Basic Mathematics

Class 21 | Limits of functions at $+\infty$

Assume from now on that $f \in \mathbb{R} \to \mathbb{R}$ ($D_f \subseteq \mathbb{R}$) and that for $\forall K > 0$

$$(K, +\infty) \cap D_f \neq \emptyset.$$

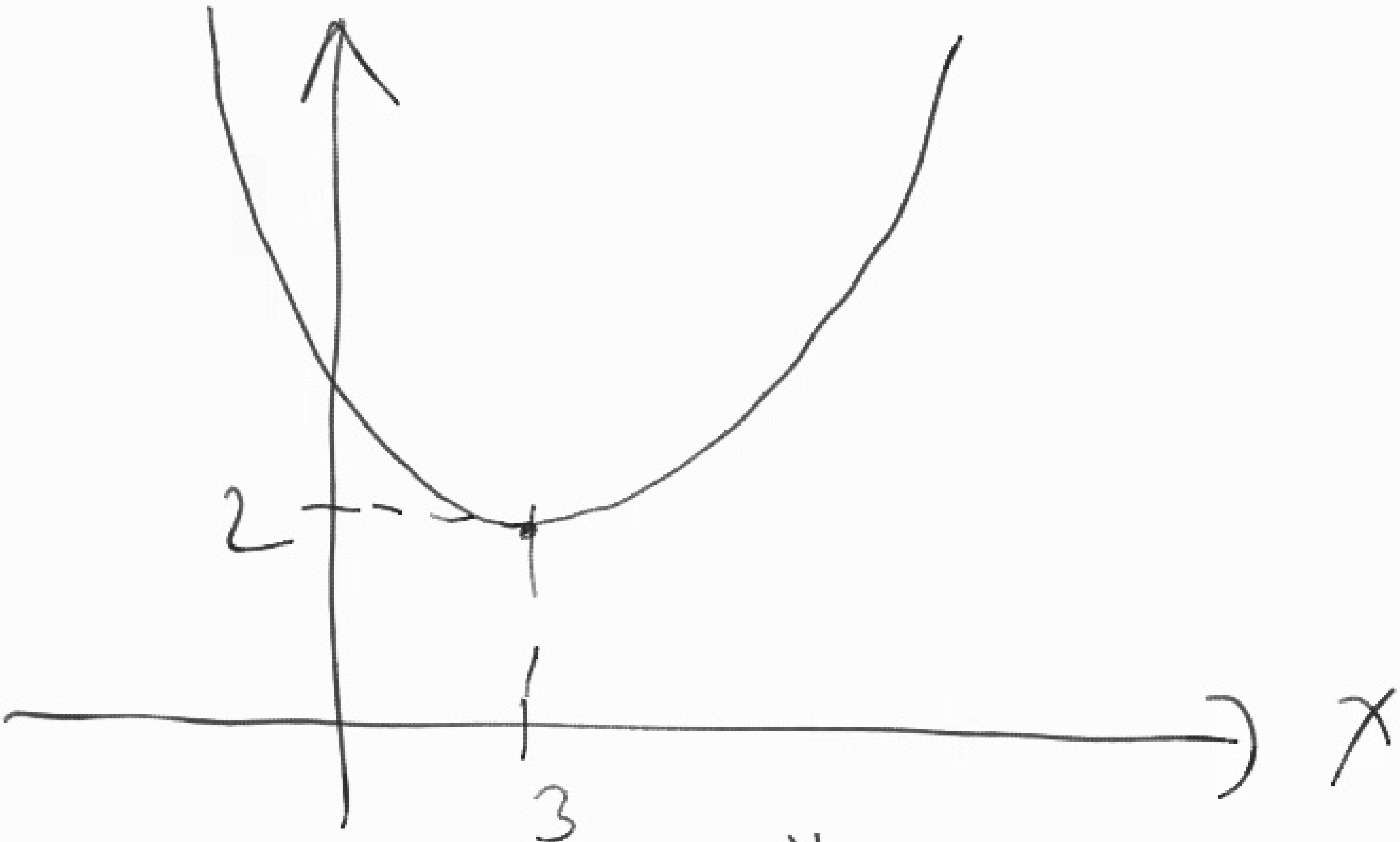
We will discuss the cases:

a) $\lim\limits_{x \to +\infty} f(x) = +\infty$

b) $\lim\limits_{x \to +\infty} f(x) = -\infty$

c) $\lim\limits_{x \to +\infty} f(x) = A \in \mathbb{R}$

$1^\circ.$ $f(x) = x^2 - 3x + 5 \quad (x \in \mathbb{R})$



what „happens" as $x \approx +\infty$

$f(x) \approx ? \simeq +\infty$

$$\underset{\substack{\downarrow \\ +\infty}}{x^2}\Big(\underset{\substack{\downarrow \\ 1}}{1} - \underset{\substack{\downarrow \\ 0}}{\frac{3}{x}} + \underset{\substack{\downarrow \\ 0}}{\frac{5}{x^2}}\Big) \to +\infty \cdot 1 = +\infty$$

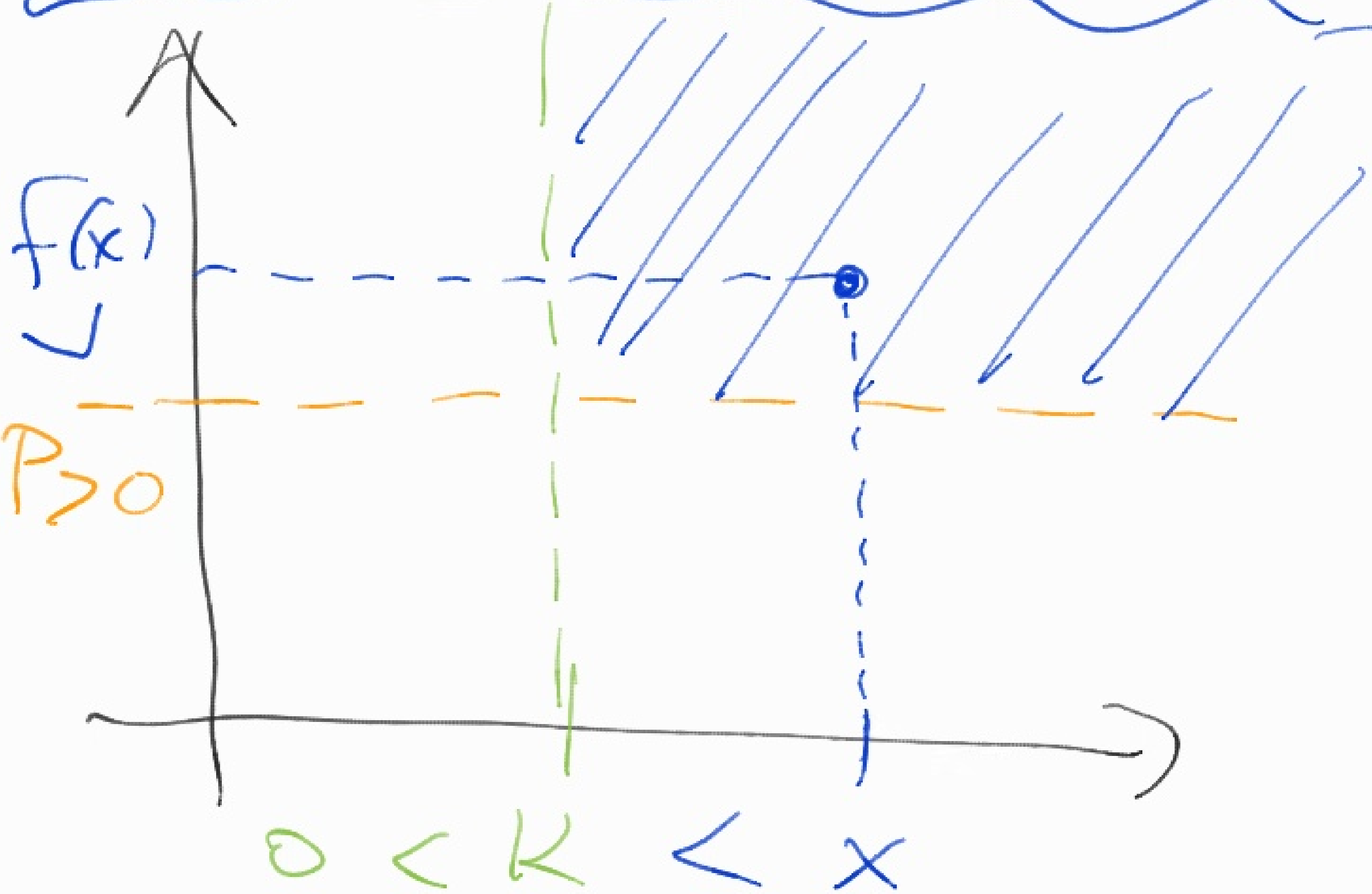
as $x \to +\infty$.

What is the definition?

$$\lim_{x \to +\infty} f(x) = +\infty \iff$$

$$\forall P > 0 \ \exists K > 0 \ \forall x \in D_f : x > K$$

$$f(x) > P$$



Now: $f(x) = x^2 - 3x + 5$

Let $P > 0$ be fixed.

We want $x^2 - 3x + 5 > K$

OPL: $x^2 - 3x + 5 > x^2 - 3x =$

$$= \frac{1}{2}x^2 + \frac{1}{2}x^2 - 3x =$$

$$= \frac{1}{2}x^2 + \frac{1}{2}x(x-6) >$$

$$> \frac{1}{2}x^2 > P \ ?$$

if $x > 6$

if $x^2 > 2P$

or $x > \sqrt{2P}$

So $\forall P > 0 \ \exists K := \sqrt{2p} + 6 > 0$

so that $\forall x > K \Longrightarrow$

$$f(x) > \frac{1}{2}x^2 > \frac{1}{2}K^2 > P$$

$$\Longrightarrow \lim_{x \to +\infty} (x^2 - 3x + 5) = +\infty$$

2) $$\lim_{x \to +\infty} \left(\frac{4x^3 - 22x^2 + x + 1}{x^2 + x + 7} \right)$$

$= +\infty$

We have to prove that

$$\forall P>0\ \exists K>0\ \underbrace{\forall x>K\ (x\in D_f)}_{x\in \underbrace{D_f\cap(K,+\infty)}_{\neq 0}}$$

$$f(x)=\frac{4x^3-22x^2+x+1}{x^2+x+7}>P$$

Let's prove it.

Fix a $\boxed{P>0}$; $\underline{OPL}$

$$\frac{4x^3-22x^2+x+1}{x^2+x+7}\ \underset{\text{if } x>1}{>}\ \frac{4x^3-22x^2}{x^2+x^2+7x^2}$$

$$= \frac{2x^3 + 2x^3 - 22x^2}{9x^2} =$$

$$= \frac{2x^3 + 2x^2(x-11)}{9x^2} \geq$$

$$\geq \frac{2x^3}{9x^2} = \frac{2x}{9} > P$$

if $x \geq 11$

$\Uparrow$

If $x > \frac{9P}{2}$

So if $x > \frac{9P}{2}$ and $x \geq 11$

$$\Rightarrow K := \frac{9P}{2} + 11$$

$\Rightarrow \forall P>0 \ \exists K := \frac{9P}{2} + 11 > 0$

$\forall x > K, \ x \in D_f \Rightarrow$

$f(x) \geq \frac{2x}{9} > \frac{2K}{9} > P.$

b) Prove by definition

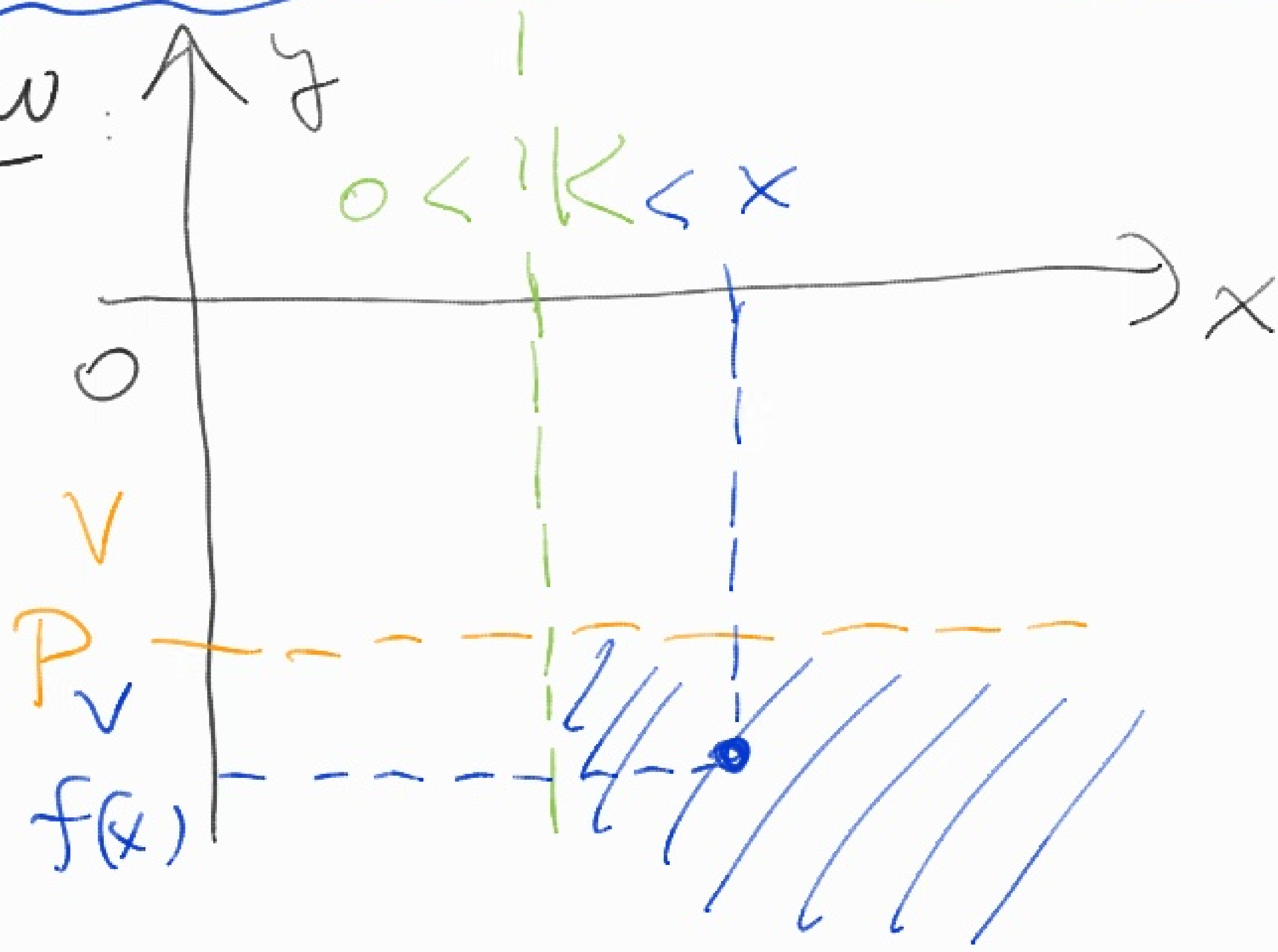
that:

$$\boxed{\lim_{x \to +\infty} \frac{-3x^4 + 5x^2 - x + 7}{2x^2 - 3x + 5} = -\infty}$$

We need the definition :

$$\boxed{\lim_{x \to +\infty} f(x) = -\infty \iff \forall P<0\ \exists K>0\ \forall x>K \text{ and } x\in D_f : \quad f(x) < P}$$

<u>Now</u> :



For now

$$\forall P<0 \ \exists K>0 \ \forall x \in (K,+\infty) \cap D_f$$

$$f(x) < P$$

Let's prove it:

Fix a $\boxed{P>0}$;

$$f(x) = \frac{-3x^4+5x^2-x+7}{2x^2-3x+5} < P$$

$\cdot(-1)$

$$\frac{3x^4-5x^2+x-7}{2x^2-3x+5} > \underbrace{-P}_{+}$$

and see technic from a)

$$\frac{3x^4-5x^2+x-7}{2x^2-3x+5} \geq$$

$$\underset{\text{if } x\geq 1}{\geq} \frac{3x^4-5x^2-7}{2x^2+5} =$$

$$= \frac{3x^4-\left(\overbrace{5x^2+7}^{\leq 5x^2+7x^2 = 12x^2}\right)}{2x^2+5} \geq$$

$$\underset{x\geq 1}{\geq} \frac{3x^4-12x^2}{7x^2} =$$

$$= \frac{2x^4 + x^4 - 12x^2}{7x^2} =$$

$$= \frac{2x^4 + x^2(x^2 - 12)}{7x^2} \geq$$

$$\geq \frac{2x^4}{7x^2} = \frac{2x^2}{7} > -P$$

$$?$$

$$\Uparrow$$

$$x^2 > \frac{-7P}{2}$$

if $x^2 \geq 12$

$$x \geq \sqrt{12}$$

$$x \geq \underset{\substack{\shortparallel \\ 4}}{\sqrt{16}} > \sqrt{12}$$

so $x > \sqrt{\frac{-7P}{2}}$ and

$x \geq 4$ then all is true

here $\Rightarrow$

$\forall P < 0 \ \exists K := \sqrt{\frac{-7P}{2}} + 4 > 0$

$\forall x > K \ , x \in D_f \Rightarrow$

$-f(x) \geq \frac{2x^2}{7} > \frac{2K^2}{7} > -P$

$\Leftrightarrow f(x) < P$ ✓

c) Prove by definition that:

$$\lim_{x \to +\infty} \frac{2x^2 - 3x + 100}{x^2 + x + 10} = 2$$

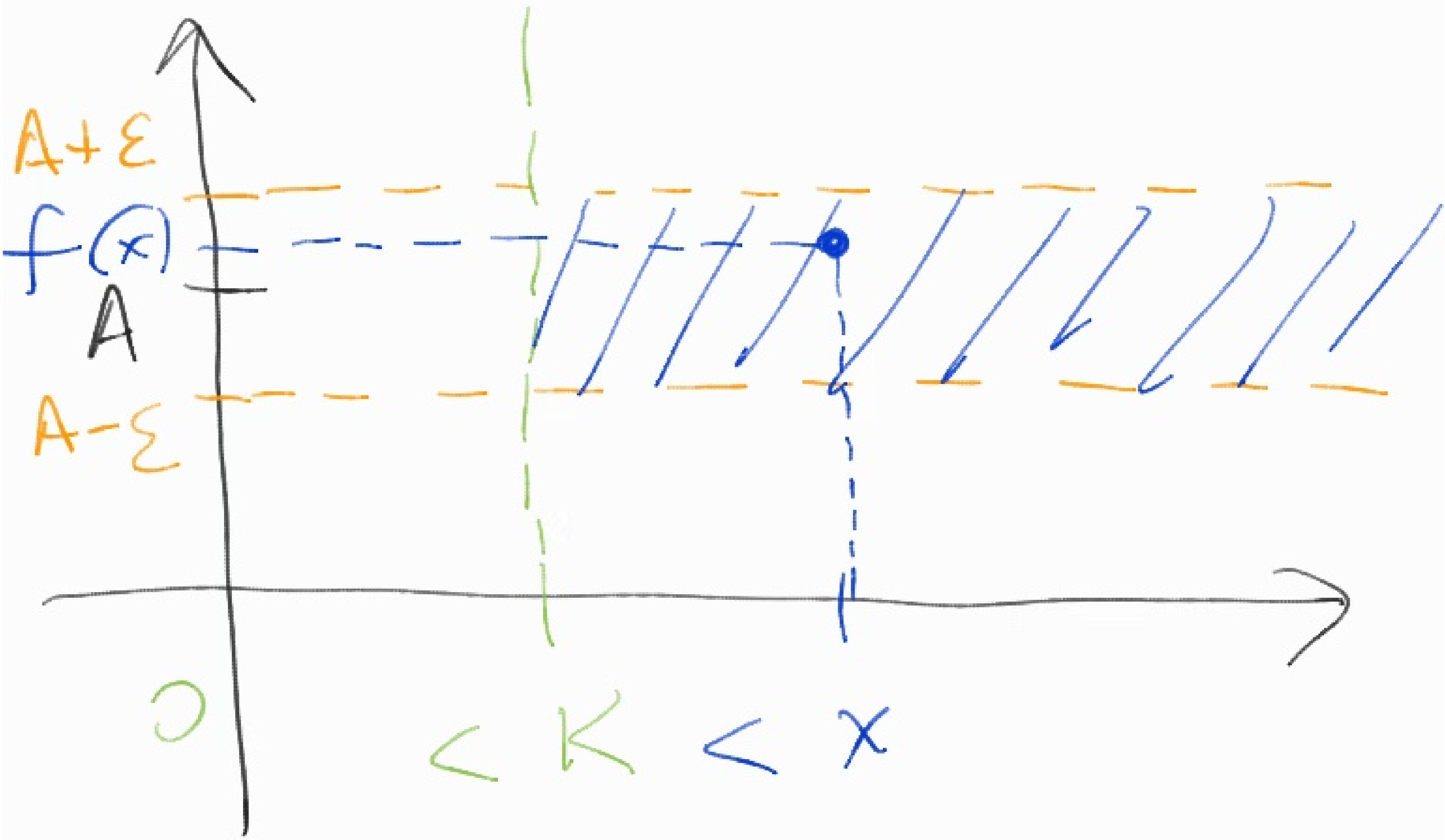
Def: $\lim_{x \to +\infty} f(x) = A \in \mathbb{R}$

"finite limit at $+\infty$"

$\iff \forall \varepsilon > 0 \ \exists K > 0 :$

$\forall x > K$ and $x \in D_f$:

$|f(x) - A| < \varepsilon$



<u>Now</u>: Fix an $\boxed{\varepsilon > 0}$,

See

$$|f(x) - A| = \left| \frac{2x^2 - 3x + 100}{x^2 + x + 10} - 2 \right| =$$

$$= \frac{|\cancel{2x^2} - 3x + 100 - \cancel{2x^2} - 2x - 20|}{|x^2 + x + 10|}$$

$$= \frac{|-5x+80|}{|\underbrace{x^2+x+1}_{+\ \forall x\in\mathbb{R}}|} = \frac{|5x-80|}{x^2+x+1} =$$

$\boxed{\text{OPU}}$

$$= \frac{5x-80}{x^2+x+1} \leq \frac{5x}{x^2}$$

if $5x-80>0$

$x > \frac{80}{5} = 16$

$$= \frac{5}{x} < \varepsilon \Leftarrow x > \frac{5}{\varepsilon}$$

So if $x > \frac{5}{\varepsilon}$ and $x > 16$

$\Rightarrow$ $K := \frac{5}{\varepsilon} + 16 > 0$

For all $\forall \varepsilon > 0 \quad \exists K := \frac{5}{\varepsilon} + 16 > 0$

so that $\forall x \in ((K, +\infty) \cap D_f$

$|f(x) - 2| \leq \frac{5}{x} < \frac{5}{K} < \varepsilon$

THE END