08/29/23 Recitation 1

Looking into Three-Dimensional Coordinate Systems: In lecture, we've started to talk about $\mathbb{R}^3$ and how it compares to $\mathbb{R}^2$. For example, the equation $x = 0$ represents a vertical line (the y-axis) in $\mathbb{R}^2$ but represents the yz-plane in $\mathbb{R}^3$. Continuing to make connections to $\mathbb{R}^3$, let us observe the distance formula.
In $\mathbb{R}^2$ given points $(x_1, y_1), (x_2, y_2)$, we know the distance is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In $\mathbb{R}^3$ given points $(x_1, y_1.z_1), (x_2, y_2, z_3)$, we know the distance is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Eg) Find the distance between points $A = (1, -5, 4)$ and $B = (4, -1, 1)$

$$\begin{aligned} d(A, B) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(4 - 1)^2 + (-1 - (-5))^2 + (1 - 4)^2} \\ &= \sqrt{(3)^2 + (4)^2 + (-3)^2} \\ &= \sqrt{34} \end{aligned}$$

Writing Equations in $\mathbb{R}^3$: Now let us take a look at how to write equations in $\mathbb{R}^3$ - particularly looking at planes and spheres. Observe these two examples:

1. Write an equation of the plane passing through point $(21, 2, 59)$ that is parallel to the xz-plane.
   - When a plane is parallel to the xz-plane, it means only the x and z coordinates may vary. Thus taking the y-value, we get the equation $y = 2$.
2. Write an equation of the plane passing through points $(2, 125, 9), (21, 25, 9), (5, 7, 9)$ that is parallel to the xy-plane.
   - Similarly, when a plane is parallel to the xy-plane, it means only the x and y coordinates may vary. Conveniently for us, we see that the z-value in all 3 coordinates are equal - and taking that value, we get the equation $z = 9$.

The takeaway is that in general, when finding equations that are parallel to a plane, we want to identify the axis ($x, y,$ or z) not included in the plane and represent the plane using that variable and a corresponding fixed point (from a point).

Now let us divert our attention to spheres.

Definition: A *sphere* is the set of all points in space equidistant from a fixed point, the center of the sphere. For center (a, b, c) and radius r, we represent the sphere by the equation:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Eg) Find the equation of the sphere with diameter PQ where $P = (2, -1, -3)$ and $Q = (-2, 5, -1)$.

First find the center C, which lies at the midpoint of the diameter

$$C = (\frac{2 + (-2)}{2}, \frac{-1 + 5}{2}, \frac{-3 + (-1)}{2}) = (0, 2, -2)$$

Next, find the radius using the distance formula (half the length of the diameter)

$$r = \frac{1}{2}\sqrt{(-2-2)^2 + (5-(-1))^2 + (-1-(-3))^2} = \frac{1}{2}\sqrt{56} = \sqrt{\frac{56}{4}} = \sqrt{14}$$

Putting it all together, we have that the equation of the sphere is:

$$x^2 + (y-2)^2 + (z+2)^2 = 14$$

Vectors in $\mathbb{R}^3$: As we introduced in lecture, vectors are quantities with magnitude and direction. Here are properties of vectors (pulled straight from the textbook)

RULE: PROPERTIES OF VECTORS IN SPACE

Let $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle$

Vector addition: $\mathbf{v} + \mathbf{w} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$

Vector subtraction: $\mathbf{v} - \mathbf{w} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$

Vector magnitude: $\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$

Unit vector in the direction of v: $\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\langle x_1, y_1, z_1 \rangle = \left\langle \frac{x_1}{\|\mathbf{v}\|}, \frac{y_1}{\|\mathbf{v}\|}, \frac{z_1}{\|\mathbf{v}\|} \right\rangle$, if $\mathbf{v} \neq \mathbf{0}$

Let us first observe constructing a vector from two points in $\mathbb{R}^3$. Consider two points $P = (0, 2, 1)$ and $Q = (2, 5, 9)$. In general we can represent the vector $\overrightarrow{PQ}$ as $< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$. Using this component form, we find:

$$\begin{aligned} \overrightarrow{PQ} &=< 2 - 0, 5 - 2, 9 - 1 > \\ &=< 2, 3, 8 > \end{aligned}$$

We can also represent the solution in standard unit form via $\overrightarrow{PQ} = 2i + 3j + 8k$. Recall that i, j, k are unit vectors s.t. $i =< 1, 0, 0 >, j =< 0, 1, 0 >, k =< 0, 0, 1 >$.

Let us wrap up by solving a vector operation question. Consider vectors $v =< 0, 2, 1 >$ and $u =< 2, 5, 9 >$. We want to find a unit vector in the direction of $3v + u$. First let us evaluate $3v + u$:

$$3v + u = 3\cdot < 0, 2, 1 > + < 2, 5, 9 >=< 3 \cdot 0 + 2, 3 \cdot 2 + 5, 3 \cdot 1 + 9 >=< 2, 11, 12 >$$

In order to find a unit vector in the same direction, we can divide the vector by it's magnitude:

$$\frac{3v + u}{||3v + u||} = \frac{1}{\sqrt{2^2 + 11^2 + 12^2}} \cdot < 2, 11, 12 >=< \frac{2}{\sqrt{269}}, \frac{11}{\sqrt{269}}, \frac{12}{\sqrt{269}} >$$

[Sorry, 269 is a prime number]