Modellek és algoritmusok

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$I_*(f)$

$I^*(f)$

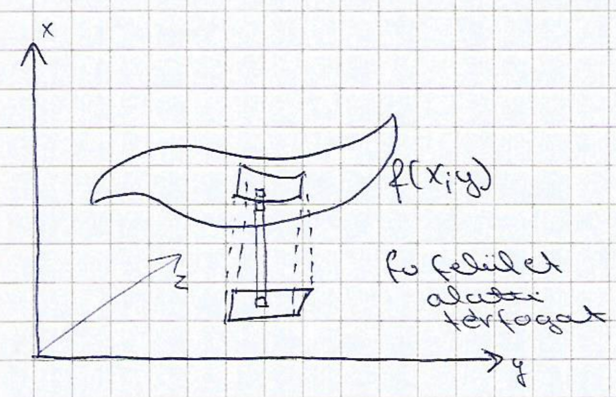
$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

$$\lim_{(x_{i+1}-x_i)\to 0} I_*(f) = \lim_{(x_{i+1}-x_i)\to 0} I^*(f) = \int_a^b f(x)\,dx$$

$$\sum_{i=0}^{n-1} f(\xi_i)(x_{i+1}-x_i) = \sum_{i=0}^{n-1} f(\xi_i)\underbrace{\Delta x_i}_{\text{távolságok}} \xrightarrow{x_{i+1}-x_i \to 0} \int_a^b f(x)\,dx$$

$\xi_i \in \{x_i, x_{i+1}\}$

Többdimenzió



$I_*(f)$

$I^*(f)$

$I = \{a,b\} \times \{c,d\}$ áll téglalap

$a = x_0 < x_1 < \ldots < x_n = b$

$c = y_0 < y_1 < \ldots < y_m = d$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(\xi_i,\eta_j)(x_{i+1}-x_i)(y_{j+1}-y_j) = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(\xi_i,\eta_j)\Delta x_i \Delta y_j$$

$\xi_i \in \{x_i, x_{i+1}\}$

$\eta_i \in \{y_j, y_{j+1}\}$

$x_{i+1}-x_i \to 0$

$y_{j+1}-y_j \to 0$

$I_*(f) = I^*(f)$

$$\downarrow$$

$$\int_I f(x,y)\,dx\,dy$$

Egydimenzióban:

Newton-Leibniz: $\int_a^b f(x)dx = F(b) - F(a) \qquad F' = f.$

Többdimenzióban:

Fubini-tétel: $I = I_1 \times I_2 = [a,b] \times [c,d]$

$$f \in R(I) \Rightarrow \int_I f(x,y)\,dx\,dy = \int_{I_1}\left(\int_{I_2} f(x,y)\,dy\right)dx =$$

$$= \int_{I_2}\left(\int_{I_1} f(x,y)\,dx\right)dy = \int_a^b \int_c^d f(x,y)\,dy\,dx =$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

közelíthető téglákkal

$A \subset \mathbb{R}^3$ "szép" halmaz

$\varrho(x,y,z)$ tömegeloszlás fv.

$\iiint_A 1\,dx\,dy\,dz =$ A halmaz térfogata

$\iiint_A \varrho(x,y,z)\,dx\,dy\,dz =$ A tömege

$\iiint_A x\,\varrho(x,y,z)\,dx\,dy\,dz =$ A súlypontjának x koordinátája

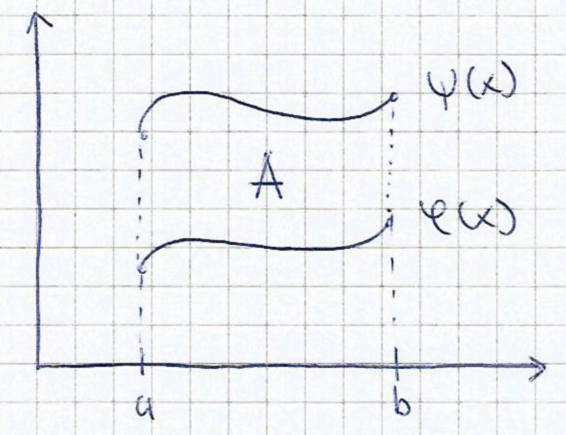
1. $f(x,y) = x^2 \cdot y \qquad I = [0,1] \times [-1,2]$

$\iint_I f = ? \qquad \iint_I f(x,y)\,dx\,dy = \int_0^1\left(\int_{-1}^2 x^2 y\,dy\right)dx$

$$\int_{-1}^2 x^2 y\,dy = x^2 \int_{-1}^2 y\,dy = x^2 \left[\frac{y^2}{2}\right]_{-1}^2 = x^2 \cdot \left(\frac{4}{2} - \frac{1}{2}\right) = x^2 \cdot \frac{3}{2}$$

$$\int_0^1 \frac{3}{2}x^2\,dx = \frac{3}{2}\int_0^1 x^2\,dx = \frac{3}{2}\cdot\left[\frac{x^3}{3}\right]_0^1 = \frac{3}{2}\cdot\frac{1}{3} = \frac{1}{2}$$

Normáltartomány: fv-el határolt terület



$\forall x \in [a,b] \quad \varphi(x) < \psi(x)$

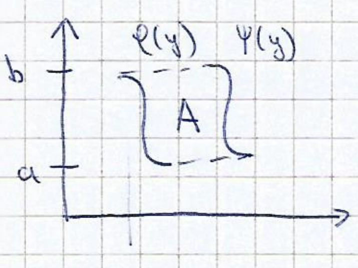
$A = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi(x) \le y \le \psi(x)\}$

$f: A \to \mathbb{R}$ korlátos

$f, \varphi, \psi \in C \Rightarrow f \in R(A)$

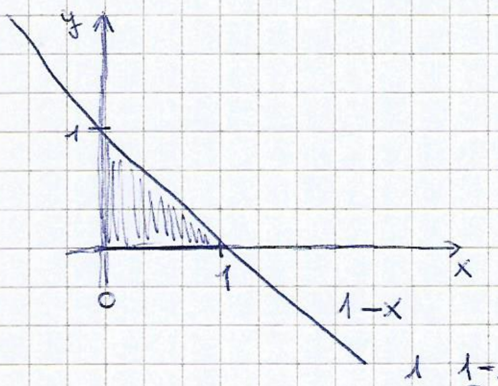
$$\iint_A f(x,y)\,dx\,dy = \int_a^b \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\,dx$$

Fordítva is lehet:



$$\iint_A f(x,y)\,dx\,dy = \int_a^b \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx\,dy$$

2. $f(x,y) = x + y \qquad A = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \quad 0 \le y \le 1-x\}$



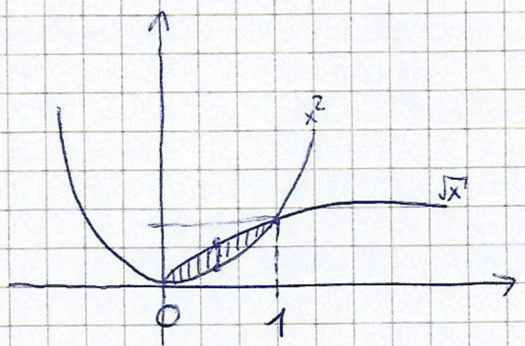
$$\iint_A x+y\,dx\,dy = \int_0^1 \int_0^{1-x} x+y\,dy\,dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_{y=0}^{1-x} dx =$$

$$\int_0^1 \left(\left(x(1-x) + \frac{(1-x)^2}{2}\right) - \left(x\cdot 0 + \frac{0^2}{2}\right)\right) = \int_0^1 x - x^2 + \frac{x^2-2x+1}{2}\,dx =$$

$$\int_0^1 \frac{2x-2x^2+x^2-2x+1}{2}\,dx = \int_0^1 \frac{1-x^2}{2}\,dx = \frac{1}{2}\int_0^1 1-x^2\,dx = \frac{1}{2}\left[x - \frac{x^3}{3}\right]_0^1 =$$

$$\frac{1}{2}\left[1 - \frac{1}{3}\right] = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

③ $\iint_A xy^2\,dx\,dy = ?$ A az x^2 és $\sqrt{x}$ fv-ek által határolt, korlátos terület



$x^2 = \sqrt{x}$ $\quad x \geq 0$

$x^4 = x$ $\quad x \neq 0$

$x^3 = 1$

$x = 1$

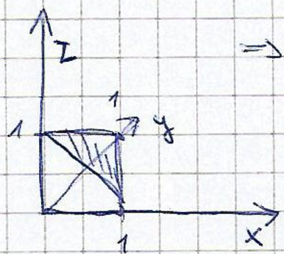
$$\int_0^1 \int_{x^2}^{\sqrt{x}} xy^2\,dy\,dx = \int_0^1 x\left[\frac{y^3}{3}\right]_{y=x^2}^{\sqrt{x}} dx = \int_0^1 x\left(\frac{\sqrt{x}^3}{3} - \frac{(x^2)^3}{3}\right)dx =$$

$$\frac{1}{3}\int_0^1 x\left(x^{\frac{3}{2}} - x^6\right)dx = \frac{1}{3}\int_0^1 x^{5/2} - x^7\,dx = \frac{1}{3}\left[\frac{x^{7/2}}{7/2} - \frac{x^8}{8}\right]_0^1 =$$

$$\left(\frac{2}{7}\cdot 1 - \frac{1}{8}\right) - (0) = \frac{1}{3}\left(\frac{16}{56} - \frac{7}{56}\right) = \frac{1}{3}\cdot\frac{9}{56} = \frac{3}{56}$$

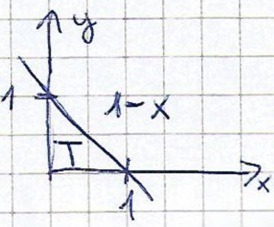
④ $A = \{$ koordinátasíkok és az $x+y+z=1$ által körbezárt korl.+zárt kompakt térrész $\}$

$$\iiint_A \frac{1}{(1+x+y+z)^3}\,dx\,dy\,dz = \iiint_A (1+x+y+z)^{-3}\,dx\,dy\,dz$$



$\Rightarrow z = 1 - x - y$

$$\iiint_A (1+x+y+z)^{-3}\,dx\,dy\,dz = \iint_T \left(\int_0^{1-x-y} (1+x+y+z)^{-3}\,dz\right)dx\,dy$$



$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+x+y+z)^{-3} \, dz \, dy \, dx$$

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$$\int_0^{1-x-y} (1+x+y+z)^{-3} \, dz = \left[\frac{(1+x+y+z)^{-2}}{-2} \right]_0^{1-x-y} =$$

$$-\frac{1}{2}\left[(1+x+y+z)^{-2}\right] = -\frac{1}{2}\left((1+x+y+(1-x-y))^{-2} - (1+x+y+0)^{-2} \right) =$$

$$-\frac{1}{2}\left(2^{-2} - (1+x+y)^{-2}\right) = -\frac{1}{8} + \frac{1}{2}(1+x+y)^{-2}$$

$$\int_0^{1-x} -\frac{1}{8} + \frac{1}{2}(1+x+y)^{-2} \, dy = \left[-\frac{y}{8} + \frac{1}{2}\frac{(1+x+y)^{-1}}{-1} \right]_0^{1-x}$$

$$= \left[-\frac{y}{8} - \frac{1}{2}(1+x+y)^{-1} \right]_0^{1-x} = \left(-\frac{(1-x)}{8} - \frac{1}{2}(1+x+(1-x))^{-1} \right) -$$

$$\left(-\frac{0}{8} - \frac{1}{2}(1+x+0)^{-1} \right) = \left(\frac{x-1}{8} - \frac{1}{2}\cdot 2^{-1} \right) - \left(-\frac{1}{2}(1+x)^{-1} \right) =$$

$$\frac{x-1}{8} - \frac{1}{4} + \frac{1}{2}\cdot\frac{1}{1+x}$$

$$\int_0^1 -\frac{1-x}{8} - \frac{1}{4} + \frac{1}{2}\cdot\frac{1}{1+x} \, dx = \int_0^1 -\frac{3}{8} + \frac{x}{8} + \frac{1}{2}\cdot\frac{1}{1+x} \, dx =$$

$$\int_0^1 -\frac{3}{8}x + \frac{x^2}{16} + \frac{1}{2}\cdot\ln(x+1) = -\frac{3}{8} + \frac{1}{16} + \frac{1}{2}\cdot\ln 2 - \overbrace{\left(0 + 0 + \frac{1}{2}\cdot\underbrace{\ln 1}_{0}\right)}^{0} =$$

$$-\frac{6}{16} + \frac{1}{16} + \frac{1}{2}\ln 2 = \frac{\ln 2}{2} - \frac{5}{16}$$