1. Define the differentiability and the derivative of a function at a point

3.1. Differentiation of functions

3.1. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in intD_f$. f is differentiable at "a" $\overset{\text{df}}{\Leftrightarrow}$

$$\exists \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \in \mathbb{R}.$$

In this case $f'(a) := \lim_a \frac{f(x) - f(a)}{x - a}$. This number is called the derivative of f at the point "a".

Let us denote the set of functions that are differentiable at "a" by $D(a)$.

2. State the theorem about the connection between the differentiability and the continuity

3.4. Theorem $f \in D(a) \Rightarrow f \in C(a)$.

Proof. The difference $f(x) - f(a)$ tends to 0 since:

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a) \to f'(a) \cdot 0 = 0 \quad (x \to a)$$

3. State the derivatives of the following functions: c, ax + b, x n , e x , sin x

1. $f(x) := c$ where $c \in \mathbb{R}$ is fixed (the constant function). Then $\forall x \in \mathbb{R}$:

$$f'(x) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

2. $f(x) = ax + b$ where $a, b \in \mathbb{R}$ are fixed (the linear function). Then $\forall x \in \mathbb{R}$:

$$f'(x) = \lim_{h \to 0} \frac{a \cdot (x + h) + b - a \cdot x - b}{h} = \lim_{h \to 0} \frac{ax + ah + b - ax - b}{h} = \lim_{h \to 0} a = a,$$

especially $(x)' = 1$.

3. $f(x) = x^n$ where $n \in \mathbb{N}$ is fixed. Then – using the binomial theorem – $\forall x \in \mathbb{R}$:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \\ &= \lim_{h \to 0} \frac{\binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1} \cdot h + \binom{n}{2} \cdot x^{n-2} \cdot h^2 + \ldots + \binom{n}{n} \cdot h^n - x^n}{h} = \\ &= \lim_{h \to 0} \left(\binom{n}{1} \cdot x^{n-1} + \binom{n}{2} \cdot x^{n-2} \cdot h + \ldots + \binom{n}{n} \cdot h^{n-1} \right) = \\ &= \binom{n}{1} \cdot x^{n-1} = n \cdot x^{n-1}. \end{aligned}$$

4. $f(x) = e^x$ (the exponential function). Then $\forall x \in \mathbb{R}$:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h \to 0} e^x \cdot \frac{e^h - 1}{h} = \\ &= e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x. \end{aligned}$$

Here we have used the familiar limit $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$ (see: Practice).

5. $f(x) = \sin x$ (the sinus function). Then $\forall x \in \mathbb{R}$:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} = \\ &= \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h} - \sin x \cdot \frac{1 - \cos h}{h^2} \cdot h \right) = \cos x. \end{aligned}$$

Here we have used the familiar limits $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ (see: Practice).

6. $(\cos x)' = -\sin x$ $(x \in \mathbb{R})$ can be proved similarly.

4. State the derivative of the sum

1. Sum

3.5. Theorem *Let $f, g \in D(x)$. Then $f+g \in D(x)$ and*

$$(f+g)'(x) = f'(x) + g'(x).$$

Proof. It can be proved that $x \in intD_{f+g}$. To see the derivative of $f+g$ let us compute as follows:

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h} = \\ &= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} = \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f'(x)+g'(x).\end{aligned}$$

5. State the derivative of the product

3.6. Theorem *Let $f, g \in D(x)$. Then $fg \in D(x)$ and*

$$(fg)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x).$$

Proof. It can be proved that $x \in intD_{fg}$. To see the derivative of fg let us compute as follows:

$$\begin{aligned}(fg)'(x) &= \lim_{h\to 0} \frac{(fg)(x+h)-(fg)(x)}{h} = \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h} = \\ &= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x+h) + f(x)\cdot g(x+h) - f(x)\cdot g(x)}{h} = \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x)\cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \\ &= f'(x)g(x) + f(x)g'(x).\end{aligned}$$

6. State the derivative of the quotient

3. Quotient

3.7. Theorem *Let $f, g \in D(x)$, $g(x) \neq 0$. Then $\frac{f}{g} \in D(x)$ and*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}.$$

Proof. It can be proved that $x \in intD_{f/g}$. To see the derivative of $\frac{f}{g}$ let us compute as follows:

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \lim_{h\to 0} \frac{(\frac{f}{g})(x+h) - (\frac{f}{g})(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \\ &= \lim_{h\to 0} \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h))}{h\cdot g(x)\cdot g(x+h)} = \\ &= \lim_{h\to 0} \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x) + f(x)\cdot g(x) - f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)} = \\ &= \left[\lim_{h\to 0} \frac{1}{g(x)\cdot g(x+h)}\right]\cdot\left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\cdot g(x) - f(x)\cdot \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right] = \\ &= \frac{1}{g(x)\cdot g(x)}\cdot [f'(x)\cdot g(x) - f(x)\cdot g'(x)] = \\ &= \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}.\end{aligned}$$

Special case: the reciprocical: Apply the Quotient Rule with the constant function $f(x) = 1$. We obtain: $\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{(g(x))^2}$.

7. State the derivative of the composition (Chain Rule)

3.8. Theorem *Let $g \in \mathbb{R} \to \mathbb{R}$, $g \in D(x)$, $f \in \mathbb{R} \to \mathbb{R}$, $f \in D(g(x))$. Then $f \circ g \in D(x)$ and*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) .$$

8. State the derivative of the inverse (Inverse Rule)

5. Inverse Rule without proof

3.9. Theorem *Let $I \subseteq R$ be an open interval, $f : I \to \mathbb{R}$, $f \in D$, be an (strictly) increasing function. Furthermore suppose that $f'(x) \neq 0$ $(x \in I)$. Then $f^{-1} \in D(J)$ where $J = R_f$ (we know that R_f is an open interval) and*

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \quad (y \in J) .$$

9. State the derivatives of tan x, ln x, a x

Using the Differentiation Rules we can deduce the derivatives of some basic functions.

1. $f(x) := \operatorname{tg} x$ (the tangent function). Then – using the Quotient Rule – $\forall x \in D_{\operatorname{tg}} = \mathbb{R} \setminus \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$:

$$\operatorname{tg}' x = \left(\frac{\sin x}{\cos x}\right)' = \frac{\sin' x \cdot \cos x - \sin x \cdot \cos' x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x.$$

2. $g(x) := \ln x$ $(x > 0)$ (the natural logarithm function).
   Let $f(x) = e^x = \exp x$ $(x \in \mathbb{R})$. Then $f^{-1}(y) = \ln y$ $(y \in \mathbb{R}^+)$. All the assumptions of the Inverse Rule are satisfied, so:

$$\ln'(y) = (f^{-1})'(y) = \frac{1}{\exp'(\ln(y))} = \frac{1}{\exp(\ln(y))} = \frac{1}{y} \quad (y \in J = \mathbb{R}^+) .$$

   If "y" is exchanged for "x": $(\ln x)' = \frac{1}{x}$ $(x \in \mathbb{R}^+)$.

10. Define the arcsin function. State its derivative

4.11. Definition $\arcsin := \sin^{-1}_{|[-\frac{\pi}{2}, \frac{\pi}{2}]} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

4.12. Remark.

$$\arcsin y = x \Leftrightarrow x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \text{and} \quad \sin x = y .$$

The derivative of arcsin:
Let $f(x) = \sin x$ $(x \in I := \left(-\frac{\pi}{2}, \frac{\pi}{2}\right))$. In this case $f^{-1}(y) = \arcsin y$ $(y \in (-1 1))$. We can apply the theorem about the derivative of inverse function:

$$\arcsin' y = (f^{-1})'(y) = \frac{1}{\sin'(\arcsin y)} = \frac{1}{\cos(\arcsin y)} .$$

Since $\arcsin y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ it is obvious that $\cos(\arcsin y) > 0$, so we can continue as follows:

$$\arcsin' y = \frac{1}{\sqrt{1 - (\sin(\arcsin y))^2}} = \frac{1}{\sqrt{1 - y^2}} .$$

So $\arcsin' y = \frac{1}{\sqrt{1 - y^2}}$ $(y \in (-1, 1))$. After replacing "y" by "x":

$$\arcsin' x = \frac{1}{\sqrt{1 - x^2}} \quad (x \in (-1, 1)) .$$

11. Define the arctan function. State its derivative

4.13. Definition $\arccos := \cos^{-1}_{|[0,\pi]} : [-1,1] \to [0,\pi]$

$$\operatorname{arc\,tg} := \operatorname{tg}^{-1}_{|(-\frac{\pi}{2},\frac{\pi}{2})} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\operatorname{arc\,ctg} := \operatorname{ctg}^{-1}_{|(0,\pi)} : \mathbb{R} \to (0,\pi)$$

Their derivatives can be computed like the one of the arcsin. The results:

$$\arccos' x = -\frac{1}{\sqrt{1-x^2}} \quad (x \in (-1,1));$$

$$\operatorname{arc\,tg}' x = \frac{1}{1+x^2} \quad (x \in \mathbb{R});$$

$$\operatorname{arc\,ctg}' x = -\frac{1}{1+x^2} \quad (x \in \mathbb{R}).$$

12. Define the concepts of local minimum and of local maximum

4.1. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D_f$. We say that f has at "a"

1. local minimum $\overset{\text{df}}{\Leftrightarrow} \exists r > 0 \; \forall x \in B(a,r) \cap D_f : \; f(x) \geq f(a)$;
2. strict local minimum $\overset{\text{df}}{\Leftrightarrow} \exists r > 0 \; \forall x \in B(a,r) \cap D_f \setminus \{a\} : \; f(x) > f(a)$;
3. local maximum $\overset{\text{df}}{\Leftrightarrow} \exists r > 0 \; \forall x \in B(a,r) \cap D_f : \; f(x) \leq f(a)$;
4. strict local maximum $\overset{\text{df}}{\Leftrightarrow} \exists r > 0 \; \forall x \in B(a,r) \cap D_f \setminus \{a\} : \; f(x) < f(a)$;

Here "a" is the point of the local extremum and $f(a)$ is the local extreme value.

13. State the First Derivative Test (First Order Necessary Condition) for local extremum

4.2. Theorem *[First Derivative Test for local extremum]*
Let $f \in \mathbb{R} \to \mathbb{R}$, $f \in D(a)$ and suppose that f has a local extremum at a. Then $f'(a) = 0$.

Proof. Suppose indirectly that $f'(a) \neq 0$. Then either $f'(a) > 0$ or $f'(a) < 0$. Take for example the case $f'(a) > 0$ (the other case can be discussed similarly). By the definition of the derivative:

$$0 < f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

It follows from the definition of the limit that

$$\exists \delta > 0 \; \forall x \in (a-\delta, a+\delta) \setminus \{a\} : \quad \frac{f(x) - f(a)}{x - a} > \frac{f'(a)}{2} > 0.$$

Since $(a-\delta, a+\delta) \setminus \{a\} = (a-\delta, a) \cup (a, a+\delta)$ let us discuss two cases: $x < a$, $x > a$.

First let $x \in (a-\delta, a)$. In this case $x - a < 0$, so from the sign of fraction follows that $f(x) - f(a) < 0$ that is $f(x) < f(a)$.

Similarly, if $x \in (a, a+\delta)$ then $x - a > 0$, so – by the sign of the fraction – $f(x) - f(a) > 0$ that is $f(x) > f(a)$.

Since any neigbourhood of "a" contain both types of these points the function f has no extreme value at "a". $\square$

4.3. Remarks. 1. The reverse of the theorem is not true, see for example the function $f(x) = x^3$ $(x \in \mathbb{R})$ at $a = 0$.

2. If $f \in \mathbb{R} \to \mathbb{R}$, $f \in D$ then the points of local extrema are contained in the set of the roots of the equation $f'(x) = 0$. The roots of $f'(x) = 0$ are called critical points or stationary points.

14. State Rolle's Mean Value Theorem

4.4. Theorem *[Rolle]*
Let $f : [a, b] \to \mathbb{R}$, $f \in C$, $f \in D$. *Suppose that* $f(a) = f(b)$.
Then $\exists \xi \in (a, b) : \; f'(\xi) = 0$.

15. State Cauchy's Mean Value Theorem

4.5. Theorem *[Cauchy] Let* $f, g : [a, b] \to \mathbb{R}$, $f, g \in C$, $f, g \in D$. *Suppose that* $g'(x) \neq 0$ $(x \in (a, b))$. *Then*

$$\exists \xi \in (a, b) : \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

16. State Lagrange's Mean Value Theorem

4.6. Theorem *[Lagrange] Let* $f : [a, b] \to \mathbb{R}$, $f \in C$, $f \in D$. *Then*

$$\exists \xi \in (a, b) : \quad \frac{f(b) - f(a)}{b - a} = f'(\xi).$$

17. State the First Derivative Test for monotonicity

4.7. Theorem *[First Derivative Test for monotonity] Let* $I \subseteq \mathbb{R}$ *be an interval (of any type),* $f : I \to \mathbb{R}$, $f \in C$, $f \in D$. *Then*

1. *If* $\forall x \in intI : \; f'(x) > 0$ *then* f *is strictly increasing (on* I*).*
2. *If* $\forall x \in intI : \; f'(x) < 0$ *then* f *is strictly decreasing (on* I*).*
3. *If* $\forall x \in intI : \; f'(x) = 0$ *then* f *is constant (on* I*).*

18. State the theorem of L'Hospital's Rule

5.1. Theorem *[L'Hospital Rule] Let* $-\infty \leq a < b \leq +\infty$, $f, g : (a, b) \to \mathbb{R}$, $f, g \in D$, $g'(x) \neq 0$ $(x \in (a, b))$. *Suppose that*

either $\lim\limits_{a+0} f = \lim\limits_{a+0} g = 0$ *or* $\lim\limits_{a+0} f = \lim\limits_{a+0} g = +\infty$ *and that* $\exists \lim\limits_{a+0} \frac{f'}{g'}$.
Then

$$\lim_{a+0} \frac{f}{g} = \lim_{a+0} \frac{f'}{g'}.$$

19. Define the two times differentiability and the second derivative of a function at a point

5.2. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in intD_f$. We say that f is 2 times differentiable at "a" (its notation is: $f \in D^2(a)$) if

$$\exists r > 0 \; \forall x \in B(a, r) : \quad f \in D(x) \quad \text{and} \quad f' \in D(a).$$

In this case the number $f''(a) := (f')'(a)$ is called the second derivative of f at the point "a".

Similarly can be defined the 3., 4., ... derivatives with recursion. Their notations are:

$$f'''(a),\; f''''(a),\; \dots \quad \text{or} \quad f^{(3)}(a),\; f^{(4)}(a),\; \dots$$

Generally if f is k times differentable at "a" then we denote this fact by $f \in D^k(a)$ and the k-th order derivative by $f^{(k)}(a)$.

20. Define the Taylor-polynomial

5.4. Definition (Taylor polynomial) Let $f \in \mathbb{R} \to \mathbb{R}$, $f \in D^n(a)$. The polynomial

$$T_n(x) := f(a) + \frac{f'(a)}{1!} \cdot (x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n =$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \cdot (x-a)^k \qquad (x \in \mathbb{R})$$

is called the n-th Taylor-polynomial of f relative to the center a.

21. State the connection between the derivatives of a function and of its Taylor-polynomials.

5.5. Remarks. 1. It is obvious that the degree of T_n is at most n that is $T_n \in \mathcal{P}_n$.

2. Obviously $T_n(a) = f(a)$.

3. $T_n'(x) = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} \cdot k \cdot (x-a)^{k-1}$. Hence we have $T_n'(a) = f'(a)$.

4. Similarly – using mathematical induction – one can prove that $T_n^{(j)}(a) = f^{(j)}(a)$ $(j = 0, \ldots, n)$.

22. State the theorem about the Taylor-formula

5.6. Theorem *[Taylor's formula]*
Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, $f \in D^{n+1}$, $a \in I$. Then for every $x \in I \setminus \{a\}$ there exists a number ξ between a and x such that:

$$f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot (x-a)^{n+1}.$$

The right-hand side of this equation is called the Lagrangian remainder term.

23. Define the concepts of strict concavity (strictly concave up, strictly concave down)

5.8. Definition Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$.
(a) f is called to be concave up (or: convex) if

$$\forall x, y \in I, x < y \; \forall 0 < \lambda < 1: \quad f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y);$$

(b) f is called to be concave down (or: concave) if

$$\forall x, y \in I, x < y \; \forall 0 < \lambda < 1: \quad f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda) f(y).$$

The following theorem can be proved:

5.9. Theorem *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $f \in C$, $f \in D$. Then*
(a) f is concave up if and only if f' is strictly increasing;
(b) f is concave down if and only if f' is strictly decreasing.

Remark that the statement of the theorem could be the definition of concavity for such functions ($f : I \to \mathbb{R}$, $f \in C$, $f \in D$).
Using the First Derivative Test for the monotonity of f' we obtain

5.10. Theorem *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $f \in C$, $f \in D^2$. Then*
(a) if $\forall x \in \text{int} I:$ $f''(x) > 0$ then f is concave up;
(b) if $\forall x \in \text{int} I:$ $f''(x) < 0$ then f is concave down.

24. State the theorem about the characterization of strict concavity using the first derivative. What is the corollary of this theorem using second derivatives?

Remark that the statement of the theorem could be the definition of concavity for such functions ($f : I \to \mathbb{R}$, $f \in C$, $f \in D$).
Using the First Derivative Test for the monotonity of f' we obtain

5.10. Theorem *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $f \in C$, $f \in D^2$. Then*
(a) if $\forall x \in \text{int} I:$ $f''(x) > 0$ then f is concave up;
(b) if $\forall x \in \text{int} I:$ $f''(x) < 0$ then f is concave down.

On the graph of a differentiable function the points where the concavity changes are of special importance. These points will be called points of inflection.

25. Define the point of inflection

5.10. Theorem *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $f \in C$, $f \in D^2$. Then*
(a) if $\forall x \in intI: \; f''(x) > 0$ then f is concave up;
(b) if $\forall x \in intI: \; f''(x) < 0$ then f is concave down .

On the graph of a differentiable function the points where the concavity changes are of special importance. These points will be called points of inflection.

26. Define the antiderivative of a function

6.1. Definition Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, $F : I \to \mathbb{R}$. The function F is called to be an antiderivative of f if $F \in D$ and

$$\forall x \in I: \quad F'(x) = f(x).$$

27. State the theorem about the set of all antiderivatives of a function

6.3. Definition The set of all antiderivatives of the function f is called the indefinite integral of f and is denoted by: $\int f, \int f(x)\, dx$.

Remark that in the practice sometimes the individual antiderivatives are named also indefinite integral, for example:

$$\int 3x^2 \, dx = x^3 .$$

The indefinite integral in practice is written not with set notations but in the following way:

$$\int 3x^2 \, dx = x^3 + C.$$

The next question that we are concerned is: Which functions have antiderivatives? Later we will prove the following theorem:

6.4. Theorem *If $I \subseteq \mathbb{R}$ is an open interval and $f : I \to \mathbb{R}$ is a continuous function then f has antiderivative.*

28. State the 5 simple integration rules

6.5. Theorem *[sum] Let $I \subseteq \mathbb{R}$ be open interval, $f, g : I \to \mathbb{R}$. If f and g have antiderivatives, so does $f + g$. Moreover*

$$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx .$$

Proof.

$$(\int f + \int g)' = (\int f)' + (\int g)' = f + g .$$

□

6.6. Theorem *[scalar multiple] Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, $\lambda \in \mathbb{R}$. If f has antiderivative, so does λf, moreover*

$$\int \lambda \cdot f(x)\, dx = \lambda \cdot \int f(x)\, dx .$$

Proof.

$$(\lambda \cdot \int f)' = \lambda \cdot (\int f)' = \lambda \cdot f .$$

□

6.7. Theorem *[linear substitution] Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ and $F : I \to \mathbb{R}$ be an antiderivative of f. Furthermore let $a, \; b \in \mathbb{R}$, $a \neq 0$ and $J := \{x \in \mathbb{R} \mid ax + b \in I\}$. Then J is an open interval and*

$$\int f(ax+b)\, dx = \frac{F(ax+b)}{a} \qquad (x \in J).$$

Proof. Obviously J is an open interval. Moreover:

$$\left(\frac{F(ax+b)}{a}\right)' = \frac{1}{a} \cdot F'(ax+b) \cdot a = f(ax+b). \quad (x \in J) .$$

□

6.8. Theorem *[integrals of type $f^\alpha \cdot f'$] Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be continuous, $\alpha \in \mathbb{R}$. Suppose that the power $(f(x))^\alpha$ is defined for every $x \in I$. Then*

a) if $\alpha \neq -1$ then

$$\int (f(x))^\alpha \cdot f'(x)\, dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} \qquad (x \in I);$$

b) if $\alpha = -1$ then

$$\int (f(x))^{-1} \cdot f'(x)\, dx = \int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| \qquad (x \in I).$$

Proof.

a) If $\alpha \neq -1$ then:

$$\left(\frac{f^{\alpha+1}}{\alpha+1}\right)' = \frac{1}{\alpha+1} \cdot (\alpha+1) \cdot f^{\alpha} \cdot f' = f^{\alpha} \cdot f';$$

b) If $\alpha = -1$ then:

$$(\ln|f|)' = \frac{1}{f} \cdot f' = \frac{f'}{f}.$$

29. State the rule of Integration by Parts for antiderivatives

7.1. Theorem *[integration by parts] Let $I \subseteq \mathbb{R}$ be an open interval, $f, g : I \to \mathbb{R}$, $f, g, \in D$, f', $g' \in C$. Then*

$$\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx \qquad (x \in I)$$

Proof. Apply the product rule of derivative:

$$(f \cdot g - \int (f' \cdot g))' = (f \cdot g)' - (\int (f' \cdot g))' = f' \cdot g + f \cdot g' - f' \cdot g = f \cdot g'.$$

30. State the rule of Substitution Form I. for antiderivatives

7.2. Theorem *[Substitution, form I.] Let I, $J \subseteq \mathbb{R}$ be open intervals, $f : J \to \mathbb{R}$, $f \in C$, $g : I \to J$ $g \in D$, $g' \in C$. Then*

$$\int (f \circ g) \cdot g' = (\int f) \circ g.$$

Proof. Apply the Chain Rule:

$$((\int f) \circ g)' = ((\int f)' \circ g) \cdot g' = (f \circ g) \cdot g'.$$

31. State the rule of Substitution Form II. for antiderivatives.

7.4. Theorem *[Substitution, form II.] Let I, $J \subseteq \mathbb{R}$ be open intervals, $f : I \to \mathbb{R}$, $f \in C$, $g : J \to I$, $g \in D$, $g' \in C$. Then*

$$\int f = \left(\int (f \circ g) \cdot g'\right) \circ g^{-1}.$$

32. Define the partition of an interval.

8.2. Definition Let $P \in \mathcal{P}[a, b]$ Then the length of the longest subinterval is called the norm of the partition P:

$$||P|| := \max\{x_i - x_{i-1} \mid i = 1, \ldots, n\}.$$

It is obvious that for every $\delta > 0$ there exists a partition P „finer" than δ that is $||P|| < \delta$.

33. Define the lower and the upper sums. Explain the notations used in them

8.3. Definition Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \ldots, x_n\} \in \mathcal{P}[a, b]$. Let

$$m_i := \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}, \qquad M_i := \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} \qquad (i = 1, \ldots, n).$$

We introduce the following sums:

a) lower sum: $s(f, P) := \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1})$,

b) upper sum: $S(f, P) := \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1})$.

34. Define the lower and the upper integral

8.8. Definition A function $f : [a,b] \to \mathbb{R}$ is called to be integrable if it is bounded and $I_*(f) = I^*(f)$. This common value of the lower and upper integral is called the integral of f from a to b and is denoted by

$$\int_a^b f, \quad \int_a^b f(x)\, dx\,.$$

In this connection the number a is called the lower limit of the integral and the number b is called the upper limit of the integral.

35. Define the integrability and the integral of a function

8.9. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $[a,b] \subseteq D_f$. We say that f is integrable over the interval $[a,b]$ if the restricted function $f_{|[a,b]}$ is integrable. The integral of f from a to b is denoted by the previous way and is defined as

$$\int_a^b f := \int_a^b f(x)\, dx := \int_a^b f_{|[a,b]}\,.$$

The set of integrable functions over $[a,b]$ is denoted by $R[a,b]$.

From the definition it follows that in the case $f(x) \geq 0$ $(x \in [a,b])$ the geometrical meaning of the integral is the area of the planar region „under the graph of f" that is the area of the region

$$R := \left\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\, 0 \leq y \leq f(x)\right\}\,.$$

36. Give an example for bounded but non-integrable function

Let $f : [a,b] \to \mathbb{R}$ be the following function:

$$f(x) := \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \cap [a,b] \\ 0 & \text{if} \quad x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [a,b]. \end{cases}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,\, b]$. Since every subinterval $[x_{i-1},\, x_i]$ contains rational and irrational numbers too we have

$$m_i := \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\} = 0,$$

$$M_i := \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} = 1 \qquad (i = 1, \ldots, n)\,.$$

Thus

$$s(f,\, P) := \sum_{i=1}^n m_i \cdot (x_i - x_{i-1}) = 0, \qquad S(f,\, P) := \sum_{i=1}^n M_i \cdot (x_i - x_{i-1}) = b - a\,.$$

Consequently

$$I_*(f) = \sup_P s(f,\, P) = 0, \qquad I^*(f) = \inf\{S(f,\, P) = b - a\,.$$

They are not equal, so $f \notin R[a,b]$.

37. Define the "backward" integration

It is convenient and useful to extend the integration if its lower limit is greater than or equal to its upper limit.

8.13. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $f \in R[a,b]$. Then

$$\int_b^a f(x)\, dx := -\int_a^b f(x)\, dx\,.$$

8.14. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D_f$. Then

$$\int_a^a f(x)\, dx := 0\,.$$

So we have defined the definite integral $\int_a^b f(x)\, dx$ for any pair $a,\, b \in \mathbb{R}$. For any pair $a,\, b \in \mathbb{R}$ let us denote the set of functions for which the integral $\int_a^b f(x)\, dx$ exists (independently of $a < b$, $a = b$, $a > b$) by $R[a,b]$.

38. List the first 6 properties of the definite integral

8.15. Theorem *[Addition] Let* $a, b \in \mathbb{R}$, $f, g \in R[a,b]$. *Then* $f+g \in R[a,b]$ *and* $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

8.16. Theorem *[Constant Multiple] Let* $a, b \in \mathbb{R}$, $f \in R[a,b]$, $c \in \mathbb{R}$. *Then* $cf \in R[a,b]$ *and* $\int_a^b cf = c \cdot \int_a^b f$.

8.17. Theorem *[Interval Additivity] Let* $a, b, c \in \mathbb{R}$ *and put them in nondecreasing order:* $A \le B \le C$. *Then*

$$f \in R[A,C] \quad \Leftrightarrow \quad f \in R[A,B] \quad \text{and} \quad f \in R[B,C].$$

In this case:

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

8.18. Corollary. If $a < b$ and $f \in R[a,b]$ then for every $[c,d] \subseteq [a,b]$: $f \in R[c,d]$.

In the following theorems $a < b$ is assumed.

8.19. Theorem *[Monotonity] Let* $a, b \in \mathbb{R}$, $a < b$, $f, g \in R[a,b]$. *Suppose that* $f(x) \le g(x)$ $(x \in [a,b])$. *Then*

$$\int_a^b f \le \int_a^b g.$$

8.20. Theorem *[„Triangle" inequality] Let* $a, b \in \mathbb{R}$, $a < b$, $f \in R[a,b]$. *Then* $|f| \in R[a,b]$ *and*

$$\left| \int_a^b f \right| \le \int_a^b |f|.$$

8.21. Theorem *[Mean value Theorem]*
Let $a, b \in \mathbb{R}$, $a < b$, $f, g \in R[a,b]$, $g(x) \ge 0$ $(x \in [a,b])$. *Let*

$$m := \inf\{f(x) \mid a \le x \le b\}, \qquad M := \sup\{f(x) \mid a \le x \le b\}.$$

Then

$$m \cdot \int_a^b g \le \int_a^b fg \le M \cdot \int_a^b g.$$

Moreover if f *is continuous on* $[a,b]$ *then*

$$\exists \xi \in [a,b]: \qquad \int_a^b fg = f(\xi) \cdot \int_a^b g.$$

39. State the Mean Value Theorem for integral and state its important special case

4.4. Theorem *[Rolle]*
Let $f : [a,b] \to \mathbb{R}$, $f \in C$, $f \in D$. *Suppose that* $f(a) = f(b)$.
Then $\exists \xi \in (a,b): \; f'(\xi) = 0$.

40. State the theorem about the modification the function at a point

41. Define the Integral Function

9.4. Definition Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ a function and suppose that f is integrable over every closed and bounded subinterval of I (this is the case for example if f is continuous). Fix a point $a \in I$. The function

$$F : I \to \mathbb{R}, \quad F(x) := \int_a^x f(t)\, dt \qquad (x \in I)$$

is called the integral function of f (vanishing at a).
The property „vanishing at a" expresses the triviality $F(a) = 0$.

42. State the theorem about the continuity of the Integral Function

9.5. Theorem *[the continuity of the Integral Function] Using the notations of the previous definition:*

$F : I \to \mathbb{R}$ *is continuous.*

43. State the theorem about the differentiability of the Integral Function

9.6. Theorem *[the right-hand differentiability of the Integral Function]*

Using the notations of the definition of the Integral Function:

Suppose that $x \in I$ but x is not the right endpoint of I. Suppose that f is „continuous from the right" at x that is

$$\lim_{h \to 0+} f(x+h) = f(x) .$$

Then

$$F'_+(x) := \lim_{h \to 0+} \frac{F(x+h-F(x)}{h} = f(x) .$$

In words: F is differentiable from the right and its right-hand derivative equals $f(x)$.

9.7. Theorem *[the left-hand differentiability of the Integral Function] Using the notations of the definition of the Integral Function:*

Suppose that $x \in I$ but x is not the left endpoint of I. Suppose that f is „continuous from the left" at x that is

$$\lim_{h \to 0-} f(x+h) = f(x) .$$

Then

$$F'_-(x) := \lim_{h \to 0-} \frac{F(x+h-F(x)}{h} = f(x) .$$

In words: F is differentiable from the left and its left-hand derivative equals $f(x)$.

44. State the Fundamental Theorem of Calculus (the Newton-Leibniz Formula)

10.1. Theorem *[Newton-Leibniz's Formula] Let $[a, b] \subseteq \mathbb{R}$ be a closed bounded interval and $f \in R[a, b]$. Suppose that there exists a function $F : [a, b] \to \mathbb{R}$ for which:*

1. *F is continuous on $[a, b]$;*
2. *F is differentiable on (a, b);*
3. *$F'(x) = f(x) \quad (x \in (a, b))$.*

Then

$$\int_a^b f(x)\, dx = F(b) - F(a) .$$

45. State the rule of Integration by Parts for definite integrals

10.3. Theorem *[Integration by Parts] Let $I \subseteq \mathbb{R}$ be an open interval, $f, g : I \to \mathbb{R}$, $f, g \in D$, $f', g' \in C$. Then for every closed and bounded subinterval $[a, b] \subset I$ holds*

$$\int_a^b f(x) \cdot g'(x)\, dx = [f(x) \cdot g(x)]_a^b - \int_a^b f'(x) \cdot g(x)\, dx .$$

46. State the rule of Substitution Form I. for definite integral

10.5. Theorem *[Substitution, Form I.] Let I, $J \subseteq \mathbb{R}$ be open intervals, $g : I \to J$, $g \in D$, $g' \in C$. Further let $f : J \to \mathbb{R}$ be a continuous function. Then for every closed and bounded subinterval $[a, b] \subset I$ we have*

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u) du .$$

Remark that the integral on the right side may be "backward".

47. State the rule of Substitution Form II. for definite integral

10.6. Theorem *[Substitution, Form II.] Let $I, J \subseteq \mathbb{R}$ be open intervals, $g : I \to J$, $g \in D$, $g' \in C$, g is strictly monotone. Further let $f : J \to \mathbb{R}$ be a continuous function. Then for every closed and bounded subinterval $[a, b] \subset J$ we have*

$$\int_a^b f(x)\, dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t)) \cdot g'(t)\, dt\,.$$

Remark that the integral on the right side may be "backward".

48. Define the Improper Integral

11.1. Definition Let $-\infty \le a < b \le +\infty$ and I be the interval whose left-hand endpoint is a and the right-hand endpoint is b. (I can be any type of intervals.) Let $f : I \to \mathbb{R}$ be a function and suppose that f is integrable over every closed and bounded subinterval of I (this is the case for example if f is continuous). We say that the improper integral

$$\int_a^b f(x)\, dx$$

is convergent if for some $c \in (a, b)$ the following limits exist and are finite:

$$\lim_{t \to a+} \int_t^c f(x)\, dx\,, \qquad \lim_{s \to b-} \int_c^s f(x)\, dx\,.$$

In this case the value of the improper integral is

$$\int_a^b f(x)\, dx := \lim_{t \to a+} \int_t^c f(x)\, dx + \lim_{s \to b-} \int_c^s f(x)\, dx\,.$$

It can be proved that the convergence and the value of the improper integral are independent of c.

49. State the formula for area between two function graphs (with sketch) and the formula for arc length of a function graph (with sketch)

12.1. Theorem *[the area of an x-normal region] Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions and suppose that*

$$\forall x \in [a, b] : \quad f(x) \le g(x)\,.$$

Then the area of the region

$$\{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ f(x) \le y \le g(x)\} \subset \mathbb{R}^2$$

is equal to

$$\int_a^b (f(x) - g(x))\, dx\,.$$

Remark that the region in the theorem is called an x-normal region in $\mathbb{R}^2$.

12.2. Theorem *[arc length of a function graph] Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, $f \in D$, $f' \in C$. If $[a, b] \subset I$ then the arc length of the curve $\{(x, f(x)) \mid a \le x \le b\}$ is*

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx\,.$$

50. Sketch a solid of revolution. State (without proof) the formula for its volume and for its surface area

12.3. Theorem *[volume of a solid of revolution] Let $f : [a, b] \to \mathbb{R}$ be continuous function and suppose that $f(x) \ge 0$ ($x \in [a, b]$). Revolve its graph about the x-axis. Then the volume of the solid of revolution*

$$\{(x, y, z) \in \mathbb{R}^3 \mid a \le x \le b,\ y^2 + z^2 \le (f(x))^2\}$$

is equal to

$$\pi \cdot \int_a^b (f(x))^2\, dx\,.$$

12.4. Theorem *[area of the surface of revolution] Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, $f \in D$, $f' \in C$. Suppose that $f(x) \ge 0$ ($x \in [a, b]$). If $[a, b] \subset I$ then the area of the surface of revolution*

$$\{(x, y, z) \in \mathbb{R}^3 \mid a \le x \le b,\ y^2 + z^2 = (f(x))^2\}$$

is equal to

$$2\pi \cdot \int_a^b f(x) \cdot \sqrt{1 + (f'(x))^2}\, dx\,.$$

51. Define the space R_n and the following operations in it: addition, scalar multiple, scalar product

Why $\mathbb{R}^n$ is important in the multivariable analysis? „Multivariable" means, that a multivariable function has a finite number of real variables - say n variables. Thus its domain can be regarded as a collection of ordered n-tuples, and forms a subset of $\mathbb{R}^n$. The Reader can consider that how connects the case $n = 1$ to the one-variable analysis studied in the subjects Analysis-1 and Analysis-2.

We review shortly the most important properties of $\mathbb{R}^n$.

For a fixed $n \in \mathbb{N}$ $\mathbb{R}^n$ is the set of all possible ordered n-tuples whose terms (components) are in $\mathbb{R}$:

$$\mathbb{R}^n := \{x = (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}.$$

Notice that in the case $n = 2$ the notation (x, y) is often used instead of (x_1, x_2). Similarly in the case $n = 3$ the notation (x, y, z) may be used instead of (x_1, x_2, x_3).

Let $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

- Addition: $x + y := (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \quad (x, y \in \mathbb{R}^n)$;
- Scalar Multiplication: $\lambda x := (\lambda x_1, \lambda x_2, \ldots, \lambda x_n) \quad (x \in \mathbb{R}^n, \lambda \in \mathbb{R})$;
- Scalar Product $\langle x, y\rangle := x_1y_1 + x_2y_2 + \ldots + x_ny_n = \sum_{i=1}^{n} x_iy_i \quad (x, y \in \mathbb{R}^n)$;

52. Define the norm and the distance in Rn.

- Norm (length):

$$\|x\| := \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2} \quad (x \in \mathbb{R}^n);$$

We have defined the distance in $\mathbb{R}^n$ as follows

$$d(x, y) := \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \quad (x, y \in \mathbb{R}^n).$$

53. Define the concept of neighbourhood (environment) in Rn. Sketch the neighbourhoods in R2.

2.1. Definition The neighbourhood (or ball or environment) of the point $a \in \mathbb{R}^n$ with radius $r > 0$ is the set

$$B(a,r) := \{x \in \mathbb{R}^n \mid \|x - a\| < r\}.$$

In the case $n = 2$ the ball is the open circular disk

$$B(a, r) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - a_1)^2 + (x_2 - a_2)^2 < r^2\}.$$

54. Define: interior point, exterior point, boundary point.

2.5. Definition Let $\emptyset \neq H \subset \mathbb{R}$, $a \in \mathbb{R}^n$. Then

1. a is an interior point of H, if $\exists r > 0: \quad B(a,r) \subseteq H$.
2. a is an exterior point of H, if $\exists r > 0: B(a,r) \cap H = \emptyset$.
   In other words: $\exists r > 0: B(a,r) \subseteq \overline{H}$.
   Here $\overline{H}$ denotes the complement of H that is $\overline{H} = \mathbb{R} \setminus H$.
3. a is a boundary point of H, if
   $\forall r > 0: \quad B(a,r) \cap H \neq \emptyset$ and $B(a,r) \cap \overline{H} \neq \emptyset$.

55. Define: the interior of a set, the exterior of a set, the boundary of a set,

2.7. Definition 1. The set of the interior points of H is called the interior of H and is denoted by $\operatorname{int} H$. So

$$\operatorname{int} H := \{a \in \mathbb{R} \mid \exists r > 0: \quad B(a,r) \subseteq H\} \subseteq H.$$

2. The set of the exterior points of H is called the exterior of H and is denoted by $\operatorname{ext} H$. So

$$\operatorname{ext} H := \{a \in \mathbb{R} \mid \exists r > 0: \quad B(a,r) \subseteq \overline{H}\} \subseteq \overline{H}.$$

3. The set of the boundary points of H is called the boundary of H and is denoted by ∂H. So

$$\partial H := \{a \in \mathbb{R} \mid \forall r > 0: \quad B(a,r) \cap H \neq \emptyset \quad \text{and} \quad B(a,r) \cap \overline{H} \neq \emptyset\} \subset \mathbb{R}.$$

56. Define: open set, closed set.

2.9. Definition The set $H \cup \partial H$ is called the closure of H and is denoted by $\operatorname{clos} H$. So $\operatorname{clos} H := H \cup \partial H$.

It is obvious that $\overline{\operatorname{clos} H} = \operatorname{int} \overline{H}$ and $\overline{\operatorname{int} H} = \operatorname{clos} \overline{H}$. This is based on the simple fact that $\partial H = \partial \overline{H}$.

2.10. Definition Let $H \subseteq \mathbb{R}^n$. Then

1. H is called an open set $\overset{\text{df}}{\Leftrightarrow} \partial H \subseteq \overline{H}$.
2. H is called a closed set $\overset{\text{df}}{\Leftrightarrow} \partial H \subseteq H$.

2.11. Remarks.

1. H is open if and only if it does not contain any boundary point and it is closed if and only if it contains all of its boundary points.
2. $\emptyset$ and $\mathbb{R}^n$ are open and closed sets at the same time. There is no other set in $\mathbb{R}^n$ that is open and closed at the same time.
3. H is open $\Leftrightarrow \overline{H}$ is closed, H is closed $\Leftrightarrow \overline{H}$ is open.
4. H is open $\Leftrightarrow H \subseteq \operatorname{int} H \Leftrightarrow H = \operatorname{int} H$.
5. H is closed $\Leftrightarrow \operatorname{clos} H \subseteq H \Leftrightarrow H = \operatorname{clos} H$.

57. When is named a vector sequence to be bounded?

3.3. Definition The sequence $a^{(k)} \in \mathbb{R}^n$ $(k \in \mathbb{N})$ is called bounded if

$$\exists M > 0 \ \forall k \in \mathbb{N}: \quad \|a^{(k)}\| \leq M.$$

It is obvious that the vector sequence $(a^{(k)})$ is bounded if and only if the real number sequence $(\|a^{(k)}\|)$ is bounded. Moreover, using the inequalities (1.2) one can prove, that a vector sequence is bounded if and only if its every coordinate sequence is bounded. That is

$$(a^{(k)}) \text{ is bounded} \quad \Leftrightarrow \quad (a_i^{(k)}) \text{ is bounded} \quad (i = 1, \ldots, n).$$

58. When is named a vector sequence to be convergent, and what is the definition of its limit?

3.4. Definition The vector sequence $a : \mathbb{N} \to \mathbb{R}^n$ is called convergent if

$$\exists A \in \mathbb{R}^n \ \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall k \geq N: \quad a^{(k)} \in B(A, \varepsilon).$$

The definition can be written using inequalities as follows:

$$\exists A \in \mathbb{R}^n \ \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall k \geq N: \quad \|a^{(k)} - A\| < \varepsilon.$$

A vector sequence is called divergent if it is not convergent.

It can be proved (using the T_2-property of the neighbourhoods) that the vector A in the above definition is unique. It is called the limit (or limit vector) of the vector sequence $(a^{(k)})$, and it is denoted in one of the following ways:

$$\lim a = A, \quad \lim a^{(k)} = A, \quad \lim_{k \to \infty} a^{(k)} = A, \quad a^{(k)} \to A \quad (k \to \infty).$$

3.6. Theorem *Let $a^{(k)} \in \mathbb{R}^n$ $(k \in \mathbb{N})$ be a vector sequence and $A \in \mathbb{R}^n$ be a vector. Then*

$$\lim_{k \to \infty} a^{(k)} = A \quad \Longleftrightarrow \quad \lim_{k \to \infty} \|a^{(k)} - A\| = 0$$

59. Define the coordinate sequence. State the theorem about the coordinate-wise convergence of a vector sequence.

In the following theorem we reduce the convergence of a vector sequence back to the convergence of its coordinate sequences.

3.7. Theorem *Let $a^{(k)} \in \mathbb{R}^n$ $(k \in \mathbb{N})$ be a vector sequence and $A \in \mathbb{R}^n$ be a vector. Then*

$$\lim_{k \to \infty} a^{(k)} = A \quad \Longleftrightarrow \quad \lim_{k \to \infty} a_i^{(k)} = A_i \quad (i = 1, \ldots, n).$$

Proof. Applying the inequalities (1.2) for the vectors $a^{(k)} - A$ we obtain

$$|a_i^{(k)} - A_i| \leq \|a^{(k)} - A\| \leq \sum_{i=1}^{n} |a_i^{(k)} - A_i| \qquad (i = 1, \ldots, n),$$

60. State the theorem about the connection between the closeness of sets and the convergence

3.9. Theorem *Let $\emptyset \neq H \subseteq \mathbb{R}$. Then H is closed if and only if*

$$\forall a_k \in H \ (k \in \mathbb{N}) \text{ convergent sequence}: \ \lim_{k\to\infty} a_k \in H.$$

61. Define the compact set. What is the connection between the compactness and the closeness and boundedness of sets?

3.10. Definition Let $\emptyset \neq H \subseteq \mathbb{R}^n$. H is called a compact set if

$$\forall a_k \in H \ (k \in \mathbb{N}) \text{ sequence } \quad \exists (a_{k_m},\, m \in \mathbb{N}) \text{ subsequence}:$$

$$(a_{k_m},\, m \in \mathbb{N}) \text{ is convergent and } \lim_{m\to\infty} a_{k_m} \in H.$$

The $\emptyset$ is called to be compact by definition.

3.11. Theorem *Let $\emptyset \neq H \subseteq \mathbb{R}^n$. Then H is compact if and only if it is closed and bounded.*

3.12. Remark. The theorem is not valid in infinite dimensional normed spaces. Every compact set is closed and bounded but there exists a closed and bounded set, that is not compact (see: Functional Analysis).

62. Define the concept of coordinate-function of a function of type R*n* ! R*m*.

As in the one variable case the concept of limits expresses where tend the function values to if the variable tends to a certain point. The first problem is to discuss the points where the variable can tend. These points are the so called accumulation points of the domain of the function.

63. Define the accumulation point.

4.1. Definition (Accumulation point) Let $\emptyset \neq H \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}^n$. We say that a is an accumulation point of H if

$$\forall r > 0 : (B(a,r) \setminus \{a\}) \cap H \neq \emptyset.$$

The set of accumulation points of H is denoted by H' that is

$$H' := \{a \in \mathbb{R}^n \mid a \text{ is an accumulation point of } H\}.$$

The points of the set $H \setminus H'$ are called isolated points of H.

64. Define the limit of a function of type R*n* ! R*m*.

4.3. Definition Let $f \in \mathbb{R}^n \to \mathbb{R}^m$, $a \in D_f'$. We say that f has limit at the point a if

$$\exists A \in \mathbb{R}^m \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in (B(a,\delta) \setminus \{a\}) \cap D_f : \quad f(x) \in B(A,\varepsilon).$$

Using the T_2-property of neighbourhoods it can be proved that the vector A in this definition is unique. This unique A is called the limit of the function f at the point a. The notations are:

$$A = \lim_a f, \quad A = \lim_{x\to a} f(x), \quad f(x) \to A \ (x \to a).$$

65. The limit of the constant, of the identity and of the canonical projections.

1. (the constant function) Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = c$ where $c \in \mathbb{R}^m$ is a fixed vector. Then for any $a \in \mathbb{R}^n$: $\lim_{x\to a} c = c$, because for any $\varepsilon > 0$ any $\delta > 0$ is good:

$$\forall x \in \Big(B(a,\delta) \setminus \{a\}\Big) \cap D_f : \quad f(x) = c \in B(c,\varepsilon).$$

2. (identity function) Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) := x$. Let $a \in \mathbb{R}^n$. Then $\lim_{x\to a} x = a$, because for any $\varepsilon > 0$ let $\delta := \varepsilon$. It will be good, since

$$\forall x \in \Big(B(a,\delta) \setminus \{a\}\Big) \cap D_f : \quad f(x) = x \in B(a,\delta) = B(a,\varepsilon).$$

3. (canonical projections) Let $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) := x_i$ where $i \in \{1,\ldots,n\}$ is fixed and $x = (x_1,\ldots,x_n)$. Let $a = (a_1,\ldots,a_n) \in \mathbb{R}^n$. Then $\lim_{x\to a} f(x) = a_i$ because for any $\varepsilon > 0$ let $\delta := \varepsilon$. This is good since if $0 < \|x - a\| < \delta$, then by (1.2):

$$|f(x) - a_i| = |x_i - a_i| = \|(x-a)_i\| \leq \|x - a\| < \delta = \varepsilon.$$

66. State the theorem about the coordinate-wise limit of a function of type R*n* ! R*m*.

4.8. Theorem *[limit by coordinates] Suppose that $m \geq 2$ and let* $f \in \mathbb{R}^n \to \mathbb{R}^m$, $a \in D'_f$, $A = (A_1, \dots, A_m) \in \mathbb{R}^m$. *Then*

$$\lim_a f = A \quad \Leftrightarrow \quad \lim_a f_i = A_i \quad (i = 1, \dots, m).$$

4.9. Remark. With short notation our theorem is:

$$\lim_{x \to a} f(x) = \left(\lim_{x \to a} f_1(x), \dots, \lim_{x \to a} f_m(x) \right).$$

67. Define the continuity of a function of type R*n* ! R*m* at a point.

4.13. Definition Let $f \in \mathbb{R}^n \to \mathbb{R}^m$, $a \in D_f$. f is continuous at a if
$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in B(a, \delta) \cap D_f : \quad f(x) \in B(f(a), \varepsilon)$.
Let us denote the set of functions that are continuous at a by $C(a)$.

From the definition it follows immediately that

- if a is an isolated point of D_f then f is continuous at a.
- if a is an accumulation point of D_f then

$$f \quad \text{is continuous at } a \quad \Leftrightarrow \quad \lim_{x \to a} f(x) = f(a).$$

68. When is called a function of type R*n* ! R*m* to be continuous?

4.14. Definition Let $f \in \mathbb{R}^n \to \mathbb{R}^m$. Then f is continuous if it is continuous at every point of its domain, that is

$$\forall a \in D_f : \quad f \in C(a).$$

Using the results of Examples 4.5, the constant function, the identity function and the canonical projections are continuous.

69. State the two theorems in connection with continuous functions defined on compact sets in R*n*(compactness of the image, Weierstrass-minimax).

4.16. Theorem *[the compactness of the image]*
Let $f \in \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function and suppose that D_f is compact. Then R_f is compact.

4.18. Theorem *[the minimax theorem of Weierstrass] Let $f \in \mathbb{R}^n \to \mathbb{R}$ be a continuous function and D_f be compact. Then $\exists \min f$ and $\exists \max f$.*

70. Define the differentiability and the derivative of a function type R*n* ! R*m* at a point.

5.1. Definition Let $f \in \mathbb{R}^n \to \mathbb{R}^m$, $a \in \operatorname{int} D_f$. We say that f is differentiable at the point a (denoted by $f \in D(a)$) if

$$\exists A \in \mathbb{R}^{m \times n} : \quad \lim_{h \to 0} \frac{f(a+h) - f(a) - A \cdot h}{\|h\|} = 0.$$

71. State the theorem about the connection between the differentiability and the continuity for functions of type R*n* ! R*m*

5.5. Theorem *If $f \in D(a)$ then $f \in C(a)$.*

Proof.

$$f(a+h) - f(a) = \frac{f(a+h) - f(a) - f'(a) \cdot h}{||h||} \cdot ||h|| + f'(a) \cdot h \to 0 \quad (h \to 0).$$

So $\lim_{h \to 0} f(a+h) = f(a)$ which implies $f \in C(a)$.

72. Define the (first order) partial derivatives of a function of type R*n* ! R*m* at a point

73. What are the columns of the derivative matrix of a function type R*n* ! R*m*.

6.1. Theorem *Let $f \in \mathbb{R}^n \to \mathbb{R}^m$, $f \in D(a)$, $j \in \{1, \dots, n\}$. Then $\exists \partial_j f(a)$ and it is identical with the j-th column of $f'(a)$.*

Proof. $f \in D(a)$ implies that $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a) - f'(a) \cdot h}{\|h\|} = 0$. Apply this relation with the vectors $h = t \cdot e_j$ where e_j is the j-th standard unit vector, and $t \in \mathbb{R}$, $t \to 0$. Then

$$\begin{aligned}
0 &= \lim_{t \to 0} \frac{f(a + t \cdot e_j) - f(a) - f'(a) \cdot t \cdot e_j}{\|t \cdot e_j\|} = \\
&= \lim_{t \to 0} \frac{f(a + t \cdot e_j) - f(a) - f'(a) e_j \cdot t}{|t| \cdot \|e_j\|} = \\
&= \lim_{t \to 0} \frac{f(a + t \cdot e_j) - f(a) - (\text{the } j\text{-th column of } f'(a)) \cdot t}{|t|}
\end{aligned}$$

This means – by the definition of the derivative – that $F \in D(0)$, and that

$$\partial_j f(a) = F'(0) = \text{ the } j\text{-th column of } f'(a)\,.$$

74. What are the rows of the derivative matrix of a function type R*n* ! R*m*.

6.2. Theorem *Let $f \in \mathbb{R}^n \to \mathbb{R}^m$, $f = (f_1, \dots, f_m)$ and $a \in \operatorname{int} D_f$. Then*

$$f \in D(a) \quad \Leftrightarrow \quad f_i \in D(a) \ (i = 1, \dots, m)\,.$$

In this case:

$$f_i'(a) = \text{ the } i\text{-th row of } f'(a) \quad (i = 1, \dots, m)\,.$$

Proof. Let $A \in \mathbb{R}^{m \times n}$. Then the fact

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - A \cdot h}{\|h\|} = 0$$

is equivalent with

$$\lim_{h \to 0} \frac{f_i(a+h) - f_i(a) - (Ah)_i}{\|h\|} = 0$$

(see: limit by coordinates). But $(Ah)_i =$ (the i-th row of A) $\cdot\, h$, so the above relation is equivalent with

$$\lim_{h \to 0} \frac{f_i(a+h) - f_i(a) - (\text{the } i\text{-th row of } A) \cdot h}{\|h\|} = 0\,.$$

Using these equivalencies in both directions, the statement of the theorem follows immediately. □

75. What are the entries of the derivative matrix for functions of type R*n* ! R*m* (*j*-th entry in the *i*-th row?).

Using the two previous theorems we obtain:

$$(f'(a))_{ij} = \text{ the } j\text{-th column of the } i\text{-th row } = \partial_j f_i(a)\,.$$

76. Define the concepts of local minimum and of local maximum for functions of type R*n* ! R

8.1. Definition Let $f \in \mathbb{R}^n \to \mathbb{R}$, $a \in D_f$. We say that f has at a

1. local minimum $\overset{\text{df}}{\Leftrightarrow} \exists r > 0 \ \forall x \in B(a,r) \cap D_f : \ f(x) \geq f(a)$;
2. strict local minimum $\overset{\text{df}}{\Leftrightarrow} \exists r > 0 \ \forall x \in B(a,r) \cap D_f \setminus \{a\} : \ f(x) > f(a)$;
3. local maximum $\overset{\text{df}}{\Leftrightarrow} \exists r > 0 \ \forall x \in B(a,r) \cap D_f : \ f(x) \leq f(a)$;
4. strict local maximum $\overset{\text{df}}{\Leftrightarrow} \exists r > 0 \ \forall x \in B(a,r) \cap D_f \setminus \{a\} : \ f(x) < f(a)$;

Here a is the place of the local extremum and $f(a)$ is the local extreme value.

77. State the First Derivative Test (First Order Necessary Condition) of local extremum for functions of type R*n* ! R

8.2. Theorem *[First Derivative Test] Let $f \in \mathbb{R}^n \to \mathbb{R}$, $f \in D(a)$ and suppose that f has a local extremum at a.*

Then $f'(a) = 0$ (or: $\nabla f(a) = 0$).

78. Define the quadratic form.

8.4. Definition Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, that is a symmetric 2-array. The function

$$Q : \mathbb{R}^n \to \mathbb{R}, \qquad Q(x) := Ax^2 = \sum_{i,j=1}^{n} a_{ij} \cdot x_i \cdot x_j$$

is called quadratic form determined by the symmetric matrix A. A is called the matrix of Q.

79. Classification of quadratic forms: when is called a quadratic form to be positive definite, negative definite, indefinite?

8.8. Definition Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form represented by the symmetric matrix $A \in \mathbb{R}^{n\times n}$. We say that Q is

(a) positive definite if $\forall x \in \mathbb{R}^n \setminus \{0\} : \quad Q(x) > 0,$

(b) negative definite if $\forall x \in \mathbb{R}^n \setminus \{0\} : \quad Q(x) < 0,$

(c) positive semidefinite if $\forall x \in \mathbb{R}^n : \quad Q(x) \geq 0,$

(d) negative semidefinite if $\forall x \in \mathbb{R}^n : \quad Q(x) \leq 0,$

(e) indefinite, if $\exists x, y \in \mathbb{R}^n : \quad Q(x) > 0, \ Q(y) < 0.$

80. How can we classify a two-variable quadratic form?

8.10. Theorem *[classification of the 2-variable quadratic forms]*

Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{R}^{2\times 2}$ *and* $Q : \mathbb{R}^n \to \mathbb{R}$ *be the quadratic form given by A, that is*

$$Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 \qquad (x = (x_1, x_2) \in \mathbb{R}^2)\,.$$

Then Q is

- *positive definite if* $\det A = ac - b^2 > 0$ *and* $a > 0$,
- *negative definite if* $\det A = ac - b^2 > 0$ *and* $a < 0$.
  (The case $\det A = ac - b^2 > 0$ *and* $a = 0$ *is impossible.)*
- *indefinite if* $\det A = ac - b^2 < 0$.
- *semidefinite but not definite if* $\det A = ac - b^2 = 0$.

The semidefinite case is in detail as follows. Suppose that $\det A = ac - b^2 = 0$. *Then Q is*

- *positive semidefinite but not positive definite if* $a > 0$ *or if* $a = 0, c > 0$,
- *negative semidefinite but not negative definite if* $a < 0$ *or if* $a = 0, c < 0$,
- *the identical 0-function if* $a = c = 0$.

81. Define the two times differentiability and the second derivative (Hesse matrix) of a function type Rn ! R at a point.

7.1. Definition Let $f \in \mathbb{R}^n \to \mathbb{R}$, $a \in \operatorname{int} D_f$. We say that f is 2 times differentiable at a (its notation is: $f \in D^2(a)$) if

$$\exists r > 0 \ \forall x \in B(a,r) : \quad f \in D(x) \quad \text{and} \quad f' \in D(a)\,.$$

The derivative function f' can be regarded as a vector valued function with the coordinate functions $\partial_j f$ $(j = 1, \ldots, n)$. So the equivalent definition of the 2 times differentiability can be given as follows.

7.2. Definition Let $f \in \mathbb{R}^n \to \mathbb{R}$, $a \in \operatorname{int} D_f$. We say that f is 2 times differentiable at a if

$$\exists r > 0 \ \forall x \in B(a,r) : \quad f \in D(x) \quad \text{and} \quad \partial_j f \in D(a) \quad (j = 1, \ldots, n)\,.$$

Suppose that $f \in D^2(a)$. Since its derivative function is

$$f' = (\partial_1 f, \partial_2 f, \ldots, \partial_n f) : \mathbb{R}^n \to \mathbb{R}^n\,,$$

its second derivative is the derivative matrix of f' at a:

$$f''(a) = (f')'(a) = \begin{bmatrix} \partial_1\partial_1 f(a) & \partial_2\partial_1 f(a) & \ldots & \partial_n\partial_1 f(a) \\ \partial_1\partial_2 f(a) & \partial_2\partial_2 f(a) & \ldots & \partial_n\partial_2 f(a) \\ \vdots & & & \vdots \\ \partial_1\partial_n f(a) & \partial_2\partial_n f(a) & \ldots & \partial_n\partial_n f(a) \end{bmatrix} \in \mathbb{R}^{n\times n}\,.$$

This matrix is called the Hesse-matrix of f at a. The ij-th entry of the Hesse-matrix is

$$(f''(a))_{ij} = \partial_j\partial_i f(a) \quad (i, j = 1, \ldots, n)\,.$$

7.3. Definition The entries of the Hesse-matrix $f''(a)$ are called the second order partial derivatives of f at a.

82. State the Young theorem.

7.5. Theorem *[Theorem of Young] If $f \in D^2(a)$, then the Hesse-matrix $f''(a)$ is symmetric that is*

$$\partial_j \partial_i f(a) = \partial_i \partial_j f(a) \quad (i, j = 1, \ldots, n).$$

83. State the theorem about the second order conditions of local extrema.

9.1. Theorem *[the definite case] Let $f \in \mathbb{R}^n \to \mathbb{R}$, $f \in C^2(a)$, $f'(a) = 0$. Then*

(a) If $f''(a)$ is positive definite then f attains local minimum at a;

(b) If $f''(a)$ is negative definite then f attains local maximum at a;

9.2. Theorem *[the indefinite case] Let $f \in \mathbb{R}^n \to \mathbb{R}$, $f \in C^2(a)$, $f'(a) = 0$. If $f''(a)$ is indefinite then f has no local extremum at a.*

9.3. Corollary. (Second Order Necessary Condition) Let $f \in \mathbb{R}^n \to \mathbb{R}$, $f \in C^2(a)$, $f'(a) = 0$. If f has local extremum at a then $f''(a)$ is semidefinite.

84. Define the intervals in R2. Give by a picture the partition of the interval [*a; b*] × [*c; d*]

85. Define the lower and the upper sums for functions *f* : [*a; b*] × [*c; d*] ! R. Explain the notations used in them

10.3. Definition Let $I \subset \mathbb{R}^n$ be an interval and $f : I \to \mathbb{R}$ be a bounded function and $P \in \mathcal{P}(I)$. Let

$$m_J := \inf\{f(x) \mid x \in J\}, \qquad M_J := \sup\{f(x) \mid x \in J\} \qquad (J \in P).$$

We introduce the following sums:

a) lower sum: $s(f, P) := \sum_{J \in P} m_J \cdot \mu(J)$,

b) upper sum: $S(f, P) := \sum_{J \in P} M_J \cdot \mu(J)$.

86. Define the lower and the upper integral for functions *f* : [*a; b*] × [*c; d*] ! R

10.6. Definition The number $I_*(f) := \sup\{s(f, P) \mid P \in \mathcal{P}(I)\}$ is called the lower integral of f. Respectively the number $I^*(f) := \inf\{S(f, P) \mid P \in \mathcal{P}(I)\}$ is called the upper integral of f.

87. Define the integrability and the integral for functions *f* : [*a; b*] × [*c; d*] ! R

10.7. Definition A function $f : I \to \mathbb{R}$ is called to be Riemann-integrable if it is bounded and $I_*(f) = I^*(f)$. This common value of the lower and upper integral is called the Riemann-integral of f.

We will use simply „integrable" and „integral" instead of „Riemann-integrable" and „Riemann-integral" respectively since no other integral concept occurs in our subject.

88. Write the formula for computation of the double integral for functions *f* : [*a; b*]×[*c; d*] ! R

Let $I = [a, b] \times [c, d] \subset \mathbb{R}^2$ be an interval and the function f be continuous on I. Then

$$\iint_I f(x, y)\, d(x, y) = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

89. Define the separable case of the double integral, and give the formula for computation of the integral in separable case

In the separable case let $f(x,y) = g(x) \cdot h(y)$ with continuous g and h. Then

$$\iint_I f(x,y)\,d(x,y) = \iint_I g(x)h(y)\,d(x,y) =$$

$$= \int_a^b \int_c^d g(x)h(y)\,dy\,dx = \int_a^b g(x) \cdot \left(\int_c^d h(y)\,dy\right)\,dx =$$

$$= \left(\int_c^d h(y)\,dy\right) \cdot \int_a^b g(x)\,dx = \left(\int_a^b g(x)\,dx\right) \cdot \left(\int_c^d h(y)\,dy\right).$$

90. Plot an *x*-normal region in R2, and give the formula for computation the double integral over *x*-normal regions

Let $T = [a,b] \subset \mathbb{R}$ be a closed bounded interval, $\varphi, \psi : [a,b] \to \mathbb{R}$ be continuous functions with $\varphi(u) \le \psi(u)$ $(u \in [a,b])$. Then the x-normal region is:

$$H = \{(x,y) \in \mathbb{R}^2 \mid x \in [a,b],\ \varphi(x) \le y \le \psi(x)\} \subset \mathbb{R}^2,$$

and for every function $f \in \mathbb{R}^2 \to \mathbb{R}$ that is continuous on H holds

$$\iint_H f(x,y)\,d(x,y) = \int_a^b \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\,dx.$$

Furthermore the y-normal region is:

$$H = \{(x,y) \in \mathbb{R}^2 \mid y \in [a,b],\ \varphi(y) \le x \le \psi(y)\} \subset \mathbb{R}^2,$$

and for every function $f \in \mathbb{R}^2 \to \mathbb{R}$ that is continuous on H holds

$$\iint_H f(x,y)\,d(x,y) = \int_a^b \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx\,dy.$$

91. Plot a *y*-normal region in R2, and give the formula for computation the double integral over *y*-normal regions

92. Define the polar coordinates in the plane

$$\Phi'(r,\varphi) = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r\cdot(\cos^2\varphi + \sin^2\varphi) = r.$$

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \Phi(r,\varphi) := (r\cos\varphi,\ r\sin\varphi) \qquad ((r,\varphi) \in \mathbb{R}^2).$$

Applying the integral transformation formula we obtain:

$$\iint_H f(x,y)\,d(x,y) = \iint_T f(r\cos\varphi, r\sin\varphi)\cdot r\,d(r,\varphi)$$

where $H = \Phi[T]$.

93. Give the formula of the Polar Transformation of Double Integrals

Applying the integral transformation formula we obtain:

$$\iint_H f(x,y)\,d(x,y) = \iint_T f(r\cos\varphi, r\sin\varphi)\cdot r\,d(r,\varphi)$$

where $H = \Phi[T]$.

94. Define the Cylindrical coordinates in the space (cylindrical coordinates)

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \Phi(r,\varphi,h) := (r\cos\varphi,\ r\sin\varphi,\ h) \qquad ((r,\varphi,h) \in \mathbb{R}^3).$$

$$\Phi'(r,\varphi,h) = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\cdot\begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\,.$$

95. Give the formula of the Cylindrical Transformation of Triple Integrals (triple integrals in cylindrical coordinates)

Applying the integral transformation formula we obtain:

$$\iiint_H f(x,y,z)\; d(x,y,z) = \iiint_T f(r\cos\varphi,\; r\sin\varphi,\; h)\cdot r\; d(r,\varphi,h)$$

where $H = \Phi[T]$.

96. Define the polar coordinates in the space (spherical coordinates)

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \Phi(r,\;\varphi,\;\vartheta) := (r\sin\vartheta\cos\varphi,\; r\sin\vartheta\sin\varphi,\; r\cos\vartheta) \qquad ((r,\varphi,\vartheta)\in\mathbb{R}^3).$$

$$\Phi'(r,\varphi,\vartheta) = \begin{vmatrix} \sin\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi & r\cos\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi & r\sin\vartheta\cos\varphi & r\cos\vartheta\sin\varphi \\ \cos\vartheta & 0 & -r\sin\vartheta \end{vmatrix} =$$

$$= (\cos\vartheta)\cdot\begin{vmatrix} -r\sin\vartheta\sin\varphi & r\cos\vartheta\cos\varphi \\ r\sin\vartheta\cos\varphi & r\cos\vartheta\sin\varphi \end{vmatrix} + (-r\sin\vartheta)\cdot\begin{vmatrix} \sin\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \end{vmatrix} =$$

$$= (\cos\vartheta)\cdot(-r^2\sin^2\varphi\sin\vartheta\cos\vartheta - r^2\cos^2\varphi\sin\vartheta\cos\vartheta) -$$

$$- (r\sin\vartheta)\cdot(r\sin^2\vartheta\cos^2\varphi + r\sin^2\vartheta\sin^2\varphi) =$$

$$= -(\cos\vartheta)\cdot r^2\sin\vartheta\cos\vartheta - (r\sin\vartheta)\cdot r\sin^2\vartheta =$$

$$= -r^2\sin\vartheta\cos^2\vartheta - r^2\sin\vartheta\sin^2\vartheta = -r^2\sin\vartheta\,.$$

97. Give the formula of the Polar Transformation of Triple Integrals (triple integrals in spherical coordinates)

Applying the integral transformation formula we obtain:

$$\iiint_H f(x,y,z)\; d(x,y,z) = \iiint_H f(r\sin\vartheta\cos\varphi,\; r\sin\vartheta\sin\varphi,\; r\cos\vartheta)\cdot r^2\sin\vartheta\; d(r,\varphi,\vartheta)$$

where $H = \Phi[T]$.