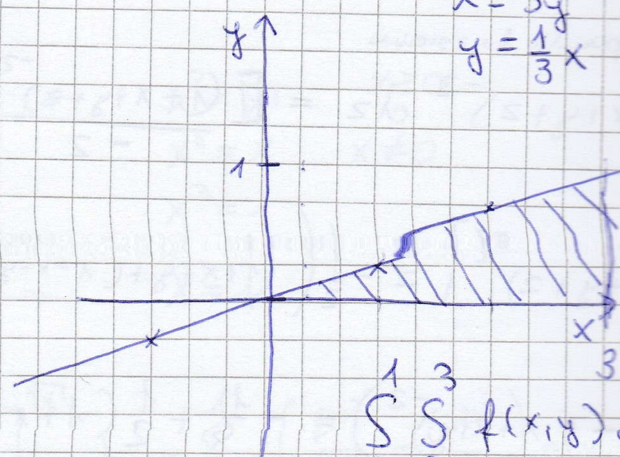
2. gyakorlat 2018.02.23.

$$\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy$$

$x = 3y$

$y = \frac{1}{3}x$



$$\int (f\circ g)\cdot g' = F\circ g \qquad F' = f$$

$$\int e^{x^2}\,dx = ?$$

$$\int e^{x^2}\,2x\,dx = e^{x^2} + c$$

$$\int_0^1\int_{3y}^3 f(x,y)\,dx\,dy =$$

$$\int_0^3\int_0^{x/3} f(x,y)\,dy\,dx$$

$$\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3\int_0^{x/3} e^{x^2}\,dy\,dx =$$

$$= \int_0^3 e^{x^2}\left(\int_0^{x/3} 1\,dy\right)dx = \int_0^3 \left[y\right]_0^{x/3} e^{x^2}\,dx = \int_0^3 e^{x^2}\cdot\left(x/3 - 0\right)dx =$$

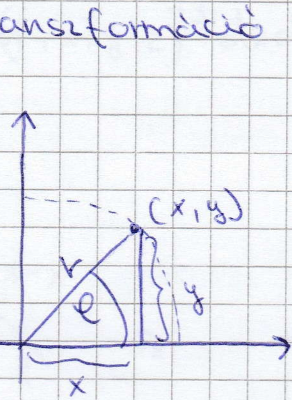
$$= \frac{1}{6}\int_0^3 2x\,e^{x^2}\,dx = \frac{1}{6}\left[e^{x^2}\right]_0^3 = \frac{1}{6}\left(e^{3^2} - e^{0^2}\right) = \frac{1}{6}e^9 - 1$$

$g \in \mathbb{R}^n \to \mathbb{R}^n \quad g \in C^1 \quad I \subset D_g$ kompakt $\quad g|_{\text{int } I}$ injektív

$f : g(I) \to \mathbb{R} \quad f \in C : \quad \int_{g(I)} f = \int_I f\circ g\,|\det g'|$ Jacobi-determináns

(1D-ban: $\int_{g(I)} f = \int_I (f\circ g)\cdot g'$)

Polártranszformáció



$r = \sqrt{x^2+y^2}$

$\operatorname{tg}\varphi = \frac{y}{x} \Rightarrow \varphi = \operatorname{arctg}(y/x)$

$\cos\varphi = \frac{x}{r} \Rightarrow x = r\cdot\cos\varphi \qquad r \in [0, +\infty)$

$\varphi \in [0, 2\pi)$

$\sin \cancel{\cos}\varphi = \frac{y}{r} \Rightarrow y = r\sin\varphi$

$$\iint_A f(x,y)\,dx\,dy = \iint_{g^{-1}(A)} (f\circ g)\,|\det g'| = \iint_{g^{-1}(A)} f(r,\varphi)\,|\det g'|\,dr\,d\varphi$$

$$g(r,\varphi)=\begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} \qquad r = r(x,y) = \sqrt{x^2+y^2}$$

$$\varphi = \varphi(x,y) = \operatorname{arctg}(y/x)$$

$$g(r,\varphi)=\begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} \qquad g'(r,\varphi) = \begin{pmatrix} \partial_1 g_1 & \partial_2 g_1 \\ \partial_1 g_2 & \partial_2 g_2 \end{pmatrix}\Bigg|_{(r,\varphi)} =$$

$$= \begin{pmatrix} \dfrac{\partial(r\cos\varphi)}{\partial r_{\text{ r szerint}}} & \dfrac{\partial(r\cos\varphi)}{\partial\varphi} \\[2ex] \dfrac{\partial(r\sin\varphi)}{\partial r} & \dfrac{\partial(r\sin\varphi)}{\partial\varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & r(-\sin\varphi) \\ \sin\varphi & r\cos\varphi \end{pmatrix} =$$

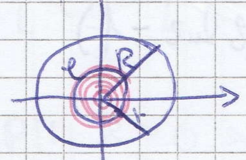
$$\det g' = \cos\varphi\cdot r\cos\varphi - r(-\sin\varphi)\sin\varphi =$$

$$r\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1} = r$$

$$\boxed{\iint_A f(x,y)\,dx\,dy = \iint_{g^{-1}(A)} f(r,\varphi)\cdot|r|\,dr\,d\varphi}$$

1. r sugarú kör területe

$$T_0 \overset{R \text{ sugarú kör területe}}{=} \iint_{x^2+y^2\le R^2} 1\,dx\,dy$$



$x^2+y^2=R^2$

körlap belseje

$x^2+y^2\le R^2$

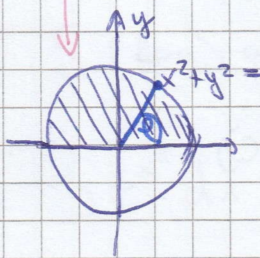
$$\iint_{x^2+y^2\le R^2} 1\,dx\,dy \underset{\text{p.t.}}{=} \int_0^R\int_0^{2\pi} 1\cdot \overset{|\det g'|}{r}\,d\varphi\,dr = \int_0^{2\pi}\int_0^R r\,dr\,d\varphi =$$

$$\int_0^{2\pi} d\varphi \int_0^R r\,dr = [\varphi]_0^{2\pi}\cdot\left[\frac{r^2}{2}\right]_0^R = 2\pi\cdot\frac{R^2}{2} = \pi\cdot R^2$$

i) $\iint\limits_{\substack{x^2+y^2\le 4\\ y\ge 0}} e^{x^2+y^2}\,dx\,dy$

$r\in[0,2]$

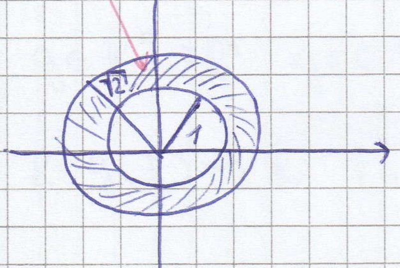
$\varphi\in[0,\pi]$



$$\iint\limits_{\substack{x^2+y^2\le 4\\ y\ge 0}} e^{x^2+y^2}\,dx\,dy = \underset{r^2=x^2+y^2}{\int_0^2\int_0^\pi e^{r^2}\cdot r\,d\varphi\,dr} = \int_0^\pi d\varphi\int_0^2 e^{r^2} r\,dr =$$

$$=\int_0^\pi d\varphi\cdot\frac{1}{2}\underbrace{\int_0^2 2re^{r^2}\,dr}_{\int(F\circ g)g'} = [\varphi]_0^\pi\cdot\frac{1}{2}\left[e^{r^2}\right]_0^2 = \pi\cdot\frac{1}{2}\cdot(e^4-1) = \frac{\pi}{2}(e^4-1)$$

ii) $\iint\limits_{1\le x^2+y^2\le 2}\ln(x^2+y^2)\,dx\,dy = \int_1^{\sqrt{2}}\int_0^{2\pi}\ln(r^2)\cdot r\,d\varphi\,dr = \int_0^{2\pi}d\varphi\int_1^{\sqrt{2}} r\ln(r^2)\,dr =$



$$=\int_0^{2\pi}d\varphi\,\frac{1}{2}\int_1^{\sqrt{2}} 2r\ln(r^2)\,dr$$

$\int(F\circ g)g' = F\circ g \quad F'=f$

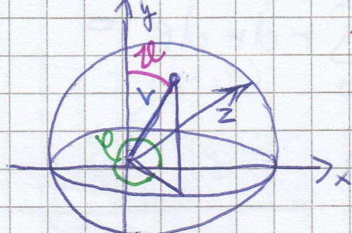
3.gy (anal 2)

$\int\ln x\,dx = x(\ln x-1)$

$$=[\varphi]_0^{2\pi}\cdot\frac{1}{2}\left[r^2(\ln r^2-1)\right]_1^{\sqrt{2}} =$$

$$=(2\pi-0)\cdot\frac{1}{2}\left(2(\ln 2-1)-1(\ln 1-1)\right) = \frac{2\pi}{2}(2\ln 2-2+1) = \pi(2\ln 2-1)$$

Gömbi (polár) koordináta rendszer (3D polár transzf.-ó)



ϑ: azimut

φ: polárszög

$x=\sin\vartheta\,(r\cos\varphi)$

$y=\sin\vartheta\,(r\sin\varphi)$

$z=r\cos\vartheta$

$\vartheta\in[0,\pi]$

$\varphi\in[0,2\pi]$

$r\in[0,+\infty)$

$$g(r,\vartheta,\varphi)=\begin{pmatrix} r\sin\vartheta\cos\varphi\\ r\sin\vartheta\sin\varphi\\ r\cos\vartheta\end{pmatrix}$$

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_{g^{-1}(A)} (f\circ g)(r,\vartheta,\varphi)\,|\det g'|\,dr\,d\vartheta\,d\varphi$$

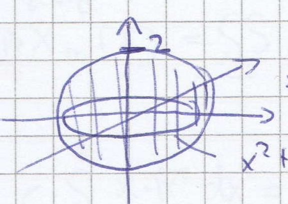
$$g' = \begin{pmatrix} \partial_1 g_1 & \partial_2 g_1 & \partial_3 g_1 \\ \partial_1 g_2 & \partial_2 g_2 & \partial_3 g_2 \\ \partial_1 g_3 & \partial_2 g_3 & \partial_3 g_3 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ \cos\vartheta & r(-\sin\vartheta) & 0 \end{pmatrix}$$

$\det g'$ = utolsó sor szerint kifejt: $\underset{\substack{\vee\!\!| \\ 0}}{r^2}\ \underset{\substack{\vee\!\!| \\ 0}}{\sin\vartheta}$

$|\det g'| = r^2 \sin\vartheta$

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_{g^{-1}(A)} f(r\sin\vartheta\cos\varphi, r\sin\vartheta\sin\varphi, r\cos\vartheta)\cdot r^2\sin\vartheta\,dr\,d\vartheta\,d\varphi$$

Pl: $\displaystyle\iiint_{x^2+y^2+z^2\le 4} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz =$



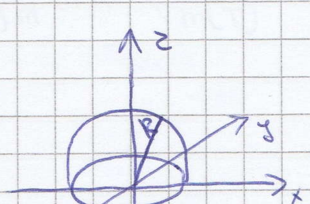
$r\in[0,2]$
$\vartheta\in[0,\pi]$
$\varphi\in[0,2\pi]$

$$= \int_0^2\int_0^\pi\int_0^{2\pi} r\cdot r^2\sin\vartheta\,d\varphi\,d\vartheta\,dr = \int_0^2\int_0^\pi\int_0^{2\pi} r^3\sin\vartheta\,d\varphi\,d\vartheta\,dr =$$

$$\int_0^{2\pi} d\varphi \int_0^2\int_0^\pi r^3\sin\vartheta\,d\vartheta\,dr = \int_0^{2\pi} d\varphi \int_0^\pi \sin\vartheta\,d\vartheta \int_0^2 r^3\,dr =$$

$$[\varphi]_0^{2\pi}\,[-\cos\vartheta]_0^\pi\,\left[\frac{r^4}{4}\right]_0^2 = 2\pi\,(-1)(\cancel{\ }(-1)-1)\cdot\frac{16}{4} = 2\pi\cdot 2\cdot\frac{16}{4} = \underline{\underline{16\pi}}$$

Pl: $R>0$ sugarú félgömb térfogata?



$x^2+y^2+z^2\le R^2$
$z\ge 0$

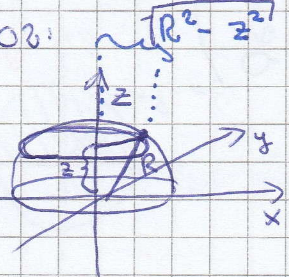
$r\in[0,R]$
$\varphi\in[0,2\pi]$
$\vartheta\in[0,\frac{\pi}{2}]$

$$\iiint_{\substack{x^2+y^2+z^2\le R^2\\ z\ge 0}} 1\,dx\,dy\,dz = \int_0^R\int_0^{\frac{\pi}{2}}\int_0^{2\pi} r^2\sin\vartheta\,d\varphi\,d\vartheta\,dr =$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin\vartheta\, d\vartheta \int_0^R r^2\, dr = [\varphi]_0^{2\pi}\cdot[-\cos\vartheta]_0^{\pi/2}\cdot\left[\frac{r^3}{3}\right]_0^R =$$

$$2\pi\cdot(-1)(0-1)\cdot\frac{R^3}{3} = \frac{2\pi R^3}{3}$$

mo2:



$$\iiint_{\substack{x^2+y^2+z^2\le R^2\\ z\ge 0}} 1\, dx\,dy\,dz = \int_0^R \underbrace{\iint_{x^2+y^2\le R^2-z^2} 1\, dx\,dy}_{R^2-z^2 \text{ sugarú kör területe}}\, dz =$$

$$= \int_0^R (R^2-z^2)\pi\, dz = \pi\left[R^2z-\frac{z^3}{3}\right]_0^R = \pi\left(R^2R-\frac{R^3}{3}\right) = \pi\left(R^3-\frac{R^3}{3}\right) =$$

$$\frac{\pi 2R^3}{3}$$

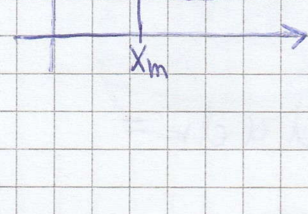
3. gyakorlat 2018.03.02

$\rho(x,y,z)$ tömeg eloszlás "anyag sűrűség fv."

$$\iiint_S 1\, dx\,dy\,dz = \text{térfogat}(S)$$

$$\iiint_S \rho(x,y,z)\, dx\,dy\,dz = \text{tömeg}(S)$$

$$\frac{1}{m}\iiint_S \begin{matrix}x\\y\\z\end{matrix}\, \rho(x,y,z)\, dx\,dy\,dz = \text{súlypont}_{\begin{matrix}x\\y\\z\end{matrix}}(S)$$ koordináta



$$\iint_T 1 = V(T)$$

$$\iint_T \rho(x,y)\, dx\,dy = m(T)$$

$$x_m(T) = \iint_T x\,\rho(x,y)\, dx\,dy$$

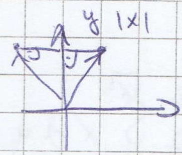
$$y_m(T) = \iint_T y\,\rho(x,y)\, dx\,dy$$

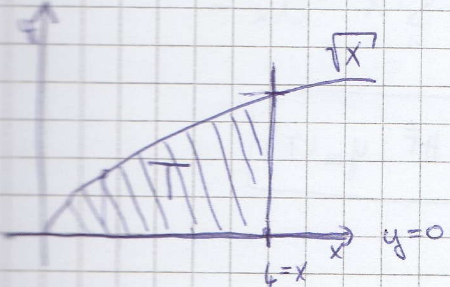
$y=\sqrt{x}$ $y=0$ $x=4$ (határos) síktartomány tömege t.k.p.

tömegközéppont

a, $\varrho(x,y) = c$

b, $\varrho(x,y) = c|x|$





a, $\underset{\text{tömeg}}{m(T)} = \iint\limits_T \varrho(x,y)\,dx\,dy =$

$$\int_0^4\int_0^{\sqrt{x}} \varrho(x,y)\,dy\,dx = \int_0^4\int_0^{\sqrt{x}} c\,dy\,dx =$$

$$c\int_0^4\int_0^{\sqrt{x}} dy\,dx = c\int_0^4 [y]_0^{\sqrt{x}}\,dx = c\cdot\int_0^4(\sqrt{x}-0)\,dx = c\cdot\int_0^4 x^{1/2}\,dx =$$

$$c\left[\frac{x^{3/2}}{3/2}\right]_0^4 = c\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 = c\cdot\frac{2}{3}\cdot 4^{3/2} = \frac{16c}{3}$$

t.k.p.

$$m\,x_m(T) = \iint\limits_T x\,\varrho(x,y)\,dx\,dy = \int_0^4\int_0^{\sqrt{x}} x\cdot c\cdot dy\,dx = c\cdot\int_0^4 x\underbrace{\int_0^{\sqrt{x}} dy}_{[y]_0^{\sqrt{x}}}\,dx =$$

$$c\cdot\int_0^4 x\cdot\sqrt{x}\,dx = c\cdot\int_0^4 x^{3/2}\,dx = c\cdot\int_0^4\frac{x^{5/2}}{5/2}\,dx = c\cdot\frac{2}{5}\left[x^{5/2}\right]_0^4 = c\cdot\frac{2}{5}\cdot 4^{5/2} = c\cdot\frac{64}{5}$$

$$\Rightarrow x_m(T) = \frac{1}{m(T)}\,c\cdot\frac{64}{5} = \frac{3}{16\cdot c}\cdot c\cdot\frac{64}{5} = \frac{4\cdot 3}{5} = \frac{12}{5}$$

$$m(T)\cdot y_m(T) = \iint\limits_T y\,\varrho(x,y)\,dx\,dy = \int_0^4\int_0^{\sqrt{x}} y\cdot c\,dy\,dx = c\cdot\frac{1}{2}\int_0^4 [y^2]_0^{\sqrt{x}}\,dx =$$

$$c\cdot\frac{1}{2}\int_0^4 x\,dx = c\cdot\frac{1}{4}\left[x^2\right]_0^4 = c\cdot\frac{1}{4}\cdot 16 = 4c$$

$$y_m(T) = \frac{1}{m(T)}\cdot 4c = \frac{3}{16c}\cdot 4c = \frac{3}{4}$$

b, $\rho(x,y) = c|x| \quad (x \geq 0)$

cx

$$m(T) = \iint_T \rho(x,y)\,dx\,dy = \int_0^4 \int_0^{\sqrt{x}} \rho(x,y)\,dy\,dx = c \cdot \int_0^4 \int_0^{\sqrt{x}} x\,dy\,dx \overset{\text{volt}}{=} c \cdot \frac{64}{5}$$

$$m(T) \cdot x_m(T) = \int_0^4 \int_0^{\sqrt{x}} x c x\,dy\,dx = c \cdot \int_0^4 x^2 \underbrace{\int_0^{\sqrt{x}} dy}_{\sqrt{x}}\,dx = c \int_0^4 x^{5/2}\,dx =$$

$\boxed{\text{HF: } y_m(T)}$

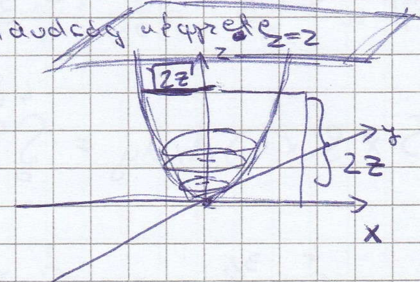
$$c \cdot \frac{2}{7} \left[x^{7/2}\right]_0^4 = c \cdot \frac{2}{7} \cdot 4^{7/2} = \frac{256c}{7}$$

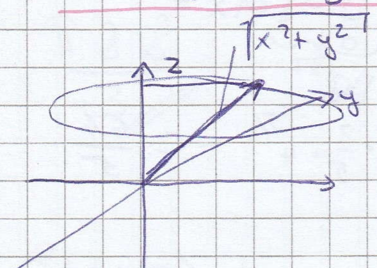
$$\Rightarrow x_m(T) = \frac{1}{m(T)} x_m(T) = \frac{256c}{7} \cdot \frac{5}{64c} = \frac{20}{7}$$

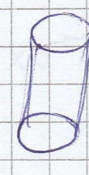
1. $x^2+y^2 = 2z$ és $z=2$ felületek által közbezárt paraboloid térfogata?

Tömeg, ha $\rho(x,y,z) = x^2+y^2$

z tengelytől mért távolság négyzete







$$\iiint_V 1\,dx\,dy\,dz = \int_0^2 \left(\iint_{x^2+y^2 \leq 2z} 1\,dx\,dy\right) dz = \int_0^2 \left(\int_0^{\sqrt{2z}} \int_0^{2\pi} 1 \cdot r\,d\varphi\,dr\right) dz =$$

$$\int_0^2 \int_0^{\sqrt{2z}} r\,dr \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\,dz = \int_0^2 2\pi \left[\frac{r^2}{2}\right]_0^{\sqrt{2z}} dz = 2\pi \int_0^2 \left(\frac{2z}{2} - 0\right) dz =$$

$$2\pi \left[\frac{z^2}{2}\right]_0^2 = 2\pi \frac{4}{2} = 4\pi$$

b, $m(V) = ?$ $\rho(x,y,z) = x^2+y^2$

$$m(V) = \iiint_V \rho(x,y,z)\,dx\,dy\,dz = \int_0^2 \left(\iint_{x^2+y^2 \leq 2z} x^2+y^2\,dx\,dy\right) dz =$$

$$= \int_0^2 \left(\int_0^{2\pi} \int_0^{\sqrt{2z}} r^2\, r\,dr\,d\varphi\right) dz = \int_0^2 \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \int_0^{\sqrt{2z}} r^3\,dr\,dz = 2\pi \int_0^2 \left[\frac{r^4}{4}\right]_0^{\sqrt{2z}} dz =$$

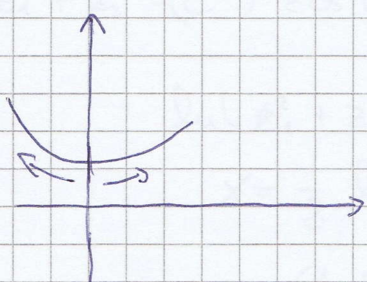
$$2\pi \int_0 \frac{z^2}{4}\, dz = 2\pi \left[\frac{z}{3}\right]_0 = 2\pi \frac{\cdot}{3} = \frac{\cdot}{3}$$

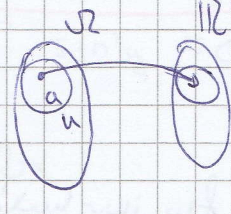
Inverz-fv. tétel:

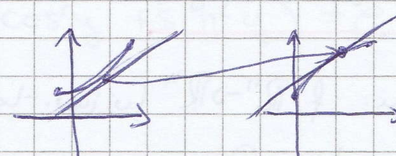
$n \in \mathbb{N}^+$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ nyílt, $f: \Omega \to \mathbb{R}^n$, $f \in C^1$.

T.f.h. $a \in \Omega$ $\det f'(a) \neq 0$, akkor $\exists\, u \in \Omega$ és $\exists\, V \subset \mathbb{R}^n$ nyílt

$f|_u : \underbrace{u \to v \text{ bijekció}}_{\text{invertálható } f}$ és $f^{-1}|_u \in C^1$ $(f^{-1})'(y) = \left(f'(f^{-1}(y))\right)^{-1}$







$f \in \mathbb{R}^n \to \mathbb{R}^n$ $\quad f'(x) \in \mathbb{R}^{n\times n}$

$f^{-1} \in \mathbb{R}^n \to \mathbb{R}^n$ $\quad (f^{-1})'(y) \in \mathbb{R}^{n\times n}$

8.

$$f(x,y) = \begin{pmatrix} \cos(y-x) \\ x e^{xy} \end{pmatrix}$$

Biz. be, h. f invertálható a $(0,\frac{\pi}{2})$ körül.

Számoljuk ki a $(f^{-1})'(f(0,\frac{\pi}{2})) = ?$

$$\partial_x f(x,y) = \begin{pmatrix} -\sin(y-x)\,(-1) \\ 1\cdot e^{xy} + x \cdot e^{xy} \cdot y \end{pmatrix}$$

$$\partial_y f(x,y) = \begin{pmatrix} -\sin(y-x)\cdot 1 \\ x \cdot e^{xy} \cdot x \end{pmatrix}$$

$$f'(x,y) = \begin{pmatrix} \sin(y-x) & -\sin(y-x) \\ e^{xy} + xy e^{xy} & x^2 e^{xy} \end{pmatrix}$$

$$f'(0,\tfrac{\pi}{2}) = \begin{pmatrix} \sin(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ e^0 + 0 & 0^2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$\det f'(0,\frac{\pi}{2}) = 1\cdot 0 - (-1)\cdot 1 = 1 \neq 0 \xRightarrow{\text{inverz fv tétel}}$ f invertálható a $(0,\frac{\pi}{2})$ környezetében

$$(f^{-1})'(\underbrace{f(0,\tfrac{\pi}{2})}_{y \text{ tételben}}) = f'(f^{-1}(f(\overset{y}{0,\tfrac{\pi}{2}})))^{-1} = (f'(0,\pi/2))^{-1} =$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \frac{1}{\underset{\det}{1}} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det a} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

3 $f(x,y) = \begin{pmatrix} e^x \cdot \cos y \\ e^x \cdot \sin y \end{pmatrix}$ Inv.-e globálisan ill. lokálisan?

globálisan: $f: \mathbb{R}^n \to \mathbb{R}^n$ fv. inv.-ható

lokálisan: $\forall a \in D_f \ \exists u \in \mathbb{R}^n$ nyílt, $f|_u$ inv.-ható

globálisan nem: $f(0,0) = \begin{pmatrix} e^0 \cos 0 \\ e^0 \sin 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f(0,2\pi) = \begin{pmatrix} e^0 \cos 2\pi \\ e^0 \sin 2\pi \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow \exists x,y \in D_f \quad f(x) = f(y)$ de $x \neq y \Rightarrow f$ nem inv. globálisan

$$\partial_x f(x,y) = \begin{pmatrix} e^x \cdot \cos y \\ e^x \cdot \sin y \end{pmatrix} \qquad \partial_y f(x,y) = \begin{pmatrix} e^x(-\sin y) \\ e^x(\cos y) \end{pmatrix}$$

$$f'(x,y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^x \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix}$$

$$\det f'(x,y) = e^x \det \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix} = e^x(\cos^2 y + \sin^2 y) = e^x > 0$$

~~impl~~ inv. fv. tétel

$\Rightarrow$ bármely $x,y \in \mathbb{R}$ $\det f'(x,y) \neq 0 \Rightarrow$

$\forall (x,y) \in \mathbb{R}^2$ környezetében f inv.-ható $\Rightarrow$

f lokálisan invertálható

Inv.
Impl. f.v. T. $\Rightarrow (f^{-1})'(f(x,y)) = (f'(x,y))^{-1}$

De! Kiszámítható explicit az inverz! (A)

$$f(x,y) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix} \qquad \frac{z_2}{z_1} = \frac{e^x \sin y}{e^x \cos y} = \frac{\sin y}{\cos y} = \operatorname{tg} y$$

$$\operatorname{arctg} \frac{z_2}{z_1} = y$$

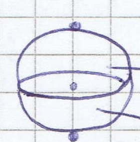
$$z_1^2 + z_2^2 = (e^x)^2 \cos^2 y + (e^x)^2 \cdot \sin^2 y = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}$$

$$\ln(z_1^2 + z_2^2) = 2x$$

$$x = \frac{1}{2} \ln(z_1^2 + z_2^2)$$

$$f^{-1}(z_1, z_2) = \begin{pmatrix} \frac{1}{2} \ln(z_1^2 + z_2^2) \\ \operatorname{arctg}\left(\frac{z_2}{z_1}\right) \end{pmatrix} \quad (z_1 \neq 0)$$

4. gyakorlat 2018.03.09.



$x^2 + y^2 + z^2 = 1$

$$F(x,y,z) = \underbrace{x^2 + y^2 + z^2 - 1 = 0}_{\text{impl. egyenlet}} \qquad \overset{\text{expl (kifejezett) fv.}}{z = \underbrace{f(x,y)}_{\text{implicit fv.}}}$$

Impl. fv. t: $\Omega \subset \mathbb{R}^2$ nyílt $f: \Omega \to \mathbb{R}$ $f \in C^1$ $(a,b) \in \Omega$

$f(a,b) = 0$ $\partial_2 f(a,b) \neq 0 \Rightarrow \exists \varphi: K(a) \to K(b)$; $f(x, \varphi(x)) = 0$

$(x \in K(a))$, $\underset{\substack{\Downarrow \\ \text{implicit fv.}}}{\varphi \in C^1}$ $\varphi'(x) = -\dfrac{\partial_1 f(x, \varphi(x))}{\partial_2 f(x, \varphi(x))}$ $(x \in K(a))$

1) $x^2 + 2xy - y^2 = 7$ y kifejezhető x implicit fv-e lenne a $(\boxed{2}, 1)$ pont körül $\varphi'(\boxed{2}) = ?$ (A)

$f(x,y) = x^2 + 2xy - y^2 - 7 = 0$
$f(2,1) = 2^2 + 2 \cdot 2 \cdot 1 - 1^2 - 7 = 0$
$0 = 0$ ✓

$\partial_y(x^2+2xy-y^2-7) = 0\cdot x^2+2x-2y = 2x-2y$

$\partial_y f(2,1) = 2\cdot 2 - 2\cdot 1 = 2 \neq 0$

$\Longrightarrow \quad \exists\, \varphi: K(2)\to K(1),\ \varphi\in C^1 \quad f(x,\varphi(x))=0 \quad x\in K(2)$

$$\varphi'(2) = -\frac{\partial_x f(2,\varphi(2))}{\partial_y f(2,\varphi(2))} = -\frac{6}{2} = -3$$

$\partial_x f(x,y) = 2x + 2y$

$\partial_x f(2,1) = 2\cdot 2 + 2\cdot 1 = 6$

2. Biz be, hogy $\exists\, y:\mathbb{R}\to\mathbb{R}$ $\quad \ln(x) + y(x)e^{y^2(x)} = 1 \quad (x>0) \quad y'(e)=?$

$f(x,y) = \ln x + ye^{y^2} - 1 = 0$ Találni egy pontot, ami kielégíti az imp. egyenletet.

$f(e,y) = \ln e + ye^{y^2} - 1 = 1 + ye^{y^2} - 1 = ye^{y^2} = 0 \iff y = 0$

$\Longrightarrow f(e,0) = 0$

$\partial_y f(x,y) = 1\cdot e^{y^2} + y\cdot e^{y^2}\cdot 2y = (1+2y^2)e^{y^2}$

$\partial_y f(e,0) = (1+2\cdot 0^2)e^{0^2} = 1 \neq 0 \Longrightarrow$

$\exists\, \varphi: K(e)\to K(0) \quad \varphi\in C^1 \quad \varphi'(e) = -\dfrac{\partial_1 f(e,0)}{\partial_2 f(e,0)}$

$\partial_x f(x,y) = \dfrac{1}{x}$

$\partial_x f(e,0) = \dfrac{1}{e}$

$\varphi'(e) = -\dfrac{1}{e}\cdot\dfrac{1}{1} = -\dfrac{1}{e}$

Tétel: impl. fv.

$\Omega_1 \subset \mathbb{R}^n \quad \Omega_2 \subset \mathbb{R}^m \quad f: \Omega_1 \times \Omega_2 \to \mathbb{R}^m$

$f \in C^1 \quad (a,b) \in \Omega_1 \times \Omega_2 \quad \det \underbrace{\partial_2 f(a,b)}_{\in \mathbb{R}^{m\times m}} \neq 0$ Jacobi: Ω_2-beli változó szerint

$\Longrightarrow \exists K(a) \subset \Omega_1 \quad \exists K(b) \subset \Omega_2 \quad \varphi: K(a) \to K(b)$

$f(x, \varphi(x)) = 0 \quad (\forall x \in K(a))$

$$\underbrace{\varphi'(x)}_{\mathbb{R}^{m\times n}} = -\underbrace{(\partial_2 f(x,\varphi(x)))^{-1}}_{\mathbb{R}^{m\times m}} \underbrace{(\partial_1 f(x,\varphi(x)))}_{\mathbb{R}^{m\times n}}$$

3 $\left.\begin{aligned} x_1^2 - x_2 x_3 &= 0 \\ 3x_1^3 - x_2 - 2x_3 &= 0 \end{aligned}\right\}$ x_2 és x_3 kifejezhető-e az x_1 impl. fv-eként az 1 körül? $\varphi'(1) = ?$

$$f(\underbrace{x_1}_{\Omega_1\text{ beli}}; \underbrace{x_2, x_3}_{\Omega_2\text{ beli}}) = \begin{pmatrix} x_1^2 - x_2 x_3 \\ 3x_1^3 - x_2 - 2x_3 \end{pmatrix} \qquad f: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$$

$$f(1; x_2, x_3) = \begin{pmatrix} 1 - x_2 x_3 \\ 3 - x_2 - 2x_3 \end{pmatrix} \overset{!}{=} 0 \Longrightarrow \begin{aligned} x_2 &= 1 \\ x_3 &= 1 \end{aligned} \quad ✓$$

1, $f(1; 1, 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2,

$$\partial_2 f(x_1; (x_2, x_3)) = \begin{pmatrix} \partial_2 f_1 & \partial_3 f_1 \\ \partial_2 f_2 & \partial_3 f_2 \end{pmatrix} = \begin{pmatrix} -x_3 & -x_2 \\ -1 & -2 \end{pmatrix}$$

$$\partial_2 f(1; 1, 1) = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$$

$$\det \partial_2 f(1; 1, 1) = \underbrace{(-1)(-2)}_{2} \underbrace{-}_{-} \underbrace{((-1)(-1))}_{1} = 1 \neq 0$$

$\Longrightarrow$

$\Rightarrow \exists\, \varphi: K(1) \to K(1,1) \qquad f(x_1, \varphi_1(x_1)\ \varphi_2(x_1)) = 0\ (\forall x_1 \in K(1))$

$\varphi'(1) = -(\partial_2 f(1;1,1))^{-1} \partial_1 f(1;1,1)$

$\partial_1 f(x_1; x_2, x_3) = \begin{pmatrix} \partial_1 f_1 \\ \partial_1 f_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 9x_1^2 \end{pmatrix} \Rightarrow \partial_1 f(1;1,1) = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$

$(\partial_2 f(1;1,1))^{-1} = \frac{1}{\underbrace{1}_{det}} \cdot \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$

$\varphi'(1) = -\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = -\begin{pmatrix} -4+9 \\ 2-9 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$

④ $\left.\begin{aligned} xe^{u+v} + 2uv &= 1 \\ ye^{u-v} - \frac{u}{1+v} &= 2x \end{aligned}\right\}$ $\exists\,(x,y) \to (u,v)$ imp fv? Mi a Jakobi az $(1,2)$-ben?

$f(x,y;u,v) = \begin{pmatrix} xe^{u+v} + 2uv - 1 \\ ye^{u-v} - \frac{u}{1+v} - 2x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$f(1,2;u,v) = \begin{pmatrix} e^{\overset{u=0}{\overset{v=0}{u+v}}} + 2uv - 1 \\ 2e^{u-v} - \frac{u}{1+v} - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

legyen $u=0$
$v=0$

1, $f(1,2;0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2, $\partial_2 f_{\text{vagy } x,y}(1,2;u,v) = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{pmatrix} 1e^{u+v} + 2v & 1\cdot e^{u+v} + 2u \\ 2e^{u-v} - \frac{1}{1+v} & -2e^{u-v} + \frac{u}{(1+v)^2} \end{pmatrix}$

$\partial_2 f(1,2;0,0) = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} =: A$

$\det \partial_2 f(1,2;0,0) = 1(-2) - 1\cdot 1 = -3 \neq 0 \Rightarrow \exists$ impl. fv.

$$\partial_1 f(x,y;u,v) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^{u+v} & 0 \\ -2x & e^{u-v} \end{pmatrix}$$

$$\partial_1 f(1,2;0,0) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} =: B$$

$\cancel{A^{-1}} \neq$ Impl. fv. tétel $\exists\, \varphi: K((1,2)) \to K((0,0))$

$f(x,y;\varphi_1(x,y),\varphi_2(x,y)) = 0 \quad \varphi \in C^1$

$$\varphi'(1,2) = -\underbrace{\partial_2 f(1,2;0,0)^{-1}}_{A^{-1}} \cdot \underbrace{\partial_1 f(1,2;0,0)}_{B}$$

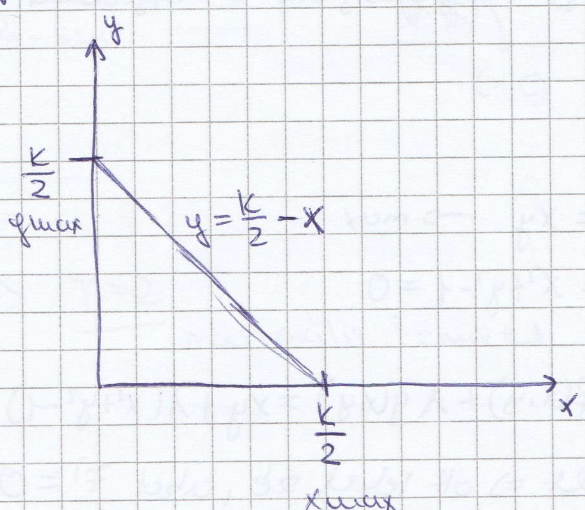
$$A^{-1} \cdot B = -\frac{1}{-3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & -1 \\ -3 & 1 \end{pmatrix}$$

Feltételes szélsőérték

1. Adott kerületű téglalapot megforgatunk → henger

Hogy válasszuk a téglalap lapjait, hogy a henger térfogata maximális legyen?

$$\underbrace{\left(\tfrac{y}{2}\right)^2 \cdot \pi}_{\text{körkerülete}} \cdot \overset{\text{magasság}}{x} = V = \text{Henger térfogata}$$



$K = 2(x+y)$
↑
adott

$\Rightarrow \quad \frac{K}{2} = x+y$

$x = \frac{K}{2} - y$

$y = \frac{K}{2} - x$

$f(x,y) = \frac{x y^2 \pi}{4}$ legyen max

Keressük x,y-t amelyre az $\underbrace{y = \frac{K}{2} - x \text{ és } x \in [0, \frac{K}{2}]}_{\text{kompakt}}$

$f(x,y)\ \frac{x y^2 \pi}{4}$ fv. folytonos

↳ visszavezethető 1 vál. fv.-re

$\Rightarrow \exists$ absz max a $K = 2(x+y)$ feltétele vonatkozóan

$$f(x) = \frac{x\left(\frac{k}{2}-x\right)^2 \pi}{4} \qquad x \in \left[0, \frac{k}{2}\right]$$

$\Downarrow$ lehetséges sz.é. helyek: $x=0$, $x=\frac{k}{2}$ + ahol $f'(x)=0$ HF

Feltételes sz.é: nem mindig vezethető vissza alacsonyabb dimenzióba ($y=f(x)$ nem mindig tehető meg)

~~Szükséges~~ Probléma: $n,m \in \mathbb{N}^+$ $m \leq n$ $\emptyset = U \in \mathbb{R}^n$ nyílt $f: U \to \mathbb{R}$

$g = (g_1, \dots, g_m): U \to \mathbb{R}^m$: $H = \{x \in U \mid \underbrace{g(x)=0}_{\text{feltétel}}\}$

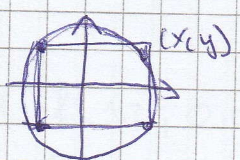
Keressük $f|_H$ szélsőértékeit.

Szükséges feltétel: $f, g_i \in C^1$ $(i=1,\dots,m)$ f-nek c-ben lokális szélsőérték helye van $\{g=0\}$ feltételre nézve ($f|_H$ –"–) és $g_i'(c)$ lin. ftlenek

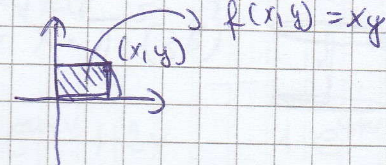
$\Rightarrow \exists \lambda_1, \dots, \lambda_m \in \mathbb{R}$ $\qquad F(x) = \underbrace{f(x)}_{\text{subject}} + \sum_{i=1}^{m} \lambda_i \underbrace{g_i(x)}_{\text{constraint}}$

Lagrange-multiplikátor

$x^2+y^2=1$ körbe írt téglalapok közül melyik a max területű?



Elég vizsgálni a negyedkört.



Eredeti terület $T = 2x \cdot 2y = 4xy$

$f(x,y) = xy \to \max$

$g(x,y) = x^2+y^2-1 = 0$

$F(x,y) = f(x,y) + \lambda g(x,y) = xy + \lambda(x^2+y^2-1)$

Szüks. felt $\Rightarrow$ ott lehet sz.é, ahol $F'=0$

$F'(x,y) = (\partial_1 F(x,y) \;\; \partial_2 F(x,y)) = (0,0)$

$\partial_1 F(x,y) = y + \lambda 2x = 0 \rightarrow y = -\lambda 2x$

$\partial_2 F(x,y) = x + \lambda 2y = 0 \rightarrow y^2 = 4\lambda^2 x^2$

$x^2 + y^2 - 1 = 0 \rightarrow \cancel{x^2 + 4\lambda^2 x^2 - 1 = 0}$

$\rightarrow x + \lambda \cdot 2 \underbrace{(-\lambda 2x)}_{y} = 0$

$x + (-4\lambda^2)x = 0$

$(1 - 4\lambda^2)x = 0 \quad \forall x$ -re

$\downarrow$

$1 - 4\lambda^2 = 0$

$1 = 4\lambda^2$

$\frac{1}{4} = \lambda^2$

$\lambda = \pm \frac{1}{2}$

$x^2 + 4\overbrace{\lambda^2 x^2}^{y^2} - 1 = 0$

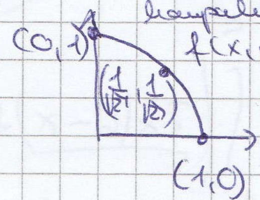
$x^2 + 4 \cdot \frac{1}{4} x^2 - 1 = 0$

$2x^2 - 1 = 0$

$x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}} \Rightarrow y = -\overbrace{\left(\pm \frac{1}{2}\right)}^{\lambda} \cdot 2 \overbrace{\left(\pm \frac{1}{\sqrt{2}}\right)}^{x} = \pm \frac{1}{\sqrt{2}}$

Mivel negyedkörön vagyunk $x \geq 0$ és $y \geq 0$

$\Rightarrow$ csak $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$-ben lehet lok. sz. é.



kompakt $f(x,y)$-nak $\exists$ absz.
- max
- lok max
- végpontok

$f(0,1) = 0$

$f(1,0) = 0$

$f\left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right) = \frac{1}{2} \rightarrow$ absz. max. $\Rightarrow$ $\underline{T = 2}$ maximális terület

5. gyakorlat 2018. 03. 23

hétfő: 14-16 konzultáció

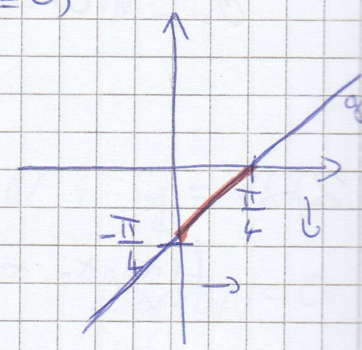
5/3 $f(x,y)=\cos^2 x+\cos^2 y$ $\qquad g(x,y)=x-y-\frac{\pi}{4}=0$

$(x\geq 0,\ y\leq 0)$

$F(x,y)=f(x,y)+\lambda g(x,y)=$

$\cos^2 x+\cos^2 y+\lambda\left(x-y-\frac{\pi}{4}\right)$

$y=x-\frac{\pi}{4}$



$$\left.\begin{aligned}
\partial_x F(x,y)&=2\cos x(-\sin x)+\lambda=-\sin 2x+\lambda=0\\
\partial_y F(x,y)&=2\cos y\cdot(-\sin y)+(-\lambda)=-\sin 2y-\lambda=0\\
x-y-\frac{\pi}{4}&=0
\end{aligned}\right\}$$

$\boxed{\lambda=?}$

$-\sin 2x+\lambda-\sin 2y-\lambda=0$ $\qquad x-y-\frac{\pi}{4}=0\Rightarrow y=x-\frac{\pi}{4}$

$-\sin 2x=\sin 2y$

$\sin 2x=-\sin 2y$ → páros fv

$\sin 2x=\sin(-2y)$

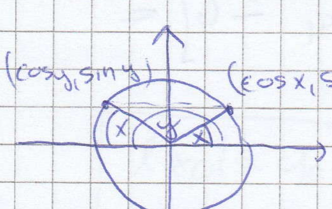
$\sin 2x=\sin\left(-2\left(x-\frac{\pi}{4}\right)\right)$

$\sin 2x=\sin\left(\frac{\pi}{2}-2x\right)$

$2x=\frac{\pi}{2}-2x+k2\pi$ $\qquad 2x=\pi-\left(\frac{\pi}{2}-2x\right)+k2\pi$

$2x=\frac{\pi}{2}+2x+k2\pi$

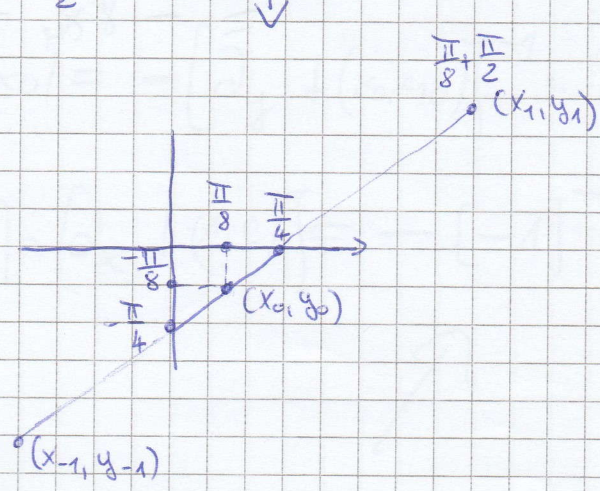
$0=\frac{\pi}{2}+k2\pi$ ↯



$4x=\frac{\pi}{2}+k2\pi$

$\boxed{x=\frac{\pi}{8}+k\frac{\pi}{2}}$

$y_k=\frac{\pi}{8}+k\frac{\pi}{2}-\frac{\pi}{4}=-\frac{\pi}{8}+k\frac{\pi}{2}$

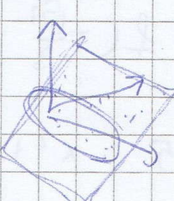


$\Rightarrow$ loki sz.é csak (x_0, y_0)-ban lehet

Három pontban kell kiértékelni f-et:

$f(\frac{\pi}{4}, 0)$, $f(0, -\frac{\pi}{4})$, $f(\frac{\pi}{8}, -\frac{\pi}{8})$ $\Rightarrow$ min, max ...

„tartomány pereme"



5/4 $f(x,y) = x + 2y$ Absz. sz.é. $2x^2 + y^2 = 4$ elipszisen

$g(x,y) = 2x^2 + y^2 - 4 = 0$

$F(x,y) = f(x,y) + \lambda g(x,y) =$

$= x + 2y + \lambda(2x^2 + y^2 - 4)$

$\partial_x F(x,y) = 1 + 4\lambda x$ $\qquad$ $\partial_y F(x,y) = 2 + 2\lambda y$

$1 + 4\lambda x = 0 \iff 1 = -4\lambda x$, $\frac{1}{-4\lambda} = x$ $(\lambda \neq 0)$ $\qquad$ $2 + 2\lambda y = 0$, $2 = -2\lambda y$, $-\frac{1}{\lambda} = y$

$2 + 2\lambda y = 0$

$2x^2 + y^2 - 4 = 0$

$2\left(\frac{1}{-4\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 - 4 = 0$

$\frac{2}{16\lambda^2} + \frac{1}{\lambda^2} - 4 = 0$

$\frac{1}{8\lambda^2} + \frac{1}{\lambda^2} = 4$

$1 + 8 = 32\lambda^2$

$32\lambda^2 - 9 = 0$

$\frac{9}{32} = \lambda^2 \Rightarrow \lambda = \pm\frac{3}{\sqrt{32}} = \pm\frac{3}{4\sqrt{2}}$

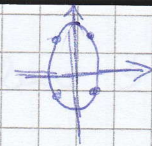
$1 + 4\lambda x = 0 \Rightarrow 1 + 4\left(\pm\frac{3}{4\sqrt{2}}\right)x = 0 \iff 1 \pm \frac{3}{\sqrt{2}}x = 0$

$\iff 1 = \mp\frac{3}{\sqrt{2}}x \iff \boxed{x = \pm\frac{\sqrt{2}}{3}}$

$2 + 2\lambda y = 0 \Rightarrow 2 + 2\left(\pm\frac{3}{4\sqrt{2}}\right)y = 0 \iff 2 \pm \frac{3}{2\sqrt{2}}y = 0$

$\iff 2 = \mp\frac{3}{2\sqrt{2}}y \iff \boxed{y = \pm 4\frac{\sqrt{2}}{3}}$

4 pontban lehet lok. szé. $\left(\pm\frac{\sqrt{2}}{3}, \pm\frac{4\sqrt{2}}{3}\right)$



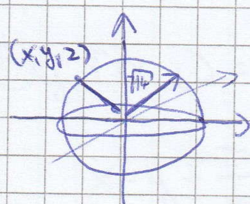
$f(x,y) = x+2y$

$\Rightarrow f\left(\frac{\sqrt{2}}{3}, \frac{4\sqrt{2}}{3}\right) = \frac{9\sqrt{2}}{3} = 3\sqrt{2}$ max

$f\left(-\frac{\sqrt{2}}{3}, -\frac{4\sqrt{2}}{3}\right) = -\frac{9\sqrt{2}}{3} = -3\sqrt{2}$ min

5/5 $x^2+y^2+z^2=14$ gömbnek mely pontja van legtávolabb / legközelebb pontól / ponthoz?

a i $(1,5,-10)$ ii, $(1,2,2)$ iii, $(-2,1,0)$



2 pont távolsága távolság f

$(x+2)^2+(y-1)^2+(z-0)^2 = f(x,y,z)$

$g(x,y,z) = x^2+y^2+z^2-14=0$

$F(x,y,z) = (x+2)^2+(y-1)^2+z^2+\lambda(x^2+y^2+z^2-14)$

$\partial_x F(x,y,z) = 2(x+2)+2\lambda x = 0$

$\partial_y F(x,y,z) = 2(y-1)+2\lambda y = 0$

$\partial_z F(x,y,z) = 2z+2\lambda z = 0$

$\boxed{\lambda=?}$

$(1+\lambda)2z = 0 \rightarrow \boxed{z=0}$

$\searrow \lambda=-1 \overset{\text{pl. }1}{\Rightarrow} 2x+4 -2x = 0$
$4=0$ ↯

ez nem lehet

$2x+4+2\lambda x=0$

$2x(1+\lambda) = -4 \quad (\lambda\neq -1)$

$x = -\frac{2}{1+\lambda}$

$2y-2+2\lambda y=0$

$2y(1+\lambda)=2$

$y=\frac{1}{1+\lambda}$

$x^2+y^2+z^2-14=0 \Leftrightarrow \left(\frac{2}{1+\lambda}\right)^2+\left(\frac{1}{1+\lambda}\right)^2+0^2-14=0$

$4+1 = 14(1+\lambda)^2$

$\frac{5}{14} = (1+\lambda)^2$

$\pm\sqrt{\frac{5}{14}} = 1+\lambda \Rightarrow \lambda = -1\pm\sqrt{\frac{5}{14}}$

$$\Rightarrow x = -\frac{2}{1-1\pm\sqrt{\frac{5}{14}}} = \mp 2\sqrt{\frac{14}{5}}$$

$$y = \frac{1}{1-1\pm\sqrt{\frac{5}{14}}} = \pm\sqrt{\frac{14}{5}}$$

$$z = 0$$

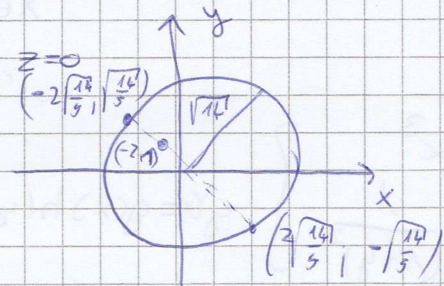
lehetséges sz.é. helyek 2db $\mp\pm$

Lehet ellenőrizni hogy rajta vagyunk-e a gömbön.

$x^2+y^2+z^2=14$-be visszahelyettesíteni az értékeket

$f\left(-2\sqrt{\frac{14}{5}}, \sqrt{\frac{14}{5}}, 0\right)$ egyik min másik max

$f\left(2\sqrt{\frac{14}{5}}; -\sqrt{\frac{14}{5}}, 0\right)$



Szeparábilis differenciálegyenletek $\boxed{d.e}$

explicit (az egyik oldala 0) elsőrendű (1x-es der.) közönséges (1 változó szerinti der.) d.e

$n \in \mathbb{N}^+$ $D \subset \mathbb{R} \times \mathbb{R}^n$ tartomány, $f: D \to \mathbb{R}^n$ folyt.

Keressük $I \subset \mathbb{R}$ intv-ot $\varphi: I \to \mathbb{R}^n$ diffható

$(x, \varphi(x)) \in D$ $(\forall x \in I)$ $\varphi'(x) = f(x, \varphi(x))$ $(x \in I)$

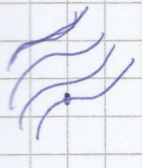
Ha $\varphi(\tau) = \xi$ adott $\Rightarrow$ kezdeti érték probléma $\boxed{k.é.p}$

Szeparábilis d.e.

$I, J \in \mathbb{R}$ nyílt intv. $g: I \to \mathbb{R}$ $h: J \to \mathbb{R}$, $g, h \in C$ $y'(x) = g(x)\,h(y(x))$

$(0 \notin R_h)$

(k.é.p) $\{(\tau, \xi) \in I \times J : y(\tau) = \xi\}$



$$\frac{y'(x)}{h(y(x))} = g(x)$$

$$\int \frac{y'(x)}{h(y(x))}\,dx = \int g(x)\,dx$$

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

$$y'(x) = \frac{dy}{dx}$$

$$\frac{dy}{dx} = g(x)\,h(y)$$ x-től független fv.

$$\frac{1}{h(y)}\,dy = g(x)\,dx$$

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

6/1 i,

$$y'(x) = \frac{x^3}{(1+y(x))^2}$$

$y \in (-\infty, -1) \cup \overbrace{(-1, +\infty)}^{J}$ pl.

ne legyen -1 (0-val osztás kiküszöbölése)

$x \in \mathbb{R} =: I$

$$\left.\begin{aligned} g(x) &= x^3 \\ h(y) &= \frac{1}{(1+y)^2} \end{aligned}\right\} y'(x) = g(x)\,h(y)$$

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

$$\underbrace{\int (1+y)^2\,dy}_{\frac{(1+y)^3}{3}} = \int x^3\,dx = \frac{x^4}{4} + C$$

$$\frac{(1+y)^3}{3} = \frac{x^4}{4} + C$$

$$(1+y)^3 = \frac{3x^4}{4} + C'$$

$$1 + y = \sqrt[3]{\frac{3x^4}{4} + C'}$$

$$y(x) = y = \sqrt[3]{\frac{3x^4}{4} + C'} - 1$$

x függvényében

legyen: hogy értelmezve legyen $y(x) = \ldots$

$I := (0, +\infty)$

$J := (-1, +\infty)$

$C' \geq 0$

6/1 ii,

$y' = y + y^2 \quad "(\mathcal{I} = (0, +\infty))"$

$y'(x) = y(x) + y^2(x)$

$h(y) = y + y^2 \quad g(x) = 1 \qquad \frac{dy}{dx} = g(x)\, h(y)$

$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx$

parciális

$\ln(y) - \ln(y+1) = \int \frac{1}{y} - \frac{1}{y+1} = \int \frac{1}{y(1+y)} = \int \frac{1}{y+y^2}\, dy = \int 1\, dx = x + c$

$\ln\left(\frac{y}{y+1}\right) = x + c$

konstans legyen c'

$\frac{y}{y+1} = e^{x+c} = e^x e^c = c' e^x$

$\frac{y+1-1}{y+1} = 1 - \frac{1}{y+1} = c' e^x$

$1 - c' e^x = \frac{1}{y+1}$

$y + 1 = = \frac{1}{1 - c' e^x}$

$c' < 0$

$y(x) = y = \frac{1}{1 - c' e^x} - 1 \qquad \mathcal{I} = (-1, +\infty)$

Visszavezetés szep. d.e.-re

$y'(x) = F(ax + by(x) + c) \quad (ab \neq 0)$

$u(x) = ax + by(x) + c$

$u'(x) = a + b\, y'(x) = a + bF(u(x))$

2/i

$y'(x) = \cos(x + y(x)) \qquad y(0) = \frac{\pi}{2}$ (k.c.p)

$u(x) = x + y(x)$

$u'(x) = 1 + F(u(x)) = \underbrace{1 + \cos(u(x))}_{h(u)}$

$g(x) = 1$

$$\int \frac{1}{h(u)} du = \int g(x)\,dx$$
$$= \int 1\,dx = x + C$$

$$\int \frac{1}{h(u)} du = \int \frac{1}{1+\cos(u)} du = \int \frac{1}{2\cos^2(\frac{u}{2})} du = \frac{1}{2} \frac{1}{\cos^2(\frac{u}{2})} = \frac{1}{2} \frac{tg(\frac{u}{2})}{1/2} = tg(\frac{u}{2})$$

$$\frac{1+\cos(\frac{u}{2}\cdot 2)}{2} = \cos^2(\frac{u}{2})$$

cissiaivni u-t

$$= tg\left(\frac{x+y(x)}{2}\right) = x + C$$

$$\frac{x+y(x)}{2} = arctg(x+C)$$

$$y(x) = 2\,arctg(x+C) - x$$

2. $y'(x) = 2y(x) + x + 1$

$u(x) = 2y(x) + x + 1$

$u'(x) = \underbrace{1 + 2u(x)}_{h(u)} \Rightarrow \int \frac{1}{h(u)} du = \int 1\,dx$

$g(x) = 1 \qquad \int \frac{1}{1+2u} du = x + C$

...

3. $y'(x) = \sqrt{y(x) - 2x}$

$u(x) = y(x) - 2x$

$u'(x) = \underbrace{-2 + \sqrt{u(x)}}_{h(u)}$

$\int \frac{1}{h(u)}\, du = \int 1\, dx$

$\int \frac{1}{\sqrt{u} - 2}\, du = x + c$

$\sqrt{u} := v \Rightarrow v^2 = u$ deriváld u-t v szerint $\frac{du}{dv}$

4. $y'(x) = -2\underbrace{(2x + 3y(x))}_{\text{csak a lineáris részt}}{}^2$

$u(x) = 2x + 3y(x)$

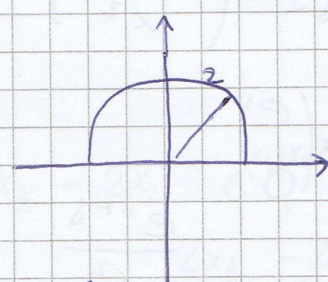
$u'(x) = \underbrace{2 + 3(-2)u^2(x)}_{h(u)}$

$\int \frac{1}{h(u)}\, du = \int 1\, dx = x + c$

$\int \frac{1}{2 - 6u^2}\, du$

2018.03.26.

1.



$f(x,y) = \frac{x^2 y}{x^2 + y^2}$ $\quad y \geq 0 \quad x^2 + y^2 \leq 4$

$\Updownarrow$

$y^2 \leq 4 - x^2$

$|y| \leq \sqrt{4 - x^2}$

$y \geq 0 \Rightarrow 0 \leq y \leq \sqrt{4 - x^2}$

$T = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4,\ y \geq 0\}$

$r \in [0, 2]$

$\varphi \in [0, \pi]$

1. mo:
$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} f(x,y)\, dy\, dx$$

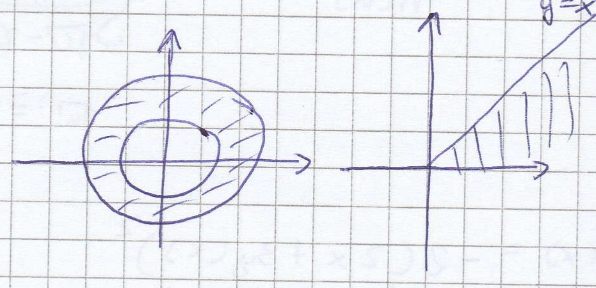
2. mo:
$$\iint_T f(x,y)\, dx\, dy = \int_0^2 \int_0^{\pi} f(r\cos\varphi, r\sin\varphi) \cdot r\, d\varphi\, dr =$$

$$\int_0^2\int_0^\pi \frac{r^2\cos^2\varphi\, r\sin\varphi}{r^2}\cdot r\,d\varphi\,dr = \int_0^2\int_0^\pi r^2\cos^2\varphi\sin\varphi\,d\varphi\,dr =$$

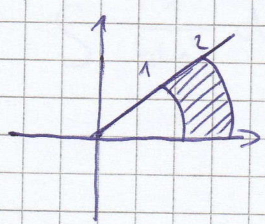
$$= \int_0^2 r^2\,dr \cdot \int_0^\pi \underbrace{\cos^2\varphi}_{f^2}\,\underbrace{\sin\varphi}_{(-f')}\,d\varphi = \left[\frac{r^3}{3}\right]_0^2 \int \left[\frac{\cos^3\varphi}{3}\right]_0^\pi (-1) = \left(\frac{8}{3}-\frac{0}{3}\right)$$

$$\left(-\frac{1}{3}-\frac{1}{3}\right)(-1) = \frac{8}{3}\cdot\frac{2}{3} = \frac{16}{9}$$

2



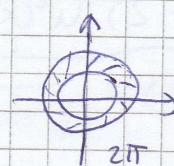
pl: $1\le x^2+y^2\le 4 \qquad 0\le y\le x$



$r\in[1,2]$

$\varphi\in[0;\frac{\pi}{4}]$

pl: $f(x,y)=\frac{x}{x^2+y^2} \qquad 1\le x^2+y^3\le 3$



$$\int_1^{\sqrt{3}}\int_0^{2\pi} \frac{r\cos\varphi}{r^2}\cdot r\,d\varphi\,dr = \int_1^{\sqrt{3}} 1\,dr\int_0^{2\pi}\cos\varphi\,d\varphi = [r]_1^{\sqrt{3}}\cdot[\sin\varphi]_0^{2\pi} = 0$$

2

$$f(x,y) = \left(\overbrace{\frac{e^{x+y}}{x^2}}^{f_1},\ \overbrace{y\sin\left(xy+\frac{\pi}{2}\right)}^{f_2}\right)$$

$a=(1,0)$

∃-e f-nek a körny-ben lok. inverze?

$(f^{-1})'(f(a))$

$$f(1,0)=\begin{pmatrix}e\\0\end{pmatrix}$$

$$\frac{e^{x+y}}{x^2}$$

$\partial_y f(x,y) =$

$$\partial_x f_1(x,y) = \partial_x\left(e^x e^y x^{-2}\right) = e^{x+y}\cdot x^{-2} + e^{x+y}\cdot -2x^{-3} \Rightarrow \partial_1 f_1(a) = -e$$

$$\partial_y f_1(x,y) = \partial_y\left(e^{x+y}\cdot x^{-2}\right) = \cancel{e^{x+y}}\ e^x\cdot x^{-2}\,\partial_y(e^y) = e^{x+y}\cdot x^{-2}$$

$$\Rightarrow \partial_y f_1(a) = e^{1+0}\cdot 1^{-2} = e$$

$\partial_1 f_2(x,y) = y\cos(xy+\frac{\pi}{2})\,y \qquad \Rightarrow \partial_1 f_2(a) = 0$

$\partial_2 f_2(x,y) = 1\cdot\sin(xy+\frac{\pi}{2}) + y\cdot\cos(xy+\frac{\pi}{2})\cdot x \Rightarrow$

$\partial_2 f_2(a) = \sin(0+\frac{\pi}{2}) + 0 = 1$

$f'(a) = \begin{pmatrix} -e & e \\ 0 & 1 \end{pmatrix} \qquad \det f'(a) = -e - 0 = -e \neq 0$

$\xRightarrow{\text{inv. fv. t.}}$ ~~$\exists\, e \cdot k$~~

$\exists\, u \in \mathbb{R}^2 \ \exists\, v \in \mathbb{R}^2 \quad f|_u : u \to v$ bijektív $a \in A \quad f(a) \in V \quad f^{-1}|_u \in C^1$

$(f^{-1})'\, f(a) = [f'(a)]^{-1}$ $\quad a \in u$, $f(a) \in V$

$[f'(a)]^{-1} = \frac{1}{-e}\begin{bmatrix} -e & e \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{-e}\begin{bmatrix} 1 & -e \\ 0 & -e \end{bmatrix} = \begin{bmatrix} 1/-e & 1 \\ 0 & 1 \end{bmatrix}$

3

$e^{x_1 + x_2 x_3} - x_2^2 = 0 \qquad x_2(x_1) = ?$

$2x_1x_3 + x_2 - x_3^2 = 0 \qquad x_3(x_1) = ?$

(1-ben az impl. fv. értéke $(-1,1)$)

$f(x_1,x_2,x_3) = \begin{pmatrix} e^{x_1+x_2x_3} - x_2^2 \\ 2x_1x_3 + x_2 - x_3^2 \end{pmatrix}$

$\partial_{(x_2,x_3)} f(x_1,x_2,x_3) = [\partial_{x_2} \ \ \partial_{x_3}] = \begin{bmatrix} e^{x_1+x_2x_3}\cdot x_3 - 2x_2 & e^{x_1+x_2x_3}\cdot x_2 \\ 1 & 2x_1 - 2x_3 \end{bmatrix}$

$b = \varphi(a)$

$a = 1 \Rightarrow b = (-1,1) \qquad f(a;\varphi(a)) = f(a,b) = f(1;-1,1) =$

$= \begin{pmatrix} e^{1+(-1)1} - (-1)^2 \\ 2\cdot 1 1 + (-1) - (-1)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓

$\partial_{(x_2,x_3)} f(a;b) = \partial_{(x_2,x_3)} f(1;-1,1) = \begin{bmatrix} e^0\cdot 1 - 2(-1) & e^0\cdot(-1) \\ 1 & 2-2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$

$\det \partial_{(x_2,x_3)} f(1;-1,1) = 3\cdot 0 + 1 = 1 \neq 0$

$\Rightarrow \exists \varphi : \mathbb{R} \to \mathbb{R}^2 \ (\varphi(a) = b) \quad f(x_1, \varphi(x_1), \varphi_2(x_1)) = 0 \quad \forall x_1 \in K(a)$

$$\varphi'(1) = \partial_1 \underset{1\ -1\ 1}{f(x_1,x_2,x_3)} \cdot - \left[\partial_2 \underset{1\ -1\ 1}{f(x_1,x_2,x_3)}\right]^{-1} =$$

$$\partial_{x_1} f(x_1,x_2,x_3) = \begin{pmatrix} e^{x_1+x_2x_3} \\ 2x_3 \end{pmatrix}$$

$$\partial_{x_1} f(1;-1,1) = \begin{pmatrix} e^{1+(-1)\cdot 1} \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\varphi'(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot - \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \frac{1}{2} \cdot - \frac{1}{1} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \\ 1/2 & -3/2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

4. $f(x,y) = 3x + 2y$ absz. sz. e. $\frac{x^2}{2} + \frac{y^2}{4} = 1$

$g(x,y) = \frac{x^2}{2} + \frac{y^2}{4} - 1 = 0$

$F(x,y) = f(x,y) + \lambda g(x,y)$

lok. sz. é F nek elsörendü szüks. felt.-e:

$$\left.\begin{aligned} \partial_x F(x,y) &= 0 \\ \partial_y F(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned}\right\}$$

$F(x,y) = 3x + 2y + \lambda\left(\frac{x^2}{2} + \frac{y^2}{4} - 1\right)$ $\boxed{\lambda \neq 0}$

$$\left.\begin{aligned} \partial_x F(x,y) &= 3 + \frac{\lambda}{2} 2x = 3 + \lambda x = 0 \\ \partial_y F(x,y) &= 2 + \frac{\lambda}{4} \cdot 2y = 2 + \frac{\lambda y}{2} = 0 \end{aligned}\right\} \Rightarrow \lambda x = -3 \quad \boxed{x = -\frac{3}{\lambda}}$$

$\frac{\lambda y}{2} = -2$

$\lambda y = -4$

$\boxed{y = -\frac{4}{\lambda}}$

$$g(x,y) = \frac{\left(-\frac{3}{\lambda}\right)^2}{2} + \frac{\left(\frac{4}{\lambda}\right)^2}{4} - 1 = 0$$

$$\frac{\frac{9}{\lambda^2}}{2} + \frac{\frac{16}{\lambda^2}}{4} - 1 = 0$$

$$\frac{9}{2\lambda^2} + \frac{16}{4\lambda^2} - 1 = 0$$

$$\frac{9}{2\lambda^2} + \frac{8}{2\lambda^2} = 1$$

$$9 + 8 = 2\lambda^2$$

$$17 = 2\lambda^2$$

$$\lambda^2 = \frac{17}{2}$$

$$\lambda = \pm\sqrt{\frac{17}{2}}$$

$$\Rightarrow \quad x = -\frac{3}{\lambda} = -\frac{3}{\pm\sqrt{\frac{17}{2}}} = \pm 3\left(\sqrt{\frac{2}{17}}\right)$$

$$y = -\frac{4}{\lambda} = -\frac{4}{\pm\sqrt{\frac{17}{2}}} = \pm 4\sqrt{\frac{2}{17}}$$

Rajta vagyunk-e az elipszisen?

Visszahely. $g(x,y)$-ba. Ell.!

$\Rightarrow$ csak a $\left(-3\frac{\sqrt{2}}{\sqrt{17}}; -4\frac{\sqrt{2}}{\sqrt{17}}\right)$

$\left(3\frac{\sqrt{2}}{\sqrt{17}}; 4\frac{\sqrt{2}}{\sqrt{17}}\right)$ pontokban lehet

$$f\left(-3\frac{\sqrt{2}}{\sqrt{17}}; -4\frac{\sqrt{2}}{\sqrt{17}}\right) = -9\frac{\sqrt{2}}{\sqrt{17}} - 8\frac{\sqrt{2}}{\sqrt{17}} = -17\frac{\sqrt{2}}{\sqrt{17}} = -\sqrt{34} \rightarrow \min$$

$$f\left(3\frac{\sqrt{2}}{\sqrt{17}}, 4\frac{\sqrt{2}}{\sqrt{17}}\right) = 9\sqrt{\frac{2}{17}} + 8\frac{\sqrt{2}}{\sqrt{17}} = 17\frac{\sqrt{2}}{\sqrt{17}} = \sqrt{17}\sqrt{2} = \sqrt{34} \rightarrow \max$$

5. $y' = xe^{x^2-y} + \frac{2x}{e^y} \qquad y(1) = 1 - \ln 2$

$$y' = xe^{x^2} \cdot e^{-y} + 2x \cdot e^{-y} = \underbrace{e^{-y}}_{h(y)} \underbrace{\left(xe^{x^2} + 2x\right)}_{g(x)}$$

$$= \int \frac{1}{h(y)} dy = g(x)\,dx$$

$$e^y = \int e^y \, dy = \int x e^{x^2} + 2x \, dx$$

$$\frac{1}{2}\int 2x e^{x^2} dx + \int 2x \, dx = \frac{1}{2} \cancel{\ln} e^{x^2} + x^2 + c$$

$$e^y = \frac{1}{2} e^{x^2} + x^2 + c$$

$$y(x) = \ln\left(\frac{1}{2} e^{x^2} + x^2 + c\right)$$

$$\left(\begin{array}{c} \frac{1}{2} e^{x^2} + x^2 + c > 0 \\ y \in \mathbb{R} \end{array} \right)$$

K.É.P: $\quad y(1) = 1 - \ln 2$

$$\ln\left(\frac{1}{2} e^{1^2} + 1^2 + c\right) = 1 - \ln 2 = \ln e - \ln 2 = \ln\left(\frac{e}{2}\right)$$

$$\ln\left(\frac{e}{2} + 1 + c\right) = \ln\left(\frac{e}{2}\right)$$

$$\frac{e}{2} + 1 + c = \frac{e}{2}$$

$$\boxed{c = -1}$$

k.é.p. mo:

$$y(x) = \ln\left(\frac{1}{2} e^{x^2} + x^2 - 1\right)$$

$$x \in (1; +\infty) := I$$

$$y \in \mathbb{R} := J$$