sampletest2-dm1.latex

November 4, 2015

Second sample test - Discrete mathematics 1.

1. How many ways can 10 prizes be awarded to 30 contestants, if
   (a) The prizes are different, and the contestant get at most one prize;
   (b) The prizes are different, and the contestants may receive more prizes;
   (c) The prizes are the same, and the contestants get at most one prize;
   (d) The prizes are the same, and the contestants may receive more prizes;
   (e) How many ways can 10 prizes be divided among 5 contestants, if every one gets at least one prize and the prizes are the same?
2. How many ways can 10 small balls be placed in 100 numbered boxes, if
   (a) the balls are also numbered;
   (b) the balls are numbered, and in every non-empty box there has to be two of them;
   (c) the balls are the same, and in every non-empty box there has to be two of them;
   (d) the balls are the same, and they can not be placed in neighboring boxes;
   (e) the balls are numbered (form 1 to 10), and they have to be placed so the parity of the balls and the boxes has to be the same?
3. How many 100 digit numbers are there in the binary number system, where each digit appears at least once? And how many in the decimal number system?
4. Solve the following congruences:
   (a) $3x \equiv 4 \pmod 7$;
   (b) $2x \equiv 5 \pmod{14}$;
   (c) $6x \equiv 4 \pmod{10}$;
   (d) $34x \equiv 2 \pmod{55}$;
   (e) $x^2 \equiv 3 \pmod 5$.
5. What are the last two digits of $1789^{1969^{2013}}$ (in the decimal number system)? Justify your answer.
6. (a) Solve the following (simultaneous) system of congruences:
$$x \equiv 2 \pmod 5$$
$$x \equiv 3 \pmod 6$$
$$x \equiv 1 \pmod 7$$
   (b) There is an island with seven and nine headed dragons. How many dragons live on the island if the sum of all heads is 107?

Solutions

1. (a) $\frac{30!}{20!} = 30 \cdot 29 \cdots 21$. The first prize goes to one of the 30 contestants, the second to one of the 29, etc. (Variations without repetition)
   (b) 30^{10}. All of the prizes can go to one of the 30 contestants (no restrictions). (Variation with repetition).
   (c) $\binom{30}{10}$. 10 winners are chosen from 30 contestants and are given one prize. (Combination without repetition).
   (d) $\binom{30+10-1}{10} = \binom{39}{10}$. If we select one "winner" multiple times it means he/she receives multiple gifts. (Combination with repetition)

(e) $\binom{5+5-1}{5}=\binom{9}{5}$. We give one prize to each, and we are left with 5 prizes to divide among 5 contestants, so that they can receive 0 or more prizes. So the remaining 5 prizes choose from 5 contestants with repetition. (Combination with repetition).

2. (a) 100^{10}. The first ball can be placed in one out of 100 boxes, and similarly with all the balls so $100 \cdot 100 \cdots 100 = 10010$. (Variations with repetition).

   (b) $\binom{100}{5}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}$. If every non-empty box has to have two balls, then there will be 5 boxes. First we choose 5 boxes out of 100 which $\binom{100}{5}$ ways, then, for each such choice we choose 2 balls from 10 to put in the first box $\binom{10}{2}$ ways, 2 from 8 for the second box, 2 from 6 for the third, 2 from 4 for the fourth, and the last two we put in the last selected box. OR $\binom{100}{5}\cdot\frac{10!}{(2!)^5}$, which is the same result, only now the second part is a permutation with repetition.

   (c) $\binom{100}{5}$. See the explanation of the previous problem.

   (d) $\binom{11+81-1}{81}=\binom{91}{81}$. For simplicity, assume each box can contain at most one ball, then the solution is $\binom{1}{2}$. We can deduce this solution by making an alternating sequence of non-empty and empty boxes which starts and ends with a non-empty box (i.e. NE,E,NE,E,..,E,NE). There are 10 non-empty boxes so in the sequence we have 9 empty boxes and all together $100-10-9=81$ unused boxes which we can put at the beginning or at the end of the sequence or beside any of the nine empty boxes (it doesn't matter on which side). So each of the not used 81 boxes has to choose one of the 11 places. (Combination with repetition).

   (e) $\binom{50}{5}\binom{50}{5}$ The problem implies that each box can contain at most one ball, and the "same parity" means if the box has an even number then the ball also has to have an even number or both the numbers have to be odd. So we have 50 odd boxes and 50 even boxes and 5 even balls and 5 odd balls.

3. Binary: $2^{99}-1$. All possibilities (must not start with zero) minus the one case with all one's.

   Decimal: Use the logical sieve: $\frac{9}{10}\left(\binom{10}{10}10^{100}-\binom{10}{9}9^{100}+\binom{10}{8}8^{100}-\cdots-\binom{10}{1}1^{100}+\binom{10}{0}0^{100}\right)$ The $\frac{9}{10}$ multiplier is because 1/9 of the numbers starts with zero. The rest of the terms come as follows: $\binom{10}{10}10^{100}$ we select all 10 digits (don't worry if not all are used), then subtract when only 9 digits were used $\binom{10}{9}9^{100}$ (10 choose 9 ways to select the digits, and \$$9^{100}$ to build the numbers for the selected digit), then add back $\binom{10}{8}8^{100}$ numbers with only 8 digits and so on.

4. (a) There is one incongruent solution because $\gcd(3,7)=1 \mid 4$ and that is $x \equiv 6 \pmod 7$ because:

$$\begin{aligned}3x &\equiv 4 \pmod 7\\ 3x &\equiv 4-7=-3 \pmod 7\\ x &\equiv -1 \equiv 6 \pmod 7\end{aligned}$$

   (b) $2x \equiv 5 \pmod{14}$ has no solutions because $\gcd(2,14)=2 \nmid 5$

   (c) There are 2 incongruent solutions because $\gcd(6,10)=2 \mid 4$, namely $x_0 \equiv 4 \pmod{10}$ and $x_1 \equiv 9 \pmod{10}$ because:

$$\begin{aligned}6x &\equiv 4 \pmod{10}\\ 3x &\equiv 2 \pmod 5 \text{ note that we divided the modulus also}\\ 3x &\equiv 2+2\cdot 5=12 \pmod 5\\ x &\equiv 4 \pmod 5 \text{ or } x=4+5t \text{ for } t\in\mathbb{Z}\end{aligned}$$

   (d) To solve this we can use the extended euclidean algorithm: $34x \equiv 2 \pmod{55}$ is the same as $-55y=34x-2$ i.e. $34x+55y=2$

$$\begin{array}{lll} & 1 & 0\\ & 0 & 1\\ 34=0\cdot 55+34 & 1 & 0\\ 55=1\cdot 34+21 & -1 & 1\\ 34=1\cdot 21+13 & 2 & -1\\ 21=1\cdot 13+8 & -3 & 2\\ 13=1\cdot 8+5 & 5 & -3\\ 8=1\cdot 5+3 & -8 & 5\\ 5=1\cdot 3+2 & 13 & -8\\ 3=1\cdot 2+1 & -21 & 13\\ 2=2\cdot 1+0 & & \end{array}$$

   Now we have $34\cdot(-21)+55\cdot 13=1$, multiplied by 2 we get \[34 · (-42) + 55 · 26 = 2\$. So $x=-42\equiv 13$ (mod ()55) is a good solution, let's check: $34\cdot 13=442\equiv 2 \pmod{55}$ is true!

   (e) We did not learn any special way to solve non-linear congruences, so we have to use brute force.

   - $0^2=0\not\equiv 3 \pmod 5$

- $1^2 = 1 \not\equiv 3 \pmod 5$
- $2^2 = 4 \not\equiv 3 \pmod 5$
- $3^2 = 4 \not\equiv 3 \pmod 5$
- $4^2 = 1 \not\equiv 3 \pmod 5$

So $x^2 \equiv 3 \pmod 5$ has no solutions.

5. The last two digits are 09. First we can reduce $1789 \bmod 100 = 89$, and we want to find $x \equiv 89^{1969^{2013}}$. We use the Euler-Fermat theorem, so we check that $\gcd(100, 89) = 1$ is true (it is true, so we can use the theorem). The theorem say $a^{\varphi(m)} \equiv 1 \pmod m$ where $a = 89$ and $m = 100$. First calculate $\varphi(100) = \varphi(2^2 \cdot 5^2) = 2^{2-1} \cdot (2-1) \cdot 5^{2-1} \cdot (5-1) = 2 \cdot 1 \cdot 5 \cdot 4 = 40$, so $89^{40} \equiv 1 \pmod{100}$. If we find $r = 1969^{2013} \bmod 40$ we are finished because then $1969^{2013} = q \cdot 40 + r$ and $89^{1969^{2013}} = 89^{q \cdot 40 + r} = (89^{40})^q \cdot 89^r \equiv 89^r \pmod{100}$.

   So now we are looking for the solution of $1969^{2013} \equiv r \pmod{40}$. First we reduce $1969 \bmod 40 = 9$. So again we use Euler-Fermat, $\varphi(40) = \varphi(2^3 \cdot 5) = 2^{3-1} \cdot (2-1) \cdot 5^{1-1} \cdot (5-1) = 4 \cdot 1 \cdot 1 \cdot 4 = 16$, so $9^{16} \equiv 1 \pmod{40}$. And now we calculate $r' = 2013 \bmod 16 = 13$.

   Substituting backwards $r \equiv 9^{13} \pmod{40}$. Luckily $9^3 = 729 \equiv 9 \pmod{40}$, so $r = 9^{13} = 9^{4 \cdot 3} \cdot 9 \equiv 9 \pmod{40}$. Again, substituting back, we have $x \equiv 89^9 \pmod{100}$ which we can calculate as follows:

$$
\begin{aligned}
89^2 &\equiv 21 \pmod{100} \\
89^4 &\equiv 21^2 \equiv 41 \pmod{100} \\
89^8 &\equiv 41^2 \equiv 81 \pmod{100} \\
89^9 &\equiv 81 \cdot 89 \equiv 9 \pmod{100}
\end{aligned}
$$

6. (a) The solution is $x \equiv 57 \pmod{210}$.

   - First solve

$$
\begin{aligned}
x &\equiv 2 \pmod 5 \\
x &\equiv 3 \pmod 6
\end{aligned}
$$

     and now $c_1 = 2, m_1 = 5, c_2 = 3, m_2 = 6$:
     - Solve: $m_1 x_1 + m_2 x_2 = 1$ for $x_1, x_2 = ?$. We use the extended Euclidean algorithm on $5x_1 + 6x_2 = 1$:

$$
\begin{array}{lrr}
 & 1 & 0 \\
 & 0 & 1 \\
5 = 0 \cdot 6 + 5 & 1 & 0 \\
6 = 1 \cdot 5 + 1 & -1 & 1 \\
5 = 5 \cdot 1 + 0 & &
\end{array}
$$

       So $x_1 = -1, x_2 = 1$ and using the formula $c_{1,2} = m_1 x_1 c_2 + m_2 x_2 c_1$ our solution is $c_{1,2} = 5 \cdot (-1) \cdot 3 + 6 \cdot 1 \cdot 2 = -15 + 12 = -3$ $x \equiv -3 \pmod{30}$
   - Now we substitute the first two congruences with our previous solution, so we have

$$
\begin{aligned}
x &\equiv -3 \pmod{30} \\
x &\equiv 1 \pmod 7
\end{aligned}
$$

     and now $c_{1,2} = -3, m_{1,2} = 30, c_3 = 1, m_3 = 7$:
     - Solve: $m_{1,2} x_{1,2} + m_3 x_3 = 1$ for $x_{1,2}, x_3 = ?$. We use the extended Euclidean algorithm on $30x_{1,2} + 7x_3 = 1$:

$$
\begin{array}{lrr}
 & 1 & 0 \\
 & 0 & 1 \\
30 = 4 \cdot 4 + 2 & 1 & -4 \\
7 = 3 \cdot 2 + 1 & -3 & 13 \\
2 = 2 \cdot 1 + 0 & &
\end{array}
$$

       So $x_{1,2} = -3, x_3 = 13$ and using the formula $c_{1,2,3} = m_{1,2} x_{1,2} c_3 + m_3 x_3 c_{1,2}$ our final solution is $c_{1,2,3} = 30 \cdot (-3) \cdot 1 + 7 \cdot 13 \cdot (-3) = -153$ ergo $x \equiv 57 \pmod{210}$.

   (b) There are 13 (5 seven headed and 8 nine headed) or 15 dragons (14 seven headed and 8 nine headed). Say x is the number of seven headed dragons and y is the number of nine headed dragons, then we have to solve the Diophantine equation $7x + 9y = 107$. Now $a = 7, b = 9, c = 107$ and we use the Euclidean algorithm:

$$
\begin{array}{lrr}
 & 1 & 0 \\
 & 0 & 1 \\
7 = 0 \cdot 9 + 7 & 1 & 0 \\
9 = 1 \cdot 7 + 2 & -1 & 1 \\
7 = 3 \cdot 2 + 1 & 4 & -3 \\
2 = 2 \cdot 1 + 0 & &
\end{array}
$$

From the last complete row we can see that

$$7 \cdot 4 + 9 \cdot -3 = 1 \tag{1}$$

which gives us $d = \gcd(a, b) = 1$. Because $d = 1 \mid 107 = c$ is true, there are solutions. To find them by multiply the equation (1) with $\frac{c}{d} = 107$ so we have

$$7 \cdot 428 + 9 \cdot (-321) = 107 \tag{2}$$

From this we see that one possible solution is $x_0 = 428$ and $y_0 = -321$ and from the formula we know all the solutions can be obtained as $x_t = x_0 + t \cdot \frac{b}{d}$ and $y_t = y_0 - t \cdot \frac{a}{d}$ for different values of $t \in \mathbb{Z}$ so we have

$$x_t = 428 + 9t \qquad\qquad y_t = -321 - 7t$$

Now since we can't have a negative number of dragons on the island, we have to find a t for which both x_t and y_t are non-negative. Because $x_t = 428 + 9t$ let's $\lfloor \frac{428}{9} \rfloor = 47$, ergo $x_{-47} = 428 - 47 \cdot 9 = 5$ is the smallest non-negative value for x_t the next non-negative value is $x_{-46} = x_{-47} + 9 = 14$ (note $x_{-48} = 428 - 48 \cdot 9 = -4$ is negative). Now for $y_{-47} = -321 - (-47) \cdot 7 = 8$ and $y_{-46} = -321 - (-46) \cdot 7 = 1$ (again note that $y_{-45} = -6$ is negative). Ergo the pairs $(x_{-47}, y =_{-47}) = (5, 8)$ and $(x_{-46}, y_{-46}) = (14, 1)$ are the two acceptable (non-negative) solutions.