Basic Math || Class 17

Relationship with the inverse matrix.

1) Find the inverse matrix of the following matrices A if they exist. Is A regular or singular? USE E.B.T.

a) $A := \begin{bmatrix} 5 & 2 & -3 \\ 3 & 1 & -2 \\ 2 & -3 & -4 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

b) $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ -3 & 1 & -7 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

a)

	a_1	a_2	a_3	e_1	e_2	e_3	
		A			I_3		
e_1	5	2	−3	1	0	0	+
e_2	3	$\boxed{1}$	−2	0	1	0	$\mid \cdot(-2)$ $\mid \cdot 3$
e_3	2	−3	−4	0	0	1	+
e_1	−1	0	$\boxed{1}$	1	−2	0	$\mid \cdot 2$ $\mid \cdot 10$
a_2	3	1	−2	0	1	0	+
e_3	11	0	−10	0	3	1	+
a_3	−1	0	1	1	−2	0	+
a_2	1	1	0	2	−3	0	+
e_3	$\boxed{1}$	0	0	10	−17	1	$\mid(-1)$
a_3	0	0	1	11	−19	1	
a_2	0	1	0	−8	14	−1	
a_1	1	0	0	10	−17	1	

The algorithm STOPS here (We could change basis e_1, e_2, e_3 to a_3, a_2, a_1)

Here in the last table we change the order of the rows into the original order a_1, a_2, a_3 $\Longrightarrow$

$$
\begin{array}{c|ccc|ccc}
 & a_1 & a_2 & a_3 & e_1 & e_2 & e_3 \\
\hline
a_1 & 1 & 0 & 0 & 10 & -17 & 1 \\
a_2 & 0 & 1 & 0 & -8 & 14 & -1 \\
a_3 & 0 & 0 & 1 & 11 & -19 & 1 \\
\hline
\end{array}
$$

I_3 $\qquad$ $\underline{\underline{A^{-1}}}$

So A has an inverse matrix so it is regular and its inverse is:

$$A^{-1} = \begin{bmatrix} 10 & -17 & 1 \\ -8 & 14 & -1 \\ 11 & -19 & 1 \end{bmatrix}$$

What is rank(A) = ?

We could change all 3 vectors a_1, a_2, a_3 so

$\boxed{\text{rank} A = 3}$

b)

	a_1	a_2	a_3	e_1	e_2	e_3	
e_1	1	1	−1	1	0	0	+
e_2	2	0	3	0	1	0	
e_3	−3	$\boxed{1}$	−7	0	0	1	$\cdot(-1)$
e_1	4	0	6	1	0	−1	+
e_2	$\boxed{2}$	0	3	0	1	0	$:2$ $\cdot(-2)$
a_2	−3	1	−7	0	0	1	+ $\cdot(3/2)$
e_1	0	0	0	1	−2	−1	
a_1	1	0	3/2	0	1/2	0	
a_2	0	1	−5/2	0	3/2	1	

$$\frac{9}{2} - 7 = \frac{9-14}{2} = -\frac{5}{2}$$

In the line of e_1 we have 0 at column a_3 so the algorithm STOPS

We could change e_2, e_3 to a_1, a_2 $\Rightarrow$

$$\boxed{\operatorname{rank}(A) = 2} \Rightarrow$$

A does not have inverse so A is singular, so $\nexists A^{-1}$.

We can also find from this table the solutions of the homogenous system $Ax = 0$. So the last table

	a_1	a_2	a_3	b	
e_1	0	0	0	$\boxed{0}$	✓
a_1	1	0	$3/2$	0	
a_2	0	1	$-5/2$	0	

Fixed variables: x_1, x_2.

Free variable: x_3.

solutions:

$$\begin{cases} x_1 \qquad + \frac{3}{2} x_3 = 0 \\ \qquad x_2 - \frac{5}{2} x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -\frac{3}{5} x_3 \\ x_2 = \frac{5}{2} x_3 \\ x_3 \in \mathbb{R} \quad \text{free} \end{cases} \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{bound.}$$

<u>Vector form:</u>

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3/5\, x_3 \\ 5/2 \;\; x_3 \\ x_3 \end{pmatrix} =$$

$$= x_3 \begin{pmatrix} -3/5 \\ 5/2 \\ 1 \end{pmatrix} \quad (x \in \mathbb{R}) \Rightarrow$$

$$S_h = \operatorname{Span}\left(\begin{pmatrix} -3/5 \\ 5/2 \\ 1 \end{pmatrix} \right) \Rightarrow$$

$\dim S_h = 1$, and

$(-3/5, 5/2, 1)$ is a basis in

S_h.

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② $A := \begin{bmatrix} 5 & 1 & -7 & -2 \\ 0 & 2 & 1 & 1 \\ 1 & 5 & 1 & 2 \\ -3 & -1 & 4 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$

a) $\operatorname{rank}(A) = ?$

b) is it regular or singular?

c) $b_1 = (-1, 3, 7, 0) \in \mathbb{R}^4$

$b_2 = (0, 5, 7, -1) \in \mathbb{R}^4$

Solve the systems:

$Ax = b_1$ and $Ax = b_2$

	a_1	a_2	a_3	a_4	b_1	b_2	
e_1	5	1	-7	-2	-1	0	+
e_2	0	2	1	1	3	5	
e_3	$\boxed{1}$	5	1	2	7	7	$\cdot(-5)$ $\quad \cdot 3$
e_4	-3	-1	4	1	0	-1	+
e_1	0	-24	-12	-12	-36	-35	+
e_2	0	2	$\boxed{1}$	1	3	5	$\cdot 12$ $\quad \cdot(-1)$
a_1	1	5	1	2	7	7	+ $\quad \cdot(-7)$
e_4	0	14	7	7	21	20	+

	a_1	a_2	a_3	a_4	b_1	b_2	
e_1	0	0	0	0	0	25	$\neq 0$
a_3	0	2	1	1	3	5	
a_1	1	3	0	1	4	2	
e_4	0	0	0	0	0	-15	$\neq 0$

No more options to change e_1, e_4 $\Longrightarrow$ the algorithm STOPS here, so this is the final table. Conclusions:

$Ax=b_1$ can be solved since in the 0 lines in column b_1 are $\boxed{0}$-s

In the 0 lines of e_1, e_4 in column of b_2 there are no $\boxed{0's}$ (it's enough if one of them is not 0) $\Rightarrow$ $Ax = b_2$ has no solutions, so it is inconsistent

- we changed e_2, e_3 to a_3, a_1 $\Rightarrow$

$$\boxed{\operatorname{rank}(A) = 2}$$

and the system $Ax = b_1$ can be solved, so:

Fixed variables: x_1, x_3
(bound —||—)

Free variables: x_2, x_4

$\Rightarrow$ Solutions:

$$\left.\begin{aligned} 2x_2 + x_3 + x_4 &= 3 \\ x_1 + 3x_2 \qquad + x_4 &= 4 \end{aligned}\right\}$$

$$\Rightarrow \begin{cases} x_3 = 3 - 2x_2 - x_4 \\ x_1 = 4 - 3x_2 - x_4 \end{cases}$$

$(x_2, x_4 \in \mathbb{R})$ Scalar form of solutions.

Vector form:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 - 3x_2 - x_4 \\ x_2 \\ 3 - 2x_2 - x_4 \\ x_4 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 4 \\ 0 \\ 3 \\ 0 \end{pmatrix}}_{x_B} + x_2 \cdot \underbrace{\begin{pmatrix} -3 \\ 1 \\ -2 \\ 0 \end{pmatrix}}_{v_2} + x_4 \cdot \underbrace{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{v_4}$$

basis solution

$x_2, x_4 \in \mathbb{R}$;

So the set of all solutions

$$S = \{ x_B + x_2 v_2 + x_4 v_4 \mid x_2, x_4 \in \mathbb{R} \} =$$

$$= x_B + \{ x_2 v_2 + x_4 v_3 \mid x_2, x_4 \in \mathbb{R} \} =$$

$$= x_B + \mathrm{Span}(v_2, v_4) =$$

$$= x_B + S_h$$

So

$$S_h = \mathrm{Span}(v_2, v_4) \Rightarrow$$

$$\dim S_h = n - r = 4 - 2 = 2$$

$\Rightarrow$ v_2, v_4 is a basis in S_h (homogenous solutions of $AX = 0$)

<u>So</u> $\dim \operatorname{Ker} A =$

$= \dim S_h = 2$

$\operatorname{rank}(A) = 2 < 4 \Rightarrow$

A is <u>singular</u> $\Rightarrow$

$\nexists A^{-1}$. We can also conclude, that

$\det A = 0 \Leftrightarrow \nexists A^{-1}$

THE END