CS495
Optimization

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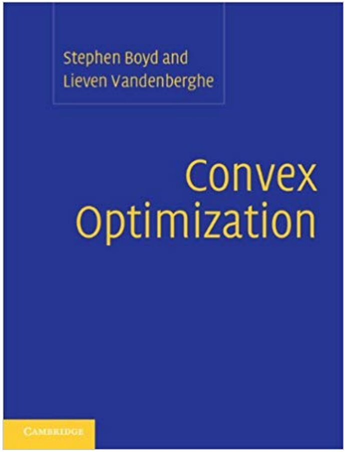
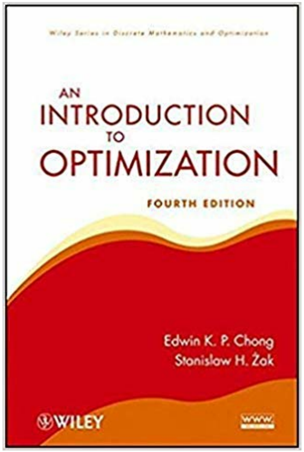
Lectures follow:
Boyd and Vandenberghe (2004)



Boyd, S., & Vandenberghe, L. (2004). Convex Optimization. Cambridge: Cambridge University Press.

Book and Stanford course: http://web.stanford.edu/~boyd/cvxbook/

Some examples from:
Chong and Zak (2013)



Chong, E. K., & Zak, S. (2001). An introduction to optimization: Wiley-Interscience.

Other complementary rigorous books for optimization/convex analysis are:
Bagirov (2014)
Bertsekas (2003)
Borwein (2006)
Dattorro (2009)
Hiriart (2001)
Rockafellar (1997)

Luenberger (1968) is a very interesting treatment for the optimization problem from the point of view of functional analysis and converging sequences.

Course Objectives

- Developing rigorous mathematical treatment for mathematical optimization.
- Building intuition, in particular to practical problems.
- Developing computer practice to using optimization SW.

Prerequisites

1. Discrete Mathematics
2. Calculus (single variable)
3. Calculus (multi variable)
4. Linear Algebra

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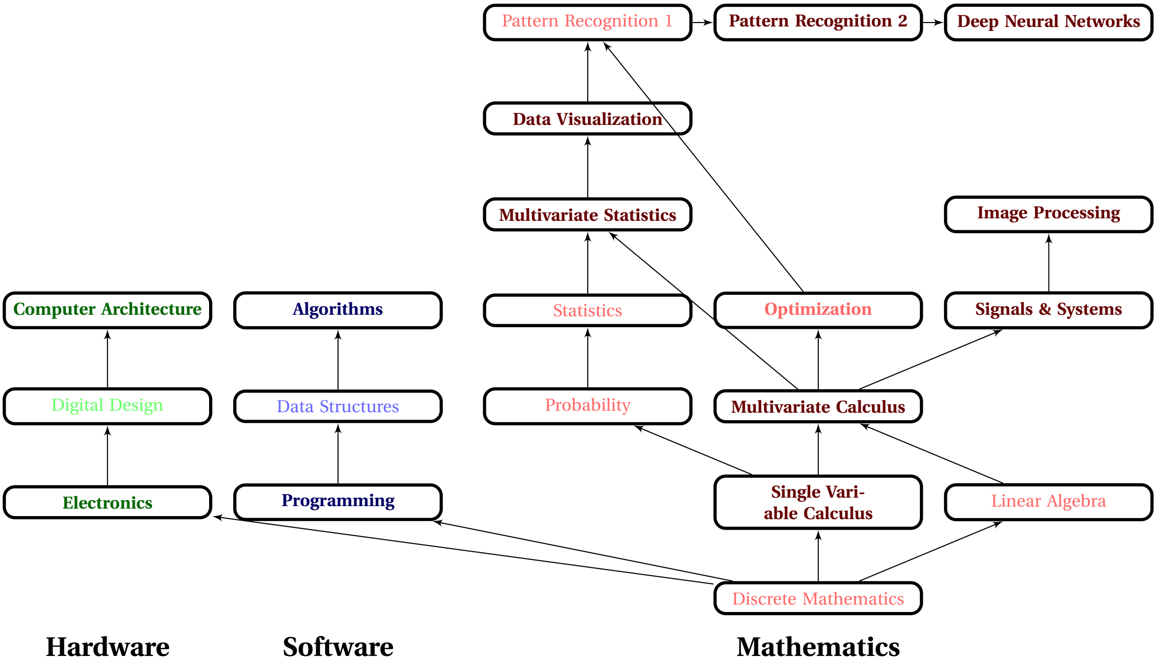
5. Some Real Analysis and some Topology: Wade (2000); Kreyszig (1978) are rigorous and wonderfully lucid; Rudin (1976) is the reference of references but very terse.


Pattern Recognition 1
Pattern Recognition 2
Deep Neural Networks
Data Visualization
Multivariate Statistics
Image Processing
Computer Architecture
Algorithms
Statistics
Optimization
Signals & Systems
Digital Design
Data Structures
Probability
Multivariate Calculus
Electronics
Programming
Single Variable Calculus
Linear Algebra
Discrete Mathematics
Hardware
Software
Mathematics

Chapter 1
Introduction
Snapshot on Optimization

Contents

Chapter 1

Introduction

1.1 Mathematical Optimization

Definition 1. *A mathematical optimization problem or just optimization problem, has the form (Boyd and Vandenberghe, 2004):*

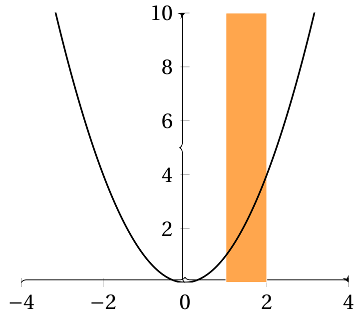
$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f_0(x) \\
\text{subject to:} \quad & f_i(x) \le 0, \qquad i = 1, \ldots, m \\
& h_i(x) = 0, \qquad i = 1, \ldots, p,
\end{aligned}
$$

$$
\begin{aligned}
x &= (x_1, \ldots, x_n) \in \mathbf{R}^n, && (\textit{optimization variable}) \\
f_0 &: \mathbf{R}^n \mapsto \mathbf{R}, && (\textit{objective (cost/utility) function}) \\
f_i &: \mathbf{R}^n \mapsto \mathbf{R}, && (\textit{inequality constraints (functions)}) \\
h_i &: \mathbf{R}^n \mapsto \mathbf{R}, && (\textit{equality constraints (functions)}) \\
\mathcal{D} &: \bigcap_{i=1}^{m} \mathbf{dom}\, f_i \cap \bigcap_{i=1}^{p} \mathbf{dom}\, h_i && (\textit{feasible set}) \\
&= \left\{x \mid x \in \mathbf{R}^n \wedge f_i(x) \le 0 \wedge h_i(x) = 0\right\} \\
x^* &: \left\{x \mid x \in \mathcal{D} \wedge f_0(x) \le f_0(z)\ \forall z \in \mathcal{D}\right\} && (\textit{solution})
\end{aligned}
$$

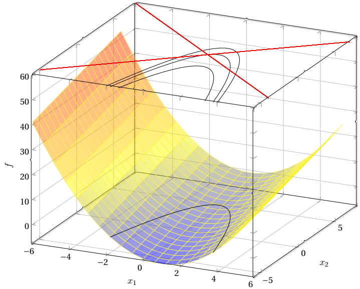
- minimize $f_0 \equiv$ maximize $-f_0$.
- $f_i \le 0 \equiv -f_i \ge 0$.
- 0s can be replaced of course by constants b_i, c_i
- unconstrained problem when $m = p = 0$.

Example 2. *:*

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & x^2 \\
\text{subject to:} \quad & x \le 2 \wedge x \ge 1.
\end{aligned}
$$



$x^* = 1$.
If the constraints are relaxed, then $x^* = 0$.



$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to:} & f_i(x) \le 0, & i = 1, \dots, m \\ & h_i(x) = 0, & i = 1, \dots, p, \end{array}$$

$x = (x_1, \dots, x_n) \in \mathbf{R}^n$, (*optimization variable*)

$f_0 : \mathbf{R}^n \mapsto \mathbf{R}$, (*objective (cost/utility) function*)

$f_i : \mathbf{R}^n \mapsto \mathbf{R}$, (*inequality constraints (functions)*)

$h_i : \mathbf{R}^n \mapsto \mathbf{R}$, (*equality constraints (functions)*)

$$\mathcal{D} : \bigcap_{i=1}^{m} \mathbf{dom}\, f_i \cap \bigcap_{i=1}^{p} \mathbf{dom}\, h_i \qquad (\textit{feasible set})$$

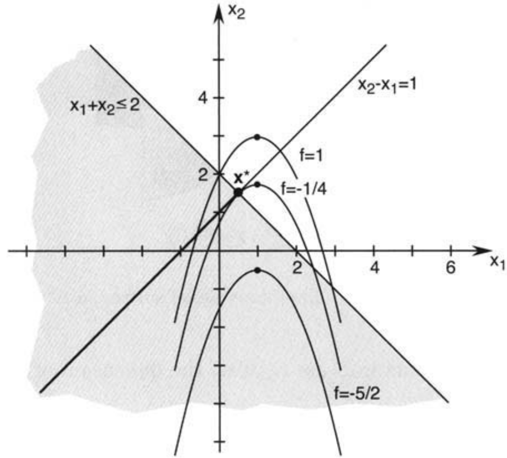
$$= \{x \mid x \in \mathbf{R}^n \land f_i(x) \le 0 \land h_i(x) = 0\}$$

$$x^* : \{x \mid x \in \mathcal{D} \land f_0(x) \le f_0(z)\ \forall z \in \mathcal{D}\} \qquad (\textit{solution})$$

Example 3. (*Chong and Zak, 2013, Ex. 20.1, P. 454*):

$$\begin{array}{ll} \underset{x}{\text{minimize}} & (x_1 - 1)^2 + x_2 - 2 \\ \text{subject to:} & x_2 - x_1 = 1 \\ & x_1 + x_2 \le 2. \end{array}$$

No global minimizer: $\partial z / \partial x_2 = 1 \ne 0$. *However,* $z|_{(x_2 - x_1 = 1)} = (x_1 - 1)^2 + (x_1 - 1)$, *which attains a minima at* $x_1 = 1/2$.



$x* = (1/2, 3/2)'$. (Let's see animation)

1.1.1 Motivation and Applications

- *optimization problem* is an abstraction of how to make "best" possible choice of $x \in \mathbf{R}^n$.
- *constrains* represent trim requirements or specifications that limit the possible choices.
- *objective function* represents the *cost* to minimize or the *utility* to maximize for each x.

Examples:

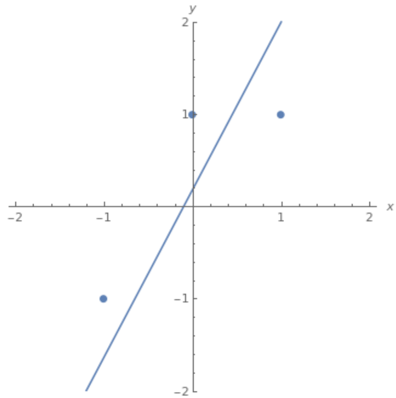
	Any problem	*Portfolio Optimization*	*Device Sizing*	*Data Science*
$x \in \mathbf{R}^n$	choice made	investment in capitals	dimensions	parameters
$f_i,\ h_i$	firm requirements /conditions	overall budget	engineering constraints	regularizer
f_0	cost (or utility)	overall risk	power consumption	error

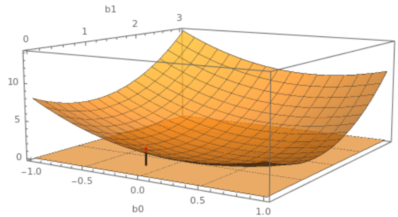
- Amazing variety of practical problems. In particular, data science: two sub-fields: construction and assessment.
- The construction of: Least Mean Square (LMS), Logistic Regression (LR), Support Vector Machines (SVM), Neural Networks(NN), Deep Neural Networks (DNN), etc.
- Many techniques are for solving the optimization problem:
  - Closed form solutions: convex optimization problems
  - Numerical solutions: Newton's methods, Gradient methods, Gradient descent, etc.
  - "Intelligent" methods: particle swarm optimization, genetic algorithms, etc.

Example 4 (Machine Learning: construction). *: Let's suppose that the best regression function is $Y = \beta_0 + \beta_1 X$, then for the training dataset (x_i, y_i) we need to minimize the MSE.*

- Half of ML field is construction: NN, SVM, etc.
- In DNN it is an optimization problem of millions of parameters.
- Let's see animation.
- Where are Probability, Statistics, and Linear Algebra here? Let's re-visit the chart.
- Is the optimization problem solvable:
  - closed form? (LSM)
  - numerically and guaranteed? (convex and linear)
  - numerically but not guaranteed? (non-convex):
    - numerical algorithms, e.g., GD,
    - local optimization,
    - heuristics, swarm, and genetics,
    - brute-force with exhaustive search

$$\underset{\beta_o,\beta_1}{\text{minimize}} \sum_i (\beta_o + \beta_1 x_i - y_i)^2$$





1.1.2 Solving Optimization Problems

- A *solution method* for a class of optimization problems is an algorithm that computes a solution.
- Even when the *objective function* and constraints are smooth, e.g., polynomials, the solution is very difficult.
- There are three classes where solutions exist, theory is very well developed, and amazingly found in many practical problems:

  Linear $\subset$ Quadratic $\subset$ Convex $\subset$ Non-linear (not linear and not known to be convex!)

- For the first three classes, the problem can be solved very reliably in hundreds or thousands of variables!

1.2 Least-Squares and Linear Programming

1.2.1 Least-Squares Problems

A *least-squares* problem is an optimization problem with no constraints (i.e., $m = p = 0$), and an objective in the form:

$$\underset{x}{\text{minimize}}\, f_0(x) = \sum_{i=1}^{k} (a_i' x - b_i)^2 = \| A_{k\times n} x_{n\times 1} - b_{k\times 1} \|^2.$$

The solution is given in **closed form** by:

$$x = (A'A)^{-1} A' b$$

- Good algorithms in many SC SW exist; it is a very mature technology.
- Solution time is $O(n^2 k)$.
- Easily solvable even for hundreds or thousands of variables.
- More on that in the Linear Algebra course.
- Many other problems reduce to typical LS problem:
  - Weighted LS (to emphasize some observations)

$$\underset{x}{\text{minimize}}\, f_0(x) = \sum_{i=1}^{k} w_i (a_i' x - b_i)^2.$$

  - Regularization (to penalize for over-fitting)

$$\underset{x}{\text{minimize}}\, f_0(x) = \sum_{i=1}^{k} (a_i' x - b_i)^2 + \rho \sum_{j=1}^{n} x_j^2.$$

1.2.2 Linear Programming

A *linear programming* problem is an optimization problem with objective and all constraint functions are linear:

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f_0(x) = C'x & \\
\text{subject to:} \quad & a_i'x \le b_i, & i = 1, \ldots, m \\
& h_i'x = g_i, & i = 1, \ldots, p,
\end{aligned}
$$

- **No** closed form solution as opposed to LS.
- Very robust, reliable, and effective set of methods for numerical solution; e.g., Dantzig's simplex, and interior point.
- Complexity is $\simeq O(n^2 m)$.
- Similar to LS, we can solve a problem of thousands of variables.
- Example is *Chebyshev minimization* problem:

$$\underset{x}{\text{minimize}}\, f_0(x) = \max_{i=1,\ldots,k} |a_i'x - b_i|,$$

- The objective is different from the LS: minimize the maximum error. **Ex:**
- After some tricks, requiring familiarity with optimization, it is equivalent to a LP:

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & t & \\
\text{subject to:} \quad & a_i'x - t \le b_i, & i = 1, \ldots, k \\
& -a_i'x - t \le -b_i, & i = 1, \ldots, k
\end{aligned}
$$

1.3 Convex Optimization

A *convex optimization* problem is an optimization problem with objective and all constraint function are convex:

$$
\begin{aligned}
&\underset{x}{\text{minimize}} && f_0(x) \\
&\text{subject to:} && f_i(x) \leq 0, && && i = 1, \ldots, m \\
& && h_i(x) = 0, && && i = 1, \ldots, p, \\
& && f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y), && \alpha + \beta = 1, && 0 \leq \alpha,\ 0 \leq \beta, && 0 \leq i \leq m \\
& && h_i(x) = a_i' x + b_i && && 0 \leq i \leq p
\end{aligned}
$$

- The LP and LS are special cases; however, only LS has closed-form solution.
- Very robust, reliable, and effective set of methods, including *interior point methods*.
- Complexity is almost: $O\big(\max(n^3, n^2 m, F)\big)$, where F is the cost of evaluating 1st and 2nd derivatives of f_i and h_i.
- Similar to LS and LP, we can solve a problem of thousands of variables.
- However, it is not as very mature technology as the LP and LS yet.
- There are many practical problems that can be re-formulated as convex problem **BUT** requires mathematical skills; but once done the problem is solved. **Hint:** realizing that the problem is convex requires more mathematical maturity than those required for LP and LS.

1.4 Nonlinear Optimization

A *non-linear optimization* problem is an optimization problem with objective and constraint functions are non-linear **BUT** not known to be convex (**so far**). Even simple-looking problems in 10 variables can be extremely challenging. Several approaches for solutions:

Local Optimization : starting at initial point in space, using differentiablity, then navigate

- does not guarantee global optimal.
- affected heavily by initial point.
- depends heavily on numerical algorithm and their parameters.
- More art than technology.
- In contrast to convex optimization, where a lot of art and mathematical skills are required to formulate the problem as convex; then numerical solution is straightforward.

Global Optimization : the true global solution is found; the compromise is complexity.

- The complexity goes exponential with dimensions.
- Sometimes it is worth it when: the cost is huge, not in real time, and dimensionality is low.

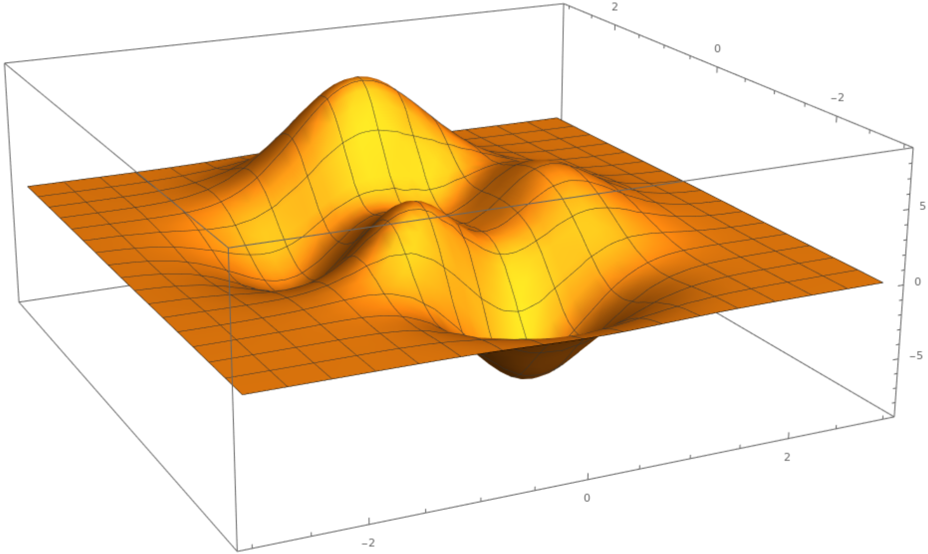
Role of Convex Optimization :

- Approximate the non-linear function to a convex one, finding the exact solution, then using it as a starting point for the original problem. (Also does not guarantee optimality)
- Setting bounds on the global solution.

Evolutionary Computations : Genetic Algorithm (GA), Simulated Annealing (SA), Particle Swarm Optimization (PSO), etc.

Example 5 (Nonlinear Objective Function). : *(Chong and Zak, 2013, Ex. 14.3, P.290)*

$$f(x,y) = 3(1-x)^2 e^{-x^2-(y+1)^2} - 10e^{-x^2-y^2}\left(-x^3 + \frac{x}{5} - y^5\right) - \frac{1}{3}e^{-(x+1)^2-y^2}$$



Part I

Theory

Chapter 2

Convex sets

2.1 Affine and convex sets

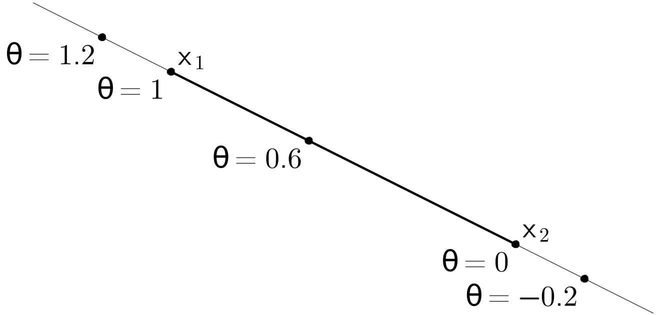
2.1.1 Lines and line segments

Definition 6 (line and line segment). *Suppose $x_1 \neq x_2 \in \mathbf{R}^n$. Points of the form*

$$y = \theta x_1 + (1-\theta)x_2$$
$$= x_2 + \theta(x_1 - x_2),$$

where $\theta \in \mathbf{R}$, form the line passing through x_1 and x_2.

- As usual, this is a definition for high dimensions taken from a proof for $n \leq 3$.
- We have done it many times: angle, norm, cardinality of sets, etc.
- if $0 \leq \theta \leq 1$ this forms a line segment.



2.1.2 Affine sets

Definition 7 (Affine sets). *A set $C \subset \mathbf{R}^n$ is affine if the line through any two distinct points in C lies in C. I.e., $\forall x_1, x_2 \in C$ and $\theta \in \mathbf{R}$, we have $\theta x_1 + (1-\theta)x_2 \in C$. In other words, C contains any linear combination (summing to one) of any two points in C.*

Examples: what about line, line segment, circle, disk, strip, first quadrant?

Corollary 8. *Suppose C is an affine set, and $x_1, \ldots, x_k \in C$, then C contains every general affine combination of the form $\theta_1 x_1 + \ldots + \theta_k x_k$, where $\theta_1 + \ldots + \theta_k = 1$.*

Wrong Proof. Suppose $y_1, y_2 \in C$, then

$$x = \sum_{i=1}^{k} \theta_i x_i = \sum_{i=1}^{k} \theta_i(\alpha_i y_1 + (1-\alpha_i)y_2);$$

and the summation of the coefficients will be

$$\sum_{i=1}^{k} \theta_i \alpha_i + \sum_{i=1}^{k} \theta_i(1-\alpha_i) = \sum_{i=1}^{k} \theta_i(\alpha_i + 1 - \alpha_i) = \sum_{i=1}^{k} \theta_i = 1.$$

Where is the bug? ■

Correct Proof. **base:** $k = 3$.

$$\begin{aligned} x &= \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \\ &= (1-\theta_3)\left(\frac{\theta_1}{1-\theta_3}x_1 + \frac{\theta_2}{1-\theta_3}x_2\right) + \theta_3 x_3. \\ &= (1-\theta_3)(\cdot \in C) + \theta_3(\cdot \in C). \end{aligned}$$

induction: suppose it is true for some $k \geq 3$; i.e., $\sum_{i=1}^{k} \theta_i x_i \in C$ when $\sum_{i=1}^{k} \theta_i = 1$. Then

$$\begin{aligned} x &= \sum_{i=1}^{k+1} \theta_i x_i \\ &= \sum_{i=1}^{k} \theta_i x_i + \theta_{k+1} x_{k+1} \\ &= (1-\theta_{k+1}) \sum_{i=1}^{k} \theta_i / (1-\theta_{k+1}) x_i + \theta_{k+1} x_{k+1} \\ &= (1-\theta_{k+1})(\cdot \in C) + \theta_{k+1}(\cdot \in C), \\ &\qquad \text{(from the induction hypothesis)} \end{aligned}$$

which completes the proof. ■

Definition 9 (Subspace from Linear Algebra)**.** *a set $V \subset \mathbf{R}^n$ of vector (here points) is a subspace if it is closed under sums and scalar multiplication. I.e., $\forall v_1, v_2 \in V$ and $\forall \alpha, \beta \in \mathbf{R}$ we have $\alpha v_1 + \beta v_2 \in V$.*

Remember:

- $\alpha + \beta$ not necessarily equals 1
- $\alpha = 0, \beta = 0 \to \mathbf{0} \in V$.
- Any subspace V is the solution set of $A_{m\times n} x_{n\times 1} = 0$, which is $\mathcal{N}(A)$ (the null space of A). Geometry? I.e., $V = \{x \mid Ax = 0\}$
- $\mathbf{rank}(A) = n - \dim(V)$.

Corollary 10. .

1. *If C is affine, then $V = C - x_0 = \{x - x_0 \mid x, x_0 \in C\}$ is a subspace.*
2. *If V is a subspace, then $C = V + x_0 = \{x + x_0 \mid x \in V\}$ is affine $\forall x_0$.*
3. *An affine set C can be represented as the solution set of a nonhomogeneous linear system $Ax = b$, where $V = C - x_0$ is $\mathcal{N}(A)$.*
4. *The solution set of any nonhomogeneous system is an affine set. (Ex. 2.1)*

Proof.

1. Suppose $x_1, x_2, x_0 \in C$, an affine set. Both $x_1 - x_0$ and $x_2 - x_0$, by construction, $\in V$; then

$$
\begin{aligned}
x &= \alpha(x_1 - x_0) + \beta(x_2 - x_0) + x_0 \\
&= \alpha x_1 + \beta x_2 + (1 - \alpha - \beta) x_0 \in C
\end{aligned}
$$

Then $x - x_0 = \alpha(x_1 - x_0) + \beta(x_2 - x_0) \in V$; hence V is a subspace.

2. Suppose $x_1, x_2 \in V$, a subspace. Both $x_1 + x_0$ and $x_2 + x_0$, by construction, $\in C$; then

$$
\begin{aligned}
x &= \theta(x_1 + x_0) + (1 - \theta)(x_2 + x_0) \\
&= \theta x_1 + (1 - \theta) x_2 + x_0 = (\cdot \in V) + x_0 \in C
\end{aligned}
$$

3. If C is affine and $x_0 \in C$, then

$$
\begin{aligned}
C - x_0 &= \{x \mid Ax = 0\} \qquad \text{(since it is a subspace)} \\
C &= \{x + x_0 \mid A(x + x_0) = Ax_0\} \\
C &= \{c \mid Ac = b\}.
\end{aligned}
$$

4. $C = \{x \mid Ax = b\}$; if $x_0 \in C$ then $Ax_0 = b$ and

$$
C - x_0 = \{x - x_0 \mid A(x - x_0) = b - Ax_0 = 0\}.
$$

Hence, $C - x_0$ is a subspace and C is affine.

■

Proof of the book. Suppose $x_1, x_2 \in C$, where $C = \{x \mid Ax = b\}$. Then

$$A(\theta x_1 + (1-\theta)x_2) = \theta Ax_1 + (1-\theta)Ax_2 = \theta b + (1-\theta)b = b,$$

which means $\theta x_1 + (1-\theta)x_2 \in C$ as well. ■

Remark 1. *:*

- *The dimension of affine is defined to be the dimension of the associate subspace.*
- *affine is a subspace plus offset.*
- *every subspace is affine but not the vice versa; i.e., subspace is a special case of affine.*

Definition 11 (affine hull)**.** *The "smallest" set of **all** affine combinations of **some** set C (not necessarily affine) is called the affine hull (**aff** C):*

$$\mathbf{aff}\ C = \{\sum_{i=1}^{k} \theta_i x_i \mid x_i \in C,\ \sum_i \theta_i = 1\}.$$

Corollary 12. **aff** C *is affine.*

Proof. For $x_1 = \sum_i \alpha_i x_i$, $\sum_i \alpha_i = 1$, and $x_2 = \sum_i \beta_i x_i$, $\sum_i \beta_i = 1$, we have

$$\theta x_1 + (1-\theta)x_2 = \theta \sum_i \alpha_i x_i + (1-\theta)\sum_i \beta_i x_i = \sum_i (\theta\alpha_i + (1-\theta)\beta_i)x_i$$

$$\sum_i (\theta\alpha_i + (1-\theta)\beta_i) = \theta \sum_i \alpha_i + (1-\theta)\sum_i \beta_i = \theta + (1-\theta) = 1.$$

Hence, **aff** C is affine as well. ■

Example 13. *Construct the affine hull of the set* $C = \{(-1,0),(1,0),(0,1)\}$

$$\begin{aligned}\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 =& (1-\theta_3)\Big(\frac{\theta_1}{1-\theta_3}x_1 + \frac{\theta_2}{1-\theta_3}x_2\Big) + \theta_3 x_3 \\ =& (1-\alpha_3)\big((1-\alpha_2)x_1 + \alpha_2 x_2\big) + \alpha_3 x_3 \qquad = (1-\alpha_2)(1-\alpha_3)x_1 + \alpha_2(1-\alpha_3)x_2 + \alpha_3 x_3,\end{aligned}$$

$$\begin{array}{lll} \theta_3 = \alpha_3 & \theta_2 = \alpha_2(1-\alpha_3) & \theta_1 = 1-\theta_2-\theta_3 = (1-\alpha_2)(1-\alpha_3) \\ \alpha_3 = \theta_3 & \alpha_2 = \theta_2/(1-\theta_3) & \alpha_1 = 1-\alpha_2 = \theta_1/(1-\theta_3). \end{array}$$

HW: Derive expressions for α_i and θ_i for n-point combination.

2.1.3 Affine dimension and relative interior

Definition 14 (some basic topology in $\mathbf{R}^n$)**.** *:*

1. *The ball of radious r and center x in the norm $\|\cdot\|$.*

$$B(x,r) = \{y \mid \|y - x\| \le r\}.$$

2. *An element $x \in C \subseteq \mathbf{R}^n$ is called an interior point of C if $\exists\, \varepsilon > 0$ for which*

$$B(x,\epsilon) = \{y \mid \|y - x\|_2 \le \varepsilon\} \subseteq C.$$

I.e., $\exists$ a ball centered at x that lies entirely in C.

3. *The set of all points interior to C is called the interior of C and is denoted* **int** C.
4. *A set C is open if* **int** $C = C$. *I.e., every point in C is an interior point.*
5. *A set C is closed if its complement is open*

$$\mathbf{R}^n \setminus C = \{x \in \mathbf{R}^n \mid x \notin C\}$$

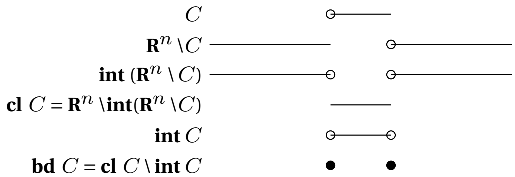
6. *The closure of a set C is defined as*

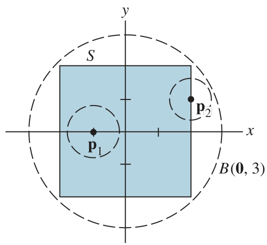
$$\mathbf{cl}\ C = \mathbf{R}^n \setminus \mathbf{int}(\mathbf{R}^n \setminus C).$$

7. *The boundary C is defined as*

$$\mathbf{bd}\ C = \mathbf{cl}\ C \setminus \mathbf{int}\ C.$$

Corollary 15. *A boundary point (a point $x \in$ **bd** C) satisfies: $\forall \epsilon > 0, \exists\, y \in C$ and $z \notin C$ s.t. $y, z \in B(x,\epsilon)$.*





Definition 16 (alter. equiv. def.)**.** *:*

- **int** C *and* **bd** C *are defined as 2,3, corollary.* *(It is obvious that:* **int** $C \cap$ **bd** $C = \phi$.*)*
- C *is open if* **int** $C = C \quad \equiv \quad C \cap$ **bd** $C = \phi$.
- C *is closed if* **bd** $C \subseteq C$.
- **cl** $C =$ **bd** $C \cup$ **int** C.

Definition 17. *We define the affine dimension of a set C as the dimension of its affine hull.*

Example 18. *The unit circle in $\mathbf{R}^2$, i.e., $\{x \mid x_1^2 + x_2^2 = 1\}$ has affine hull of whole $\mathbf{R}^2$. So its affine dimension is 2. However, it has a dimensionality of 1 since it is parametric in just one parameter (manifold).*

Definition 19. *We define the relative interior of the set C, denoted **relint** C, as its interior relative to **aff** C*

$$\textbf{relint } C = \{x \in C \mid B(x,r) \cap \textbf{aff } C \subseteq C \textit{ for some } r > 0\},$$

*and its relative boundary, denoted **relbd** C is defined as*

$$\textbf{relbd } C = \textbf{cl } C \setminus \textbf{relint } C.$$

Example 20. *Consider a square in the (x_1, x_2)-plane in $\mathbf{R}^3$, defined as:*

$$C = \{x \in \mathbf{R}^3 \mid -1 \le x_1 \le 1,\ -1 \le x_2 \le 1,\ x_3 = 0\}.$$

Then:

$$\begin{aligned}
\textbf{int } C &= \Phi \\
\textbf{cl } C &= \mathbf{R}^n \setminus \textbf{int}\left(\mathbf{R}^n \setminus C\right) = C \\
\textbf{bd } C &= \textbf{cl } C \setminus \textbf{int } C = C \\
\textbf{aff } C &= \{x \in \mathbf{R}^3 \mid x_3 = 0\} \\
\textbf{relint } C &= \{x \in \mathbf{R}^3 \mid -1 < x_1 < 1,\ -1 < x_2 < 1,\ x_3 = 0\} \\
\textbf{relbd } C &= \{x \in \mathbf{R}^3 \mid \max\{|x_1|, |x_2|\} = 1,\ x_3 = 0\}
\end{aligned}$$

2.1.4 Convex sets

Definition 21 (convex set). *A set C is convex if the line segment between any two points in C lies in C; i.e., if for any $x_1, x_2 \in C$ and any θ with $0 \le \theta \le 1$, we have*

$$\theta x_1 + (1-\theta)x_2 \in C.$$

Corollary 22. *Suppose C is convex set, and $x_1, \ldots, x_k \in C$, then C contains every general convex combination (also called mixture); i.e.,*

$$\sum_i \theta_i x_i \in C,\ \sum_i \theta_i = 1,\ \theta_i \ge 0.$$

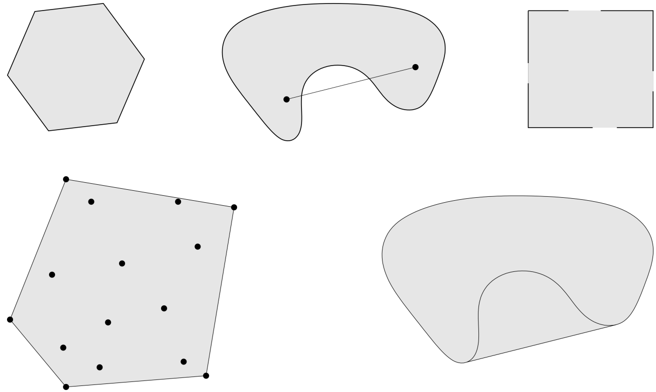
Proof. identical to proof of corollary 8. ■

Definition 23 (convex hull). *The "smallest" set of **all** convex combinations of **some** set C (not necessarily convex) is called the convex hull (**conv** C)*

$$\mathbf{conv}\, C = \left\{\sum_{i=1}^{k} \theta_i x_i \mid x_i \in C,\ \sum_i \theta_i = 1,\ \theta_i \ge 0\right\}.$$

Corollary 24. **conv** *C is convex.*

Proof. identical to proof of corollary 12. ■



Example 25. *Revisit example 13.*

Example 26 (Applications). *: Suppose $X \in C$ is a r.v., C is convex. Then $\mathrm{E}\, X \in C$ if it exists:*

$$\mathrm{E}\, X = \sum_{i=1}^{n} p_i x_i$$

$$\mathrm{E}\, X = \sum_{i=1}^{\infty} p_i x_i$$

$$\mathrm{E}\, X = \int_C f_X(x) x\, dx \qquad \text{(Riemann sum)}$$

2.1.5 Cones

Definition 27. *A set C is called a cone (or nonnegative homogeneous) if $\forall\ x \in C,\ \theta \geq 0$ we have $\theta x \in C$; and it is a convex cone if it is convex in addition to being a cone.*

Definition 28. *A point of the form $\sum_{i=1}^{k} \theta_i x_i,\ \theta_i \geq 0$ is called a conic combination.*

Corollary 29. *A set C is a convex cone if and only if it contains all conic combinations of its elements; i.e.,*

$$\sum_i \theta_i x_i \in C\ \forall x_i \in C \text{ and } \theta_i \geq 0.$$

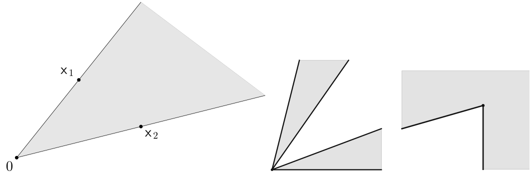
Proof.
Sufficiency: is obvious. Choosing $\sum_i \theta_i = 1$ implies C is convex; and setting $\theta_i = 0\ \forall i > 1$ implies C is cone.
Necessity: Since C is convex cone, then $\forall x_i \in C, \theta_i \geq 0$ we have:

$$\theta_i x_i \in C \qquad \text{(cone)}$$

$$\sum_i (1/n)(\theta_i x_i) \in C \qquad \text{(convex)}$$

$$n \sum_i (1/n)(\theta_i x_i) = \sum_i \theta_i x_i \in C \qquad \text{(cone)}$$

■

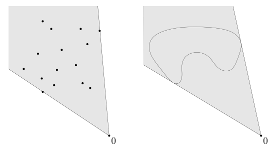


Definition 30. *A conic hull of a set C is the minimum set of all conic combination:*

$$\textbf{cone}\ C = \{\sum_i \theta_i x_i \mid x_i \in C,\ \theta_i \geq 0,\ i = 1, \ldots, n\}.$$

Corollary 31. **cone** *C is convex cone.*

Proof. If $y \in \textbf{cone}\ C, \alpha \geq 0$, then $\alpha y = \alpha \sum_i \theta_i x_i = \sum_i (\alpha\theta_i) x_i \in \textbf{cone}\ C$. And if $y_1, y_2 \in \textbf{cone}\ C$ then $\alpha y_1 + (1-\alpha) y_2 = \alpha \sum_i \theta_i x_i + (1-\alpha)\sum_i \mu_i x_i = \sum_i (\alpha\theta_i + (1-\alpha)\mu_i) x_i \in \textbf{cone}\ C$ ■



2.2 Some important examples

Fast Revision

- Each of the sets: ϕ, x_0 (a singleton), $\mathbf{R}^n$ are affine and convex.
- Any line is affine. If it passes through zero, it is a subspace and a convex cone.
- Any subspace is convex cone but not vise versa.
- A line segment is convex, but not affine (unless it reduces to a singleton).
- A *ray*, $\{x_0 + \theta v \mid \theta \geq 0, v \neq 0\}$ is convex but not affine. It is convex cone if $x_0 = 0$.

2.2.1 Hyperplanes and halfspaces

Definition 32. *A hyperplane is a set of the form*

$$\begin{aligned} \mathcal{S} &= \{x \mid a'x = b\}, & a, b \in \mathbf{R}^n,\ a \neq 0 \\ &\equiv \{x \mid a'(x - x_0) = 0\}, & a'x_0 = b. \end{aligned}$$

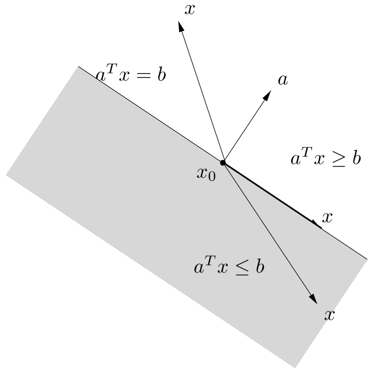
- Vectors with inner product with a is b: $\frac{a'}{\|a\|}x = \frac{b}{\|a\|}$. I.e., from $\mathbf{0}$, walk a distance $\frac{b}{\|a\|}$ (either + or -) in the direction of a, then draw perpendicular line.

Definition 33. *A closed halfspace is the region generated by the hyperplane and defined as:*

$$\begin{aligned} \mathcal{H} &= \{x \mid a'x \leq b\}, & a, b \in \mathbf{R}^n,\ a \neq 0 \\ &\equiv \{x \mid a'(x - x_0) \leq 0\}, & a'x_0 = b. \end{aligned}$$

- region of all vectors with projection $< b/\|a\|$.
- Vectors with obtuse angle with a: ($\cos\theta = \frac{a'x}{\|a\|\|x\|}$).
- Line passing with p_0 and $\perp$ on $\mathcal{S}$:

$$\begin{aligned} x &= p_0 + \theta\overline{a} && \text{(parametric eq.)} \\ a'x_0 &= a'p_0 + \theta_0\|a\| \\ \theta_0 &= (b - a'p_0)/\|a\| && (x_0 \text{ pt. of intersection.}) \\ x_0 - p_0 &= \frac{(b - a'p_0)}{\|a\|}\overline{a}. \end{aligned}$$



Corollary 34. *$\mathcal{S}$ is affine, $\mathcal{H}$ is convex and not affine,* **int** *$\mathcal{H} = \mathcal{H} \setminus \mathcal{S}$, and* **bd** *$\mathcal{H} = \mathcal{S}$.*

Proof. $\mathcal{S}$ is affine done. $\mathcal{H}$ is convex: take $0 \leq \theta \leq 1$ $\theta a'x_1 + (1-\theta)a'x_2 \leq \theta b + (1-\theta)b = b$. (why not affine?!)

$$\begin{aligned} y &= x + ru, \quad 0 \leq \|u\| \leq 1 && (y \in B(x, r)) \\ a'y &= a'x + ra'u = b - (b - a'x) + r\|a\|\|u\|\cos(a, u) \end{aligned}$$

If $b = a'x$, i.e., $x \in \mathcal{S}$, $a'u > 0$ or < 0 (depending on the angle) and hence $a'y > b$ or $< b$. Then $\mathcal{S} \subseteq$ **bd** $\mathcal{H}$.

If $a'x < b$, i.e., $x \in \mathcal{H} \setminus \mathcal{S}$, $\exists r < \frac{b - a'x}{\|a\|}$, s.t. $a'y < b$. Hence: **int** $\mathcal{H} = \mathcal{H} \setminus \mathcal{S}$ and **bd** $\mathcal{H} = \mathcal{S}$. ■

2.2.2 Euclidean balls and ellipsoids

Definition 35. *A Euclidean ball in* $\mathbf{R}^n$ *is the set:*

$$\begin{aligned} B(x_c, r) &= \{x = x_c + ru \mid \|u\|_2 \le 1\} \\ &= \{x \mid \|x - x_c\|_2 / r \le 1\} \\ &= \{x \mid (x - x_c)'(x - x_c)/r^2 \le 1\}. \end{aligned}$$

Definition 36. *Ellipsoid in* $\mathbf{R}^n$ *is the set:*

$$\begin{aligned} \mathcal{E} &= \{x = x_c + Au \mid \|u\|_2 \le 1,\ A \succ 0\} \\ &= \{x \mid \|A^{-1}(x - x_c)\|_2 \le 1, A \succ 0\} \\ &= \{x \mid (x - x_c)'(A^{-1})'A^{-1}(x - x_c) \le 1\} \end{aligned}$$

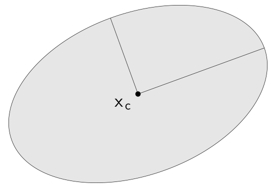
Spectral decomposition for $A = A'$.

$$\begin{aligned} Au &= (\lambda_1 v_1 v_1' + \lambda_2 v_2 v_2' + \cdots + \lambda_n v_n v_n')u \\ &= \lambda_1 v_1 (v_1' u) + \lambda_2 v_2 (v_2' u) + \cdots + \lambda_n v_n (v_n' u), \end{aligned}$$

which reduces to a Ball when $\lambda_i = r$.

Remark 2. *A does not have to be symmetric; put:*

$$\begin{aligned} P^{-1} &= (A^{-1})'A^{-1} = V\Sigma^{-1/2}\Sigma^{-1/2}V' && \textit{symmetric} \\ P^{1/2}u_2 &= Au_1 && \textit{is bijection} \\ \|u_2\|^2 &= u_1' A' P^{-1/2} P^{-1/2} A u_1 = \|u_1\|^2 \end{aligned}$$



Remark 3 (Contours of $\mathcal{N}(\mu, \Sigma)$). *:*

$$f_X(x) = \frac{1}{((2\pi)^p |\Sigma|)^{1/2}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

Corollary 37. *An ellipsoid, hence a ball, is convex*

Proof. For $x_1, x_2 \in \mathcal{E}, 0 \le \theta \le 1$,

$$\begin{aligned} x_1 &= x_c + Au_1, \|u_1\| \le 1 \\ x_2 &= x_c + Au_2, \|u_2\| \le 1 \\ x &= \theta(x_c + Au_1) + (1 - \theta)(x_c + Au_2) \\ &= x_c + A(\theta u_1 + (1 - \theta)u_2) \\ \|u\| &= \|(\theta u_1 + (1 - \theta)u_2)\| \\ &\le \theta\|u_1\| + (1 - \theta)\|u_2\| \\ &\le \theta + (1 - \theta) = 1. \end{aligned}$$

■

2.2.3 Norm balls and norm cones

Definition 38 (Norm)**.** *Let* $x, y \in \mathbf{R}^n$, $t \in \mathbf{R}$; *a function* $f : \mathbf{R}^n \mapsto \mathbf{R}_+$ *with* **dom** $f = \mathbf{R}^n$ *is called a norm if*

1. $f(x) = 0 \to x = 0$ *(definite)*
2. $f(tx) = |t| f(x)$ *(homogeneous)*
3. $f(x + y) \le f(x) + f(y)$ *(triangle inequality)*

$f(0) = 0$ *is implied from (2)* *(positive definite)*

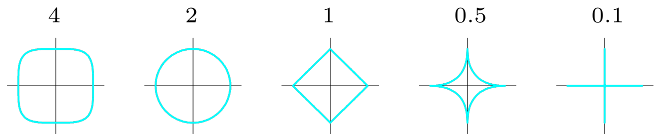
Definition 39 (L^p-norm ($\|\cdot\|_p$))**.** *is defined as*

$$\|x\|_p = \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} = \big(|x_1|^p + \cdots + |x_n|^p\big)^{1/p}.$$

Proof of $\|\cdot\|_p$ *is a norm.* :

1. $\Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} = 0 \to \sum_{i=1}^{n} |x_i|^p = 0 \to x_i = 0.$
2. $\|tx\|_p = \Big(\sum_{i=1}^{n} |tx_i|^p\Big)^{1/p} = |t| \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} = |t| \|x\|_p$
3. $\|x + y\|_p \le \|x\|_p + \|y\|_p$:
   (Minkowski's inequality proof (Kreyszig, 1978))

counter example for $p < 1$: $\|(0,1)\|_{1/2} + \|(1,0)\|_{1/2} = 1 + 1 = 2$, whereas $\|(1,1)\|_{1/2} = (1+1)^{1/(1/2)} = 4.$ ∎



$\{x \in \mathbf{R}^2 \mid \|x\|_p = c,\ p = 4, 2, 1, 0.5, 0.1\}$

- L_1-norm, Manhatan dist., Taxicab, abs. value

$$\|x\|_1 = \Big(\sum_{i=1}^{n} |x_i|\Big).$$

- L_2-norm, Euclidean distance (most meaningful)

$$\|x\|_2 = \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2}.$$

- L_∞-norm

$$\|x\|_\infty = \lim_{p\to\infty} \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} = \max_i |x_i|$$

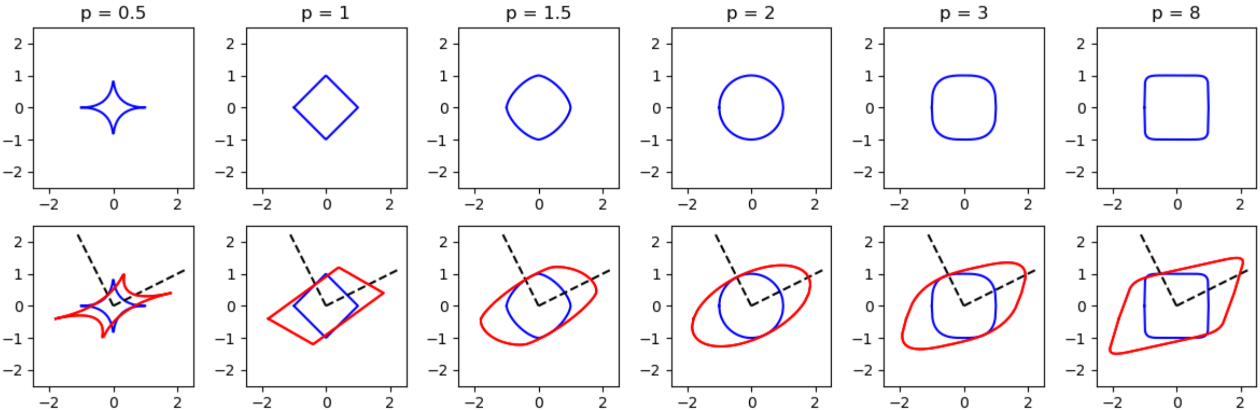
Corollary 40 (properties of $\|\cdot\|_p$)**.** :

1. $|x_i| < \|x\|_p \ \forall p < \infty.$
2. L_P*-norm is monotonic in* p.

Proof. is HW. ∎

Definition 41 (A general norm ellipsoid in $\mathbf{R}^n$)**.** *is the set generated by a norm ball, for any norm $\|\cdot\|$, of radius r, centered at x_c, and transformed by any symmetric matrix $A \succ 0$:*

$$\mathcal{E} = \left\{x = x_c + Au \mid \|u\|_p \le 1,\ A \succ 0\right\} \equiv \left\{x \mid \|A^{-1}(x - x_c)\|_p \le 1,\ A \succ 0\right\}. \tag{2.1}$$



- $A = \lambda_1 v_1 v_1' + \lambda_2 v_2 v_2',\ v_1 = (2,1)'/\sqrt{5},\ v_2 = (-1,2)'/\sqrt{5},\ \lambda_1 = 2,\ \lambda_2 = 1.$
- The unit ball intersect with the ellipsoid at v_2; why? The ellipsoids, of course, no longer have unit L_p-norm.

Corollary 42. *The general ellipsoid* (2.1) *is convex.*

Definition 43 (Metric)**.** *Let $x, y, z \in S$, a function $\delta : S \times S \mapsto \mathbf{R}_+$ is called a metric on S if:*

1. $\delta(x,y) = 0 \leftrightarrow x = y$ *(positive definite)*
2. $\delta(x,y) = \delta(y,x)$ *(symmetric)*
3. $\delta(x,y) \leq \delta(x,z) + \delta(z,y)$ *(triangle inequality)*

Lemma 44. *: $\delta(x,y) = f(x-y)$ is a metric ($\delta(x,0) = f(x)$):*

$$\delta(x,y) = f(x-y) = 0 \leftrightarrow x - y = 0 \leftrightarrow x = y$$

$$\delta(x,y) = f(x-y) = f(-1(y-x)) = f(y-x) = \delta(y,x)$$

$$\begin{aligned}\delta(x,y) &= f(x-y) = f((x-z)+(z-y)) \\ &\leq f(x-z) + f(z-y) = \delta(x,z) + \delta(y,z)\end{aligned}$$

Definition 45 (Loss)**.** *Let $x \in S_1$ (called set of nature), and $y \in S_2$ (called set of actions); then a function $L : S_1 \times S_2 \mapsto \mathbf{R}_+$ is called a loss incurred from assigning the action y based on the truth of nature x. For details on loss and utility theory see Berger (1993).*

Remark 4. *:*

- *Not any metric defines a norm; e.g., $\delta(x,y) = I_{x \neq y}$: First, prove it is a metric (**HW**). Then: $\delta(x,0) = f(x) = 1 \neq 10 = f(10x) = \delta(10x,0) \quad \forall x \neq 0$.*
- *Why? metric suites any set even categorical.*
- *Loss do not have to follow metric properties at all; e.g., $L(P,N) \neq L(N,P)$ in medical classification prediction.*

Norm balls and norm cones

Definition 46. *The norm cone associated with any norm $\|x\|$, $x \in \mathbf{R}^n$ is the set*

$$C = \left\{ (x,t)' \in \mathbf{R}^{n+1} \mid \|x\| \le t,\ t > 0 \right\}$$

Example 47. *The second-order cone is the norm cone for the Euclidean norm; i.e.,*

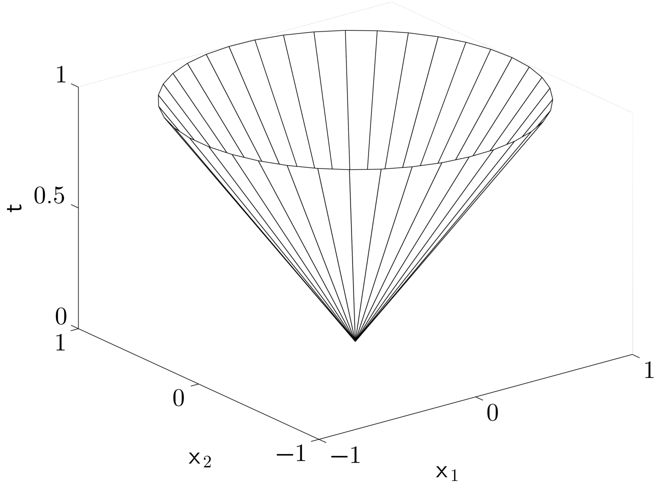
$$\begin{aligned} C &= \left\{ (x,t)' \in \mathbf{R}^{n+1} \mid \|x\|_2 \le t \right\} \\ &= \left\{ (x,t)' \mid x'x \le t, t > 0 \right\} \end{aligned}$$

Corollary 48. *The norm cone is convex.*

Proof. : given $p_i = (x_i, t_i), \|x_i\| \le t_i,\ i = 1, 2$, then

$$\begin{aligned} p &= \theta(x_1, t_1) + (1-\theta)(x_2, t_2) = (x, t) \\ &= (\theta x_1 + (1-\theta)x_2,\ \theta t_1 + (1-\theta)t_2) \\ \|x\| &= \|\theta x_1 + (1-\theta)x_2\| \\ &\le \theta\|x_1\| + (1-\theta)\|x_2\| \\ &\le \theta t_1 + (1-\theta)t_2 = t. \end{aligned}$$

■



To imagine it: pay attention to that the radius is the same as the height t. Therefore, the cross section is convex and in the z-direction is convex as well. E.g.,

$$C = \left\{ (x, t^2) \in \mathbf{R}^{n+1} \mid \|x\| \le t,\ t > 0 \right\}$$

is convex; however:

$$C = \left\{ (x, \sqrt{t}) \in \mathbf{R}^{n+1} \mid \|x\| \le t,\ t > 0 \right\}$$

is not.

2.2.4 Polyhedra

(Remember the early definition 1.2.2).

Definition 49. *A polyhedron is defined as the solution set of a finite number of linear qualities and inequalities:*

$$\mathcal{P} = \left\{x \mid a_j' x \le b_j,\ j = 1, \cdots, m, \quad c_j' x = d_j,\ j = 1, \cdots, p\right\}.$$

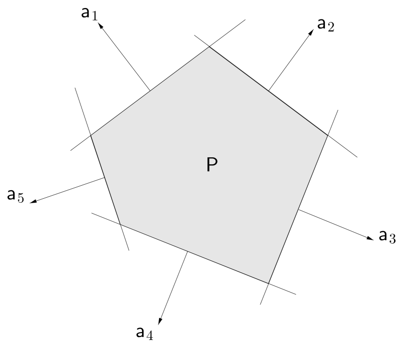
For short notation, we write

$$\mathcal{P} = \left\{x \mid Ax \preceq b,\ Cx = d\right\}, \quad A = \begin{pmatrix} a_1' \\ \vdots \\ a_m' \end{pmatrix}, \quad C = \begin{pmatrix} c_1' \\ \vdots \\ c_m' \end{pmatrix}.$$

The polyhedron is called polytope if it is bounded.

Example 50. *The nonnegative orthant*

$$\mathbf{R}_+^n = \left\{x \in \mathbf{R}^n \mid x_i \ge 0,\ i = 1, \cdots, n\right\} = \left\{x \in \mathbf{R}^n \mid x \succeq 0\right\}$$



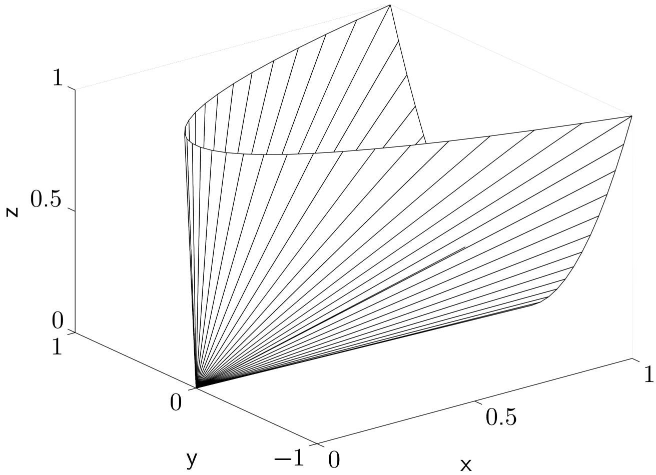
Corollary 51. *The polyhedron is convex.*

Proof. $x_1, x_2 \in \mathcal{P}$, $x = \theta x_1 + (1-\theta)x_2$, $0 \le \theta \le 1$. Then:

$$\begin{aligned} a_j' x &= \theta a_j' x_1 + (1-\theta) a_j' x_2 \\ &\le \theta b_j + (1-\theta) b_j = b_j \\ c_j' x &= \theta c_j' x_1 + (1-\theta) c_j' x_2 \\ &= \theta d_j + (1-\theta) d_j = d_j. \end{aligned}$$

Hence, all conditions are satisfied; the proof is complete. ■

2.2.5 The positive semidefinite cone



2.3 Operations that preserve convexity

2.4 Generalized inequalities

2.5 Separating and supporting hyperplanes

2.6 Dual cones and generalized inequalities

Part II

Applications

Part III

Algorithms

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