IPM-DM — Exam work — March 8th, 2018 — Gregor Koporec

Exam work should be done without external help. I am always available for possible discussions and/or a necessary hint.

Send your solutions to `gasper.fijavz@fri.uni-lj.si` in a readable .pdf.

The exam work is due end of June 2018. However, try to do it sooner.

Gašper Fijavž

1. A complete bipartite graph $K_{n,n}$ on $2n$ vertices can be described as follows:

$$V(G) = \{u_0, u_1, u_2, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$$

   Vertices u_i and v_j are adjacent for every choice of $i, j \in \{0, \ldots, n-1\}$. A preference relation at u_i ($i = 0, \ldots, n-1$) can be described as a sequence $v_{i+1}, v_{i+2}, \ldots, v_{i+n}$, which means that u_i would prefer to be paired with v_{i+1}, then with v_{i+2}, up until the vertex $v_{i+n} = v_i$, which is the least favorable partner for u_i. A preference relation at v_i is also described with a sequence $u_{i+1}, u_{i+2}, \ldots, u_{i+n} = u_i$.

   Addition in indices is taken modulo n, in case of a too large sum one should subtract n. In $K_{11,11}$ the vertex u_{7+10} is indeed equal to $u_{7+10 \pmod{11}} = u_{7+10-11} = u_6$.

   (a) Show that $\{u_i v_{i+1}; i \in \{0, \ldots, n-1\}\}$ is a stable matching.
   (b) Show that the matching of *horizontal* edges $\{u_i v_i; i \in \{0, \ldots, n-1\}\}$ is not stable.
   (c) Show that no matching, which contains a horizontal edge $u_i v_i$ can be stable.
   (d) Are all the remaining perfect matchings stable?

2. Let G be a bipartite graph and assume we are, for every vertex $v \in V(G)$ given its preference relation on its neighbors.

   (a) Let M_1, M_2 be matchings so that M_1 has strictly more edges than M_2. Show that there exists an odd path (ie. having an odd number of edges)

$$P = v_0 e_1 v_1 e_2 v_2 \ldots v_{n-1} e_n v_n$$

   so that, (i) the endvertices v_0 and v_n touch a single edge from the union $M_1 \cup M_2$, respectively, and (ii) that every edge $e_1, e_2, \ldots, e_n \in M_1 \cup M_2$.

   (b) Show that all *stable* matchings in G contain the same number of edges (by, say, observing an odd path implied above).

3. We say that G is *k-(vertex)-connected*, if (i) it contains at least $k+1$ vertices and (ii) for every pair of vertices $u \neq v$ there exist at least k internally disjoint $u - v$ paths (ie. no two paths from such a collection share a common vertex — apart from u and v). Vertex connectivity of G, $\kappa(G)$, is the maximal integer k, for which G is k-connected.

   We treat edge connectivity analoguosly: G is *ℓ-edge-connected*, if (i) it contains at least *two* vertices and if (ii) for every pair of vertices $u \neq v$ there exist at least ℓ edge disjoint $u - v$ paths (ie. no two paths from such a collection share a common edge, but these may share common vertices). Edge connectivity of G, $\lambda(G)$, is the maximal integer ℓ, for which G is ℓ-edge-connected.

   (a) Show that for every graph G we have $\kappa(G) \leq \lambda(G)$.

(b) Let κ and λ be integers satisfying $1 \leq \kappa \leq \lambda$. Give an example of G (a rough description is sufficient) so that $\kappa(G) = \kappa$ and $\lambda(G) = \lambda$.

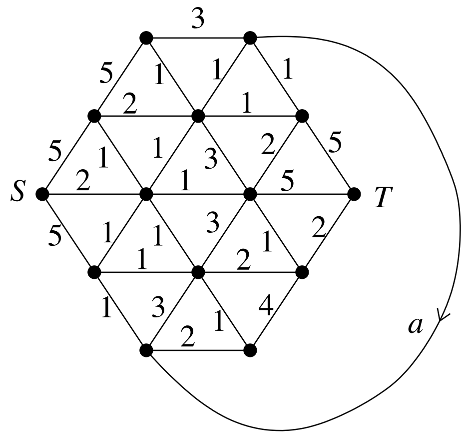
4. Let G be a k-connected graph.

   (a) Assume first $k = 2$. Show that for arbitrary pair of vertices u, v there exists a cycle containing both.

   (b) Let $k \geq 2$. Show that for every k-tuple of vertices $u_1, \ldots, u_k$ there exists a cycle containing all of them.

Hints for 3 and 4 are in Diestel's Graph Theory monograph (`http://diestel-graph-theory.com/`).

5. Find a maximal $s - t$ flow and a minimal $s - t$ cut in the below network. The edges (except for the curved one) are oriented left-to-right. Assume also that $a = 1$.



   Does the maximal flow change its value, if we increase a up to 7?

6. The network N is shown below.

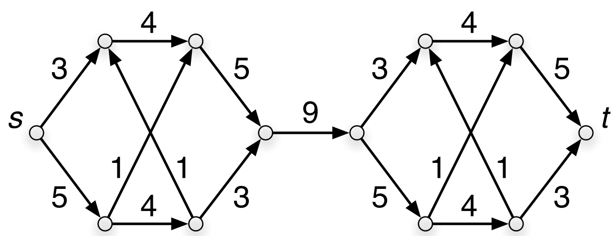
   (a) Find a maximal $s - t$ flow.

   (b) Find a minimal $s - t$ cut.

   Assume that one can increase the capacities cumulatively by 4 (you can increment the capacity of a single edge by 4, or you can increase the capacities of four edges by 1 each, or similar) which results in an altered network N'.

   (c) Which edges should have their capacities increased and by how much, so that the maximal $s - t$ flow in N' is as large as possible?

   (d) Find the resulting maximal $s - t$ flow and a minimal $s - t$ cut in such N'.



7. Let N be a network. How to find the *cheapest* maximal flow? Assume $N = (V, \vec{E}, c, s, t)$ is equipped with costs associated with flow along edges:

$$p : \vec{E} \to [0, \infty)$$

Let f be an $s - t$ flow in N. The *price* of flow f, $P(f)$, is defined as

$$P(f) = \sum_{e \in \vec{E}} p(e) \cdot f(e),$$

where $f(e)$ is the flow along a directed edge e.

(a) Describe a procedure that will find the cheapest maximal $s - t$ flow — the one having the smallest price among all maximal flows.

(b) What happens when all the capacities and prices are equal to 1?

8. A school is organizing an excursion for its 400 pupils. A bus company, which is to provide the transport, has 10 buses with 50 seats and 8 buses with 40 seats; but only has 9 available drivers on the desired day of the excursion. The price of a bigger bus is 800€, smaller ones cost 600€. How to organize the excursion in a cheapest possible way? Describe the problem as a linear program and solve it. Is the optimal solution non-integral?

9. We are trying to solve a linear program

$$\begin{array}{rr} & \max\ \mathbf{c}^T\mathbf{x} \\ \hline \text{where} & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

in standard form with the following data:

$$\mathbf{c} = \begin{bmatrix} 1.80307 \\ 2.402 \\ -1.53337 \\ -0.277813 \\ 3.83748 \\ 1.33938 \end{bmatrix}, A = \begin{bmatrix} 4.38569 & 5.02868 & 5.71122 & 3.07443 & 3.20015 & 5.91664 \\ 3.03715 & 5.56542 & 4.84196 & 1.76601 & 3.35549 & 1.12137 \\ 5.89878 & 5.35258 & 4.1234 & 1.0886 & 3.65847 & 2.14506 \\ 3.54887 & 5.49185 & 2.24427 & 4.31597 & 2.20345 & 3.96759 \\ 4.85858 & 2.28729 & 3.49223 & 1.89316 & 3.65843 & 5.37064 \\ 1.45508 & 1.32773 & 3.81647 & 2.60464 & 2.09959 & 2.20636 \\ 3.91769 & 2.25205 & 2.97618 & 5.11776 & 3.25921 & 2.10699 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2.84408 \\ 3.16529 \\ 2.31004 \\ 1.92566 \\ 1.69553 \\ 1.98222 \\ 3.75266 \end{bmatrix}$$

(a) Find an optimal solution.

(b) Write down and solve also the dual problem.

(c) Suppose that we can increase the right hand side of the *fifth* constraint by 0.1. How does the optimal solution change and what is the connection to the optimal solution of the dual problem?

(d) Suppose the one can alter the right hand sides of the constraints by $\mathbf{d} = (d_1, d_2, d_3, d_4, d_5, d_6, d_7)^T$, where $d_i \geq 0$ and $d_1 + \ldots + d_7 \leq 1$. What is the optimal solution in such a case? Write down the problem as a linear program and find the optimal change (the change that give the biggest increment en the optimal solution $\mathbf{x}$) of the right hand sides as well as the optimal solution of the perturbed problem $\mathbf{x}$ in such a case.