Basic Mathematics ; Class 12

VECTOR SPACES / SUBSPACES

I. lecture part: Linear-Algebra
- Mr Csurgo Pages 23-28.

II Practice:

① $V := \mathbb{R}^5$, $+$, $\lambda\cdot$ componentwise the usual operations.

$x := (-3, 4, 1, 5, 2)$

$y := (2, 0, 4, -3, -1)$

$z := (7, -1, 0, 2, 3)$

and $A := \begin{bmatrix} 5 & 1 & -4 & -2 & 1 \\ 0 & 2 & 4 & -3 & -1 \end{bmatrix}$

<u>Evaluate:</u>

$$x+y = \begin{pmatrix} -3 \\ 4 \\ 1 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 4 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3+2 \\ 4+0 \\ 1+4 \\ 5-3 \\ 2-1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -1 \\ 4 \\ 5 \\ 2 \\ 1 \end{pmatrix}}}$$

$$y-z = \begin{pmatrix} 2 \\ 0 \\ 4 \\ -3 \\ -1 \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \\ 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2-7 \\ 0+1 \\ 4-0 \\ -3-2 \\ -1-3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -5 \\ 1 \\ 4 \\ -5 \\ -4 \end{pmatrix}}}$$

$$4 \cdot x = 4 \cdot \begin{pmatrix} -3 \\ 4 \\ 1 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \cdot (-3) \\ 4 \cdot 4 \\ 4 \cdot 1 \\ 4 \cdot 5 \\ 4 \cdot 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -12 \\ 16 \\ 4 \\ 20 \\ 8 \end{pmatrix}}}$$

$$x+3y-2z=\begin{pmatrix}-3\\4\\1\\5\\2\end{pmatrix}+3\cdot\begin{pmatrix}2\\0\\4\\-3\\-1\end{pmatrix}-2\begin{pmatrix}7\\-1\\0\\2\\3\end{pmatrix}$$

$$=\begin{pmatrix}-3+2\cdot 3-2\cdot 7\\4+3\cdot 0-2\cdot(-1)\\1+3\cdot 4-2\cdot 0\\5+3\cdot(-3)-2\cdot 2\\2+3\cdot(-1)-2\cdot 3\end{pmatrix}=\underline{\underline{\begin{pmatrix}-11\\6\\13\\-8\\-7\end{pmatrix}}}$$

this is a linear combination of vectors x, y, z with coefficients $1, 3, -2$.

$$A \cdot x = \begin{bmatrix} 5 & 1 & -4 & -2 & 1 \\ 0 & 2 & 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ 1 \\ 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -15 + \cancel{4} - \cancel{4} - 10 + 2 \\ 0 + 8 + 4 - 15 - 2 \end{bmatrix} =$$

$$= \begin{bmatrix} -23 \\ -5 \end{bmatrix} \in \mathbb{R}^{2 \times 1} \cong \underline{\underline{\mathbb{R}^2}}$$

Subspaces If $(V, +, \lambda\cdot)$ is a vector space and $\emptyset \neq W \subseteq V$ is a subspace of $V \Longleftrightarrow$

a) $\forall x, y \in W \Rightarrow x + y \in W$

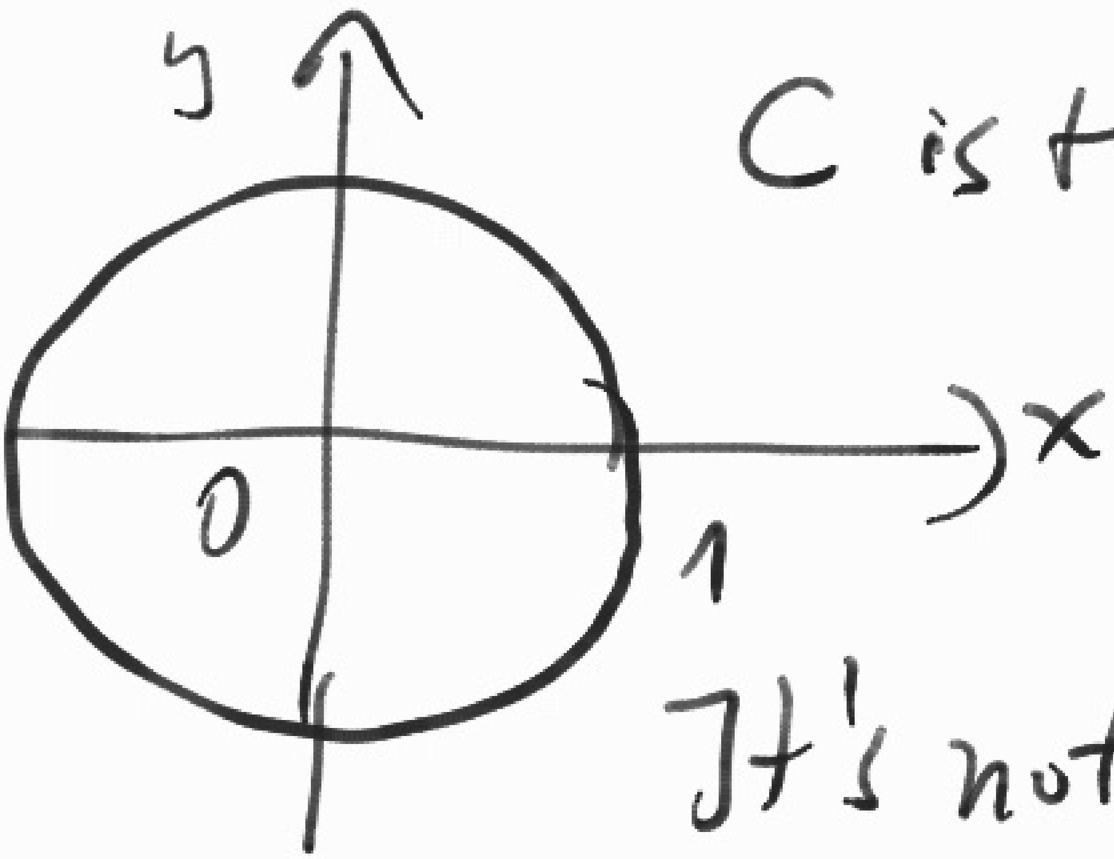
b) $\forall \lambda \in \mathbb{K}, \forall x \in W \Rightarrow \lambda \cdot x \in W$

Corollary: If W is a subspace $\Longrightarrow$ $\boxed{0 \in W}$ must be true.

Exercises:

(1) Are the following sets subspaces of $\mathbb{R}^2 =: V$?

a) $C := \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$

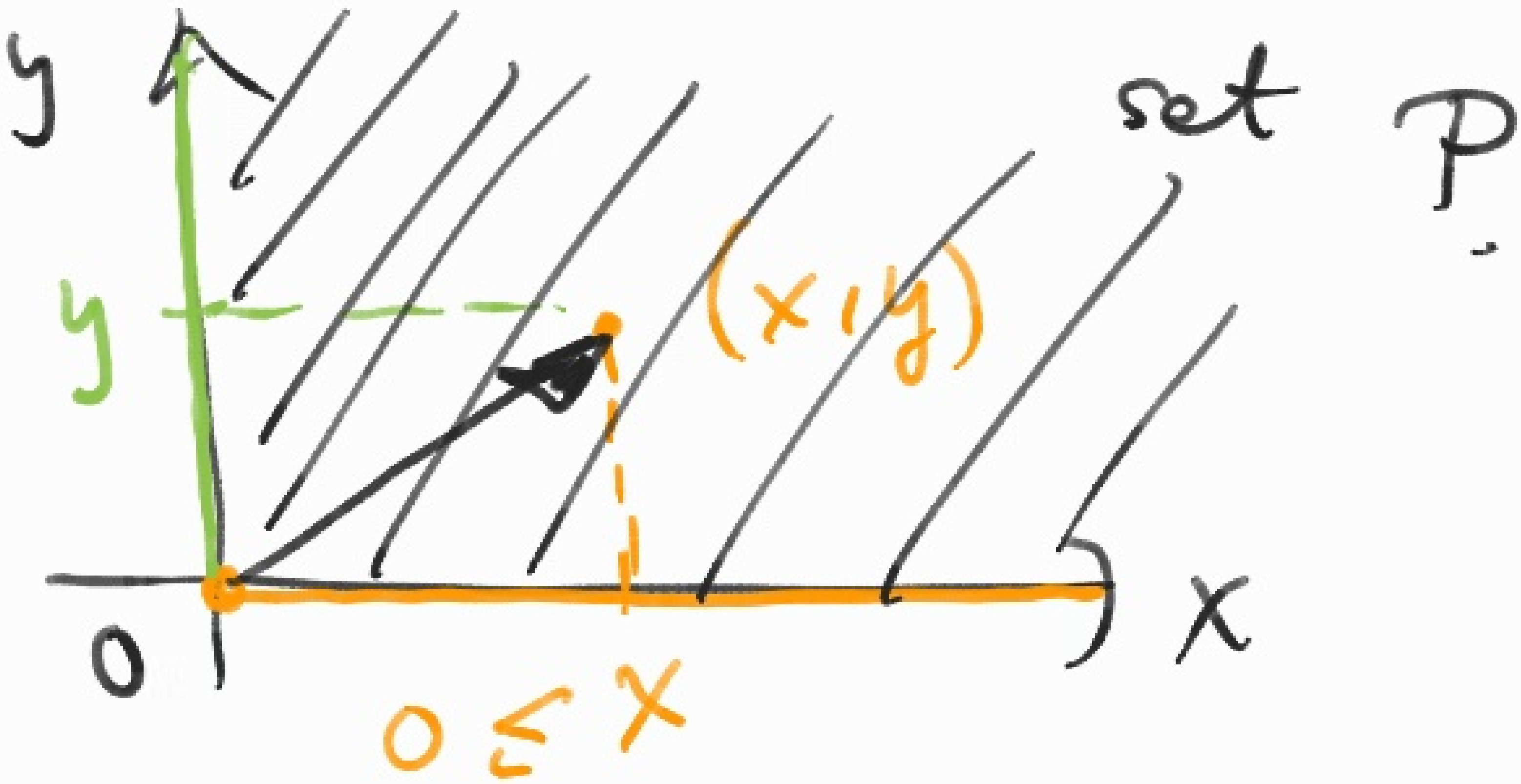


C is the unit circle in $\mathbb{R}^2$.

It's not subspace because $(0,0) \notin C$

$(0^2+0^2=0 \neq 1)$.

b) $P := \{(x,y) \in \mathbb{R}^2 \mid x \geq 0$ and $y \geq 0\}$.



$(0,0) \in P$ so P <u>can be</u> vector space. Is it closed under addition? If $(x,y) \in P$, $(u,v) \in P \Longrightarrow (x,y)+(u,v) \in P$?

$$\left.\begin{array}{l} \text{So } x \geq 0,\ y \overset{?}{\geq} 0 \text{ and} \\ \quad u \geq 0,\ v \geq 0 \end{array}\right\} \Longrightarrow$$

$x+y \geq 0$ and $u+v \geq 0 \Longrightarrow$

$(x,y)+(u,v)\in P$ ✓

But If $(x,y)\in P$ with

$x,y\geq 0$ for example:

$$\underset{\lambda}{\underset{\shortparallel}{(-1)}}\cdot(x,y)=(-x,-y)\notin P.$$

Ex: $(x,y)=(1,2)\in P$ but

$(-1)\cdot(1,2)=(-1,-2)\notin P.$

So P is not a subspace.

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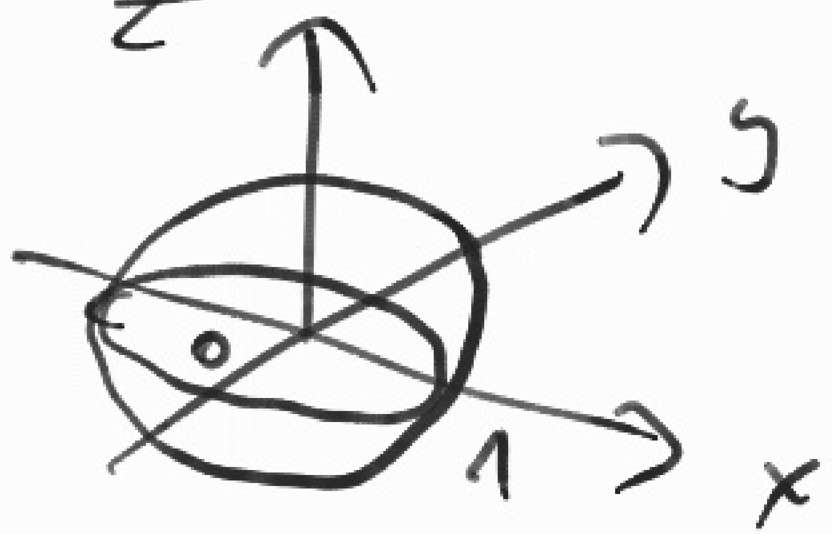
(2) Are the following sets subspaces of $\mathbb{R}^3$?

a) $S_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$



the unit sphere in $\mathbb{R}^3$.

It's not a subspace becouse

$$(0,0,0) \notin S_1$$

$$(0^2+0^2+0^2 = 0 \neq 1)$$

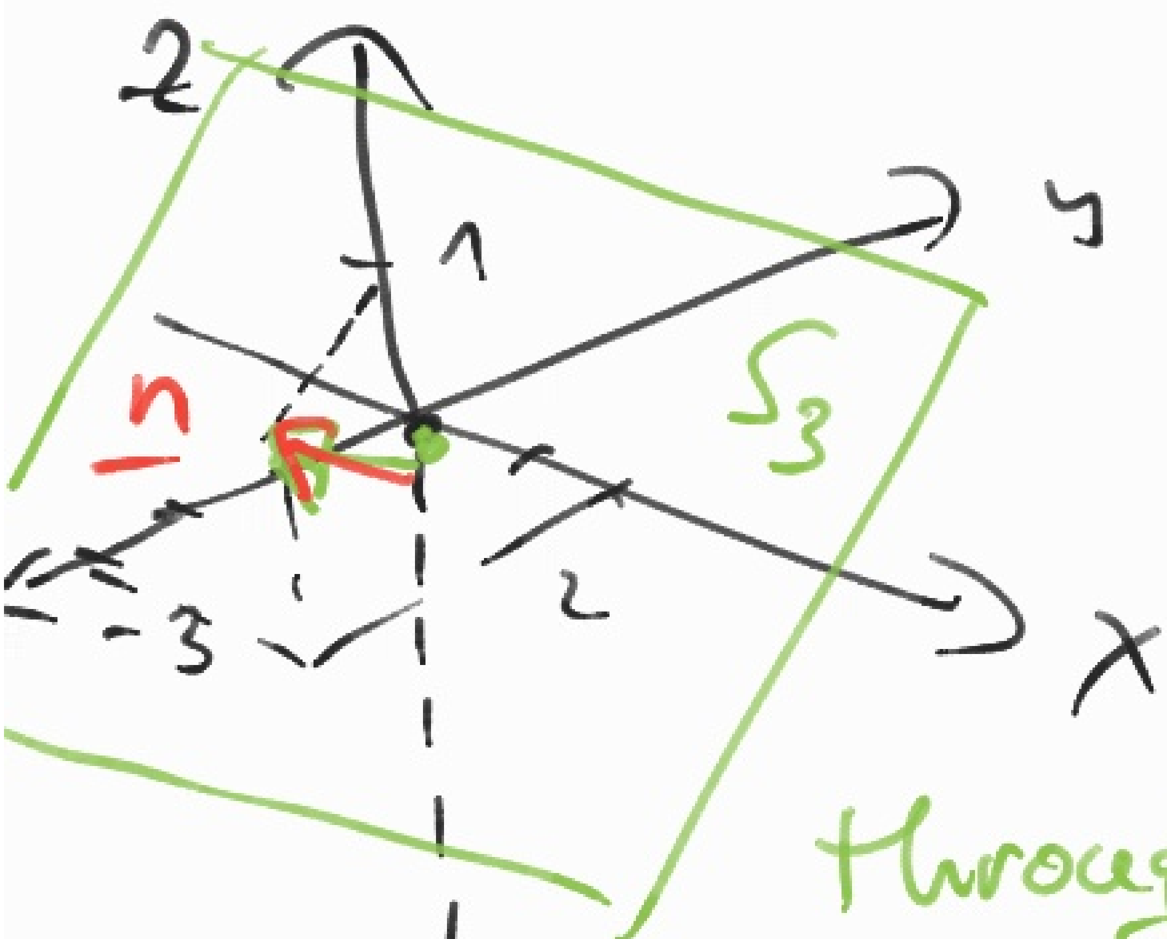
b) $S_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0\}$.

Is not subspace

exp: $(1,2,1) \in S_2$ but

$$(-1)\cdot(1,2,1) = (-1,-2,-1) \notin S_2.$$

(c) $S_3 = \{(x,y,z) \in \mathbb{R}^3 \mid 2x - 3y + 1z = 0\}$



$(0,0,0) \in S_3$ ✓

$2\cdot 0 - 3\cdot 0 + 0 = 0$ ✓

is a plane that goes through $(0,0,0)$ and has the normal vector $\underline{n}\,(2,-3,1)$

Is it a subspace?

we have to check, that if $a, b \in S_3 \Rightarrow a+b \overset{?}{\in} S_3$

<u>so</u> $a = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S_3$, $b = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in S_3$

$\Updownarrow \qquad\qquad \Updownarrow$

$$2x - 3y + z = 0 \quad (1) \qquad 2u - 3v + w = 0 \quad (2)$$

what is

$$a + b = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x+u \\ y+v \\ z+w \end{pmatrix}$$

<u>We</u> evaluate

$$2(x+u) - 3(y+v) + (z+w) =$$

$$= (2x - 3y + z) + \underbrace{(2u - 3v + w)}_{=(2)}$$

and (1) $\wedge$ (2) $\Rightarrow$ (Δ) = 0.

$$a+b = \begin{pmatrix} x+u \\ y+v \\ z+w \end{pmatrix} \in S_3 \checkmark$$

<u>**AND**</u> If $\lambda \in \mathbb{R}$ any number and $a \in S_3 \Rightarrow$

$\lambda a \overset{?}{\in} S_3$ <u>so</u>

$$\lambda a = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} \in S_3 \Leftrightarrow$$

$$2(\lambda x) - 3(\lambda y) + 1\cdot(\lambda z) = 0$$

$$\parallel$$

$$\lambda \underbrace{(2x - 3y + z)}_{=0} = 0 \checkmark$$

$\Leftarrow (a \in S_3)$

So S_3 is a subspace in $\mathbb{R}^3$

d) $S_4 := \{(x,y,z) \in \mathbb{R}^3 \mid 2x - 3y + z = 5\}$

S_4 is not a subspace, since $(0,0,0) \notin S_4$

$$2 \cdot 0 - 3 \cdot 0 + 0 = 0 \neq 5$$

c) $S_5 = \{(x-y, 3x, 2x+y) \in \mathbb{R}^3 : x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3$

• So if $a, b \in S_5 \Rightarrow$

$$a = \begin{pmatrix} x - y \\ 3x \\ 2x + y \end{pmatrix}, \text{ with some } x, y \in \mathbb{R}$$

$$b = \begin{pmatrix} u - v \\ 3u \\ 2u + v \end{pmatrix}, \text{ with some } u, v \in \mathbb{R}.$$

$$\Rightarrow a + b = \begin{pmatrix} x - y \\ 3x \\ 2x + y \end{pmatrix} + \begin{pmatrix} u - v \\ 3u \\ 2u + v \end{pmatrix} =$$

$$= \begin{pmatrix} (x+u) - (y+v) \\ 3(x+u) \\ 2(x+u) + (y+v) \end{pmatrix} = \begin{pmatrix} \alpha - \beta \\ 3\alpha \\ 2\alpha + \beta \end{pmatrix}$$

<u>Where</u>: $\alpha = x + u \in \mathbb{R}$

$\beta = y + v \in \mathbb{R}$

$\Rightarrow a+b \in S_5$ ✓

Similarly if $\lambda \in \mathbb{R}$,

$$\underline{a} = \begin{pmatrix} x-y \\ 3x \\ 2x+y \end{pmatrix} \in S_5 \text{ with } x, y \in \mathbb{R}$$

Then

$$\lambda \cdot \underline{a} = \begin{pmatrix} \lambda(x-y) \\ \lambda(3x) \\ \lambda(2x+y) \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda x - \lambda y \\ 3 \cdot (\lambda x) \\ 2(\lambda x) + \lambda y \end{pmatrix} = \begin{pmatrix} \alpha - \beta \\ 3\alpha \\ 2\alpha + \beta \end{pmatrix} \in S_5$$

where $\alpha = \lambda x \in \mathbb{R}$
$\beta = \lambda y \in \mathbb{R}$ ✓

$\Rightarrow \lambda a \in S_5$. So S_5 is a subspace in $\mathbb{R}^3$

THE END