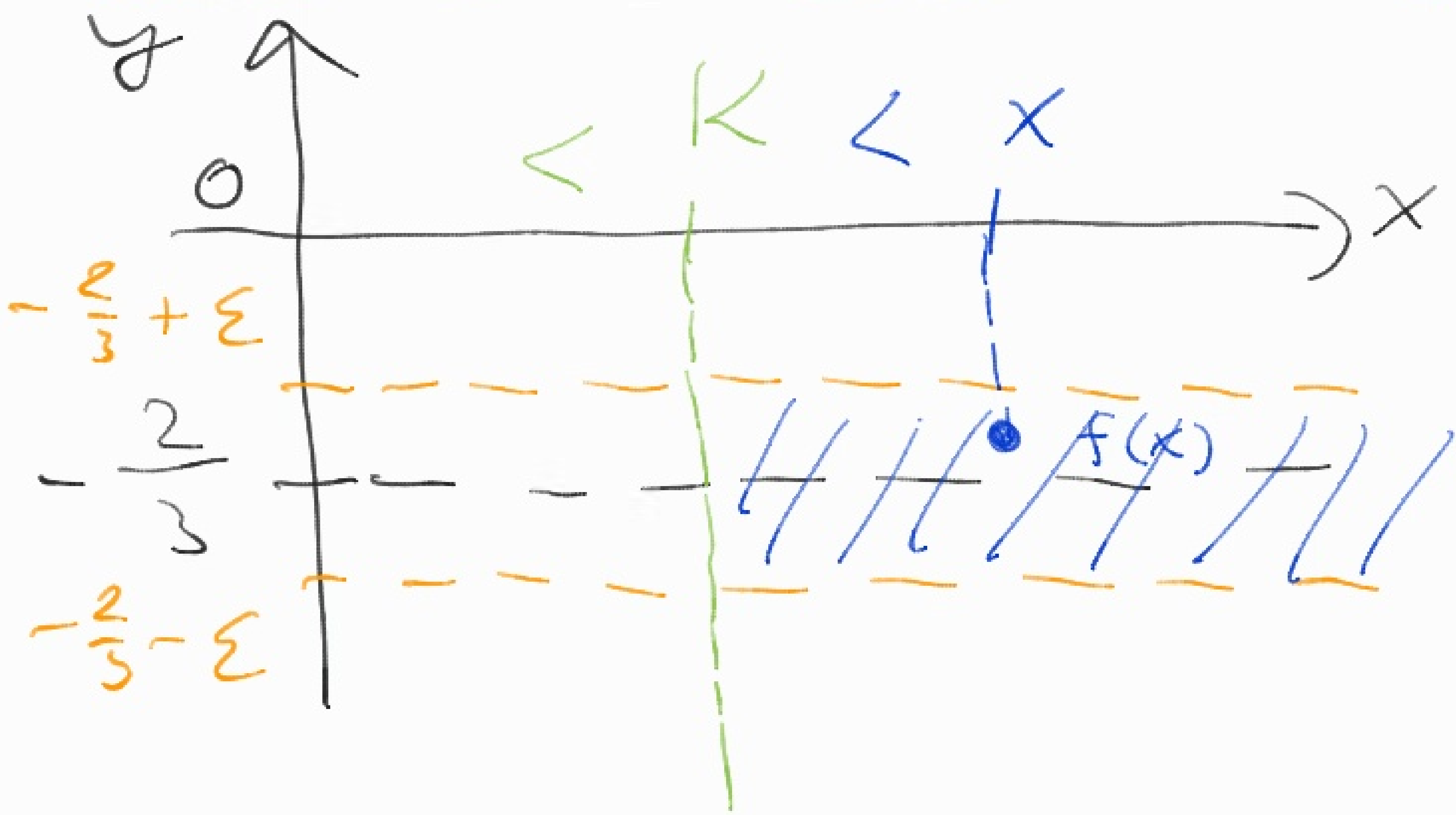
Basic Mathematics, Class 22

I Limits of functions at $+\infty$

① Prove by definition, that

$$\lim_{x \to +\infty} \frac{x^2+3x-1-2x^3}{3x^3+x^2+x+5} = -\frac{2}{3}$$



we need to prove the following statement:

$$\forall \varepsilon > 0 \ \exists K > 0 \ \forall x > K \text{ and } x \in D_f$$
$$\left| f(x) - \left(-\frac{2}{3}\right) \right| < \varepsilon$$

So let's fix an $\varepsilon > 0$ and consider the difference:

$$\left| f(x) + \frac{2}{3} \right| = \left| \frac{x^2 + 3x - 1 - 2x^3}{3x^3 + x^2 + x + 5} + \frac{2}{3} \right|$$

$$= \frac{\left| 3x^2 + 9x - 3 \cancel{- 6x^3} \cancel{+ 6x^3} + 2x^2 + 2x + 10 \right|}{\left| 3\left(3x^3 + x^2 + x + 5\right) \right|}$$

$$= \frac{5x^2 + 11x + 2}{9x^3 + 3x^2 + 3x + 15} \leq \text{OPC}$$

$x > 0$ we can assume

$(x \to +\infty)$

$$\le \frac{5x^2+11x^2+2x^2}{9x^3} = \frac{18x^2}{9x^3} =$$

if $x \ge 1$ $\quad = \frac{2}{x} < \varepsilon \iff$

$$x > \frac{2}{\varepsilon}$$

So if $x > \frac{2}{3} + 1 =: K$

$\forall \varepsilon > 0 \ \exists K := \frac{2}{\varepsilon} + 1 > 0$ so that if $x > K$ and $x \in D_f$:

$$\left| f(x) - \left(-\frac{2}{3}\right) \right| \le \frac{2}{x} < \frac{2}{K} < \varepsilon .$$

II Composition of functions.

① Give $f \circ g$ and $g \circ f$

if :

$$f(x) = x^2 - 1 \quad , \; x \in [-1, 2)$$

$$g(x) = \sqrt{x+1} \quad , \; x \in [-1, +\infty)$$

What do we want to define by $f \circ g$?

Its a function

$$(f \circ g)(x) = f(g(x)) =$$

$$f(\sqrt{x+1}) = (\sqrt{x+1})^2 - 1 =$$

$$= x + 1 - 1 = x \; ,$$

but for what inputs?

$$x \in D_{f\circ g} = \{ x \in D_g \mid g(x) \in D_f \}$$

If $D_{f\circ g} = \emptyset \Rightarrow$ No $f\circ g$ exists.

If $D_{f\circ g} \neq \emptyset$ then $f\circ g$ exists.

Now

$$D_{f\circ g} = \{ x \in [-1, +\infty) \mid \sqrt{x+1} \in [-1, 2) \}$$

We solve $-1 \leq \sqrt{x+1} < 2$ on interval $[-1, +\infty)$.

So $\underbrace{-1}_{-} \le \underbrace{\sqrt{x+1}}_{+/0} < \underbrace{2}_{+}$ $\Rightarrow ()^2$

This is true for all $x \in [-1, +\infty)$

$x+1 < 4$

$x < 4$

$$\Rightarrow \boxed{D_{f \circ g} = [-1, 4)} \text{ and}$$

$$(f \circ g)(x) = x \quad \forall x \in [-1, 4)$$

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5) What is $g \circ f$?

First $D_{g \circ f} = \{x \in D_f \mid f(x) \in D_g\}$

$= \{x \in [-1, 2) \mid x^2 - 1 \in [-1, +\infty)\}$

We have to solve now:

$$x^2 - 1 \geq -1 \quad \underline{\text{and}}$$

$$x \in [-1, 2) \text{ (only)}.$$

$\Longleftrightarrow$ $x^2 \geq 0$ which is true for all $x \in \mathbb{R}$ $\Rightarrow$

If $\boxed{x \in [-1, 2) = D_{g \circ f}}$

$$(g \circ f)(x) = g(f(x)) =$$

$$= g(x^2 - 1) = \sqrt{x^2 - 1 + 1} =$$

$$= \sqrt{x^2} = |x| \quad x \in [-1, 2)$$

We can see here, that

$f \circ g \neq g \circ f$

III Invertible functions

(1) Consider function

$$f(x) = \frac{2x-1}{x+3} \quad \left(x \in (0;+\infty)\right)$$

a) Is f invertable?

b) Give f^{-1} $\left(D_{f^{-1}}, R_{f^{-1}}, f^{-1}(x) = ?\right)$

Def: If $f \in \mathbb{R} \to \mathbb{R}$ $(\Longleftrightarrow$ $\emptyset \neq D_f \subseteq \mathbb{R}$, f has real outputs$)$

f is invertible $(\Longleftrightarrow)$

if $\forall\ x \neq t \in D_f \implies f(x) \neq f(t)$

$(f(x) - f(t) \neq 0)$ or another

form:

if: $x, t \in D_f$:

$$f(x) = f(t) \implies x = t$$

Now if $x, t \in (0, +\infty)$ and

$f(x) = f(t) \implies \frac{2x-1}{x+3} = \frac{2t-1}{t+3}$

$\implies (2x-1)(t+3) = (x+3)(2t-1)$

$$\cancel{2xt} + 6x - t - \cancel{3} = \cancel{2xt} - x + 6t - \cancel{3}$$

$7x = 7t \implies x = t$

$\implies f$ is injective / invertible.

$$D_{f^{-1}} := R_f = \{ y \in \mathbb{R} \mid$$

$$\exists x \in D_f \text{ so that } y = f(x) \}$$

$$= \left\{ y \in \mathbb{R} \;\middle|\; \exists x \in (0, \infty) : y = \frac{2x-1}{x+3} \right\}$$

$$y = \frac{2x-1}{x+3} \iff xy + 3y = 2x - 1$$

$$(x>0)$$

$$2x - xy = 1 + 3y$$

$$x(2-y) = 1 + 3y \text{ so } \underline{\text{if}}$$

$$2 - y \neq 0 \quad y \neq 2 \implies$$

$$x = \frac{1+3y}{2-y} = f^{-1}(y)$$

with $x \geq 0$ so

$\frac{1+3y}{2-y} > 0$ and $y \neq 2$

so if $2-y > 0 \iff \boxed{y<2}$

$\Rightarrow 1+3y > 0$ $\Downarrow$

$\boxed{y > -\frac{1}{3}} \Rightarrow -\frac{1}{3} < y < 2$

And if $2-y<0 \Rightarrow$

if $y>2 \Rightarrow 1+3y<0$

$\Downarrow$ $\boxed{y < -\frac{1}{3}}$

$\Leftarrow$

$y \in \varnothing$

We get that:

$$D_{f^{-1}} = R_f = \left(-\frac{1}{3}; 2\right)$$

$\underset{\sim\sim}{\text{and}}$ $$f^{-1}(y) = \frac{1+3y}{2-y} \quad \left(-\frac{1}{3} < y < 2\right)$$

$$R_{f^{-1}} = D_f = (0; +\infty)$$

$$\boxed{\begin{aligned} &(2)\ f(x) = -x^2 - 4x - 3 \\ &\quad a)\ x \in D_f := \mathbb{R} \\ &\quad b)\ x \in D_f := [-2; +\infty) \end{aligned}}$$

Sol:

a) f is a parabola symmetric on $x=-2$ so for example

$$f(-1) = f(-3)$$
$$\begin{array}{ccc} \shortparallel & & \shortparallel \\ -1+4-3 & & -9+12-3 \\ \shortparallel & & \shortparallel \\ 0 & & 0 \end{array}$$

$-1 \neq -3 \in D_f$ and $f(-1)=f(-3)=$
$=0 \implies f$ is not invertible in case a)

b) Assume that $x, t \in [-2, +\infty)$
$x \neq t \implies$ Evaluate

$$\underbrace{f(x) - f(t)} = -x^2 - 4x - 3 -$$

$$- \left(-t^2 - 4t - 3\right) = t^2 - x^2 + 4t - 4x =$$

$$= (t-x)(t+x) + 4(t-x) =$$

$$= \underbrace{(t-x)}_{\neq 0} \underbrace{(t+x+4)}_{\neq 0} \underbrace{\neq 0} \Rightarrow$$

since $x \neq t$

$$\left.\begin{array}{l} t \geq -2 \\ x \geq -2 \\ x \neq t \end{array}\right\} \Rightarrow t+x+4 > -4+4 = 0$$

$\underbrace{f(x) \neq f(t)}$ so f is $\underbrace{\text{injective}}$

or $\underbrace{\text{invertable}}$.

<u>Find</u> $D_{f^{-1}} = R_f =$

$= \{ y \in \mathbb{R} \mid \exists x \in [-2, +\infty) : -x^2 - 4x - 3 = y \}$

$$-x^2-4x-3=y$$

$$1-(x^2+4x+4)=y$$

$$1-(x+2)^2=y$$

$1-y=(x+2)^2 \leftarrow$ it can be solved only when

$1-y\geq 0$ so $y\leq 1$

$\Rightarrow \quad x+2=\pm\sqrt{1-y}$

$x_1=-2+\sqrt{1-y}$ or $x_2=2-\sqrt{1-y}$

which is good for us?

We need $x\geq -2$

$-2+\sqrt{1-y} \geq -2$ ✓ and

if $y \leq 1$

$-2-\sqrt{1-y} \leq -2$

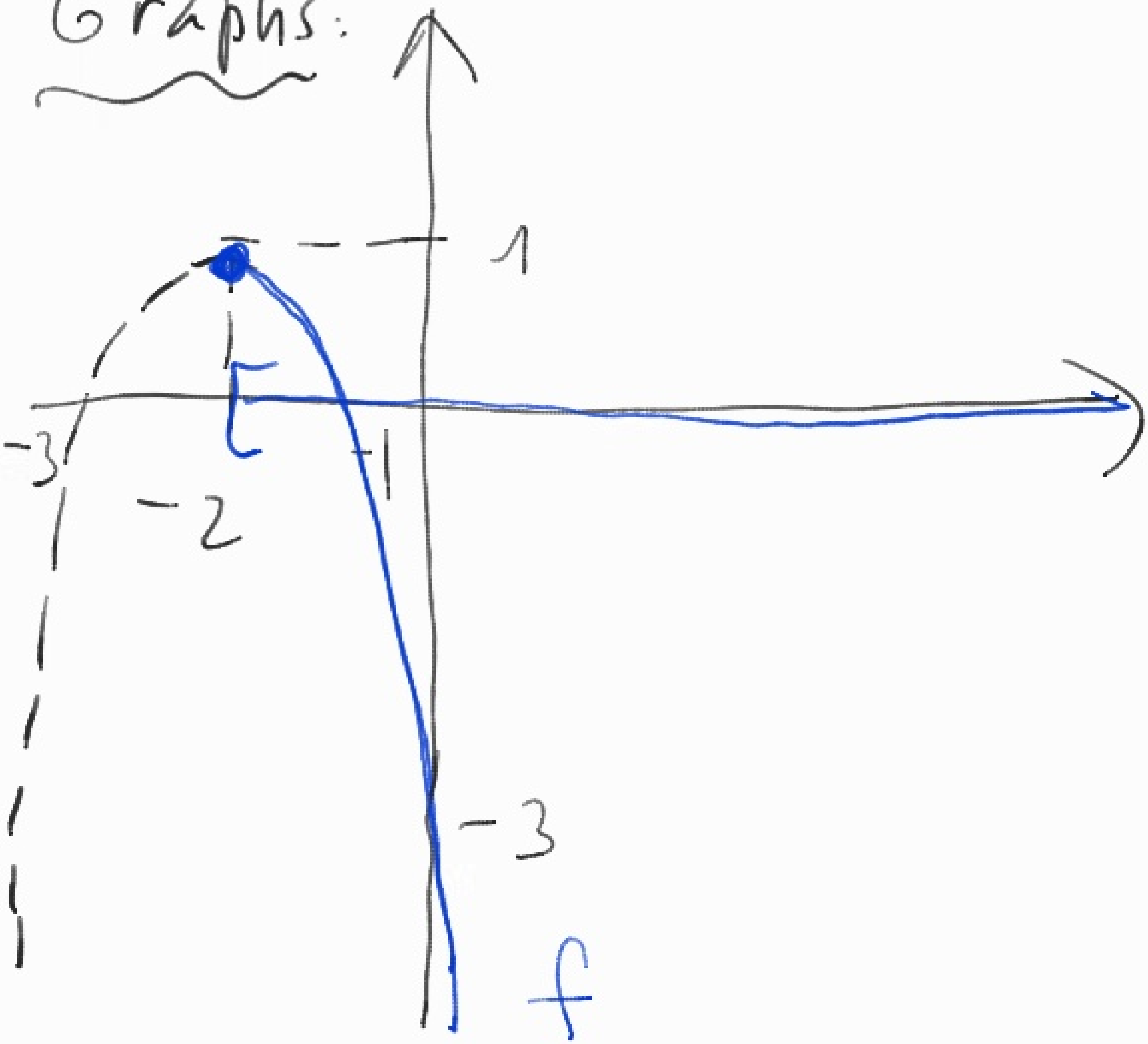
if $y \leq 1$

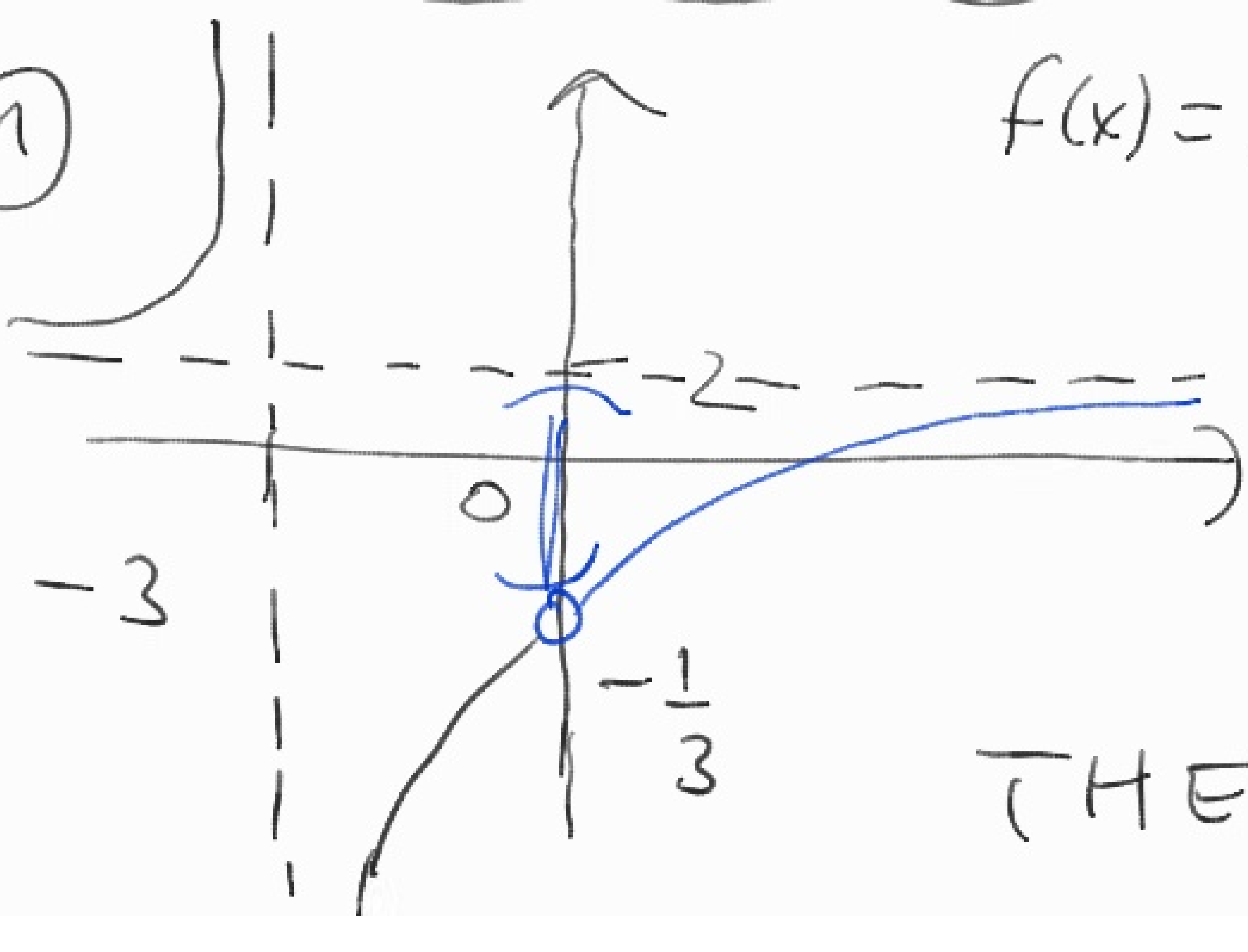
$\Rightarrow D_{f^{-1}} = (-\infty, 1]$ and

$f^{-1}(y) = -2+\sqrt{1-y} \quad (y \leq 1)$

$R_{f^{-1}} = D_f = [-2; +\infty)$

Graphs:





(1) $f(x) = 2 - \frac{7}{x+3}$

THE END