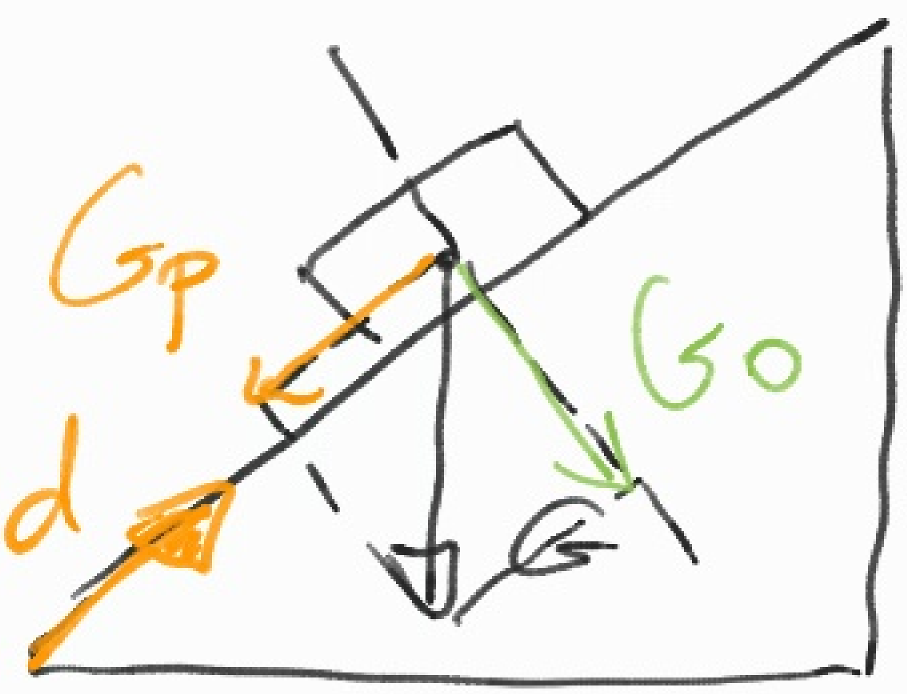
# Basic Math | Class 11 |

## Real Euclidian spaces

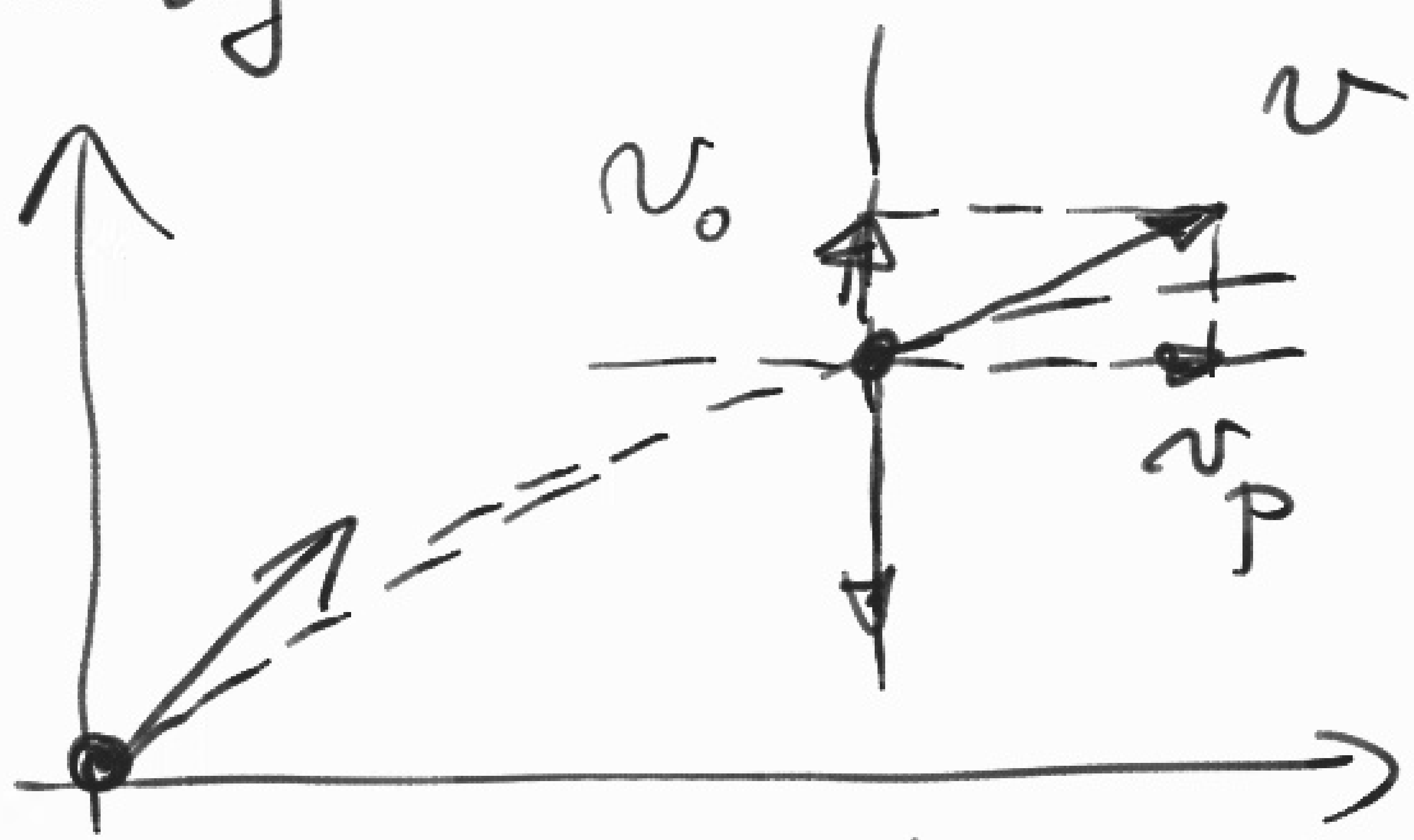
### 1) Motivation

a) Remember in phisics we had:




We had to decompose vector $G$ to parallel and orthogonal components regarding to a directional vector $d$. $(G_p, G_0)$

The same when an object of mass $m$ is throughn away




to decompose the velocity vector to $v_p$, $v_0$ components.

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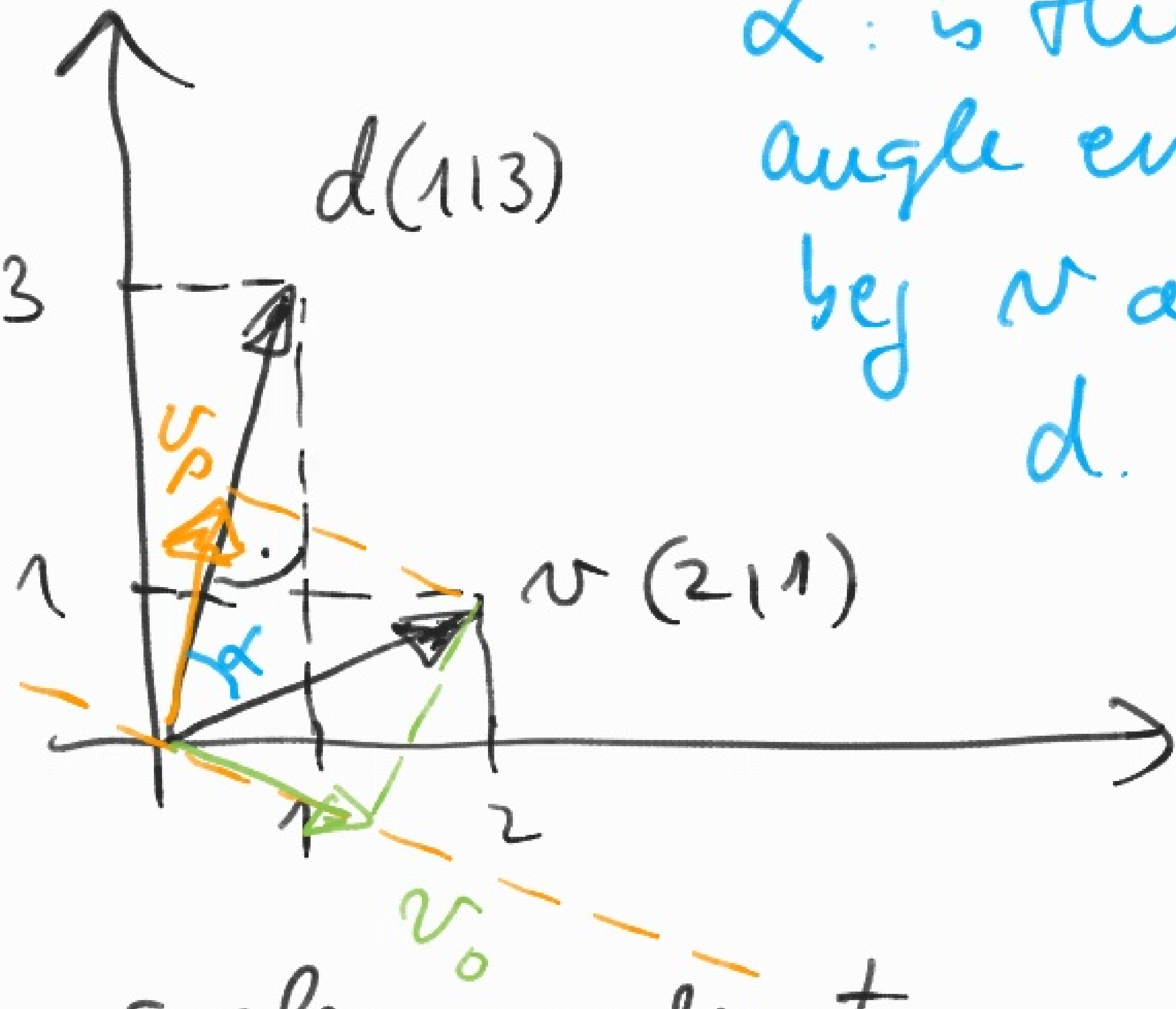
b) Consider vector $v(2,1)$ and $d(1,3)$. Decompose $v$

to $v_p, v_o$ relotud to $d$.




$\alpha$: is the angle enclosed by $v$ and $d$.

The scalar product

$$v \cdot d = |v| \cdot |d| \cdot \cos\alpha$$

<u>So</u> $\cos\alpha = \dfrac{|v_p|}{|v|}$

$$v_p = |v_p| \cdot \frac{d}{|d|} = |v| \cdot \cos\alpha \cdot \frac{d}{|d|}$$

$$|d| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

On the other side

$$v \cdot d = |v| \cdot |d| \cdot \cos\alpha$$

$$\cos\alpha = \frac{v \cdot d}{|v||d|}$$

$$v_p = \frac{\cancel{|v|}}{|d|} \cdot \frac{v \cdot d}{\cancel{|v|}|d|} \cdot d =$$

$$\boxed{= \frac{\langle v, d \rangle}{\langle d, d \rangle} d} =$$

$$= \frac{\langle (2,1);(1,3) \rangle}{\langle (1,3),(1,3) \rangle}(1,3) =$$

$$= \frac{2\cdot 1 + 1\cdot 3}{1\cdot 1 + 3\cdot 3}(1,3) = \frac{5}{10}(1,3) =$$

$$= \underline{\underline{\left(\frac{1}{2},\frac{3}{2}\right)}} = v_p.$$

<u>And</u> $v_p + v_0 = v \implies$

$$v_0 = v - v_p = (2,1) - \left(\frac{1}{2},\frac{3}{2}\right)$$

$$= \underline{\underline{\left(\frac{3}{2},-\frac{1}{2}\right)}}.$$

Generalization.

$x, y \in \mathbb{R}^n$; $\langle x, y \rangle := \sum_{k=1}^{n} x_k \cdot y_k$

is the "Scalar product" of $x, y$.

$x \perp y \iff \langle x, y \rangle = 0$.

See Euclidian spaces chapter in the lecture part.

(2) $x = (1, -2, -3, 5)$

$y = (-1, 2, -1, 0)$

$z = (2, -1, 1, 3)$

Compute:

a) $\langle x|y\rangle = \left\langle \begin{pmatrix} 1 \\ -2 \\ -3 \\ 5 \end{pmatrix} ; \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \right\rangle =$

$= 1\cdot(-1) + (-2)\cdot 2 + (-3)\cdot(-1) +$

$+ 5\cdot 0 = -1 - 4 + 3 = \underline{\underline{-2}}$

b) $\dfrac{\langle x,z\rangle\cdot y - \langle y,z\rangle x}{\langle y,y\rangle} = (*)$

$\langle x,z\rangle = \left\langle \begin{pmatrix} 1 \\ -2 \\ -3 \\ 5 \end{pmatrix} ; \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix} \right\rangle =$

$= 2 + 2 - 3 + 15 = \underline{\underline{16}}$

$\langle y,z\rangle = \left\langle \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix} ; \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix} \right\rangle =$

$= -2 - 2 - 1 = \underline{\underline{-5}}$

$$\langle y, y\rangle = \left\langle \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix} ; \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \right\rangle =$$

$$= 1 + 4 + 1 = \underline{\underline{6}}$$

$$\underline{\text{So}} \cdot (*) = \frac{16y + 5z}{6} =$$

$$= \frac{1}{6} \left( \begin{pmatrix} -16 \\ 32 \\ -16 \\ 0 \end{pmatrix} + \begin{pmatrix} 10 \\ -5 \\ 5 \\ 15 \end{pmatrix} \right) =$$

$$= \frac{1}{6} \begin{pmatrix} -6 \\ 27 \\ -11 \\ 15 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -1 \\ 9/2 \\ -11/6 \\ 5/2 \end{pmatrix}}}.$$

③
$$\left.\begin{array}{l} u_1=(1,1,1,1) \\ u_2=(1,-1,-1,1) \\ u_3=(0,0,0,0) \\ u_4=(-1,0,0,1) \end{array}\right\} \Rightarrow$$

$u_1, u_2, u_3, u_4$ is an OS (orthogonal system) which means here, that $\langle u_i, u_j \rangle = 0$ ($\forall\, i,j \in \{1,2,3,4\}$ $i \neq j$)

$\langle u_i, u_3 \rangle = 0$ ✓ $i = 1,2,4$ ✓

$\langle u_1, u_2 \rangle = 1 - 1 - 1 + 1 = 0$ ✓

$\langle u_1, u_4 \rangle = -1 + 1 = 0$ ✓

$\langle u_2, u_4 \rangle = -1 + 1 = 0$ ✓

There vectors are dependent (0 vector is included) $\Rightarrow$ if we leave it out $\Rightarrow$ $u_1, u_2, u_4$ are O.S. no 0 vector in $\Rightarrow$ they are also independent $\Rightarrow$ $\text{rank}(u_1, u_2, u_3, u_4) = 3$

4. Decompose vector $x = (2, 1, 3, 1) \in \mathbb{R}^4$ for parallel and orthogonal components by the subspace $W = \text{Span}((1, -1, -1, 1), (1, 1, 1, 1), (-1, 0, 0, 1))$

we can see that if
$W = \text{Span}(u_1, u_2, u_3)$
from previous exercise.
$\Rightarrow$ $u_1, u_2, u_3$ are O.S.

The decomposition formula says that

$P(X)$ = orthogonal projection of $x$ on $W$ =

$$= \frac{\langle x, u_1 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 + \frac{\langle x, u_2 \rangle}{\langle u_2, u_2 \rangle} \cdot u_2 + \frac{\langle x, u_3 \rangle}{\langle u_3, u_3 \rangle} \cdot u_3$$

$$x = (2,1,3,1)$$

$$u_1 = (1,-1,-1,1)$$

$$u_2 = (1,1,1,1)$$

$$u_3 = (-1,0,0,1)$$

$$\Rightarrow \langle x, u_1 \rangle = 2 - 1 - 3 + 1 = -1$$

$$\langle x, u_2 \rangle = 2 + 1 + 3 + 1 = 7$$

$$\langle x, u_3 \rangle = -2 + 0 + 0 + 1 = -1$$

$$\langle u_1, u_1 \rangle = 4$$

$$\langle u_2, u_2 \rangle = 4$$

$$\langle u_3, u_3 \rangle = 2$$

$$\Rightarrow x_p = P(x) = -\frac{1}{4} u_1 + \frac{7}{4} u_2 - \frac{1}{2} u_3$$

$$= -\frac{1}{4}\cdot\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} + \frac{7}{4}\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1\\0\\0\\1\end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{1}{4} + \frac{7}{4} + \frac{1}{2} \\ \frac{1}{4} + \frac{7}{4} \\ \frac{1}{4} + \frac{7}{4} \\ -\frac{1}{4} + \frac{7}{4} - \frac{1}{2} \end{pmatrix} = \underline{\underline{\begin{pmatrix}2\\2\\2\\1\end{pmatrix}}}$$

$$\Rightarrow \quad x_0 = x_\perp = x - P(x) =$$

$$= \begin{pmatrix}2\\1\\3\\1\end{pmatrix} - \begin{pmatrix}2\\2\\2\\1\end{pmatrix} = \underline{\underline{\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}}}$$

5 Use the Gram-Schmidt Orthogonalization process to transform the linearly independent vectors

$b_1 = (1,1,1,1)$

$b_2 = (3,3,-1,-1)$

$b_3 = (-2,0,6,8)$

into an equivalent O.S.

What is the rank of $b_1, b_2, b_3$?

Reminder $\boxed{\text{G-S process}}$

Let $b_1, b_2, \dots, b_n \in V$ to be a finite linearly independent system. The following process transforms it into an OS $u_1, \dots, u_n \in V \setminus \{0\}$ so that $\forall k \in \{1, \dots, n\}$

$$\mathrm{Span}(b_1, \dots, b_k) = \mathrm{Span}(u_1, \dots, u_k)$$

(the two systems are equivalent)

$$u_1 := b_1$$

$$u_2 = b_2 - \frac{\langle b_2 | u_1 \rangle}{\langle u_1 | u_1 \rangle} \cdot u_1$$

$$u_3 = b_3 - \frac{\langle b_3, u_1 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 - \frac{\langle b_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

$\vdots$

$$u_n = b_n - \frac{\langle b_n, u_1 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 - \cdots - \frac{\langle b_n, u_{n-1} \rangle}{\langle u_{n-1}, u_{n-1} \rangle} \cdot u_{n-1}$$

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Now: G-Sch. O. P.

$u_1 := b_1 = (1, 1, 1, 1)$

$$u_2 := b_2 - \frac{\langle b_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 =$$

$$= \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix} \parallel$$

$$\underline{\underline{\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}}} = u_2$$

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$$u_3 = b_3 - \frac{\langle b_3, u_1 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 - \frac{\langle b_3, u_2 \rangle}{\langle u_2, u_2 \rangle} \cdot u_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-16}{4} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \\ -3 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \\ -4 \\ -4 \end{pmatrix} =$$

$$= \underline{\underline{\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = u_3}}$$

<u>So</u> $u_1, u_2, u_3 =$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}; \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \text{ O.S.}$$

$\Longrightarrow$ it's also lin. indep.

$\Rightarrow$ $u_1, u_2, u_3$ is indep

$\operatorname{rank}(u_1, u_2, u_3) = 3 \Rightarrow$

$\operatorname{rank}(b_1, b_2, b_3) = 3$ as well.

THE END