21-259 Recitation Selected Solutions

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Contents

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Basics

This class is **Calculus in Three Dimensions**, thus the class will require you to think in 3-D.

- Get used to drawing in 3-D.
  My preference is to draw the usual vertical and horizontal axes.
  Label the horizontal axis y and the vertical axis z.
  Then draw a diagonal from the bottom left to the top right, passing through the origin. Label this axis x. Note that you can interchange any of the axis labels as necessary (which will come in handy when we start doing 3-D integration).
- In $\mathbb{R}^3$, the (Euclidean) distance (also referred to as the L_2 norm) between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Planes and Spheres

In this course, you will often deal with planes and spheres (e.g., spherical coordinates).

- You will learn the canonical formula for a plane later (finding it involves cross products). For now, we will find equations for planes parallel to another plane.

Ex. Write an equation of the plane passing through point $(21, 2, 59)$ that is parallel to the xz-plane.

When a plane is parallel to the xz-plane, it means only the x and z coordinates may vary.

Thus taking the y-value, we get the equation $y = 2$.

Ex. Write an equation of the plane passing through points $(2, 125, 9), (21, 25, 9), (5, 7, 9)$ that is parallel to the xy-plane.

When a plane is parallel to the xy-plane, it means only the x and y coordinates may vary.

Conveniently, the z-value in all 3 coordinates are equal; we get the equation $z = 9$.

- A sphere is the set of all points in space equidistant from a fixed point, the center of the sphere. For center (a, b, c) and radius r, we represent the sphere by the ("canonical") equation:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Ex. Find the equation of the sphere with diameter $\overline{PQ}$ where $P = (2, -1, -3)$ and $Q = (-2, 5, -1)$.

First, find the center, which is at the midpoint of the diameter $\overline{PQ}$:

$$C = \left(\frac{2 + (-2)}{2}, \frac{-1 + 5}{2}, \frac{-3 + (-1)}{2} \right) = (0, 2, -2)$$

Next, find the radius using the distance formula (half the length of the diameter)

$$r = \frac{1}{2} \left\| \overline{PQ} \right\| = \frac{1}{2} \sqrt{(-2 - 2)^2 + (5 - (-1))^2 + (-1 - (-3))^2} = \frac{1}{2}\sqrt{56} = \sqrt{\frac{56}{4}} = \sqrt{14}$$

Thus the sphere is given by $x^2 + (y - 2)^2 + (z + 2)^2 = 14$

Vector Notation

Vectors are quantities with magnitude and direction. In $\mathbb{R}^3$, the standard unit vectors are

$$\hat{i} = \langle 1,0,0\rangle \quad \hat{j} = \langle 0,1,0\rangle \quad \hat{k} = \langle 0,0,1\rangle$$

There are several notations. Fix points in $\mathbb{R}^3$ $P = (0,2,1)$ and $Q = (2,5,9)$
Let's represent the vector $\overrightarrow{PQ} = \langle x_Q - x_P, y_Q - y_P, z_Q - z_P\rangle$.

- Component Form: $\overrightarrow{PQ} = \langle 2-0, 5-2, 9-1\rangle = \langle 2,3,8\rangle$
- Using the unit vectors: $\overrightarrow{PQ} = 2\hat{i} + 3\hat{j} + 8\hat{k}$

Ex: Consider the vectors $v = \langle 0,2,1\rangle$ and $u = \langle 2,5,9\rangle$. Find a unit vector in the direction of $3v+u$.

You will need the following rules (taken from your textbook)

RULE: PROPERTIES OF VECTORS IN SPACE

Let $\mathbf{v} = \langle x_1, y_1, z_1\rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2\rangle$ be vectors, and let k be a scalar.

Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1, kz_1\rangle$

Vector addition: $\mathbf{v} + \mathbf{w} = \langle x_1, y_1, z_1\rangle + \langle x_2, y_2, z_2\rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2\rangle$

Vector subtraction: $\mathbf{v} - \mathbf{w} = \langle x_1, y_1, z_1\rangle - \langle x_2, y_2, z_2\rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2\rangle$

Vector magnitude: $\|\mathbf{v}\| = \sqrt{x_1{}^2 + y_1{}^2 + z_1{}^2}$

Unit vector in the direction of v: $\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\langle x_1, y_1, z_1\rangle = \left\langle \frac{x_1}{\|\mathbf{v}\|}, \frac{y_1}{\|\mathbf{v}\|}, \frac{z_1}{\|\mathbf{v}\|}\right\rangle$, if $\mathbf{v} \neq \mathbf{0}$

Solution: first find $3v+u$, then find unit vector in that direction.

$$\begin{aligned} 3v + u &= \langle 3(0), 3(2), 3(1)\rangle + \langle 2,5,9\rangle && \text{(scalar multiplication)} \\ &= \langle 0,6,3\rangle + \langle 2,5,9\rangle = \langle 2,11,12\rangle \\ \frac{1}{\|3v+u\|}(3v+u) &= \frac{\langle 2,11,12\rangle}{\sqrt{2^2+11^2+12^2}} = \langle \frac{2}{\sqrt{269}}, \frac{11}{\sqrt{269}}, \frac{12}{269}\rangle && \text{(unit vector in direction)} \end{aligned}$$

Observe that 269 is prime, so you can't simplify the denominator. Thus we're done. Not all numbers will be pretty, but as a tip: make sure your answers are feasible. If you are taking the length of a vector with 1-digit coordinates and get a 5-digit number, that's probably wrong.
DON'T FORGET THE SQUARE ROOT $\sqrt{}$ when taking norms!!!!

Computing 3x3 Determinants

Useful for computing cross products. You are free to use whatever trick you want so long as you show your work. (note vertical bars means determinant, while square brackets mean matrix).

Cofactor Method

Key notes: $+, -, +$ (**DON'T FORGET THE MIDDLE IS NEGATIVE**) and the 2x2 sub-determinants are the elements not in the same row or column.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei - afg - bdi + bfg + cdh - ceg$$

For cross products between two vectors $u = \langle u_x, u_y, u_z \rangle$ and $v = \langle v_x, v_y, v_z \rangle$, you just compute the 3x3 determinant $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$

Diagonal Method

Recall that the 2x2 determinant is computed using the diagonals: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, where the diagonals going from right to left are positive and diagonals going from left to right are negative. We extend this idea to 3x3 determinants:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \xRightarrow{expand} \begin{matrix} & & a & b & c & & \\ & f & d & e & f & d & \\ h & i & g & h & i & g & h \end{matrix} \xRightarrow{compute} aei + bfg + cdh - afh - bdi - ceg$$

which is the same result as derived from the cofactor method.

09-02-2025

Section 2.5, Checkpoint 2.47

Find an equation of the plane containing the lines L1 and L2:

$$L1 : x = -y = z$$

$$L2 : \frac{x-3}{2} = y = z - 2$$

On the original handout, I mistyped and had $L_2 = x - 32$. This has been corrected, as the error makes the question unsolvable since the lines would be skew.

Worked Solution

Let $L_1 : x = -y = z = t$. The parameterized form of L_1 is thus $\langle t, -t, t\rangle = (0, 0, 0) + \langle 1, -1, 1\rangle t$

Let $L_2 : \frac{x-3}{2} = y = z - 2 = t$. The parameterized form of L_2 is thus $\langle 2t + 3, t, t + 2\rangle = (3, 0, 2) + \langle 2, 1, 1\rangle t$.

A normal vector to the plane that contains both lines is the cross product of the two lines' direction vectors, which was $\vec{n} = \langle -2, 1, 3\rangle$ *(work done in recitation).*

Recall the equation of a plane is $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$, where $\vec{r} = (x, y, z)$, $\vec{n}$ is a normal vector of the plane, and $\vec{r_0}$ is some arbitrary point in the plane. *Note I said **a** normal vector, since we could scale $\vec{n}$ arbitrarily.*

We have $\vec{n}$, so we just need to get $\vec{r_0}$. *In recitation, I mentioned that we can pick any arbitrary point on either line to create the equation of the plane that contains L_1 and L_2. This holds true, and I will arbitrarily pick some points below and prove that the resulting plane equation is the same.*

- In recitation, we picked the easy point $(0, 0, 0)$, which we know to lie in L_1.

$$\begin{aligned}\vec{n} \cdot (\vec{r} - \vec{r_0}) &= \langle -2, 1, 3\rangle \cdot (x - 0, y - 0, z - 0)\\ &= -2x + y + 3z = 0\end{aligned}$$

- But what if we had picked $(3, 0, 2)$, which was our anchor point for L_2?

$$\begin{aligned}\vec{n} \cdot (\vec{r} - \vec{r_0}) &= \langle -2, 1, 3\rangle \cdot (x - 3, y - 0, z - 2)\\ &= -2(x - 3) + y + 3(z - 2) = 0\\ &= -2x + 6 + y + 3z - 6 = 0\\ &= -2x + y + 3z = 0 && \text{(the 6's cancel!)}\end{aligned}$$

- As an extension, derive the plane equation you get if you choose the intersection point of L_1 and L_2! You'll see that you still reach the same plane equation.

09-04-2025

Note: We do NOT cover the foci of paraboloids or ellipsoids.

Section 2.6, Example 2.59 Identifying Equations of Quadric Surfaces

Identify the surfaces represented by the given equations.

(a) $16x^2 + 9y^2 + 16z^2 = 144$ **Solution** Observe that all the terms are squared, and all the coefficients are positive (but not equal) so this is an ellipsoid.

(b) $9x^2 - 18x + 4y^2 + 16y - 36z + 25 = 0$

Solution We have two squared terms, and z term of degree one. We could rearrange this to get $36z = 9x^2 - 18x + 4y^2 + 16y + 25$, which appears to be a elliptic paraboloid.

Section 2.6, Checkpoint 2.54

Identify the surface represented by the equation $9x^2 + y^2 - z^2 + 2z - 10 = 0$.

Solution We have three squared terms, two positive coefficients and one negative coefficient. So this is a double cone.

Section 2.6, Exercise 350

Find the equation of the quadric surface with points $P(x, y, z)$ that are equidistant from point $Q(0, 2, 0)$ and plane of equation $y = -2$. Identify the surface.

Solution First, the distance from a point (x, y, z) to the plane $y = -2$ is just $|y|$. *This is because the plane $y = -2$ has normal vector $\hat{j}$; projecting $\overrightarrow{PQ}$ onto $\hat{j}$ just yields the y component.*

To do this, we want $\langle x, y, z\rangle$ that satisfies

$$\begin{aligned}
\sqrt{(x-0)^2 + (y-2)^2 + (z-0)^2} &= y - (-2) \\
\sqrt{x^2 + (y-2)^2 + z^2} &= y + 2 \\
x^2 + (y-2)^2 + z^2 &= (y+2)^2 \\
x^2 + y^2 - 4y + 4 + z^2 &= y^2 + 4y + 4 \\
x^2 + z^2 &= 8y
\end{aligned}$$

This is the form of an elliptic paraboloid.

09-09-2025

Section 3.1, Exercise 8

Find the limit of the following vector valued function at the indicated value of t.

$$\lim_{t\to 4}\left\langle \sqrt{t-3}, \frac{\sqrt{t}-2}{t-4}, \tan\left(\frac{\pi}{t}\right)\right\rangle$$

Solution We can just substitute for the x and z components. The middle one requires more thought; direct substitution gives $\frac{0}{0}$, so we apply L'Hopital's rule.

$$\lim_{t\to 4}\frac{\sqrt{t}-2}{t-4} = \lim_{t\to 4}\frac{\frac{1}{2\sqrt{t}}}{1} = \frac{1}{4}$$

So the limit is

$$\left\langle 1, \frac{1}{4}, 1\right\rangle$$

THEOREM 3.3

Properties of the Derivative of Vector-Valued Functions

Let **r** and **u** be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

i.	$\frac{d}{dt}[c\mathbf{r}(t)]$	$=$	$c\mathbf{r}'(t)$	Scalar multiple
ii.	$\frac{d}{dt}[\mathbf{r}(t)\pm\mathbf{u}(t)]$	$=$	$\mathbf{r}'(t)\pm\mathbf{u}'(t)$	Sum and difference
iii.	$\frac{d}{dt}[f(t)\,\mathbf{u}(t)]$	$=$	$f'(t)\,\mathbf{u}(t)+f(t)\,\mathbf{u}'(t)$	Scalar product
iv.	$\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)]$	$=$	$\mathbf{r}'(t)\cdot\mathbf{u}(t)+\mathbf{r}(t)\cdot\mathbf{u}'(t)$	Dot product
v.	$\frac{d}{dt}[\mathbf{r}(t)\times\mathbf{u}(t)]$	$=$	$\mathbf{r}'(t)\times\mathbf{u}(t)+\mathbf{r}(t)\times\mathbf{u}'(t)$	Cross product
vi.	$\frac{d}{dt}[\mathbf{r}(f(t))]$	$=$	$\mathbf{r}'(f(t))\cdot f'(t)$	Chain rule
vii.	If $\mathbf{r}(t)\cdot\mathbf{r}(t)$	$=$	c, then $\mathbf{r}(t)\cdot\mathbf{r}'(t)=0$.	

10-02-2025

Given $z = f(x, y)$ is continuous and differentiable on a closed, bounded set D, the strategy to find absolute extrema of f on D is to

1. Determine the critical points of f in D
2. Calculate f at each of these critical points
3. Determine the maximum and minimum values of f on the boundary of its domain
4. Find the absolute maximum and minimum values of f by comparing the values from steps 2 and 3.

10-07-2025 - Midterm 2 tomorrow

Use the method of Lagrange Multipliers to find the maximum and minimum values of the function subject to the given constraints.

Section 4.8, Exercise 369

Minimize $f(x,y) = x^2 + y^2$ on the hyperbola $xy = 1$.

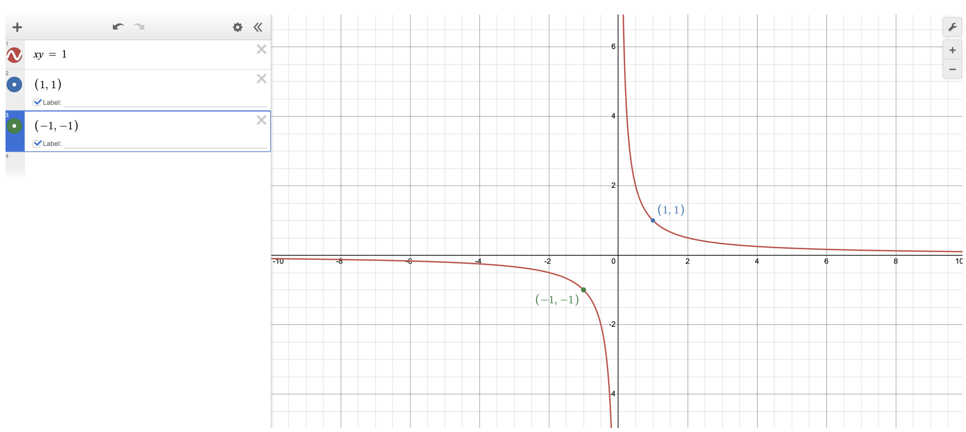
Solution Our objective is f, and our constraint is $g(x,y) = xy = 1$.

$$\begin{aligned} \nabla f &= \lambda \nabla g(x,y) \\ (2x, 2y) &= \lambda(y, x) \end{aligned}$$

So our system to solve is $\begin{cases} (1) \quad 2x = \lambda y \\ (2) \quad 2y = \lambda x \\ (3) \quad xy = 1 \end{cases}$ which we do by solving for λ:

$$\begin{aligned} \lambda &= \frac{2y}{x} && \text{(from eqn. (2))} \\ 2x &= \frac{2y^2}{x} && \text{(from subst. into eqn. (1))} \\ 2x^2 &= 2y^2 \\ x &= \pm y \end{aligned}$$

Combine $x = \pm y$ with $xy = 1$, we get $(x,y) = (1,1)$ or $(x,y) = (-1,-1)$. A quick intuition check; a hyperbola $xy = 1$ will have asymptotes along the x and y axes. And note as x or y goes to infinity, $x^2 + y^2$ also goes to infinity. So clearly the points we have solved for can't be the maximums, so they're probably the minimums. Graphing $xy = 1$ on Desmos reveals that our answer makes sense:



Section 4.8, Exercise 374

Maximize $U(x, y) = 8x^{\frac{4}{5}}y^{\frac{1}{5}}$ under constraint $4x + 2y = 12$.

Solution We solve for y in terms of x, then treat this as a calc 1 optimization problem. *The arithmetic might not be right, please let me know if this is incorrect. The approach is sound though.*

$$\begin{aligned} 4x + 2y = 12 &\implies 2x + y = 6 \\ &\implies y = 6 - 2x \\ &\implies U(x) = 8x^{4/5} \cdot (6 - 2x)^{1/5} \end{aligned}$$

Now we maximize using calc 1 techniques:

$$\begin{aligned} \frac{dU}{dx} &= 8 \cdot \left(\frac{4}{5}x^{-1/5} \cdot (6 - 2x)^{1/5} + (-2) \cdot \frac{1}{5} \cdot (6 - 2x)^{-4/5} \cdot x^{4/5} \right) = 0 \\ &\implies \frac{4}{5}x^{-1/5} \cdot (6 - 2x)^{1/5} + (-2) \cdot \frac{1}{5} \cdot (6 - 2x)^{-4/5} \cdot x^{4/5} = 0 \\ &\implies \frac{4}{5}x^{-1/5} \cdot (6 - 2x)^{1/5} = \frac{2}{5} \cdot (6 - 2x)^{-4/5} \cdot x^{4/5} \\ &\implies 4 \cdot (6 - 2x)^{1/5} = 2 \cdot (6 - 2x)^{-4/5} \cdot x \\ &\implies 6 - 2x = \frac{x}{2} \\ &\implies 6 = \frac{5x}{2} \\ &\implies x = \frac{12}{5} \implies y = 6 - 2 \cdot \frac{12}{5} = \frac{6}{5} \end{aligned}$$

Now is this is a max or min? We find the hessian of U:

$$\begin{aligned} U_x &= \frac{32}{5} \cdot x^{-1/5} \cdot y^{1/5} \\ U_y &= \frac{8}{5} \cdot x^{4/5} \cdot y^{-4/5} \\ U_{xx} &= -\frac{32}{5} \cdot x^{-6/5} \cdot y^{1/5} \\ U_{yy} &= -\frac{32}{5} \cdot x^{4/5} \cdot y^{-9/5} \\ U_{xy} &= \frac{32}{25} \cdot x^{-1/5} \cdot y^{-4/5} \\ U_{yx} &= \frac{32}{25} \cdot x^{-1/5} \cdot y^{-4/5} \qquad \text{(Clairaut's)} \end{aligned}$$

You can evaluate the hessian, find the determinant, and realize that this point must be a maximum (determinant positive, upper left element negative). Therefore, $U\left(\frac{12}{5}, \frac{6}{5}\right)$ is a maximum, and we just plug in to find the value.

Section 4.8, Exercise 379

Maximize $f(x, y, z) = x^2 + y^2 + z^2$ under constraints $x + y + z = 9$ and $x + 2y + 3z = 20$.

Solution Objective is f, constraints g_1, g_2. Lagrange multipliers gives us

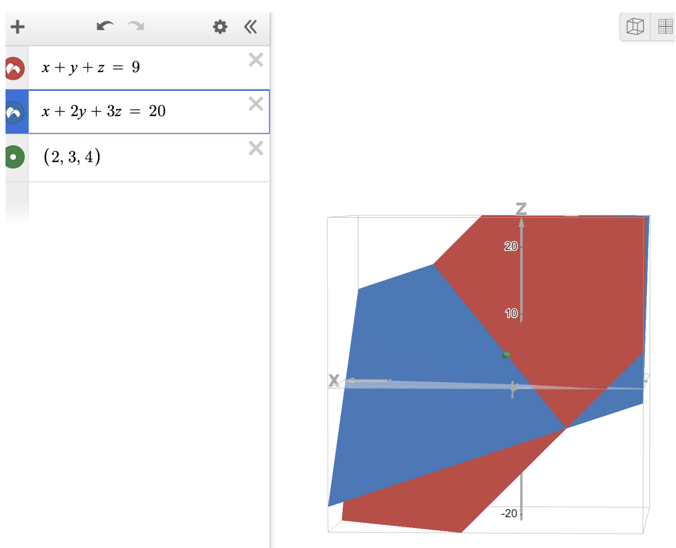
$$\nabla f = \lambda \cdot \nabla g_1 + \mu \cdot \nabla g_2$$
$$(2x, 2y, 2z) = (\lambda, \lambda, \lambda) + (\mu, 2\mu, 3\mu)$$

So our system is: $\begin{cases} (1) \quad 2x = \lambda + \mu \\ (2) \quad 2y = \lambda + 2\mu \\ (3) \quad 2z = \lambda + 3\mu \\ (4) \quad x + y + z = 9 \\ (5) \quad x + 2y + 3z = 20 \end{cases}$

$$\begin{aligned} & 2x + 2y + 2z = 3\lambda + 6\mu = 18 && \text{(eqns. 1, 2, 3, 4)} \\ \implies & \lambda = 6 - 2\mu \\ \implies & 2y = 6 - 2\mu + 2\mu = 6 \implies y = 3 && \text{(eqn. 2)} \\ \implies & x + z = 6 \cap x + 3z = 14 && \text{(eqns. 4, 5)} \\ \implies & 2x = 8 \implies z = 4 \implies x = 2 \end{aligned}$$

So we've found the solution $(x, y, z) = (2, 3, 4)$, $f(2, 3, 4) = 29$, but this is a minimum, so there is no maximum.

Think of it this way, our constraints are two intersecting planes so our solution space is a line. The function we maximize is the distance from the origin, squared. Therefore, there are points along this line infinitely far from the origin, so the point we made is probably the minimum of f under these constraints. Graphing shows us this intuition is correct:



10-09-2025 - Post Midterm 2, No Content

10-21-2025

10-23-2025

10-28-2025

10-30-2025

11-06-2025

11-11-2025 - Midterm 3 tomorrow

11-13-2025 - Post Midterm 3, No Content

11-18-2025

11-20-2025

11-25-2025

12-02-2025 - Final Review

12-04-2025 - Final Review