

Contents

1 Basic	1	8 Polynomial	19
1.1 Default Code	1	8.1 FWHT	19
1.2 .vimrc	2	8.2 FFT	19
1.3 Windows Setup	2	8.3 NTT	19
1.4 Debug	2	8.4 Polynomial Operation	20
1.5 Fast IO	2	8.5 Generating Function	21
1.6 Random	2	8.5.1 Ordinary Generating Function	21
1.7 Checker	2	8.5.2 Exponential Generating Function	21
1.8 PBDS Tree	2	9 String	21
2 Data Structure	2	9.1 KMP Algorithm	21
2.1 Heavy-Light Decomposition	2	9.2 Manacher Algorithm	21
2.2 Link Cut Tree	3	9.3 MCP	21
2.3 Treap	3	9.4 Suffix Array	22
2.4 KD Tree	4	9.5 Suffix Automaton	22
3 Flow & Matching	4	9.6 Z-value Algorithm	22
3.1 Dinic	4	9.7 Main Lorentz	22
3.2 Bounded Flow	5	9.8 AC Automaton	23
3.3 Gomory Hu	5	10 Formula	23
3.4 Hungarian Algorithm	5	10.1 Recurrences	23
3.5 ISAP Algorithm	6	10.2 Geometry	23
3.6 Bipartite Matching	6	10.2.1 Rotation Matrix	23
3.7 Max Simple Graph Matching	6	10.2.2 Triangles	23
3.8 MCMF	7	10.2.3 Quadrilaterals	24
3.9 Min Cost Circulation	7	10.2.4 Spherical coordinates	24
3.10 SW Mincut	8	10.2.5 Green's Theorem	24
3.11 Stable Marriage	8	10.2.6 Point-Line Duality	24
4 Geometry	8	10.3 Trigonometry	24
4.1 Geometry Template	8	10.4 Derivatives/Integrals	24
4.2 Convex Hull	9	10.5 Sums	24
4.3 Minimum Enclosing Circle	9	10.6 Series	24
4.4 Minkowski Sum	9	1 Basic	
4.5 Polar Angle Comparator	9	1.1 Default Code	
4.6 Half Plane Intersection	9		
4.7 Dynamic Convex Hull	9		
4.8 3D Point	10		
4.9 ConvexHull3D	10		
4.10 Circle Operations	11		
4.11 Delaunay Triangulation	11		
4.12 Voronoi Diagram	12		
4.13 Polygon Union	12		
4.14 Tangent Point to Convex Hull	12		
5 Graph	12		
5.1 Block Cut Tree	12		
5.2 2-SAT	13		
5.3 Dominator Tree	13		
5.4 Virtual Tree	14		
5.5 Directed Minimum Spanning Tree	14		
5.6 Vizing	14		
5.7 Maximum Clique	15		
5.8 Number of Maximal Clique	15		
5.9 Minimum Mean Cycle	15		
5.10 Minimum Steiner Tree	15		
6 Math	16		
6.1 Extended Euclidean Algorithm	16		
6.2 Floor & Ceil	16		
6.3 Legendre	16		
6.4 Simplex	16		
6.5 Floor Sum	17		
6.6 DiscreteLog	17		
6.7 Miller Rabin & Pollard Rho	17		
6.8 XOR Basis	18		
6.9 Linear Equation	18		
6.10 Chinese Remainder Theorem	18		
6.11 Sqrt Decomposition	18		
7 Misc	18		
7.1 Cyclic Ternary Search	18		
7.2 Matroid	19		

```
//Challenge: Accepted
//#pragma GCC optimize("Ofast")
#include <bits/stdc++.h>
using namespace std;

#define io ios_base::sync_with_stdio(0);cin.tie(0);cerr.tie(0)
#define iter(v) v.begin(),v.end()
#define SZ(v) (int)v.size()
#define pb emplace_back
#define ff first
#define ss second

using ll = long long;
using pii = pair<int, int>;
using pll = pair<ll, ll>;

#ifdef zisk
void debug(){cerr << "\n";}
template<class T, class ... U>
void debug(T a, U ... b){cerr << a << " ", debug(b...);}
template<class T> void pary(T l, T r){
    while (l != r) cerr << *l << " ", l++;
    cerr << "\n";
}
#else
#define debug(...) void()
#define pary(...) void()
#endif
template<class A, class B>
ostream& operator<< (ostream& o, pair<A,B> p)
{ return o << '(' << p.ff << ', ' << p.ss << ')'; }

int main(){
    io;
}
```

1.2 .vimrc

```
sy on
se nu rnu bs=2 sw=4 ts=4 hls ls=2 si acd bo=all mouse=a
map <F9> :w<bar>!g++ "%" -o %:r -std=c++17 -Wall -Wextra -
    Wshadow -O2 -Dzisk -g -fsanitize=undefined,address<CR>
map <F8> :!./%:r<CR>
inoremap {<CR> {<CR>}<ESC>kO
```

1.3 Windows Setup

1. Open Sublime Text
2. Alt+Shift+2, Ctrl+K, Ctrl+Shift+Up
3. Create input.txt and output.txt
4. Preferences -> Settings -> 左邊最下面複製到右邊，把 Vintage 刪掉
5. Tools -> Build System -> New Build System

```
compile.bat $file_base_name
```

```
Save at AppData/Roaming/Sublime Text 3/Packages/User/test Tools
-> Build System -> test
```

compile.bat:

```
g++ %1.cpp -o %1 -std=c++17 -O2 ^
-Wall -Wextra -Wshadow -Dzisk ^
-Wl,--stack,268435456 -g -gdb3 -D_GLIBCXX_ASSERTIONS ^
&& %1.exe < input.txt > output.txt
```

1.4 Debug

- Compile flags: -D_GLIBCXX_ASSERTIONS, -D_GLIBCXX_DEBUG (slow), -gdb3
- GDB: break <function>, continue, bt

1.5 Fast IO

```
// from JAW
inline int my_getchar() {
    const int N = 1<<20;
    static char buf[N];
    static char *p = buf, *end = buf;
    if(p == end) {
        if((end = buf + fread(buf, 1, N, stdin)) == buf)
            return EOF;
        p = buf;
    }
    return *p++;
}

inline int readint(int &x) {
    static char c, neg;
    while((c = my_getchar()) < '-') {
        if(c == EOF) return 0;
    }
    neg = (c == '-') ? -1 : 1;
    x = (neg == 1) ? c - '0' : 0;
    while((c = my_getchar()) >= '0') x = (x << 3) + (x << 1)
        + (c - '0');
    x *= neg;
    return 1;
}

const int kBufSize = 524288;
char inbuf[kBufSize];
char buf[kBufSize]; size_t size_;
inline void Flush_() { write(1, buf, size_); size_ = 0; }
inline void CheckFlush_(size_t sz) { if (sz + size_ >
    kBufSize) Flush_(); }

inline void PutInt(int a) {
    static char tmp[22] = "01234567890123456789\n";
    CheckFlush_(10);
```

```
if(a < 0){
    *(buf_ + size_) = '-';
    a = ~a + 1;
    size_++;
}
int tail = 20;
if (!a) {
    tmp[--tail] = '0';
} else {
    for (; a; a /= 10) tmp[--tail] = (a % 10) ^ '0';
}
memcpy(buf_ + size_, tmp + tail, 21 - tail);
size_ += 21 - tail;
}

int main(){
    Flush_();
    return 0;
}
```

1.6 Random

```
mt19937 rng(chrono::system_clock::now().time_since_epoch().
    count());
```

1.7 Checker

```
#!/usr/bin/env bash
set -e
while ;; do
    python3 gen.py > test.txt
    diff <./a.exe < test.txt) <(/b.exe < test.txt)
done
```

1.8 PBDS Tree

```
#include <bits/extc++.h>
using namespace __gnu_pbds;
using Tree = tree<int, null_type, less<, rb_tree_tag,
    tree_order_node_statistics_update>;
// .find_by_order(x)
// .order_of_key(x)
```

2 Data Structure

2.1 Heavy-Light Decomposition

```
struct Heavy_light_Decomposition { // 1-base
    int n, up[maxn], dep[maxn], to[maxn], siz[maxn], pa[maxn]
        ];
    int C, ti[maxn], ord[maxn], wdown[maxn], edge[maxn], et =
        0;
    vector<pii> G[maxn];
    void init(int _n) {
        n = _n, C = 0, et = 1;
        for (int i = 1; i <= n; i++)
            G[i].clear(), to[i] = 0;
    }
    void add_edge(int a, int b, int w) {
        G[a].push_back(pii(b, et)), G[b].push_back(pii(a, et));
        edge[et++] = w;
    }
    void dfs(int u, int f, int d) {
        siz[u] = 1, pa[u] = f, dep[u] = d;
        for (auto &v: G[u])
            if (v.ff != f) {
                dfs(v.ff, u, d+1), siz[u] += siz[v];
                if (siz[to[u]] < siz[v]) to[u] = v;
            }
    }
    void cut(int u, int link) {
```

```

    ti[u] = C;
    ord[C++] = u, up[u] = link;
    if (!to[u]) return;
    cut(to[u], link);
    for (auto v:G[u]) {
        if (v.ff != pa[u] && v.ff != to[u]) cut(v.ff, v.ff);
    }
}
void build() { dfs(1, 1, 1), cut(1, 1); }
int query(int a, int b) {
    int ta = up[a], tb = up[b], re = 0;
    while (ta != tb)
        if (dep[ta] < dep[tb])
            /*query*/, tb = up[b = pa[tb]];
        else /*query*/, ta = up[a = pa[ta]];
    if (a == b) return re;
    if (ti[a] > ti[b]) swap(a, b);
    /*query*/
    return re;
}
};

```

2.2 Link Cut Tree

```

struct Splay { // LCT + PATH add
    static Splay nil;
    Splay *ch[2], *f;
    int rev;
    int sz;
    ll val, sum, tag;
    Splay() : rev(0), sz(1), val(1), sum(1), tag(0) {
        f = ch[0] = ch[1] = &nil;
    }
    bool isr() { return f->ch[0] != this && f->ch[1] != this; }
    int dir() { return f->ch[0] == this ? 0 : 1; }
    void setCh(Splay *c, int d) {
        ch[d] = c;
        if (c != &nil) c->f = this;
        pull();
    }
    void push() {
        for(int i = 0; i < 2; i++){
            if(ch[i] == &nil) continue;
            if(rev) swap(ch[i]->ch[0], ch[i]->ch[1]), ch[i]->rev
                ^= 1;
            if(tag != 0){
                ch[i]->tag += tag;
                ch[i]->val += tag;
                ch[i]->sum += tag * ch[i]->sz;
            }
        }
        tag = 0;
        rev = 0;
    }
    void pull() {
        // take care of the nil!
        sz = 1;
        sum = val;
        for(int i = 0; i < 2; i++){
            if(ch[i] == &nil) continue;
            ch[i]->f = this;
            sz += ch[i]->sz;
            sum += ch[i]->sum;
        }
    }
    void rotate(){
        Splay *p = f;
        int d = dir();
        if (!p->isr()) p->f->setCh(this, p->dir());
        else f = p->f;
        p->setCh(ch[!d], d);
        setCh(p, !d);
        p->pull(), pull();
    }
}

```

```

void update(){
    if(f != &nil) f->update();
    push();
}
void splay(){
    update();
    for(Splay* fa; fa = f, !isr(); rotate())
        if(!fa->isr()) (fa->dir() == dir() ? fa : this)->
            rotate();
}
Splay *access(Splay* q = &nil){
    splay();
    setCh(q, 1);
    pull();
    if (f != &nil) return f->access(this);
    else return q;
}
void root_path(){access(), splay();}
void chroot() {root_path(), swap(ch[0], ch[1]), rev = 1,
    push(), pull();}
void split(Splay* y){chroot(), y->root_path();}
void link(Splay* y){root_path(), y->chroot(), setCh(y, 1)
    ;}
void cut(Splay* y) {split(y), y->push(), y->ch[0] = y->ch
    [0]->f = &nil;}
Splay *get_root(){
    root_path();
    auto q = this;
    for(; q->ch[0] != &nil; q = q->ch[0]) q->push();
    return q;
}
Splay *lca(Splay* y){
    access(), y->root_path();
    return y->f == &nil ? &nil : y->f;
}
bool conn(Splay* y){return get_root() == y->get_root();}
} Splay::nil;

```

2.3 Treap

```

struct node {
    int data, sz;
    node *l, *r;
    node(int k) : data(k), sz(1), l(0), r(0) {}
    void up() {
        sz = 1;
        if (l) sz += l->sz;
        if (r) sz += r->sz;
    }
    void down() {}
};
int sz(node *a) { return a ? a->sz : 0; }
node *merge(node *a, node *b) {
    if (!a || !b) return a ? a : b;
    if (rand() % (sz(a) + sz(b)) < sz(a))
        return a->down(), a->r = merge(a->r, b), a->up(),
            a;
    return b->down(), b->l = merge(a, b->l), b->up(), b;
}
void split(node *o, node *&a, node *&b, int k) {
    if (!o) return a = b = 0, void();
    o->down();
    if (o->data <= k)
        a = o, split(o->r, a->r, b, k), a->up();
    else b = o, split(o->l, a, b->l, k), b->up();
}
void split2(node *o, node *&a, node *&b, int k) {
    if (sz(o) <= k) return a = o, b = 0, void();
    o->down();
    if (sz(o->l) + 1 <= k)
        a = o, split2(o->r, a->r, b, k - sz(o->l) - 1);
    else b = o, split2(o->l, a, b->l, k);
    o->up();
}
node *kth(node *o, int k) {
    if (k <= sz(o->l)) return kth(o->l, k);
}

```

```

    if (k == sz(o->l) + 1) return o;
    return kth(o->r, k - sz(o->l) - 1);
}
int Rank(node *o, int key) {
    if (!o) return 0;
    if (o->data < key)
        return sz(o->l) + 1 + Rank(o->r, key);
    else return Rank(o->l, key);
}
bool erase(node *o, int k) {
    if (!o) return 0;
    if (o->data == k) {
        node *t = o;
        o->down(), o = merge(o->l, o->r);
        delete t;
        return 1;
    }
    node *t = k < o->data ? o->l : o->r;
    return erase(t, k) ? o->up(), 1 : 0;
}
void insert(node *o, int k) {
    node *a, *b;
    split(o, a, b, k),
        o = merge(a, merge(new node(k), b));
}
void interval(node *o, int l, int r) {
    node *a, *b, *c;
    split2(o, a, b, l - 1), split2(b, b, c, r);
    // operate
    o = merge(a, merge(b, c));
}

```

2.4 KD Tree

```

namespace kdt {
    int root, lc[maxn], rc[maxn], xl[maxn], xr[maxn],
        yl[maxn], yr[maxn];
    point p[maxn];
    int build(int l, int r, int dep = 0) {
        if (l == r) return -1;
        function<bool(const point &, const point &)> f =
            [dep](const point &a, const point &b) {
                if (dep & 1) return a.x < b.x;
                else return a.y < b.y;
            };
        int m = (l + r) >> 1;
        nth_element(p + l, p + m, p + r, f);
        xl[m] = xr[m] = p[m].x;
        yl[m] = yr[m] = p[m].y;
        lc[m] = build(l, m, dep + 1);
        if (~lc[m]) {
            xl[m] = min(xl[m], xl[lc[m]]);
            xr[m] = max(xr[m], xr[lc[m]]);
            yl[m] = min(yl[m], yl[lc[m]]);
            yr[m] = max(yr[m], yr[lc[m]]);
        }
        rc[m] = build(m + 1, r, dep + 1);
        if (~rc[m]) {
            xl[m] = min(xl[m], xl[rc[m]]);
            xr[m] = max(xr[m], xr[rc[m]]);
            yl[m] = min(yl[m], yl[rc[m]]);
            yr[m] = max(yr[m], yr[rc[m]]);
        }
        return m;
    }
    bool bound(const point &q, int o, long long d) {
        double ds = sqrt(d + 1.0);
        if (q.x < xl[o] - ds || q.x > xr[o] + ds ||
            q.y < yl[o] - ds || q.y > yr[o] + ds)
            return false;
        return true;
    }
    long long dist(const point &a, const point &b) {
        return (a.x - b.x) * 111 * (a.x - b.x) +
            (a.y - b.y) * 111 * (a.y - b.y);
    }
}

```

```

void dfs(
    const point &q, long long &d, int o, int dep = 0) {
    if (!bound(q, o, d)) return;
    long long cd = dist(p[o], q);
    if (cd != 0) d = min(d, cd);
    if ((dep & 1) && q.x < p[o].x ||
        !(dep & 1) && q.y < p[o].y) {
        if (~lc[o]) dfs(q, d, lc[o], dep + 1);
        if (~rc[o]) dfs(q, d, rc[o], dep + 1);
    } else {
        if (~rc[o]) dfs(q, d, rc[o], dep + 1);
        if (~lc[o]) dfs(q, d, lc[o], dep + 1);
    }
}
void init(const vector<point> &v) {
    for (int i = 0; i < v.size(); ++i) p[i] = v[i];
    root = build(0, v.size());
}
long long nearest(const point &q) {
    long long res = 1e18;
    dfs(q, res, root);
    return res;
}
} // namespace kdt

```

3 Flow & Matching

3.1 Dinic

```

struct MaxFlow { // 1-base
    struct edge {
        int to, cap, flow, rev;
    };
    vector<edge> g[maxn];
    int s, t, dis[maxn], ind[maxn], n;

    void init(int _n) {
        n = _n + 2;
        s = _n + 1, t = _n + 2;
        for (int i = 0; i <= n; ++i) g[i].clear();
    }
    void reset() {
        for (int i = 0; i <= n; ++i)
            for (auto &j : g[i]) j.flow = 0;
    }
    void add_edge(int u, int v, int cap) {
        g[u].pb(edge{v, cap, 0, (int)g[v].size()});
        g[v].pb(edge{u, 0, 0, (int)g[u].size() - 1});
        //change g[v] to cap for undirected graphs
    }
    bool bfs() {
        fill(dis, dis+n+1, -1);
        queue<int> q;
        q.push(s), dis[s] = 0;
        while (!q.empty()) {
            int cur = q.front(); q.pop();
            for (auto &e : g[cur]) {
                if (dis[e.to] == -1 && e.flow != e.cap) {
                    q.push(e.to);
                    dis[e.to] = dis[cur] + 1;
                }
            }
        }
        return dis[t] != -1;
    }
    int dfs(int u, int cap) {
        if (u == t || !cap) return cap;
        for (int &i = ind[u]; i < (int)g[u].size(); ++i) {
            edge &e = g[u][i];
            if (dis[e.to] == dis[u] + 1 && e.flow != e.cap) {
                int df = dfs(e.to, min(e.cap - e.flow, cap));
                if (df) {
                    e.flow += df;
                    g[e.to][e.rev].flow -= df;
                    return df;
                }
            }
        }
    }
}

```

```

    }
}
}
dis[u] = -1;
return 0;
}
int maxflow() {
    int flow = 0, df;
    while (bfs()) {
        fill(ind, ind+n+1, 0);
        while ((df = dfs(s, inf))) flow += df;
    }
    return flow;
}
}
}flow;

```

3.2 Bounded Flow

```

struct Dinic { // 1-base
    struct edge {
        int to, cap, flow, rev;
    };
    vector<edge> g[maxn];
    int n, s, t, dis[maxn], ind[maxn], cnt[maxn];
    const int inf = 1e9;

    void init(int _n) {
        n = _n + 2;
        s = _n + 1, t = _n + 2;
        for (int i = 0; i <= n; ++i) g[i].clear(), cnt[i] = 0;
    }
    //reset, bfs, dfs same as Dinic
    void add_edge(int u, int v, int lcap, int rcap) {
        cnt[u] -= lcap, cnt[v] += lcap;
        g[u].pb(edge{v, rcap, lcap, (int)g[v].size()});
        g[v].pb(edge{u, 0, 0, (int)g[u].size() - 1});
    }
    int maxflow(int _s, int _t) {
        s = _s, t = _t;
        int flow = 0, df;
        while (bfs()) {
            fill(ind, ind+n+1, 0);
            while ((df = dfs(s, inf))) flow += df;
        }
        return flow;
    }
    bool feasible() {
        int sum = 0;
        for (int i = 1; i <= n - 2; ++i)
            if (cnt[i] > 0)
                add_edge(n - 1, i, 0, cnt[i]), sum += cnt[i];
            else if (cnt[i] < 0) add_edge(i, n, 0, -cnt[i]);
        if (sum != maxflow(n - 1, n)) sum = -1;
        for (int i = 1; i <= n - 2; ++i)
            if (cnt[i] > 0)
                g[n - 1].pop_back(), g[i].pop_back();
            else if (cnt[i] < 0)
                g[i].pop_back(), g[n].pop_back();
        return sum != -1;
    }
    int boundedflow(int _s, int _t) {
        add_edge(_t, _s, 0, inf);
        if (!feasible()) return -1; // infeasible flow
        int x = g[_t].back().flow;
        g[_t].pop_back(), g[_s].pop_back();
        /* Minimum feasible flow */
        int y = maxflow(_t, _s);
        return x-y;

        /* Maximum feasible flow
        int y = maxflow(_s, _t);
        return x+y;
        */
    }
};

```

3.3 Gomory Hu

```

MaxFlow Dinic;
int g[MAXN];
void GomoryHu(int n) { // 0-base
    fill_n(g, n, 0);
    for (int i = 1; i < n; ++i) {
        Dinic.reset();
        add_edge(i, g[i], Dinic.maxflow(i, g[i]));
        for (int j = i + 1; j <= n; ++j)
            if (g[j] == g[i] && ~Dinic.dis[j])
                g[j] = i;
    }
}

```

3.4 Hungarian Algorithm

```

struct KM { //1-base, max perfect matching in  $O(n^3)$ 
    int n;
    int c[maxn][maxn];
    int lx[maxn], ly[maxn], mx[maxn], my[maxn], slack[maxn];
    bool vx[maxn], vy[maxn];
    bool dfs(int p, bool ch) {
        if (vx[p]) return 0;
        vx[p] = 1;
        for (int v = 1; v <= n; v++) {
            slack[v] = min(slack[v], lx[p] + ly[v] - c[p][v]);
            if (lx[p] + ly[v] - c[p][v] > 0) continue;
            vy[v] = 1;
            if (!my[v] || dfs(my[v], ch)) {
                if (ch) mx[p] = v, my[v] = p;
                return 1;
            }
        }
        return 0;
    }
    ll solve() {
        for (int i = 1; i <= n; i++) {
            lx[i] = -inf;
            for (int j = 1; j <= n; j++) lx[i] = max(lx[i], a[i][j]);
        }
        for (int i = 1; i <= n; i++) {
            for (int j = 1; j <= n; j++) vx[j] = vy[j] = 0;
            for (int j = 1; j <= n; j++) slack[j] = inf;
            if (dfs(i, 1)) continue;
            bool aug = 0;
            while (!aug) {
                for (int j = 1; j <= n; j++) {
                    if (!vy[j] && slack[j] == 0) {
                        vy[j] = 1;
                        if (dfs(my[j], 0)) {
                            aug = 1;
                            break;
                        }
                    }
                }
            }
            if (aug) break;
            int delta = inf;
            for (int j = 1; j <= n; j++) {
                if (!vy[j]) delta = min(delta, slack[j]);
            }
            for (int j = 1; j <= n; j++) {
                if (vx[j]) lx[j] -= delta;
                if (vy[j]) ly[j] += delta;
                else {
                    slack[j] -= delta;
                    if (slack[j] == 0 && !my[j]) aug = 1;
                }
            }
        }
        for (int j = 1; j <= n; j++) vx[j] = vy[j] = 0;
        dfs(i, 1);
    }
    ll ans = 0;
    for (int i = 1; i <= n; i++) ans += lx[i] + ly[i];
}

```

```

    return ans;
}
};

```

3.5 ISAP Algorithm

```

struct Maxflow { //to be modified
    static const int MAXV = 20010;
    static const int INF = 1000000;
    struct Edge {
        int v, c, r;
        Edge(int _v, int _c, int _r)
            : v(_v), c(_c), r(_r) {}
    };
    int s, t;
    vector<Edge> G[MAXV * 2];
    int iter[MAXV * 2], d[MAXV * 2], gap[MAXV * 2], tot;
    void init(int x) {
        tot = x + 2;
        s = x + 1, t = x + 2;
        for (int i = 0; i <= tot; i++) {
            G[i].clear();
            iter[i] = d[i] = gap[i] = 0;
        }
    }
    void addEdge(int u, int v, int c) {
        G[u].push_back(Edge(v, c, SZ(G[v])));
        G[v].push_back(Edge(u, 0, SZ(G[u]) - 1));
    }
    int dfs(int p, int flow) {
        if (p == t) return flow;
        for (int &i = iter[p]; i < SZ(G[p]); i++) {
            Edge &e = G[p][i];
            if (e.c > 0 && d[p] == d[e.v] + 1) {
                int f = dfs(e.v, min(flow, e.c));
                if (f) {
                    e.c -= f;
                    G[e.v][e.r].c += f;
                    return f;
                }
            }
        }
        if (--gap[d[p]] == 0) d[s] = tot;
        else {
            d[p]++;
            iter[p] = 0;
            ++gap[d[p]];
        }
        return 0;
    }
    int solve() {
        int res = 0;
        gap[0] = tot;
        for (res = 0; d[s] < tot; res += dfs(s, INF))
            ;
        return res;
    }
} flow;

```

3.6 Bipartite Matching

```

//min vertex cover: take unmatched vertex in L and find
//alternating tree,
//ans is not reached in L + reached in R
// O(VE)
int n; //1-base, max matching
int mx[maxn], my[maxn];
bool adj[maxn][maxn], vis[maxn];
bool dfs(int u) {
    if (vis[u]) return 0;
    vis[u] = 1;
    for (int v = 1; v <= n; v++) {
        if (!adj[u][v]) continue;
        if (!my[v] || (my[v] && dfs(my[v]))) {
            mx[u] = v, my[v] = u;

```

```

        return 1;
    }
}
return 0;
}
// O(E sqrt(V)), O(E Log V) for random sparse graphs
struct Bipartite_Matching { // 0-base
    int l, r;
    int mp[maxn], mq[maxn];
    int dis[maxn], cur[maxn];
    vector<int> G[maxn];
    bool dfs(int u) {
        for (int &i = cur[u]; i < SZ(G[u]); ++i) {
            int e = G[u][i];
            if (!mq[e] || (dis[mq[e]] == dis[u] + 1 && dfs(mq[e])))
                return mp[mq[e] = u] = e, 1;
        }
        dis[u] = -1;
        return 0;
    }
    bool bfs() {
        int rt = 0;
        queue<int> q;
        fill_n(dis, l, -1);
        for (int i = 0; i < l; ++i)
            if (!mp[i])
                q.push(i), dis[i] = 0;
        while (!q.empty()) {
            int u = q.front();
            q.pop();
            for (int e : G[u])
                if (!mq[e])
                    rt = 1;
                else if (!dis[mq[e]]) {
                    q.push(mq[e]);
                    dis[mq[e]] = dis[u] + 1;
                }
        }
        return rt;
    }
    int matching() {
        int rt = 0;
        fill_n(mp, l, -1);
        fill_n(mq, r, -1);
        while (bfs()) {
            fill_n(cur, l, 0);
            for (int i = 0; i < l; ++i)
                if (!mp[i] && dfs(i))
                    ++rt;
        }
        return rt;
    }
    void add_edge(int s, int t) {
        G[s].pb(t);
    }
    void init(int _l, int _r) {
        l = _l, r = _r;
        for (int i = 0; i < l; ++i)
            G[i].clear();
    }
} match;

```

3.7 Max Simple Graph Matching

```

struct GenMatch { // 1-base
    int V, pr[N];
    bool el[N][N], inq[N], inp[N], inb[N];
    int st, ed, nb, bk[N], djs[N], ans;
    void init(int _V) {
        V = _V;
        for (int i = 0; i <= V; ++i) {
            for (int j = 0; j <= V; ++j) el[i][j] = 0;
            pr[i] = bk[i] = djs[i] = 0;
            inq[i] = inp[i] = inb[i] = 0;
        }
    }

```

```

}
void add_edge(int u, int v) {
    el[u][v] = el[v][u] = 1;
}
int lca(int u, int v) {
    fill_n(inp, V + 1, 0);
    while (1)
        if (u = djs[u], inp[u] = true, u == st) break;
        else u = bk[pr[u]];
    while (1)
        if (v = djs[v], inp[v]) return v;
        else v = bk[pr[v]];
    return v;
}
void upd(int u) {
    for (int v; djs[u] != nb;) {
        v = pr[u], inb[djs[u]] = inb[djs[v]] = true;
        u = bk[v];
        if (djs[u] != nb) bk[u] = v;
    }
}
void blo(int u, int v, queue<int> &qe) {
    nb = lca(u, v), fill_n(inb, V + 1, 0);
    upd(u), upd(v);
    if (djs[u] != nb) bk[u] = v;
    if (djs[v] != nb) bk[v] = u;
    for (int tu = 1; tu <= V; ++tu)
        if (inb[djs[tu]])
            if (djs[tu] = nb, !inq[tu])
                qe.push(tu), inq[tu] = 1;
}
void flow() {
    fill_n(inq + 1, V, 0), fill_n(bk + 1, V, 0);
    iota(djs + 1, djs + V + 1, 1);
    queue<int> qe;
    qe.push(st), inq[st] = 1, ed = 0;
    while (!qe.empty()) {
        int u = qe.front();
        qe.pop();
        for (int v = 1; v <= V; ++v)
            if (el[u][v] && djs[u] != djs[v] &&
                pr[u] != v) {
                if ((v == st) ||
                    (pr[v] > 0 && bk[pr[v]] > 0)) {
                    blo(u, v, qe);
                } else if (!bk[v]) {
                    if (bk[v] = u, pr[v] > 0) {
                        if (!inq[pr[v]]) qe.push(pr[v]);
                    } else {
                        return ed = v, void();
                    }
                }
            }
    }
}
void aug() {
    for (int u = ed, v, w; u > 0;)
        v = bk[u], w = pr[v], pr[v] = u, pr[u] = v,
        u = w;
}
int solve() {
    fill_n(pr, V + 1, 0), ans = 0;
    for (int u = 1; u <= V; ++u)
        if (!pr[u])
            if (st = u, flow(), ed > 0) aug(), ++ans;
    return ans;
}
};

```

3.8 MCMF

```

struct MCMF { // 0-base
    struct edge {
        ll from, to, cap, flow, cost, rev;
    } * past[maxn];
    vector<edge> G[maxn];

```

```

    bitset<maxn> inq;
    ll dis[maxn], up[maxn], s, t, mx, n;
    bool BellmanFord(ll &flow, ll &cost) {
        fill(dis, dis + n, inf);
        queue<ll> q;
        q.push(s), inq.reset(), inq[s] = 1;
        up[s] = mx - flow, past[s] = 0, dis[s] = 0;
        while (!q.empty()) {
            ll u = q.front();
            q.pop(), inq[u] = 0;
            if (!up[u]) continue;
            for (auto &e : G[u])
                if (e.flow != e.cap &&
                    dis[e.to] > dis[u] + e.cost) {
                    dis[e.to] = dis[u] + e.cost, past[e.to] = &e;
                    up[e.to] = min(up[u], e.cap - e.flow);
                    if (!inq[e.to]) inq[e.to] = 1, q.push(e.to);
                }
        }
        if (dis[t] == inf) return 0;
        flow += up[t], cost += up[t] * dis[t];
        for (ll i = t; past[i]; i = past[i]->from) {
            auto &e = *past[i];
            e.flow += up[t], G[e.to][e.rev].flow -= up[t];
        }
        return 1;
    }
    ll MinCostMaxFlow(ll _s, ll _t, ll &cost) {
        s = _s, t = _t, cost = 0;
        ll flow = 0;
        while (BellmanFord(flow, cost));
        return flow;
    }
    void init(ll _n, ll _mx) {
        n = _n, mx = _mx;
        for (int i = 0; i < n; ++i) G[i].clear();
    }
    void add_edge(ll a, ll b, ll cap, ll cost) {
        G[a].pb(edge{a, b, cap, 0, cost, G[b].size()});
        G[b].pb(edge{b, a, 0, 0, -cost, G[a].size() - 1});
    }
};

```

3.9 Min Cost Circulation

```

//to be modified
struct Edge { int to, cap, rev, cost; };
vector<Edge> g[kN];
int dist[kN], pv[kN], ed[kN];
bool mark[kN];
int NegativeCycle(int n) {
    memset(mark, false, sizeof(mark));
    memset(dist, 0, sizeof(dist));
    int upd = -1;
    for (int i = 0; i <= n; ++i) {
        for (int j = 0; j < n; ++j) {
            int idx = 0;
            for (auto &e : g[j]) {
                if (e.cap > 0 && dist[e.to] > dist[j] + e.cost) {
                    dist[e.to] = dist[j] + e.cost;
                    pv[e.to] = j, ed[e.to] = idx;
                    if (i == n) {
                        upd = j;
                        while (!mark[upd]) mark[upd] = true, upd = pv[upd];
                        return upd;
                    }
                }
            }
            idx++;
        }
    }
    return -1;
}
int Solve(int n) {
    int rt = -1, ans = 0;

```

```

while ((rt = NegativeCycle(n)) >= 0) {
    memset(mark, false, sizeof(mark));
    vector<pair<int, int>> cyc;
    while (!mark[rt]) {
        cyc.emplace_back(pv[rt], ed[rt]);
        mark[rt] = true;
        rt = pv[rt];
    }
    reverse(cyc.begin(), cyc.end());
    int cap = kInf;
    for (auto &i : cyc) {
        auto &e = g[i.first][i.second];
        cap = min(cap, e.cap);
    }
    for (auto &i : cyc) {
        auto &e = g[i.first][i.second];
        e.cap -= cap;
        g[e.to][e.rev].cap += cap;
        ans += e.cost * cap;
    }
}
return ans;
}

```

3.10 SW Mincut

```

// stoer wagner algorithm: global min cut
const int maxn = 505;
struct SW { // O(V^3) 0-based
    int n, vis[maxn], del[maxn];
    int edge[maxn][maxn], wei[maxn];
    void init(int _n) {
        n = _n;
        fill(del, del+n, 0);
        for (int i = 0; i < n; i++) fill(edge[i], edge[i] + n, 0);
    }
    void addEdge(int u, int v, int w) {
        edge[u][v] += w, edge[v][u] += w;
    }
    void search(int &s, int &t) {
        fill(vis, vis+n, 0);
        fill(wei, wei+n, 0);
        s = t = -1;
        while (1) {
            int ma = -1, cur = 0;
            for (int i = 0; i < n; ++i)
                if (!del[i] && !vis[i] && ma < wei[i])
                    cur = i, ma = wei[i];
            if (mx == -1) break;
            vis[cur] = 1, s = t, t = cur;
            for (int i = 0; i < n; ++i)
                if (!vis[i] && !del[i]) wei[i] += edge[cur][i];
        }
    }
    int solve() {
        int ret = INF;
        for (int i = 0, x=0, y=0; i < n-1; ++i) {
            search(x, y), ret = min(res, wei[y]), del[y] = 1;
            for (int j = 0; j < n; ++j)
                edge[x][j] = (edge[j][x] += edge[y][j]);
        }
        return ret;
    }
};

```

3.11 Stable Marriage

- 1: Initialize $m \in M$ and $w \in W$ to free
- 2: **while** \exists free man m who has a woman w to propose to **do**
- 3: $w \leftarrow$ first woman on m 's list to whom m has not yet proposed
- 4: **if** \exists some pair (m', w) **then**
- 5: **if** w prefers m to m' **then**
- 6: $m' \leftarrow$ free
- 7: $(m, w) \leftarrow$ engaged

```

8:         end if
9:     else
10:          $(m, w) \leftarrow$  engaged
11:     end if
12: end while

```

4 Geometry

4.1 Geometry Template

```

using ld = ll;
using pdd = pair<ld, ld>;
using Line = pair<pdd, pdd>;
#define X first
#define Y second
// ld eps = 1e-7;

pdd operator+(pdd a, pdd b)
{ return {a.X + b.X, a.Y + b.Y}; }
pdd operator-(pdd a, pdd b)
{ return {a.X - b.X, a.Y - b.Y}; }
pdd operator*(ld i, pdd v)
{ return {i * v.X, i * v.Y}; }
pdd operator*(pdd v, ld i)
{ return {i * v.X, i * v.Y}; }
pdd operator/(pdd v, ld i)
{ return {v.X / i, v.Y / i}; }
ld dot(pdd a, pdd b)
{ return a.X * b.X + a.Y * b.Y; }
ld cross(pdd a, pdd b)
{ return a.X * b.Y - a.Y * b.X; }
ld abs2(pdd v)
{ return v.X * v.X + v.Y * v.Y; }
ld abs(pdd v)
{ return sqrt(abs2(v)); }
int sgn(ld v)
{ return v > 0 ? 1 : (v < 0 ? -1 : 0); }
// int sgn(ld v){ return v > eps ? 1 : ( v < -eps ? -1 : 0)
// ; }

int ori(pdd a, pdd b, pdd c)
{ return sgn(cross(b - a, c - a)); }
bool collinearity(pdd a, pdd b, pdd c)
{ return ori(a, b, c) == 0; }
bool btw(pdd p, pdd a, pdd b)
{ return collinearity(p, a, b) && sgn(dot(a - p, b - p)) <= 0; }

bool seg_intersect(Line a, Line b){
    pdd p1, p2, p3, p4;
    tie(p1, p2) = a; tie(p3, p4) = b;
    if(btw(p1, p3, p4) || btw(p2, p3, p4) || btw(p3, p1, p2)
        || btw(p4, p1, p2))
        return true;
    return ori(p1, p2, p3) * ori(p1, p2, p4) < 0 &&
        ori(p3, p4, p1) * ori(p3, p4, p2) < 0;
}

pdd intersect(Line a, Line b){
    pdd p1, p2, p3, p4;
    tie(p1, p2) = a; tie(p3, p4) = b;
    ld a123 = cross(p2 - p1, p3 - p1);
    ld a124 = cross(p2 - p1, p4 - p1);
    return (p4 * a123 - p3 * a124) / (a123 - a124);
}

pdd perp(pdd p1)
{ return pdd(-p1.Y, p1.X); }
pdd projection(pdd p1, pdd p2, pdd p3)
{ return p1 + (p2 - p1) * dot(p3 - p1, p2 - p1) / abs2(p2 - p1); }
pdd reflection(pdd p1, pdd p2, pdd p3)
{ return p3 + perp(p2 - p1) * cross(p3 - p1, p2 - p1) / abs2(p2 - p1) * 2; }
pdd linearTransformation(pdd p0, pdd p1, pdd q0, pdd q1, pdd r) {
    pdd dp = p1 - p0, dq = q1 - q0, num(cross(dp, dq), dot(dp, dq));
}

```



```

    return q0 + pdd(cross(r - p0, num), dot(r - p0, num)) /
        abs2(dp);
} // from line p0--p1 to q0--q1, apply to r

```

4.2 Convex Hull

```

vector<int> getConvexHull(vector<pdd>& pts){
    vector<int> id(SZ(pts));
    iota(iter(id), 0);
    sort(iter(id), [&](int x, int y){ return pts[x] < pts[y];
        });
    vector<int> hull;
    for(int tt = 0; tt < 2; tt++){
        int sz = SZ(hull);
        for(int j : id){
            pdd p = pts[j];
            while(SZ(hull) - sz >= 2 &&
                cross(pts[hull.back()] - pts[hull[SZ(hull) - 2]],
                    p - pts[hull[SZ(hull) - 2]]) <= 0)
                hull.pop_back();
            hull.pb(j);
        }
        hull.pop_back();
        reverse(iter(id));
    }
    return hull;
}

```

4.3 Minimum Enclosing Circle

```

using ld = long double;
pair<pdd, ld> circumcenter(pdd a, pdd b, pdd c);
pair<pdd, ld> MinimumEnclosingCircle(vector<pdd> &pts){
    random_shuffle(iter(pts));
    pdd c = pts[0];
    ld r = 0;
    for(int i = 1; i < SZ(pts); i++){
        if(abs(pts[i] - c) <= r) continue;
        c = pts[i]; r = 0;
        for(int j = 0; j < i; j++){
            if(abs(pts[j] - c) <= r) continue;
            c = (pts[i] + pts[j]) / 2;
            r = abs(pts[i] - c);
            for(int k = 0; k < j; k++){
                if(abs(pts[k] - c) > r)
                    tie(c, r) = circumcenter(pts[i], pts[j], pts[k]);
            }
        }
    }
    return {c, r};
}

```

4.4 Minkowski Sum

```

void reorder_poly(vector<pdd>& pnts){
    int mn = 0;
    for(int i = 1; i < (int)pnts.size(); i++)
        if(pnts[i].Y < pnts[mn].Y || (pnts[i].Y == pnts[mn].Y
            && pnts[i].X < pnts[mn].X))
            mn = i;
    rotate(pnts.begin(), pnts.begin() + mn, pnts.end());
}

```

```

vector<pdd> minkowski(vector<pdd> P, vector<pdd> Q){
    reorder_poly(P);
    reorder_poly(Q);
    int psz = P.size();
    int qsz = Q.size();
    P.pb(P[0]); P.pb(P[1]); Q.pb(Q[0]); Q.pb(Q[1]);
    vector<pdd> ans;
    int i = 0, j = 0;
    while(i < psz || j < qsz){
        ans.pb(P[i] + Q[j]);
    }
}

```

```

    int t = sgn(cross(P[i + 1] - P[i], Q[j + 1] - Q[j]));
    if(t >= 0) i++;
    if(t <= 0) j++;
}
return ans;
}

```

4.5 Polar Angle Comparator

```

// -1: a // b (if same), 0/1: a < b
int cmp(pll a, pll b, bool same = true){
#define is_neg(k) (sgn(k.Y) < 0 || (sgn(k.Y) == 0 && sgn(k.X) < 0))
    int A = is_neg(a), B = is_neg(b);
    if(A != B)
        return A < B;
    if(sgn(cross(a, b)) == 0)
        return same ? abs2(a) < abs2(b) : -1;
    return sgn(cross(a, b)) > 0;
}

```

4.6 Half Plane Intersection

```

// from 8BQube
pll area_pair(Line a, Line b)
{ return pll(cross(a.Y - a.X, b.X - a.X), cross(a.Y - a.X,
    b.Y - a.X)); }
bool isin(Line l0, Line l1, Line l2) {
    // Check inter(l1, l2) strictly in l0
    auto [a02X, a02Y] = area_pair(l0, l2);
    auto [a12X, a12Y] = area_pair(l1, l2);
    if (a12X - a12Y < 0) a12X *= -1, a12Y *= -1;
    return (__int128) a02Y * a12X - (__int128) a02X * a12Y >
        0; // C^4
}
/* Having solution, check size > 2 */
/* --^-- Line.X --^-- Line.Y --^-- */
vector<Line> halfPlaneInter(vector<Line> arr) {
    sort(iter(arr), [&](Line a, Line b) -> int {
        if (cmp(a.Y - a.X, b.Y - b.X, 0) != -1)
            return cmp(a.Y - a.X, b.Y - b.X, 0);
        return ori(a.X, a.Y, b.Y) < 0;
    });
    deque<Line> dq(1, arr[0]);
    for (auto p : arr) {
        if (cmp(dq.back().Y - dq.back().X, p.Y - p.X, 0) == -1)
            continue;
        while (SZ(dq) >= 2 && !isin(p, dq[SZ(dq) - 2], dq.back()))
            dq.pop_back();
        while (SZ(dq) >= 2 && !isin(p, dq[0], dq[1]))
            dq.pop_front();
        dq.pb(p);
    }
    while (SZ(dq) >= 3 && !isin(dq[0], dq[SZ(dq) - 2], dq.back()))
        dq.pop_back();
    while (SZ(dq) >= 3 && !isin(dq.back(), dq[0], dq[1]))
        dq.pop_front();
    return vector<Line>(iter(dq));
}

```

4.7 Dynamic Convex Hull

```

struct Line{
    ll a, b, l = MIN, r = MAX;
    Line(ll a, ll b): a(a), b(b) {}
    ll operator()(ll x) const{
        return a * x + b;
    }
    bool operator<(Line b) const{
        return a < b.a;
    }
}

```

```

    bool operator<(ll b) const{
        return r < b;
    }
};

ll ceil(ll a, ll b){
    if(b < 0) a *= -1, b *= -1;
    if(a > 0) return (a + b - 1) / b;
    else return a / b;
}

ll intersect(Line a, Line b){
    return ceil(a.b - b.b, b.a - a.a);
}

struct DynamicConvexHull{
    multiset<Line, less<>> ch;

    void add(Line ln){
        auto it = ch.lower_bound(ln);
        while(it != ch.end()){
            Line tl = *it;
            if(tl(tl.r) <= ln(tl.l)){
                it = ch.erase(it);
            }
            else break;
        }
        auto it2 = ch.lower_bound(ln);
        while(it2 != ch.begin()){
            Line tl = *prev(it2);
            if(tl(tl.l) <= ln(tl.l)){
                it2 = ch.erase(prev(it2));
            }
            else break;
        }
        it = ch.lower_bound(ln);
        if(it != ch.end()){
            Line tl = *it;
            if(tl(tl.l) >= ln(tl.l)) ln.r = tl.l - 1;
            else{
                ll pos = intersect(ln, tl);
                tl.l = pos;
                ln.r = pos - 1;
                ch.erase(it);
                ch.insert(tl);
            }
        }
        it2 = ch.lower_bound(ln);
        if(it2 != ch.begin()){
            Line tl = *prev(it2);
            if(tl(tl.r) >= ln(tl.r)) ln.l = tl.r + 1;
            else{
                ll pos = intersect(tl, ln);
                tl.r = pos - 1;
                ln.l = pos;
                ch.erase(prev(it2));
                ch.insert(tl);
            }
        }
        if(ln.l <= ln.r) ch.insert(ln);
    }

    ll query(ll pos){
        auto it = ch.lower_bound(pos);
        if(it == ch.end()) return 0;
        return (*it)(pos);
    }
};

```

4.8 3D Point

// Copy from 8BQube

```

struct Point {
    double x, y, z;
    Point(double _x = 0, double _y = 0, double _z = 0): x(_x)
        , y(_y), z(_z){}

```

```

    Point(pdd p) { x = p.X, y = p.Y, z = abs2(p); }
};

Point operator-(Point p1, Point p2)
{ return Point(p1.x - p2.x, p1.y - p2.y, p1.z - p2.z); }
Point operator+(Point p1, Point p2)
{ return Point(p1.x + p2.x, p1.y + p2.y, p1.z + p2.z); }
Point operator*(Point p1, double v)
{ return Point(p1.x * v, p1.y * v, p1.z * v); }
Point operator/(Point p1, double v)
{ return Point(p1.x / v, p1.y / v, p1.z / v); }
Point cross(Point p1, Point p2)
{ return Point(p1.y * p2.z - p1.z * p2.y, p1.z * p2.x - p1.
    x * p2.z, p1.x * p2.y - p1.y * p2.x); }
double dot(Point p1, Point p2)
{ return p1.x * p2.x + p1.y * p2.y + p1.z * p2.z; }
double abs(Point a)
{ return sqrt(dot(a, a)); }
Point cross3(Point a, Point b, Point c)
{ return cross(b - a, c - a); }
double area(Point a, Point b, Point c)
{ return abs(cross3(a, b, c)); }
double volume(Point a, Point b, Point c, Point d)
{ return dot(cross3(a, b, c), d - a); }
//Azimuthal angle (Longitude) to x-axis in interval [-pi,
    pi]
double phi(Point p) { return atan2(p.y, p.x); }
//Zenith angle (Latitude) to the z-axis in interval [0, pi]
double theta(Point p) { return atan2(sqrt(p.x * p.x + p.y *
    p.y), p.z); }
Point masscenter(Point a, Point b, Point c, Point d)
{ return (a + b + c + d) / 4; }
pdd proj(Point a, Point b, Point c, Point u) {
    // proj. u to the plane of a, b, and c
    Point e1 = b - a;
    Point e2 = c - a;
    e1 = e1 / abs(e1);
    e2 = e2 - e1 * dot(e2, e1);
    e2 = e2 / abs(e2);
    Point p = u - a;
    return pdd(dot(p, e1), dot(p, e2));
}
Point rotate_around(Point p, double angle, Point axis) {
    double s = sin(angle), c = cos(angle);
    Point u = axis / abs(axis);
    return u * dot(u, p) * (1 - c) + p * c + cross(u, p) * s;
}

```

4.9 ConvexHull3D

```

struct convex_hull_3D {
    struct Face {
        int a, b, c;
        Face(int ta, int tb, int tc): a(ta), b(tb), c(tc) {}
    }; // return the faces with pt indexes
    vector<Face> res;
    vector<Point> P;
    convex_hull_3D(const vector<Point> &_P): res(), P(_P) {
        // all points coplanar case will WA, O(n^2)
        int n = SZ(P);
        if (n <= 2) return; // be careful about edge case
        // ensure first 4 points are not coplanar
        swap(P[1], *find_if(iter(P), [&](auto p) { return sgn(
            abs2(P[0] - p)) != 0; }));
        swap(P[2], *find_if(iter(P), [&](auto p) { return sgn(
            abs2(cross3(p, P[0], P[1])) != 0; }));
        swap(P[3], *find_if(iter(P), [&](auto p) { return sgn(
            volume(P[0], P[1], P[2], p)) != 0; }));
        vector<vector<int>> flag(n, vector<int>(n));
        res.emplace_back(0, 1, 2); res.emplace_back(2, 1, 0);
        for (int i = 3; i < n; ++i) {
            vector<Face> next;
            for (auto f : res) {
                int d = sgn(volume(P[f.a], P[f.b], P[f.c], P[i]));
                if (d <= 0) next.pb(f);
                int ff = (d > 0) - (d < 0);
            }

```

```

        flag[f.a][f.b] = flag[f.b][f.c] = flag[f.c][f.a] = ff
        ;
    }
    for (auto f : res) {
        auto F = [&](int x, int y) {
            if (flag[x][y] > 0 && flag[y][x] <= 0)
                next.emplace_back(x, y, i);
        };
        F(f.a, f.b); F(f.b, f.c); F(f.c, f.a);
    }
    res = next;
}
}
bool same(Face s, Face t) {
    if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.a])) != 0)
        return 0;
    if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.b])) != 0)
        return 0;
    if (sgn(volume(P[s.a], P[s.b], P[s.c], P[t.c])) != 0)
        return 0;
    return 1;
}
int polygon_face_num() {
    int ans = 0;
    for (int i = 0; i < SZ(res); ++i)
        ans += none_of(res.begin(), res.begin() + i, [&](Face g) {
            return same(res[i], g);
        });
    return ans;
}
double get_volume() {
    double ans = 0;
    for (auto f : res)
        ans += volume(Point(0, 0, 0), P[f.a], P[f.b], P[f.c]);
    return fabs(ans / 6);
}
double get_dis(Point p, Face f) {
    Point p1 = P[f.a], p2 = P[f.b], p3 = P[f.c];
    double a = (p2.y - p1.y) * (p3.z - p1.z) - (p2.z - p1.z)
        * (p3.y - p1.y);
    double b = (p2.z - p1.z) * (p3.x - p1.x) - (p2.x - p1.x)
        * (p3.z - p1.z);
    double c = (p2.x - p1.x) * (p3.y - p1.y) - (p2.y - p1.y)
        * (p3.x - p1.x);
    double d = 0 - (a * p1.x + b * p1.y + c * p1.z);
    return fabs(a * p.x + b * p.y + c * p.z + d) / sqrt(a * a
        + b * b + c * c);
}
// n^2 delaunay: facets with negative z normal of
// convexhull of (x, y, x^2 + y^2), use a pseudo-point
// (0, 0, inf) to avoid degenerate case

```

4.10 Circle Operations

```

// from 8BQube
const double PI=acos(-1);
vector<pdd> circleLineIntersection(pdd c, double r, pdd a,
    pdd b) {
    pdd p = a + (b - a) * dot(c - a, b - a) / abs2(b - a);
    double s = cross(b - a, c - a), h2 = r * r - s * s / abs2
        (b - a);
    if (sgn(h2) < 0) return {};
    if (sgn(h2) == 0) return {p};
    pdd h = (b - a) / abs(b - a) * sqrt(h2);
    return {p - h, p + h};
}
double _area(pdd pa, pdd pb, double r){
    if(abs(pa)<abs(pb)) swap(pa, pb);
    if(abs(pb)<eps) return 0;
    double S, h, theta;
    double a=abs(pb),b=abs(pa),c=abs(pb-pa);
    double cosB = dot(pb,pb-pa) / a / c, B = acos(cosB);
    double cosC = dot(pa,pb) / a / b, C = acos(cosC);
    if(a > r){
        S = (C/2)*r*r;
        h = a*b*sin(C)/c;
    }
    if (h < r && B < PI/2) S -= (acos(h/r)*r*r - h*sqrt(r*r
        -h*h));
    }
    else if(b > r){
        theta = PI - B - asin(sin(B)/r*a);
        S = .5*a*r*sin(theta) + (C-theta)/2*r*r;
    }
    else S = .5*sin(C)*a*b;
    return S;
}
double areaPolyCircle(const vector<pdd> poly,const pdd &O,
    const double r){
    double S=0;
    for(int i=0;i<SZ(poly);++i)
        S+=_area(poly[i]-O,poly[(i+1)%SZ(poly)]-O,r)*ori(0,poly
            [i],poly[(i+1)%SZ(poly)]);
    return fabs(S);
}
bool CCinter(Cir &a, Cir &b, pdd &p1, pdd &p2) {
    pdd o1 = a.O, o2 = b.O;
    double r1 = a.R, r2 = b.R, d2 = abs2(o1 - o2), d = sqrt(
        d2);
    if(d < max(r1, r2) - min(r1, r2) || d > r1 + r2) return
        0;
    pdd u = (o1 + o2) * 0.5 + (o1 - o2) * ((r2 * r2 - r1 * r1
        ) / (2 * d2));
    double A = sqrt((r1 + r2 + d) * (r1 - r2 + d) * (r1 + r2
        - d) * (-r1 + r2 + d));
    pdd v = pdd(o1.Y - o2.Y, -o1.X + o2.X) * A / (2 * d2);
    p1 = u + v, p2 = u - v;
    return 1;
}
vector<Line> CCTang( const Cir& c1 , const Cir& c2 , int
    sign1 ){
    vector<Line> ret;
    double d_sq = abs2( c1.O - c2.O );
    if (sgn(d_sq) == 0) return ret;
    double d = sqrt(d_sq);
    pdd v = (c2.O - c1.O) / d;
    double c = (c1.R - sign1 * c2.R) / d; // cos t
    if (c * c > 1) return ret;
    double h = sqrt(max( 0.0, 1.0 - c * c )); // sin t
    for (int sign2 = 1; sign2 >= -1; sign2 -= 2) {
        pdd n = pdd(v.X * c - sign2 * h * v.Y,
            v.Y * c + sign2 * h * v.X);
        pdd p1 = c1.O + n * c1.R;
        pdd p2 = c2.O + n * (c2.R * sign1);
        if (sgn(p1.X - p2.X) == 0 and
            sgn(p1.Y - p2.Y) == 0)
            p2 = p1 + perp(c2.O - c1.O);
        ret.pb(Line(p1, p2));
    }
    return ret;
}

```

```

    if (h < r && B < PI/2) S -= (acos(h/r)*r*r - h*sqrt(r*r
        -h*h));
    }
    else if(b > r){
        theta = PI - B - asin(sin(B)/r*a);
        S = .5*a*r*sin(theta) + (C-theta)/2*r*r;
    }
    else S = .5*sin(C)*a*b;
    return S;
}
double areaPolyCircle(const vector<pdd> poly,const pdd &O,
    const double r){
    double S=0;
    for(int i=0;i<SZ(poly);++i)
        S+=_area(poly[i]-O,poly[(i+1)%SZ(poly)]-O,r)*ori(0,poly
            [i],poly[(i+1)%SZ(poly)]);
    return fabs(S);
}
bool CCinter(Cir &a, Cir &b, pdd &p1, pdd &p2) {
    pdd o1 = a.O, o2 = b.O;
    double r1 = a.R, r2 = b.R, d2 = abs2(o1 - o2), d = sqrt(
        d2);
    if(d < max(r1, r2) - min(r1, r2) || d > r1 + r2) return
        0;
    pdd u = (o1 + o2) * 0.5 + (o1 - o2) * ((r2 * r2 - r1 * r1
        ) / (2 * d2));
    double A = sqrt((r1 + r2 + d) * (r1 - r2 + d) * (r1 + r2
        - d) * (-r1 + r2 + d));
    pdd v = pdd(o1.Y - o2.Y, -o1.X + o2.X) * A / (2 * d2);
    p1 = u + v, p2 = u - v;
    return 1;
}
vector<Line> CCTang( const Cir& c1 , const Cir& c2 , int
    sign1 ){
    vector<Line> ret;
    double d_sq = abs2( c1.O - c2.O );
    if (sgn(d_sq) == 0) return ret;
    double d = sqrt(d_sq);
    pdd v = (c2.O - c1.O) / d;
    double c = (c1.R - sign1 * c2.R) / d; // cos t
    if (c * c > 1) return ret;
    double h = sqrt(max( 0.0, 1.0 - c * c )); // sin t
    for (int sign2 = 1; sign2 >= -1; sign2 -= 2) {
        pdd n = pdd(v.X * c - sign2 * h * v.Y,
            v.Y * c + sign2 * h * v.X);
        pdd p1 = c1.O + n * c1.R;
        pdd p2 = c2.O + n * (c2.R * sign1);
        if (sgn(p1.X - p2.X) == 0 and
            sgn(p1.Y - p2.Y) == 0)
            p2 = p1 + perp(c2.O - c1.O);
        ret.pb(Line(p1, p2));
    }
    return ret;
}

```

4.11 Delaunay Triangulation

```

/* Delaunay Triangulation:
Given a sets of points on 2D plane, find a
triangulation such that no points will strictly
inside circumcircle of any triangle. */
struct Edge {
    int id; // oidx[id]
    list<Edge>::iterator twin;
    Edge(int _id = 0):id(_id) {}
};
struct Delaunay { // 0-base
    int n, oidx[N];
    list<Edge> head[N]; // result udir. graph
    p11 p[N];
    void init(int _n, p11 _p[]) {
        n = _n, iota(oidx, oidx + n, 0);
        for (int i = 0; i < n; ++i) head[i].clear();
        sort(oidx, oidx + n, [&](int a, int b)
            { return _p[a] < _p[b]; });
        for (int i = 0; i < n; ++i) p[i] = _p[oidx[i]];
    }
}

```

```

    divide(0, n - 1);
}
void addEdge(int u, int v) {
    head[u].push_front(Edge(v));
    head[v].push_front(Edge(u));
    head[u].begin()->twin = head[v].begin();
    head[v].begin()->twin = head[u].begin();
}
void divide(int l, int r) {
    if (l == r) return;
    if (l + 1 == r) return addEdge(l, l + 1);
    int mid = (l + r) >> 1, nw[2] = {l, r};
    divide(l, mid), divide(mid + 1, r);
    auto gao = [&](int t) {
        pll pt[2] = {p[nw[0]], p[nw[1]]};
        for (auto it : head[nw[t]]) {
            int v = ori(pt[1], pt[0], p[it.id]);
            if (v > 0 || (v == 0 && abs2(pt[t ^ 1] - p[it.id])
                < abs2(pt[1] - pt[0])))
                return nw[t] = it.id, true;
        }
        return false;
    };
    while (gao(0) || gao(1));
    addEdge(nw[0], nw[1]); // add tangent
    while (true) {
        pll pt[2] = {p[nw[0]], p[nw[1]]};
        int ch = -1, sd = 0;
        for (int t = 0; t < 2; ++t)
            for (auto it : head[nw[t]])
                if (ori(pt[0], pt[1], p[it.id]) > 0 && (ch ==
                    -1 || in_cc({pt[0], pt[1], p[ch]}, p[it.
                        id])))
                    ch = it.id, sd = t;
        if (ch == -1) break; // upper common tangent
        for (auto it = head[nw[sd]].begin(); it != head[nw[sd]
            ].end(); )
            if (seg_strict_intersect(pt[sd], p[it->id], pt[sd ^
                1], p[ch]))
                head[it->id].erase(it->twin), head[nw[sd]].erase(
                    it++);
            else ++it;
        nw[sd] = ch, addEdge(nw[0], nw[1]);
    }
}
} tool;

```

4.12 Voronoi Diagram

*// all coord. is even, you may want to call halfPlaneInter
after then*

```

vector<vector<Line>> vec;
void build_voronoi_line(int n, pll *arr) {
    tool.init(n, arr); // Delaunay
    vec.clear(), vec.resize(n);
    for (int i = 0; i < n; ++i)
        for (auto e : tool.head[i]) {
            int u = tool.oidx[i], v = tool.oidx[e.id];
            pll m = (arr[v] + arr[u]) / 2LL, d = perp(arr[v] -
                arr[u]);
            vec[u].pb(Line(m, m + d));
        }
}

```

4.13 Polygon Union

// from 8BQube

```

ld rat(pll a, pll b) {
    return sgn(b.X) ? (ld)a.X / b.X : (ld)a.Y / b.Y;
} // all poly. should be cconv
ld polyUnion(vector<vector<pll>> &poly) {
    ld res = 0;
    for (auto &p : poly)
        for (int a = 0; a < SZ(p); ++a) {
            pll A = p[a], B = p[(a + 1) % SZ(p)];

```

```

        vector<pair<ld, int>> segs = {{0, 0}, {1, 0}};
        for (auto &q : poly) {
            if (&p == &q) continue;
            for (int b = 0; b < SZ(q); ++b) {
                pll C = q[b], D = q[(b + 1) % SZ(q)];
                int sc = ori(A, B, C), sd = ori(A, B, D);
                if (sc != sd && min(sc, sd) < 0) {
                    ld sa = cross(D - C, A - C), sb = cross(D - C,
                        B - C);
                    segs.pb(sa / (sa - sb), sgn(sc - sd));
                }
                if (!sc && !sd && &q < &p && sgn(dot(B - A, D - C
                    )) > 0) {
                    segs.pb(rat(C - A, B - A), 1);
                    segs.pb(rat(D - A, B - A), -1);
                }
            }
        }
        sort(iter(segs));
        for (auto &s : segs) s.X = clamp(s.X, 0.0, 1.0);
        ld sum = 0;
        int cnt = segs[0].second;
        for (int j = 1; j < SZ(segs); ++j) {
            if (!cnt) sum += segs[j].X - segs[j - 1].X;
            cnt += segs[j].Y;
        }
        res += cross(A, B) * sum;
    }
    return res / 2;
}

```

4.14 Tangent Point to Convex Hull

// from 8BQube

```

/* The point should be strictly out of hull
return arbitrary point on the tangent line */
pii get_tangent(vector<pll> &C, pll p) {
    auto gao = [&](int s) {
        return cyc_tsearch(SZ(C), [&](int x, int y)
            { return ori(p, C[x], C[y]) == s; });
    };
    return pii(gao(1), gao(-1));
} // return (a, b), ori(p, C[a], C[b]) >= 0

```

5 Graph

5.1 Block Cut Tree

```

struct BCC{
    vector<int> v, e, cut;
};
struct BlockCutTree{ // 0-based, allow multi edges but not
    allow loops
    int n, m, cnt = 0;
    // n:|V|, m:|E|, cnt:|bcc|
    vector<BCC> bcc;
    vector<vector<pii>> g; // original graph
    vector<pii> edges; // 0-based
    vector<vector<int>> vbcc;
    // vbcc[i] = BCCs containing vertex i, vbcc[i].size()>1
    // iff i is an articulation
    vector<int> ebcc;
    // edge i is a bridge iff bcc[ebcc[i]].e.size() == 1
    // block cut tree:
    // adj[bcc i]: bcc[i].cut
    // adj[cut i]: vbcc[i]

```

```

BlockCutTree(int _n, vector<pii> _edges):
    n(_n), m(SZ(_edges)), g(_n), edges(_edges), vbcc(_n),
    ebcc(SZ(_edges)){
    for(int i = 0; i < m; i++){
        auto [u, v] = edges[i];
        g[u].pb(pii(v, i)); g[v].pb(pii(u, i));
    }
}

```

```

}

void build(){
    vector<int> in(n, -1), low(n, -1);
    vector<vector<int>> up(n);
    vector<int> stk;
    int ts = 0;
    auto _dfs = [&](auto dfs, int now, int par, int pe) ->
        void{
        if(pe != -1) up[now].pb(pe);
        in[now] = low[now] = ts++;
        stk.pb(now);
        for(auto [v, e] : g[now]){
            if(e == pe) continue;
            if(in[v] != -1){
                if(in[v] < in[now]) up[now].pb(e);
                low[now] = min(low[now], in[v]);
                continue;
            }
            dfs(dfs, v, now, e);
            low[now] = min(low[now], low[v]);
        }
        if((now != par && low[now] >= in[par]) || (now == par
            && SZ(g[now]) == 0)){
            bcc.pb();
            while(true){
                int v = stk.back();
                stk.pop_back();
                vbcc[v].pb(cnt);
                bcc[cnt].v.pb(v);
                for(int e : up[v]){
                    ebcc[e] = cnt;
                    bcc[cnt].e.pb(e);
                }
                if(v == now) break;
            }
            if(now != par){
                vbcc[par].pb(cnt);
                bcc[cnt].v.pb(par);
            }
            cnt++;
        }
    };
    for(int i = 0; i < n; i++){
        if(in[i] == -1) _dfs(_dfs, i, i, -1);
    }
    for(int i = 0; i < cnt; i++){
        for(int j : bcc[i].v)
            if(SZ(vbcc[j]) > 1) bcc[i].cut.pb(j);
    }
};

```

5.2 2-SAT

```

struct SAT{ // 0-based, [n, 2n) is neg of [0, n)
    int n;
    vector<vector<int>> g, rg;
    bool ok = true;
    vector<bool> ans;

    void init(int _n){
        n = _n;
        g.resize(2 * n);
        rg.resize(2 * n);
        ans.resize(n);
    }
    int neg(int v){
        return v < n ? v + n : v - n;
    }
    void addEdge(int u, int v){
        g[u].pb(v);
        rg[v].pb(u);
    }
    void addClause(int a, int b){
        addEdge(neg(a), b);
        addEdge(neg(b), a);
    }
};

```

```

}
void build(){
    vector<bool> vst(2 * n, false);
    vector<int> tmp, scc(2 * n, -1);
    int cnt = 1;
    function<void(int)> dfs = [&](int now){
        vst[now] = true;
        for(int i : rg[now]){
            if(vst[i]) continue;
            dfs(i);
        }
        tmp.pb(now);
    };
    for(int i = 0; i < 2 * n; i++){
        if(!vst[i]) dfs(i);
    }
    reverse(all(tmp));
    function<void(int, int)> dfs2 = [&](int now, int id){
        scc[now] = id;
        for(int i : g[now]){
            if(scc[i] != -1) continue;
            dfs2(i, id);
        }
    };
    for(int i : tmp){
        if(scc[i] == -1) dfs2(i, cnt++);
    }
    debug(scc);
    for(int i = 0; i < n; i++){
        if(scc[i] == scc[neg(i)]){
            ok = false;
            return;
        }
        if(scc[i] < scc[neg(i)]) ans[i] = true;
        else ans[i] = false;
    }
};

```

5.3 Dominator Tree

```

// copy from 8BQube
struct dominator_tree { // 1-base
    vector<int> G[N], rG[N];
    int n, pa[N], dfn[N], id[N], Time;
    int semi[N], idom[N], best[N];
    vector<int> tree[N]; // dominator_tree
    void init(int _n) {
        n = _n;
        for (int i = 1; i <= n; ++i)
            G[i].clear(), rG[i].clear();
    }
    void add_edge(int u, int v) {
        G[u].pb(v), rG[v].pb(u);
    }
    void dfs(int u) {
        id[dfn[u] = ++Time] = u;
        for (auto v : G[u])
            if (!dfn[v]) dfs(v), pa[dfn[v]] = dfn[u];
    }
    int find(int y, int x) {
        if (y <= x) return y;
        int tmp = find(pa[y], x);
        if (semi[best[y]] > semi[best[pa[y]]])
            best[y] = best[pa[y]];
        return pa[y] = tmp;
    }
    void tarjan(int root) {
        Time = 0;
        for (int i = 1; i <= n; ++i) {
            dfn[i] = idom[i] = 0;
            tree[i].clear();
            best[i] = semi[i] = i;
        }
        dfs(root);
    }
};

```

```

for (int i = Time; i > 1; --i) {
    int u = id[i];
    for (auto v : rG[u])
        if (v = dfn[v]) {
            find(v, i);
            semi[i] = min(semi[i], semi[best[v]]);
        }
    tree[semi[i]].pb(i);
    for (auto v : tree[pa[i]]) {
        find(v, pa[i]);
        idom[v] =
            semi[best[v]] == pa[i] ? pa[i] : best[v];
    }
    tree[pa[i]].clear();
}
for (int i = 2; i <= Time; ++i) {
    if (idom[i] != semi[i]) idom[i] = idom[idom[i]];
    tree[id[idom[i]]].pb(id[i]);
}
}
};

```

5.4 Virtual Tree

// copy from 8BQube

```

vector<int> vG[N];
int top, st[N];

```

```

void insert(int u) {
    if (top == -1) return st[++top] = u, void();
    int p = LCA(st[top], u);
    if (p == st[top]) return st[++top] = u, void();
    while (top >= 1 && dep[st[top - 1]] >= dep[p])
        vG[st[top - 1]].pb(st[top]), --top;
    if (st[top] != p)
        vG[p].pb(st[top]), --top, st[++top] = p;
    st[++top] = u;
}

```

```

void reset(int u) {
    for (int i : vG[u]) reset(i);
    vG[u].clear();
}

```

```

void solve(vector<int> &v) {
    top = -1;
    sort(ALL(v),
        [&](int a, int b) { return dfn[a] < dfn[b]; });
    for (int i : v) insert(i);
    while (top > 0) vG[st[top - 1]].pb(st[top]), --top;
    // do something
    reset(v[0]);
}

```

5.5 Directed Minimum Spanning Tree

```

const ll INF = LLONG_MAX;
struct edge{
    int u = -1, v = -1;
    ll w = INF;
    int id = -1;
};

// 0-based, E[i].id = i
bool DMST(int n, vector<edge> &E, int root, vector<edge> &
    sol){
    vector<int> id(n), vis(n);
    vector<edge> in(n);
    for(edge e : E)
        if(e.u != e.v && e.w < in[e.v].w && e.v != root)
            in[e.v] = e;
    for(int i = 0; i < n; i++)
        if(i != root && in[i].u == -1) return false; // no sol
    int cnt = 0;
    fill(iter(id), -1); fill(iter(vis), -1);

```

```

for(int u = 0; u < n; u++){
    int v = u;
    while(vis[v] != u && id[v] == -1 && in[v].u != -1)
        vis[v] = u, v = in[v].u;
    if(v != root && id[v] == -1){
        for(int x = in[v].u; x != v; x = in[x].u)
            id[x] = cnt;
        id[v] = cnt++;
    }
}
if(!cnt) return sol = in, true; // no cycle
for(int u = 0; u < n; u++)
    if(id[u] == -1) id[u] = cnt++;
vector<edge> nE;
for(int i = 0; i < SZ(E); i++){
    edge tmp = E[i];
    tmp.u = id[tmp.u], tmp.v = id[tmp.v];
    if(in[E[i].v].w != INF) tmp.w -= in[E[i].v].w;
    nE.pb(tmp);
}
vector<edge> tsol;
if(!DMST(cnt, nE, id[root], tsol)) return false;
sol.resize(n);
for(int i = 0; i < cnt; i++){
    if(i == id[root]) continue;
    int t = tsol[i].id;
    sol[E[t].v] = E[t];
}
for(int i = 0; i < n; i++)
    if(sol[i].id == -1) sol[i] = in[i];
return true;
}

```

5.6 Vizing

// find D+1 edge coloring of a graph with max deg D

```

struct vizing { // returns edge coloring in adjacent matrix
    G. 1 - based
    const int N = 105;
    int C[N][N], G[N][N], X[N], vst[N], n; // ans: G[i][j]
    void init(int _n) { n = _n; // n = |V|+1
        for (int i = 0; i <= n; ++i)
            for (int j = 0; j <= n; ++j)
                C[i][j] = G[i][j] = 0;
    }
    void solve(vector<pii> &E) {
        auto update = [&](int u)
        { for (X[u] = 1; C[u][X[u]]; ++X[u]); };
        auto color = [&](int u, int v, int c) {
            int p = G[u][v];
            G[u][v] = G[v][u] = c;
            C[u][c] = v, C[v][c] = u;
            C[u][p] = C[v][p] = 0;
            if (p) X[u] = X[v] = p;
            else update(u), update(v);
            return p;
        };
        auto flip = [&](int u, int c1, int c2) {
            int p = C[u][c1];
            swap(C[u][c1], C[u][c2]);
            if (p) G[u][p] = G[p][u] = c2;
            if (!C[u][c1]) X[u] = c1;
            if (!C[u][c2]) X[u] = c2;
            return p;
        };
        fill_n(X + 1, n, 1);
        for (int t = 0; t < SZ(E); ++t) {
            int u = E[t].X, v0 = E[t].Y, v = v0, c0 = X[u], c =
                c0, d;
            vector<pii> L;
            fill_n(vst + 1, n, 0);
            while (!G[u][v0]) {
                L.emplace_back(v, d = X[v]);
                if (!C[v][c]) for (int a = SZ(L) - 1; a >= 0; --a)
                    c = color(u, L[a].X, c);
            }

```

```

    else if (!C[u][d]) for (int a = SZ(L) - 1; a >= 0;
        --a) color(u, L[a].X, L[a].Y);
    else if (vst[d]) break;
    else vst[d] = 1, v = C[u][d];
}
if (!G[u][v0]) {
    for (; v; v = flip(v, c, d), swap(c, d));
    if (int a; C[u][c0]) {
        for (a = SZ(L) - 2; a >= 0 && L[a].Y != c; --a);
        for (; a >= 0; --a) color(u, L[a].X, L[a].Y);
    }
    else --t;
}
}
}
};

```

5.7 Maximum Clique

```

struct MaxClique { // fast when N <= 100
    bitset<N> G[N], cs[N];
    int ans, sol[N], q, cur[N], d[N], n;
    void init(int _n) {
        n = _n;
        for (int i = 0; i < n; ++i) G[i].reset();
    }
    void add_edge(int u, int v) {
        G[u][v] = G[v][u] = 1;
    }
    void pre_dfs(vector<int> &r, int l, bitset<N> mask) {
        if (l < 4) {
            for (int i : r) d[i] = (G[i] & mask).count();
            sort(ALL(r), [&](int x, int y) { return d[x] > d[y]; });
        }
        vector<int> c(SZ(r));
        int lft = max(ans - q + 1, 1), rgt = 1, tp = 0;
        cs[1].reset(), cs[2].reset();
        for (int p : r) {
            int k = 1;
            while ((cs[k] & G[p]).any()) ++k;
            if (k > rgt) cs[++rgt + 1].reset();
            cs[k][p] = 1;
            if (k < lft) r[tp++] = p;
        }
        for (int k = lft; k <= rgt; ++k)
            for (int p = cs[k]._Find_first(); p < N; p = cs[k]._Find_next(p))
                r[tp] = p, c[tp] = k, ++tp;
        dfs(r, c, l + 1, mask);
    }
    void dfs(vector<int> &r, vector<int> &c, int l, bitset<N> mask) {
        while (!r.empty()) {
            int p = r.back();
            r.pop_back(), mask[p] = 0;
            if (q + c.back() <= ans) return;
            cur[q++] = p;
            vector<int> nr;
            for (int i : r) if (G[p][i]) nr.pb(i);
            if (!nr.empty()) pre_dfs(nr, l, mask & G[p]);
            else if (q > ans) ans = q, copy_n(cur, q, sol);
            c.pop_back(), --q;
        }
    }
    int solve() {
        vector<int> r(n);
        ans = q = 0, iota(ALL(r), 0);
        pre_dfs(r, 0, bitset<N>(string(n, '1')));
        return ans;
    }
};

```

5.8 Number of Maximal Clique

```

struct BronKerbosch { // 1-base
    int n, a[N], g[N][N];
    int S, all[N][N], some[N][N], none[N][N];
    void init(int _n) {
        n = _n;
        for (int i = 1; i <= n; ++i)
            for (int j = 1; j <= n; ++j) g[i][j] = 0;
    }
    void add_edge(int u, int v) {
        g[u][v] = g[v][u] = 1;
    }
    void dfs(int d, int an, int sn, int nn) {
        if (S > 1000) return; // pruning
        if (sn == 0 && nn == 0) ++S;
        int u = some[d][0];
        for (int i = 0; i < sn; ++i) {
            int v = some[d][i];
            if (g[u][v]) continue;
            int tsu = 0, tnn = 0;
            copy_n(all[d], an, all[d + 1]);
            all[d + 1][an] = v;
            for (int j = 0; j < sn; ++j)
                if (g[v][some[d][j]])
                    some[d + 1][tsu++] = some[d][j];
            for (int j = 0; j < nn; ++j)
                if (g[v][none[d][j]])
                    none[d + 1][tnn++] = none[d][j];
            dfs(d + 1, an + 1, tsu, tnn);
            some[d][i] = 0, none[d][nn++] = v;
        }
    }
    int solve() {
        iota(some[0], some[0] + n, 1);
        S = 0, dfs(0, 0, n, 0);
        return S;
    }
};

```

5.9 Minimum Mean Cycle

```

// from 8BQube
ll road[N][N]; // input here
struct MinimumMeanCycle {
    ll dp[N + 5][N], n;
    pll solve() {
        ll a = -1, b = -1, L = n + 1;
        for (int i = 2; i <= L; ++i)
            for (int k = 0; k < n; ++k)
                for (int j = 0; j < n; ++j)
                    dp[i][j] =
                        min(dp[i - 1][k] + road[k][j], dp[i][j]);
        for (int i = 0; i < n; ++i) {
            if (dp[L][i] >= INF) continue;
            ll ta = 0, tb = 1;
            for (int j = 1; j < n; ++j)
                if (dp[j][i] < INF &&
                    ta * (L - j) < (dp[L][i] - dp[j][i]) * tb)
                    ta = dp[L][i] - dp[j][i], tb = L - j;
            if (ta == 0) continue;
            if (a == -1 || a * tb > ta * b) a = ta, b = tb;
        }
        if (a != -1) {
            ll g = __gcd(a, b);
            return pll(a / g, b / g);
        }
        return pll(-1LL, -1LL);
    }
    void init(int _n) {
        n = _n;
        for (int i = 0; i < n; ++i)
            for (int j = 0; j < n; ++j) dp[i + 2][j] = INF;
    }
};

```

5.10 Minimum Steiner Tree

```
// from 8BQube
//  $O(V^3 + V^2 \log V)$ 
struct SteinerTree { // 0-base
    static const int T = 10, N = 105, INF = 1e9;
    int n, dst[N][N], dp[1 << T][N], tdst[N];
    int vcost[N]; // the cost of vertexs
    void init(int _n) {
        n = _n;
        for (int i = 0; i < n; ++i) {
            for (int j = 0; j < n; ++j) dst[i][j] = INF;
            dst[i][i] = vcost[i] = 0;
        }
    }
    void add_edge(int ui, int vi, int wi) {
        dst[ui][vi] = min(dst[ui][vi], wi);
    }
    void shortest_path() {
        for (int k = 0; k < n; ++k)
            for (int i = 0; i < n; ++i)
                for (int j = 0; j < n; ++j)
                    dst[i][j] =
                        min(dst[i][j], dst[i][k] + dst[k][j]);
    }
    int solve(const vector<int> &ter) {
        shortest_path();
        int t = SZ(ter);
        for (int i = 0; i < (1 << t); ++i)
            for (int j = 0; j < n; ++j) dp[i][j] = INF;
        for (int i = 0; i < n; ++i) dp[0][i] = vcost[i];
        for (int msk = 1; msk < (1 << t); ++msk) {
            if (!(msk & (msk - 1))) {
                int who = __lg(msk);
                for (int i = 0; i < n; ++i)
                    dp[msk][i] =
                        vcost[ter[who]] + dst[ter[who]][i];
            }
            for (int i = 0; i < n; ++i)
                for (int submsk = (msk - 1) & msk; submsk;
                    submsk = (submsk - 1) & msk)
                    dp[msk][i] = min(dp[msk][i],
                        dp[submsk][i] + dp[msk ^ submsk][i] -
                        vcost[i]);
            for (int i = 0; i < n; ++i) {
                tdst[i] = INF;
                for (int j = 0; j < n; ++j)
                    tdst[i] =
                        min(tdst[i], dp[msk][j] + dst[j][i]);
            }
            for (int i = 0; i < n; ++i) dp[msk][i] = tdst[i];
        }
        int ans = INF;
        for (int i = 0; i < n; ++i)
            ans = min(ans, dp[(1 << t) - 1][i]);
        return ans;
    }
};
```

6 Math

6.1 Extended Euclidean Algorithm

```
//  $ax + ny = 1, ax + ny == ax == 1 \pmod n$ 
void extgcd(ll x, ll y, ll &g, ll &a, ll &b) {
    if (y == 0) g = x, a = 1, b = 0;
    else extgcd(y, x % y, g, b, a), b -= (x / y) * a;
}
```

6.2 Floor & Ceil

```
int ifloor(int a, int b) {
    return a / b - (a % b && (a < 0) ^ (b < 0));
}
int iceil(int a, int b) {
    return a / b + (a % b && (a < 0) ^ (b > 0));
}
```

```
}
```

6.3 Legendre

```
// the Jacobi symbol is a generalization of the Legendre
// symbol,
// such that the bottom doesn't need to be prime.
//  $(n/p) \rightarrow$  same as Legendre
//  $(n/ab) = (n/a)(n/b)$ 
// work with long long
int Jacobi(int a, int m) {
    int s = 1;
    for (; m > 1; ) {
        a %= m;
        if (a == 0) return 0;
        const int r = __builtin_ctz(a);
        if ((r & 1) && ((m + 2) & 4)) s = -s;
        a >>= r;
        if (a & m & 2) s = -s;
        swap(a, m);
    }
    return s;
}

// 0: a == 0
// -1: a isn't a quad res of p
// else: return X with  $X^2 \% p == a$ 
// doesn't work with long long
int QuadraticResidue(int a, int p) {
    if (p == 2) return a & 1;
    const int jc = Jacobi(a, p);
    if (jc == 0) return 0;
    if (jc == -1) return -1;
    int b, d;
    for (; ; ) {
        b = rand() % p;
        d = (1LL * b * b + p - a) % p;
        if (Jacobi(d, p) == -1) break;
    }
    int f0 = b, f1 = 1, g0 = 1, g1 = 0, tmp;
    for (int e = (1LL + p) >> 1; e; e >>= 1) {
        if (e & 1) {
            tmp = (1LL * g0 * f0 + 1LL * d * (1LL * g1 * f1 % p))
                % p;
            g1 = (1LL * g0 * f1 + 1LL * g1 * f0) % p;
            g0 = tmp;
        }
        tmp = (1LL * f0 * f0 + 1LL * d * (1LL * f1 * f1 % p)) %
            p;
        f1 = (2LL * f0 * f1) % p;
        f0 = tmp;
    }
    return g0;
}
```

6.4 Simplex

```
#pragma once

typedef double T; // Long double, Rational, double + mod<P>
// ...
typedef vector<T> vd;
typedef vector<vd> vvd;

const T eps = 1e-8, inf = 1/.0;
#define MP make_pair
#define ltj(X) if(s == -1 || MP(X[j], N[j]) < MP(X[s], N[s]))
    s=j

struct LPSolver {
    int m, n;
    vi N, B;
    vvd D;

    LPSolver(const vvd& A, const vd& b, const vd& c) :
```



```

m(sz(b)), n(sz(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
    rep(i,0,m) rep(j,0,n) D[i][j] = A[i][j];
    rep(i,0,m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[
        i];}
    rep(j,0,n) { N[j] = j; D[m][j] = -c[j]; }
    N[n] = -1; D[m+1][n] = 1;
}

void pivot(int r, int s) {
    T *a = D[r].data(), inv = 1 / a[s];
    rep(i,0,m+2) if (i != r && abs(D[i][s]) > eps) {
        T *b = D[i].data(), inv2 = b[s] * inv;
        rep(j,0,n+2) b[j] -= a[j] * inv2;
        b[s] = a[s] * inv2;
    }
    rep(j,0,n+2) if (j != s) D[r][j] *= inv;
    rep(i,0,m+2) if (i != r) D[i][s] *= -inv;
    D[r][s] = inv;
    swap(B[r], N[s]);
}

bool simplex(int phase) {
    int x = m + phase - 1;
    for (;;) {
        int s = -1;
        rep(j,0,n+1) if (N[j] != -phase) ltj(D[x]);
        if (D[x][s] >= -eps) return true;
        int r = -1;
        rep(i,0,m) {
            if (D[i][s] <= eps) continue;
            if (r == -1 || MP(D[i][n+1] / D[i][s], B[i])
                < MP(D[r][n+1] / D[r][s], B[r])) r = i;
        }
        if (r == -1) return false;
        pivot(r, s);
    }
}

T solve(vd &x) {
    int r = 0;
    rep(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
    if (D[r][n+1] < -eps) {
        pivot(r, n);
        if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
        rep(i,0,m) if (B[i] == -1) {
            int s = 0;
            rep(j,1,n+1) ltj(D[i]);
            pivot(i, s);
        }
    }
    bool ok = simplex(1); x = vd(n);
    rep(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
    return ok ? D[m][n+1] : inf;
}
};

```

6.5 Floor Sum

```

// from 8BQube
ll floor_sum(ll n, ll m, ll a, ll b) {
    ll ans = 0;
    if (a >= m)
        ans += (n - 1) * n * (a / m) / 2, a %= m;
    if (b >= m)
        ans += n * (b / m), b %= m;
    ll y_max = (a * n + b) / m, x_max = (y_max * m - b);
    if (y_max == 0) return ans;
    ans += (n - (x_max + a - 1) / a) * y_max;
    ans += floor_sum(y_max, a, m, (a - x_max % a) % a);
    return ans;
}
// sum^{n-1}_0 floor((a * i + b) / m) in Log(n + m + a + b)

```

6.6 DiscreteLog

```

int DiscreteLog(int s, int x, int y, int m) {
    constexpr int kStep = 32000;
    unordered_map<int, int> p;
    int b = 1;
    for (int i = 0; i < kStep; ++i) {
        p[y] = i;
        y = 1LL * y * x % m;
        b = 1LL * b * x % m;
    }
    for (int i = 0; i < m + 10; i += kStep) {
        s = 1LL * s * b % m;
        if (p.find(s) != p.end()) return i + kStep - p[s];
    }
    return -1;
}

int DiscreteLog(int x, int y, int m) {
    if (m == 1) return 0;
    int s = 1;
    for (int i = 0; i < 100; ++i) {
        if (s == y) return i;
        s = 1LL * s * x % m;
    }
    if (s == y) return 100;
    int p = 100 + DiscreteLog(s, x, y, m);
    if (fpow(x, p, m) != y) return -1;
    return p; //returns: x^p = y (mod m)
}

```

6.7 Miller Rabin & Pollard Rho

```

// n < 4,759,123,141      3 : 2, 7, 61
// n < 1,122,004,669,633  4 : 2, 13, 23, 1662803
// n < 3,474,749,660,383  6 : primes <= 13
// n < 2^64              7 :
// 2, 325, 9375, 28178, 450775, 9780504, 1795265022
ll mul(ll a, ll b, ll n){
    return (__int128)a * b % n;
}

bool Miller_Rabin(ll a, ll n) {
    if ((a = a % n) == 0) return 1;
    if (n % 2 == 0) return n == 2;
    ll tmp = (n - 1) / ((n - 1) & (1 - n));
    ll t = __lg(((n - 1) & (1 - n))), x = 1;
    for (; tmp; tmp >>= 1, a = mul(a, a, n))
        if (tmp & 1) x = mul(x, a, n);
    if (x == 1 || x == n - 1) return 1;
    while (--t)
        if ((x = mul(x, x, n)) == n - 1) return 1;
    return 0;
}

bool prime(ll n){
    vector<ll> tmp = {2, 325, 9375, 28178, 450775, 9780504,
        1795265022};
    for(ll i : tmp)
        if(!Miller_Rabin(i, n)) return false;
    return true;
}

map<ll, int> cnt;
void PollardRho(ll n) {
    if (n == 1) return;
    if (prime(n)) return ++cnt[n], void();
    if (n % 2 == 0) return PollardRho(n / 2), ++cnt[2], void();
    ll x = 2, y = 2, d = 1, p = 1;
#define f(x, n, p) ((mul(x, x, n) + p) % n)
    while (true) {
        if (d != n && d != 1) {
            PollardRho(n / d);
            PollardRho(d);
            return;
        }
        if (d == n) ++p;
        x = f(x, n, p), y = f(f(y, n, p), n, p);
        d = gcd(abs(x - y), n);
    }
}

```

6.8 XOR Basis

```
const int digit = 60; // [0, 2^digit)
struct Basis{
    int total = 0, rank = 0;
    vector<ll> b;
    Basis(): b(digit) {}
    bool add(ll v){ // Gauss Jordan Elimination
        total++;
        for(int i = digit - 1; i >= 0; i--){
            if(!(1LL << i & v)) continue;
            if(b[i] != 0){
                v ^= b[i];
                continue;
            }
            for(int j = 0; j < i; j++){
                if(1LL << j & v) v ^= b[j];
            }
            for(int j = i + 1; j < digit; j++){
                if(1LL << i & b[j]) b[j] ^= v;
            }
            b[i] = v;
            rank++;
            return true;
        }
        return false;
    }
    ll getmax(ll x = 0){
        for(ll i : b) x = max(x, x ^ i);
        return x;
    }
    ll getmin(ll x = 0){
        for(ll i : b) x = min(x, x ^ i);
        return x;
    }
    bool can(ll x){
        return getmin(x) == 0;
    }
    ll kth(ll k){ // kth smallest, 0-indexed
        vector<ll> tmp;
        for(ll i : b) if(i) tmp.pb(i);
        ll ans = 0;
        for(int i = 0; i < SZ(tmp); i++){
            if(1LL << i & k) ans ^= tmp[i];
        }
        return ans;
    }
};
```

6.9 Linear Equation

```
vector<int> RREF(vector<vector<ll>> &mat){
    int N = mat.size(), M = mat[0].size();
    int rk = 0;
    vector<int> cols;
    for (int i = 0; i < M; i++) {
        int cnt = -1;
        for (int j = N-1; j >= rk; j--){
            if(mat[j][i] != 0) cnt = j;
        }
        if(cnt == -1) continue;
        swap(mat[rk], mat[cnt]);
        ll lead = mat[rk][i];
        for (int j = 0; j < M; j++) mat[rk][j] /= lead;
        for (int j = 0; j < N; j++) {
            if(j == rk) continue;
            ll tmp = mat[j][i];
            for (int k = 0; k < M; k++)
                mat[j][k] -= mat[rk][k] * tmp;
        }
        cols.pb(i);
        rk++;
    }
    return cols;
}
struct LinearEquation{
    bool ok;
    vector<ll> par; //particular solution (Ax = b)
    vector<vector<ll>> homo; //homogenous (Ax = 0)
    vector<vector<ll>> rref;
```

```
//first M columns are matrix A
//last column of eq is vector b
void solve(const vector<vector<ll>> &eq){
    int M = (int)eq[0].size() - 1;
    rref = eq;
    auto piv = RREF(rref);
    int rk = piv.size();
    if(piv.size() && piv.back() == M){
        ok = 0; return;
    }
    ok = 1;
    par.resize(M);
    vector<bool> ispiv(M);
    for (int i = 0; i < rk; i++) {
        par[piv[i]] = rref[i][M];
        ispiv[piv[i]] = 1;
    }
    for (int i = 0; i < M; i++) {
        if (ispiv[i]) continue;
        vector<ll> h(M);
        h[i] = 1;
        for (int j = 0; j < rk; j++) h[piv[j]] = -rref[j][i];
        homo.pb(h);
    }
}
```

6.10 Chinese Remainder Theorem

```
pll solve_crt(ll x1, ll m1, ll x2, ll m2){
    ll g = gcd(m1, m2);
    if ((x2 - x1) % g) return {0, 0}; // no sol
    m1 /= g; m2 /= g;
    ll _, p, q;
    extgcd(m1, m2, _, p, q); // p <= C
    ll lcm = m1 * m2 * g;
    ll res = ((__int128)p * (x2 - x1) % lcm * m1 % lcm + x1)
        % lcm;
    // be careful with overflow, C^3
    return {(res + lcm) % lcm, lcm}; // (x, m)
}
```

6.11 Sqrt Decomposition

```
// for all i in [l, r], floor(n / i) = x
for(int l = 1, r; l <= n; l = r + 1){
    int x = ifloor(n, l);
    r = ifloor(n, x);
}
// for all i in [l, r], ceil(n / i) = x
for(int l, r = n; r >= 1; r = l - 1){
    int x = iceil(n, r);
    l = iceil(n, x);
}
```

7 Misc

7.1 Cyclic Ternary Search

```
/* bool pred(int a, int b);
f(0) ~ f(n - 1) is a cyclic-shift U-function
return idx s.t. pred(x, idx) is false forall x*/
int cyc_tsearch(int n, auto pred) {
    if (n == 1) return 0;
    int l = 0, r = n; bool rv = pred(1, 0);
    while (r - l > 1) {
        int m = (l + r) / 2;
        if (pred(0, m) ? rv : pred(m, (m + 1) % n)) r = m;
        else l = m;
    }
    return pred(1, r % n) ? l : r % n;
}
```

7.2 Matroid

我們稱一個二元組 $M = (E, \mathcal{I})$ 為一個擬陣，其中 $\mathcal{I} \subseteq 2^E$ 為 E 的子集所形成的非空集合，若：

- 若 $S \in \mathcal{I}$ 以及 $S' \subsetneq S$ ，則 $S' \in \mathcal{I}$
- 對於 $S_1, S_2 \in \mathcal{I}$ 滿足 $|S_1| < |S_2|$ ，存在 $e \in S_2 \setminus S_1$ 使得 $S_1 \cup \{e\} \in \mathcal{I}$

除此之外，我們有以下的定義：

- 位於 \mathcal{I} 中的集合我們稱之為獨立集 (independent set)，反之不在 \mathcal{I} 中的我們稱為相依集 (dependent set)
- 極大的獨立集為基底 (base)、極小的相依集為迴路 (circuit)
- 一個集合 Y 的秩 (rank) $r(Y)$ 為該集中最大的獨立子集，也就是 $r(Y) = \max\{|X| \mid X \subseteq Y \text{ 且 } X \in \mathcal{I}\}$

性質：

1. $X \subseteq Y \wedge Y \in \mathcal{I} \implies X \in \mathcal{I}$
2. $X \subseteq Y \wedge X \notin \mathcal{I} \implies Y \notin \mathcal{I}$
3. 若 B 與 B' 皆是基底且 $B \subseteq B'$ ，則 $B = B'$
若 C 與 C' 皆是迴路且 $C \subseteq C'$ ，則 $C = C'$
4. $e \in E \wedge X \subseteq E \implies r(X) \leq r(X \cup \{e\}) \leq r(X) + 1$ i.e. 加入一個元素後秩不會降底，最多增加 1
5. $\forall Y \subseteq E, \exists X \subseteq Y, r(X) = |X| = r(Y)$

一些等價的性質：

1. 對於所有 $X \subseteq E$ ， X 的極大獨立子集都有相同的大小
2. 對於 $B_1, B_2 \in \mathcal{B} \wedge B_1 \neq B_2$ ，對於所有 $e_1 \in B_1 \setminus B_2$ ，存在 $e_2 \in B_2 \setminus B_1$ 使得 $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$
3. 對於 $X, Y \in \mathcal{I}$ 且 $|X| < |Y|$ ，存在 $e \in Y \setminus X$ 使得 $X \cup \{e\} \in \mathcal{B}$
4. 如果 $r(X \cup \{e_1\}) = r(X \cup \{e_2\}) = r(X)$ ，則 $r(X \cup \{e_1, e_2\}) = r(X)$ 。
如果 $r(X \cup \{e\}) = r(X)$ 對於所有 $e \in E'$ 都成立，則 $r(X \cup E') = r(X)$ 。

擬陣交

```
Data: 兩個擬陣  $M_1 = (E, \mathcal{I}_1)$  以及  $M_2 = (E, \mathcal{I}_2)$ 
Result:  $I$  為最大的位於  $\mathcal{I}_1 \cap \mathcal{I}_2$  中的獨立集
 $I \leftarrow \emptyset$ 
 $X_1 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_1\}$ 
 $X_2 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_2\}$ 
while  $X_1 \neq \emptyset$  且  $X_2 \neq \emptyset$  do
    if  $e \in X_1 \cap X_2$  then
         $I \leftarrow I \cup \{e\}$ 
    else
        構造交換圖  $\mathcal{D}_{M_1, M_2}(I)$ 
        在交換圖上找到一條  $X_1$  到  $X_2$  且沒有捷徑的路徑  $P$ 
         $I \leftarrow I \Delta P$ 
    end if
     $X_1 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_1\}$ 
     $X_2 \leftarrow \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}_2\}$ 
end while
```

8 Polynomial

8.1 FWHT

```
/* x: a[j], y: a[j + (L >> 1)]
or: (y += x * op), and: (x += y * op)
xor: (x, y = (x + y) * op, (x - y) * op)
invop: or, and, xor = -1, -1, 1/2 */
void fwt(int *a, int n, int op) { //or
    for (int L = 2; L <= n; L <= 1)
        for (int i = 0; i < n; i += L)
            for (int j = i; j < i + (L >> 1); ++j)
                a[j + (L >> 1)] += a[j] * op;
}
const int N = 21;
int f[N][1 << N], g[N][1 << N], h[N][1 << N], ct[1 << N];
void subset_convolution(int *a, int *b, int *c, int L) {
    // c_k = \sum_{i+j=k, i&j=0} a_i * b_j
    int n = 1 << L;
    for (int i = 1; i < n; ++i)
        ct[i] = ct[i & (i - 1)] + 1;
    for (int i = 0; i < n; ++i)
        f[ct[i]][i] = a[i], g[ct[i]][i] = b[i];
    for (int i = 0; i <= L; ++i)
```

```
        fwt(f[i], n, 1), fwt(g[i], n, 1);
    for (int i = 0; i <= L; ++i)
        for (int j = 0; j <= i; ++j)
            for (int x = 0; x < n; ++x)
                h[i][x] += f[j][x] * g[i - j][x];
    for (int i = 0; i <= L; ++i) fwt(h[i], n, -1);
    for (int i = 0; i < n; ++i) c[i] = h[ct[i]][i];
}
```

8.2 FFT

```
using val_t = complex<double>;
template<int MAXN>
struct FFT {
    const double PI = acos(-1);
    val_t w[MAXN];
    FFT() {
        for (int i = 0; i < MAXN; ++i) {
            double arg = 2 * PI * i / MAXN;
            w[i] = val_t(cos(arg), sin(arg));
        }
    }
    void bitrev(vector<val_t> &a, int n) //same as NTT
    void trans(vector<val_t> &a, int n, bool inv = false) {
        bitrev(a, n);
        for (int L = 2; L <= n; L <= 1) {
            int dx = MAXN / L, dl = L >> 1;
            for (int i = 0; i < n; i += L) {
                for (int j = i, x = 0; j < i + dl; ++j, x += dx) {
                    val_t tmp = a[j + dl] * (inv ? conj(w[x]) : w[x]);
                    a[j + dl] = a[j] - tmp;
                    a[j] += tmp;
                }
            }
        }
        if (inv) {
            for (int i = 0; i < n; ++i) a[i] /= n;
        }
    }
    //multiplying two polynomials A * B:
    //fft.trans(A, siz, 0), fft.trans(B, siz, 0):
    //A[i] *= B[i], fft.trans(A, siz, 1);
};
```

8.3 NTT

```
//(2^16)+1, 65537, 3
//7*17*(2^23)+1, 998244353, 3
//1255*(2^20)+1, 1315962881, 3
//51*(2^25)+1, 1711276033, 29
template<int MAXN, ll P, ll RT> //MAXN must be 2^k
struct NTT {
    ll w[MAXN];
    ll mpow(ll a, ll n);
    ll minv(ll a) { return mpow(a, P - 2); }
    NTT() {
        ll dw = mpow(RT, (P - 1) / MAXN);
        w[0] = 1;
        for (int i = 1; i < MAXN; ++i) w[i] = w[i - 1] * dw % P;
    }
    void bitrev(vector<ll> &a, int n) {
        int i = 0;
        for (int j = 1; j < n - 1; ++j) {
            for (int k = n >> 1; (i ^ k) < k; k >>= 1);
            if (j < i) swap(a[i], a[j]);
        }
    }
    void operator()(vector<ll> &a, int n, bool inv = false) {
        //0 <= a[i] < P
        bitrev(a, n);
        for (int L = 2; L <= n; L <= 1) {
            int dx = MAXN / L, dl = L >> 1;
            for (int i = 0; i < n; i += L) {
```

```

    for (int j = i, x = 0; j < i + dl; ++j, x += dx) {
        ll tmp = a[j + dl] * w[x] % P;
        if ((a[j + dl] = a[j] - tmp) < 0) a[j + dl] += P;
        if ((a[j] += tmp) >= P) a[j] -= P;
    }
}
}
if (inv) {
    reverse(a.begin()+1, a.begin()+n);
    ll invn = minv(n);
    for (int i = 0; i < n; ++i) a[i] = a[i] * invn % P;
}
};

```

8.4 Polynomial Operation

```

// Copy from 8BQube
#define fi(s, n) for (int i = (int)(s); i < (int)(n); ++i)
template<int MAXN, ll P, ll RT> // MAXN = 2^k
struct Poly : vector<ll> { // coefficients in [0, P)
    using vector<ll>::vector;
    static NTT<MAXN, P, RT> ntt;
    int n() const { return (int)size(); } // n() >= 1
    Poly(const Poly &p, int m) : vector<ll>(m) {
        copy_n(p.data(), min(p.n(), m), data());
    }
    Poly& irev() { return reverse(data(), data() + n()), *this; }
    Poly& isz(int m) { return resize(m), *this; }
    Poly& iadd(const Poly &rhs) { // n() == rhs.n()
        fi(0, n()) if (((*this)[i] += rhs[i]) >= P) (*this)[i] -= P;
        return *this;
    }
    Poly& imul(ll k) {
        fi(0, n()) (*this)[i] = (*this)[i] * k % P;
        return *this;
    }
    Poly Mul(const Poly &rhs) const {
        int m = 1;
        while (m < n() + rhs.n() - 1) m <= 1;
        Poly X(*this, m), Y(rhs, m);
        ntt(X, m), ntt(Y, m);
        fi(0, m) X[i] = X[i] * Y[i] % P;
        ntt(X, m, true);
        return X.isz(n() + rhs.n() - 1);
    }
    Poly Inv() const { // (*this)[0] != 0, 1e5/95ms
        if (n() == 1) return {ntt.minv((*this)[0])};
        int m = 1;
        while (m < n() * 2) m <= 1;
        Poly Xi = Poly(*this, (n() + 1) / 2).Inv().isz(m);
        Poly Y(*this, m);
        ntt(Xi, m), ntt(Y, m);
        fi(0, m) {
            Xi[i] *= (2 - Xi[i] * Y[i]) % P;
            if ((Xi[i] %= P) < 0) Xi[i] += P;
        }
        ntt(Xi, m, true);
        return Xi.isz(n());
    }
    Poly& shift_inplace(const ll &c) { //to be tested
        int n = this->n();
        vector<ll> fc(n), ifc(n);
        fc[0] = ifc[0] = 1;
        for (int i = 1; i < n; ++i){
            fc[i] = fc[i-1] * i % P;
            ifc[i] = minv(fc[i]);
        }
        for (int i = 0; i < n; ++i) (*this)[i] = (*this)[i] *
            fc[i] % P;
        Poly g(n);
        ll cp = 1;
        for (int i = 0; i < n; ++i) g[i] = cp * ifc[i] % P, cp
            = cp * c % P;
    }
};

```

```

    *this = (*this).irev().Mul(g).isz(n).irev();
    for (int i = 0; i < n; i++) (*this)[i] = (*this)[i] *
        ifc[i] % P;
    return *this;
}
Poly shift(const ll &c) const { return Poly(*this).
    shift_inplace(c); }
Poly Sqrt() const { // Jacobi((*this)[0], P) = 1, 1e5/235
    ms
    if (n() == 1) return {QuadraticResidue((*this)[0], P)};
    Poly X = Poly(*this, (n() + 1) / 2).Sqrt().isz(n());
    return X.iadd(Mul(X.Inv()).isz(n())).imul(P / 2 + 1);
}
pair<Poly, Poly> DivMod(const Poly &rhs) const { // (rhs
    .)back() != 0
    if (n() < rhs.n()) return {{0}, *this};
    const int m = n() - rhs.n() + 1;
    Poly X(rhs); X.irev().isz(m);
    Poly Y(*this); Y.irev().isz(m);
    Poly Q = Y.Mul(X.Inv()).isz(m).irev();
    X = rhs.Mul(Q), Y = *this;
    fi(0, n()) if ((Y[i] -= X[i]) < 0) Y[i] += P;
    return {Q, Y.isz(max(1, rhs.n() - 1))};
}
Poly Dx() const {
    Poly ret(n() - 1);
    fi(0, ret.n()) ret[i] = (i + 1) * (*this)[i + 1] % P;
    return ret.isz(max(1, ret.n()));
}
Poly Sx() const {
    Poly ret(n() + 1);
    fi(0, n()) ret[i + 1] = ntt.minv(i + 1) * (*this)[i] %
        P;
    return ret;
}
Poly _tmul(int nn, const Poly &rhs) const {
    Poly Y = Mul(rhs).isz(n() + nn - 1);
    return Poly(Y.data() + n() - 1, Y.data() + Y.n());
}
vector<ll> _eval(const vector<ll> &x, const vector<Poly>
    &up) const {
    const int m = (int)x.size();
    if (!m) return {};
    vector<Poly> down(m * 2);
    // down[1] = DivMod(up[1]).second;
    // fi(2, m * 2) down[i] = down[i / 2].DivMod(up[i]).
        second;
    down[1] = Poly(up[1]).irev().isz(n()).Inv().irev().
        _tmul(m, *this);
    fi(2, m * 2) down[i] = up[i ^ 1]._tmul(up[i].n() - 1,
        down[i / 2]);
    vector<ll> y(m);
    fi(0, m) y[i] = down[m + i][0];
    return y;
}
static vector<Poly> _tree1(const vector<ll> &x) {
    const int m = (int)x.size();
    vector<Poly> up(m * 2);
    fi(0, m) up[m + i] = {{x[i] ? P - x[i] : 0}, 1};
    for (int i = m - 1; i > 0; --i) up[i] = up[i * 2].Mul(
        up[i * 2 + 1]);
    return up;
}
vector<ll> Eval(const vector<ll> &x) const { // 1e5, 1s
    auto up = _tree1(x); return _eval(x, up);
}
static Poly Interpolate(const vector<ll> &x, const vector
    <ll> &y) { // 1e5, 1.4s
    const int m = (int)x.size();
    vector<Poly> up = _tree1(x), down(m * 2);
    vector<ll> z = up[1].Dx()._eval(x, up);
    fi(0, m) z[i] = y[i] * ntt.minv(z[i]) % P;
    fi(0, m) down[m + i] = {z[i]};
    for (int i = m - 1; i > 0; --i) down[i] = down[i * 2].
        Mul(up[i * 2 + 1]).iadd(down[i * 2 + 1].Mul(up[i *
            2]));
    return down[1];
}

```

```

}
Poly Ln() const { // (*this)[0] == 1, 1e5/170ms
    return Dx().Mul(Inv()).Sx().isz(n());
}
Poly Exp() const { // (*this)[0] == 0, 1e5/360ms
    if (n() == 1) return {1};
    Poly X = Poly(*this, (n() + 1) / 2).Exp().isz(n());
    Poly Y = X.Ln(); Y[0] = P - 1;
    fi(0, n()) if ((Y[i] = (*this)[i] - Y[i]) < 0) Y[i] +=
        P;
    return X.Mul(Y).isz(n());
}
// M := P(P - 1). If k >= M, k := k % M + M.
Poly Pow(ll k) const {
    int nz = 0;
    while (nz < n() && !(*this)[nz]) ++nz;
    if (nz * min(k, (ll)n()) >= n()) return Poly(n());
    if (!k) return Poly(Poly {1}, n());
    Poly X(data() + nz, data() + nz + n() - nz * k);
    const ll c = ntt.mpow(X[0], k % (P - 1));
    return X.Ln().imul(k % P).Exp().imul(c).irev().isz(n())
        .irev();
}
static ll LinearRecursion(const vector<ll> &a, const
    vector<ll> &coef, ll n) { // a_n = \sum c_{-j} a_{(n-j)}
    const int k = (int)a.size();
    assert((int)coef.size() == k + 1);
    Poly C(k + 1), W(Poly {1}, k), M = {0, 1};
    fi(1, k + 1) C[k - i] = coef[i] ? P - coef[i] : 0;
    C[k] = 1;
    while (n) {
        if (n % 2) W = W.Mul(M).DivMod(C).second;
        n /= 2, M = M.Mul(M).DivMod(C).second;
    }
    ll ret = 0;
    fi(0, k) ret = (ret + W[i] * a[i]) % P;
    return ret;
}
};
#undef fi
using Poly_t = Poly<131072 * 2, 998244353, 3>;
template<> decltype(Poly_t::ntt) Poly_t::ntt = {};

```

8.5 Generating Function

8.5.1 Ordinary Generating Function

- $C(x) = A(rx)$: $c_n = r^n a_n$ 的一般生成函數。
- $C(x) = A(x) + B(x)$: $c_n = a_n + b_n$ 的一般生成函數。
- $C(x) = A(x)B(x)$: $c_n = \sum_{i=0}^n a_i b_{n-i}$ 的一般生成函數。
- $C(x) = A(x)^k$: $c_n = \sum_{i_1+i_2+\dots+i_k=n} a_{i_1} a_{i_2} \dots a_{i_k}$ 的一般生成函數。
- $C(x) = xA(x)'$: $c_n = na_n$ 的一般生成函數。
- $C(x) = \frac{A(x)}{1-x}$: $c_n = \sum_{i=0}^n a_i$ 的一般生成函數。
- $C(x) = A(1) + x \frac{A(1)-A(x)}{1-x}$: $c_n = \sum_{i=n}^{\infty} a_i$ 的一般生成函數。

常用展開式

- $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$
- $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$, $\binom{a}{n} = \frac{a(a-1)(a-2)\dots(a-n+1)}{n!}$.

常見生函

- 卡特蘭數: $f(x) = \frac{1-\sqrt{1-4x}}{2x}$

8.5.2 Exponential Generating Function

a_0, a_1, \dots 的指數生成函數:

$$\hat{A}(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} = a_0 + a_1 x + \frac{a_2}{2!} x^2 + \frac{a_3}{3!} x^3 + \dots$$

- $\hat{C}(x) = \hat{A}(x) + \hat{B}(x)$: $c_n = a_n + b_n$ 的指數生成函數
- $\hat{C}(x) = \hat{A}^{(k)}(x)$: $c_n = a_{n+k}$ 的指數生成函數
- $\hat{C}(x) = x\hat{A}(x)$: $c_n = na_n$ 的指數生成函數
- $\hat{C}(x) = \hat{A}(x)\hat{B}(x)$: $c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k}$ 的指數生成函數
- $\hat{C}(x) = \hat{A}(x)^k$: $\sum_{i_1+i_2+\dots+i_k=n} \binom{n}{i_1, i_2, \dots, i_k} a_{i_1} a_{i_2} \dots a_{i_k}$ 的指數生成函數
- $\hat{C}(x) = \exp(A(x))$: 假設 $A(x)$ 是一個分量 (component) 的生成函數, 那 $\hat{C}(x)$ 是將 n 個有編號的東西分成若干個分量的指數生成函數

9 String

9.1 KMP Algorithm

```

void kmp(string s){
    int siz = s.size();
    vector<int> f(siz, 0);
    f[0] = 0;
    for (int i = 1; i < siz; i++) {
        f[i] = f[i-1];
        bool zero = 0;
        while (s[f[i]] != s[i]) {
            if (f[i] == 0) {
                zero = 1;
                break;
            }
            f[i] = f[f[i]-1];
        }
        if (!zero) f[i]++;
    }
}

```

9.2 Manacher Algorithm

```

vector<int> manacher(string s) {
    int n = s.size();
    vector<int> v(n);
    int pnt = -1, len = 1;
    for (int i = 0; i < n; i++) {
        int cor = 2 * pnt - i;
        if (cor >= 0) v[i] = min(v[cor], cor - pnt + len);
        while (i+v[i] < n && i-v[i] >= 0 && s[i+v[i]] == s[i-v[i]]) v[i]++;
        if (i + v[i] >= pnt + len) pnt = i, len = v[i];
    }
    for (int i = 0; i < n; i++) v[i] = 2 * v[i] - 1;
    return v;
}

```

9.3 MCP

```

string mcp(string s) { //Duval algorithm for Lyndon
    factorization
    s += s;
    int n = s.size(), i = 0, ans = 0;
    while (i < n/2) {
        ans = i;
        int j = i+1, k=i;
        while (j < n && s[k] <= s[j]) {
            if (s[k] < s[j]) k = i;
            else k++;
            j++;
        }
        while (i <= k) i += j - k;
    }
    return s.substr(ans, n/2);
}

```

9.4 Suffix Array

```

struct SuffixArray { //tested
    vector<int> sa, lcp, rank; //lcp[i] is lcp of sa[i] and
                             sa[i-1]
    SuffixArray(string& s, int lim=256) { // or basic_string<
        int>
        int n = s.size() + 1, k = 0, a, b;
        vector<int> x(n, 0), y(n), ws(max(n, lim));
        rank.resize(n);
        for (int i = 0; i < n-1; i++) x[i] = (int)s[i];
        sa = lcp = y, iota(sa.begin(), sa.end(), 0);
        for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim =
            p) {
            p = j, iota(y.begin(), y.end(), n - j);
            for (int i = 0; i < n; i++) if (sa[i] >= j) y[p++] = sa
                [i] - j;
            for (int &i : ws) i = 0;
            for (int i = 0; i < n; i++) ws[x[i]]++;
            for (int i = 1; i < lim; i++) ws[i] += ws[i - 1];
            for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
            swap(x, y), p = 1, x[sa[0]] = 0;
            for (int i = 1; i < n; i++) a = sa[i - 1], b = sa[i], x[
                b] =
                (y[a] == y[b] && y[a + j] == y[b + j]) ? p - 1 : p
                    ++;
        }
        for (int i = 1; i < n; i++) rank[sa[i]] = i;
        for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
            for (k && k--, j = sa[rank[i] - 1];
                s[i + k] == s[j + k]; k++);
    }
};

```

9.5 Suffix Automaton

```

// from 8BQube
// at most 2n-1 states, 3n-4 edges

// to find longest common substring for multiple strings
// S_1, ..., S_k
// assign a special (distinct) character D_i to each string
// let T = S_1 D_1 ... S_k D_k, then build SAM of T
// answer is state with max length reachable to all D_i
const int maxn = 1000010;
struct SAM { //1 base
    vector<int> adj[maxn];
    int tot, root, lst, par[maxn], mx[maxn], fi[maxn], iter;
    //mx: maxlen of node, mx[par[i]]+1: minlen of node
    //fi: first endpos
    //corresponding substring of node can be found by fi and
        mx
    int nxt[maxn][33];
    int newNode() {
        int res = ++tot;
        fill(nxt[res], nxt[res] + 33, 0);
        par[res] = mx[res] = 0;
        fi[res] = iter;
        return res;
    }
    void init() {
        tot = 0;
        iter = 0;
        root = newNode();
        par[root] = 0, mx[root] = 0;
        lst = root;
    }
    void push(int c) {
        int p = lst;
        int np = newNode();
        mx[np] = mx[p] + 1;
        for (; p && nxt[p][c] == 0; p = par[p])
            nxt[p][c] = np;
        if (p == 0) par[np] = root;
        else {
            int q = nxt[p][c];

```

```

            if (mx[p] + 1 == mx[q]) par[np] = q;
            else {
                int nq = newNode();
                mx[nq] = mx[p] + 1;
                for (int i = 0; i < 33; i++)
                    nxt[nq][i] = nxt[q][i];
                par[nq] = par[q];
                fi[nq] = fi[q];
                par[q] = np;
                par[np] = nq;
                for (; p && nxt[p][c] == q; p = par[p])
                    nxt[p][c] = nq;
            }
        }
        lst = np;
    }

    void push(string str) {
        for (int i = 0; str[i]; i++) {
            iter++;
            push(str[i] - 'a' + 1);
        }
    }
    ll get_diff_strings(){
        ll tot = 0;
        for (int i = 1; i <= tot; i++) tot += mx[i] - mx[par[i
            ]];
        return tot;
    }
    bool in[maxn];
    int cnt[maxn]; //cnt is number of occurrences of node
    void count() {
        for (int i = 1; i <= tot; ++i)
            ++in[par[i]];
        queue<int> q;
        for (int i = 1; i <= tot; ++i)
            if (!in[i]) q.push(i);
        while (!q.empty()) {
            int u = q.front();
            q.pop();
            cnt[par[u]] += cnt[u];
            if (!--in[par[u]])
                q.push(par[u]);
        }
    }
} sam;

```

9.6 Z-value Algorithm

```

vector<int> z_function(string const& s) {
    int n = s.size();
    vector<int> z(n);
    for (int i = 1, l = 0, r = 0; i < n; i++) {
        if (i <= r) z[i] = min(r-i+1, z[i-l]);
        while (i + z[i] < n && s[z[i]] == s[i+z[i]])
            z[i]++;
        if (i + z[i] - 1 > r)
            l = i, r = i + z[i] - 1;
    }
    return z;
}

```

9.7 Main Lorentz

```

vector<int> z_function(string const& s);
int get_z(vector<int> const& z, int i) {
    return (0 <= i && i < SZ(z)) ? z[i] : 0;
}
vector<pair<int, int>> rep;

void convert_to_rep(int shift, bool left, int cntr, int l,
    int k1, int k2) {
    int lef = max(1, l-k2), rig = min(l, k1);
    int minl, maxl;
    if (left) {

```

```

    rig = min(rig, l-1);
    minl = shift + cntr - rig, maxl = shift+cntr-lef;
} else {
    minl = shift + cntr - 1 - rig + 1, maxl = shift + cntr
        - 1 - lef + 1;
}
//left endpoint: [minl, maxl], Length: 2*l
}

void find_rep(string s, int shift = 0) {
    int n = s.size();
    if (n == 1) return;

    int nu = n / 2;
    int nv = n - nu;
    string u = s.substr(0, nu);
    string v = s.substr(nu);
    string ru(u.rbegin(), u.rend());
    string rv(v.rbegin(), v.rend());

    find_rep(u, shift);
    find_rep(v, shift + nu);

    vector<int> z1 = z_function(ru);
    vector<int> z2 = z_function(v + '#' + u);
    vector<int> z3 = z_function(ru + '#' + rv);
    vector<int> z4 = z_function(v);

    for (int cntr = 0; cntr < n; cntr++) {
        int l, k1, k2;
        if (cntr < nu) {
            l = nu - cntr;
            k1 = get_z(z1, nu - cntr);
            k2 = get_z(z2, nv + 1 + cntr);
        } else {
            l = cntr - nu + 1;
            k1 = get_z(z3, nu + 1 + nv - 1 - (cntr - nu));
            k2 = get_z(z4, (cntr - nu) + 1);
        }
        if (k1 + k2 >= 1)
            convert_to_rep(shift, cntr < nu, cntr, l, k1, k2);
    }
}

```

9.8 AC Automaton

```

const int maxn = 300005, maxc = 26;
struct AC_Automaton { //1-base
    int nx[maxn][maxc], fl[maxn], cnt[maxn], pri[maxn], tot;
    //pri: bfs order of trie (0-base)
    int newnode() {
        tot++;
        fill(nx[tot], nx[tot] + maxc, -1);
        return tot;
    }
    void init() { tot = 0, newnode(); }
    int input(string &s) { // return the end_node of string
        int X = 1;
        for (char c : s) {
            if (!~nx[X][c - 'a']) nx[X][c - 'a'] = newnode();
            X = nx[X][c - 'a'];
        }
        return X;
    }
    void make_fl() { //fail link
        queue<int> q;
        q.push(1), fl[1] = 0;
        for (int t = 0; !q.empty(); ) {
            int R = q.front();
            q.pop(), pri[t++] = R;
            for (int i = 0; i < maxc; ++i)
                if (~nx[R][i]) {
                    int X = nx[R][i], Z = fl[R];
                    for (; Z && !~nx[Z][i];) Z = fl[Z];
                    fl[X] = Z ? nx[Z][i] : 1, q.push(X);
                }
        }
    }
}

```

```

    }
}
void get_v(string &s) {
    //number of times prefix appears in strings
    int X = 1;
    fill(cnt, cnt + tot+1, 0);
    for (char c : s) {
        while (X && !~nx[X][c - 'a']) X = fl[X];
        X = X ? nx[X][c - 'a'] : 1, ++cnt[X];
    }
    for (int i = tot-1; i > 0; --i)
        cnt[fl[pri[i]]] += cnt[pri[i]];
    }
} ac;

```

10 Formula

10.1 Recurrences

If $a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}$, and r_1, \dots, r_k are distinct roots of $x^k + c_1 x^{k-1} + \dots + c_k$, there are d_1, \dots, d_k s.t.

$$a_n = d_1 r_1^n + \dots + d_k r_k^n.$$

Non-distinct roots r become polynomial factors, e.g. $a_n = (d_1 n + d_2) r^n$.

10.2 Geometry

10.2.1 Rotation Matrix

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

- rotate 90° : $(x, y) \rightarrow (-y, x)$
- rotate -90° : $(x, y) \rightarrow (y, -x)$

10.2.2 Triangles

Side lengths: a, b, c

$$\text{Semiperimeter: } p = \frac{a+b+c}{2}$$

$$\text{Area: } A = \sqrt{p(p-a)(p-b)(p-c)}$$

$$\text{Circumradius: } R = \frac{abc}{4A}$$

$$\text{Inradius: } r = \frac{A}{p}$$

Length of median (divides triangle into two equal-area triangles): $m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$

$$\text{Length of bisector (divides angles in two): } s_a = \sqrt{bc \left(1 - \left(\frac{a}{b+c} \right)^2 \right)}$$

$$\text{Law of sines: } \frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R}$$

$$\text{Law of cosines: } a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$\text{Law of tangents: } \frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$$

Incenter:

$$P_1 = (x_1, y_1), P_2 = (x_2, y_2), P_3 = (x_3, y_3)$$

$$s_1 = P_2 P_3, s_2 = P_1 P_3, s_3 = P_1 P_2$$

$$s_1 P_1 + s_2 P_2 + s_3 P_3$$

$$\frac{s_1 + s_2 + s_3}{3}$$

Circumcenter:

$$P_0 = (0, 0), P_1 = (x_1, y_1), P_2 = (x_2, y_2)$$

$$x_c = \frac{1}{2} \times \frac{y_2(x_1^2 + y_1^2) - y_1(x_2^2 + y_2^2)}{-x_2 y_1 + x_1 y_2}$$

$$y_c = \frac{1}{2} \times \frac{x_2(x_1^2 + y_1^2) - x_1(x_2^2 + y_2^2)}{-x_1 y_2 + x_2 y_1}$$

Check if (x_0, y_0) is in the circumcircle:

$$\begin{vmatrix} x_1 - x_0 & y_1 - y_0 & (x_1^2 + y_1^2) - (x_0^2 + y_0^2) \\ x_2 - x_0 & y_2 - y_0 & (x_2^2 + y_2^2) - (x_0^2 + y_0^2) \\ x_3 - x_0 & y_3 - y_0 & (x_3^2 + y_3^2) - (x_0^2 + y_0^2) \end{vmatrix}$$

0: on edge, > 0: inside, < 0: outside

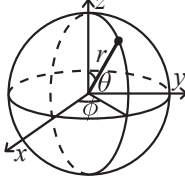
10.2.3 Quadrilaterals

With side lengths a, b, c, d , diagonals e, f , diagonals angle θ , area A and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180° , $ef = ac + bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

10.2.4 Spherical coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z/\sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \text{atan2}(y, x) \end{aligned}$$

10.2.5 Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_{L^+} (P dx + Q dy)$$

$$\text{Area} = \frac{1}{2} \oint_L x dy - y dx$$

Circular sector:

$$x = x_0 + r \cos \theta$$

$$y = y_0 + r \sin \theta$$

$$\begin{aligned} A &= r \int_{\alpha}^{\beta} (x_0 + \cos \theta) \cos \theta + (y_0 + \sin \theta) \sin \theta d\theta \\ &= r(r\theta + x_0 \sin \theta - y_0 \cos \theta)|_{\alpha}^{\beta} \end{aligned}$$

10.2.6 Point-Line Duality

$$p = (a, b) \leftrightarrow p^* : y = ax - b$$

- $p \in l \iff l^* \in p^*$
- p_1, p_2, p_3 are collinear $\iff p_1^*, p_2^*, p_3^*$ intersect at a point
- p lies above $l \iff l^*$ lies above p^*
- lower convex hull \leftrightarrow upper envelope

10.3 Trigonometry

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\sin n\pi = 0 \quad \cos n\pi = (-1)^n$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(2\alpha) = 2 \cos \alpha \sin \alpha$$

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$$

$$= 2 \cos^2 \alpha - 1$$

$$= 1 - 2 \sin^2 \alpha$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

$$\cos \alpha \sin \beta = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$$

$$\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$(V + W) \tan(\alpha - \beta)/2 = (V - W) \tan(\alpha + \beta)/2$$

where V, W are lengths of sides opposite angles α, β .

$$a \cos x + b \sin x = r \cos(x - \phi)$$

$$a \sin x + b \cos x = r \sin(x + \phi)$$

where $r = \sqrt{a^2 + b^2}$, $\phi = \text{atan2}(b, a)$.

10.4 Derivatives/Integrals

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan x = 1 + \tan^2 x$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$\int \tan ax = -\frac{\ln |\cos ax|}{a}$$

$$\int x \sin ax = \frac{\sin ax - ax \cos ax}{a^2}$$

$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2} \text{erf}(x)$$

$$\int x e^{ax} = \frac{e^{ax}}{a^2} (ax - 1)$$

$$\int \sin^2(x) = \frac{x}{2} - \frac{1}{4} \sin 2x$$

$$\int \sin^3 x = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x$$

$$\int \cos^2(x) = \frac{x}{2} + \frac{1}{4} \sin 2x$$

$$\int \cos^3 x = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x$$

$$\int x \sin x = \sin x - x \cos x$$

$$\int x \cos x = \cos x + x \sin x$$

$$\int x e^x = e^x (x - 1)$$

$$\int x^2 e^x = e^x (x^2 - 2x + 2)$$

$$\int x^2 \sin x = 2x \sin x - (x^2 - 2) \cos x$$

$$\int x^2 \cos x = 2x \cos x + (x^2 - 2) \sin x$$

$$\int e^x \sin x = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\int e^x \cos x = \frac{1}{2} e^x (\sin x + \cos x)$$

$$\int x e^x \sin x = \frac{1}{2} e^x (x \sin x - x \cos x + \cos x)$$

$$\int x e^x \cos x = \frac{1}{2} e^x (x \sin x + x \cos x - \sin x)$$

10.5 Sums

$$c^a + c^{a+1} + \dots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

10.6 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$