Basic Mathematics | Class 13

Generated Subspaces

① $W := \mathrm{Span}\big(\underbrace{(1,2,-1)}_{a};\underbrace{(-3,1,1)}_{b}\big) \subseteq \mathbb{R}^3$.

a) What are the elements of W?

b) Give some example elements of W.

c) $x := (2,4,0)$ $y := (5,-4,-1)$

Is $x \in W$? $y \in W$?

Mo: $\mathrm{Span}(a,b) \overset{\text{def}}{=}$

$= \{\alpha a + \beta b \in \mathbb{R}^3 \mid \alpha,\beta \in \mathbb{R}\} =$

$$= \left\{ \alpha \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\} =$$

$$= \left\{ \begin{pmatrix} \alpha - 3\beta \\ 2\alpha + \beta \\ -\alpha + \beta \end{pmatrix} \in \mathbb{R}^3 \middle| \alpha, \beta \in \mathbb{R} \right\}.$$

b) For example, if:

$$\alpha = 1, \ \beta = 2 \Rightarrow \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} \in W.$$

<u>or</u> $$\alpha = -1, \ \beta = 1 \Rightarrow \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} \in W.$$

c) $\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \in W \iff \exists \alpha, \beta \in \mathbb{R}$

so that $\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha - 3\beta \\ 2\alpha + \beta \\ -\alpha + \beta \end{pmatrix}$

$$\iff \begin{cases} \alpha - 3\beta = 2 \\ 2\alpha + \beta = 4 \\ -\alpha + \beta = 0 \Rightarrow \beta = \alpha \end{cases} \Rightarrow$$

$$\alpha - 3\alpha = 2 \Rightarrow -2\alpha = 2$$

$$\left.\begin{array}{l} \boxed{\alpha = -1} \\ 3\alpha = 4 \quad \boxed{\alpha = \frac{3}{4}} \end{array}\right\} \lightning$$

So $x \notin W$.

For $y = \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} \in W \Leftrightarrow$

$\exists \alpha, \beta \in \mathbb{R}$:

$$\begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha - 3\beta \\ 2\alpha + \beta \\ -\alpha + \beta \end{pmatrix} \Leftrightarrow$$

$$\begin{cases} \alpha - 3\beta = 5 \\ 2\alpha + \beta = -4 \\ -\alpha + \beta = -1 \end{cases}$$

$\oplus \Rightarrow -2\beta = 4$

$\boxed{\beta = -2}$

$\Rightarrow \boxed{\alpha = \beta + 1 = -1}$

and $2\alpha + \beta = -2 - 2 = -4$ ✓

So $y \in W$, $y = -1 \cdot a - 2b$;

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(2) Consider the vectors:

$u = (1, 2, -1)$, $v = (6, 4, 2)$;

$x = (9, 2, 7)$; $y = (4, -1, 8)$

a) Compute : $-2u + 3v$

b) $\text{Span}(u, v) = ?$

c) Is $x \in \text{Span}(u, v)$?

d) Is $y \in \text{Span}(u, v)$?

Sol: a) $-2u + 3v =$

$$= -2 \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -2+18 \\ -4+12 \\ 2+6 \end{pmatrix} =$$

$$= \underline{\underline{\begin{pmatrix} 16 \\ 8 \\ 8 \end{pmatrix}}}$$

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b) $\underline{\mathrm{Span}(u, v)} = \{\alpha u + \beta v \mid \alpha, \beta \in \mathbb{R}\} =$

$$= \left\{ \alpha \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$$

$$= \underline{\left\{ \begin{pmatrix} \alpha + 6\beta \\ 2\alpha + 4\beta \\ -\alpha + 2\beta \end{pmatrix} \in \mathbb{R}^3 \middle| \alpha, \beta \in \mathbb{R} \right\}}$$

c) $x \in \operatorname{Span}(u, v) \iff$

$\exists \alpha, \beta \in \mathbb{R} \quad x = \alpha u + \beta v \iff$

$$\iff \exists \alpha, \beta \in \mathbb{R}: x = \begin{pmatrix} 9 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} \alpha + 6\beta \\ 2\alpha + 4\beta \\ -\alpha + 2\beta \end{pmatrix}$$

$$\iff \begin{cases} \alpha + 6\beta = 9 \quad (1) \\ 2\alpha + 4\beta = 2 \\ -\alpha + 2\beta = 7 \end{cases} \implies$$

$$8\beta = 16 \implies \boxed{\beta = 2}$$

$$-\alpha + 2 \cdot 2 = 7 \qquad \boxed{\alpha = -3}$$

In equation 2:

$$2(-3) + 4 \cdot 2 = 2 \checkmark$$

$$\implies$$

So:

$$X = -3u + 2v \in \mathrm{Span}(u, v)$$

d) $y \in \mathrm{Span}(u, v) \Leftrightarrow$

$\exists \alpha, \beta \in \mathbb{R} : y = \alpha u + \beta v \Leftrightarrow$

$$y = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix} = \begin{pmatrix} \alpha + 6\beta \\ 2\alpha + 4\beta \\ -\alpha + 2\beta \end{pmatrix} \Leftrightarrow$$

$$\begin{cases} \alpha + 6\beta = 4 \\ 2\alpha + 4\beta = -1 \\ -\alpha + 2\beta = 8 \end{cases} \Rightarrow 8\beta = 12 \quad \boxed{\beta = \frac{3}{2}}$$

$\oplus$ $| \cdot 2$

$\Downarrow$

$8\beta = 15$

$\boxed{\beta = \frac{15}{8}}$ $\lightning$

$\Rightarrow \boxed{y \notin \mathrm{Span}(u, v)}$

3. Consider the subspaces:

a) $S_5 = \left\{ \begin{pmatrix} x-y \\ 3x \\ 2x+y \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x, y \in \mathbb{R} \right\}$

b) $S_3 := \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; 2x - 3y + z = 0 \right\}$

c) $W_1 = \left\{ \begin{pmatrix} x-y+5z \\ 3x-z \\ 2x+y-7z \\ -x \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x, y, z \in \mathbb{R} \right\}$

d) $W_2 := \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; x + 3y = 0 \right\}$

$\boxed{\text{Determine \underline{finite} generator systems to each of them.}}$

Solution:

a) The elements of S_5 can be written in the following form:

$$\begin{pmatrix} x-y \\ 3x \\ 2x+y \end{pmatrix} = \begin{pmatrix} x \\ 3x \\ 2x \end{pmatrix} + \begin{pmatrix} -y \\ 0 \\ y \end{pmatrix} =$$

$$= x \underbrace{\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}_{a} + y \cdot \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{b} = x \cdot a + y b$$

$\underline{\text{So}}$ $S_5 = \left\{ x \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}$

$\overset{\text{def}}{=} \operatorname{Span}\big((1,3,2), (-1,0,1) \big)$

$\underline{\text{So}}$ $(1,3,2), (-1,0,1)$ is a gen. system in S_5.

b) $2x - 3y + z = 0 \implies$

$z = -2x + 3y \implies$

$S_3 = \left\{ \begin{pmatrix} x \\ y \\ -2x+3y \end{pmatrix} \in \mathbb{R}^3 \middle| \, x, y \in \mathbb{R} \right\}$

$=$

$$= \left\{ \begin{pmatrix} x \\ 0 \\ -2x \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 3y \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$$

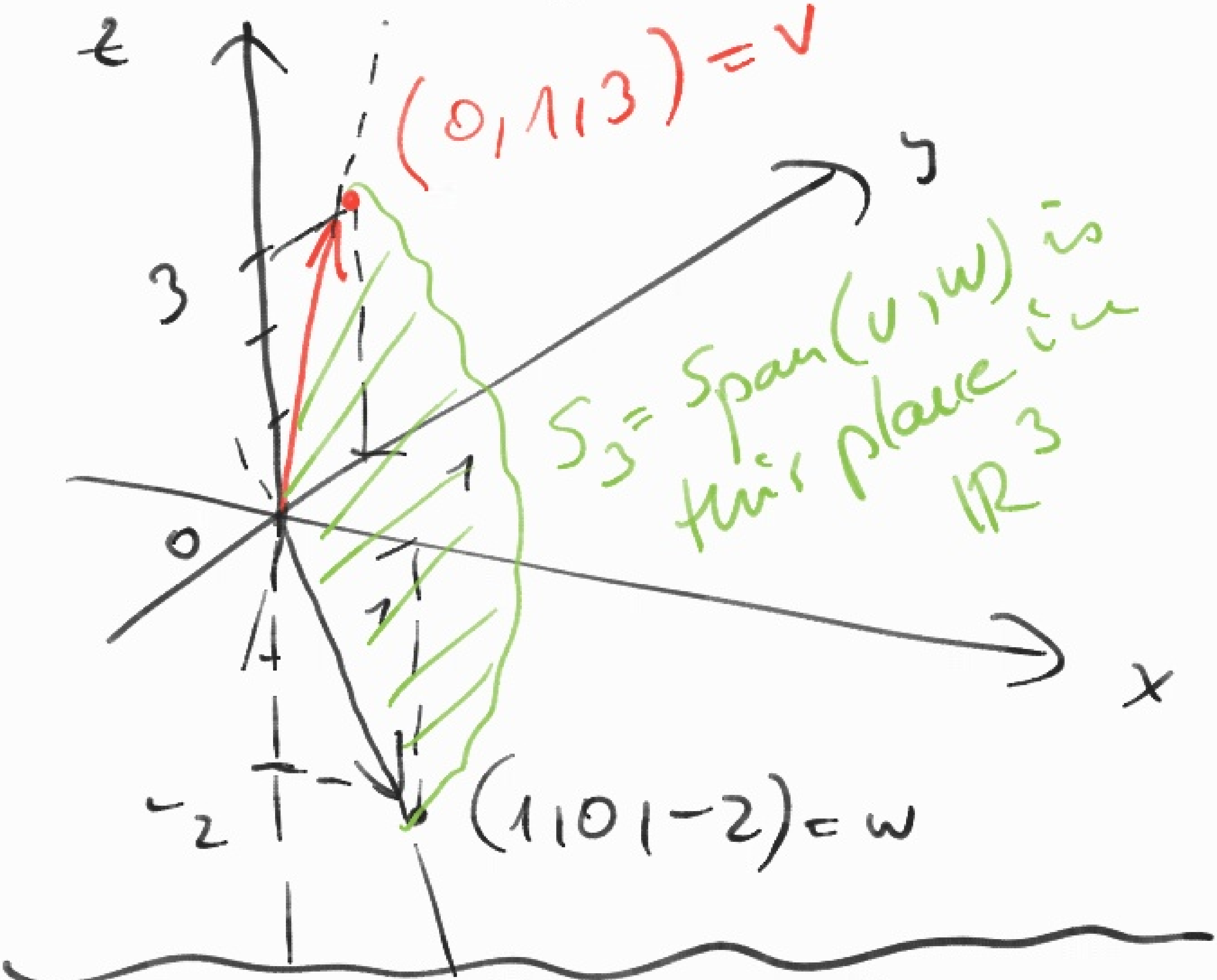
$$= \left\{ x \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$$

$$= \mathrm{Span}\left((1,0,-2), (0,1,3) \right)$$

<u>So</u> $(1,0,-2), (0,1,3)$ is a gen. system in S_3.

<u>Remark</u>:

What is S_3?



c) $$W_1 = \left\{ \begin{pmatrix} x-y+5z \\ 3x-z \\ 2x+y-7z \\ -x \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x,y,z \in \mathbb{R} \right\} =$$

$$= \left\{ \begin{pmatrix} x \\ 3x \\ 2x \\ -x \end{pmatrix} + \begin{pmatrix} -y \\ 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 5z \\ -z \\ -7z \\ 0 \end{pmatrix} \middle| \; x,y,z \in \mathbb{R} \right\}$$

$$= \left\{ x \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 5 \\ -1 \\ -7 \\ 0 \end{pmatrix} \middle| \; x,y,z \in \mathbb{R} \right\} =$$

$$= \operatorname{Span}\left(\begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ -7 \\ 0 \end{pmatrix} \right)$$

<u>So</u> gen. system: are the vectors $\begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ -7 \\ 0 \end{pmatrix}$.

$\boxed{4}$ Determine a finite gen. system for the following subspaces in $\mathbb{R}^3$:

a) $W_1 = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; [2,-3,5] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$

Sol: The condition here means that:

$$2x - 3y + 5z = 0 \Rightarrow$$

$$y = \frac{2x + 5z}{3} \Rightarrow$$

$$W_1 = \left\{ \begin{pmatrix} x \\ \frac{2x+5z}{3} \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x, z \in \mathbb{R} \right\}$$

$$= \left\{ x \cdot \begin{pmatrix} 1 \\ \frac{2}{3} \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 5/3 \\ 1 \end{pmatrix} \;\middle|\; z, x \in \mathbb{R} \right\} =$$

$$= \mathrm{Span}\left((1, 2/3, 0); (0, 5/3, 1) \right)$$

$$\overset{?}{=} \mathrm{Span}\left((3,2,0); (0,5,3) \right)$$

So gen system: $(3,2,0)$

$(0,5,3)$;

b) $W_2 = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x - 2y + 3z = 0 \\ 2x \quad\;\; - z = 0 \end{array} \right\}$

$\Rightarrow \; z = 2x$

$x - 2y + 3(2x) = 0$

$7x - 2y = 0 \quad y = \frac{7}{2}x$

$\Rightarrow W_2 = \left\{ \begin{pmatrix} x \\ 7/2\, x \\ 2x \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\} =$

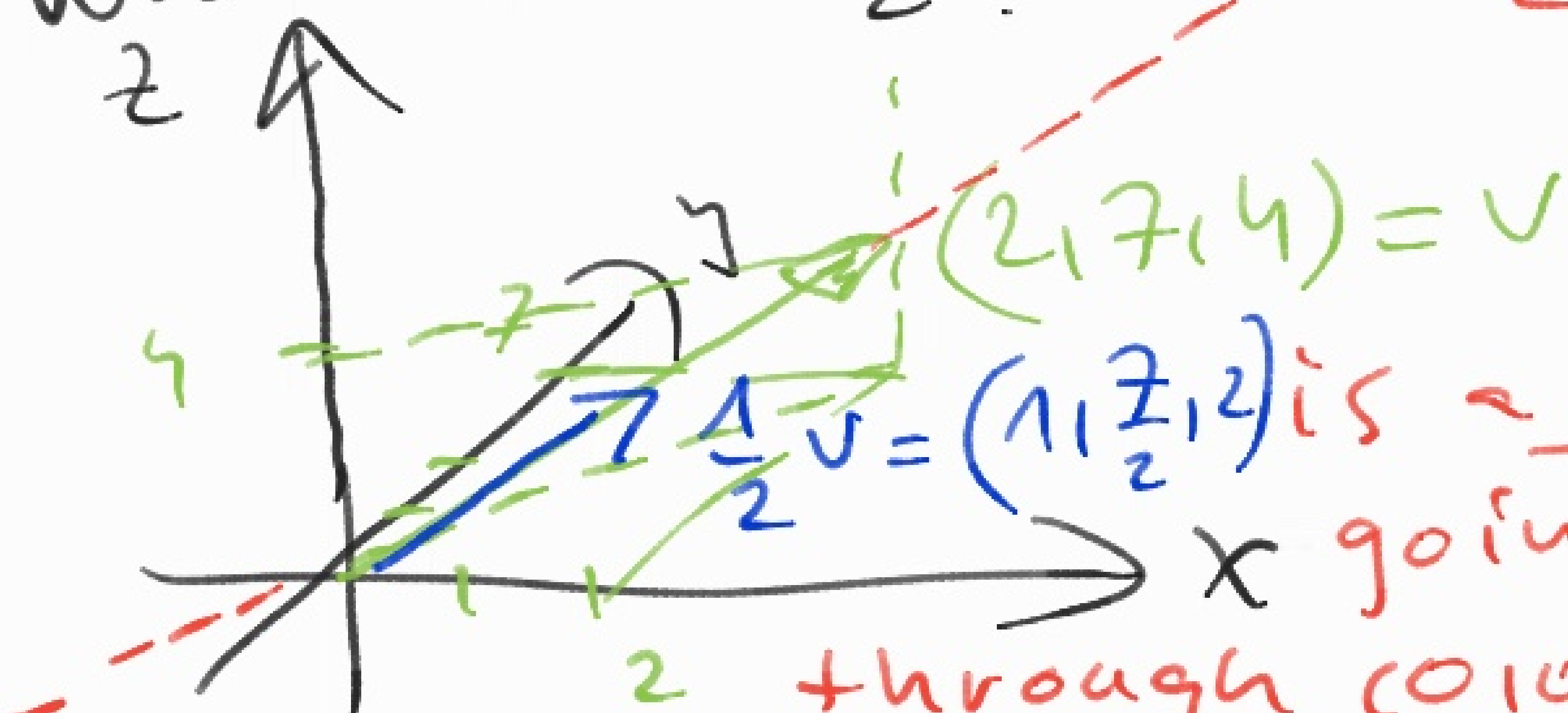
$$= \left\{ x \cdot \begin{pmatrix} 1 \\ 7/2 \\ 2 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\} =$$

$$= \operatorname{Span}\left((1, 7/2, 2) \right) \overset{?}{=}$$

$$= \operatorname{Span}\left((2, 7, 4) \right)$$

Gen. system is $(2, 7, 4)$.

What is W_2?



is a <u>line</u> going through $(0,0,0)$ and direction vector v.

c) $W_3 = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \begin{bmatrix} 2 & -1 & -2 \\ -4 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \begin{array}{l} 2x - y - 2z = 0 \wedge \\ -4x + 2y + 4z = 0 \end{array} \right\}$$

$\uparrow$

Observe that:

$$-4x + 2y + 4z = 0 \;\Big|\; : (-2)$$

$2x - y - 2z = 0$ so we only have one equation here $\Longrightarrow$

$$W_3 = \{(x,y,z) \in \mathbb{R}^3 \mid 2x - y - 2z = 0\}$$

$$\Downarrow$$

$$y = 2x - 2z$$

$$= \left\{ \begin{pmatrix} x \\ 2x-2z \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x, z \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ 2x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2z \\ z \end{pmatrix} \;\middle|\; x, z \in \mathbb{R} \right\} =$$

$$= \left\{ x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; x, z \in \mathbb{R} \right\}$$

$$\Rightarrow$$

$$W_3 = \mathrm{Span}\left(\begin{pmatrix}1\\2\\0\end{pmatrix}; \begin{pmatrix}0\\-2\\1\end{pmatrix}\right)$$

$\Rightarrow$ gen. system. $= \begin{pmatrix}1\\2\\0\end{pmatrix}; \begin{pmatrix}0\\-2\\1\end{pmatrix}$

What is W_3?