Basic Math | Class 14

15th of April 2020.

Linear independence

① Determine whether the following vector systems in $\mathbb{R}^4$ are linearly indep. or dependent:

a) $v_1 = (1,2,2,-1)$

$v_2 = (4,3,9,-4)$

$v_3 = (5,8,9,-5)$

We have to check if

$\alpha v_1 + \beta v_2 + \gamma v_3 = 0 \Rightarrow$

$\alpha = \beta = \gamma = 0$ only

(lin. independent case)

<u>or</u> $\exists\, \alpha, \beta, \gamma$ not all of them $= 0$ so that

$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$

(lin. dependent case)

<u>So</u>

$$\alpha \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 3 \\ 9 \\ -4 \end{pmatrix} + \gamma \begin{pmatrix} 5 \\ 8 \\ 9 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \alpha + 4\beta + 5\gamma = 0 \\ 2\alpha + 3\beta + 8\gamma = 0 \\ 2\alpha + 9\beta + 9\gamma = 0 \\ -\alpha - 4\beta - 5\gamma = 0 \quad \Big| \cdot 2 \end{cases} \quad + \quad +$$

$$\begin{cases} \alpha + 4\beta + 5\gamma = 0 \\ -5\beta - 2\gamma = 0 \\ \beta - \gamma = 0 \Rightarrow \beta = \gamma \end{cases}$$

$$-5\beta - 2\beta = 0 \Rightarrow -7\beta = 0 \Rightarrow$$

$$\boxed{\beta = 0} \quad \boxed{\gamma = 0} \quad \Rightarrow \boxed{\alpha = 0}$$

$\Rightarrow$ only the trivial linear combination gives 0 vector

so v_1, v_2, v_3 are lin. indep.

b) $v_1 = (1, 2, 3, 1)$

$v_2 = (2, 2, 1, 3)$

$v_3 = (-1, 2, 7, -3)$

Suppose that : $\alpha v_1 + \beta v_2 + \gamma v_3 = 0$

$$\alpha \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 2 \\ 1 \\ 3 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 2 \\ 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \alpha + 2\beta - \gamma = 0 \;|\cdot 2 \;|\cdot(-3) \;|\cdot 7$$

$$\begin{cases} 2\alpha + 2\beta + 2\gamma = 0 \;\oplus \Rightarrow \\ 3\alpha + \beta + 7\gamma = 0 \quad 4\alpha + 6\beta = 0 \\ \oplus \\ \alpha + 3\beta - 3\gamma = 0 \;\oplus \Rightarrow \end{cases}$$

$\Rightarrow 10\alpha + 15\beta = 0 \quad -2\alpha - 3\beta = 0$

$\Rightarrow \alpha = -\frac{3}{2}\beta \qquad \Rightarrow \boxed{\alpha = \frac{-3}{2}\beta}$

$$\Rightarrow \gamma = \alpha + 2\beta = -\frac{3}{2}\beta + 2\beta =$$

$$= \frac{1}{2}\beta$$

<u>So</u> $\alpha = -\frac{3}{2}\beta$

$\beta \in \mathbb{R}$ (free variable)

$\gamma = \frac{1}{2}\beta$

For example for

$$\beta = 2 \Rightarrow \begin{cases} \alpha = -3 \\ \beta = 2 \\ \gamma = 1 \end{cases}$$

$$-3v_1 + 2v_2 + 1 \cdot v_3 = 0$$

Coefficients are not all 0
so v_1, v_2, v_3 are linearly dependent.

2. For 1b: From a linearly dependent system we can omit one vector so that the generated subspace does not change! Which vector can be omitted?
Here v_1, v_2, v_3 are dependent and:
$$-3v_1 + 2v_2 + 1 \cdot v_3 = 0$$

We can omit one vector whose coefficient is not 0 in any dependency equation.

<u>here</u> we can omit v_1, or v_2, or v_3. <u>So</u> that

$$\mathrm{Span}(v_1, v_2, v_3) = \mathrm{Span}(v_2, v_3)$$
$$= \mathrm{Span}(v_1, v_3) = \mathrm{Span}(v_1, v_2)$$

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3 $v_1 = (1, -2, 1) \in \mathbb{R}^3$

$v_2 = (2, 1, 0) \in \mathbb{R}^3$

Expand this lin. indep.

System by a vector $v_3 \in \mathbb{R}^3$ so that:

a) v_1, v_2, v_3 is linearly dependent

b) v_1, v_2, v_3 is linearly independent.

<u>Sol</u>: a) we need

$$v_3 \in \operatorname{Span}(v_1, v_2)$$

For example

$$v_3 := 1 \cdot v_1 + 2 \cdot v_2 =$$

$$= 1\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \underline{\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}} \text{ or}$$

$v_3 = -v_1 + 3v_2 =$

$$= -\begin{pmatrix}1\\-2\\1\end{pmatrix} + 3\cdot\begin{pmatrix}2\\1\\0\end{pmatrix} =$$

$$= \begin{pmatrix}-1+6\\2+3\\-1+0\end{pmatrix} = \begin{pmatrix}5\\5\\-1\end{pmatrix}$$

or $v_3 = v_1$ is ok

$v_3 = v_2$ is ok ...

b) we need $v_3 \notin \mathrm{Span}(v_1, v_2)$

So: $\mathrm{Span}(v_1, v_2) =$

$$= \left\{ \alpha \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \alpha + 2\beta \\ -2\alpha + \beta \\ \alpha \end{pmatrix} \in \mathbb{R}^3 \middle| \alpha, \beta \in \mathbb{R} \right\}.$$

If $\alpha = 1 \Rightarrow$

$$\begin{pmatrix} 1 + 2\beta \\ -2 + \beta \\ 1 \end{pmatrix}$$ <u>and</u> is good.

$$v_3 := \begin{pmatrix} 1 + 2 \cdot 1 \\ -2 + 0 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}}}$$

or

$$v_3 = \begin{pmatrix} 1+2\beta \\ -2+\beta \\ 2 \end{pmatrix}$$

$\beta \in \mathbb{R}$ is also good

$$\beta = 1 \Rightarrow \quad v_3 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$