BASIC MATHEMATICS

PRACTICE 8, MATHEMATICAL INDUCTION.

(1) **THEOREM** (Principle of Mathematical Induction OR complete induction).

Let $A(n)$ be a statement on $n \in \mathbb{N}$ natural numbers. If:

(a) $A(0)$ is true and

(b) For all $n \in \mathbb{N}$ when

$$A(n) \text{ is true} \Longrightarrow A(n+1) \text{ is true}$$

THEN: $A(n)$ is true for all $n \in \mathbb{N}$.

Remarks: i) Condition (a) is called the first step or starting step, and can start not only from 0 is also possible to have an $n_0 \in \mathbb{N}$ so that $A(n_0)$ is true.

ii) In step b) we suppose that $A(n)$ is true for some n and prove that this implies that $A(n+1)$ is true as well.

② Exercises:

1° Prove that for all $n \in \mathbb{N}^+$ we have:

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \dots + n^2 \underset{A(n)}{=} \frac{n(n+1)(2n+1)}{6}$$

Proof: By induction.

STEP 1: check for the first allowed n, here for

$\boxed{n=1}$: $1^2 = \frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 1)}{6} \iff$

$1 = \frac{2 \cdot 3}{6} \iff 1 = 1$ ✓

STEP 2: Suppose $A(n)$ is true for some $n \in \mathbb{N}^+$ and we have to prove $A(n+1)$ so:

$$1^2 + 2^2 + \dots + (n+1)^2 = \frac{(n+1) \cdot ((n+1)+1)(2(n+1)+1)}{6}$$

or: $1^2 + 2^2 + \dots + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$

$\boxed{\text{STEP 3}}$:

Proof: $\underbrace{\boxed{1^2+\cdots+n^2}}_{\text{inductional supposing}}+(n+1)^2=\boxed{\dfrac{n(n+1)(2n+1)}{6}}+(n+1)^2$

$$=(n+1)\cdot\left[\frac{n(2n+1)}{6}+(n+1)\right]=$$

$$=(n+1)\,\frac{2n^2+n+6n+6}{6}=\frac{(n+1)(2n^2+7n+6)}{6}$$

$$=\frac{(n+1)\cdot(n+2)(2n+3)}{6}\ \checkmark$$

So by induction $A(n)$ is true for all $n\in\mathbb{N}^+$.

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(20) $\displaystyle\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{n}{n+1}\quad(n\in\mathbb{N}^+)$

$A(n)$

$$\iff \frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{n(n+1)}\stackrel{?}{=}\frac{n}{n+1}$$

STEP 1: FOR $n=1$: $\dfrac{1}{1\cdot2}=\dfrac{1}{1+1}$ ✓

STEP 2: Suppose for some $n\in\mathbb{N}^+$ is true $A(n)$

We have to prove $A(n+1)$ so:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{(n+1)(n+2)}=\frac{n+1}{n+2}$$

(so we put $n \to n+1$ everywhere)

STEP 3: Proof: we start with the left-hand side =

$$\boxed{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}}+\frac{1}{(n+1)(n+2)}=$$

ind. supposing ($A(n)$ is true)

$$=\boxed{\frac{n}{n+1}}+\frac{1}{(n+1)(n+2)}=\frac{n(n+2)+1}{(n+1)(n+2)}=$$

$$=\frac{n^2+2n+1}{(n+1)(n+2)}=\frac{(n+1)^{\cancel{2}}}{\cancel{(n+1)}(n+2)}=\frac{n+1}{n+2}\ \checkmark$$

So by induction $A(n)$ is true for all $n\in\mathbb{N}^+$.

Remark: observe that here is given what is the sum of numbers $\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \dots, \frac{1}{n(n+1)}$, it's $\frac{n}{n+1}$. For induction we need the statement.

But how can we find this result?

Idea: $\frac{1}{k(k+1)} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

So: $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) =$

$$= \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

we call this a telescopic sum.

This is another proof of course.

$\boxed{\text{INEQUALITIES}}$

30° Prove that:

$$\boxed{A(n):\ 2\sqrt{n+1} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1 \quad \text{For all } n \in \mathbb{N}^+}$$

So

$$A(n):\ 2\sqrt{n+1}-2 < \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}} \le 2\sqrt{n}-1$$

For: $n = 1, 2, 3, \dots$

$\boxed{\text{STEP 1}}$: For $\boxed{n=1} \Rightarrow$

$$2\sqrt{2}-2 < \frac{1}{\sqrt{1}} \le 2\cdot\sqrt{1}-1 \iff$$

$$2\sqrt{2}-2 < 1 \le 1 \checkmark$$

$\iff$

$$2\sqrt{2} < 3 \iff 8 < 9 \checkmark$$

$\boxed{\text{STEP 2}}$ Suppose $A(n)$ is true for some $n \in \mathbb{N}^+$. We have to prove that this implies that $A(n+1)$ is true as well, so: $n \to n+1$

$$2\sqrt{n+2}-2 \overset{(a)}{<} \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n+1}} \overset{(b)}{\le} 2\sqrt{n+1}-1.$$

We do separately parts (a) and (b)

(a) The sum =

$$\boxed{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}+\frac{1}{\sqrt{n+1}} > \boxed{2\sqrt{n+1}-2}+\frac{1}{\sqrt{n+1}}$$

inductional supposing part (a)

So is enough to prove that

$$2\sqrt{n+1}-\cancel{2}+\frac{1}{\sqrt{n+1}} \geq 2\sqrt{n+2}-\cancel{2}$$

$$\overset{()^2}{\Longleftrightarrow} \quad 4(n+1)+4+\frac{1}{n+1} \geq 4(n+2)$$

$$\Longleftrightarrow \quad 4(n+1)^2+4(n+1)+1 \geq 4(n+2)(n+1)$$

$$\cancel{4n^2}+\cancel{8n}+4+\cancel{4n}+5 \geq \cancel{4n^2}+\cancel{12n}+8$$

$$9 \geq 8 \quad \checkmark$$

So by induction (a) is true for all $n \in \mathbb{N}^+$.

Part (b)

The sum $= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n+1}} =$

$= \boxed{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}} + \frac{1}{\sqrt{n+1}} \leq \boxed{2\sqrt{n}-1} + \frac{1}{\sqrt{n+1}} \leq$

(b) ind. supp.

is enough to show that

$\leq 2\sqrt{n+1} - 1$

$\boxed{\text{So}}$:

$2\sqrt{n} - \cancel{1} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1} - \cancel{1} \quad \Big| \cdot \sqrt{n+1}$

$2\sqrt{n(n+1)} + 1 \leq 2(n+1) \iff$

$\iff \underbrace{2\sqrt{n^2+n}}_{+\geq 0} \leq \underbrace{2n+1}_{+} \quad \Big| ()^2 \iff$

$\iff \cancel{4n^2} + \cancel{4n} \leq \cancel{4n^2} + \cancel{4n} + 1 \iff$

$0 \leq 1$ ✓ So (b) is also true for all $n \in \mathbb{N}^+$.

(4/b)

$$A(n)$$

4° $(2n)! < 2^{2n} \cdot (n!)^2 \quad (n \in \mathbb{N}^+)$

$\boxed{\text{STEP 1}}$: $\boxed{n=1}$ $\Longrightarrow$

$$(2\cdot 1)! < 2^2 \cdot (1!)^2 \iff$$

$$2 < 4 \checkmark$$

$\boxed{\text{STEP 2}}$ Suppose $A(n)$ is true for some $n \in \mathbb{N}^+$.

We need to conclude from this

$\Longrightarrow A(n+1)$:

$$(2n+2)! < 2^{2n+2} \cdot ((n+1)!)^2$$

Proof:

$\boxed{\text{STEP 3}}$ THE LEFT-HAND SIDE =

$$= (2n+2)! = (2n)!(2n+1)(2n+2) <$$

$$\underset{\text{ind. supp.}}{<} 2^{2n} \cdot (n!)^2 \cdot (2n+1)(2n+2) \underset{\substack{\text{is enough}\\\text{to prove}\\\text{that}}}{\leq} 2^{2n+2}((n+1)!)^2$$

This last inequality $:\Longleftrightarrow$

$$\cancel{4^n}\cdot\cancel{(n!)^2}\cdot(2n+1)\cdot 2\cancel{(n+1)} \le 4\cdot\cancel{4^n}\cdot\cancel{(n!)^2}(n+1)^{\cancel{2}}$$

(we divide by ⊕ numbers)

$$\Longleftrightarrow\quad 2(2n+1)\le 4(n+1) \Longleftrightarrow$$

$$2n+1\le 2n+2 \Longleftrightarrow 1\le 2 \checkmark$$

$\Longrightarrow$ by induction $A(n)$ is true for all $n\in\mathbb{N}^+$.

5° (4/c)

For all $2\le n\in\mathbb{N}$ (so $n=2,3,4,5,\dots$)

$$A(n):\ \frac{1}{2\sqrt{n}} < \prod_{k=1}^{n}\frac{2k-1}{2k} < \frac{1}{\sqrt{3n+1}}$$

Proof: By induction:

$\boxed{\text{STEP 1}}$ For $\boxed{n=2}$

$$\frac{1}{2\sqrt{2}} < \prod_{k=1}^{2}\frac{2k-1}{2k} < \frac{1}{\sqrt{7}}$$

So: $\frac{1}{\cancel{2}\sqrt{2}} < \frac{1}{\cancel{2}} \cdot \frac{3}{4} < \frac{1}{\sqrt{7}}$

$\Updownarrow \qquad\qquad \Updownarrow$

$4 < 3\sqrt{2} \qquad\qquad 3\sqrt{7} < 8$

$\Updownarrow \qquad\qquad \Updownarrow$

$16 < 18$ ✓ $\qquad\qquad 63 < 64$ ✓.

$\boxed{\text{STEP 2}}$ Suppose that for SOME $n \in \mathbb{N}$ $n \geq 2$ we have:

$$\frac{1}{2\sqrt{n}} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}.$$

We need to prove that this implies $\Longrightarrow$ for $(n+1)$ as well:

$$\frac{1}{2\sqrt{n+1}} \underset{a}{<} \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n+1}{2n+2} \underset{b}{<} \frac{1}{\sqrt{3n+4}}$$

$\boxed{\text{STEP 3}}$ Proof of $A(n) \Rightarrow A(n+1)$!

Part (a):

The product $=$

$$\underbrace{\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}}_{}\cdot\frac{2n+1}{2n+2} > \boxed{\frac{1}{2\sqrt{n}}}\cdot\frac{2n+1}{2n+2} \geq \frac{1}{2\sqrt{n+1}}$$

ind. supposing

$\oplus$ $\underline{\frac{2n+1}{2n+2} > 0}$

is enough now

If we prove that:

$$\frac{1}{\cancel{2}\sqrt{n+1}} \leq \frac{1}{\cancel{2}\sqrt{n}}\cdot\frac{2n+1}{2n+2}$$

then we are done.

$$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}\cdot\frac{2n+1}{2(n+1)} \quad \Big| \cdot\sqrt{n+1}$$

$$1 \leq \frac{1}{\sqrt{n}}\cdot\frac{2n+1}{2\cdot\sqrt{n+1}} \iff$$

$$2\cdot\sqrt{n^2+n} \leq 2n+1 \iff ()^2$$

$$+ \qquad\qquad +$$

$$\iff 4n^2+4n \leq 4n^2+4n+1 \iff$$

$$0 \leq 1 \;\checkmark$$

$\boxed{\text{Part (b)}}$. The product again $=$

$$\boxed{\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}}\cdot\frac{2n+1}{2n+2}\underset{\text{ind. supp}}{\leq}\boxed{\frac{1}{\sqrt{3n+1}}}\cdot\frac{2n+1}{2n+2}\leq$$

$\oplus\ \dfrac{2n+1}{2n+2}>0$

$$\leq (\text{enough}) \leq \frac{1}{\sqrt{3n+4}} \ (\Leftarrow \text{what we need})$$

$$\Longleftrightarrow \underbrace{\sqrt{3n+4}\cdot(2n+1)}_{+} \leq \underbrace{(2n+2)\cdot\sqrt{3n+1}}_{+} \quad |()^2$$

$$\Longleftrightarrow (3n+4)(2n+1)^2 \leq (2n+2)^2\cdot(3n+1)$$

$$\Longleftrightarrow (3n+4)(4n^2+4n+1) \leq (4n^2+8n+4)(3n+1)$$

$$\Longleftrightarrow \cancel{12n^3}+12n^2+3n+16n^2+16n+\cancel{4} \leq$$

$$\leq \cancel{12n^3}+4n^2+24n^2+8n+12n+\cancel{4}$$

$0 \leq 20n$ which is true if $n \geq 2$ as well. $\Longrightarrow$

So by induction $A(n)$ is true $\forall 2 \leq n \in \mathbb{N}$

4/d $\forall n = 1, 2, 3, 4, \ldots$

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$$

Proof: By induction:

STEP 1 $n=1$: $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 \Leftrightarrow$

$$\frac{6+4+3}{12} = \frac{13}{12} > 1 \checkmark$$

STEP 2 Suppose that for some $n \in \mathbb{N}^+$

we have

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$$

We need to prove using this

for $(n+1)$ so

$$\frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n+4} > 1.$$

STEP 3 The sum:

$$\boxed{\frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n+1}} + \frac{1}{3n+2} + \frac{1}{3n+3} + \frac{1}{3n+4} >$$

$$> 1 - \frac{1}{n+1} + \frac{1}{3n+2} + \frac{1}{3n+3} + \frac{1}{3n+4} \geq 1$$

inductional supposing

is enough to prove this

To finish we have to prove that:

$$\cancel{1-}\frac{1}{n+1}+\frac{1}{3n+2}+\frac{1}{3n+3}+\frac{1}{3n+4}\geq \cancel{1}$$

$$\frac{1}{3n+2}+\frac{1}{3n+4}\geq \frac{2}{3}\cdot\frac{1}{n+1}$$

$\cdot\,(3n+2)>0$
$\cdot\,(3n+4)>0$
$\cdot\,(n+1)>0$
$\cdot\,3$

$$3(n+1)(3n+4)+(3n+2)\cdot 3(n+1)\geq$$
$$\geq 2\cdot(3n+2)(3n+4)$$

$$\cancel{9n^2}+21n+12+\cancel{9n^2}+9n+\cancel{6n}+6\geq$$
$$\geq \cancel{18n^2}+\cancel{36n}+16$$

$18\geq 16$ ✓ $\Rightarrow$ By induction we are done.