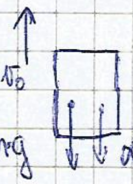
Rakéta 2018.04.06. 6. gyakorlat



$m>0$

mg ↓ ↓ $\alpha \cdot v^2$ - erőhatások. Meddig emelkedik a rakéta?

0 időpontbeli sebesség: $v(0) = v_0$

$a(t) = \frac{\Delta v}{\Delta t}$, $v(t)$

$a(t) = v'(t)$

$$F = ma = \underbrace{-mg - \alpha v^2}_{\text{visszahúzó erő}}$$

$$m v'(t) = -mg - \alpha v^2$$

$$v'(t) = -g - \frac{\alpha}{m} v^2$$

$$\frac{dv}{dt} = -g - \frac{\alpha}{m} v^2$$

$$\int -\frac{1}{g + \frac{\alpha}{m} v^2} dv = \int dt = t + C = -\frac{1}{g} \int \frac{1}{1 + \left(\sqrt{\frac{\alpha}{mg}} v\right)^2} dv = -\frac{1}{g} \cdot \frac{\operatorname{arctg} \sqrt{\frac{\alpha}{mg}} v}{\sqrt{\frac{\alpha}{mg}}} =$$

$$\boxed{\overbrace{-\frac{1}{g} \cdot \sqrt{\frac{mg}{\alpha}}}^{\sqrt{\frac{m}{\alpha g}}} \cdot \operatorname{arctg}\left(\sqrt{\frac{\alpha}{mg}} v\right) = t + C}$$

$$\operatorname{arctg}\left(\sqrt{\frac{\alpha}{mg}} v\right) = -\sqrt{\frac{\alpha g}{m}} (t + C)$$

$$\sqrt{\frac{\alpha}{mg}} v = \operatorname{tg}\left(-\sqrt{\frac{\alpha g}{m}} (t + C)\right)$$

$$v(t) = v = \sqrt{\frac{mg}{\alpha}} \operatorname{tg}\left(-\sqrt{\frac{\alpha g}{m}} (t + C)\right)$$

$C = ?$ $v(0) = v_0$

$t = 0$ $v(0) = v_0$

$$\Rightarrow C = -\sqrt{\frac{m}{\alpha g}} \operatorname{arctg}\left(\sqrt{\frac{\alpha}{mg}} v_0\right)$$

$$\rightarrow v(t) = \sqrt{\frac{mg}{\alpha}} \operatorname{tg}\left(\operatorname{arctg} \sqrt{\frac{\alpha}{mg}} v_0 - \sqrt{\frac{\alpha g}{m}} t\right)$$

Meddig emelkedik? Míg a sebesség nem nulla.

$v(t)=0 \iff tg(\ldots)=0 \iff -\sqrt{\frac{\alpha g}{m}}(t+c)=0 \iff t=-c$

$\iff \boxed{t=\sqrt{\frac{m}{\alpha g}}\operatorname{arctg}\left(\sqrt{\frac{\alpha}{mg}}\,v_0\right)}$

Szeparábilis d.e. $y'(x)=g(x)h(y) \iff \frac{dy}{dx}=g(x)h(y)$

$\int \frac{1}{h(y)}dy=\int g(x)dx$

Elsőrendű lineáris d.e

$y'(x)=g(x)y(x)+h(x)$

Homogén $\iff h(x)=0$

Inhomogén $\iff h(x)\neq 0$

Homogén e. mo-a: $y'(x)=g(x)y(x)$

$y'=g(x)y \rightarrow$ szep.d.e. $\frac{dy}{dx}=g(x)y$

$\ln y=\int\frac{1}{y}dy=\int g(x)dx$

$\rightarrow y_0(x)=e^{\int g(x)dx}$ $\boxed{y_0=e^{\int g}}$

Inhomogén egyenlet mo.-a (partikuláris m.o.)

$y_p'(x)=g(x)y_p(x)+h(x)$

T.f.h. y_p és $\tilde{y}_p$ mo-a az inhomogén d.e-nek

$(y_p-\tilde{y}_p)'=y_p'-\tilde{y}_p'=gy_p+h-(g\tilde{y}_p+h)=g(y_p-\tilde{y}_p)$

$y_p-\tilde{y}_p$ a homogén egyenlet mo.-a

$y_p-\tilde{y}_p=Cy_0 \Rightarrow$ Teljes mo.: $y(x)=C\underset{\substack{\uparrow\\ \text{homogén egyenlet mo-a}}}{y_0(x)}+\underset{\substack{\uparrow\\ \text{inhomogén rész egy partikuláris mo.-a}}}{y_p(x)}$

y_p-t hogyan határozzuk meg?

Állandók variálásának módszere:

$$\underline{y_p}(x) = C(x)\, \overset{\substack{\text{homogén} \\ \uparrow}}{y_0}(x)$$

$$y_p' = g\underline{y_p} + h = g\,c\,y_0 + h$$

$$y_p' = (c\,y_0)' = c'y_0 + c\,y_0' = c'y_0 + c\,g\,y_0 - c\,g\,y_0 + h$$

$$c'y_0 = h$$

$$c' = \frac{h}{y_0} \Rightarrow c = \int \frac{h}{y_0}$$

$$\boxed{y_p} = c\,y_0 = y_0 \int \frac{h}{y_0}$$

x fv-eben,
x szerint $\int$-va

1a) $y'(x) + \frac{2}{x}y(x) = x^3 \qquad y(1) = 1$

$$y' = gy + h$$

$$y' - gy = h$$

$$\boxed{\begin{array}{l} y_0 = e^{\int g} \\ y_p = y_0 \int \frac{h}{y_0} \\ \hline y = C\,y_0 + y_p \end{array}}$$

$$g(x) = \frac{2}{x}$$

$$\int g(x)\,dx = \int -\frac{2}{x}\,dx = -2\ln(x)$$

$$y_0(x) = e^{\int g(x)\,dx} = e^{-2\ln(x)} = e^{\ln\{x\}^{-2}} = x^{-2} = \frac{1}{x^2}$$

$$\frac{h(x)}{y_0(x)} = \frac{x^3}{1/x^2} = x^5$$

$$\int \frac{h(x)}{y_0(x)}\,dx = \int x^5\,dx = \frac{x^6}{6}$$

$$y_p(x) = y_0(x) \int \frac{h(x)}{y_0(x)}\,dx = \frac{1}{x^2} \cdot \frac{x^6}{6} = \frac{x^4}{6}$$

Teljes mo: $y(x) = c\,y_0(x) + y_p(x) = c \cdot \frac{1}{x^2} + \frac{x^4}{6}$

$y(1) = 1 \qquad c = ?$

$$c \cdot \frac{1}{1^2} + \frac{1^4}{6} = 1$$

$$c + \frac{1}{6} = 1 \iff c = \frac{5}{6}$$

$$\underline{y(x) = \frac{5 + x^6}{6x^2}} \qquad (x \in (0, +\infty))$$

Ell: deriválni

1b $\quad y'(x)\sin(x) - y(x)\cos(x) = -1 \qquad y\left(\frac{\pi}{2}\right) = 1$

$$y'(x) - \frac{\cos x}{\sin x} y(x) = -\frac{1}{\sin x} \qquad x \in (0, \pi) \Rightarrow \sin x \neq 0$$

$$y'(x) - \operatorname{ctg} x \, y(x) = -\frac{1}{\sin x}$$

$$y' - g\, y = h$$

$$g(x) = \operatorname{ctg}(x)$$

$$\int g(x)\,dx = \int \operatorname{ctg}(x)\,dx = \int \overset{f^{-1} \cdot g'}{\frac{\cos x}{\sin x}} = \ln(\sin x)$$

$$y_0(x) = e^{\int g(x)dx} = e^{\ln \sin x} = \sin x$$

$$\frac{h(x)}{y_0(x)} = -\frac{1}{\sin x} \cdot \frac{1}{\sin x} = -\frac{1}{\sin^2 x}$$

$$\int \frac{h(x)}{y_0(x)}\,dx = -\int \frac{1}{\sin^2 x}\,dx \overset{\text{TRÜKK}}{=} -\int \frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}}\,dx = -\int \overset{f^{-2} \cdot f'}{\frac{\frac{1}{\cos^2 x}}{\operatorname{tg}^2 x}}\,dx = -\frac{\operatorname{tg}^{-1}(x)}{-1} =$$

$$\frac{1}{\operatorname{tg}(x)} \qquad y_p(x) = y_0(x) \int \frac{h(x)}{y_0(x)} = \sin x \cdot \frac{1}{\underbrace{\operatorname{tg}(x)}_{\frac{\sin(x)}{\cos(x)}}} = \underline{\cos x}$$

$$y(x) = c\, y_0(x) + y_p(x) = \underline{c \cdot \sin x + \cos x}$$

$$y\left(\frac{\pi}{2}\right) = 1 \iff c \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} + \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} = 1 \iff c + 0 = 1 \iff \boxed{c = 1}$$

$$y(x) = \sin x + \cos x$$

1.c $y'(x) + \frac{2-3x^2}{x^3}\, y(x) = 1$

$$y'(x) - \frac{3x^2-2}{x^3}\, y(x) = 1$$

$$y' - gy = h$$

$$\int g(x)dx = \int \frac{3x^2-2}{x^3}dx = \int \frac{3}{x} - \frac{2}{x^3}\, dx = 3\ln(x) - 2\cdot\frac{x^{-2}}{-2} = 3\ln x + \frac{1}{x^2}$$

$$\Rightarrow y_0(x) = e^{\int g(x)dx} = e^{3\ln x + \frac{1}{x^2}} = e^{\ln x^3 + \frac{1}{x^2}} = \underline{x^3 e^{1/x^2}}$$

$$\frac{h(x)}{y_0(x)} = \frac{1}{x^3 e^{1/x^2}} = x^{-3}e^{-x^{-2}}$$

$$\int \frac{h(x)}{y_0(x)}dx = \int \underset{\cancel{(\int f' \cdot g)}}{e^{-x^{-2}}\cdot x^{-3}} = \frac{1}{2}\int 2x^{-3}e^{-x^{-2}}dx = \frac{1}{2}e^{-x^{-2}}$$

$$(e^{-x^{-2}})' = e^{-x^{-2}}(-1)(-2)x^{-3} = 2e^{-x^{-2}}x^{-3}$$

$$y_p(x) = y_0(x)\int\frac{h(x)}{y_0(x)} = x^3 e^{1/x^2}\cdot\frac{1}{2}e^{-x^{-2}} = \frac{x^3}{2}$$

$$y(x) = C\, y_0(x) + y_p(x) = \underline{C\cdot e^{x^{-2}}\cdot x^3 + \frac{x^3}{2}}$$

1.d,

$y'(x) + y(x)\,ctg(x) = 5e^{\cos x}$ $\qquad y(\frac{\pi}{2}) = -4$

$$y'(x) - (-ctg(x))\,y(x) = 5e^{\cos x}$$

$$y' - gy = h$$

$$\int g(x)dx = -\int ctg(x)\,dx = \cancel{-\frac{1}{\sin^2 x}} - \ln\sin x$$

$$y_0(x) = e^{\int g(x)dx} = \cancel{e^{-1/\sin^2 x}}\; e^{-\ln\sin x} = e^{\ln(\sin x)^{-1}} = (\sin x)^{-1} = \frac{1}{\sin x}$$

$$\int \frac{h(x)}{y_0(x)}dx = \int \frac{5e^{\cos x}}{1/\sin x}\, dx\ 5\int \sin x \cdot e^{\cos x} = -5\int(-\sin x)e^{\cos x}dx =$$

$$(e^{\cos x})' = e^{\cos x}(-\sin x)$$

$$= -5e^{\cos x}$$

$$y_p(x) = y_0(x)\int\frac{h(x)}{y_0(x)} = \frac{1}{\sin x}\cdot(-5e^{\cos x})$$

$$y(x) = c\,y_0(x) + y_p(x) = c\cdot\frac{1}{\sin x} - \frac{5e^{\cos x}}{\sin x} = \frac{c-5e^{\cos x}}{\sin x}$$

$$y\left(\frac{\pi}{2}\right) = -4 \Leftrightarrow \frac{c-5e^{\overbrace{\cos\left(\frac{\pi}{2}\right)}^{0}}}{\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}} = -4 \Leftrightarrow c-5 = -4$$

$$\boxed{c=1}$$

$$y(x) = \frac{1-5e^{\cos x}}{\sin x}$$

Ell: $y'(x) = ?$