Contents

1 Basic

1.1 Default Code

```cpp
//Challenge: Accepted
//#pragma GCC optimize("Ofast")
#include <bits/stdc++.h>
using namespace std;

#define io ios_base::sync_with_stdio(0);cin.tie(0);cerr.tie
    (0)
#define iter(v) v.begin(),v.end()
#define SZ(v) (int)v.size()
#define pb emplace_back
#define mp make_pair
#define ff first
#define ss second

using ll = long long;
using pii = pair<int, int>;
using pll = pair<ll, ll>;

#ifdef zisk
void debug(){cerr << "\n";}
template<class T, class ... U>
void debug(T a, U ... b){cerr << a << " ", debug(b...);}
template<class T> void pary(T l, T r){
  while (l != r) cerr << *l << " ", l++;
  cerr << "\n";
}
#else
#define debug(...) void()
#define pary(...) void()
#endif
  template<class A, class B>
ostream& operator<<(ostream& o, pair<A,B> p)
{ return o << '(' << p.ff << ',' << p.ss << ')'; }

int main(){
  io;
}
```

1.2 .vimrc

```
sy on
se nu rnu bs=2 sw=4 ts=4 hls ls=2 si acd bo=all mouse=a et
map <F9> :w<bar>!g++ "%" -o %:r -std=c++17 -Wall -Wextra -
    Wshadow -Dzisk -g -fsanitize=undefined,address<CR>
map <F8> :!./%:r<CR>
map <C-a> <ESC>ggVG
inoremap {<CR> {<CR>}<ESC>ko
```

2 Data Structure

2.1 Heavy-Light Decomposition

```
struct Heavy_light_Decomposition { // 1-base
  int n, up[maxn], dep[maxn], to[maxn], siz[maxn], pa[maxn
      ];
  int C, ti[maxn], ord[maxn], wdown[maxn], edge[maxn], et =
       0;
  vector<pii> G[maxn];
  void init(int _n) {
    n = _n, C = 0, et = 1;
    for (int i = 1;i <= n;i++)
      G[i].clear(), to[i] = 0;
  }
  void add_edge(int a, int b, int w) {
    G[a].push_back(pii(b, et)), G[b].push_back(pii(a, et));
    edge[et++] = w;
  }
  void dfs(int u, int f, int d) {
    siz[u] = 1, pa[u] = f, dep[u] = d;
    for (auto &v: G[u])
      if (v.ff != f) {
        dfs(v.ff, u, d+1), siz[u] += siz[v];
        if (siz[to[u]] < siz[v]) to[u] = v;
      }
  }
  void cut(int u, int link) {
    ti[u] = C;
    ord[C++] = u, up[u] = link;
    if (!to[u]) return;
    cut(to[u], link);
    for (auto v:G[u]) {
      if (v.ff != pa[u] && v.ff != to[u]) cut(v.ff, v.ff);
    }
  }
  void build() { dfs(1, 1, 1), cut(1, 1); }
  int query(int a, int b) {
    int ta = up[a], tb = up[b], re = 0;
    while (ta != tb)
      if (dep[ta] < dep[tb])
        /*query*/, tb = up[b = pa[tb]];
      else /*query*/, ta = up[a = pa[ta]];
    if (a == b) return re;
    if (ti[a] > ti[b]) swap(a, b);
    /*query*/
    return re;
  }
};
```

2.2 Li-Chao Tree

```
struct LiChao_min {
  struct line {
    ll m, c;
    line(ll _m = 0, ll _c = 0) {
      m = _m;
      c = _c;
    }
    ll eval(ll x) { return m * x + c; }
  };
  struct node {
    node *l, *r;
    line f;
    node(line v) {
      f = v;
      l = r = NULL;
    }
  };
  typedef node *pnode;
  pnode root;
  int sz;
#define mid ((l + r) >> 1)
  void insert(line &v, int l, int r, pnode &nd) {
    if (!nd) {
      nd = new node(v);
      return;
    }
    ll trl = nd->f.eval(l), trr = nd->f.eval(r);
    ll vl = v.eval(l), vr = v.eval(r);
    if (trl <= vl && trr <= vr) return;
    if (trl > vl && trr > vr) {
      nd->f = v;
      return;
    }
    if (trl > vl) swap(nd->f, v);
    if (nd->f.eval(mid) < v.eval(mid))
      insert(v, mid + 1, r, nd->r);
    else swap(nd->f, v), insert(v, l, mid, nd->l);
  }
  ll query(int x, int l, int r, pnode &nd) {
    if (!nd) return inf;
    if (l == r) return nd->f.eval(x);
    if (mid >= x)
      return min(
          nd->f.eval(x), query(x, l, mid, nd->l));
    return min(
        nd->f.eval(x), query(x, mid + 1, r, nd->r));
  }
  /* -sz <= query_x <= sz */
  void init(int _sz) {
    sz = _sz + 1;
    root = NUll;
  }
  void add_line(ll m, ll c) {
    line v(m, c);
    insert(v, -sz, sz, root);
  }
  ll query(ll x) { return query(x, -sz, sz, root); }
};
```

2.3 Link Cut Tree

```
struct Splay { // subtree-sum, path-max
  static Splay nil;
  Splay *ch[2], *f;
  int val, rev, size, vir, id, type;
  pii ma;
  Splay(int _val = 0, int _id = 0)
    : val(_val), rev(0), size(0), vir(0), id(_id) {
      ma = make_pair(val, id);
      f = ch[0] = ch[1] = &nil;
      type = 0;
    }
  bool isr() { //is root
    return f->ch[0] != this && f->ch[1] != this;
  }
  int dir() { return f->ch[0] == this ? 0 : 1; }
  void setCh(Splay *c, int d) {
    ch[d] = c;
    if (c != &nil) c->f = this;
    pull();
  }
  void push() {
    if (!rev) return;
    swap(ch[0], ch[1]);
    if (ch[0] != &nil) ch[0]->rev ^= 1;
    if (ch[1] != &nil) ch[1]->rev ^= 1;
    rev = 0;
  }
  void pull() {
    // take care of the nil!
    size = ch[0]->size + ch[1]->size + vir + type;
    ma = max(make_pair(val, id), max(ch[0]->ma, ch[1]->ma))
        ;
    if (ch[0] != &nil) ch[0]->f = this;
    if (ch[1] != &nil) ch[1]->f = this;
  }
} Splay::nil;
Splay *nil = &Splay::nil;
void rotate(Splay *x) {
  Splay *p = x->f;
  int d = x->dir();
  if (!p->isr()) p->f->setCh(x, p->dir());
  else x->f = p->f;
  p->setCh(x->ch[!d], d);
```

```
  x->setCh(p, !d);
  p->pull(), x->pull();
}
void splay(Splay *x) {
  vector<Splay *> splayVec;
  for (Splay *q = x;; q = q->f) {
    splayVec.pb(q);
    if (q->isr()) break;
  }
  reverse(iter(splayVec));
  for (auto it : splayVec) it->push();
  while (!x->isr()) {
    if (x->f->isr()) rotate(x);
    else if (x->dir() == x->f->dir())
      rotate(x->f), rotate(x);
    else rotate(x), rotate(x);
  }
}
Splay *access(Splay *x) {
  Splay *q = nil;
  for (; x != nil; x = x->f){
    splay(x);
    x->vir -= q->size; x->vir += x->ch[1]->size;
    x->setCh(q, 1); x->pull();
    q = x;
  }
  return q;
}
void root_path(Splay *x) { access(x), splay(x); }
void chroot(Splay *x) {
  root_path(x), x->rev ^= 1;
  x->push(), x->pull();
}
void split(Splay *x, Splay *y) {
  chroot(x), root_path(y);
}
void link(Splay *x, Splay *y) {
  chroot(x), root_path(y);
  x->f = y; y->vir += x->size;
}
void cut(Splay *x, Splay *y) {
  split(x, y);
  y->push();
  y->ch[0] = y->ch[0]->f = nil;
  y->pull();
}
Splay *get_root(Splay *x) {
  for (root_path(x); x->ch[0] != nil; x = x->ch[0])
    x->push();
  splay(x);
  return x;
}
bool conn(Splay *x, Splay *y) {
  return get_root(x) == get_root(y);
}
Splay *lca(Splay *x, Splay *y) {
  access(x), root_path(y);
  if (y->f == nil) return y;
  return y->f;
}
void change(Splay *x, int val) {
  splay(x), x->val = val, x->pull();
}
pii query(Splay *x, Splay *y) {
  split(x, y);
  return y->ma;
}
```

2.4 Treap

```
struct node {
  int data, sz;
  node *l, *r;
  node(int k) : data(k), sz(1), l(0), r(0) {}
  void up() {
    sz = 1;
    if (l) sz += l->sz;
    if (r) sz += r->sz;
  }
  void down() {}
};
int sz(node *a) { return a ? a->sz : 0; }
node *merge(node *a, node *b) {
  if (!a || !b) return a ? a : b;
  if (rand() % (sz(a) + sz(b)) < sz(a))
    return a->down(), a->r = merge(a->r, b), a->up(),
           a;
  return b->down(), b->l = merge(a, b->l), b->up(), b;
}
void split(node *o, node *&a, node *&b, int k) {
  if (!o) return a = b = 0, void();
  o->down();
  if (o->data <= k)
    a = o, split(o->r, a->r, b, k), a->up();
  else b = o, split(o->l, a, b->l, k), b->up();
}
void split2(node *o, node *&a, node *&b, int k) {
  if (sz(o) <= k) return a = o, b = 0, void();
  o->down();
  if (sz(o->l) + 1 <= k)
    a = o, split2(o->r, a->r, b, k - sz(o->l) - 1);
  else b = o, split2(o->l, a, b->l, k);
  o->up();
}
node *kth(node *o, int k) {
  if (k <= sz(o->l)) return kth(o->l, k);
  if (k == sz(o->l) + 1) return o;
  return kth(o->r, k - sz(o->l) - 1);
}
int Rank(node *o, int key) {
  if (!o) return 0;
  if (o->data < key)
    return sz(o->l) + 1 + Rank(o->r, key);
  else return Rank(o->l, key);
}
bool erase(node *&o, int k) {
  if (!o) return 0;
  if (o->data == k) {
    node *t = o;
    o->down(), o = merge(o->l, o->r);
    delete t;
    return 1;
  }
  node *&t = k < o->data ? o->l : o->r;
  return erase(t, k) ? o->up(), 1 : 0;
}
void insert(node *&o, int k) {
  node *a, *b;
  split(o, a, b, k),
    o = merge(a, merge(new node(k), b));
}
void interval(node *&o, int l, int r) {
  node *a, *b, *c;
  split2(o, a, b, l - 1), split2(b, b, c, r);
  // operate
  o = merge(a, merge(b, c));
}
```

3 Flow Matching

3.1 Bounded Flow

```
struct Dinic { // 1-base
  struct edge {
    int to, cap, flow, rev;
  };
  vector<edge> g[maxN];
  int n, s, t, dis[maxN], ind[maxN], cnt[maxN];
  const int inf = 1e9;

  void init(int _n) {
```

```
    n = _n + 2;
    s = _n + 1, t = _n + 2;
    for (int i = 0; i <= n; ++i) g[i].clear(), cnt[i] = 0;
  }
  void reset() {
    for (int i = 0; i <= n; ++i)
      for (auto &j : g[i]) j.flow = 0;
  }
  void add_edge(int u, int v, int lcap, int rcap) {
    cnt[u] -= lcap, cnt[v] += lcap;
    g[u].pb(edge{v, rcap, lcap, (int)g[v].size()});
    g[v].pb(edge{u, 0, 0, (int)g[u].size() - 1});
  }
  void add_edge(int u, int v, int cap) {
    g[u].pb(edge{v, cap, 0, (int)g[v].size()});
    g[v].pb(edge{u, 0, 0, (int)g[u].size() - 1});
    //change g[v] to cap for undirected graphs
  }
  bool bfs() {
    fill(dis, dis+n+1, -1);
    queue<int> q;
    q.push(s), dis[s] = 0;
    while (!q.empty()) {
      int cur = q.front(); q.pop();
      for (auto &e : g[cur]) {
        if (dis[e.to] == -1 && e.flow != e.cap) {
          q.push(e.to);
          dis[e.to] = dis[cur] + 1;
        }
      }
    }
    return dis[t] != -1;
  }
  int dfs(int u, int cap) {
    if (u == t || !cap) return cap;
    for (int &i = ind[u]; i < (int)g[u].size(); ++i) {
      edge &e = g[u][i];
      if (dis[e.to] == dis[u] + 1 && e.flow != e.cap) {
        int df = dfs(e.to, min(e.cap - e.flow, cap));
        if (df) {
          e.flow += df;
          g[e.to][e.rev].flow -= df;
          return df;
        }
      }
    }
    dis[u] = -1;
    return 0;
  }
  int maxflow(int _s, int _t) {
    s = _s, t = _t;
    int flow = 0, df;
    while (bfs()) {
      fill(ind, ind+n+1, 0);
      while ((df = dfs(s, inf))) flow += df;
    }
    return flow;
  }
  bool feasible() {
    int sum = 0;
    for (int i = 1; i <= n - 2; ++i)
      if (cnt[i] > 0)
        add_edge(n - 1, i, cnt[i]), sum += cnt[i];
      else if (cnt[i] < 0) add_edge(i, n, -cnt[i]);
    if (sum != maxflow(n - 1, n)) sum = -1;
    for (int i = 1; i <= n - 2; ++i)
      if (cnt[i] > 0)
        g[n - 1].pop_back(), g[i].pop_back();
      else if (cnt[i] < 0)
        g[i].pop_back(), g[n].pop_back();
    return sum != -1;
  }
  int boundedflow(int _s, int _t) {
    add_edge(_t, _s, inf);
    if (!feasible()) return -1; // infeasible flow
    int x = g[_t].back().flow;
    g[_t].pop_back(), g[_s].pop_back();
    int y = maxflow(_t, _s);
    return x-y;
  }
};
```

3.2 Dinic

```
struct MaxFlow { // 1-base
  struct edge {
    int to, cap, flow, rev;
  };
  vector<edge> g[maxn];
  int s, t, dis[maxn], ind[maxn], n;

  void init(int _n) {
    n = _n + 2;
    s = _n + 1, t = _n + 2;
    for (int i = 0; i <= n; ++i) g[i].clear();
  }
  void reset() {
    for (int i = 0; i <= n; ++i)
      for (auto &j : g[i]) j.flow = 0;
  }
  void add_edge(int u, int v, int cap) {
    g[u].pb(edge{v, cap, 0, (int)g[v].size()});
    g[v].pb(edge{u, 0, 0, (int)g[u].size() - 1});
    //change g[v] to cap for undirected graphs
  }
  bool bfs() {
    fill(dis, dis+n+1, -1);
    queue<int> q;
    q.push(s), dis[s] = 0;
    while (!q.empty()) {
      int cur = q.front(); q.pop();
      for (auto &e : g[cur]) {
        if (dis[e.to] == -1 && e.flow != e.cap) {
          q.push(e.to);
          dis[e.to] = dis[cur] + 1;
        }
      }
    }
    return dis[t] != -1;
  }
  int dfs(int u, int cap) {
    if (u == t || !cap) return cap;
    for (int &i = ind[u]; i < (int)g[u].size(); ++i) {
      edge &e = g[u][i];
      if (dis[e.to] == dis[u] + 1 && e.flow != e.cap) {
        int df = dfs(e.to, min(e.cap - e.flow, cap));
        if (df) {
          e.flow += df;
          g[e.to][e.rev].flow -= df;
          return df;
        }
      }
    }
    dis[u] = -1;
    return 0;
  }
  int maxflow() {
    int flow = 0, df;
    while (bfs()) {
      fill(ind, ind+n+1, 0);
      while ((df = dfs(s, inf))) flow += df;
    }
    return flow;
  }
}flow;
```

3.3 Gomory Hu

```
MaxFlow Dinic;
int g[MAXN];
void GomoryHu(int n) { // 0-base
```

```
  fill_n(g, n, 0);
  for (int i = 1; i < n; ++i) {
    Dinic.reset();
    add_edge(i, g[i], Dinic.maxflow(i, g[i]));
    for (int j = i + 1; j <= n; ++j)
      if (g[j] == g[i] && ~Dinic.dis[j])
        g[j] = i;
  }
}
```

3.4 Hungarian Algorithm

```
int c[maxn][maxn]; //hungarian algorithm in O(n^3)
                   //1 base
int lx[maxn], ly[maxn], mx[maxn], my[maxn];
bool vx[maxn], vy[maxn];
int slack[maxn];
int tot;
bool dfs(int n, bool ch) {
  if (vx[n]) return false;
  vx[n] = 1;
  for (int v = 1;v <= tot;v++) {
    slack[v] = min(slack[v], lx[n] + ly[v] - c[n][v]);
    if (lx[n] + ly[v] - c[n][v] > 0) continue;
    vy[v] = 1;
    if (!my[v] || dfs(my[v], ch)) {
      if (ch) mx[n] = v, my[v] = n;
      return true;
    }
  }
  return false;
}
int main() {
  for (int i = 1;i <= n;i++) {
    for (int j = 1;j <= n;j++) vx[j] = vy[j] = 0;
    for (int j = 1;j <= n;j++) slack[j] = 1<<30;
    if (dfs(i, 1)) continue;
    bool aug = 0;
    while (!aug) {
      for (int j = 1;j <= n;j++) {
        if (!vy[j] && slack[j] == 0) {
          vy[j] = 1;
          if (dfs(my[j], 0)) {
            aug = 1;
            break;
          }
        }
      }
      if (aug) break;
      int delta = 1<<30;
      for (int j = 1;j <= n;j++) {
        if (!vy[j]) delta = min(delta, slack[j]);
      }
      for (int j = 1;j <= n;j++) {
        if (vx[j]) lx[j] -= delta;
        if (vy[j]) ly[j] += delta;
        else {
          slack[j] -= delta;
          if (slack[j] == 0 && !my[j]) aug = 1;
        }
      }
    }
    for (int j = 1;j <= n;j++) vx[j] = vy[j] = 0;
    dfs(i, 1);
  }
}
```

3.5 ISAP Algorithm

```
struct Maxflow { //to be modified
  static const int MAXV = 20010;
  static const int INF = 1000000;
  struct Edge {
    int v, c, r;
    Edge(int _v, int _c, int _r)
        : v(_v), c(_c), r(_r) {}
  };
  int s, t;
  vector<Edge> G[MAXV * 2];
  int iter[MAXV * 2], d[MAXV * 2], gap[MAXV * 2], tot;
  void init(int x) {
    tot = x + 2;
    s = x + 1, t = x + 2;
    for (int i = 0; i <= tot; i++) {
      G[i].clear();
      iter[i] = d[i] = gap[i] = 0;
    }
  }
  void addEdge(int u, int v, int c) {
    G[u].push_back(Edge(v, c, SZ(G[v])));
    G[v].push_back(Edge(u, 0, SZ(G[u]) - 1));
  }
  int dfs(int p, int flow) {
    if (p == t) return flow;
    for (int &i = iter[p]; i < SZ(G[p]); i++) {
      Edge &e = G[p][i];
      if (e.c > 0 && d[p] == d[e.v] + 1) {
        int f = dfs(e.v, min(flow, e.c));
        if (f) {
          e.c -= f;
          G[e.v][e.r].c += f;
          return f;
        }
      }
    }
    if ((--gap[d[p]]) == 0) d[s] = tot;
    else {
      d[p]++;
      iter[p] = 0;
      ++gap[d[p]];
    }
    return 0;
  }
  int solve() {
    int res = 0;
    gap[0] = tot;
    for (res = 0; d[s] < tot; res += dfs(s, INF))
      ;
    return res;
  }
} flow;
```

3.6 KM Algorithm

```
int n, m; //1-base, max matching
int mx[maxn], my[maxn];
bool adj[maxn][maxn], vis[maxn];
bool dfs(int n) {
  if (vis[n]) return false;
  vis[n] = 1;
  for (int v = 1;v <= n;v++) {
    if (!adj[n][v]) continue;
    if (!my[v] || (my[v] && dfs(my[v]))) {
      mx[n] = v, my[v] = n;
      return true;
    }
  }
  return false;
}
//min vertex cover: take unmatched vertex in L and find
    alternating tree,
//ans is not reached in L + reached in R
```

3.7 Max Simple Graph Matching

```
struct GenMatch { // 1-base
  int V, pr[N];
  bool el[N][N], inq[N], inp[N], inb[N];
  int st, ed, nb, bk[N], djs[N], ans;
  void init(int _V) {
```

```
    V = _V;
    for (int i = 0; i <= V; ++i) {
      for (int j = 0; j <= V; ++j) el[i][j] = 0;
      pr[i] = bk[i] = djs[i] = 0;
      inq[i] = inp[i] = inb[i] = 0;
    }
  }
  void add_edge(int u, int v) {
    el[u][v] = el[v][u] = 1;
  }
  int lca(int u, int v) {
    fill_n(inp, V + 1, 0);
    while (1)
      if (u = djs[u], inp[u] = true, u == st) break;
      else u = bk[pr[u]];
    while (1)
      if (v = djs[v], inp[v]) return v;
      else v = bk[pr[v]];
    return v;
  }
  void upd(int u) {
    for (int v; djs[u] != nb;) {
      v = pr[u], inb[djs[u]] = inb[djs[v]] = true;
      u = bk[v];
      if (djs[u] != nb) bk[u] = v;
    }
  }
  void blo(int u, int v, queue<int> &qe) {
    nb = lca(u, v), fill_n(inb, V + 1, 0);
    upd(u), upd(v);
    if (djs[u] != nb) bk[u] = v;
    if (djs[v] != nb) bk[v] = u;
    for (int tu = 1; tu <= V; ++tu)
      if (inb[djs[tu]])
        if (djs[tu] = nb, !inq[tu])
          qe.push(tu), inq[tu] = 1;
  }
  void flow() {
    fill_n(inq + 1, V, 0), fill_n(bk + 1, V, 0);
    iota(djs + 1, djs + V + 1, 1);
    queue<int> qe;
    qe.push(st), inq[st] = 1, ed = 0;
    while (!qe.empty()) {
      int u = qe.front();
      qe.pop();
      for (int v = 1; v <= V; ++v)
        if (el[u][v] && djs[u] != djs[v] &&
            pr[u] != v) {
          if ((v == st) ||
              (pr[v] > 0 && bk[pr[v]] > 0)) {
            blo(u, v, qe);
          } else if (!bk[v]) {
            if (bk[v] = u, pr[v] > 0) {
              if (!inq[pr[v]]) qe.push(pr[v]);
            } else {
              return ed = v, void();
            }
          }
        }
    }
  }
  void aug() {
    for (int u = ed, v, w; u > 0;)
      v = bk[u], w = pr[v], pr[v] = u, pr[u] = v,
        u = w;
  }
  int solve() {
    fill_n(pr, V + 1, 0), ans = 0;
    for (int u = 1; u <= V; ++u)
      if (!pr[u])
        if (st = u, flow(), ed > 0) aug(), ++ans;
    return ans;
  }
};
```

3.8 MCMF

```
struct MCMF { // 0-base
  struct edge {
    ll from, to, cap, flow, cost, rev;
  } * past[maxn];
  vector <edge> G[maxn];
  bitset <maxn> inq;
  ll dis[maxn], up[maxn], s, t, mx, n;
  bool BellmanFord(ll &flow, ll &cost) {
    fill(dis, dis + n, inf);
    queue<ll> q;
    q.push(s), inq.reset(), inq[s] = 1;
    up[s] = mx - flow, past[s] = 0, dis[s] = 0;
    while (!q.empty()) {
      ll u = q.front();
      q.pop(), inq[u] = 0;
      if (!up[u]) continue;
      for (auto &e : G[u])
        if (e.flow != e.cap &&
            dis[e.to] > dis[u] + e.cost) {
          dis[e.to] = dis[u] + e.cost, past[e.to] = &e;
          up[e.to] = min(up[u], e.cap - e.flow);
          if (!inq[e.to]) inq[e.to] = 1, q.push(e.to);
        }
    }
    if (dis[t] == inf) return 0;
    flow += up[t], cost += up[t] * dis[t];
    for (ll i = t; past[i]; i = past[i]->from) {
      auto &e = *past[i];
      e.flow += up[t], G[e.to][e.rev].flow -= up[t];
    }
    return 1;
  }
  ll MinCostMaxFlow(ll _s, ll _t, ll &cost) {
    s = _s, t = _t, cost = 0;
    ll flow = 0;
    while (BellmanFord(flow, cost));
    return flow;
  }
  void init(ll _n, ll _mx) {
    n = _n, mx = _mx;
    for (int i = 0; i < n; ++i) G[i].clear();
  }
  void add_edge(ll a, ll b, ll cap, ll cost) {
    G[a].pb(edge{a, b, cap, 0, cost, G[b].size()});
    G[b].pb(edge{b, a, 0, 0, -cost, G[a].size() - 1});
  }
};
```

3.9 Min Cost Circulation

```
//to be modified
struct Edge { int to, cap, rev, cost; };
vector<Edge> g[kN];
int dist[kN], pv[kN], ed[kN];
bool mark[kN];
int NegativeCycle(int n) {
  memset(mark, false, sizeof(mark));
  memset(dist, 0, sizeof(dist));
  int upd = -1;
  for (int i = 0; i <= n; ++i) {
    for (int j = 0; j < n; ++j) {
      int idx = 0;
      for (auto &e : g[j]) {
        if (e.cap > 0 && dist[e.to] > dist[j] + e.cost) {
          dist[e.to] = dist[j] + e.cost;
          pv[e.to] = j, ed[e.to] = idx;
          if (i == n) {
            upd = j;
            while (!mark[upd]) mark[upd] = true, upd = pv[
                upd];
            return upd;
          }
        }
        idx++;
      }
    }
```

```
  }
  return -1;
}
int Solve(int n) {
  int rt = -1, ans = 0;
  while ((rt = NegativeCycle(n)) >= 0) {
    memset(mark, false, sizeof(mark));
    vector<pair<int, int>> cyc;
    while (!mark[rt]) {
      cyc.emplace_back(pv[rt], ed[rt]);
      mark[rt] = true;
      rt = pv[rt];
    }
    reverse(cyc.begin(), cyc.end());
    int cap = kInf;
    for (auto &i : cyc) {
      auto &e = g[i.first][i.second];
      cap = min(cap, e.cap);
    }
    for (auto &i : cyc) {
      auto &e = g[i.first][i.second];
      e.cap -= cap;
      g[e.to][e.rev].cap += cap;
      ans += e.cost * cap;
    }
  }
  return ans;
}
```

3.10 SW Mincut

```
// stoer wagner algorithm: global min cut
const int maxn = 505;
struct SW { // O(V^3) 0-based
  int n, vis[maxn], del[maxn];
  int edge[maxn][maxn], wei[maxn];
  void init(int _n) {
    n = _n;
    fill(del, del+n, 0);
    for (int i = 0;i < n;i++) fill(edge[i], edge[i] + n, 0)
        ;
  }
  void addEdge(int u, int v, int w) {
    edge[u][v] += w, edge[v][u] += w;
  }
  void search(int &s, int &t) {
    fill(vis, vis+n, 0);
    fill(wei, wei+n, 0);
    s = t = -1;
    while (1) {
      int ma = -1, cur = 0;
      for (int i = 0; i < n; ++i)
        if (!del[i] && !vis[i] && ma < wei[i])
          cur = i, ma = wei[i];
      if (mx == -1) break;
      vis[cur] = 1, s = t, t = cur;
      for (int i = 0; i < n; ++i)
        if (!vis[i] && !del[i]) wei[i] += edge[cur][i];
    }
  }
  int solve() {
    int ret = INF;
    for (int i = 0, x=0, y=0; i < n-1; ++i) {
      search(x, y), ret = min(res, wei[y]), del[y] = 1;
      for (int j = 0; j < n; ++j)
        edge[x][j] = (edge[j][x] += edge[y][j]);
    }
    return ret;
  }
};
```

4 Geometry

4.1 Geometry Template

```
using ld = ll;
using pdd = pair<ld, ld>;
using Line = pair<pdd, pdd>;
#define X first
#define Y second
// ld eps = 1e-7;

pdd operator+(pdd a, pdd b)
{ return {a.X + b.X, a.Y + b.Y}; }
pdd operator-(pdd a, pdd b)
{ return {a.X - b.X, a.Y - b.Y}; }
pdd operator*(ld i, pdd v)
{ return {i * v.X, i * v.Y}; }
pdd operator*(pdd v, ld i)
{ return {i * v.X, i * v.Y}; }
pdd operator/(pdd v, ld i)
{ return {v.X / i, v.Y / i}; }
ld dot(pdd a, pdd b)
{ return a.X * b.X + a.Y * b.Y; }
ld cross(pdd a, pdd b)
{ return a.X * b.Y - a.Y * b.X; }
ld abs2(pdd v)
{ return v.X * v.X + v.Y * v.Y; };
ld abs(pdd v)
{ return sqrt(abs2(v)); };
int sgn(ld v)
{ return v > 0 ? 1 : (v < 0 ? -1 : 0); }
// int sgn(NumType v){ return v > eps ? 1 : ( v < -eps ? -1
     : 0); }
int ori(pdd a, pdd b, pdd c)
{ return sgn(cross(b - a, c - a)); }
bool collinearity(pdd a, pdd b, pdd c)
{ return ori(a, b, c) == 0; }
bool btw(pdd p, pdd a, pdd b)
{ return collinearity(p, a, b) && sgn(dot(a - p, b - p)) <=
     0; }

bool seg_intersect(Line a, Line b){
  pdd p1, p2, p3, p4;
  tie(p1, p2) = a; tie(p3, p4) = b;
  if(btw(p1, p3, p4) || btw(p2, p3, p4) || btw(p3, p1, p2)
      || btw(p4, p1, p2))
    return true;
  return ori(p1, p2, p3) * ori(p1, p2, p4) < 0 &&
    ori(p3, p4, p1) * ori(p3, p4, p2) < 0;
}
pdd intersect(Line a, Line b){
  pdd p1, p2, p3, p4;
  tie(p1, p2) = a; tie(p3, p4) = b;
  ld a123 = cross(p2 - p1, p3 - p1);
  ld a124 = cross(p2 - p1, p4 - p1);
  return (p4 * a123 - p3 * a124) / (a123 - a124);
}
pdd perp(pdd p1)
{ return pdd(-p1.Y, p1.X); }
pdd projection(pdd p1, pdd p2, pdd p3)
{ return (p2 - p1) * dot(p3 - p1, p2 - p1) / abs2(p2 - p1);
     }
```

4.2 Convex Hull

```
vector<int> getConvexHull(vector<pdd>& pts){
  vector<int> id(SZ(pts));
  iota(iter(id), 0);
  sort(iter(id), [&](int x, int y){ return pts[x] < pts[y];
      });
  vector<int> hull;
  for(int tt = 0; tt < 2; tt++){
    int sz = SZ(hull);
    for(int j : id){
      pdd p = pts[j];
      while(SZ(hull) - sz >= 2 &&
          cross(pts[hull.back()] - pts[hull[SZ(hull) - 2]],
            p - pts[hull[SZ(hull) - 2]]) <= 0)
        hull.pop_back();
      hull.pb(j);
```

```
    }
    hull.pop_back();
    reverse(iter(id));
  }
  return hull;
}
```

4.3 Minimum Enclosing Circle

```
using ld = long double;
pair<pdd, ld> circumcenter(pdd a, pdd b, pdd c);
pair<pdd, ld> MinimumEnclosingCircle(vector<pdd> &pts){
  random_shuffle(iter(pts));
  pdd c = pts[0];
  ld r = 0;
  for(int i = 1; i < SZ(pts); i++){
    if(abs(pts[i] - c) <= r) continue;
    c = pts[i]; r = 0;
    for(int j = 0; j < i; j++){
      if(abs(pts[j] - c) <= r) continue;
      c = (pts[i] + pts[j]) / 2;
      r = abs(pts[i] - c);
      for(int k = 0; k < j; k++){
        if(abs(pts[k] - c) > r)
          tie(c, r) = circumcenter(pts[i], pts[j], pts[k]);
      }
    }
  }
  return {c, r};
}
```

4.4 Minkowski Sum

```
void reorder_poly(vector<pdd>& pnts){
  int mn = 0;
  for(int i = 1; i < (int)pnts.size(); i++)
    if(pnts[i].Y < pnts[mn].Y || (pnts[i].Y == pnts[mn].Y
        && pnts[i].X < pnts[mn].X))
      mn = i;
  rotate(pnts.begin(), pnts.begin() + mn, pnts.end());
}

vector<pdd> minkowski(vector<pdd> P, vector<pdd> Q){
  reorder_poly(P);
  reorder_poly(Q);
  int psz = P.size();
  int qsz = Q.size();
  P.pb(P[0]); P.pb(P[1]); Q.pb(Q[0]); Q.pb(Q[1]);
  vector<pdd> ans;
  int i = 0, j = 0;
  while(i < psz || j < qsz){
    ans.pb(P[i] + Q[j]);
    int t = sgn(cross(P[i + 1] - P[i], Q[j + 1] - Q[j]));
    if(t >= 0) i++;
    if(t <= 0) j++;
  }
  return ans;
}
```

4.5 Polar Angle Comparator

```
// -1: a // b (if same), 0/1: a < b
int cmp(pll a, pll b, bool same = true){
#define is_neg(k) (sgn(k.Y) < 0 || (sgn(k.Y) == 0 && sgn(k.
    X) < 0))
  int A = is_neg(a), B = is_neg(b);
  if(A != B)
    return A < B;
  if(sgn(cross(a, b)) == 0)
    return same ? abs2(a) < abs2(b) : -1;
  return sgn(cross(a, b)) > 0;
}
```

4.6 Half Plane Intersection

```
pll area_pair(Line a, Line b)
{ return pll(cross(a.Y - a.X, b.X - a.X), cross(a.Y - a.X,
    b.Y - a.X)); }
bool isin(Line l0, Line l1, Line l2) {
  // Check inter(l1, l2) strictly in l0
  auto [a123, a124] = area_pair(l0, l1);
  if (a123 - a124 < 0) swap(a123, a124), swap(l1.X, l1.Y);
  auto [b123, b124] = area_pair(l0, l2);
  if (b123 - b124 < 0) swap(l2.X, l2.Y);
  auto [c123, c124] = area_pair(l2, l1);
  if (c123 - c124 < 0) c123 *= -1, c124 *= -1;
  return c123 * (a123 - a124) < a123 * (c123 - c124); // C
      ^4
}
/* Having solution, check size > 2 */
/* --^-- Line.X --^-- Line.Y --^-- */
vector<Line> halfPlaneInter(vector<Line> arr) {
  sort(iter(arr), [&](Line a, Line b) -> int {
    if (cmp(a.Y - a.X, b.Y - b.X, 0) != -1)
      return cmp(a.Y - a.X, b.Y - b.X, 0);
    return ori(a.X, a.Y, b.Y) < 0;
  });
  deque<Line> dq(1, arr[0]);
  for (auto p : arr) {
    if (cmp(dq.back().Y - dq.back().X, p.Y - p.X, 0) == -1)
      continue;
    while (SZ(dq) >= 2 && !isin(p, dq[SZ(dq) - 2], dq.back
        ()))
      dq.pop_back();
    while (SZ(dq) >= 2 && !isin(p, dq[0], dq[1]))
      dq.pop_front();
    dq.pb(p);
  }
  while (SZ(dq) >= 3 && !isin(dq[0], dq[SZ(dq) - 2], dq.
      back()))
    dq.pop_back();
  while (SZ(dq) >= 3 && !isin(dq.back(), dq[0], dq[1]))
    dq.pop_front();
  return vector<Line>(iter(dq));
}
```

4.7 Dynamic Convex Hull

```
struct Line{
  ll a, b, l = MIN, r = MAX;
  Line(ll a, ll b): a(a), b(b) {}
  ll operator()(ll x) const{
    return a * x + b;
  }
  bool operator<(Line b) const{
    return a < b.a;
  }
  bool operator<(ll b) const{
    return r < b;
  }
};

ll iceil(ll a, ll b){
  if(b < 0) a *= -1, b *= -1;
  if(a > 0) return (a + b - 1) / b;
  else return a / b;
}

ll intersect(Line a, Line b){
  return iceil(a.b - b.b, b.a - a.a);
}

struct DynamicConvexHull{
  multiset<Line, less<>> ch;

  void add(Line ln){
    auto it = ch.lower_bound(ln);
    while(it != ch.end()){
      Line tl = *it;
```

```
      if(tl(tl.r) <= ln(tl.r)){
        it = ch.erase(it);
      }
      else break;
    }
    auto it2 = ch.lower_bound(ln);
    while(it2 != ch.begin()){
      Line tl = *prev(it2);
      if(tl(tl.l) <= ln(tl.l)){
        it2 = ch.erase(prev(it2));
      }
      else break;
    }
    it = ch.lower_bound(ln);
    if(it != ch.end()){
      Line tl = *it;
      if(tl(tl.l) >= ln(tl.l)) ln.r = tl.l - 1;
      else{
        ll pos = intersect(ln, tl);
        tl.l = pos;
        ln.r = pos - 1;
        ch.erase(it);
        ch.insert(tl);
      }
    }
    it2 = ch.lower_bound(ln);
    if(it2 != ch.begin()){
      Line tl = *prev(it2);
      if(tl(tl.r) >= ln(tl.r)) ln.l = tl.r + 1;
      else{
        ll pos = intersect(tl, ln);
        tl.r = pos - 1;
        ln.l = pos;
        ch.erase(prev(it2));
        ch.insert(tl);
      }
    }
    if(ln.l <= ln.r) ch.insert(ln);
  }

  ll query(ll pos){
    auto it = ch.lower_bound(pos);
    if(it == ch.end()) return 0;
    return (*it)(pos);
  }
};
```

4.8 3D Point

```
// Copy from 8BQube
struct Point {
  double x, y, z;
  Point(double _x = 0, double _y = 0, double _z = 0): x(_x)
      , y(_y), z(_z){}
  Point(pdd p) { x = p.X, y = p.Y, z = abs2(p); }
};
Point operator-(const Point &p1, const Point &p2)
{ return Point(p1.x - p2.x, p1.y - p2.y, p1.z - p2.z); }
Point cross(const Point &p1, const Point &p2)
{ return Point(p1.y * p2.z - p1.z * p2.y, p1.z * p2.x - p1.
    x * p2.z, p1.x * p2.y - p1.y * p2.x); }
double dot(const Point &p1, const Point &p2)
{ return p1.x * p2.x + p1.y * p2.y + p1.z * p2.z; }
double abs(const Point &a)
{ return sqrt(dot(a, a)); }
Point cross3(const Point &a, const Point &b, const Point &c
    )
{ return cross(b - a, c - a); }
double area(Point a, Point b, Point c)
{ return abs(cross3(a, b, c)); }
double volume(Point a, Point b, Point c, Point d)
{ return dot(cross3(a, b, c), d - a); }
pdd proj(Point a, Point b, Point c, Point u) {
// proj. u to the plane of a, b, and c
  Point e1 = b - a;
  Point e2 = c - a;
  e1 = e1 / abs(e1);
  e2 = e2 - e1 * dot(e2, e1);
  e2 = e2 / abs(e2);
  Point p = u - a;
  return pdd(dot(p, e1), dot(p, e2));
}
```

4.9 ConvexHull3D

```
// Copy from 8BQube
struct CH3D {
  struct face{int a, b, c; bool ok;} F[8 * N];
  double dblcmp(Point &p,face &f)
  {return dot(cross3(P[f.a], P[f.b], P[f.c]), p - P[f.a]);}
  int g[N][N], num, n;
  Point P[N];
  void deal(int p,int a,int b) {
    int f = g[a][b];
    face add;
    if (F[f].ok) {
      if (dblcmp(P[p],F[f]) > eps) dfs(p,f);
      else
        add.a = b, add.b = a, add.c = p, add.ok = 1, g[p][b
            ] = g[a][p] = g[b][a] = num, F[num++]=add;
    }
  }
  void dfs(int p, int now) {
    F[now].ok = 0;
    deal(p, F[now].b, F[now].a), deal(p, F[now].c, F[now].b
        ), deal(p, F[now].a, F[now].c);
  }
  bool same(int s,int t){
    Point &a = P[F[s].a];
    Point &b = P[F[s].b];
    Point &c = P[F[s].c];
    return fabs(volume(a, b, c, P[F[t].a])) < eps && fabs(
        volume(a, b, c, P[F[t].b])) < eps && fabs(volume(a,
         b, c, P[F[t].c])) < eps;
  }
  void init(int _n){n = _n, num = 0;}
  void solve() {
    face add;
    num = 0;
    if(n < 4) return;
    if([&](){
        for (int i = 1; i < n; ++i)
        if (abs(P[0] - P[i]) > eps)
        return swap(P[1], P[i]), 0;
        return 1;
        }() || [&](){
        for (int i = 2; i < n; ++i)
        if (abs(cross3(P[i], P[0], P[1])) > eps)
        return swap(P[2], P[i]), 0;
        return 1;
        }() || [&](){
        for (int i = 3; i < n; ++i)
        if (fabs(dot(cross(P[0] - P[1], P[1] - P[2]), P[0]
            - P[i])) > eps)
        return swap(P[3], P[i]), 0;
        return 1;
        }())return;
    for (int i = 0; i < 4; ++i) {
      add.a = (i + 1) % 4, add.b = (i + 2) % 4, add.c = (i
          + 3) % 4, add.ok = true;
      if (dblcmp(P[i],add) > 0) swap(add.b, add.c);
      g[add.a][add.b] = g[add.b][add.c] = g[add.c][add.a] =
          num;
      F[num++] = add;
    }
    for (int i = 4; i < n; ++i)
      for (int j = 0; j < num; ++j)
        if (F[j].ok && dblcmp(P[i],F[j]) > eps) {
          dfs(i, j);
          break;
        }
    for (int tmp = num, i = (num = 0); i < tmp; ++i)
```

```
    if (F[i].ok) F[num++] = F[i];
  }
  double get_area() {
    double res = 0.0;
    if (n == 3)
      return abs(cross3(P[0], P[1], P[2])) / 2.0;
    for (int i = 0; i < num; ++i)
      res += area(P[F[i].a], P[F[i].b], P[F[i].c]);
    return res / 2.0;
  }
  double get_volume() {
    double res = 0.0;
    for (int i = 0; i < num; ++i)
      res += volume(Point(0, 0, 0), P[F[i].a], P[F[i].b], P
          [F[i].c]);
    return fabs(res / 6.0);
  }
  int triangle() {return num;}
  int polygon() {
    int res = 0;
    for (int i = 0, flag = 1; i < num; ++i, res += flag,
        flag = 1)
      for (int j = 0; j < i && flag; ++j)
        flag &= !same(i,j);
    return res;
  }
  Point getcent(){
    Point ans(0, 0, 0), temp = P[F[0].a];
    double v = 0.0, t2;
    for (int i = 0; i < num; ++i)
      if (F[i].ok == true) {
        Point p1 = P[F[i].a], p2 = P[F[i].b], p3 = P[F[i].c
            ];
        t2 = volume(temp, p1, p2, p3) / 6.0;
        if (t2>0)
          ans.x += (p1.x + p2.x + p3.x + temp.x) * t2, ans.
              y += (p1.y + p2.y + p3.y + temp.y) * t2, ans.
              z += (p1.z + p2.z + p3.z + temp.z) * t2, v +=
               t2;
      }
    ans.x /= (4 * v), ans.y /= (4 * v), ans.z /= (4 * v);
    return ans;
  }
  double pointmindis(Point p) {
    double rt = 99999999;
    for(int i = 0; i < num; ++i)
      if(F[i].ok == true) {
        Point p1 = P[F[i].a], p2 = P[F[i].b], p3 = P[F[i].c
            ];
        double a = (p2.y - p1.y) * (p3.z - p1.z) - (p2.z -
            p1.z) * (p3.y - p1.y);
        double b = (p2.z - p1.z) * (p3.x - p1.x) - (p2.x -
            p1.x) * (p3.z - p1.z);
        double c = (p2.x - p1.x) * (p3.y - p1.y) - (p2.y -
            p1.y) * (p3.x - p1.x);
        double d = 0 - (a * p1.x + b * p1.y + c * p1.z);
        double temp = fabs(a * p.x + b * p.y + c * p.z + d)
             / sqrt(a * a + b * b + c * c);
        rt = min(rt, temp);
      }
    return rt;
  }
};
```

4.10 Circle Operations

```
// from 8BQube
const double PI=acos(-1);
vector<pdd> circleLineIntersection(pdd c, double r, pdd a,
    pdd b) {
  pdd p = a + (b - a) * dot(c - a, b - a) / abs2(b - a);
  double s = cross(b - a, c - a), h2 = r * r - s * s / abs2
      (b - a);
  if (h2 < 0) return {};
  if (h2 == 0) return {p};
  pdd h = (b - a) / abs(b - a) * sqrt(h2);
  return {p - h, p + h};
}
double _area(pdd pa, pdd pb, double r){
  if(abs(pa)<abs(pb)) swap(pa, pb);
  if(abs(pb)<eps) return 0;
  double S, h, theta;
  double a=abs(pb),b=abs(pa),c=abs(pb-pa);
  double cosB = dot(pb,pb-pa) / a / c, B = acos(cosB);
  double cosC = dot(pa,pb) / a / b, C = acos(cosC);
  if(a > r){
    S = (C/2)*r*r;
    h = a*b*sin(C)/c;
    if (h < r && B < PI/2) S -= (acos(h/r)*r*r - h*sqrt(r*r
        -h*h));
  }
  else if(b > r){
    theta = PI - B - asin(sin(B)/r*a);
    S = .5*a*r*sin(theta) + (C-theta)/2*r*r;
  }
  else S = .5*sin(C)*a*b;
  return S;
}
double areaPolyCircle(const vector<pdd> poly,const pdd &O,
    const double r){
  double S=0;
  for(int i=0;i<SZ(poly);++i)
    S+=_area(poly[i]-O,poly[(i+1)%SZ(poly)]-O,r)*ori(O,poly
        [i],poly[(i+1)%SZ(poly)]);
  return fabs(S);
}
bool CCinter(Cir &a, Cir &b, pdd &p1, pdd &p2) {
  pdd o1 = a.O, o2 = b.O;
  double r1 = a.R, r2 = b.R, d2 = abs2(o1 - o2), d = sqrt(
      d2);
  if(d < max(r1, r2) - min(r1, r2) || d > r1 + r2) return
      0;
  pdd u = (o1 + o2) * 0.5 + (o1 - o2) * ((r2 * r2 - r1 * r1
      ) / (2 * d2));
  double A = sqrt((r1 + r2 + d) * (r1 - r2 + d) * (r1 + r2
      - d) * (-r1 + r2 + d));
  pdd v = pdd(o1.Y - o2.Y, -o1.X + o2.X) * A / (2 * d2);
  p1 = u + v, p2 = u - v;
  return 1;
}
vector<Line> CCtang( const Cir& c1 , const Cir& c2 , int
    sign1 ){
  vector<Line> ret;
  double d_sq = abs2( c1.O - c2.O );
  if (sgn(d_sq) == 0) return ret;
  double d = sqrt(d_sq);
  pdd v = (c2.O - c1.O) / d;
  double c = (c1.R - sign1 * c2.R) / d; // cos t
  if (c * c > 1) return ret;
  double h = sqrt(max( 0.0, 1.0 - c * c)); // sin t
  for (int sign2 = 1; sign2 >= -1; sign2 -= 2) {
    pdd n = pdd(v.X * c - sign2 * h * v.Y,
        v.Y * c + sign2 * h * v.X);
    pdd p1 = c1.O + n * c1.R;
    pdd p2 = c2.O + n * (c2.R * sign1);
    if (sgn(p1.X - p2.X) == 0 and
        sgn(p1.Y - p2.Y) == 0)
      p2 = p1 + perp(c2.O - c1.O);
    ret.pb(Line(p1, p2));
  }
  return ret;
}
```

4.11 Delaunay Triangulation

```
// from 8BQube
/* Delaunay Triangulation:
Given a sets of points on 2D plane, find a
triangulation such that no points will strictly
inside circumcircle of any triangle.
find : return a triangle contain given point
add_point : add a point into triangulation
```

```
A Triangle is in triangulation iff. its has_chd is 0.
Region of triangle u: iterate each u.edge[i].tri,
each points are u.p[(i+1)%3], u.p[(i+2)%3]
Voronoi diagram: for each triangle in triangulation,
the bisector of all its edges will split the region.
nearest point will belong to the triangle containing it
 */
const ll inf = MAXC * MAXC * 100; // lower_bound unknown
struct Tri;
struct Edge {
  Tri* tri; int side;
  Edge(): tri(0), side(0){}
  Edge(Tri* _tri, int _side): tri(_tri), side(_side){}
};
struct Tri {
  pll p[3];
  Edge edge[3];
  Tri* chd[3];
  Tri() {}
  Tri(const pll& p0, const pll& p1, const pll& p2) {
    p[0] = p0; p[1] = p1; p[2] = p2;
    chd[0] = chd[1] = chd[2] = 0;
  }
  bool has_chd() const { return chd[0] != 0; }
  int num_chd() const {
    return !!chd[0] + !!chd[1] + !!chd[2];
  }
  bool contains(pll const& q) const {
    for (int i = 0; i < 3; ++i)
      if (ori(p[i], p[(i + 1) % 3], q) < 0)
        return 0;
    return 1;
  }
} pool[N * 10], *tris;
void edge(Edge a, Edge b) {
  if(a.tri) a.tri->edge[a.side] = b;
  if(b.tri) b.tri->edge[b.side] = a;
}
struct Trig { // Triangulation
  Trig() {
    the_root = // Tri should at least contain all points
      new(tris++) Tri(pll(-inf, -inf), pll(inf + inf, -inf)
          , pll(-inf, inf + inf));
  }
  Tri* find(pll p) { return find(the_root, p); }
  void add_point(const pll &p) { add_point(find(the_root, p
      ), p); }
  Tri* the_root;
  static Tri* find(Tri* root, const pll &p) {
    while (1) {
      if (!root->has_chd())
        return root;
      for (int i = 0; i < 3 && root->chd[i]; ++i)
        if (root->chd[i]->contains(p)) {
          root = root->chd[i];
          break;
        }
    }
    assert(0); // "point not found"
  }
  void add_point(Tri* root, pll const& p) {
    Tri* t[3];
    /* split it into three triangles */
    for (int i = 0; i < 3; ++i)
      t[i] = new(tris++) Tri(root->p[i], root->p[(i + 1) %
          3], p);
    for (int i = 0; i < 3; ++i)
      edge(Edge(t[i], 0), Edge(t[(i + 1) % 3], 1));
    for (int i = 0; i < 3; ++i)
      edge(Edge(t[i], 2), root->edge[(i + 2) % 3]);
    for (int i = 0; i < 3; ++i)
      root->chd[i] = t[i];
    for (int i = 0; i < 3; ++i)
      flip(t[i], 2);
  }
  void flip(Tri* tri, int pi) {
    Tri* trj = tri->edge[pi].tri;
    int pj = tri->edge[pi].side;
    if (!trj) return;
    if (!in_cc(tri->p[0], tri->p[1], tri->p[2], trj->p[pj])
        ) return;
    /* flip edge between tri,trj */
    Tri* trk = new(tris++) Tri(tri->p[(pi + 1) % 3], trj->p
        [pj], tri->p[pi]);
    Tri* trl = new(tris++) Tri(trj->p[(pj + 1) % 3], tri->p
        [pi], trj->p[pj]);
    edge(Edge(trk, 0), Edge(trl, 0));
    edge(Edge(trk, 1), tri->edge[(pi + 2) % 3]);
    edge(Edge(trk, 2), trj->edge[(pj + 1) % 3]);
    edge(Edge(trl, 1), trj->edge[(pj + 2) % 3]);
    edge(Edge(trl, 2), tri->edge[(pi + 1) % 3]);
    tri->chd[0] = trk; tri->chd[1] = trl; tri->chd[2] = 0;
    trj->chd[0] = trk; trj->chd[1] = trl; trj->chd[2] = 0;
    flip(trk, 1); flip(trk, 2);
    flip(trl, 1); flip(trl, 2);
  }
};
vector<Tri*> triang; // vector of all triangle
set<Tri*> vst;
void go(Tri* now) { // store all tri into triang
  if (vst.find(now) != vst.end())
    return;
  vst.insert(now);
  if (!now->has_chd())
    return triang.pb(now);
  for (int i = 0; i < now->num_chd(); ++i)
    go(now->chd[i]);
}
void build(int n, pll* ps) { // build triangulation
  tris = pool; triang.clear(); vst.clear();
  random_shuffle(ps, ps + n);
  Trig tri; // the triangulation structure
  for (int i = 0; i < n; ++i)
    tri.add_point(ps[i]);
  go(tri.the_root);
}
```

4.12 Voronoi Diagram

```
// from 8BQube
vector<Line> ls[N];
pll arr[N];
Line make_line(pdd p, Line l) {
  pdd d = l.Y - l.X; d = perp(d);
  pdd m = (l.X + l.Y) / 2;
  l = Line(m, m + d);
  if (ori(l.X, l.Y, p) < 0)
    l = Line(m + d, m);
  return l;
}
double calc_area(int id) {
  // use to calculate the area of point "strictly in the
      convex hull"
  vector<Line> hpi = halfPlaneInter(ls[id]);
  vector<pdd> ps;
  for (int i = 0; i < SZ(hpi); ++i)
    ps.pb(intersect(hpi[i].X, hpi[i].Y, hpi[(i + 1) % SZ(
        hpi)].X, hpi[(i + 1) % SZ(hpi)].Y));
  double rt = 0;
  for (int i = 0; i < SZ(ps); ++i)
    rt += cross(ps[i], ps[(i + 1) % SZ(ps)]);
  return fabs(rt) / 2;
}
void solve(int n, pii *oarr) {
  map<pll, int> mp;
  for (int i = 0; i < n; ++i)
    arr[i] = pll(oarr[i].X, oarr[i].Y), mp[arr[i]] = i;
  build(n, arr); // Triangulation
  for (auto *t : triang) {
    vector<int> p;
    for (int i = 0; i < 3; ++i)
      if (mp.find(t->p[i]) != mp.end())
        p.pb(mp[t->p[i]]);
```

```
    for (int i = 0; i < SZ(p); ++i)
      for (int j = i + 1; j < SZ(p); ++j) {
        Line l(oarr[p[i]], oarr[p[j]]);
        ls[p[i]].pb(make_line(oarr[p[i]], l));
        ls[p[j]].pb(make_line(oarr[p[j]], l));
      }
  }
}
```

5 Graph

5.1 Block Cut Tree

```
struct BlockCutTree{
  vector<vector<int>> tree; // 1-based
  vector<int> node;
  vector<int> type; // 0:square, 1:circle

  bool iscut(int v){
    return type[node[v]] == 1;
  }

  vector<int> getbcc(int v){
    if(!iscut(v)) return {node[v]};
    vector<int> ans;
    for(int i : tree[node[v]])
      ans.pb(i);
    return ans;
  }

  void build(int n, vector<vector<int>>& g){
    tree.resize(2 * n + 1);
    type.resize(2 * n + 1);
    node.resize(n + 1, -1);
    vector<int> in(n + 1);
    vector<int> low(n + 1);
    stack<int> st;

    int ts = 1;
    int bcc = 1;
    auto addv = [&](int id, int v){
      if(node[v] == -1){
        node[v] = id;
        return;
      }
      if(type[node[v]] == 0){
        int o = node[v];
        node[v] = bcc++;
        type[node[v]] = 1;
        tree[o].pb(node[v]);
        tree[node[v]].pb(o);
      }
      tree[id].pb(node[v]);
      tree[node[v]].pb(id);
    };
    function<void(int, int)> dfs = [&](int now, int p){
      in[now] = low[now] = ts++;
      st.push(now);
      int child = 0;
      for(int i : g[now]){
        if(i == p) continue;
        if(in[i]){
          low[now] = min(low[now], in[i]);
          continue;
        }
        child++;
        dfs(i, now);
        low[now] = min(low[now], low[i]);

        if(low[i] >= in[now]){
          int nowid = bcc++;
          while(true){
            int x = st.top();
            st.pop();
            addv(nowid, x);
            if(x == i) break;
          }
          addv(nowid, now);
        }
      }
      if(child == 0 && now == p) addv(bcc++, now);
    };
    dfs(1, 1);
  }
};
```

5.2 2-SAT

```
struct SAT{ // 0-based, [n, 2n) is neg of [0, n)
  int n;
  vector<vector<int>> g, rg;
  bool ok = true;
  vector<bool> ans;

  void init(int _n){
    n = _n;
    g.resize(2 * n);
    rg.resize(2 * n);
    ans.resize(n);
  }
  int neg(int v){
    return v < n ? v + n : v - n;
  }
  void addEdge(int u, int v){
    g[u].pb(v);
    rg[v].pb(u);
  }
  void addClause(int a, int b){
    addEdge(neg(a), b);
    addEdge(neg(b), a);
  }
  void build(){
    vector<bool> vst(2 * n, false);
    vector<int> tmp, scc(2 * n, -1);
    int cnt = 1;
    function<void(int)> dfs = [&](int now){
      vst[now] = true;
      for(int i : rg[now]){
        if(vst[i]) continue;
        dfs(i);
      }
      tmp.pb(now);
    };
    for(int i = 0; i < 2 * n; i++){
      if(!vst[i]) dfs(i);
    }
    reverse(all(tmp));
    function<void(int, int)> dfs2 = [&](int now, int id){
      scc[now] = id;
      for(int i : g[now]){
        if(scc[i] != -1) continue;
        dfs2(i, id);
      }
    };
    for(int i : tmp){
      if(scc[i] == -1) dfs2(i, cnt++);
    }
    debug(scc);
    for(int i = 0; i < n; i++){
      if(scc[i] == scc[neg(i)]){
        ok = false;
        return;
      }
      if(scc[i] < scc[neg(i)]) ans[i] = true;
      else ans[i] = false;
    }
  }

};
```

5.3 Dominator Tree

```
// copy from 8BQube
struct dominator_tree { // 1-base
  vector<int> G[N], rG[N];
  int n, pa[N], dfn[N], id[N], Time;
  int semi[N], idom[N], best[N];
  vector<int> tree[N]; // dominator_tree
  void init(int _n) {
    n = _n;
    for (int i = 1; i <= n; ++i)
      G[i].clear(), rG[i].clear();
  }
  void add_edge(int u, int v) {
    G[u].pb(v), rG[v].pb(u);
  }
  void dfs(int u) {
    id[dfn[u] = ++Time] = u;
    for (auto v : G[u])
      if (!dfn[v]) dfs(v), pa[dfn[v]] = dfn[u];
  }
  int find(int y, int x) {
    if (y <= x) return y;
    int tmp = find(pa[y], x);
    if (semi[best[y]] > semi[best[pa[y]]])
      best[y] = best[pa[y]];
    return pa[y] = tmp;
  }
  void tarjan(int root) {
    Time = 0;
    for (int i = 1; i <= n; ++i) {
      dfn[i] = idom[i] = 0;
      tree[i].clear();
      best[i] = semi[i] = i;
    }
    dfs(root);
    for (int i = Time; i > 1; --i) {
      int u = id[i];
      for (auto v : rG[u])
        if (v = dfn[v]) {
          find(v, i);
          semi[i] = min(semi[i], semi[best[v]]);
        }
      tree[semi[i]].pb(i);
      for (auto v : tree[pa[i]]) {
        find(v, pa[i]);
        idom[v] =
          semi[best[v]] == pa[i] ? pa[i] : best[v];
      }
      tree[pa[i]].clear();
    }
    for (int i = 2; i <= Time; ++i) {
      if (idom[i] != semi[i]) idom[i] = idom[idom[i]];
      tree[id[idom[i]]].pb(id[i]);
    }
  }
};
```

5.4 Virtual Tree

```
// copy from 8BQube
vector<int> vG[N];
int top, st[N];

void insert(int u) {
  if (top == -1) return st[++top] = u, void();
  int p = LCA(st[top], u);
  if (p == st[top]) return st[++top] = u, void();
  while (top >= 1 && dep[st[top - 1]] >= dep[p])
    vG[st[top - 1]].pb(st[top]), --top;
  if (st[top] != p)
    vG[p].pb(st[top]), --top, st[++top] = p;
  st[++top] = u;
}

void reset(int u) {
  for (int i : vG[u]) reset(i);
  vG[u].clear();
}

void solve(vector<int> &v) {
  top = -1;
  sort(ALL(v),
      [&](int a, int b) { return dfn[a] < dfn[b]; });
  for (int i : v) insert(i);
  while (top > 0) vG[st[top - 1]].pb(st[top]), --top;
  // do something
  reset(v[0]);
}
```

5.5 Directed Minimum Spanning Tree

```
const ll INF = LLONG_MAX;
struct edge{
  int u = -1, v = -1;
  ll w = INF;
  int id = -1;
};

// 0-based, E[i].id = i
bool DMST(int n, vector<edge> &E, int root, vector<edge> &
    sol){
  vector<int> id(n), vis(n);
  vector<edge> in(n);
  for(edge e : E)
    if(e.u != e.v && e.w < in[e.v].w && e.v != root)
      in[e.v] = e;
  for(int i = 0; i < n; i++)
    if(i != root && in[i].u == -1) return false; // no sol
  int cnt = 0;
  fill(iter(id), -1); fill(iter(vis), -1);
  for(int u = 0; u < n; u++){
    int v = u;
    while(vis[v] != u && id[v] == -1 && in[v].u != -1)
      vis[v] = u, v = in[v].u;
    if(v != root && id[v] == -1){
      for(int x = in[v].u; x != v; x = in[x].u)
        id[x] = cnt;
      id[v] = cnt++;
    }
  }
  if(!cnt) return sol = in, true; // no cycle
  for(int u = 0; u < n; u++)
    if(id[u] == -1) id[u] = cnt++;
  vector<edge> nE;
  for(int i = 0; i < SZ(E); i++){
    edge tmp = E[i];
    tmp.u = id[tmp.u], tmp.v = id[tmp.v];
    if(in[E[i].v].w != INF) tmp.w -= in[E[i].v].w;
    nE.pb(tmp);
  }
  vector<edge> tsol;
  if(!DMST(cnt, nE, id[root], tsol)) return false;
  sol.resize(n);
  for(int i = 0; i < cnt; i++){
    if(i == id[root]) continue;
    int t = tsol[i].id;
    sol[E[t].v] = E[t];
  }
  for(int i = 0; i < n; i++)
    if(sol[i].id == -1) sol[i] = in[i];
  return true;
}
```

6 Math

6.1 Extended Euclidean Algorithm

```
// ax+ny = 1, ax+ny == ax == 1 (mod n)
void extgcd(ll x,ll y,ll &g,ll &a,ll &b) {
```

```
  if (y == 0) g=x,a=1,b=0;
  else extgcd(y,x%y,g,b,a),b-=(x/y)*a;
}
```

6.2 Floor & Ceil

```
int floor_div(int a,int b){
  return a/b-(a%b&&a<0^b<0);
}
int ceil_div(int a,int b){
  return a/b+(a%b&&a<0^b>0);
}
```

6.3 Legendre

```
// the Jacobi symbol is a generalization of the Legendre
    symbol,
// such that the bottom doesn't need to be prime.
// (n|p) -> same as legendre
// (n|ab) = (n|a)(n|b)
// work with long long
int Jacobi(int a, int m) {
  int s = 1;
  for (; m > 1; ) {
    a %= m;
    if (a == 0) return 0;
    const int r = __builtin_ctz(a);
    if ((r & 1) && ((m + 2) & 4)) s = -s;
    a >>= r;
    if (a & m & 2) s = -s;
    swap(a, m);
  }
  return s;
}

// 0: a == 0
// -1: a isn't a quad res of p
// else: return X with X^2 % p == a
// doesn't work with long long
int QuadraticResidue(int a, int p) {
  if (p == 2) return a & 1;
  const int jc = Jacobi(a, p);
  if (jc == 0) return 0;
  if (jc == -1) return -1;
  int b, d;
  for (; ; ) {
    b = rand() % p;
    d = (1LL * b * b + p - a) % p;
    if (Jacobi(d, p) == -1) break;
  }
  int f0 = b, f1 = 1, g0 = 1, g1 = 0, tmp;
  for (int e = (1LL + p) >> 1; e; e >>= 1) {
    if (e & 1) {
      tmp = (1LL * g0 * f0 + 1LL * d * (1LL * g1 * f1 % p))
          % p;
      g1 = (1LL * g0 * f1 + 1LL * g1 * f0) % p;
      g0 = tmp;
    }
    tmp = (1LL * f0 * f0 + 1LL * d * (1LL * f1 * f1 % p)) %
        p;
    f1 = (2LL * f0 * f1) % p;
    f0 = tmp;
  }
  return g0;
}
```

6.4 Simplex

```
#pragma once

typedef double T; // long double, Rational, double + mod<P
    >...
typedef vector<T> vd;
typedef vector<vd> vvd;

const T eps = 1e-8, inf = 1/.0;
#define MP make_pair
#define ltj(X) if(s == -1 || MP(X[j],N[j]) < MP(X[s],N[s]))
     s=j

struct LPSolver {
  int m, n;
  vi N, B;
  vvd D;

  LPSolver(const vvd& A, const vd& b, const vd& c) :
    m(sz(b)), n(sz(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
      rep(i,0,m) rep(j,0,n) D[i][j] = A[i][j];
      rep(i,0,m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[
          i];}
      rep(j,0,n) { N[j] = j; D[m][j] = -c[j]; }
      N[n] = -1; D[m+1][n] = 1;
    }

  void pivot(int r, int s) {
    T *a = D[r].data(), inv = 1 / a[s];
    rep(i,0,m+2) if (i != r && abs(D[i][s]) > eps) {
      T *b = D[i].data(), inv2 = b[s] * inv;
      rep(j,0,n+2) b[j] -= a[j] * inv2;
      b[s] = a[s] * inv2;
    }
    rep(j,0,n+2) if (j != s) D[r][j] *= inv;
    rep(i,0,m+2) if (i != r) D[i][s] *= -inv;
    D[r][s] = inv;
    swap(B[r], N[s]);
  }

  bool simplex(int phase) {
    int x = m + phase - 1;
    for (;;) {
      int s = -1;
      rep(j,0,n+1) if (N[j] != -phase) ltj(D[x]);
      if (D[x][s] >= -eps) return true;
      int r = -1;
      rep(i,0,m) {
        if (D[i][s] <= eps) continue;
        if (r == -1 || MP(D[i][n+1] / D[i][s], B[i])
            < MP(D[r][n+1] / D[r][s], B[r])) r = i;
      }
      if (r == -1) return false;
      pivot(r, s);
    }
  }

  T solve(vd &x) {
    int r = 0;
    rep(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
    if (D[r][n+1] < -eps) {
      pivot(r, n);
      if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
      rep(i,0,m) if (B[i] == -1) {
        int s = 0;
        rep(j,1,n+1) ltj(D[i]);
        pivot(i, s);
      }
    }
    bool ok = simplex(1); x = vd(n);
    rep(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
    return ok ? D[m][n+1] : inf;
  }
};
```

6.5 Floor Sum

```
// from 8BQube
ll floor_sum(ll n, ll m, ll a, ll b) {
  ll ans = 0;
  if (a >= m)
    ans += (n - 1) * n * (a / m) / 2, a %= m;
```

```
  if (b >= m)
    ans += n * (b / m), b %= m;
  ll y_max = (a * n + b) / m, x_max = (y_max * m - b);
  if (y_max == 0) return ans;
  ans += (n - (x_max + a - 1) / a) * y_max;
  ans += floor_sum(y_max, a, m, (a - x_max % a) % a);
  return ans;
}// sum^{n-1}_0 floor((a * i + b) / m) in log(n + m + a + b
    )
```

6.6 Miller Rabin & Pollard Rho

```
// n < 4,759,123,141      3 : 2, 7, 61
// n < 1,122,004,669,633  4 : 2, 13, 23, 1662803
// n < 3,474,749,660,383  6 : pirmes <= 13
// n < 2^64              7 :
// 2, 325, 9375, 28178, 450775, 9780504, 1795265022
ll mul(ll a, ll b, ll n){
  return (__int128)a * b % n;
}
bool Miller_Rabin(ll a, ll n) {
  if ((a = a % n) == 0) return 1;
  if (n % 2 == 0) return n == 2;
  ll tmp = (n - 1) / ((n - 1) & (1 - n));
  ll t = __lg(((n - 1) & (1 - n))), x = 1;
  for (; tmp; tmp >>= 1, a = mul(a, a, n))
    if (tmp & 1) x = mul(x, a, n);
  if (x == 1 || x == n - 1) return 1;
  while (--t)
    if ((x = mul(x, x, n)) == n - 1) return 1;
  return 0;
}
bool prime(ll n){
  vector<ll> tmp = {2, 325, 9375, 28178, 450775, 9780504,
      1795265022};
  for(ll i : tmp)
    if(!Miller_Rabin(i, n)) return false;
  return true;
}
map<ll, int> cnt;
void PollardRho(ll n) {
  if (n == 1) return;
  if (prime(n)) return ++cnt[n], void();
  if (n % 2 == 0) return PollardRho(n / 2), ++cnt[2], void
      ();
  ll x = 2, y = 2, d = 1, p = 1;
#define f(x, n, p) ((mul(x, x, n) + p) % n)
  while (true) {
    if (d != n && d != 1) {
      PollardRho(n / d);
      PollardRho(d);
      return;
    }
    if (d == n) ++p;
    x = f(x, n, p), y = f(f(y, n, p), n, p);
    d = gcd(abs(x - y), n);
  }
}
```

7 Misc

7.1 Fraction

```
struct Frac{
  ll p, q;  // p / q
  Frac(ll _p, ll _q): p(_p), q(_q) { if(q < 0) p = -p, q =
      -q; }
};
Frac operator-(Frac a) { return Frac(-a.p, a.q); }
Frac operator+(Frac a, Frac b){
  ll q = a.q * b.q;
  ll p = a.p * b.q + b.p * a.q;
  return Frac(p, q);
}
Frac inv(Frac a){ return Frac(a.q, a.p); }
Frac operator-(Frac a, Frac b) { return a + (-b); }
Frac operator*(Frac a, Frac b) { return Frac({a.p * b.p, a.
    q * b.q}); }
Frac operator/(Frac a, Frac b) { return a * inv(b); }
ostream& operator<<(ostream& o, Frac a) { return o << a.p
    << '/' << a.q; }
```

7.2 Matroid

我們稱一個二元組 $M = (E, \mathcal{I})$ 為一個擬陣，其中 $\mathcal{I} \subseteq 2^E$ 為 E 的子集所形成的**非空**集合，若：

- 若 $S \in \mathcal{I}$ 以及 $S' \subsetneq S$，則 $S' \in \mathcal{I}$
- 對於 $S_1, S_2 \in \mathcal{I}$ 滿足 $|S_1| < |S_2|$，存在 $e \in S_2 \setminus S_1$ 使得 $S_1 \cup \{e\} \in \mathcal{I}$

除此之外，我們有以下的定義：

- 位於 $\mathcal{I}$ 中的集合我們稱之為獨立集（independent set），反之不在 $\mathcal{I}$ 中的我們稱為相依集（dependent set）
- 極大的獨立集為基底（base）、極小的相依集為迴路（circuit）
- 一個集合 Y 的秩（rank）$r(Y)$ 為該集合中最大的獨立子集，也就是 $r(Y) = \max\{|X| \mid X \subseteq Y \text{ 且} X \in \mathcal{I}\}$

性質：

1. $X \subseteq Y \wedge Y \in \mathcal{I} \implies X \in \mathcal{I}$
2. $X \subseteq Y \wedge X \notin \mathcal{I} \implies Y \notin \mathcal{I}$
3. 若 B 與 B' 皆是基底且 $B \subseteq B'$，則 $B = B'$
若 C 與 C' 皆是迴路且 $C \subseteq C'$，則 $C = C'$
4. $e \in E \wedge X \subseteq E \implies r(X) \le r(X \cup \{e\}) \le r(X) + 1$ i.e. 加入一個元素後秩不會降底，最多增加 1
5. $\forall Y \subseteq E, \exists X \subseteq Y, r(X) = |X| = r(Y)$

一些等價的性質：

1. 對於所有 $X \subseteq E$，X 的極大獨立子集都有相同的大小
2. 對於 $B_1, B_2 \in \mathcal{B} \wedge B_1 \ne B_2$，對於所有 $e_1 \in B_1 \setminus B_2$，存在 $e_2 \in B_2 \setminus B_1$ 使得 $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$
3. 對於 $X, Y \in \mathcal{I}$ 且 $|X| < |Y|$，存在 $e \in Y \setminus X$ 使得 $X \cup \{e\} \in \mathcal{B}$
4. 如果 $r(X \cup \{e_1\}) = r(X \cup \{e_2\}) = r(X)$，則 $r(X \cup \{e_1, e_2\}) = r(X)$。
如果 $r(X \cup \{e\}) = r(X)$ 對於所有 $e \in E'$ 都成立，則 $r(X \cup E') = r(X)$。

擬陣交

Data: 兩個擬陣 $M_1 = (E, \mathcal{I}_1)$ 以及 $M_2 = (E, \mathcal{I}_2)$
Result: I 為最大的位於 $\mathcal{I}_1 \cap \mathcal{I}_2$ 中的獨立集
$I \leftarrow \emptyset$
$X_1 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_1\}$
$X_2 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_2\}$
while $X_1 \ne \emptyset$ 且 $X_2 \ne \emptyset$ **do**
  if $e \in X_1 \cap X_2$ **then**
    $I \leftarrow I \cup \{e\}$
  else
    構造交換圖 $\mathcal{D}_{M_1, M_2}(I)$
    在交換圖上找到一條 X_1 到 X_2 且沒有捷徑的路徑 P
    $I \leftarrow I \triangle P$
  end if
  $X_1 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_1\}$
  $X_2 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_2\}$
end while

8 Polynomial

8.1 FFT

```
using val_t = complex<double>;
template<int MAXN>
struct FFT {
  const double PI = acos(-1);
  val_t w[MAXN];
  FFT() {
    for (int i = 0; i < MAXN; ++i) {
      double arg = 2 * PI * i / MAXN;
      w[i] = val_t(cos(arg), sin(arg));
    }
  }
```

```
  void bitrev(vector<val_t> &a, int n) //same as NTT
  void trans(vector<val_t> &a, int n, bool inv = false) {
    bitrev(a, n);
    for (int L = 2; L <= n; L <<= 1) {
      int dx = MAXN / L, dl = L >> 1;
      for (int i = 0; i < n; i += L) {
        for (int j = i, x = 0; j < i + dl; ++j, x += dx) {
          val_t tmp = a[j + dl] * (inv ? conj(w[x]) : w[x])
              ;
          a[j + dl] = a[j] - tmp;
          a[j] += tmp;
        }
      }
    }
    if (inv) {
      for (int i = 0; i < n; ++i) a[i] /= n;
    }
  }
  //multiplying two polynomials A * B:
  //fft.trans(A, siz, 0), fft.trans(B, siz, 0):
  //A[i] *= B[i], fft.trans(A, siz, 1);
};
```

8.2 NTT

```
//to be modified
//(2^16)+1, 65537, 3
//7*17*(2^23)+1, 998244353, 3
//1255*(2^20)+1, 1315962881, 3
//51*(2^25)+1, 1711276033, 29
template<int MAXN, ll P, ll RT> //MAXN must be 2^k
struct NTT {
  ll w[MAXN];
  ll mpow(ll a, ll n);
  ll minv(ll a) { return mpow(a, P - 2); }
  NTT() {
    ll dw = mpow(RT, (P - 1) / MAXN);
    w[0] = 1;
    for (int i = 1; i < MAXN; ++i) w[i] = w[i - 1] * dw % P
        ;
  }
  void bitrev(vector<ll> &a, int n) {
    int i = 0;
    for (int j = 1; j < n - 1; ++j) {
      for (int k = n >> 1; (i ^= k) < k; k >>= 1);
      if (j < i) swap(a[i], a[j]);
    }
  }
  void operator()(vector<ll> &a, int n, bool inv = false) {
       //0 <= a[i] < P
    bitrev(a, n);
    for (int L = 2; L <= n; L <<= 1) {
      int dx = MAXN / L, dl = L >> 1;
      for (int i = 0; i < n; i += L) {
        for (int j = i, x = 0; j < i + dl; ++j, x += dx) {
          ll tmp = a[j + dl] * w[x] % P;
          if ((a[j + dl] = a[j] - tmp) < 0) a[j + dl] += P;
          if ((a[j] += tmp) >= P) a[j] -= P;
        }
      }
    }
    if (inv) {
      reverse(a.begin()+1, a.begin()+n);
      ll invn = minv(n);
      for (int i = 0; i < n; ++i) a[i] = a[i] * invn % P;
    }
  }
};
```

8.3 Polynomial Operation

```
// Copy from 8BQube
#define fi(s, n) for (int i = (int)(s); i < (int)(n); ++i)
template<int MAXN, ll P, ll RT> // MAXN = 2^k
struct Poly : vector<ll> { // coefficients in [0, P)
  using vector<ll>::vector;
  static NTT<MAXN, P, RT> ntt;
  int n() const { return (int)size(); } // n() >= 1
  Poly(const Poly &p, int m) : vector<ll>(m) {
    copy_n(p.data(), min(p.n(), m), data());
  }
  Poly& irev() { return reverse(data(), data() + n()), *
      this; }
  Poly& isz(int m) { return resize(m), *this; }
  Poly& iadd(const Poly &rhs) { // n() == rhs.n()
    fi(0, n()) if (((*this)[i] += rhs[i]) >= P) (*this)[i]
        -= P;
    return *this;
  }
  Poly& imul(ll k) {
    fi(0, n()) (*this)[i] = (*this)[i] * k % P;
    return *this;
  }
  Poly Mul(const Poly &rhs) const {
    int m = 1;
    while (m < n() + rhs.n() - 1) m <<= 1;
    Poly X(*this, m), Y(rhs, m);
    ntt(X.data(), m), ntt(Y.data(), m);
    fi(0, m) X[i] = X[i] * Y[i] % P;
    ntt(X.data(), m, true);
    return X.isz(n() + rhs.n() - 1);
  }
  Poly Inv() const { // (*this)[0] != 0, 1e5/95ms
    if (n() == 1) return {ntt.minv((*this)[0])};
    int m = 1;
    while (m < n() * 2) m <<= 1;
    Poly Xi = Poly(*this, (n() + 1) / 2).Inv().isz(m);
    Poly Y(*this, m);
    ntt(Xi.data(), m), ntt(Y.data(), m);
    fi(0, m) {
      Xi[i] *= (2 - Xi[i] * Y[i]) % P;
      if ((Xi[i] %= P) < 0) Xi[i] += P;
    }
    ntt(Xi.data(), m, true);
    return Xi.isz(n());
  }
  Poly Sqrt() const { // Jacobi((*this)[0], P) = 1, 1e5/235
      ms
    if (n() == 1) return {QuadraticResidue((*this)[0], P)};
    Poly X = Poly(*this, (n() + 1) / 2).Sqrt().isz(n());
    return X.iadd(Mul(X.Inv()).isz(n())).imul(P / 2 + 1);
  }
  pair<Poly, Poly> DivMod(const Poly &rhs) const { // (rhs
      .)back() != 0
    if (n() < rhs.n()) return {{0}, *this};
    const int m = n() - rhs.n() + 1;
    Poly X(rhs); X.irev().isz(m);
    Poly Y(*this); Y.irev().isz(m);
    Poly Q = Y.Mul(X.Inv()).isz(m).irev();
    X = rhs.Mul(Q), Y = *this;
    fi(0, n()) if ((Y[i] -= X[i]) < 0) Y[i] += P;
    return {Q, Y.isz(max(1, rhs.n() - 1))};
  }
  Poly Dx() const {
    Poly ret(n() - 1);
    fi(0, ret.n()) ret[i] = (i + 1) * (*this)[i + 1] % P;
    return ret.isz(max(1, ret.n()));
  }
  Poly Sx() const {
    Poly ret(n() + 1);
    fi(0, n()) ret[i + 1] = ntt.minv(i + 1) * (*this)[i] %
        P;
    return ret;
  }
  Poly _tmul(int nn, const Poly &rhs) const {
    Poly Y = Mul(rhs).isz(n() + nn - 1);
    return Poly(Y.data() + n() - 1, Y.data() + Y.n());
  }
  vector<ll> _eval(const vector<ll> &x, const vector<Poly>
      &up) const {
    const int m = (int)x.size();
    if (!m) return {};
```

```
    vector<Poly> down(m * 2);
    // down[1] = DivMod(up[1]).second;
    // fi(2, m * 2) down[i] = down[i / 2].DivMod(up[i]).
        second;
    down[1] = Poly(up[1]).irev().isz(n()).Inv().irev().
        _tmul(m, *this);
    fi(2, m * 2) down[i] = up[i ^ 1]._tmul(up[i].n() - 1,
        down[i / 2]);
    vector<ll> y(m);
    fi(0, m) y[i] = down[m + i][0];
    return y;
  }
  static vector<Poly> _tree1(const vector<ll> &x) {
    const int m = (int)x.size();
    vector<Poly> up(m * 2);
    fi(0, m) up[m + i] = {(x[i] ? P - x[i] : 0), 1};
    for (int i = m - 1; i > 0; --i) up[i] = up[i * 2].Mul(
        up[i * 2 + 1]);
    return up;
  }
  vector<ll> Eval(const vector<ll> &x) const { // 1e5, 1s
    auto up = _tree1(x); return _eval(x, up);
  }
  static Poly Interpolate(const vector<ll> &x, const vector
      <ll> &y) { // 1e5, 1.4s
    const int m = (int)x.size();
    vector<Poly> up = _tree1(x), down(m * 2);
    vector<ll> z = up[1].Dx()._eval(x, up);
    fi(0, m) z[i] = y[i] * ntt.minv(z[i]) % P;
    fi(0, m) down[m + i] = {z[i]};
    for (int i = m - 1; i > 0; --i) down[i] = down[i * 2].
        Mul(up[i * 2 + 1]).iadd(down[i * 2 + 1].Mul(up[i *
        2]));
    return down[1];
  }
  Poly Ln() const { // (*this)[0] == 1, 1e5/170ms
    return Dx().Mul(Inv()).Sx().isz(n());
  }
  Poly Exp() const { // (*this)[0] == 0, 1e5/360ms
    if (n() == 1) return {1};
    Poly X = Poly(*this, (n() + 1) / 2).Exp().isz(n());
    Poly Y = X.Ln(); Y[0] = P - 1;
    fi(0, n()) if ((Y[i] = (*this)[i] - Y[i]) < 0) Y[i] +=
        P;
    return X.Mul(Y).isz(n());
  }
  // M := P(P - 1). If k >= M, k := k % M + M.
  Poly Pow(ll k) const {
    int nz = 0;
    while (nz < n() && !(*this)[nz]) ++nz;
    if (nz * min(k, (ll)n()) >= n()) return Poly(n());
    if (!k) return Poly(Poly {1}, n());
    Poly X(data() + nz, data() + nz + n() - nz * k);
    const ll c = ntt.mpow(X[0], k % (P - 1));
    return X.Ln().imul(k % P).Exp().imul(c).irev().isz(n())
        .irev();
  }
  static ll LinearRecursion(const vector<ll> &a, const
      vector<ll> &coef, ll n) { // a_n = \sum c_j a_(n-j)
    const int k = (int)a.size();
    assert((int)coef.size() == k + 1);
    Poly C(k + 1), W(Poly {1}, k), M = {0, 1};
    fi(1, k + 1) C[k - i] = coef[i] ? P - coef[i] : 0;
    C[k] = 1;
    while (n) {
      if (n % 2) W = W.Mul(M).DivMod(C).second;
      n /= 2, M = M.Mul(M).DivMod(C).second;
    }
    ll ret = 0;
    fi(0, k) ret = (ret + W[i] * a[i]) % P;
    return ret;
  }
};
#undef fi
using Poly_t = Poly<131072 * 2, 998244353, 3>;
template<> decltype(Poly_t::ntt) Poly_t::ntt = {};
```

8.4 Generating Function

8.4.1 Ordinary Generating Function

- $C(x) = A(rx)$: $c_n = r^n a_n$ 的一般生成函數。
- $C(x) = A(x) + B(x)$: $c_n = a_n + b_n$ 的一般生成函數。
- $C(x) = A(x)B(x)$: $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ 的一般生成函數。
- $C(x) = A(x)^k$: $c_n = \sum_{i_1+i_2+\ldots+i_k=n} a_{i_1} a_{i_2} \ldots a_{i_k}$ 的一般生成函數。
- $C(x) = xA(x)'$: $c_n = na_n$ 的一般生成函數。
- $C(x) = \frac{A(x)}{1-x}$: $c_n = \sum_{i=0}^{n} a_i$ 的一般生成函數。
- $C(x) = A(1) + x\frac{A(1)-A(x)}{1-x}$: $c_n = \sum_{i=n}^{\infty} a_i$ 的一般生成函數。

常用展開式

- $\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots$
- $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$, $\binom{a}{n} = \frac{a(a-1)(a-2)\ldots(a-n+1)}{n!}$.

常見生函

- 卡特蘭數：$f(x) = \frac{1-\sqrt{1-4x}}{2x}$

8.4.2 Exponential Generating Function

$a_0, a_1, \ldots$ 的指數生成函數：

$$\hat{A}(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} = a_0 + a_1 x + \frac{a_2}{2!} x^2 + \frac{a_3}{3!} x^3 + \ldots$$

- $\hat{C}(x) = \hat{A}(x) + \hat{B}(x)$: $c_n = a_n + b_n$ 的指數生成函數
- $\hat{C}(x) = \hat{A}^{(k)}(x)$: $c_n = a_{n+k}$ 的指數生成函數
- $\hat{C}(x) = x\hat{A}(x)$: $c_n = na_n$ 的指數生成函數
- $\hat{C}(x) = \hat{A}(x)\hat{B}(x)$: $c_n = \sum_{k=0}^{n} \binom{n}{i} a_k b_{n-k}$ 的指數生成函數
- $\hat{C}(x) = \hat{A}(x)^k$: $\sum_{i_1+i_2+\cdots+i_k=n} \binom{n}{i_1, i_2, \ldots, i_k} a_i a_{i_2} \ldots a_{i_k}$ 的指數生成函數
- $\hat{C}(x) = \exp(A(x))$: 假設 $A(x)$ 是一個分量 (component) 的生成函數，那 $\hat{C}(x)$ 是將 n 個有編號的東西分成若干個分量的指數生成函數

9 String

9.1 Rolling Hash

```
int n;
string s;
vector<ll> h, rh;
vector<ll> kp;
const ll K = 26, MOD = 1000000007;

void topos(ll &a){
  a = (a % MOD + MOD) % MOD;
}

int ord(char c){
  return c - 'a';
}

pll geth(int l, int r){
  if(l > r) return mp(0, 0);
  ll ans = h[r] - h[l - 1] * kp[r - l + 1];
  topos(ans);
  return mp(ans, r - l + 1);
}

pll getrh(int l, int r){
  if(l > r) return mp(0, 0);
  l = n - l + 1;
  r = n - r + 1;
  swap(l, r);
  ll ans = rh[r] - rh[l - 1] * kp[r - l + 1];
```

```
  topos(ans);
  return mp(ans, r - l + 1);
}

pll concat(pll a, pll b){
  ll ans = a.F * kp[b.S] + b.F;
  ans %= MOD;
  return mp(ans, a.S + b.S);
}

void build(){
  n = s.size();
  s = " " + s;

  h.resize(n + 1);
  rh.resize(n + 1);
  kp.resize(n + 1);
  kp[0] = 1;
  for(int i = 1; i <= n; i++){
    kp[i] = kp[i - 1] * K % MOD;
  }
  for(int i = 1; i <= n; i++){
    h[i] = h[i - 1] * K % MOD + ord(s[i]);
    h[i] %= MOD;
    rh[i] = rh[i - 1] * K % MOD + ord(s[n - i + 1]);
    rh[i] %= MOD;
  }
}
```

9.2 KMP Algorithm

```
void kmp(string s){
  int siz = s.size();
  vector<int> f(siz, 0);
  f[0] = 0;
  for (int i = 1;i < siz;i++) {
    f[i] = f[i-1];
    bool zero = 0;
    while (s[f[i]] != s[i]) {
      if (f[i] == 0) {
        zero = 1;
        break;
      }
      f[i] = f[f[i]-1];
    }
    if (!zero) f[i]++;
  }
}
```

9.3 Manacher Algorithm

```
vector<int> manacher(string s) {
  int n = s.size();
  vector<int> v(n);
  int pnt = -1, len = 1;
  for (int i = 0;i < n;i++) {
    int cor = 2 * pnt - i;
    if (cor >= 0) v[i] = min(v[cor], cor - pnt + len);
    while (i+v[i] < n && i-v[i] >= 0 && s[i+v[i]] == s[i-v[
        i]]) v[i]++;
    if (i + v[i] >= pnt + len) pnt = i, len = v[i];
  }
  for (int i = 0;i < n;i++) v[i] = 2 * v[i] - 1;
  return v;
}
```

9.4 MCP

```
string mcp(string s) { //Duval algorithm for Lyndon
    factorization
  s += s;
  int n = s.size(), i = 0, ans = 0;
  while (i < n/2) {
    ans = i;
    int j = i+1, k=i;
    while (j < n && s[k] <= s[j]) {
      if (s[k] < s[j]) k = i;
      else k++;
      j++;
    }
    while (i <= k) i += j - k;
  }
  return s.substr(ans, n/2);
}
```

9.5 Suffix Array

```
struct SuffixArray { //tested
  vector<int> sa, lcp, rank; //lcp[i] is lcp of sa[i] and
      sa[i-1]
  SuffixArray(string& s, int lim=256) { // or basic_string<
      int>
    int n = s.size() + 1, k = 0, a, b;
    vector<int> x(n, 0), y(n), ws(max(n, lim));
    rank.resize(n);
    for (int i = 0;i < n-1;i++) x[i] = (int)s[i];
    sa = lcp = y, iota(sa.begin(), sa.end(), 0);
    for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim =
        p) {
      p = j, iota(y.begin(), y.end(), n - j);
      for (int i = 0;i < n;i++) if (sa[i] >= j) y[p++] = sa
          [i] - j;
      for (int &i : ws) i = 0;
      for (int i = 0;i < n;i++) ws[x[i]]++;
      for (int i = 1;i < lim;i++) ws[i] += ws[i - 1];
      for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
      swap(x, y), p = 1, x[sa[0]] = 0;
      for(int i = 1;i < n;i++) a = sa[i - 1], b = sa[i], x[
          b] =
        (y[a] == y[b] && y[a + j] == y[b + j]) ? p - 1 : p
            ++;
    }
    for (int i = 1;i < n;i++) rank[sa[i]] = i;
    for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
      for (k && k--, j = sa[rank[i] - 1];
          s[i + k] == s[j + k]; k++);
  }
};
```

9.6 Suffix Array Automaton

```
// from 8BQube
const int MAXM = 1000010;
struct SAM {
  int tot, root, lst, mom[MAXM], mx[MAXM];
  int nxt[MAXM][33], cnt[MAXM], in[MAXM];
  int newNode() {
    int res = ++tot;
    fill(nxt[res], nxt[res] + 33, 0);
    mom[res] = mx[res] = cnt[res] = in[res] = 0;
    return res;
  }
  void init() {
    tot = 0;
    root = newNode();
    mom[root] = 0, mx[root] = 0;
    lst = root;
  }
  void push(int c) {
    int p = lst;
    int np = newNode();
    mx[np] = mx[p] + 1;
    for (; p && nxt[p][c] == 0; p = mom[p])
      nxt[p][c] = np;
    if (p == 0) mom[np] = root;
    else {
      int q = nxt[p][c];
      if (mx[p] + 1 == mx[q]) mom[np] = q;
```

```
      else {
        int nq = newNode();
        mx[nq] = mx[p] + 1;
        for (int i = 0; i < 33; i++)
          nxt[nq][i] = nxt[q][i];
        mom[nq] = mom[q];
        mom[q] = nq;
        mom[np] = nq;
        for (; p && nxt[p][c] == q; p = mom[p])
          nxt[p][c] = nq;
      }
    }
    lst = np, cnt[np] = 1;
  }
  void push(char *str) {
    for (int i = 0; str[i]; i++)
      push(str[i] - 'a' + 1);
  }
  void count() {
    for (int i = 1; i <= tot; ++i)
      ++in[mom[i]];
    queue<int> q;
    for (int i = 1; i <= tot; ++i)
      if (!in[i]) q.push(i);
    while (!q.empty()) {
      int u = q.front();
      q.pop();
      cnt[mom[u]] += cnt[u];
      if (!--in[mom[u]])
        q.push(mom[u]);
    }
  }
} sam;
```

9.7 Z-value Algorithm

```
vector<int> z_function(string const& s) {
  int n = s.size();
  vector<int> z(n);
  for (int i = 1, l = 0, r = 0; i < n; i++) {
    if (i <= r) z[i] = min(r-i+1, z[i-l]);
    while (i + z[i] < n && s[z[i]] == s[i+z[i]])
      z[i]++;
    if (i + z[i] - 1 > r)
      l = i, r = i + z[i] - 1;
  }
  return z;
}
```

9.8 Main Lorentz

```
vector<int> z_function(string const& s);
int get_z(vector<int> const& z, int i) {
  return (0 <= i && i < SZ(z)) ? z[i] : 0;
}
vector<pair<int, int>> rep;

void convert_to_rep(int shift, bool left, int cntr, int l,
    int k1, int k2) {
  for (int l1 = max(1, l - k2); l1 <= min(l, k1); l1++) {
    if (left && l1 == l) break;
    int l2 = l - l1;
    int pos = shift + (left ? cntr - l1 : cntr - l - l1 +
        1);
    rep.emplace_back(pos, pos + 2*l - 1);
  }
}

void find_rep(string s, int shift = 0) {
  int n = s.size();
  if (n == 1) return;

  int nu = n / 2;
  int nv = n - nu;
  string u = s.substr(0, nu);
  string v = s.substr(nu);
  string ru(u.rbegin(), u.rend());
  string rv(v.rbegin(), v.rend());

  find_rep(u, shift);
  find_rep(v, shift + nu);

  vector<int> z1 = z_function(ru);
  vector<int> z2 = z_function(v + '#' + u);
  vector<int> z3 = z_function(ru + '#' + rv);
  vector<int> z4 = z_function(v);

  for (int cntr = 0; cntr < n; cntr++) {
    int l, k1, k2;
    if (cntr < nu) {
      l = nu - cntr;
      k1 = get_z(z1, nu - cntr);
      k2 = get_z(z2, nv + 1 + cntr);
    } else {
      l = cntr - nu + 1;
      k1 = get_z(z3, nu + 1 + nv - 1 - (cntr - nu));
      k2 = get_z(z4, (cntr - nu) + 1);
    }
    if (k1 + k2 >= l)
      convert_to_rep(shift, cntr < nu, cntr, l, k1, k2);
  }
}
```

9.9 AC Automaton

```
// copy from nontoi
struct AhoCorasick {
  enum { P = 26, st = 'a'};
  struct node { // zero-based
    array<int, P> ch = {0};
    int fail = 0, cnt = 0, dep = 0;
  };
  int cnt;
  vector<node> v;
  vector<int> ans;
  void init_(int mx) {
    v.clear();
    cnt = 1, v.resize(mx);
    v[0].fail = 0;
  }
  void insert(string s) {
    int p = 0, dep = 1;
    for(auto i : s) {
      int c = i - st;
      if(!v[p].ch[c]) {
        v[cnt].dep = dep;
        v[p].ch[c] = cnt ++;
      }
      p = v[p].ch[c], dep ++;
    }
    v[p].cnt ++;
  }
  void build(vector<string> s) {
    for(auto i : s) insert(i);
    queue<int> q;
    for(int i = 0; i < P; i ++) {
      if(v[0].ch[i]) q.push(v[0].ch[i]);
    }
    while(q.size()) {
      int p = q.front();
      q.pop();
      for(int i = 0; i < P; i ++) if(v[p].ch[i]) {
        int to = v[p].ch[i], cur = v[p].fail;
        while(cur && !v[cur].ch[i]) cur = v[cur].fail;
        if(v[cur].ch[i]) cur = v[cur].ch[i];
        v[to].fail = cur;
        v[to].cnt += v[cur].cnt;
        q.push(to);
      }
    }
  }
```

```
  void traverse(string s) {
    int p = 0;
    ans.assign(cnt, 0);
    for(auto i : s) {
      int c = i - st;
      while(p && !v[p].ch[c]) p = v[p].fail;
      if(v[p].ch[c]) {
        p = v[p].ch[c];
        ans[p] ++, v[p].cnt;
      }
    }
    vector<int> ord(cnt, 0);
    iota(all(ord), 0);
    sort(all(ord), [&](int a, int b) { return v[a].dep > v[
        b].dep; });
    for(auto i : ord) ans[v[i].fail] += ans[i];
    return;
  }
  int go(string s) {
    int p = 0;
    for(auto i : s) {
      int c = i - st;
      assert(v[p].ch[c]);
      p = v[p].ch[c];
    }
    return ans[p];
  }
};
```

10 Formula

10.1 Recurrences

If $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$, and $r_1, \ldots, r_k$ are distinct roots of $x^k + c_1 x^{k-1} + \cdots + c_k$, there are $d_1, \ldots, d_k$ s.t.

$$a_n = d_1 r_1^n + \cdots + d_k r_k^n.$$

Non-distinct roots r become polynomial factors, e.g. $a_n = (d_1 n + d_2) r^n$.

10.2 Geometry

10.2.1 Triangles

Side lengths: a, b, c

Semiperimeter: $p = \dfrac{a+b+c}{2}$

Area: $A = \sqrt{p(p-a)(p-b)(p-c)}$

Circumradius: $R = \dfrac{abc}{4A}$

Inradius: $r = \dfrac{A}{p}$

Length of median (divides triangle into two equal-area triangles): $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

Length of bisector (divides angles in two): $s_a = \sqrt{bc\left(1 - \left(\dfrac{a}{b+c}\right)^2\right)}$

Law of sines: $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c} = \dfrac{1}{2R}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos\alpha$

Law of tangents: $\dfrac{a+b}{a-b} = \dfrac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}$

Incenter:
$P_1 = (x_1, y_1), P_2 = (x_2, y_2), P_3 = (x_3, y_3)$
$s_1 = \overline{P_2P_3}, s_2 = \overline{P_1P_3}, s_3 = \overline{P_1P_2}$
$\dfrac{s_1P_1 + s_2P_2 + s_3P_3}{s_1 + s_2 + s_3}$

Circumcenter:
$P_0 = (0,0), P_1 = (x_1, y_1), P_2 = (x_2, y_2)$
$x_c = \frac{1}{2} \times \dfrac{y_2(x_1^2 + y_1^2) - y_1(x_2^2 + y_2^2)}{-x_2y_1 + x_1y_2}$
$y_c = \frac{1}{2} \times \dfrac{x_2(x_1^2 + y_1^2) - x_1(x_2^2 + y_2^2)}{-x_1y_2 + x_2y_1}$

Check if (x_0, y_0) is in the circumcircle:

$$\begin{vmatrix} x_1 - x_0 & y_1 - y_0 & (x_1^2 + y_1^2) - (x_0^2 + y_0^2) \\ x_2 - x_0 & y_2 - y_0 & (x_2^2 + y_2^2) - (x_0^2 + y_0^2) \\ x_3 - x_0 & y_3 - y_0 & (x_3^2 + y_3^2) - (x_0^2 + y_0^2) \end{vmatrix}$$

0: on edge, > 0: inside, < 0: outside

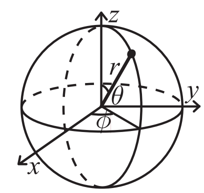
10.2.2 Quadrilaterals

With side lengths a, b, c, d, diagonals e, f, diagonals angle θ, area A and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin\theta = F\tan\theta = \sqrt{4e^2f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180°, $ef = ac + bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

10.2.3 Spherical coordinates



$$\begin{aligned} x &= r\sin\theta\cos\phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r\sin\theta\sin\phi & \theta &= \mathrm{acos}(z/\sqrt{x^2 + y^2 + z^2}) \\ z &= r\cos\theta & \phi &= \mathrm{atan2}(y, x) \end{aligned}$$

10.2.4 Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_{L^+} (Pdx + Qdy)$$

$$\text{Area} = \frac{1}{2}\oint_L x\ dy - y\ dx$$

Circular sector:

$$\begin{aligned} x &= x_0 + r\cos\theta \\ y &= y_0 + r\sin\theta \\ A &= r\int_\alpha^\beta (x_0 + \cos\theta)\cos\theta + (y_0 + \sin\theta)\sin\theta\ d\theta \\ &= r(r\theta + x_0\sin\theta - y_0\cos\theta)|_\alpha^\beta \end{aligned}$$

10.3 Trigonometry

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\sin n\pi = 0 \qquad \cos n\pi = (-1)^n$$

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(2\alpha) &= 2\cos\alpha\sin\alpha \\ \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha \end{aligned}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$$

$$\cos\alpha\sin\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$$

$$(V + W)\tan(\alpha - \beta)/2 = (V - W)\tan(\alpha + \beta)/2$$

where V, W are lengths of sides opposite angles α, β.

$$a\cos x + b\sin x = r\cos(x - \phi)$$
$$a\sin x + b\cos x = r\sin(x + \phi)$$

where $r = \sqrt{a^2 + b^2}, \phi = \operatorname{atan2}(b, a)$.

10.4 Derivatives/Integrals

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan x = 1 + \tan^2 x \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$
$$\int \tan ax = -\frac{\ln|\cos ax|}{a} \qquad \int x\sin ax = \frac{\sin ax - ax\cos ax}{a^2}$$
$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) \qquad \int xe^{ax} = \frac{e^{ax}}{a^2}(ax - 1)$$
$$\int \sin^2(x) = \frac{x}{2} - \frac{1}{4}\sin 2x \qquad \int \sin^3 x = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x$$
$$\int \cos^2(x) = \frac{x}{2} + \frac{1}{4}\sin 2x \qquad \int \cos^3 x = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x$$
$$\int x\sin x = \sin x - x\cos x \qquad \int x\cos x = \cos x + x\sin x$$
$$\int xe^x = e^x(x-1) \qquad \int x^2e^x = e^x(x^2 - 2x + 2)$$

$$\int x^2\sin x = 2x\sin x - (x^2 - 2)\cos x$$
$$\int x^2\cos x = 2x\cos x + (x^2 - 2)\sin x$$
$$\int e^x\sin x = \frac{1}{2}e^x(\sin x - \cos x)$$
$$\int e^x\cos x = \frac{1}{2}e^x(\sin x + \cos x)$$
$$\int xe^x\sin x = \frac{1}{2}e^x(x\sin x - x\cos x + \cos x)$$
$$\int xe^x\cos x = \frac{1}{2}e^x(x\sin x + x\cos x - \sin x)$$

10.5 Sums

$$c^a + c^{a+1} + \cdots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$
$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n+1)(n+1)}{6}$$
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$
$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

10.6 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, (-\infty < x < \infty)$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, (-1 < x \leq 1)$$
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \ldots, (-1 \leq x \leq 1)$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, (-\infty < x < \infty)$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, (-\infty < x < \infty)$$

10.7 Probability theory

Let X be a discrete random variable with probability $p_X(x)$ of assuming the value x. It will then have an expected value (mean) $\mu = \mathbb{E}(X) = \sum_x xp_X(x)$ and variance $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$ where σ is the standard deviation. If X is instead continuous it will have a probability density function $f_X(x)$ and the sums above will instead be integrals with $p_X(x)$ replaced by $f_X(x)$.
Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent X and Y,

$$V(aX + bY) = a^2V(X) + b^2V(Y).$$

10.7.1 Discrete distributions

Binomial distribution The number of successes in n independent yes/no experiments, each which yields success with probability p is $\text{Bin}(n, p)$, $n = 1, 2, \ldots$, $0 \leq p \leq 1$.

$$p(k) = \binom{n}{k}p^k(1-p)^{n-k}$$
$$\mu = np, \sigma^2 = np(1-p)$$

$\text{Bin}(n, p)$ is approximately $\text{Po}(np)$ for small p.

First success distribution The number of trials needed to get the first success in independent yes/no experiments, each wich yields success with probability p is $\text{Fs}(p)$, $0 \leq p \leq 1$.

$$p(k) = p(1-p)^{k-1}, k = 1, 2, \ldots$$
$$\mu = \frac{1}{p}, \sigma^2 = \frac{1-p}{p^2}$$

Poisson distribution The number of events occurring in a fixed period of time t if these events occur with a known average rate κ and independently of the time since the last event is $\text{Po}(\lambda)$, $\lambda = t\kappa$.

$$p(k) = e^{-\lambda}\frac{\lambda^k}{k!}, k = 0, 1, 2, \ldots$$
$$\mu = \lambda, \sigma^2 = \lambda$$

10.7.2 Continuous distributions

Uniform distribution If the probability density function is constant between a and b and 0 elsewhere it is $\text{U}(a, b)$, $a < b$.

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$
$$\mu = \frac{a+b}{2}, \sigma^2 = \frac{(b-a)^2}{12}$$

Exponential distribution The time between events in a Poisson process is $\text{Exp}(\lambda)$, $\lambda > 0$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$
$$\mu = \frac{1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}$$

Normal distribution Most real random values with mean μ and variance σ^2 are well described by $\mathcal{N}(\mu, \sigma^2)$, $\sigma > 0$.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$

10.8 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let $X_1, X_2, \ldots$ be a sequence of random variables generated by the Markov process. Then there is a transition matrix $\mathbf{P} = (p_{ij})$, with $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$, and $\mathbf{p}^{(n)} = \mathbf{P}^n \mathbf{p}^{(0)}$ is the probability distribution for X_n (i.e., $p_i^{(n)} = \Pr(X_n = i)$), where $\mathbf{p}^{(0)}$ is the initial distribution.
π is a stationary distribution if $\pi = \pi \mathbf{P}$. If the Markov chain is *irreducible* (it is possible to get to any state from any state), then $\pi_i = \frac{1}{\mathbb{E}(T_i)}$ where $\mathbb{E}(T_i)$ is the expected time between two visits in state i. π_j / π_i is the expected number of visits in state j between two visits in state i.
For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors, π_i is proportional to node i's degree.
A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1). $\lim_{k \to \infty} \mathbf{P}^k = \mathbf{1}\pi$.
A Markov chain is an A-chain if the states can be partitioned into two sets $\mathbf{A}$ and $\mathbf{G}$, such that all states in $\mathbf{A}$ are absorbing ($p_{ii} = 1$), and all states in $\mathbf{G}$ leads to an absorbing state in $\mathbf{A}$. The probability for absorption in state $i \in \mathbf{A}$, when the initial state is j, is $a_{ij} = p_{ij} + \sum_{k \in \mathbf{G}} a_{ik} p_{kj}$. The expected time until absorption, when the initial state is i, is $t_i = 1 + \sum_{k \in \mathbf{G}} p_{ki} t_k$.