Basic Mathematics

Class 18

Eigenvectors - Eigenvalues
Diagonalizable matrixes

let $A \in \mathbb{K}^{n \times n}$ ($n \in \mathbb{N}^+$ fix)

We say that $\lambda \in \mathbb{K}$ is an eigenvalue of A if $\exists x \in \mathbb{R}^n$ $x \neq 0$ so that $Ax = \lambda x \iff$ $(A - \lambda I)x = 0$ is a homogenous system. This has other solutions then the 0 vector $\iff \det(A - \lambda I) = 0$

① $A = \begin{bmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \in \mathbb{K}^{3\times 3}$

Find the eigenvalues, eigenvectors, eigenspaces, eigenbasis, diagonal form if it exists.

Sol:

$\boxed{\text{STEP1}}$ Eigenvalues:

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -1 & -1 \\ 3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{vmatrix} \begin{matrix} + \\ \big| \cdot (-1) \\ \end{matrix} =$$

$$= \begin{vmatrix} 2-\lambda & 0 & -1 \\ 3 & 1-\lambda & -3 \\ -1 & \lambda-1 & 2-\lambda \end{vmatrix} =$$

$$= (\lambda-1)\begin{vmatrix} 2-\lambda & 0 & -1 \\ 3 & -1 & -3 \\ -1 & 1 & 2-\lambda \end{vmatrix} \oplus =$$

$$= (\lambda-1)\cdot\begin{vmatrix} 2-\lambda & 0 & -1 \\ 3 & -1 & -3 \\ 2 & 0 & -1-\lambda \end{vmatrix} =$$

$$= (\lambda-1)\cdot(-1)\cdot\begin{vmatrix} 2-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} =$$

$$= (\lambda-1)\cdot(-1)\cdot\left(-2-2\lambda+\lambda+\lambda^2+2\right) =$$

$$= (\lambda-1)(-1)(\lambda^2-\lambda) =$$

$$= \underline{-\lambda\cdot(\lambda-1)^2} = 0 \iff$$

$\lambda_1 = 0 \quad a(0) = 1$
$\lambda_2 = 1 \quad a(1) = 2$ } algebraic multiplicity.

So the spectrum of A: $Sp(A) = \{0, 1\}$ is the set of eigenvalues.

$\boxed{\text{STEP 2}}$ Eigenvectors

a) For $\boxed{\lambda_1 = 0}$ we want vectors $x_1 \neq 0$ so that $Ax_1 = \lambda_1 x_1$ or $(A - \lambda_1 I)x_1 = 0$

$$\Longleftrightarrow \begin{bmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{= x_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow$$

$$\begin{cases} 2a - b - c = 0 \\ 3a - 2b - 3c = 0 \\ -a + b + 2c = 0 \end{cases} \quad \begin{array}{l} | \cdot (-2) \\ \oplus \Rightarrow -a - c = 0 \\ \oplus \Rightarrow a + c = 0 \end{array}$$

$$\Rightarrow \boxed{c = -a} \quad b = 2a - c = 3a$$

So $x_1 = \begin{pmatrix} a \\ 3a \\ -a \end{pmatrix} = a \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \quad (0 \neq a \in \mathbb{K})$

are the eigenvectors to $\lambda_1 = 0$.

<u>Eigenspace to λ_1:</u>

$$W_{\lambda_1} = \{ x \in \mathbb{K}^3 \mid Ax = \lambda_1 x \} =$$

$$= \left\{ a \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \middle| \, a \in \mathbb{K} \right\} =$$

0 as well

$= \text{Span}(1, 3, -1) \Longrightarrow$

$\dim W_{\lambda_1} = 1 \Rightarrow g(\lambda_1) = \dim W_{\lambda_1} =$

$= 1$ the geometric multiplicity of λ_1.

(b) To $\boxed{\lambda_2 = 1}$ $\quad Ax_2 = \lambda_2 x_2$

or $(A - \lambda_2 I)\, x_2 = 0 \Longleftrightarrow$

$$\begin{bmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow$$

$$\begin{cases} a - b - c = 0 \\ 3a - 3b - 3c = 0 \\ -a + b + c = 0 \end{cases} \Rightarrow a = b + c$$

Sol: $x_2 = \begin{pmatrix} b+c \\ b \\ c \end{pmatrix} =$

$$= \begin{pmatrix} b \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} c \\ 0 \\ c \end{pmatrix} = b\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$b, c \in \mathbb{K}$, $(b, c) \neq (0, 0)$.

Eigenspace to λ_2:

$$W_{\lambda_2} = \left\{ b\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; b, c \in \mathbb{K} \right\}$$

$= \mathrm{Span}\left((1|1|0), (1|0|1)\right) \Longrightarrow$

$\dim W_{\lambda_2} = g(\lambda_2) = 2$

$\boxed{\text{STEP 3}}$ Is there a basis of eigenvectors of A in $\mathbb{K}^3$?

There is an EB in $\mathbb{K}^3$ $\Longleftrightarrow$

$$\begin{cases} a(0) + a(1) = 3 = n \quad \underline{\text{and}} \\ a(0) = g(0) = 1 \ \checkmark \\ a(1) = g(1) = 2 \ \checkmark \end{cases}$$

<u>So</u> $\exists$ EB <u>which is</u>:

$$\underbrace{\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}}_{=: c_1}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{=: c_2}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{=: c_3}$$

$\boxed{\text{STEP 4}}$ Is A <u>diagonalizable</u>?

A has a diagonal form $\Longleftrightarrow$ $\exists C \in \mathbb{K}^{3\times 3}$ invertible so that $C^{-1} A \cdot C = D$ where D is a diagonal matrix and in its main diagonal are the eigenvalues of A.

$\Longleftrightarrow$ $\exists$ E.B. in $\mathbb{K}^3$.

<u>Now</u> ✓ <u>so</u> if:

$$C := [c_1 \;\; c_2 \;\; c_3] = \begin{bmatrix} 1 & 1 & 1 \\ +3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

<u>and</u> $C^{-1} \cdot A \cdot C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$ diagonal form of A.

② $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \in \mathbb{K}^{3\times 3}$

The same questions.

$\boxed{\text{STEP 1}}$ Eigenvalues:

$$P(\lambda) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & -1 \\ 2 & -1-\lambda & -\lambda \end{vmatrix}$$

$+$

$$= (1-\lambda)\cdot(\lambda^2 - 1 - \lambda) + 1\cdot(-1-\lambda+2\lambda) =$$

$$= (1-\lambda)(\lambda^2-\lambda-1) + (\lambda-1) =$$

$$= (1-\lambda)(\lambda^2-\lambda-2) = (1-\lambda)(\lambda+1)(\lambda-2)$$

$= 0 \iff$

$\lambda_1 = 1 \quad a(1) = 1$

$\lambda_2 = -1 \quad a(-1) = 1$

$\lambda_3 = 2 \quad a(2) = 1$

$\boxed{\text{STEP 2}}$ Eigenvectors, eigenspaces:

a) $\lambda_1 = 1$: $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Leftrightarrow \left.\begin{aligned} -b + c &= 0 \\ a - c &= 0 \\ 2a - b - c &= 0 \end{aligned}\right\} \Rightarrow \boxed{a = b = c}$$

$$x_1 = \begin{pmatrix} a \\ a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (a \in \mathbb{K} \setminus \{0\})$$

$$W_{\lambda_1} = \left\{ a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| a \in \mathbb{K} \right\} =$$

$$= \underbrace{\mathrm{Span}(1 | 1 | 1)}_{} \Rightarrow$$

$$\underbrace{\dim W_{\lambda_1} = g(\lambda_1) = g(1) = 1}_{};$$

(b) $\boxed{\lambda_2 = -1} \Rightarrow \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} =$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow$$

$$\begin{cases} 2a - b + c = 0 \\ a + 2b - c = 0 \\ 2a - b + c = 0 \end{cases} \oplus \quad 3a + b = 0$$

$b = -3a$

$c = a - 6a =$

$= -5a$

$\underline{\text{So}}$ $x_2 = \begin{pmatrix} a \\ -3a \\ -5a \end{pmatrix} = a \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$

$0 \neq a \in \mathbb{K} \Longrightarrow$

$W_{\lambda_2} = \left\{ a \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} \middle| a \in \mathbb{K} \right\} =$

$= \mathrm{Span}\,(1|-3,-5) \Longrightarrow$

$\dim W_{\lambda_2} = g(\lambda_2) = g(-1) = 1$

c) $\lambda_3 = 2 \Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left\{ \begin{matrix} -a-b+c=0 \\ a-b-c=0 \\ 2a-b-2c=0 \end{matrix} \right\} \Rightarrow b=0$

$\Rightarrow c=a$

So $x_3 = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $(0 \neq a \in \mathbb{K})$

$\Rightarrow W_{\lambda_3} = \left\{ a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, a \in \mathbb{K} \right\} =$

$= \operatorname{Span}(1|0|1) \Rightarrow$

$\dim W_{\lambda_3} = 1 = g(\lambda_3) = g(2);$

$\boxed{\text{STEP 3}}$

$a(1) + a(-1) + a(2) = 1 + 1 + 1 = 3$ ✓

$a(1) = g(1) = 1$ ✓

$a(2) = g(2) = 1$ ✓ $\Rightarrow$

$a(-1) = g(-1) = 1$ ✓

$\exists$ E.B. (eigenbasis) in $\mathbb{K}^3$ which is for expl:

$$\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{c_1} ; \underbrace{\begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}}_{c_2} ; \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{c_3}$$

$\boxed{\text{STEP 4}}$ $\exists$ EB in $\mathbb{K}^3 \implies$ A is diagonalizable so

$$C := [c_1 \ c_2 \ c_3] =$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 0 \\ 1 & -5 & 1 \end{bmatrix} \text{ and}$$

$$C^{-1} \cdot A \cdot C = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \leftarrow \text{the}$$

diagonal form of A.

<u>Remark</u>: We can change the order of vectors c_1, c_2, c_3 in matrix C; so if

$$C := \begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 1 \\ -5 & 1 & 1 \end{bmatrix} \text{ then}$$

$$C^{-1} A C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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③ $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \in \mathbb{K}^{3\times 3}$

Some questions :

$\boxed{\text{STEP 1}}$ $P(\lambda) = \det(A - \lambda I) =$

$$= \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} =$$

$+$

$$= \begin{vmatrix} 2-\lambda & -1 & 1 \\ 0 & 1-\lambda & -1 \\ 2-\lambda & -1 & 2-\lambda \end{vmatrix} =$$

$$= (2-\lambda) \begin{vmatrix} 1 & -1 & 1 \\ 0 & 1-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} =$$

$+$

$$= (2-\lambda) \begin{vmatrix} 0 & -1 & 1 \\ 1-\lambda & 1-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} =$$

$$= (2-\lambda) \cdot (-1)(1-\lambda) \cdot \begin{vmatrix} -1 & 1 \\ -1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda-1)(\lambda-2+1) =$$

$$= (2-\lambda)(\lambda-1)^2 = 0 \iff$$

$$\begin{cases} \lambda_1 = 2 & a(2) = 1 \\ \lambda_2 = 1 & a(1) = 2 \end{cases}$$

$\boxed{\text{STEP 2}}$ Eigenvectors/spaces

$$\lambda_1 = 2 : \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} -a - b + c = 0 \\ a - b - c = 0 \\ -b = 0 \Rightarrow \boxed{b = 0} \end{cases}$$

$\Rightarrow \boxed{a = c}$ so

$$x_1 = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (a \in \mathbb{K} \setminus \{0\})$$

$$W_{\lambda_1} = \left\{ a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \middle| a \in \mathbb{K} \right\} =$$

$$= \text{Span}(1, 0, 1) \Rightarrow$$

$$\dim W_{\lambda_1} = g(2) = \underset{\sim}{1}.$$

b) $\lambda_2 = 1$ $\quad \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Leftrightarrow \begin{cases} -b+c=0 \\ a-c=0 \\ -b+c=0 \end{cases} \Rightarrow \underline{a=b=c}$$

$$x_2 = \begin{pmatrix} a \\ a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (0 \neq a \in \mathbb{K})$$

$$\Rightarrow W_{\lambda_2} = \left\{ a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| a \in \mathbb{K} \right\} =$$

$$= \operatorname{Span}(1 \mid 1 \mid 1) \Rightarrow$$

$$\dim W_{\lambda_2} = g(\lambda_2) = \underline{g(1) = 1}$$

STEP 3 :

$$a(1) + a(2) = 3 \checkmark$$

$$a(2) = g(2) = 1 \checkmark$$

$$\underbrace{a(1) = 2 \neq g(1) = 1}$$

$$\Rightarrow \nexists \text{ E.B. in } \mathbb{K}^3$$

so <u>no</u> diagonal form for $\underline{\underline{A}}$.

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(4) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \in \mathbb{K}^{3\times 3}$

$\boxed{\text{STEP 1}}$ $P(\lambda) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix}$

= Sarrus rule =

$= (1-\lambda)^3 + 3(1-\lambda) + (1-\lambda) =$

$= (1-\lambda)\left[(1-\lambda)^2 + 4\right] = 0$

$\Leftrightarrow \lambda_1 = 1 \qquad a(1) = 1$

<u>or</u> $(1-\lambda)^2 + 4 = 0 \Leftrightarrow$

$(1-\lambda)^2 = -4 \Leftrightarrow (\lambda-1)^2 = -4$

$\lambda - 1 = \pm 2i \qquad \lambda_2 = 1 + 2i$

$\lambda_3 = 1 - 2i$

$a(1+2i) = a(1-2i) = 1.$

$\boxed{\text{STEP 2}}$ $\boxed{\text{Eigenvectors / Eigenspaces}}$

a) $\lambda_1 = 1$: $\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Leftrightarrow \begin{cases} -b-c=0 \\ a=0 \\ 3a=0 \end{cases} \quad \begin{cases} a=0 \\ c=-b \end{cases}$$

$$x_1 = \begin{pmatrix} 0 \\ b \\ -b \end{pmatrix} = b \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (0 \neq b \in \mathbb{K})$$

SO if $\boxed{\mathbb{K} = \mathbb{R}}$ here

Then $Sp(A) = \{1\}$;

$a(1) = 1 \neq 3 \Rightarrow$ No eigen basis!

and no diagonal form for A.

If $\boxed{\mathbb{K}=\mathbb{C}} \Rightarrow$

$Sp(A)=\{1, 1+2i, 1-2i\}$

$\left.\begin{array}{l} a(1)=1 \\ a(1+2i)=1 \\ a(1-2i)=1 \end{array}\right\}$ their sum is 3 so there may be E.B. we go on.

b) $\lambda_2 = 1+2i \Rightarrow$

$$\begin{bmatrix} -2i & -1 & -1 \\ 1 & -2i & 0 \\ 3 & 0 & -2i \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -2ia - b - c = 0 \\ a - 2ib = 0 \Rightarrow \underline{a = 2ib} \\ 3a - 2ic = 0 \end{cases}$$

$$\Leftrightarrow c = \frac{3a}{2i} = \frac{3 \cdot 2ib}{2i} = 3b$$

$\underline{\text{So}}$ $x_2 = \begin{pmatrix} 2ib \\ b \\ 3b \end{pmatrix} = b \begin{pmatrix} 2i \\ 1 \\ 3 \end{pmatrix}$

$0 \neq b \in \mathbb{C}$

$$\Rightarrow W_{\lambda_2} = \left\{ b \begin{pmatrix} 2i \\ 1 \\ 3 \end{pmatrix} \middle| \; b \in \mathbb{K} \right\}$$

$= \operatorname{Span}(2i, 1, 3) \Rightarrow$

$\dim W_{\lambda_2} = g(1+2i) = 1$

c) $\lambda_3 = 1 - 2i \implies$

$A \cdot x_3 = \lambda_3 \cdot x_3 \implies$

$\bar{A}\ \bar{x}_3 = \bar{\lambda}_3 \cdot \bar{x}_3 \implies$

$A \cdot \bar{x}_3 = \lambda_2\, \bar{x}_3$ <u>so</u>

$\bar{x}_3 = x_2$ is good $\implies$

$x_3 = \bar{x}_2 = \begin{pmatrix} -2i \\ 1 \\ 3 \end{pmatrix}$ <u>will do</u>

$W_{\lambda_3} = \left\{ b \begin{pmatrix} -2i \\ 1 \\ 3 \end{pmatrix} \middle|\, b \in \mathbb{K} \right\} =$

$= \underline{\mathrm{Span}(-2i, 1, 3)} \implies$

$$\dim W_{\lambda_3} = g(1-2i) = 1$$

$\boxed{\text{STEP 3}}$ on $\mathbb{C}$ is there

EB in $\mathbb{C}^3$:

$$\begin{cases} a(1) + a(1+2i) + a(1-2i) = 3 \checkmark \\ a(1) = g(1) = 1 \\ a(1+2i) = g(1+2i) = 1 \\ a(1-2i) = g(1-2i) = 1 \end{cases}$$

<u>OR</u> $\Longleftarrow$

$$g(1) + g(1+2i) + g(1-2i) = 3 = \dim(\mathbb{C}^3) \Longrightarrow$$

$\exists$ EB in $\mathbb{C}^3$ which is

for example:

$$\underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{c_1}, \underbrace{\begin{pmatrix} 2i \\ 1 \\ 3 \end{pmatrix}}_{c_2}, \underbrace{\begin{pmatrix} -2i \\ 1 \\ 3 \end{pmatrix}}_{c_3}$$

$$C := \begin{bmatrix} 0 & 2i & -2i \\ 1 & 1 & 1 \\ -1 & 3 & 3 \end{bmatrix}$$ <u>and</u>

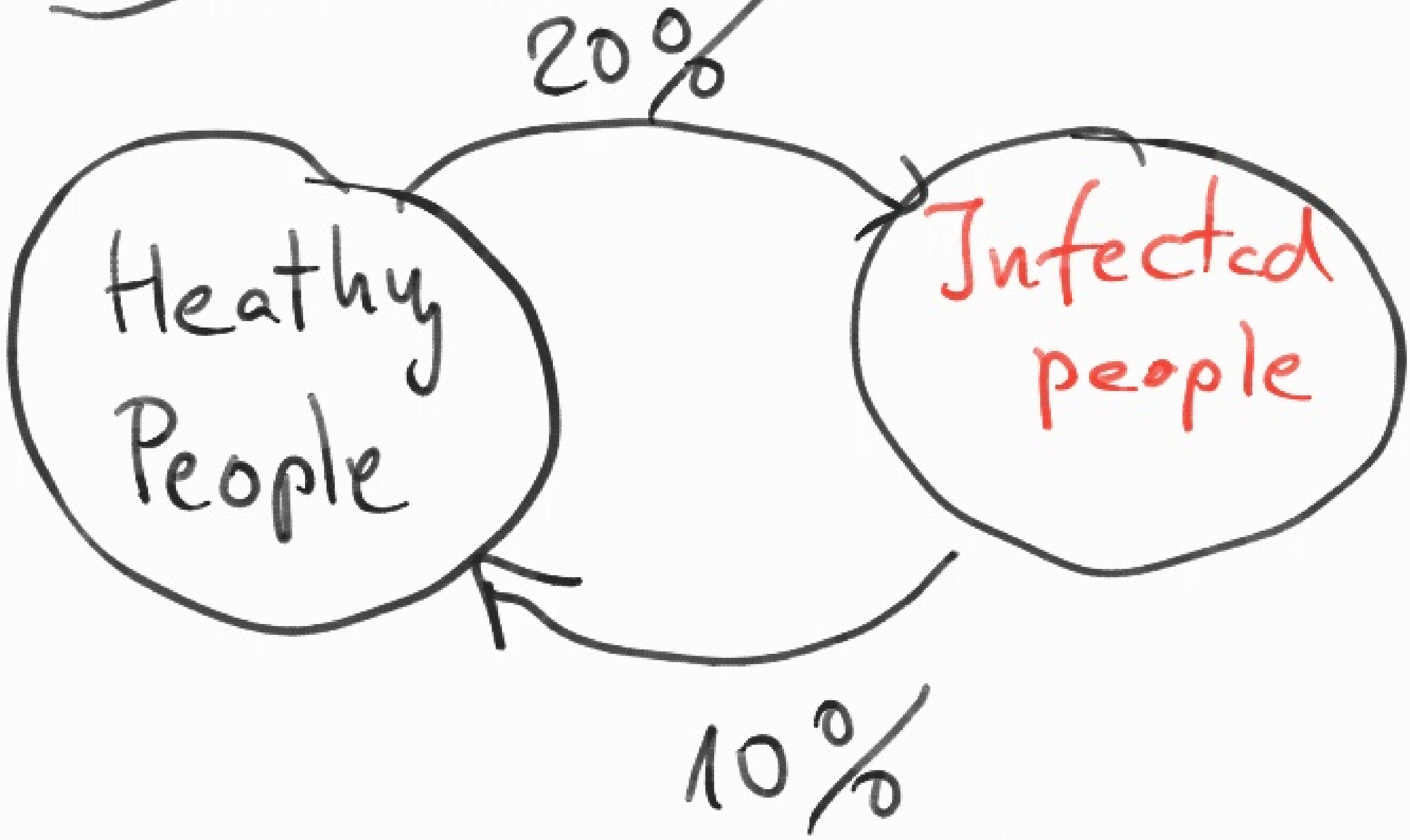
$$C^{-1} \cdot A \cdot C = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+2i & 0 \\ 0 & 0 & 1-2i \end{bmatrix}$$

the diagonal form of A.

THE END

Application:

Virus infection modell



Every day 20% of healthy people get infected and 10% of infected ones becomes healthy again with some medicine

What is going to happen in a long-run term if we have at a point

300 healthy people and 0 infected ones.

Denote by h_n, i_n the number of healthy people and infected ones at the end of day n.

So $h_0 = 300$, $i_0 = 0$

What is the situation at the end of day 1?

H J

$h_1 = 0{,}8\, h_0 + 0{,}1 \cdot i_0 =$

$= 0{,}8 \cdot 300 + 0{,}1 \cdot 0 =$

$= \underline{\underline{240}}$

$i_1 = 0{,}2 \cdot h_0 + 0{,}9 \cdot i_0 =$

$= 0{,}2 \cdot 300 + 0{,}9 \cdot 0 =$

$= \underline{\underline{60}}$

At the end of day 2:

$h_2 = 0{,}8 \cdot h_1 + 0{,}1 \cdot i_1 =$

$= 0{,}8 \cdot 240 + 0{,}1 \cdot 60 =$

$= 192 + 6 = \underline{\underline{198}}$

$$i_2 = 0{,}2 \cdot h_1 + 0{,}9 \cdot i_1 =$$

$$= 0{,}2 \cdot 240 + 0{,}9 \cdot 60 =$$

$$= 48 + 54 = \underline{\underline{96}}$$

<u>So</u>:

$$h_{n+1} = 0{,}8 \cdot h_n + 0{,}1 \cdot i_n$$

$$i_{n+1} = 0{,}2 \cdot h_n + 0{,}9 \cdot i_n$$

$$(\forall n \in \mathbb{N})$$

<u>OR</u>

$$\begin{bmatrix} h_{n+1} \\ i_{n+1} \end{bmatrix} = \begin{bmatrix} 0{,}8 & 0{,}1 \\ 0{,}2 & 0{,}9 \end{bmatrix} \cdot \begin{bmatrix} h_n \\ i_n \end{bmatrix}$$

$\Rightarrow$ If $x_n := \begin{bmatrix} u_n \\ i_n \end{bmatrix}$ $(n=0,1,2,\ldots)$

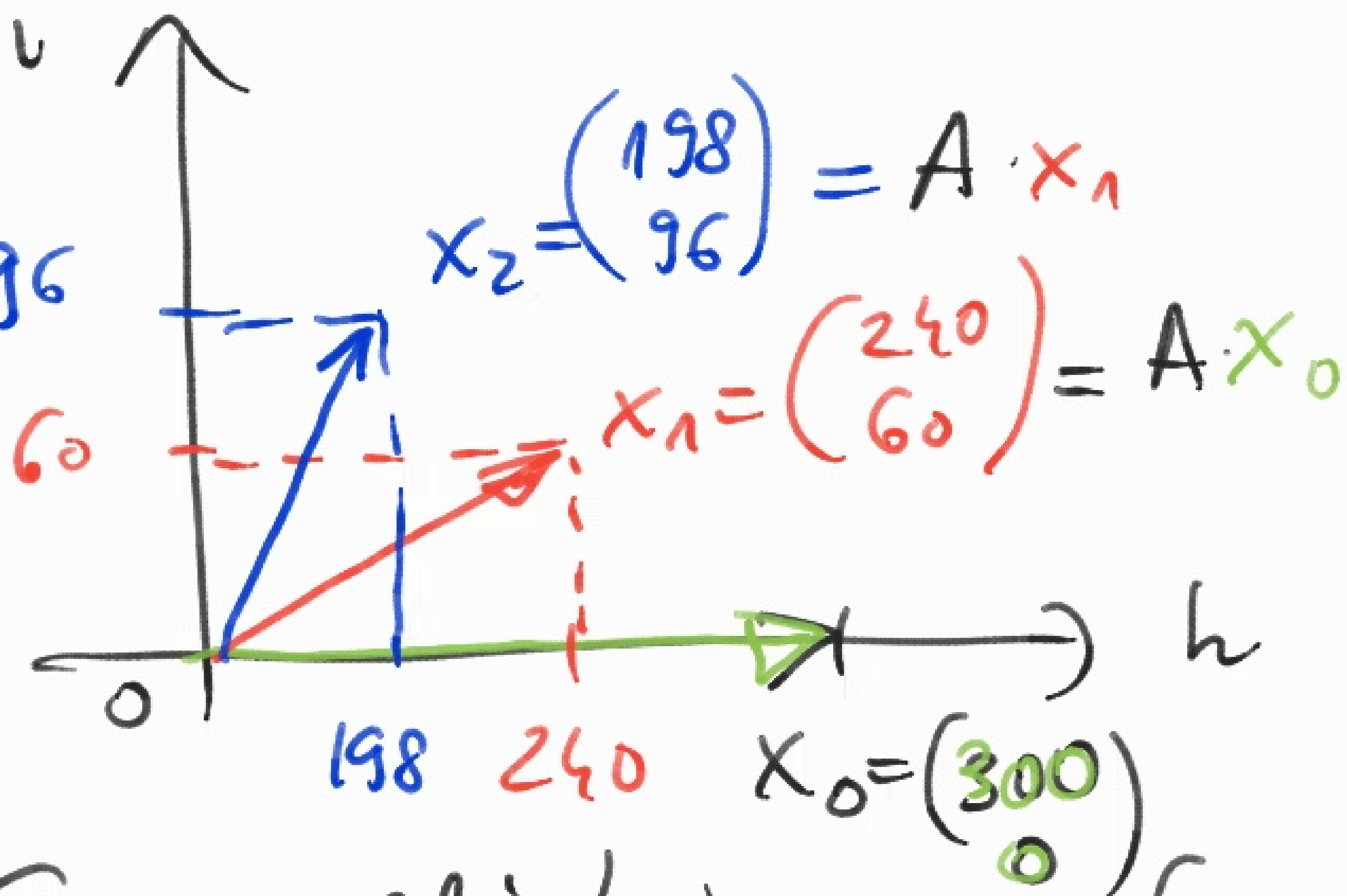
$x_n = A \cdot x_{n-1} = A\,(A \cdot x_{n-2}) =$
$= A^2 \cdot x_{n-2} = A^2 \cdot (A\,x_{n-3}) =$
$= A^3 \cdot x_{n-3} = \cdots = A^n \cdot x_0 =$
$= A^n \cdot \begin{bmatrix} 300 \\ 0 \end{bmatrix}$

So we need A^n

(The powers of a matrix)

A is called the Markov matrix of this model.

What happens with this vector series $x_0, x_1, x_2, \ldots, x_4, \ldots$ through this „process"?



So multiplying a vector by A is rotating and scaling that vector.

Is there an equilibrium state?

Is there a vector $x \neq 0$
so that $Ax = \lambda \cdot x$
with $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^2 \setminus \{(0,0)\}$.
(not rotating, just scaling)
We get to the eigenvector
eigenvalue topic.

Find these for matrix A.

$\boxed{\text{STEP 1}}$ $P(\lambda) = \det(A - \lambda \underline{I}) =$

$$= \det\left(\begin{bmatrix} 0,8 & 0,1 \\ 0,2 & 0,9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right)$$

$$= \begin{vmatrix} 0,8-\lambda & 0,1 \\ 0,2 & 0,9-\lambda \end{vmatrix} =$$

$= (0{,}8-\lambda)(0{,}9-\lambda) - 0{,}02 =$

$= 0{,}72 - 0{,}8\lambda - 0{,}9\lambda + \lambda^2 -$

$- 0{,}02 =$

$= \lambda^2 - 1{,}7\lambda + 0{,}7 = 0 \iff$

$\iff (\lambda - 1)(\lambda - 0{,}7) = 0$

$$\begin{cases} \lambda_1 = 1 & a(1) = 1 \\ \lambda_2 = 0{,}7 & a(0{,}7) = 1 \end{cases}$$

$\boxed{\text{STEP 2}}$ $\boxed{\text{Eigenvectors/spaces}}$

$\boxed{\lambda_1 = 1}$ $(A - \lambda_1 I)X = 0$

$\iff \left(\begin{bmatrix} 0{,}8 & 0{,}1 \\ 0{,}2 & 0{,}9 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) X = 0$

$$\begin{bmatrix} -0{,}2 & 0{,}1 \\ 0{,}2 & -0{,}1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -0{,}2a + 0{,}1b = 0 & \mid \cdot 10 \\ 0{,}2a - 0{,}1b = 0 & \mid \cdot 10 \end{cases}$$

$$b - 2a = 0 \qquad b = 2a$$

$$x_1 = \begin{pmatrix} a \\ 2a \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \underline{\underline{a \in \mathbb{R} \setminus \{0\}}}$$

$$W_{\lambda_1} = \left\{ a \begin{pmatrix} 1 \\ 2 \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\} =$$

$$= \underline{\operatorname{span}(1 \mid 2)} \Rightarrow \dim W_{\lambda_1} =$$

$$= g(\lambda_1) = g(1) = \underline{\underline{1}}$$

For $\boxed{\lambda_2 = 0,7}$

$$(A - 0,7\, I)\, x_2 = 0$$

$$\left(\begin{bmatrix} 0,8 & 0,1 \\ 0,2 & 0,9 \end{bmatrix} - \begin{bmatrix} 0,7 & 0 \\ 0 & 0,7 \end{bmatrix} \right) x = 0$$

$$\Leftrightarrow \begin{bmatrix} 0,1 & 0,1 \\ 0,2 & 0,2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 0,1a + 0,1b = 0 \\ 0,2a + 0,2b = 0 \end{cases}$$

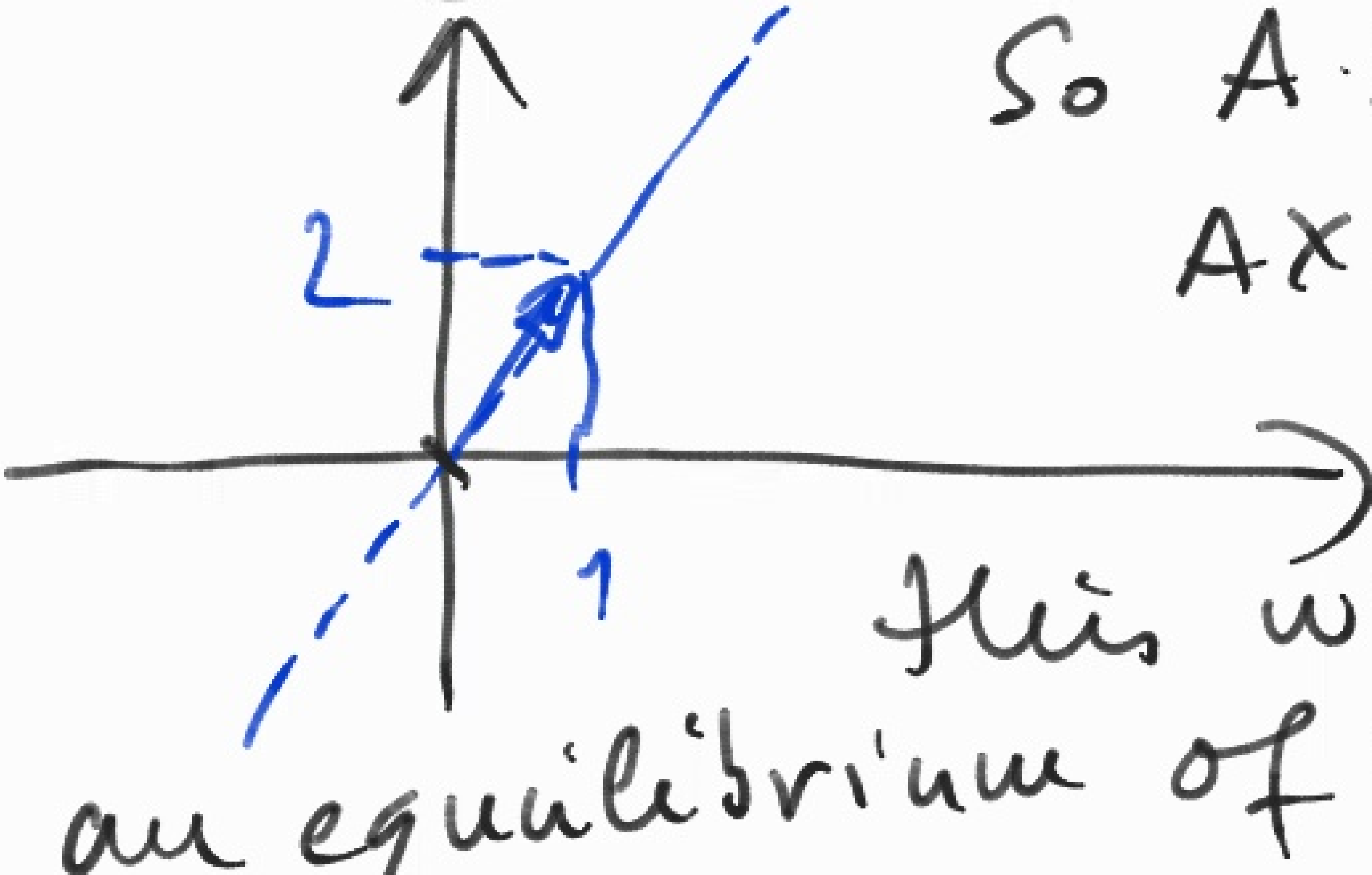
$$b = -a \Rightarrow x_2 = \begin{bmatrix} a \\ -a \end{bmatrix} =$$

$$= a \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (a \in \mathbb{R} \setminus \{0\}).$$

$$W_{\lambda_2} = \left\{ a \begin{pmatrix} 1 \\ -1 \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\} =$$

$$= \underline{\mathrm{Span}\,(1 \mid -1)} \; ; \Rightarrow$$

$$\underline{\underline{\dim W_{\lambda_2} = g(\lambda_2) = g(0{,}7) = 1}}$$



So $A \cdot x_1 = 1 \cdot x_1$

$A x_1 = x_1$

this will be an equilibrium of the <u>system</u>. After a long <u>time</u> we have $\underline{\underline{\begin{pmatrix} 100 \\ 200 \end{pmatrix}}}$

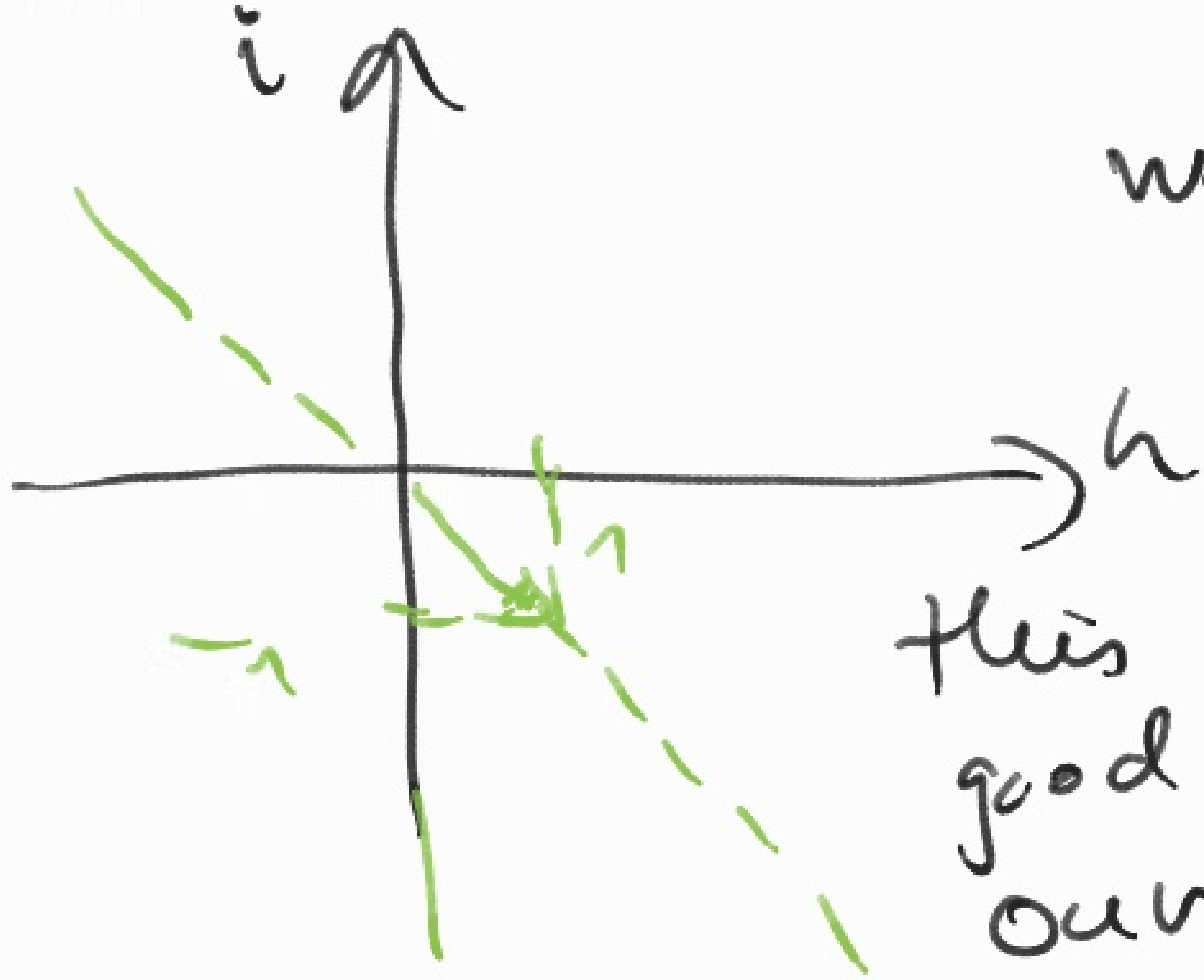
$\underline{a + 2a = 300}$

$a = 100$

$$\Rightarrow \quad A \cdot \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 0{,}8 & 0{,}1 \\ 0{,}2 & 0{,}9 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

$$= \begin{bmatrix} 80 + 20 \\ 20 + 180 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \end{bmatrix} \checkmark$$

<u>For</u> $\lambda_2 = 0{,}7$, $x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



we need

$h_n, i_n \geq 0$

so

this is no good for our problem

We want to find $\underline{\underline{A^n}}$ as $\underset{\sim}{\text{well}}$.

Suppose there is a matrix $C \in \mathbb{R}^{2\times 2}$ invertible, so that $C^{-1} A C = D$

Do we have one?

<u>STEP 3</u>: $a(1) + a(0,7) = 1+1 = \,= 2 = n$ ✓

$$\begin{cases} a(1) = g(1) = 1 \\ a(0,7) = g(0,7) = 1 \end{cases}$$

$\Rightarrow \exists$ $\underline{EB}$ in $\mathbb{R}^2$ which is : $\underset{c_1}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} ; \underset{c_2}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \Rightarrow$

$C := \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ <u>and</u>

the diagonal form of

A:

$$C^{-1} A \cdot C = D = \begin{bmatrix} 1 & 0 \\ 0 & 0{,}7 \end{bmatrix}$$

(for this C)

Evaluate

$$D^n = \begin{bmatrix} 1 & 0 \\ 0 & 0{,}7 \end{bmatrix}^n = \begin{bmatrix} 1^n & 0 \\ 0 & (0{,}7)^n \end{bmatrix}$$

$(\forall n \in \mathbb{N}^+)$ <u>and</u>

$$D^n = (C^{-1} A C)^n =$$

$$= \overbrace{(C^{-1}AC)}^{1} \cdot \overbrace{(C^{-1}AC)}^{2} \cdots \overbrace{(C^{-1}AC)}^{n} =$$

$$= C^{-1} A \cdot A \cdot C \cdot C^{-1} A C \cdots (C^{-1}AC) =$$

$$= (C^{-1} A^3 C) \overset{I}{\cdots} (C^{-1} A C) =$$

$$= C^{-1} A^n C$$

So

$$C^{-1} A^n C = \begin{bmatrix} 1 & 0 \\ 0 & (0,7)^n \end{bmatrix}$$

$$\Rightarrow A^n = C \cdot D^n \cdot C^{-1}$$

So

$$A^n = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & (0{,}7)^n \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 & (0{,}7)^n \\ 2 & -(0{,}7)^n \end{bmatrix} \cdot \frac{1}{-3} \cdot \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} =$$

$$= \frac{1}{3} \begin{bmatrix} 1 & (0{,}7)^n \\ 2 & -(0{,}7)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} =$$

$$= \frac{1}{3} \cdot \begin{bmatrix} 1 + 2(0{,}7)^n & 1 - (0{,}7)^n \\ 2 - 2(0{,}7)^n & 1 \end{bmatrix}$$

In how many days will we get to equilibrium state? ...

$$X_n = A^n \cdot X_0 = A^n \cdot \begin{bmatrix} 300 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 2 \cdot (0{,}7)^n & 1 - (0{,}7)^n \\ 2 - 2 \cdot (0{,}7)^n & 1 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 100 + 200 \cdot (0{,}7)^n \\ 200 - 200 \cdot (0{,}7)^n \end{bmatrix} \to$$

$$\to \begin{bmatrix} 100 \\ 200 \end{bmatrix} \quad (\text{as } n \to \infty)$$