Basic Math | Class 16

Rank, Systems of linear equations

(1.) Solve the following systems of equations and answer the following questions:

(1.°) Using the EBT method solve the system. Find all its solutions. Give the solutions in vector / scalar form. Free variables, bound variables?

2°. If it is consistent $\Rightarrow$ give the set of solutions $\mathcal{M} = S$.

3° Give the solution set $\mathcal{M}_h = S_h$ of the associated homogenous system.
Determine a basis in $\mathcal{M}_h$.
Find the dim S_h

4°. What is the coefficient matrix of the system? and give its rank.

$$\text{Rank } A = ?$$

CASE OF UNIQUE SOLUTION

a) $$\begin{cases} x_2 - 3x_3 = -5 \\ 4x_1 + 5x_2 - 2x_3 = 10 \\ 2x_1 + 3x_2 - x_3 = 7 \end{cases}$$

Scalar form of this system

Coefficient matrix:

$$A := \begin{bmatrix} 0 & 1 & -3 \\ 4 & 5 & -2 \\ 2 & 3 & -1 \end{bmatrix};$$

Vector b:

$$b := \begin{bmatrix} -5 \\ 10 \\ 7 \end{bmatrix}$$ right hand side.

If $x := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ then the $\underbrace{\text{vector form}}$ of this system is:

$$x_1 \cdot \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + x_3 \cdot \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 10 \\ 7 \end{pmatrix} \quad \underline{\text{and}}$$

the $\underbrace{\text{matrix form}}$ of it

$$\begin{bmatrix} 0 & 1 & -3 \\ 4 & 5 & -2 \\ 2 & 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \\ 7 \end{bmatrix}$$

$\iff A \cdot x = b.$

Remark: We can have solution (the system is CONSISTENT)

iff $\begin{pmatrix} -5 \\ 10 \\ 7 \end{pmatrix} \in \mathrm{Span}\left(\begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}; \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}; \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \right)$

or

$b \in \mathrm{Span}(a_1, a_2, a_3)$

where a_1, a_2, a_3 are the column vectors of A.

Solve the system by the GAUSS-JORDAN elimination method using the Elementary Basis Transformation (E.B.T.) This is the method we will use to solve systems of linear equations.

Let's see it.

generator element

	a_1	a_2	a_3	b	$\vert\cdot(-3)$
e_1	0	$\boxed{1}$	-3	-5	$\vert\cdot(-5)$
e_2	4	5	-2	10	+
e_3	2	3	-1	7	+

a_2	0	1	-3	-5	
e_2	4	0	13	35	+
e_3	$\boxed{2}$	0	8	22	$\vert:2$ $\vert\cdot(-2)$

a_2	0	1	-3	-5	+
e_2	0	0	$\boxed{-3}$	-9	$\vert:(-3)$
a_1	1	0	4	11	$\vert\cdot(-1)$

$\vert\cdot\frac{4}{3}$

+

	a_1	a_2	a_3	b
a_2	0	1	0	4
a_3	0	0	1	3
a_1	1	0	0	-1

We could change all the basis elements e_1, e_2, e_3 by a_2, a_3, a_1 so the algorithm stops and in this case we have a uniqe solution, namely

$x_1 = -1, \; x_2 = 4, \; x_3 = 3$

So we found

$$x = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \in \mathbb{R}^3 \text{ is}$$

the only solution.

Free variables: 0

Bound variables: 3

(x_1, x_2, x_3)

$$\boxed{\operatorname{rank}(A) = 3 = \text{number of bound variables}}$$

Solution set

$$S = \left\{ x = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \right\} = x_p + S_h$$

x_B = basis solution = $\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$

$S_h = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$. So

$Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the

homogenous system

When $b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

we get the only

solution $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$\Rightarrow \dim S_h = 0$;

By theoretic part:

If $r = \text{rank}(A)$,
$A \in \mathbb{R}^{m \times n}$ $\Rightarrow$

bond variables $= r$

$\dim S_u = n - r$

Now $n = 3$; $r = 3$

so $n - r = 0$.

b) CASE OF INFINITELY MANY SOLUTIONS:

$$
\begin{cases}
-3x_1 + x_2 + x_3 - x_4 - 2x_5 = 2 \\
2x_1 - x_2 + x_5 = 0 \\
-x_1 + x_2 + 2x_3 + x_4 - x_5 = 8 \\
x_2 + x_3 + 2x_4 = 6
\end{cases}
$$

Here $m = 4$ (number of equations)

$n = 5$ (number of unknowns) so this is a 4×5 type of system.

Coefficient matrix:

$$A = \begin{bmatrix} -3 & 1 & 1 & -1 & -2 \\ 2 & -1 & 0 & 0 & 1 \\ -1 & 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 5}$$

$$b = \begin{bmatrix} 2 \\ 0 \\ 8 \\ 6 \end{bmatrix} \in \mathbb{R}^4 ; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5$$

So matrix form of our system is

$$A \cdot X = b$$

Solve it by E.B.T.

	a_1	a_2	a_3	a_4	a_5	b	
e_1	-3	1	$\boxed{1}$	-1	-2	2	$\vert\cdot(-2)$ $\vert\cdot(-1)$
e_2	2	-1	0	0	1	0	+
e_3	-1	1	2	1	-1	8	+
e_4	0	1	1	2	0	6	

	a_1	a_2	a_3	a_4	a_5	b	
a_3	-3	1	1	-1	-2	2	+ $\vert\cdot(-2)$
e_2	2	-1	0	0	$\boxed{1}$	0	$\vert\cdot 2$ $\vert\cdot(-3)$
e_3	5	-1	0	3	3	4	+
e_4	3	0	0	3	2	4	+

	a_1	a_2	a_3	a_4	a_5	b	
a_3	1	-1	1	-1	0	2	+
a_5	2	-1	0	0	1	0	+
e_3	$\boxed{-1}$	2	0	3	0	4	$\vert\cdot(-1)$ $\vert\cdot 2$
e_4	-1	2	0	3	0	4	$\vert\cdot(-1)$ +

	a_1	a_2	a_3	a_4	a_5	b
a_3	0	1	1	2	0	6
a_5	0	3	0	6	1	8
a_1	1	-2	0	-3	0	-4
e_4	0	0	0	0	0	0

if this is 0

Here there is no 0 element. So no more basis vectors of a_2, a_4 Can exchange vector e_4 So the algorithm STOPS.

When the $\square$ part is 0 in b and

$\boxed{0 \cdots 0}$ all are 0

Then we have infinitely many solutions, so the system can be solved, it is called

CONSISTENT

system of equations.

How can we find the solutions? What is this last table mean?

$$x_2 + x_3 + 2x_4 = 6$$

$$3x_2 + 6x_4 + x_5 = 8$$

$$x_1 - 2x_2 - 3x_4 = -4$$

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = \boxed{0}$$

We could change

$e_1, e_2, e_3 \Longleftrightarrow$ to

$a_3, a_5, a_1 \Longrightarrow$

variables x_3, x_5, x_1 are bound or dependent variables

variables: x_2, x_4 are free variables (independent ones)

$\Rightarrow$ rank$(A) = r = 3$ (number of bound variables)

$\Rightarrow \dim \operatorname{Ker}(A) = n - r = 5 - 3 = 2$

(number of free variables)

$\operatorname{Ker} A = \{ X \in \mathbb{R}^n : AX = 0 \}$

$= S_h$ = the set of all <u>homogenous solutions</u>

$\dim S_h = \dim \operatorname{Ker} A =$

$= n - r = n - \operatorname{rank} A =$

$= 5 - 3 = \boxed{2}$ <u>Solutions</u>

$$\begin{cases} x_2, x_4 \in \mathbb{R} \text{ are free} \\ x_1 = -4 + 2x_2 + 3x_4 \\ x_3 = 6 - x_2 - 2x_4 \\ x_5 = 8 - 3x_2 - 6x_4 \end{cases}$$

SCALAR FORM OF solution

Solution in vector form:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -4 + 2x_2 + 3x_4 \\ x_2 \\ 6 - x_2 - 2x_4 \\ x_4 \\ 8 - 3x_2 - 6x_4 \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} -4 \\ 0 \\ 6 \\ 0 \\ 8 \end{pmatrix}}_{x_B} + x_2 \underbrace{\begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \\ -3 \end{pmatrix}}_{=: v_2} + x_4 \underbrace{\begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \\ -6 \end{pmatrix}}_{=: v_4}$$

basis solution

The set of all solutions:

$$S = \{ x_B + x_2 \cdot v_2 + x_4 \cdot v_4 \mid x_2, x_4 \in \mathbb{R} \} =$$

$$= x_B + \{ x_2 \cdot v_2 + x_4 \cdot v_4 \mid x_2, x_4 \in \mathbb{R} \} =$$

$$= x_B + \mathrm{Span}(v_2, v_3) =$$

$$= x_B + S_h,$$

where

$S_h = \{x \in \mathbb{R}^5 : Ax = 0\} =$

$= \operatorname{Ker} A =$ by theorem $=$

$= \operatorname{Span}(v_2, v_4)$.

Our theorem is also stating that vectors v_2, v_4 are linearly independent here $\Longrightarrow$

$\dim S_h = \dim \operatorname{Ker} A =$

$= \dim \operatorname{Span}(v_2, v_4) = 2 =$

$= n - r = n - \operatorname{rank} A =$

$= 5 - 3 = 2$.

So a basis in S_h is

v_2, v_4 ;

Remark If someone is choosing different generator elements, may have a solution of other form as well (but the same structure: 2 free variables 3 bound variables, rank $A = 3$; $\dim S_h = 2$ ...)

c) CASE OF INCONSISTENT SYSTEM. SOLVE:

$$\begin{cases} 2x_1 + 3x_2 - x_3 + 2x_4 = -1 \\ x_1 + 4x_2 - 4x_3 + 3x_4 = 2 \\ 4x_1 + x_2 + 5x_3 = 1 \end{cases}$$

$$A = \begin{bmatrix} 2 & 3 & -1 & 2 \\ 1 & 4 & -4 & 3 \\ 4 & 1 & 5 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 4},$$

$$b = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}; \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = ?$$

$$S = \{ x \in \mathbb{R}^4 \mid Ax = b \} = ?$$

$$S_h = \{ x \in \mathbb{R}^4 \mid Ax = 0 \} = ?$$

	a_1	a_2	a_3	a_4	b	
e_1	2	3	-1	2	-1	$+$
e_2	1	4	-4	3	2	$+$
e_3	4	$\boxed{1}$	5	0	1	$\mid \cdot(-4) \mid \cdot(-3)$
e_1	-10	0	-16	$\boxed{2}$	-4	$\mid :2 \mid \cdot\left(\frac{-3}{2}\right)$
e_2	-15	0	-24	3	-2	$+$
a_2	4	1	5	0	1	
a_4	-5	0	-8	1	-2	
e_2	0	0	0	0	$\boxed{4}$	
a_2	4	1	5	0	1	

There could be only one option to continue namely to change e_2 by a_3 or a_1 but their coefficient in e_2-s row is 0 so the algorithm **STOPS**

$$\left(\boxed{0 \quad 0 \quad 0 \quad 0}\right)$$

We check now what is in column of b at line of e_2 : $\boxed{4} \neq 0$

$\Rightarrow$ There is no solution so the system is called to be INCONSISTENT.

(If here $\boxed{}$ would be 0 we would have infinitely many solutions see point b))

So $\boxed{S = \emptyset}$; BUT

if $b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ we get the homogenous solutions, since through the whole algorithm b stops the 0 vector then.

Let's find the solutions

Final table:

	a_1	a_2	a_3	a_4	b
a_4	-5	0	-8	1	0
e_2	0	0	0	0	0
a_2	4	1	5	0	0

Dependent variables x_2, x_4 $\Longrightarrow$

$\boxed{\operatorname{rank} A = 2}$ $\Longrightarrow$

$\dim S_h = n - r = 4 - 2 = \underline{\underline{2}}$

Solutions:

$$\begin{cases} -5x_1 \quad -8x_3 + x_4 = 0 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = 0 \\ 4x_1 + x_2 + 5x_3 \qquad = 0 \end{cases}$$

Free variables: x_1, x_3

Bound —"—: x_2, x_4

$$\begin{cases} x_2 = -4x_1 - 5x_3 \\ x_4 = 5x_1 + 8x_3 \\ x_1, x_3 \in \mathbb{R} \end{cases}$$

Vector form:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ -4x_1 - 5x_3 \\ x_3 \\ 5x_1 + 8x_3 \end{pmatrix} =$$

$$= \begin{pmatrix} x_1 \\ -4x_1 \\ 0 \\ 5x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -5x_3 \\ x_3 \\ 8x_3 \end{pmatrix} =$$

$$= x_1 \cdot \underbrace{\begin{pmatrix} 1 \\ -4 \\ 0 \\ 5 \end{pmatrix}}_{v_1} + x_3 \cdot \underbrace{\begin{pmatrix} 0 \\ -5 \\ 1 \\ 8 \end{pmatrix}}_{v_3}$$

So the homogenous solutions:

$$S_h = \left\{ x_1 \cdot v_1 + x_3 \cdot v_3 \;\middle|\; x_1, x_3 \in \mathbb{R} \right\} =$$

$$= \mathrm{Span}(v_1, v_3)$$

By the theorem of linear systems:

v_1, v_3 is lin. indep. system $\Longrightarrow$

$\dim S_h = \dim \mathrm{Ker} A = 2$

and a basis in S_u or Ker A are vectors:

$$v_1, v_3 = \begin{pmatrix} 1 \\ -4 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ 1 \\ 8 \end{pmatrix},$$

THE END