Clairaut Equations and Singular Solutions

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1 Problem Statement

An equation of the form

$$y = x\frac{dy}{dx} + f(\frac{dy}{dx}) \tag{1}$$

where the continuously differentiable function *f(t)* is evaluated at $t = dy/dx$, is called a **Clairaut equation**. Interest in these equations is due to the fact that a Clairaut equation has a one-parameter family of solutions that consist of *straight lines*. Further, the **envelope** of this family—that is, the curve whose tangent lines are given by the family—is also a solution to the Clairaut equation and is called the **singular solution**.

To solve a Clairaut equation:

(a) Differentiate the Clairaut equation with respect to x and simply to show that

$$[x + f'(\frac{dy}{dx})]\frac{d^2y}{dx^2} = 0, \tag{2}$$

where

$$f'(t) = \frac{d}{dt}f(t). \tag{3}$$

(b) From equation (2), you can conclude that dy/dx = c or f'(dy/dx) = -x. Assume that dy/dx = c and substitute back into equation (1) to obtain the family of *straight line solutions*

$$y = cx + f(c). \tag{4}$$

(c) Show that another solution to equation (1) is given parametrically by

$$x = -f'(p), \tag{5}$$

$$y = f(p) - pf'(p) \tag{6}$$

where the parameter $p = dy/dx$. This solution is the *singular solution.*

(d) Use the above method to find the family of straight-line solutions and the singular solution to the equation

$$y = x(\frac{dy}{dx}) + 2(\frac{dy}{dx})^2 \tag{7}$$

Here $f(t) = 2t^2$. Sketch several of the straight-line solutions along with the singular solution on the same coordinate system. Observe that the straight-line solutions are all tangent to the singular solution.

(e) Repeat part (d) for the equation

$$x(\frac{dy}{dx})^3 - y(\frac{dy}{dx}) + 2 = 0 \tag{8}$$

2 My Work

2.1 Part (a)

$$y = x(\frac{dy}{dx}) + f(\frac{dy}{dx}) \tag{1}$$

Using Product Rule and Chain Rule for Derivatives:

$$\frac{d}{dx}(y) = \frac{dy}{dx} = \frac{d}{dx}[x(\frac{dy}{dx}) + f(\frac{dy}{dx})] \tag{2}$$

$$\frac{dy}{dx} = [\frac{d}{dx}(x)][\frac{dy}{dx}] + [x][\frac{d}{dx}(\frac{dy}{dx})] + \frac{d}{dx}[f(\frac{dy}{dx})] \tag{3}$$

Compute the derivatives and apply the rules to get

$$\frac{dy}{dx} = [1][\frac{dy}{dx}] + [x][\frac{d^2y}{dx^2}] + [f'(\frac{dy}{dx})][\frac{d^2y}{dx^2}] \tag{4}$$

Notice that dy/dx cancels out on the left and right sides.

$$0 = [x][\frac{d^2y}{dx^2}] + [f'(\frac{dy}{dx})][\frac{d^2y}{dx^2}] \tag{5}$$

Rearrange the equation.

$$0 = \frac{d^2y}{dx^2}[x + f'(\frac{dy}{dx})] \tag{6}$$

This is the same as equation (2) from the Problem Statement, Part (a).

2.2 Part (b)

Assume that $dy/dx = c$. We make this assumption because in order for our solution from part (a) to be true, either

$$\frac{d^2y}{dx^2} = 0, \tag{1}$$

which means $dy/dx = c$, where c is some constant, or

$$[x + f'(\frac{dy}{dx})] = 0 \tag{2}$$

For part (b), we assume the former condition is true. Substitute $dy/dx = c$. back into equation (1) from the Problem Statement, Part (a).

$$y = x\frac{dy}{dx} + f(\frac{dy}{dx}) = cx + f(c) \tag{3}$$

This is the family of straight-line solutions to the Clairaut equation.

2.3 Part (c)

Now, we deal with the latter condition:

$$[x + f'(\frac{dy}{dx})] = 0, \tag{1}$$

which means

$$x = -f'(\frac{dy}{dx}) \tag{2}$$

Define a new variable $p = dy/dx$. Substitute p into the previous equation to get

$$x = -f'(p) \tag{3}$$

Substitute p into the original Clairaut equation to get

$$y = x\frac{dy}{dx} + f(\frac{dy}{dx}) = [-f'(p)][p] + f(p) = f(p) - pf'(p) \tag{4}$$

This is the singular solution to the Clairaut equation.

2.4 Part (d)

Now, we finally get to solve an example problem!

$$y = x\frac{dy}{dx} + 2(\frac{dy}{dx})^2 \tag{1}$$

$$\frac{d}{dx}(y) = \frac{dy}{dx} = (1)(\frac{dy}{dx}) + (\frac{d^2y}{dx^2})(x) + 4(\frac{dy}{dx})(\frac{d^2y}{dx^2}) \tag{2}$$

$$\frac{dy}{dx} = \frac{dy}{dx} + \frac{d^2y}{dx^2}[x + 4(\frac{dy}{dx})] \tag{3}$$

$$0 = \frac{d^2y}{dx^2}[x + 4(\frac{dy}{dx})] \tag{4}$$

Solve for the singular solution:

$$x = -4(\frac{dy}{dx}) \tag{5}$$

Note that this is actually a separable differential equation!

$$-\frac{x}{4}dx = dy \tag{6}$$

$$y = -\frac{x^2}{8} + C \tag{7}$$

But let's solve it the parametric way too:

$$f(t) = 2t^2 \to f'(t) = 4t \tag{8}$$

$$y = f(p) - pf'(p) = 2(\frac{dy}{dx})^2 - (\frac{dy}{dx})(4)(\frac{dy}{dx}) = -2(\frac{dy}{dx})^2 \tag{9}$$

If we parameterize p to a more conventional variable like t, what we have in a simple 2-dimensional parametric equation:

$$x = -4t, y = -2t^2 \tag{10}$$

Which can be defined explicitly as:

$$y = -\frac{x^2}{8}. \tag{11}$$

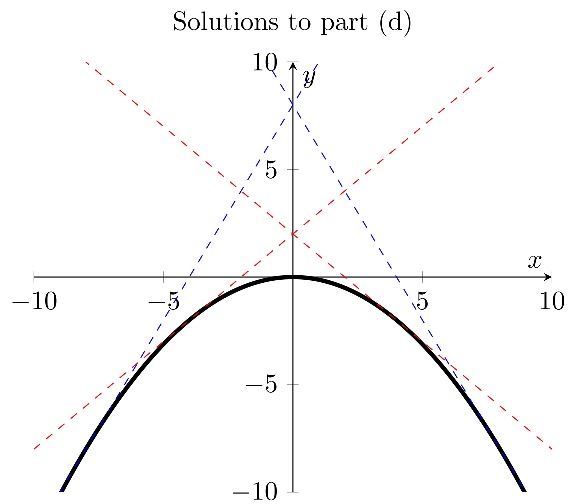
Note that this is the same as solving for the solution through separating and integrating, with constant C set to zero. Nonetheless, let's move onto the straight line solutions:

$$\frac{dy}{dx} = c \tag{12}$$

$$y = cx + 2(c)^2 \tag{13}$$

c	y
1	x + 2
2	2x + 8
-1	-x + 2
-2	-2x + 8

2.4.1 Graphing the solutions



2.4.2 Final Answer

The Clairaut equation

$$y = x\frac{dy}{dx} + 2(\frac{dy}{dx})^2 \tag{14}$$

has a single solution

$$y = -\frac{x^2}{8} \tag{15}$$

which envelopes the family of straight-line solutions given by

$$y = cx + 2c^2 \tag{16}$$

2.5 Part (e)

Now we will solve another Clairaut equation:

$$x(\frac{dy}{dx})^3 - y(\frac{dy}{dx})^2 + 2 = 0. \tag{1}$$

Rearrange to get it in the form of a Clairaut equation:

$$y(\frac{dy}{dx})^2 = x(\frac{dy}{dx})^3 + 2 \tag{2}$$

$$y = x(\frac{dy}{dx}) + 2(\frac{dy}{dx})^{-2} \tag{3}$$

Take the derivative of y:

$$\frac{d}{dx}(y) = \frac{dy}{dx} = [1][\frac{dy}{dx}] + [x][\frac{d^2y}{dx^2}] - 4(\frac{dy}{dx})^{-3}(\frac{d^2y}{dx^2}) \tag{4}$$

$$0 = \frac{d^2y}{dx^2}[x - 4(\frac{dy}{dx})^{-3}] \tag{5}$$

Solve for singular solution:

$$x = -4(\frac{dy}{dx})^{-3} \tag{6}$$

$$p = \frac{dy}{dx}, f(t) = \frac{2}{t^2} \tag{7}$$

$$y = f(p) - pf'(p) \tag{8}$$

$$y = 2(\frac{dy}{dx})^{-2} - \frac{dy}{dx}[-4(\frac{dy}{dx})^{-3}] \tag{9}$$

$$y = 6(\frac{dy}{dx})^{-2} \tag{10}$$

Reparametrizing dy/dx with t, This leaves us with two parametric equations:

$$x = 4t^{-3}, y = 6t^{-2} \tag{11}$$

Rewrite y explicitly in terms of x:

$$y = 6(\frac{x}{4})^{2/3} \tag{12}$$

Now we solve for the straight-line family of solutions:

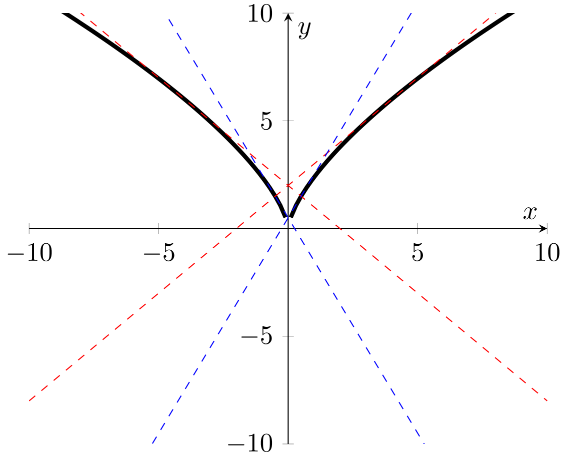
$$\frac{d^2y}{dx^2} = 0 \rightarrow \frac{dy}{dx} = c \tag{13}$$

$$y = cx + 2c^{-2} \tag{14}$$

c	y
1	x + 2
2	2x + 1/2
-1	-x + 2
-2	-2x + 1/2

2.5.1 Graphing the solutions

Solutions to part (d)



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