MATH NOTES

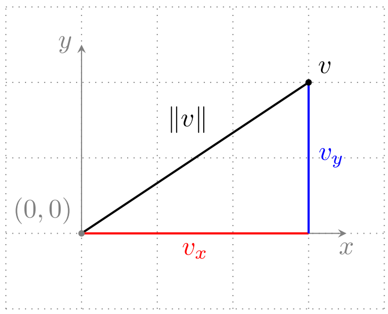
8DCC

Contents

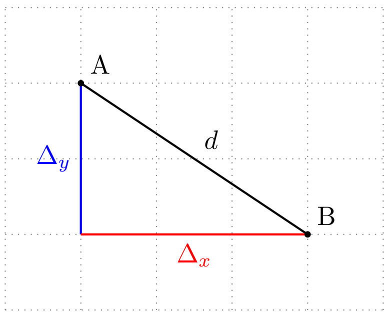
1. Geometry

1.1. Magnitude of a vector. The magnitude of a vector is the length of the vector, and it's denoted as $\|v\|$. The formula for calculating the magnitude of a two-dimensional vector is the following.

$$\|v\| = \sqrt{v_x^2 + v_y^2}$$



1.2. Distance between two points. The distance between two points is the hypotenuse of a right triangle whose two cathetus are the difference between the x and y coordinates of the two points.

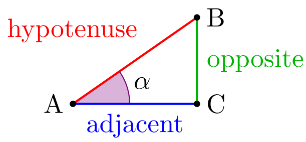


$$d = \sqrt{(B_x - A_x)^2 + (B_y - A_y)^2}$$

1.3. Unit vector. A unit vector is a vector of length 1, and it's usually denoted as u or $\hat{u}$. The normalized or unitary vector $\hat{u}$ of a vector v is a vector of length 1 with the direction of v. The following formula can be used for normalizing a vector.

$$\hat{u} = \frac{v}{\|v\|}$$

1.4. Sine and cosine. Given the following right triangle, containing the acute angle α:

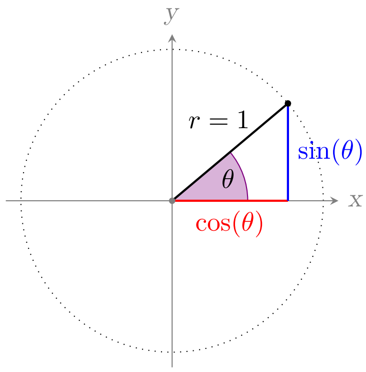


The sine and cosine of the angle can be calculated with the following formulas:

$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Alternatively, the following definition uses a **unit circle** to visualize the sine and cosine more clearly. A unit circle is a circle of radius one centered at the origin (0, 0) in the cartesian coordinate system.

By tracing a line from the origin to a point in this circle, an angle θ is formed with the positive x axis. The x and y coordinates of this point are equal to $\cos\theta$ and $\sin\theta$, respectively.



Since the radius of the circle (i.e. the hypotenuse of the formed right triangle) is one, the previous formula remains consistent:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{opposite}}{1} = \text{opposite}$$

1.5. **Dot product.** The dot product or scalar product takes two vectors and returns a scalar that represents the projection of one vector onto the other. In simpler terms, it's a way of quantifying how aligned is vector a with vector b.

The basic formula is the following:

$$a \cdot b = a_x b_x + a_y b_y$$

The dot product has a direct relationship with the angle formed by the two vectors. The dot product of two **unit vectors** is the cosine of the angle.

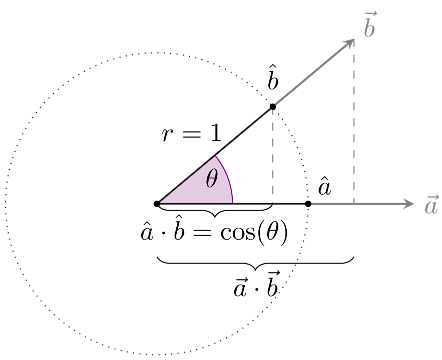
$$\hat{a} \cdot \hat{b} = \cos\theta$$

Therefore, if both vectors are **normalized** (i.e. they are unit vectors), the returned value will always be in the $[-1, 1]$ range.

To calculate the dot product of non-normalized vectors, this formula is used:

$$a \cdot b = \|a\| \|b\| \cos\theta$$

The dot product can be expressed as the shadow that a projects over b.

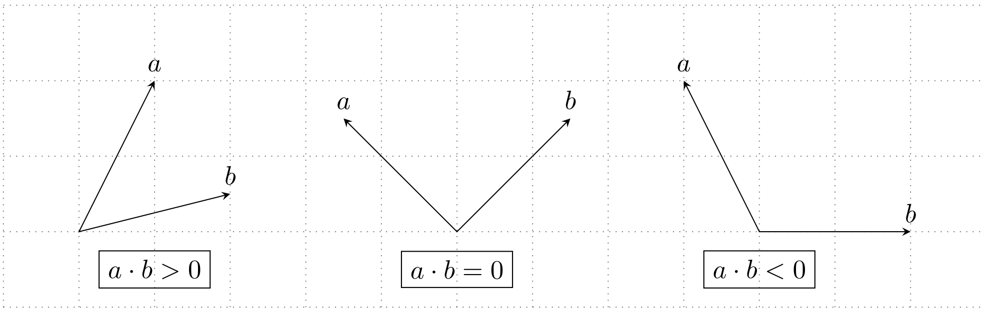


For a more detailed and interactive explanation of the dot product, see Math Insight [4].

With this in mind, the dot product can be used to calculate the angle itself.

$$\cos\theta = \frac{a \cdot b}{\|a\|\|b\|}$$
$$\theta = \cos^{-1}\left(\frac{a \cdot b}{\|a\|\|b\|}\right)$$

A lot of information can be obtained from the dot product. If the dot product is positive, a has a component in the same direction as b. If the dot product is zero, a and b are perpendicular. If it's negative, a has a component in the opposite direction of b.



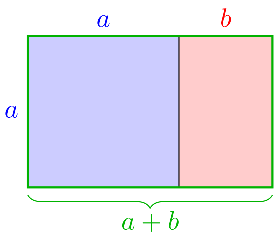
1.6. **Golden ratio.** The golden ratio is an irrational number with a value of:

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.618033988749\ldots$$

Two numbers a and b are in the golden ratio (noted φ) if their ratio ($\frac{a}{b}$) is the same as the ratio of their sum to the larger number. Assuming $a > b > 0$:

$$\frac{a}{b} = \frac{a+b}{a} = \varphi$$

A **golden rectangle** is a rectangle whose adjacent sides are in the golden ratio, and it can be used to illustrate the previous formula.



The red rectangle with short side b and long side a is itself a golden rectangle. When placed adjacent to the blue square (with sides of length a), the green rectangle is produced, with long side $a + b$ and short side a. This green rectangle is similar to the red rectangle, and therefore also a golden rectangle.

This process of adding an adjacent square to the rectangle, and producing a similar rectangle reminds of the Fibonacci or Lucas sequences. If a Fibonacci and Lucas number is divided by its immediate predecessor in the sequence, the quotient approximates to φ.

$$\frac{F_{16}}{F_{15}} = \frac{987}{610} = 1.6180327\ldots$$
$$\frac{L_{16}}{L_{15}} = \frac{2207}{1364} = 1.6180351\ldots$$

2. Physics

2.1. Gravitational force. The gravitational force of each body is calculated with the following formula.

$$F = G\frac{m_1 m_2}{r^2}$$

Where G is the gravitational constant, m_1 and m_2 are the mass of each body, and r is the distance between the objects.

The effect of a force is to accelerate the body. The relationship is the following.

$$F = ma$$

Where F is the force, m is the mass and a is the acceleration of the body. Therefore, the acceleration can be calculated from the force with the following formula.

$$a = \frac{F}{m}$$

The force has a direction. It acts towards the direction of the line joining the centres of the two bodies. We can get the X and Y coordinates of the acceleration with some trigonometry.

$$a_x = a\cos\theta$$
$$a_y = a\sin\theta$$

Where a_x and a_y are the X and Y accelerations, a is the acceleration, and θ is the angle that the line joining the centers make with the horizontal.

3. Modulus operation

The modulus of two numbers is the remainder of it's integer division. The modulus of two numbers could be defined as follows.

$$a = \lfloor a/b \rfloor \times b + a \bmod b$$

Where $\lfloor a/b \rfloor$ indicates the integer division of a and b.

3.1. **Equivalences.** These equivalences might be useful when dealing with modulus operators that only support positive values, for example.

Given the following function, that returns the modulus of two positive values,

$$\mathrm{AbsMod}(a, b) = |a| \bmod |b|$$

the following conditional formula can be used for determining the modulus of any positive and negative combination.

$$a \bmod b = \begin{cases} \mathrm{AbsMod}(a, b), & a \geq 0 \land b \geq 0 \\ b + \mathrm{AbsMod}(a, b), & a \geq 0 \land b < 0 \\ b - \mathrm{AbsMod}(a, b), & a < 0 \land b \geq 0 \\ -\mathrm{AbsMod}(a, b), & a < 0 \land b < 0 \end{cases}$$

The modulus of a and b is equal to the negation of the modulus of $-a$ and $-b$.

$$a \bmod b \iff -(-a \bmod -b)$$

This can be used for converting the divisor and dividend to negative, if needed.

$$a \bmod -b \iff -(-a \bmod b)$$
$$-a \bmod b \iff -(a \bmod -b)$$
$$-a \bmod -b \iff -(a \bmod b)$$

The modulus of a and b is equal to the divisor (b) minus the modulus of the negated dividend and the unchanged divisor.

$$a \bmod b \iff b - (-a \bmod b)$$

This can be used for converting the dividend to positive, if needed.

$$-a \bmod b \iff b - a \bmod b$$

4. Color conversion

4.1. **Value ranges.** An RGB color has values in the [0..255] range, while in an HSV color the *hue* is in the [0..360] range and the *saturation* and *value* are in the [0..1] range, although they might be represented as percentages.

4.2. **RGB to HSV.** First, the RGB values need to be normalized to the [0..1] range.

$$R' = \frac{R}{255}$$
$$G' = \frac{G}{255}$$
$$B' = \frac{B}{255}$$

Then, the maximum and minimum RGB values are calculated, along with its difference.

$$C_{max} = \max(R', G', B')$$
$$C_{min} = \min(R', G', B')$$
$$\Delta = C_{max} - C_{min}$$

To calculate the *hue*, the following conditional formula is used.

$$H = \begin{cases} 0°, & \Delta = 0 \\ 60° \times \left(\frac{G'-B'}{\Delta} \bmod 6\right), & C_{max} = R' \\ 60° \times \left(\frac{B'-R'}{\Delta} + 2\right), & C_{max} = G' \\ 60° \times \left(\frac{R'-G'}{\Delta} + 4\right), & C_{max} = B' \end{cases}$$

To calculate the *saturation*, the following formula is used.

$$S = \begin{cases} 0, & C_{max} = 0 \\ \frac{\Delta}{C_{max}}, & C_{max} \neq 0 \end{cases}$$

Finally, since C_{max} is already normalized, it can be used directly as the *value* component.

$$V = C_{max}$$

4.3. **HSV to RGB.** Calculate the *chroma* by multiplying the *saturation* and the *value*.

$$C = S \times V$$

Then, the X value is calculated, which will be used as a component in the initial RGB color below.

$$H' = \frac{H}{60°}$$
$$X = C \times (1 - |H' \bmod 2 - 1|)$$

Note that H' must be an integer for the modulus operation.

The *chroma* and X values will be used for the initial RGB values depending on the *hue* with this conditional formula.

$$(R', G', B') = \begin{cases} (C, X, 0), & 0° \leq H < 60° \\ (X, C, 0), & 60° \leq H < 120° \\ (0, C, X), & 120° \leq H < 180° \\ (0, X, C), & 180° \leq H < 240° \\ (X, 0, C), & 240° \leq H < 300° \\ (C, 0, X), & 300° \leq H < 360° \end{cases}$$

The value of H' can be used in the conditions instead of the *hue*, but I consider this form more visual.

To find the real RGB values, m has to be added to each component to match the HSV *value*.

$$m = V - C$$
$$(R, G, B) = (R' + m, G' + m, B' + m)$$

References

[1] Frank D and Nykamp DQ. *An introduction to vectors*. From Math Insight. Retrieved 23 May 2024, from http://mathinsight.org/vector_introduction

[2] Nykamp DQ. *Magnitude of a vector definition*. From Math Insight. Retrieved 17 Jun 2024, from https://mathinsight.org/definition/magnitude_vector

[3] Wikipedia. *Unit vector*. Retrieved 23 May 2024, from https://en.wikipedia.org/wiki/Unit_vector

[4] Nykamp DQ. *The dot product*. From Math Insight. Retrieved 23 May 2024, from https://mathinsight.org/dot_product