Basic Mathematics

Class 9 | Complex Numbers

Ⓐ Introduction: We know that for real numbers x there is no solution for the equation $\boxed{x^2 = -1}$ (*)

Let's extend the "number set" on which we want to solve (*).

Define the so called imaginary unit denoted by i so that

$$\boxed{i^2 := -1}$$

Using this i we define new type of numbers, called COMPLEX numbers of form: $x + iy$ where $x, y \in \mathbb{R}$.

For example: $1 + 2i$, $-3 + \pi \cdot i$; $\frac{1}{2} i$; $3 + 0i$; $-2 - 31i$, $\sqrt{2} + \sqrt{3} i$

and so on. So the set of complex numbers will be denoted by $\mathbb{C}$

So (B)

$\mathbb{C} := \{ x+iy \mid x, y \in \mathbb{R} \}$.

We consider $w \in \mathbb{C}$, then $w = x+iy$ and we call

x : the real part of w

y : the imaginary part of w.

So $\boxed{\operatorname{Re} w = x \; ; \; \operatorname{Im} w = y}$

$\operatorname{Re}(1+2i) = 1$, $\operatorname{Im}(1+2i) = 2$

$\operatorname{Re}(-7-5i) = -7$, $\operatorname{Im}(-7-5i) = -5$

$\operatorname{Re}(\sqrt{2}) = \sqrt{2}$, $\operatorname{Im}(\sqrt{2}) = 0$

The real numbers are subset of complex numbers, so $\mathbb{R} \subseteq \mathbb{C}$

because they are of form:

$$x + 0 \cdot i = x \in \mathbb{R}.$$

We use letters $z, w \ldots$ for complex numbers.

We consider two complex numbers equal : $z, w \in \mathbb{C}$

$$\boxed{z = w \iff \begin{array}{l} \mathrm{Re}(z) = \mathrm{Re}(w) \text{ and} \\ \mathrm{Im}(z) = \mathrm{Im}(w). \end{array}}$$

So

(1°) Find $x, y \in \mathbb{R}$ so that:

$$(2+i)x - (2-i)y = x - y + 2i \iff$$

$$2x - 2y + (x+y)i = x - y + 2i$$

real parts: $\begin{cases} 2x - 2y = x - y \\ x + y = 2 \end{cases} \iff$
im. parts:

$$\begin{cases} x - y = 0 \\ x + y = 2 \end{cases} \overset{+}{\Rightarrow} 2x = 2 \quad \boxed{x = 1}$$

$$\Rightarrow y = x \Rightarrow \boxed{y = 1}$$

(C) Representation of complex numbers

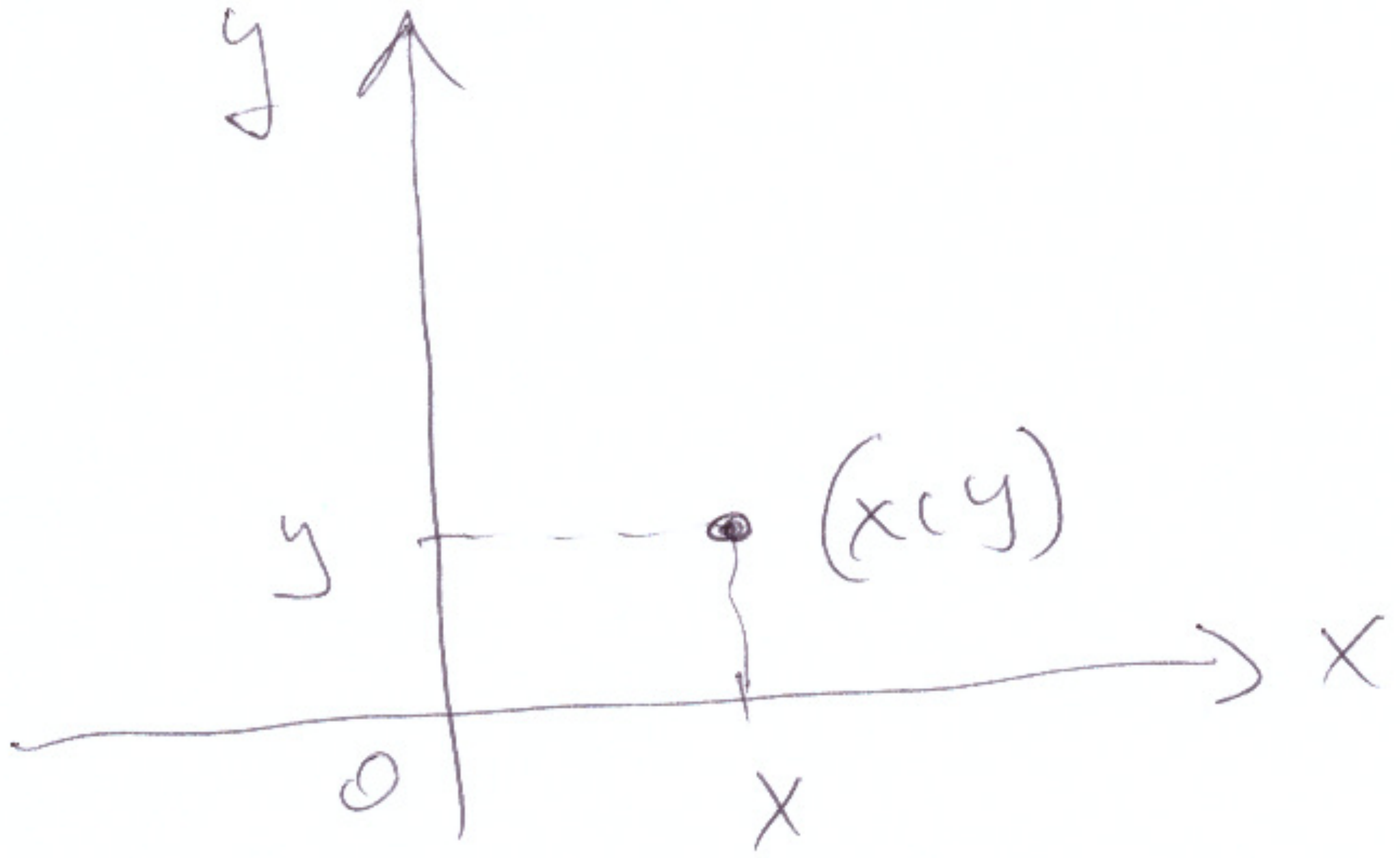
We can make a one-to-one correspondence between $\mathbb{C}$ and $\mathbb{R}^2$ (plane vectors)

So that

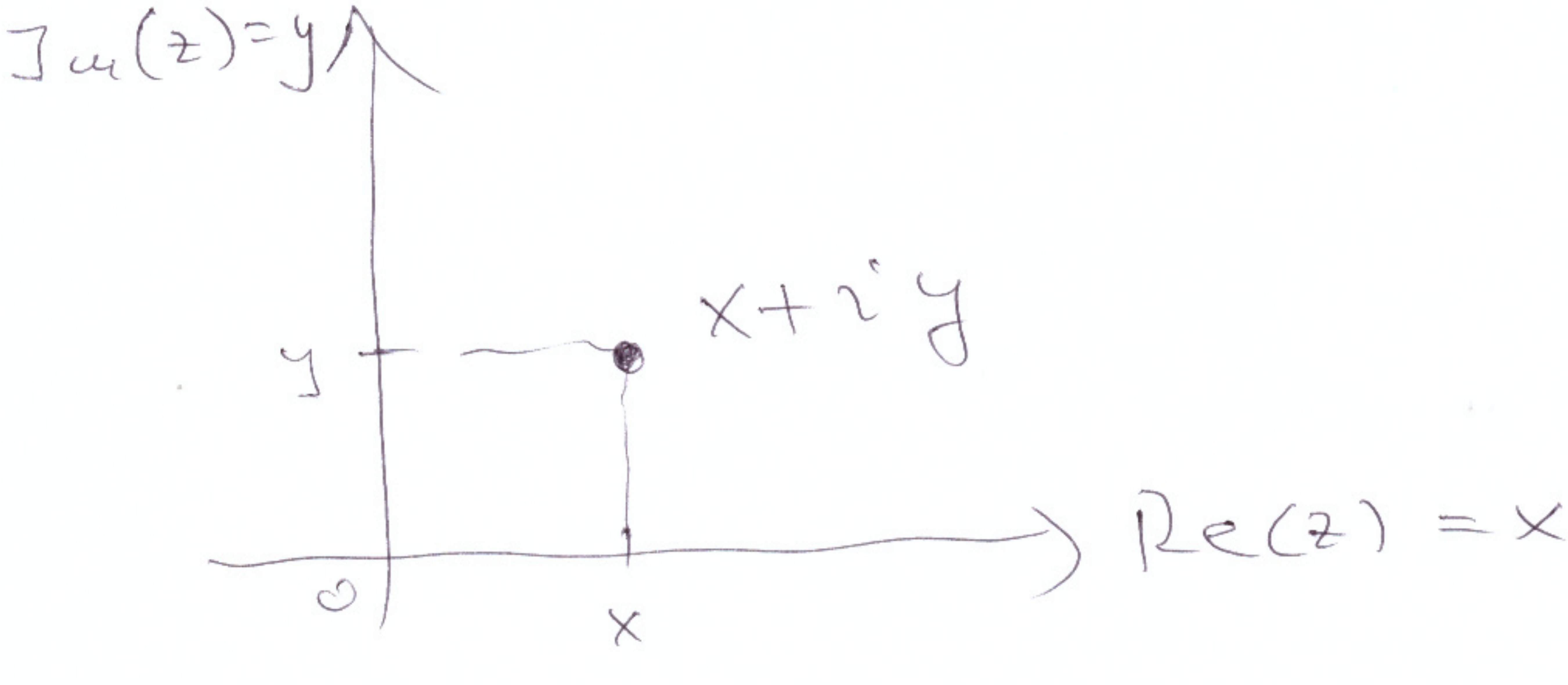
$$z = x + iy \longleftrightarrow (x, y)$$

$$\underset{\mathbb{C}}{\underset{\cap}{x+iy}} \qquad \underset{\mathbb{R}^2}{\underset{\cap}{(x,y)}}$$

Cartesian coordinate system

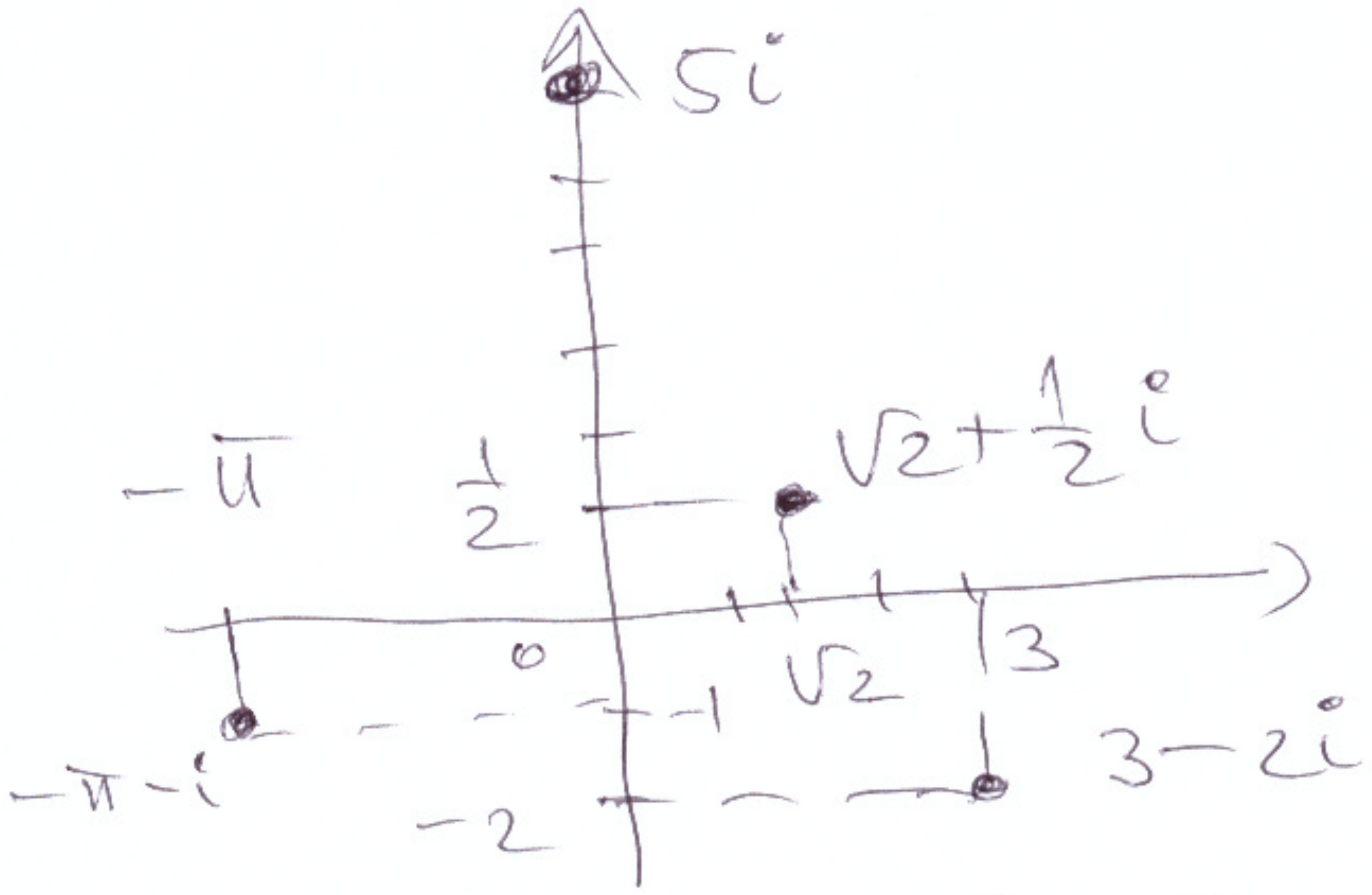


Complex plane



So: Where are in the complex plane the numbers: $3-2i$; $\sqrt{2}+\frac{1}{2}i$; $5i$?

$-\pi - i$



(D) Algebraic operations with complex numbers.

(1) Addition:

Ex: $(2+3i)+(-3+5i)=$

$=(2-3)+(3i+5i)=\underline{-1+8i}$;

(we add real parts with real parts imaginary parts by imaginary parts)

(2) Substruction, difference:

$(-1+3i)-(5+i)=$

$=-1-5+3i-i=$

$=\underline{-6+2i}$;

(3) Multiplication:

$(2+4i)(5-7i) = 10 \underline{-14i} + \underline{20i}$

$-28i^2 = 10 + 6i + 28 = \boxed{38+6i}$

$i^2 = -1$

(4) Dividing complex numbers:

$\frac{2+3i}{1-2i} = ? = \frac{2+3i}{1-2i} \cdot \frac{1+2i}{1+2i} =$

we multiply by the so called conjugate of the denominator $1-2i$

$= \frac{2+4i+3i+6i^2}{(1)^2-(2i)^2} = \frac{-4+7i}{1-4i^2} =$

$= \frac{-4+7i}{5} = -\frac{4}{5} + \frac{7}{5}i$

THE FORM : $x+iy=z$ is called the ALGEBRAIC FORM OF z.

Exercises:

(10) Evaluate, and give the algebraic form of the result:

a) $\frac{1}{2-3i} = \frac{2+3i}{(2-3i)(2+3i)} = \frac{2+3i}{4-9i^2} = \frac{2+3i}{13} =$

$= \frac{2}{13} + \frac{3}{13} i$;

b) $\frac{1+5i}{3+2i} = \frac{(1+5i)(3-2i)}{(3+2i)(3-2i)} = \frac{3-2i+15i-10i^2}{9+4}$

$= \frac{13+13i}{13} = 1+i$;

c) $(1-2i)(5+i) = 5+i-10i-2\overset{-1}{i^2} =$

$= 7-9i$;

d) $\frac{1}{1+\frac{1}{1+\frac{1}{1+i}}} = \frac{1}{1+\frac{1}{\frac{2+i}{1+i}}} =$

$= \frac{1}{1+\frac{1+i}{2+i}} = \frac{1}{\frac{2+i+1+i}{2+i}} = \frac{2+i}{3+2i} =$

$= \frac{2+i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{6-4i+3i-2i^2}{9+4} = \frac{8-i}{13} =$

$= \frac{8}{13} - \frac{1}{13} i$;

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e) $(2-i)^2 + (2+i)^3 = 4 - 4i + i^2 + 8 + 12i + 6i^2 + \underbrace{i^3}_{= i^2 \cdot i} = 12 + 8i - 1 - 6 + (-1)i =$

$= 5 + 7i$;

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f) $(3-\sqrt{2}i)^3 (3+\sqrt{2}i) = (3-\sqrt{2}i)^2 \cdot (3-\sqrt{2}i)(3+\sqrt{2}i) =$

$= (9 - 6\sqrt{2}i + 2i^2)(9 - 2i^2) =$

$= (7 - 6\sqrt{2}i) \cdot 11 = 77 - 66\sqrt{2} \cdot i$;

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g) $\frac{1+i}{3-i} + \frac{3-i}{1+i} = \frac{(1+i)^2 + (3-i)^2}{(3-i)(1+i)} =$

$= \frac{\cancel{1} + 2i + \cancel{i^2} + 9 - 6i + i^2}{3 + 2i - i^2} = \frac{8-4i}{4+2i} =$

$= \frac{4-2i}{2+i} = \frac{(4-2i)(2-i)}{(2+i)(2-i)} = \frac{8 - 4i - 4i + \overbrace{2i^2}^{-2}}{4+1} =$

$= \frac{6-8i}{5} = \frac{6}{5} - \frac{8}{5} i$;

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h) $\frac{(1+i)^2}{1-i}+\frac{(1-i)^3}{(1+i)^2}=\frac{\cancel{1}+2i+\cancel{i^2}}{1-i}+\frac{(\cancel{1}-2i+\cancel{i^2})(1-i)}{\cancel{1}+2i+\cancel{i^2}}=$

$=\frac{2i}{1-i}+\frac{(-\cancel{2i})(1-i)}{\cancel{2i}}=\frac{2i}{1-i}-(1-i)=$

$=\frac{2i(1+i)}{(1-i)(1+i)}-1+i=\frac{2i+2i^2}{1-i^2}-1+i=$

$=\frac{2i-2}{2}-1+i=i-1-1+i=\underline{-2+2i};$

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i) $\left(\frac{1+i}{1-i}\right)^{2018}+\left(\frac{1-i}{1+i}\right)^{2019}=i^{2018}+\left(\frac{1}{i}\right)^{2019}=$

$\underline{\underline{\frac{1+i}{1-i}}}=\frac{(1+i)^2}{(1-i)(1+i)}=\frac{1+2i+i^2}{1-i^2}=\frac{2i}{2}=\underline{\underline{i}}$

$=\left(i^2\right)^{1009}+\left(\frac{i}{i^2}\right)^{2019}=(-1)^{1009}+(-i)^{2019}=$

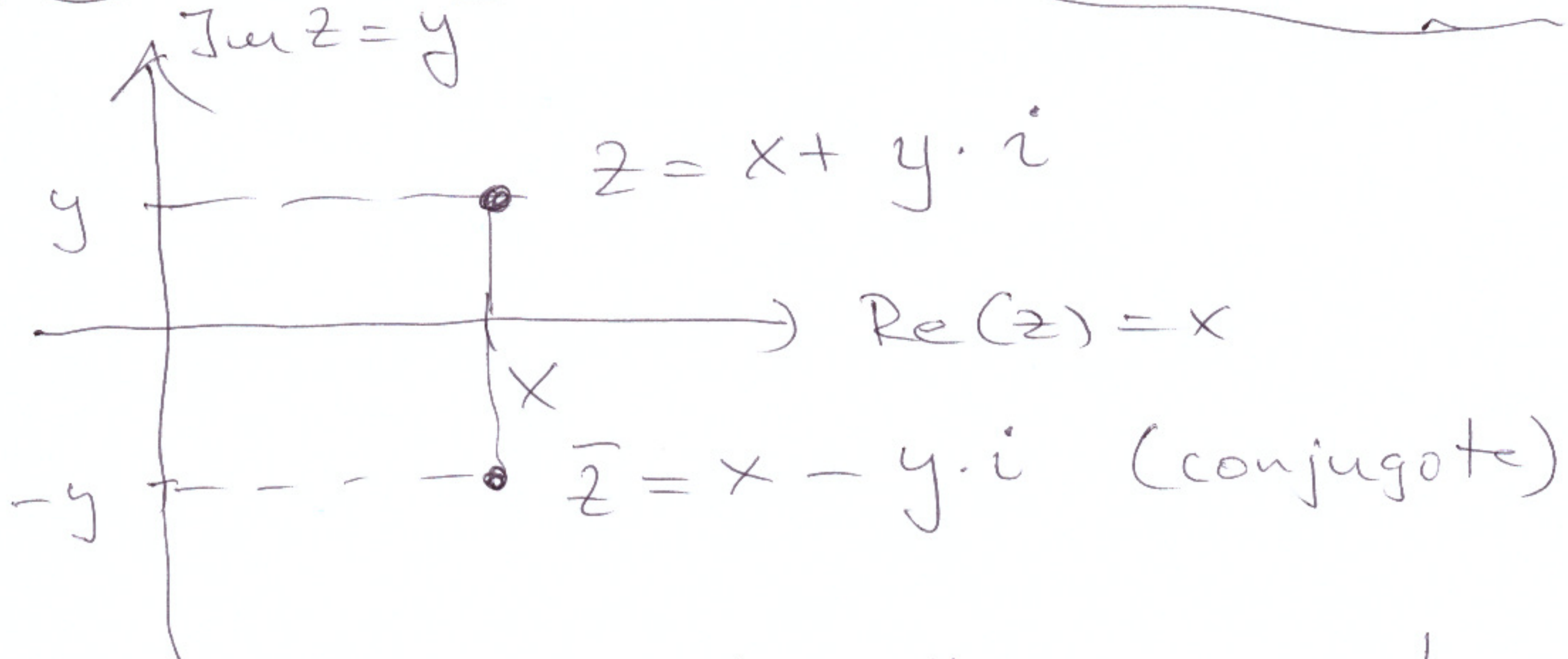
$=-1-\left(i^{2019}\right)=-1-i\cdot i^{2018}=$

$=-1-i\cdot(-1)=\underline{-1+i};$

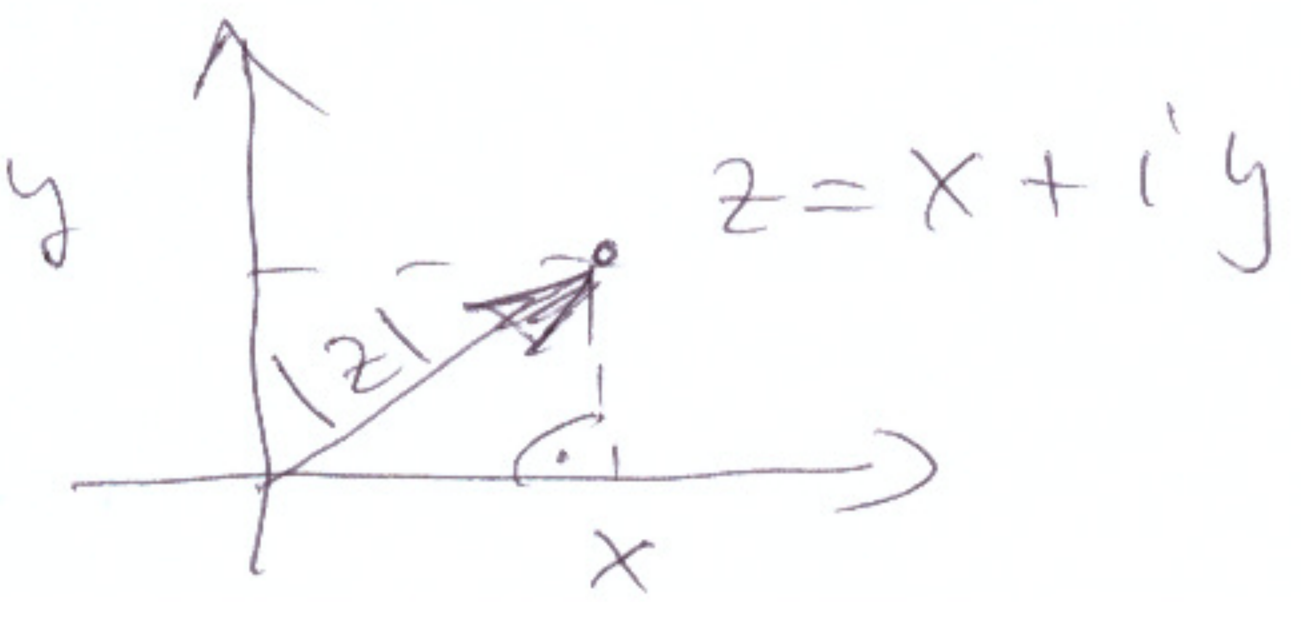
D) Conjugate and modulus of a complex number:

Def: If $z \in \mathbb{C}$ $z = x + iy$ $(x, y \in \mathbb{R})$
Then $\overline{z} := x - iy$ is the conjugate of z and

$|z| := \sqrt{x^2 + y^2}$ is the modulus or the length of. z.



Conjugation geometrically means to reflect point z on the realpart axes. (or x axes)



$$|z|^2 = x^2 + y^2$$

$$0 \leq |z| = \sqrt{x^2 + y^2}$$

A complex number

$z = x + iy$ can be also

considered as a plane-vector

(x, y)

So $|z| = |(x, y)| = \sqrt{x^2+y^2}$ is the length of vector (x, y)

Exp: $\overline{(2-3i)} = 2+3i$

$\overline{(-5-i)} = -5+i$

$\overline{2i} = -2i$

$|3-5i| = \sqrt{3^2+(-5)^2} = \sqrt{9+25} = \sqrt{34}$

$|7i| = \sqrt{7^2} = 7$

$|i| = \sqrt{0^2+1^2} = 1$;

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⑦ Evaluate $|z|$ if

$$z = \left(\sqrt{2+\sqrt{2}} + i\cdot\sqrt{2-\sqrt{2}}\right)^4$$

Solution:

$$|z| = \left| \left(\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}\right)^4 \right| =$$

$$= \left(\left| \sqrt{2+\sqrt{2}} + i \cdot \sqrt{2-\sqrt{2}} \right| \right)^4 =$$

$$= \left(\sqrt{\left(\sqrt{2+\sqrt{2}}\right)^2 + \left(\sqrt{2-\sqrt{2}}\right)^2} \right)^4 =$$

$$= \left((2+\sqrt{2}) + (2-\sqrt{2}) \right)^2 = 4^2 = \underline{\underline{16}};$$

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(8) Find all those complex numbers z for which z^3 is an imaginary number. Where are they on $\mathbb{C}$?

Sol: $z = x + iy$ with $x, y \in \mathbb{R}$

$$z^3 = (x+iy)^3 = x^3 + 3x^2 \cdot iy + 3x(iy)^2 +$$

$$+ (iy)^3 = \underline{x^3} + 3x^2y \cdot i \underline{- 3xy^2} - iy^3 =$$

$i^3 = i \cdot i^2 = -i$

$$= \underbrace{x^3 - 3xy^2}_{Re(z^3)} + i \underbrace{\left(3x^2y - y^3\right)}_{Im(z^3)}$$

So z^3 is an imaginary (pure) number $\Longleftrightarrow$

$$\begin{cases} Re(z^3) = 0 \quad \underline{\text{and}} \\ Im(z^3) \neq 0. \end{cases}$$

so:

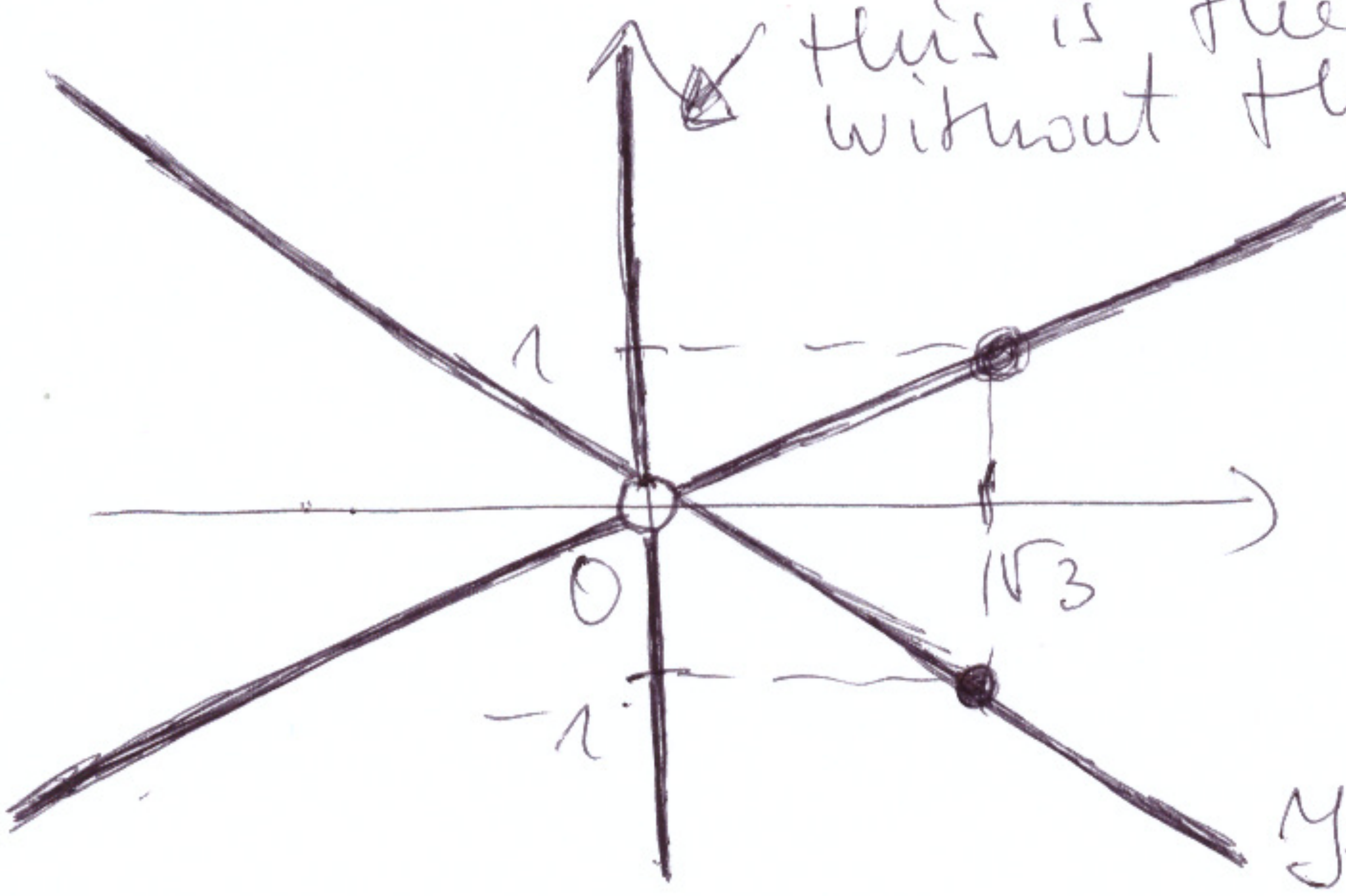
$$\begin{cases} x^3 - 3xy^2 = 0 \Longleftrightarrow x(x^2 - 3y^2) = 0 \\ 3x^2y - y^3 \neq 0 \Longleftrightarrow y(3x^2 - y^2) \neq 0. \end{cases}$$

So $\boxed{x=0}$ or $x^2 = 3y^2 \Longleftrightarrow \boxed{y = \pm\frac{1}{\sqrt{3}}x}$

with $\boxed{y \neq 0}$ and $\boxed{y \neq \pm\sqrt{3}x}$

Where are the needed points?

this is the $x=0$ line without the $(0,0)$ point



$y = \frac{1}{\sqrt{3}}x$ line

$y = -\frac{1}{\sqrt{3}}x$ line

These lines without the $(0,0)$ point don't have common points with the "forbidden" lines

$$y=\sqrt{3}x \quad \text{or} \quad y=-\sqrt{3}x \quad \text{or} \quad y=0.$$

9/b Find all $z\in\mathbb{C}$ and illustrate them on the complex plane for which we have:

$$\boxed{z^3=\overline{z}}.$$

Sol. $z=x+iy$ $(x,y\in\mathbb{R})$ $\Rightarrow$

$$(x+iy)^3=\overline{(x+iy)} \iff$$

$$x^3+3x^2\cdot iy+3x(iy)^2+(iy)^3=x-iy$$

$$x^3+3x^2yi-3xy^2-iy^3=x-iy$$

$$(x^3-3xy^2)+i(3x^2y-y^3)=x-iy$$

$\iff$ Real parts "=" and Im. parts "=" $\iff$

$$\begin{cases} x^3-3xy^2=x \\ 3x^2y-y^3=-y \end{cases}$$

$$\Leftrightarrow \begin{cases} x^3-3xy^2-x=0 \\ 3x^2y-y^3+y=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x(x^2-3y^2-1)=0 \\ y(3x^2-y^2+1)=0 \end{cases}$$

Possibilities:

(1) $\begin{cases} x=0 \\ y=0 \end{cases}$ (2) $\begin{cases} x=0 \\ 3x^2-y^2+1=0 \end{cases}$ (3) $\begin{cases} x^2-3y^2-1=0 \\ y=0 \end{cases}$

(4) $\begin{cases} x^2-3y^2-1=0 \\ 3x^2-y^2+1=0 \end{cases}$

(1) $\boxed{(x,y)=(0,0)}$

$0+0i$

(2) $y^2=1$

$y=\pm1$

$x=0$

$0+i$; $0-i$

(3) $x^2=1$

$x=\pm1$

$y=0$

$1+0i$

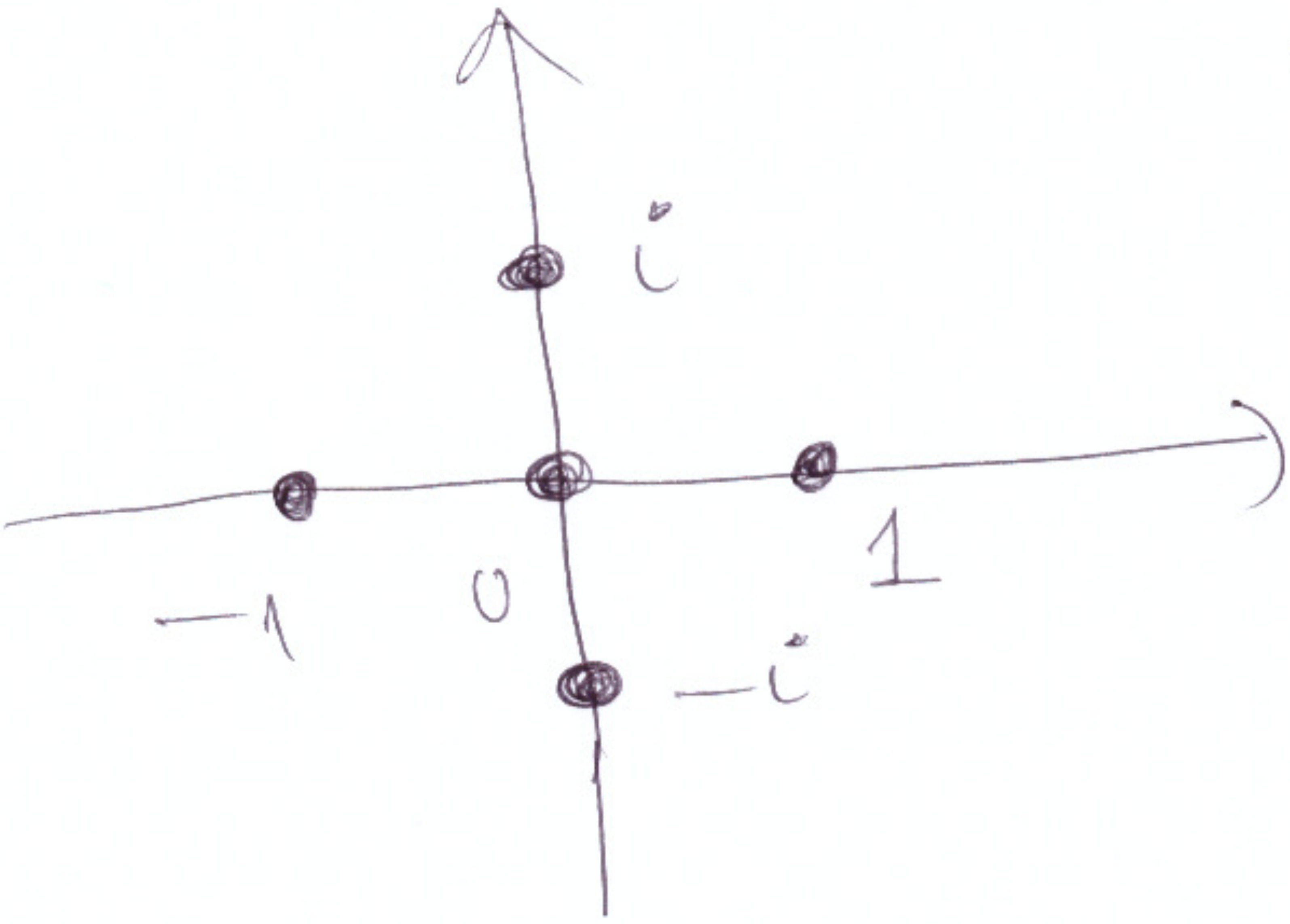
$-1+0i$

(4) $4x^2-4y^2=0 \Rightarrow y^2=x^2$

$x^2-3x^2=1 \quad -2x^2=1 \quad x\notin\mathbb{R}$ NO SOL.

So our solution is:

$z = 0 \; ; \; i \; ; \; -i \; ; \; 1 \; ; \; -1$



These five points and only these ones satisfy that $z^3 = \bar{z}$

(E) EQUATIONS

12/a. Solve:

$x^3 - 1 = 0 \quad (x \in \mathbb{C})$ now

$(x-1)(x^2+x+1) = 0$

$x - 1 = 0$

$\boxed{x_1 = 1}$

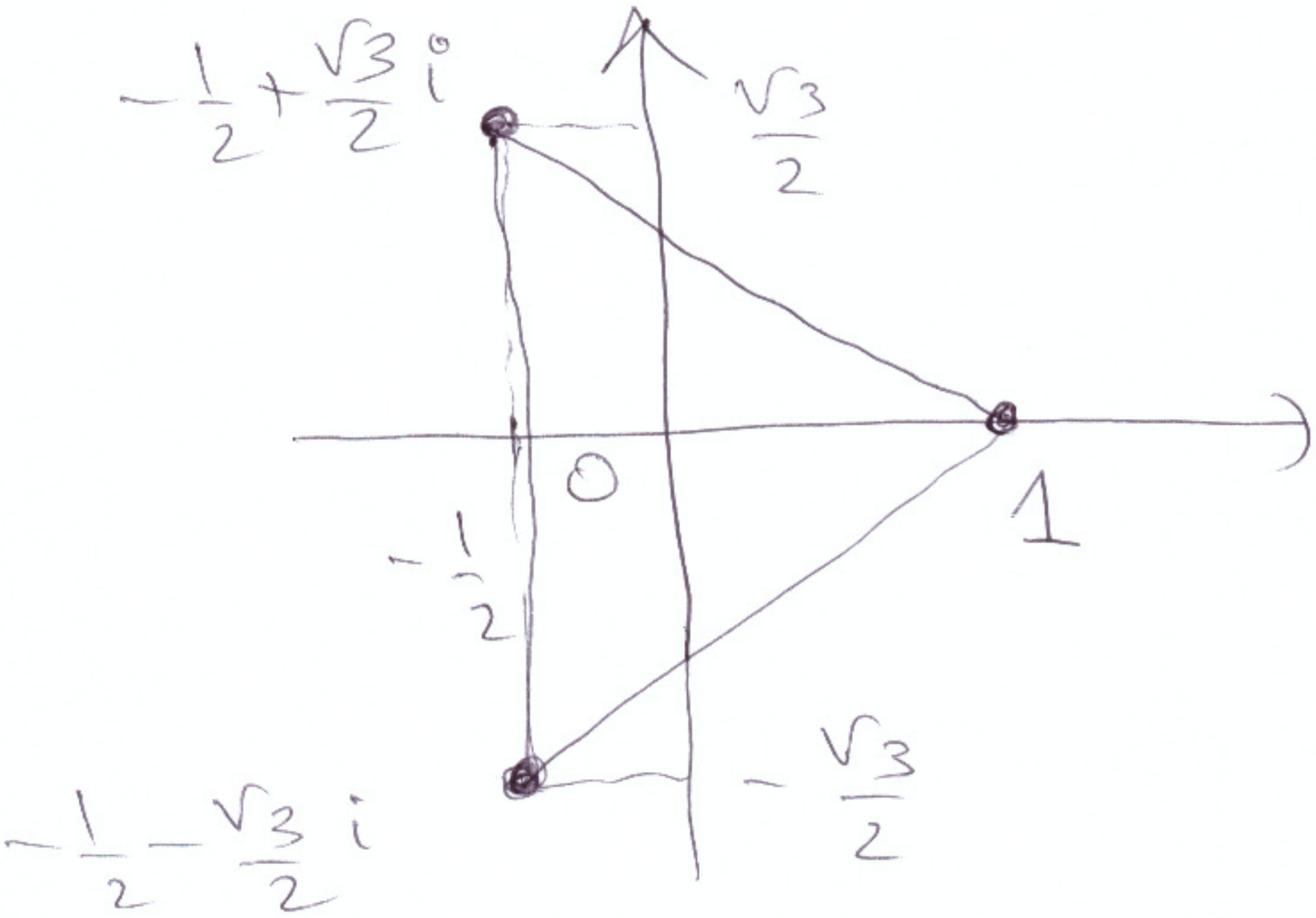
$x^2 + x + 1 = 0$

$x_{2,3} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$

$\sqrt{-3} = \sqrt{-1} \cdot \sqrt{3} = i \cdot \sqrt{3}$

So $\boxed{x_{2,3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i}$;

Solutions:



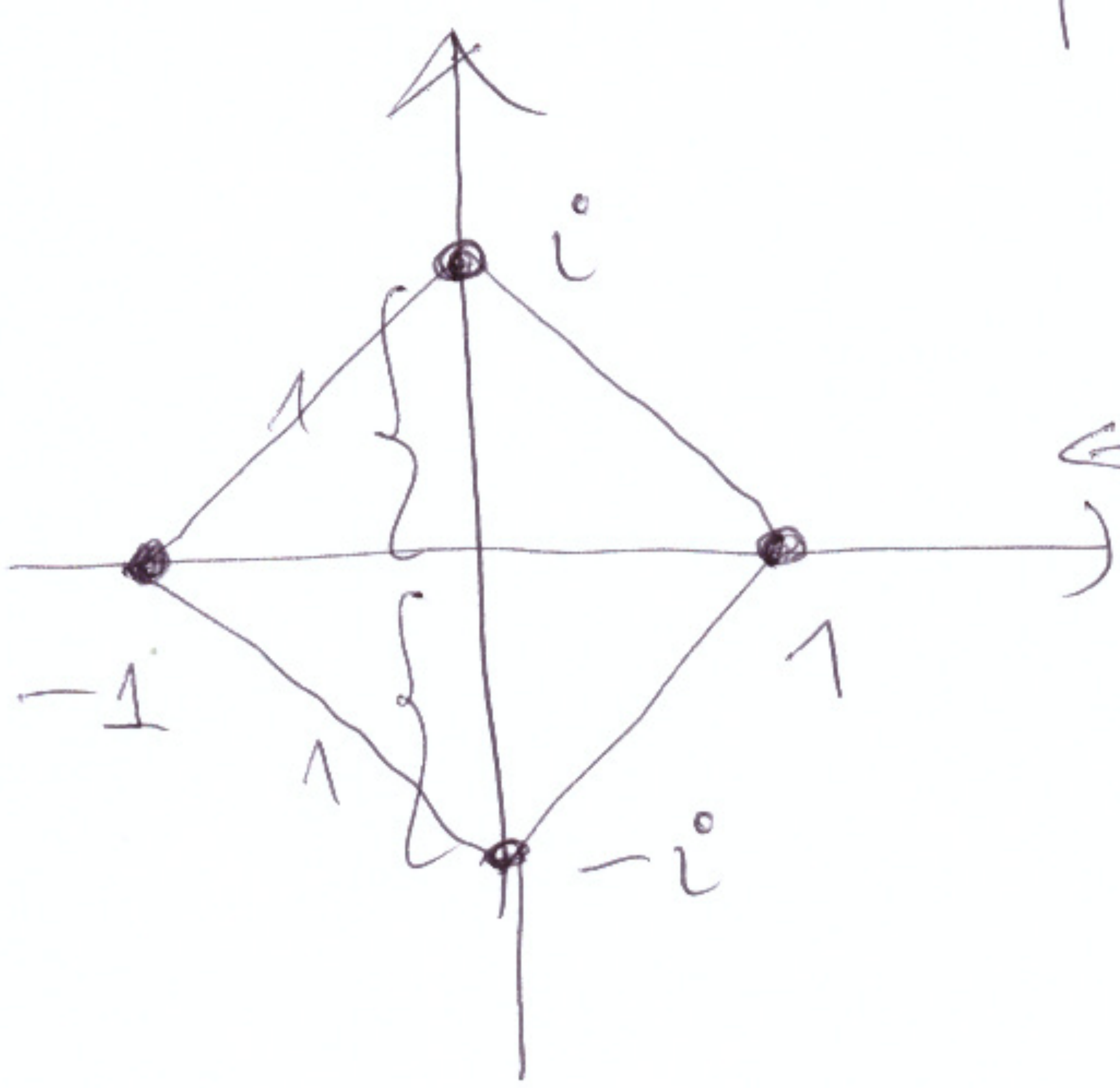
These points are the vertexes of an equilateral triangle

$\boxed{12/b}$ $z^4-1=0 \Leftrightarrow (z^2-1)(z^2+1)=0$

$(z-1)(z+1)(z^2+1)=0$

$\boxed{z_1=1}$; $\boxed{z_2=-1}$; $z^2=-1$

$\boxed{z_3=i}$ $\boxed{z_4=-i}$



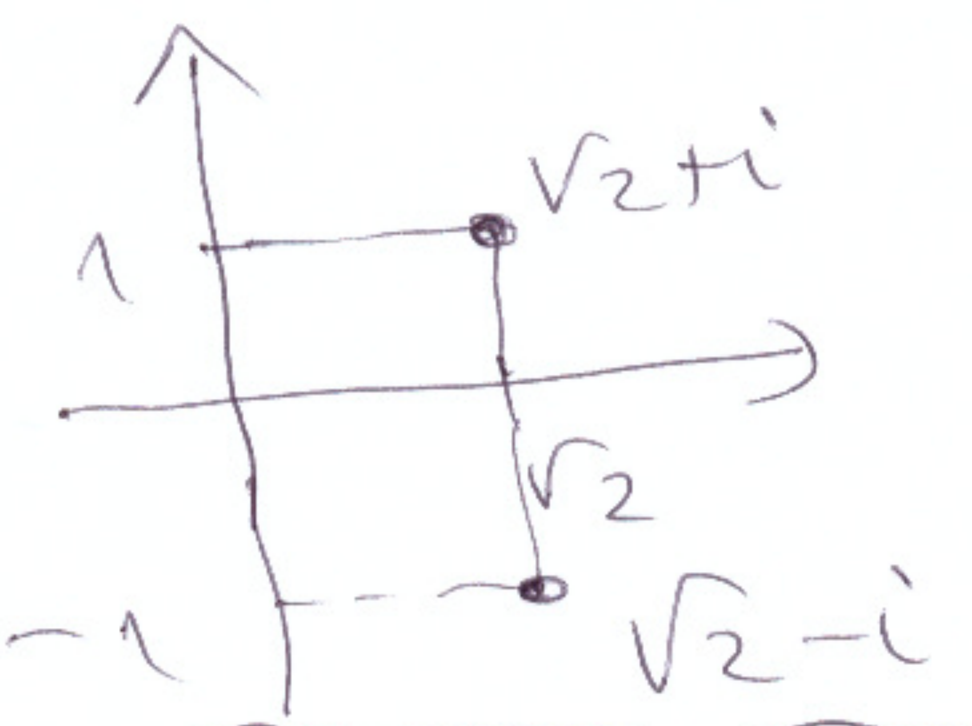
The solutions are the vertexes of a quadrat.

$\boxed{12/c}$ $x^2 - 2\sqrt{2}x + 5 = 0.$

$$x_{1,2} = \frac{2\sqrt{2} \pm \sqrt{8-20}}{2} = \frac{2\sqrt{2} \pm \sqrt{-4}}{2} =$$

$$= \frac{2\sqrt{2} \pm 2\sqrt{-1}}{2} = \frac{2\sqrt{2} \pm 2i}{2} = \underline{\sqrt{2} \pm i}$$

The two complex solutions are the <u>conjugate</u> of each other.



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$\boxed{12/d}$ $x^3 + 9x^2 + 18x + 28 = 0.$ $(*)$

First we try to find out one solution. If there is an integer solution of $(*)$ then it must divide the constant term: 28

<u>So</u> integer divisors of 28 are

$\pm 1;\ \pm 2;\ \pm 4;\ \pm 7;\ \pm 14;\ \pm 28.$

We <u>can find</u> the integer solutions between them!

Let's use HORNER table

	1	-9	18	28	
check: 2	1	-7	4	36 $\neq 0$	← not root
-2	1	-11	40	$\neq 0$	← not root
-1	1	-10	28	$\boxed{0}$	← its a root

$$\Rightarrow x^3 - 9x^2 + 18x + 28 =$$

$$= (x-(-1)) \cdot (1x^2 - 10x + 28) =$$

$$= (x+1)(x^2 - 10x + 28) = 0$$

$$\boxed{x_1 = -1} \qquad x_{2,3} = \frac{10 \pm \sqrt{100-112}}{2} =$$

$$\boxed{x_2 = \sqrt{5} + \sqrt{3}i} \qquad = \frac{10 \pm \sqrt{-12}}{2} = \frac{10 \pm \sqrt{-1} \cdot \sqrt{12}}{2} =$$

$$\boxed{x_3 = \sqrt{5} - \sqrt{3}i} \qquad = \frac{10 \pm i \cdot 2\sqrt{3}}{2} =$$

$$= 5 \pm \sqrt{3}i$$

$\boxed{12/e}$ $z^4 - 30z^2 + 289 = 0$

If $w := z^2 \Rightarrow w^2 - 30w + 289 = 0$

$$\Rightarrow w_{1,2} = \frac{30 \pm \sqrt{900 - 1156}}{2} = \frac{30 \pm \sqrt{-256}}{2} =$$

$$= \frac{30 \pm 16i}{2} = 15 \pm 8i$$

So

$z^2 = 15 + 8i$ or $z^2 = 15 - 8i$

Remark: In your future studies you will learn about the TRIGONOMETRIC form of complex numbers and about how to take n-th root of a complex number. Here we would need to take $\sqrt{15+8i}$ which has actually two values. The same is valid when we want

$$\sqrt{-256} \to 16i$$
$$\phantom{\sqrt{-256}} \to -16i$$

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What do we do now?

So if $z = x + iy \Rightarrow$

$(x+iy)^2 = 15 + 8i$

Real part = Im part

$x^2 + 2xyi - y^2 = 15 + 8i \Leftrightarrow$

$$\begin{cases} x^2 - y^2 = 15 \\ 2xy = 8 \Rightarrow y = \frac{4}{x} \quad (x \neq 0) \end{cases}$$

$x^2 - \frac{16}{x^2} = 15 \quad | \cdot x^2 \qquad x^4 - 15x^2 - 16 = 0$

$x^2 = \frac{15 \pm \sqrt{225 + 64}}{2} = \frac{15 \pm \sqrt{289}}{2} \Rightarrow$

$x^2 = \frac{15 + \sqrt{289}}{2} \qquad x^2 = \frac{15 - \sqrt{289}}{2} < 0$

and $x \in \mathbb{R}$
so no sol here.

$\Downarrow$

$$\begin{cases} x_{1/2} = \pm\sqrt{\frac{15 + \sqrt{289}}{2}} \\ y_{1/2} = \frac{4}{\pm\sqrt{\frac{15+\sqrt{289}}{2}}} = \pm \frac{4\sqrt{2}}{\sqrt{15+\sqrt{289}}} \end{cases}$$

The same way if

$(x+iy)^2 = 15 - 8i$

$$x^2 + 2xyi - y^2 = 15 - 8i$$

$$\begin{cases} x^2 - y^2 = 15 \\ 2xy = -8 \Rightarrow \quad y = -\frac{4}{x} \end{cases}$$

$$x^2 - \frac{16}{x^2} = 15 \Leftrightarrow \quad x_{3,4}^2 = \frac{15 + \sqrt{289}}{2}$$

$$x_{3,4} = \pm\sqrt{\frac{15+\sqrt{289}}{2}}$$

$$y_{3,4} = -\frac{4}{x} = \mp \frac{4\sqrt{2}}{\sqrt{15+\sqrt{289}}}$$

So our solutions will be:

$$z_1 = \sqrt{\frac{15+\sqrt{289}}{2}} + \frac{4\sqrt{2}}{\sqrt{15+\sqrt{289}}} \cdot i$$

$$z_2 = -\sqrt{\frac{15+\sqrt{289}}{2}} - \frac{4\sqrt{2}}{\sqrt{15+\sqrt{289}}}\, i$$

$$z_3 = \sqrt{\frac{15+\sqrt{289}}{2}} - \frac{4\sqrt{2}}{\sqrt{15+\sqrt{289}}} \cdot i$$

$$z_4 = -\sqrt{\frac{15+\sqrt{289}}{2}} + \frac{4\sqrt{2}}{\sqrt{15+\sqrt{289}}} \cdot i$$

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