Basic Mathematics
Practice 11
Determinants / Inverse mtx

1°. Lecture part: See

Linear-Algebra-Gorgolstm-2016

pages: 12–19

2° Practice part

(1) Evaluate the following determinants in many ways:

(a) $\det A := \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 =$

$= 4 - 6 = \underline{-2}$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2} \Rightarrow$

$$\boxed{\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

b) $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

Expanding on line 1:

$$\det A = +3 \cdot \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} + (-4) \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3(40-24) - (16-6)$$

$$-4 \cdot (8-5) = 3 \cdot 16 - 10 - 12$$

$$= 48 - 22 = \boxed{26}$$

Expand it on column 2.

$$\det A = -1 \cdot \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} + 5 \begin{vmatrix} 3 & -4 \\ 1 & 8 \end{vmatrix} - 4 \cdot \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}$$

$$= -(16-6) + 5(\underbrace{24+4}_{28}) - 4 \cdot (\underbrace{18+8}_{26}) =$$

$$= -10 + 140 - 104 =$$

$$= -10 + 36 = \boxed{26}$$

Expand it by row 3.

$$\det A = +1 \cdot \begin{vmatrix} 1 & -4 \\ 5 & 6 \end{vmatrix} - 4 \cdot \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}$$

$$+ 8 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} = 6 + 20 - 4(18+8)$$

$$+ 8(\overbrace{15-2}^{13}) = 26 - 104 + 104$$

$$= \underline{\underline{26}}$$

So generally if we have:

$$A = \begin{bmatrix} \triangle & a_{1j} & \circ \\ & a_{2j} & \\ & \vdots & \\ a_{i1}\ a_{i2} \cdots & \boxed{a_{ij}} & \cdots a_{in} \\ & \vdots & \\ \square & a_{nj} & \ast \end{bmatrix} \Rightarrow$$

$$(-1)^{i+j} \cdot a_{ij} \cdot \det \begin{bmatrix} \triangle & \circ \\ \square & \ast \end{bmatrix}$$

$$\sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det A_{ij}$$

The signs can be remembered by this chess-board rule:

$$\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

for a 5×5 determinant

Other way to evaluate $\det A$ using the properties of determinants

$$\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ \boxed{1} & 4 & 8 \end{vmatrix} \begin{matrix} + \\ + \\ \cdot(-2) \\ (-3) \end{matrix} = \begin{vmatrix} 0 & -11 & -28 \\ 0 & -3 & -10 \\ 1 & 4 & 8 \end{vmatrix} =$$

now expand by column 1.

$$= +1 \begin{vmatrix} -11 & -28 \\ -3 & -10 \end{vmatrix} - \underbrace{0 \cdot \begin{vmatrix} -7 & -20 \\ 4 & 8 \end{vmatrix}}_{0} + \underbrace{0 \cdot \begin{vmatrix} -3 & -10 \\ 4 & 8 \end{vmatrix}}_{0} = 10 - 84 = \underline{\underline{26}}.$$

ⓒ

$$\begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & 5 \\ 0 & 3 & 8 \end{vmatrix} =$$

$$= -1 \cdot \begin{vmatrix} 1 & 5 \\ 3 & 8 \end{vmatrix} = -(8-15) = \underline{\underline{+7}}.$$

exp. by column 1

(d)

$$\begin{vmatrix} 1 & 2 & 1 \\ 5 & 7 & 3 \\ 2 & 4 & 2 \end{vmatrix} \cdot(-2) = \begin{vmatrix} 1 & 2 & 1 \\ 5 & 7 & 3 \\ \boxed{0 \quad 0 \quad 0} \end{vmatrix}$$

$+$

$= 0$

by line 3

(e)

$$\begin{vmatrix} \sin^2 x & \cos^2 x & \sin 2x \\ \cos^2 x & \sin^2 x & \sin 2x \\ 1+\sin 2x & -1 & 1 \end{vmatrix} =$$

$+$

$$= \begin{vmatrix} 1 & \cos^2 x & \sin 2x \\ 1 & \sin^2 x & \sin 2x \\ \sin 2x & -1 & 1 \end{vmatrix} =$$

$\cdot(-\sin 2x)$

$+$

$$= \left|\begin{array}{cc|c} 1 & \cos^2 x & 0 \\ 1 & \sin^2 x & 0 \\ \hline \sin 2x & -1 & \underbrace{1-\sin^2 2x}_{\cos^2 2x} \end{array}\right| =$$

= expand by column 3 =

$$= +\cos^2 2x \cdot \begin{vmatrix} 1 & \cos^2 x \\ 1 & \sin^2 x \end{vmatrix} =$$

$$= \cos^2 2x \cdot (\sin^2 x - \cos^2 x) =$$

$$= -\cos^2 2x \, (\underbrace{\cos^2 x - \sin^2 x}_{\cos 2x}) =$$

$$= \underline{\underline{-\cos^3 2x}} \, .$$

(f) THE $\boxed{\text{SARRUS rule}}$ for 3x3 determinants

$$\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} \begin{matrix} 3 & 1 \\ 2 & 5 \\ 1 & 4 \end{matrix}$$

(You add
← like this
the first
two colums

$$\det A = +\left(3 \cdot 5 \cdot 8 + 1 \cdot 6 \cdot 1 + (-4) \cdot 2 \cdot 4\right) - \left(-4 \cdot 5 \cdot 1 + 3 \cdot 6 \cdot 4 + 1 \cdot 2 \cdot 8\right) =$$

$$= \underline{120 + 6} - 32 + \underline{20} - 82 - 16$$

$$= 146 - 120 = \underline{\underline{26}}$$

OR THE SAME METHOD with <u>TRIANGLES</u>:

$$\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} =$$

$$= +3 \cdot 5 \cdot 8 + 1 \cdot 1 \cdot 6 + 2 \cdot 4 \cdot 4 - 1 \cdot 5 \cdot (-4) - 3 \cdot 6 \cdot 4 - 1 \cdot 2 \cdot 8 = \dots =$$

$$= \underline{\underline{26}}$$

③ Inverse of a matrix.

(1) Is A singular / regular

$\det A = 0 \qquad \det A \neq 0$

In regular case find A^{-1}!

e.g. So if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow$

$\det A = 4 - 6 = -2 \neq 0$

so A is regular and

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^{-1} = \dfrac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Rightarrow$

Now $A^{-1} = \frac{1}{-2} \cdot \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} =$

$$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Check that $A \cdot A^{-1} = A^{-1} \cdot A = \underline{I}$

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b) $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix} \Rightarrow$

we saw that $\det A = 26 \neq 0$

$\Rightarrow$ A is regular, it has an inverse <u>and</u>

$$\boxed{A^{-1} = \frac{1}{\det A} \cdot \tilde{A}}$$

where matrix

$$\left(\tilde{A}\right)_{ij} = (-1)^{i+j} \cdot \det A_{ji} =$$

$$= a'_{ji} \quad \underset{(j,i)}{\text{cofactor}}$$

STEP 1 :

$$A^T = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{bmatrix} \Rightarrow$$

STEP 2 :

$$A^{-1} = \frac{1}{26} \begin{bmatrix} +\begin{vmatrix} 5 & 4 \\ 6 & 8 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ -4 & 8 \end{vmatrix} & +\begin{vmatrix} 1 & 5 \\ -4 & 6 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 6 & 8 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ -4 & 8 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ -4 & 6 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{26} \begin{bmatrix} 16 & -24 & 26 \\ -10 & 28 & -26 \\ 3 & -11 & 13 \end{bmatrix}$$

<u>HW</u>: $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix}$

What is A^{-1}?

ⓒ $A = \begin{bmatrix} -2 & 3 \\ -4 & 6 \end{bmatrix}$

$\det A = -12 + 12 = 0 \Rightarrow$
A is singular $\Rightarrow \nexists A^{-1}$

ⓓ $A = \begin{bmatrix} -2 & 5 \\ -3 & 1 \end{bmatrix}$

$\det A = -2 + 15 = 13 \neq 0 \Rightarrow$
A is regular, it has inverse
and

$$A^{-1} = \frac{1}{13} \cdot \begin{bmatrix} 1 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1/13 & -5/13 \\ 3/13 & -2/13 \end{bmatrix}$$

THE END.