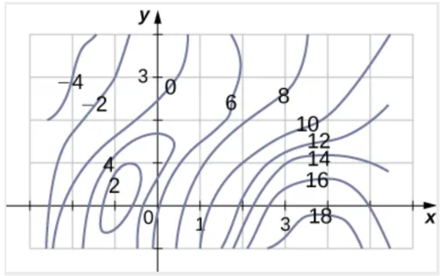
where $R = \left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, \frac{1}{2}\right]$.

15.3 Checkpoint 5.5

Find the volume of the solid bounded above by the graph of $f(x, y) = xy \sin\left(x^2 y\right)$ and below by the xy-plane on the rectangular region $R = [0, 1] \times [0, \pi]$

15.4 Checkpoint 5.6

A contour map is shown for a function $f(x, y)$ on the rectangle $R = [-3, 6] \times [-1, 4]$.



(a) Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of $\iint_R f(x, y)\, dA$

(b) Estimate the average value of the function $f(x, y)$

15.5 Checkpoint 5.12

Find the volume of the solid bounded above by $f(x, y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$, where $x \in [0, 1]$.

15.6 Checkpoint 5.15

Evaluate the improper integral

$$\iint_D \frac{y}{\sqrt{1 - x^2 - y^2}}\, dA$$

where $D = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$

15.7 Extension to Probability: Checkpoint 5.16

For those who are interested in how this class applies to future classes.