Basic Math. Class 15.

Basis Dimension

1. Consider the following vectors in $\mathbb{R}^4$:

$v_1 = (3, 0, -2, 4)$

$v_2 = (2, 1, -1, 3)$

$v_3 = (-1, 4, 2, 0)$

$v_4 = (-1, 1, 1, -1)$

Let $W := \mathrm{Span}(v_1, v_2, v_3, v_4)$

Select a basis in W.

What is $\dim W = ?$

Sol. a basis in W is a linearly independent generator system of W. So v_1, v_2, v_3, v_4 is a generator system in W, we have to check if it's independent or not. So suppose that:

$$\alpha \cdot v_1 + \beta v_2 + \gamma \cdot v_3 + \delta v_4 = 0$$

$$\Longleftrightarrow$$

$$\alpha \cdot \begin{pmatrix} 3 \\ 0 \\ -2 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 4 \\ 2 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 3\alpha + 2\beta - \gamma - \delta = 0 & \underline{\mathrm{I}} \\ \beta + 4\gamma + \delta = 0 & \underline{\mathrm{II}} \\ -2\alpha - \beta + 2\gamma + \delta = 0 & \underline{\mathrm{III}} \\ 4\alpha + 3\beta - \delta = 0 & \underline{\mathrm{IV}} \end{cases}$$

$\underline{\mathrm{IV}} \Rightarrow \delta = 4\alpha + 3\beta$ we substitute this into equations $\overline{\mathrm{I}}, \overline{\mathrm{II}}, \overline{\mathrm{III}}$

$$\Rightarrow \begin{cases} 3\alpha + 2\beta - \gamma - (4\alpha + 3\beta) = 0 & \underline{\mathrm{I}} \\ \beta + 4\gamma + (4\alpha + 3\beta) = 0 & \underline{\mathrm{II}} \\ -2\alpha - \beta + 2\gamma + (4\alpha + 3\beta) = 0 & \underline{\mathrm{III}} \end{cases}$$

$$\Rightarrow \begin{cases} -\alpha - \beta - \gamma = 0 \Rightarrow \gamma = -\alpha - \beta \\ 4\alpha + 4\beta + 4\gamma = 0 \mid :4 \\ 2\alpha + 2\beta + 2\gamma = 0 \mid :2 \end{cases}$$

$\alpha + \beta + (-\alpha - \beta) = 0$ $\underline{II}$.

$\underline{III}$ the same $\Rightarrow$

$0\alpha + 0\beta = 0$.

So $\alpha, \beta \in \mathbb{R}$ are free variables

$\gamma = -\alpha - \beta$

$\delta = 4\alpha + 3\beta$.

So for example:

$$\left.\begin{aligned} \alpha &= 1 \\ \beta &= 0 \\ \gamma &= -1 \\ \delta &= 4 \end{aligned}\right\} \Rightarrow \begin{aligned} & 1 \cdot v_1 + 0 \cdot v_2 + \\ & (-1) \cdot v_3 + 4 \cdot v_4 = 0 \\ & \Rightarrow v_1, v_2, v_3, v_3 \end{aligned}$$

are lin. dependent $\Longrightarrow$
(reducing dependent vector system theorem)
we can omit one vector (with no 0 coefficient in a dependency equation)
so here we can omit or leave v_1, v_3 or v_4 so that

$$W = \mathrm{Span}(v_1, v_2, v_3, v_4) =$$
$$= \mathrm{Span}(v_1, v_2, v_3) =$$
$$= \mathrm{Span}(v_2, v_3, v_4) = \mathrm{Span}(v_1, v_2, v_4)$$

we also can see that
v_1, v_2, v_3, v_4 is <u>not a basis</u>.

- let's omit v_4 $\Rightarrow$
$W = \text{Span}(v_1, v_2, v_3)$.

Now v_1, v_2, v_3 is a gen. system in W, we have to check if they are independent or dependent. Let $\alpha, \beta, \gamma \in \mathbb{R}$ so that $\alpha v_1 + \beta v_2 + \gamma v_3 = 0$

$$\Leftrightarrow \alpha \cdot \begin{pmatrix} 3 \\ 0 \\ -2 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Find α, β, γ. $\Rightarrow$

$$\begin{cases} 3\alpha + 2\beta - \gamma = 0 & \text{I.} \Rightarrow \gamma = 3\alpha + 2\beta \\ \beta + 4\gamma = 0 & \text{II.} \\ -2\alpha - \beta + 2\gamma = 0 & \text{III.} \\ 4\alpha + 3\beta = 0 & \text{IV} \end{cases}$$

$$\begin{cases} \beta + 4(3\alpha + 2\beta) = 0 & \text{II} \\ -2\alpha - \beta + 2(3\alpha + 2\beta) = 0 & \text{III.} \\ 4\alpha + 3\beta = 0 & \text{IV.} \end{cases}$$

$$\left.\begin{aligned} 12\alpha + 9\beta = 0 \\ 4\alpha + 3\beta = 0 \\ 4\alpha + 3\beta = 0 \end{aligned}\right\} \quad 4\alpha + 3\beta = 0 \quad \Rightarrow$$

$$\alpha = -\frac{3}{4}\beta$$

$\beta \in \mathbb{R}$ free variable

$$\gamma = 3\alpha + 2\beta = 3\left(-\frac{3}{4}\beta\right) + 2\beta =$$

$$= -\frac{9}{4}\beta + 2\beta = -\frac{1}{4}\beta$$

<u>Solutions</u> $\alpha = -\frac{3}{4}\beta$, $\beta \in \mathbb{R}$, $\underset{\gamma}{\underset{\shortparallel}{-\frac{1}{4}\beta}}$

<u>For exp.</u> if $\beta = 4 \Rightarrow$

$$\begin{cases} \alpha = -3 \\ \beta = 4 \\ \gamma = -1 \end{cases} \Rightarrow -3v_1 + 4v_2 - v_3 = 0$$

v_1, v_2, v_3 are dependent!

we can omit any of them $\Rightarrow$

$W = \text{Span}(v_1, v_2, v_3) =$

$= \text{Span}(v_1, v_2) = \text{Span}(v_2, v_3) =$

$= \text{Span}(v_1, v_3)$

Let's omit / leave out vector v_2 for example.

So $W = \text{Span}(v_1, v_3)$.

Check if v_1, v_3 is dep/indep.

Assume $\alpha, \beta \in \mathbb{R}$ and

$\alpha v_1 + \beta v_3 = 0 \iff$

$$\alpha \cdot \begin{pmatrix} 3 \\ 0 \\ -2 \\ 4 \end{pmatrix} + \beta \cdot \begin{pmatrix} -1 \\ 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \left.\begin{aligned} 3\alpha - \beta &= 0 \\ 4\beta &= 0 \\ -2\alpha + 2\beta &= 0 \\ 4\alpha &= 0 \end{aligned}\right\} \Rightarrow$$

$\boxed{\alpha = 0}$; $\boxed{\beta = 0}$ is the only solution so v_1, v_3 is linearly independent and is a gen. system in W

$\Rightarrow v_1, v_2$ is a basis in W

$\Rightarrow \boxed{\dim W = 2}$;

So W is a 2-dimensional subspace in $\mathbb{R}^4$.

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② Determine whether the following vector systems form a basis in $\mathbb{R}^4$:

a) x_1, x_2

b) x_1, x_2, x_3, x_4, x_5

c) x_1, x_2, x_3, x_4 where

$x_1 = (2, 3, -2, 7)$, $x_2 = (0, 1, 0, 1)$

$x_3 = (1, 2, -1, 0)$, $x_4 = (-1, -5, 2, 0)$

$x_5 = (3, -1, 1, 2)$

Sol: we know that vectors

$$\left.\begin{aligned} e_1 &= (1,0,0,0) \\ e_2 &= (0,1,0,0) \\ e_3 &= (0,0,1,0) \\ e_4 &= (0,0,0,1) \end{aligned}\right\}$$ are a lin. independent generator system in $\mathbb{R}^4$

So $\dim \mathbb{R}^4 = 4$, and every basis in $\mathbb{R}^4$ must have 4 element. then.

So a) x_1, x_2 cannot be basis in $\mathbb{R}^4$ (2-elements are not enough)

b) x_1, x_2, x_3, x_4, x_5 they are 5 different vectors $5 > 4 \Rightarrow$ cannot be basis in $\mathbb{R}^4$.

c) x_1, x_2, x_3, x_4 there is 4 of them so it can be basis.

Are they independent?

$$\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 = 0 \iff$$

$$\alpha \begin{pmatrix} 2 \\ 3 \\ -2 \\ 7 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} -1 \\ -5 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{cases} 2\alpha + \gamma - \delta = 0 & \text{I} \\ 3\alpha + \beta + 2\gamma - 5\delta = 0 & \text{II} \\ -2\alpha - \gamma + 2\delta = 0 & \text{III} \\ 7\alpha + \beta = 0 & \text{IV} \end{cases}$$

$\Rightarrow \underline{\mathrm{I}} + \underline{\mathrm{III}} \Rightarrow \boxed{\delta = 0}$

<u>So</u>

$$\begin{cases} 2\alpha + \gamma = 0 \\ 3\alpha + \beta + 2\gamma = 0 \\ -2\alpha - \gamma = 0 \\ 7\alpha + \beta = 0 \end{cases} \qquad \gamma = -2\alpha$$

$$\begin{cases} 3\alpha + \beta - 4\alpha = 0 \\ 7\alpha + \beta = 0 \end{cases}$$

$$\beta - \alpha = 0 \Rightarrow \beta = \alpha$$

$$7\alpha + \alpha = 0 \quad \boxed{\alpha = 0} \Rightarrow \boxed{\beta = 0}$$

$\boxed{\gamma = 0}$ which satisfies $\underline{\mathrm{I}}, \underline{\mathrm{II}}, \underline{\mathrm{III}}, \underline{\mathrm{IV}}$.

$\Rightarrow \alpha=\beta=\gamma=\delta=0$ so

x_1, x_2, x_3, x_4 are independent

$\Rightarrow$ Theorem (4 small statements)

x_1, x_2, x_3, x_4 is basis in $\mathbb{R}^4$.

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3. Select a basis from the following vector systems in $\mathbb{R}^4$ in the subspace

$W = \mathrm{Span}(v_1, v_2, v_3)$.

Find $\dim W$ as well, where

a) $v_1 = (1, 2, 2, -1)$ b) $v_1 = (1, 2, 3, 1)$

$v_2 = (4, 3, 9, -4)$ $v_2 = (2, 2, 1, 3)$

$v_3 = (5, 8, 9, -5)$ $v_3 = (-1, 2, 7, -3)$

5a) : a)

$$\alpha\begin{pmatrix}1\\2\\2\\-1\end{pmatrix}+\beta\begin{pmatrix}4\\3\\9\\-4\end{pmatrix}+\gamma\begin{pmatrix}5\\8\\9\\-5\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\left.\begin{aligned}\alpha+4\beta+5\gamma&=0\\2\alpha+3\beta+8\gamma&=0\\2\alpha+9\beta+9\gamma&=0\\-\alpha-4\beta-5\gamma&=0\end{aligned}\right\}\quad\begin{aligned}&\overline{\underline{I}}.\\&\overline{\underline{II}}.\\&\overline{\underline{III}}.\\&\overline{\underline{IV}}.\end{aligned}$$

$\overline{\underline{IV}}+\overline{\underline{I}} \Rightarrow 0=0$ Is enough only:

$$\left\{\begin{aligned}2\alpha+3\beta+8\gamma&=0 &&\overline{\underline{II}}\\2\alpha+9\beta+9\gamma&=0 &&\overline{\underline{III}}\\-\alpha-4\beta-5\gamma&=0 &&\overline{\underline{IV}}\end{aligned}\right.$$

$$\left.\begin{aligned}2\cdot\overline{\underline{IV}}+\overline{\underline{II}}&\Rightarrow -5\beta-2\gamma=0\\2\cdot\overline{\underline{IV}}+\overline{\underline{III}}&\Longrightarrow \beta+\gamma=0\end{aligned}\right\}\begin{aligned}&+\\&\\&1\cdot5\end{aligned}$$

$\Rightarrow 3\gamma = 0 \Rightarrow \boxed{\gamma = 0} \Rightarrow \beta = -\gamma = 0$

$\boxed{\beta = 0}$

$\Rightarrow \alpha = -4\beta - 5\gamma = 0 \quad \boxed{\alpha = 0}$

$\Rightarrow \alpha = 0, \beta = 0, \gamma = 0$ is the only solution $\Rightarrow v_1, v_2, v_3$ is linearly independent vector system, they also generate $W \Rightarrow v_1, v_2, v_3$ is basis in W.

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b) $v_1 = (1, 2, 3, 1)$

$v_2 = (2, 2, 1, 3)$

$v_3 = (-1, 2, 7, -3)$

and now $W := \mathrm{Span}(v_1, v_2, v_3)$
We saw in class 14/1/6
that v_1, v_2, v_3 is linearly
dependent, we got for
example that:

$$-3v_1 + 2v_2 + 1 \cdot v_3 = 0.$$

So the coefficients here
are not 0 we can omit
any of v_1, v_2, v_3 so that
their span does not change.
We can omit for example
$v_3 \Rightarrow W = \mathrm{Span}(v_1, v_2)$.

We need a basis, check if v_1, v_2 is indep/dep.

$$\alpha v_1 + \beta v_2 = 0 \iff$$

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \iff$$

$$\begin{cases} \alpha + 2\beta = 0 \quad | \cdot (-1) \\ 2\alpha + 2\beta = 0 \\ 3\alpha + \beta = 0 \\ \alpha + 3\beta = 0 \end{cases} \quad + \Rightarrow \boxed{\beta = 0}$$

$$\Downarrow$$

$$\boxed{\alpha = -2\beta = 0}$$

And all other equations are fullfilled so $\alpha = \beta = 0$ is the only solution here.

$\Rightarrow$ v_1, v_2 are lin. independent vectors, they are also generator system in $W = \mathrm{Span}(v_1, v_2)$

<u>So</u> v_1, v_2 is a basis in W.

$\Rightarrow$ $\boxed{\dim W = 2}$

$\boxed{4/a}$ Are the following vector systems basis in $\mathbb{R}^3$?

$\boxed{a)\ (1,0,0),\ (2,2,0),\ (3,3,3)}$

$\dim \mathbb{R}^3 = 3$ so they can be basis.

$$\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}+\beta\begin{pmatrix}2\\2\\0\end{pmatrix}+\gamma\begin{pmatrix}3\\3\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases}\alpha+2\beta+3\gamma=0 & \boxed{\alpha=0}\\ 2\beta+3\gamma=0 & \Rightarrow \boxed{\beta=0}\\ 3\gamma=0 & \Rightarrow \boxed{\gamma=0}\end{cases}$$

$\Rightarrow$ $(1|0|0)$, $(2|2|0)$, $(3,3,3)$ is independent, $\dim \mathbb{R}^3=3$

$\Rightarrow$ is also basis in $\mathbb{R}^3$.

Theorem

c) $(2|-3,1)$, $(4|1,1)$, $(0|-7,1)$, $(1|6,4)$, we have $4>3$ vectors $\Rightarrow$ cannot be basis

$\boxed{d)\ (2,4,-1),\ (-1,2,5)}$

only 2 vectors but $< 3 = \dim \mathbb{R}^3$ so they are not a basis in $\mathbb{R}^3$.

$\boxed{b)\ (3,1,-4),\ (2,5,6),\ (1,4,8)}$

$$\alpha \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3\alpha + 2\beta + \gamma = 0 & \oplus \\ \alpha + 5\beta + 4\gamma = 0 & \mid \cdot(-3) \quad \mid \cdot 4 \\ -4\alpha + 6\beta + 8\gamma = 0 & \oplus \end{cases}$$

$$\Rightarrow \begin{cases} -13\beta - 11\gamma = 0 & \\ 26\beta + 24\gamma = 0 & \mid : 2 \end{cases} \qquad 13\beta + 12\gamma = 0 \quad \oplus \Rightarrow$$

$(4) \Rightarrow \boxed{\gamma = 0} \Rightarrow \boxed{\beta = 0} \Rightarrow$

$$\alpha = -5\beta - 4\gamma = 0$$

$\boxed{\alpha = 0} \Rightarrow$ the given 3 vectors are independent, $\dim \mathbb{R}^3 = 3$ $\Bigg\} \Rightarrow$

they are a basis in $\mathbb{R}^3$.

THE END