$\boxed{\text{Basic Math. Practice 10}}$

Linear Algebra

I $\boxed{\text{Matrices:}}$

1°. Lecture part → see "Linear Algebra lecture-schemes from mr. István Gyöngyi."

2°. Practice part:

1) $A := \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 0 \end{bmatrix} \in \mathbb{R}^{\boxed{2} \times \boxed{3}}$

rows ↓ $\boxed{2}$, $\boxed{3}$ ↑ columns

what is $A(2,3) = (A)_{23} = 0$

$(A)_{12} = -1$

$(A)_{21} = 3$

$a_{11} = 1$; $a_{13} = 2$;

2.) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ \sqrt{2} & \boxed{i} & 1 \end{bmatrix} \in \mathbb{C}^{3\times 3}$

which element is this?

$i = (A)_{32}$ $\sqrt{2} = (A)_?$

$(A)_{23} = ?$ $a_{23} = 3$

$(A)_{22} = ?$ $A^* = \overline{A^T} = \begin{bmatrix} 1 & 2 & \sqrt{2} \\ 0 & -1 & -i \\ 1 & 3 & 1 \end{bmatrix}$.

3) Consider the matrices:

$$A := \begin{bmatrix} -2 & 1 & 3 \\ 0 & 2 & 5 \end{bmatrix}, \quad B := \begin{bmatrix} 3 & 0 & 2 \\ 1 & 3 & -1 \end{bmatrix}$$

$$C := \begin{bmatrix} 2 & 4 \\ 5 & 4 \end{bmatrix}$$

Determine:

a) $$A+B = \begin{bmatrix} -2 & 1 & 3 \\ 0 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 2 \\ 1 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 1+0 & 3+2 \\ 0+1 & 2+3 & 5-1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 4 \end{bmatrix}}}.$$

b) $$A-B = \begin{bmatrix} -2 & 1 & 3 \\ 0 & 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 2 \\ 1 & 3 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} -2-3 & 1-0 & 3-2 \\ 0-1 & 2-3 & 5+1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -5 & 1 & 1 \\ -1 & -1 & 6 \end{bmatrix}}}$$

c) $2A-3B=2\cdot\begin{bmatrix}-2 & 1 & 3\\ 0 & 2 & 5\end{bmatrix}-3\cdot\begin{bmatrix}3 & 0 & 2\\ 1 & 3 & -1\end{bmatrix}=$

$=\begin{bmatrix}-4 & 2 & 6\\ 0 & 4 & 10\end{bmatrix}+\begin{bmatrix}-9 & 0 & -6\\ -3 & -9 & 3\end{bmatrix}=$

$=\begin{bmatrix}-13 & 2 & 0\\ -3 & -5 & 13\end{bmatrix}.$

d) $A+C=\underbrace{\begin{bmatrix}-2 & 1 & 3\\ 0 & 2 & 5\end{bmatrix}}_{\in\mathbb{R}^{2\times 3}}+\underbrace{\begin{bmatrix}2 & 4\\ 5 & 4\end{bmatrix}}_{\in\mathbb{R}^{2\times 2}}$

we cannot add them

so $\nexists A+C$.

e) $A^T=\begin{bmatrix}-2 & 1 & 3\\ 0 & 2 & 5\end{bmatrix}^T=\begin{bmatrix}-2 & 0\\ 1 & 2\\ 3 & 5\end{bmatrix}$

f)

$$A \cdot B = \begin{bmatrix} -2 & 1 & 3 \\ 0 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 2 \\ 1 & 3 & -1 \end{bmatrix}$$

2×3 $\quad$ 2×3

they must be equal if we want to multiply them, so

$\nexists A \cdot B$

$$2 \times 2 : \begin{bmatrix} 2 & 4 \\ 5 & 4 \end{bmatrix}$$

g)

$$A^T \cdot C = \begin{bmatrix} -2 & 0 \\ 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -4 & -8 \\ 12 & 12 \\ 31 & 32 \end{bmatrix}$$

3×2 $\quad$ 3×2 ''

$A^T \cdot C$

h) $C^2 = \begin{bmatrix} 2 & 4 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 + 4\cdot 5 & 2\cdot 4 + 4\cdot 4 \\ 5\cdot 2 + 4\cdot 5 & 5\cdot 4 + 4\cdot 4 \end{bmatrix}$

$$= \begin{bmatrix} 24 & 24 \\ 30 & 36 \end{bmatrix}.$$

i) $C \cdot B$

$2 \times \underbrace{2 \quad 2}_{\checkmark} \times 3$

Result matrix will be of type 2×3

So : $\Rightarrow$

$$C \cdot B = \begin{bmatrix} 2 & 4 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 3+1\cdot 4 & 2\cdot 0+4\cdot 3 & 0 \\ 5\cdot 3+1\cdot 4 & 5\cdot 0+4\cdot 3 & 6 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 10 & 12 & 0 \\ 19 & 12 & 6 \end{bmatrix}}} \in \mathbb{R}^{2\times 3}.$$

4) Evaluating a polinomial at a matrix!

Let $A := \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$

$f(x) = 2x^3 - x^2 - 5x + 3 \quad (x \in \mathbb{R})$

Compute the matrix $f[A]$.

By definition:

$$f[A] = 2A^3 - A^2 - 5\cdot A + 3I,$$

~~where~~ $I := \begin{bmatrix}1&0\\0&1\end{bmatrix} \in \mathbb{R}^{2\times 2}$

(unit matrix of order 2)

We need the powers of A:

$$A^2 = \begin{array}{ccc} & \begin{bmatrix}1&2\\-1&2\end{bmatrix} & \begin{bmatrix}1&2\\-1&2\end{bmatrix} \\ \underbrace{\begin{bmatrix}1&2\\-1&2\end{bmatrix}}_{A} & \underbrace{\begin{bmatrix}-1&6\\-3&2\end{bmatrix}}_{A^2} & \underbrace{\begin{bmatrix}-7&10\\-5&-2\end{bmatrix}}_{A^3} \end{array}$$

$$A^3 := A^2\cdot A \quad (\neq A\cdot A^2) \Longrightarrow$$

$$f[A] = 2\cdot\begin{bmatrix}-7&10\\-5&-2\end{bmatrix} - \begin{bmatrix}-1&6\\-3&2\end{bmatrix} - 5\cdot\begin{bmatrix}1&2\\-1&2\end{bmatrix} +$$

$$+\, 3\begin{bmatrix}1&0\\0&1\end{bmatrix} =$$

$$= \begin{bmatrix} -14 & 20 \\ -10 & -4 \end{bmatrix} + \begin{bmatrix} 1 & -6 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} -5 & -10 \\ 5 & -10 \end{bmatrix}$$

$$+ \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -15 & 4 \\ -2 & -13 \end{bmatrix}}} = f[A];$$

5) Powers of matrices:

$$\boxed{\text{a) } A = \begin{bmatrix} 1 & 1 & a \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}, a \in \mathbb{R} \quad \text{Evaluate } A^n = ? \ (n \in \mathbb{N})}$$

$$A^0 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \ A^1 = A;$$

$$\begin{array}{ccc} & \begin{bmatrix}1&1&a\\0&1&1\\0&0&1\end{bmatrix} & \begin{bmatrix}1&1&a\\0&1&1\\0&0&1\end{bmatrix} \\ A^2=\begin{bmatrix}1&1&a\\0&1&1\\0&0&1\end{bmatrix} & \underbrace{\begin{bmatrix}1&2&2a+1\\0&1&2\\0&0&1\end{bmatrix}}_{A^2} & \underbrace{\begin{bmatrix}1&3&3a+3\\0&1&3\\0&0&1\end{bmatrix}}_{A^3}\end{array}$$

$$\begin{array}{cc} & \begin{bmatrix}1&1&a\\0&1&1\\0&0&1\end{bmatrix} \\ A^4=A^3\cdot A=\begin{bmatrix}1&3&3a+3\\0&1&3\\0&0&1\end{bmatrix} & \underbrace{\begin{bmatrix}1&4&4a+6\\0&1&4\\0&0&1\end{bmatrix}}_{A^4}\end{array}$$

let's find out

$$A^n=\begin{bmatrix}1&n&na+(1+2+\cdots+(n-1))\\0&1&n\\0&0&1\end{bmatrix}$$

So we state that:

$\forall n \in \mathbb{N}^+$:

$$A^n = \begin{bmatrix} 1 & n & na + \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$

Proof: By induction!

For $n = 1, 2, 3, 4$ ✓

Suppose for some n its true.

We need for $(n+1)$ $\Rightarrow$

$$A^{n+1} = \begin{bmatrix} 1 & n+1 & (n+1)a + \frac{(n+1)\cdot n}{2} \\ 0 & 1 & n+1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof:

$$A^{n+1} = A^n \cdot A = \begin{bmatrix} 1 & n & na + \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & a \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad *$$

$$* = \begin{bmatrix} 1 & 1+n & a + n + na + \frac{n(n-1)}{2} \\ 0 & 1 & 1+n \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & n+1 & (n+1)a + \frac{n(n+1)}{2} \\ 0 & 1 & n+1 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

6.) <u>Inverse matrix</u>

Check if the given matrix
C is the inverse of A:

$$A = \begin{bmatrix} 3 & -8 \\ 4 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$$

We need to check if

$$AC = CA = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A \cdot C = \begin{bmatrix} 3 & -8 \\ 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 18 & 50 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So C is not
the inverse of A.

How can we find it's inverse if it exists? → see next class.

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b) $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 7 & -4 \end{bmatrix}$;

$C = \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & -1 \\ -19 & 11 & 1 \end{bmatrix}$

Check if $AC = CA = I$?

$A \cdot C = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 7 & -4 \end{bmatrix} \cdot \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & -1 \\ -19 & 11 & 1 \end{bmatrix} =$

$$= \begin{bmatrix} 14-51+38 & -8+30-22 & -1+3-2 \\ 28-85+57 & -16+50-33 & -2+5-3 \\ -42-34+76 & 24+20-44 & 3+2-4 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$C \cdot A$: Homework