

CS495

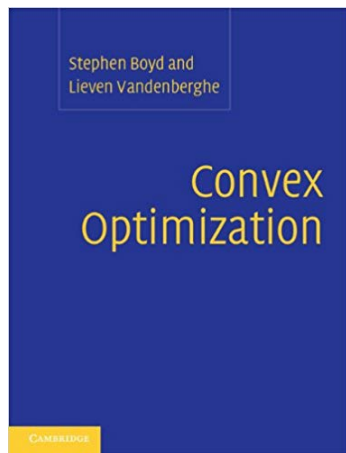
Optimization

Waleed A. Yousef, Ph.D.,
Human Computer Interaction Lab.,
Computer Science Department,
Faculty of Computers and Information,
Helwan University,
Egypt.

May 23, 2019

Lectures follow:

Boyd and Vandenberghe (2004)



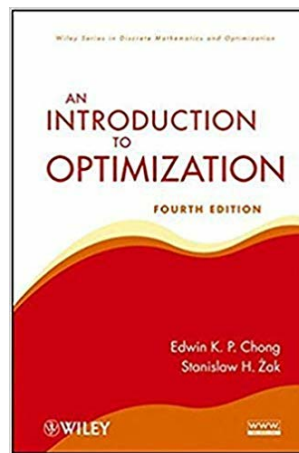
Boyd, S., & Vandenberghe, L. (2004). Convex Optimization. Cambridge: Cambridge University Press.

Book and Stanford course:

<http://web.stanford.edu/~boyd/cvxbook/>

Some examples from:

Chong and Zak (2013)



Chong, E. K., & Zak, S. (2013). An introduction to optimization: Wiley-Interscience.

Other complementary rigorous books for optimization/convex analysis are:

Bagirov (2014)

Bertsekas (2003)

Borwein (2006)

Dattorro (2009)

Hiriart (2001)

Rockafellar (1997)

Luenberger (1968) is a very interesting treatment for the optimization problem from the point of view of functional analysis and converging sequences.

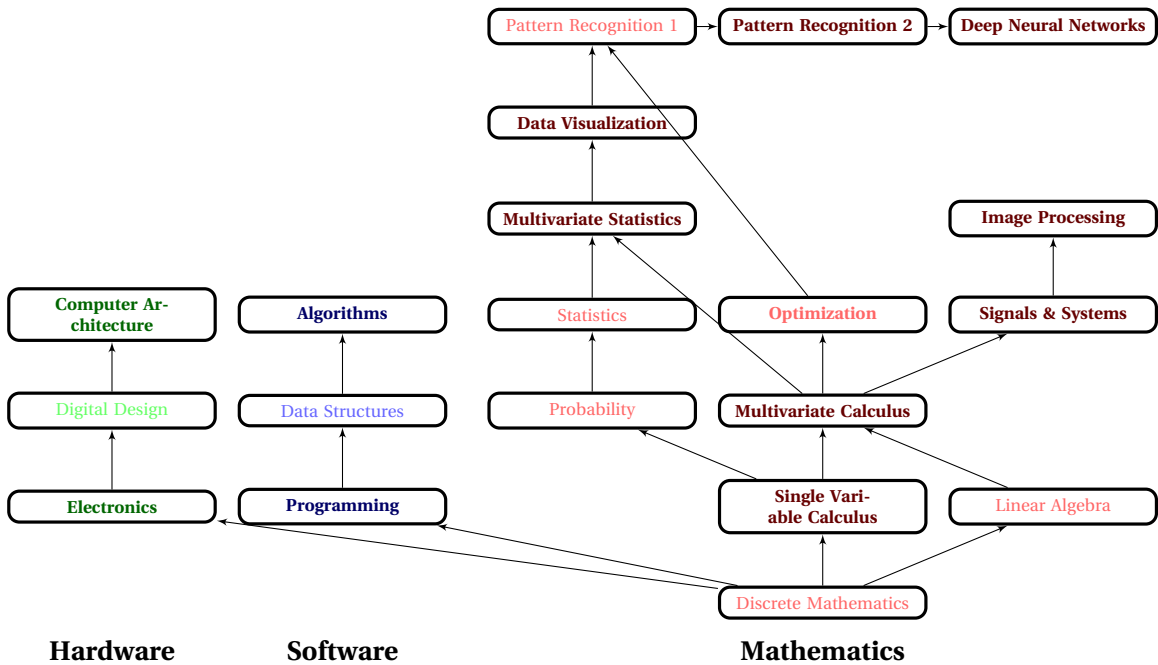
Course Objectives

- Developing rigorous mathematical treatment for mathematical optimization.
- Building intuition, in particular to practical problems.
- Developing computer practice to using optimization SW.

Prerequisites

1. Discrete Mathematics
 2. Calculus (single variable)
 3. Calculus (multi variable)
 4. Linear Algebra
-

5. Some Real Analysis and some Topology: [Wade \(2000\)](#); [Kreyszig \(1978\)](#) are rigorous and wonderfully lucid; [Rudin \(1976\)](#) is the reference of references but very terse.



Chapter 1

Introduction

Snapshot on Optimization

Contents

Contents

I	Introduction	1
1.1	Mathematical Optimization	2
1.1.1	Motivation and Applications	4
1.1.2	Solving Optimization Problems	6
1.2	Least-Squares and Linear Programming	7
1.2.1	Least-Squares Problems	7
1.2.2	Linear Programming	8
1.3	Convex Optimization	9
1.4	Nonlinear Optimization	10

I Theory 12

2	Convex sets	13
2.1	Affine and convex sets	14
2.1.1	Lines and line segments	14
2.1.2	Affine sets	15
2.1.3	Affine dimension and relative interior	19
2.1.4	Convex sets	21
2.1.5	Cones	22

2.2	Some important examples	23
2.2.1	Hyperplanes and halfspaces	24
2.2.2	Euclidean balls and ellipsoids	25
2.2.3	Norm balls and norm cones	26
2.2.4	Polyhedra	30
2.2.5	The positive semidefinite cone	31
2.3	Operations that preserve convexity	32
2.4	Generalized inequalities	33
2.5	Separating and supporting hyperplanes	34
2.6	Dual cones and generalized inequalities	35

3	Convex functions	46
3.1	Basic properties and examples	47
3.2	Operations that preserve convexity	48
3.3	The conjugate function	49
3.4	Quasiconvex functions	50
3.5	Log-concave and log-convex functions	51
3.6	Convexity with respect to generalized inequalities	52

4	Convex optimization problems	67
4.1	Optimization problems	68
4.2	Convex optimization	69
4.3	Linear optimization problems	70
4.4	Quadratic optimization problems	71
4.5	Geometric programming	72
4.6	Generalized inequality constraints	73
4.7	Vector optimization	74

5	Duality	89
5.1	The Lagrange dual function	90
5.2	The Lagrange dual problem	91
5.3	Geometric interpretation	92
5.4	Saddle-point interpretation	93
5.5	Optimality conditions	94
5.6	Perturbation and sensitivity analysis	95
5.7	Examples	96
5.8	Theorems of alternatives	97
5.9	Generalized inequalities	98

II Applications

109

6	Approximation and fitting	110
6.1	Norm approximation	111
6.2	Least-norm problems	112
6.3	Regularized approximation	113
6.4	Robust approximation	114
6.5	Function fitting and interpolation	115
7	Statistical estimation	142
7.1	Parametric distribution estimation	143
7.2	Nonparametric distribution estimation	144
7.3	Optimal detector design and hypothesis testing	145
7.4	Chebyshev and Chernoff bounds	146
7.5	Experiment design	147
8	Geometric problems	160
8.1	Projection on a set	161
8.2	Distance between sets	162
8.3	Euclidean distance and angle problems	163
8.4	Extremal volume ellipsoids	164
8.5	Centering	165
8.6	Classification	166
8.7	Placement and location	167
8.8	Floor planning	168

III Algorithms

191

9	Unconstrained minimization	192
9.1	Unconstrained minimization	193
9.2	Descent methods	194
9.3	Gradient descent method	195
9.4	Steepest descent method	196
9.5	Newton's method	197
9.6	Self-concordance	198
9.7	Implementation	199
10	Equality constrained minimization	226
10.1	Equality constrained minimization problems	227
10.2	Newton's method with equality constraints	228
10.3	Infeasible start Newton method	229
10.4	Implementation	230
11	Interior-point methods	239
11.1	Inequality constrained minimization problems	240
11.2	Logarithmic barrier function and central path	241
11.3	The barrier method	242
11.4	Feasibility and phase I methods	243
11.5	Complexity analysis via self-concordance	244
11.6	Problems with generalized inequalities	245
11.7	Primal-dual interior-point methods	246
11.8	Implementation	247

Bibliography

Chapter 1

Introduction

1.1 Mathematical Optimization

Definition 1 A mathematical optimization problem or just optimization problem, has the form (Boyd and Vandenberghe, 2004):

$$\begin{aligned} &\underset{x}{\text{minimize}} && f_0(x) \\ &\text{subject to:} && f_i(x) \leq 0, && i = 1, \dots, m \\ &&& h_i(x) = 0, && i = 1, \dots, p, \end{aligned}$$

$x = (x_1, \dots, x_n) \in \mathbf{R}^n$, (optimization variable)

$f_0 : \mathbf{R}^n \mapsto \mathbf{R}$, (objective (cost/utility) function)

$f_i : \mathbf{R}^n \mapsto \mathbf{R}$, (inequality constraints (functions))

$h_i : \mathbf{R}^n \mapsto \mathbf{R}$, (equality constraints (functions))

$$\mathcal{D} : \bigcap_{i=1}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i \quad (\text{feasible set})$$

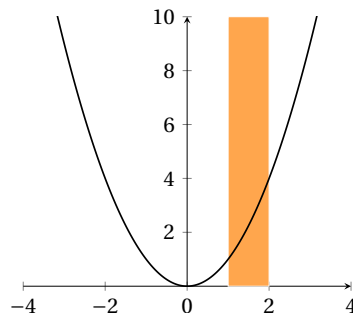
$$= \{x \mid x \in \mathbf{R}^n \wedge f_i(x) \leq 0 \wedge h_i(x) = 0\}$$

$$x^* : \{x \mid x \in \mathcal{D} \wedge f_0(x) \leq f_0(z) \forall z \in \mathcal{D}\} \quad (\text{solution})$$

- minimize $f_0 \equiv \text{maximize } -f_0$.
- $f_i \leq 0 \equiv -f_i \geq 0$.
- 0s can be replaced of course by constants b_i, c_i
- unconstrained problem when $m = p = 0$.

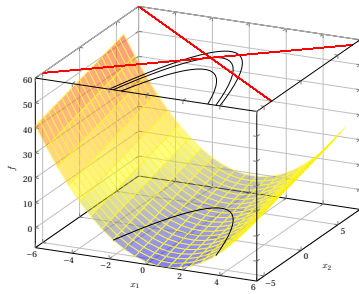
Example 2 :

$$\begin{aligned} &\underset{x}{\text{minimize}} && x^2 \\ &\text{subject to:} && x \leq 2 \wedge x \geq 1. \end{aligned}$$



$$x^* = 1.$$

If the constraints are relaxed, then $x^* = 0$.



$$\begin{aligned} &\underset{x}{\text{minimize}} && f_0(x) \\ &\text{subject to:} && f_i(x) \leq 0, && i = 1, \dots, m \\ &&& h_i(x) = 0, && i = 1, \dots, p, \end{aligned}$$

$x = (x_1, \dots, x_n) \in \mathbf{R}^n$, (optimization variable)

$f_0: \mathbf{R}^n \mapsto \mathbf{R}$, (objective (cost/utility) function)

$f_i: \mathbf{R}^n \mapsto \mathbf{R}$, (inequality constraints (functions))

$h_i: \mathbf{R}^n \mapsto \mathbf{R}$, (equality constraints (functions))

$$\mathcal{D}: \bigcap_{i=1}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i \quad (\text{feasible set})$$

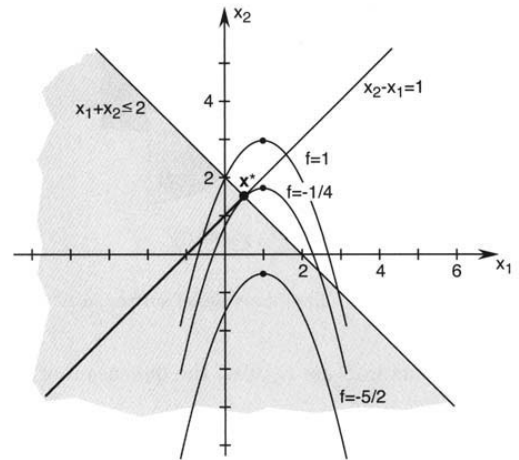
$$= \{x \mid x \in \mathbf{R}^n \wedge f_i(x) \leq 0 \wedge h_i(x) = 0\}$$

$$x^*: \{x \mid x \in \mathcal{D} \wedge f_0(x) \leq f_0(z) \forall z \in \mathcal{D}\} \quad (\text{solution})$$

Example 3 (*Chong and Zak, 2013, Ex. 20.1, P. 454*):

$$\begin{aligned} &\underset{x}{\text{minimize}} && (x_1 - 1)^2 + x_2 - 2 \\ &\text{subject to:} && x_2 - x_1 = 1 \\ &&& x_1 + x_2 \leq 2. \end{aligned}$$

No global minimizer: $\partial z / \partial x_2 = 1 \neq 0$. However, $z|_{(x_2-x_1=1)} = (x_1 - 1)^2 + (x_1 - 1)$, which attains a minimum at $x_1 = 1/2$.



$x^* = (1/2, 3/2)'$. (Let's see animation)

1.1.1 Motivation and Applications

- *optimization problem* is an abstraction of how to make “best” possible choice of $x \in \mathbf{R}^n$.
- *constrains* represent trim requirements or specifications that limit the possible choices.
- *objective function* represents the *cost* to minimize or the *utility* to maximize for each x .

Examples:

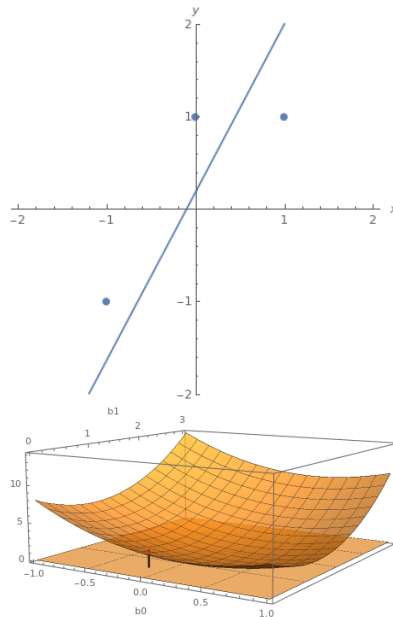
	<i>Any problem</i>	<i>Portfolio Optimization</i>	<i>Device Sizing</i>	<i>Data Science</i>
$x \in \mathbf{R}^n$	choice made	investment in capitals	dimensions	parameters
f_i, h_i	firm requirements /conditions	overall budget	engineering constraints	regularizer
f_0	cost (or utility)	overall risk	power consumption	error

- Amazing variety of practical problems. In particular, data science: two sub-fields: construction and assessment.
- The construction of: Least Mean Square (LMS), Logistic Regression (LR), Support Vector Machines (SVM), Neural Networks(NN), Deep Neural Networks (DNN), etc.
- Many techniques are for solving the optimization problem:
 - Closed form solutions: convex optimization problems
 - Numerical solutions: Newton’s methods, Gradient methods, Gradient descent, etc.
 - “Intelligent” methods: particle swarm optimization, genetic algorithms, etc.

Example 4 (Machine Learning: construction) :

Let's suppose that the best regression function is $Y = \beta_0 + \beta_1 X$, then for the training dataset (x_i, y_i) we need to minimize the MSE.

$$\underset{\beta_0, \beta_1}{\text{minimize}} \sum_i (\beta_0 + \beta_1 x_i - y_i)^2$$



- Half of ML field is construction: NN, SVM, etc.
- In DNN it is an optimization problem of millions of parameters.
- Let's see animation.
- Where are Probability, Statistics, and Linear Algebra here? Let's re-visit the chart.
- Is the optimization problem solvable:
 - closed form? (LSM)
 - numerically and guaranteed? (convex and linear)
 - numerically but not guaranteed? (non-convex):
 - * numerical algorithms, e.g., GD,
 - * local optimization,
 - * heuristics, swarm, and genetics,
 - * brute-force with exhaustive search

1.1.2 Solving Optimization Problems

- A *solution method* for a class of optimization problems is an algorithm that computes a solution.
- Even when the *objective function* and constraints are smooth, e.g., polynomials, the solution is very difficult.
- There are three classes where solutions exist, theory is very well developed, and amazingly found in many practical problems:

Linear \subset Quadratic \subset Convex \subset Non-linear (not linear and not known to be convex!)

- For the first three classes, the problem can be solved very reliably in hundreds or thousands of variables!

1.2 Least-Squares and Linear Programming

1.2.1 Least-Squares Problems

A *least-squares* problem is an optimization problem with no constraints (i.e., $m = p = 0$), and an objective in the form:

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^k (a'_i x - b_i)^2 = \|A_{k \times n} x_{n \times 1} - b_{k \times 1}\|^2.$$

The solution is given in **closed form** by:

$$x = (A' A)^{-1} A' b$$

- Good algorithms in many SC SW exist; it is a very mature technology.
- Solution time is $O(n^2 k)$.
- Easily solvable even for hundreds or thousands of variables.
- More on that in the Linear Algebra course.
- Many other problems reduce to typical LS problem:
 - Weighted LS (to emphasize some observations)

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^k w_i (a'_i x - b_i)^2.$$

- Regularization (to penalize for over-fitting)

$$\underset{x}{\text{minimize}} f_0(x) = \sum_{i=1}^k (a'_i x - b_i)^2 + \rho \sum_{j=1}^n x_j^2.$$

1.2.2 Linear Programming

A *linear programming* problem is an optimization problem with objective and all constraint functions are linear:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) = C'x & \\ \text{subject to:} & a'_i x \leq b_i, & i = 1, \dots, m \\ & h'_i x = g_i, & i = 1, \dots, p, \end{array}$$

- **No** closed form solution as opposed to LS.
- Very robust, reliable, and effective set of methods for numerical solution; e.g., Dantzig's simplex, and interior point.
- Complexity is $\simeq O(n^2 m)$.
- Similar to LS, we can solve a problem of thousands of variables.
- Example is *Chebyshev minimization* problem:

$$\underset{x}{\text{minimize}} f_0(x) = \max_{i=1, \dots, k} |a'_i x - b_i|,$$

- The objective is different from the LS: minimize the maximum error. **Ex:**
- After some tricks, requiring familiarity with optimization, it is equivalent to a LP:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & t & \\ \text{subject to:} & a'_i x - t \leq b_i, & i = 1, \dots, k \\ & -a'_i x - t \leq -b_i, & i = 1, \dots, k \end{array}$$

1.3 Convex Optimization

A *convex optimization* problem is an optimization problem with objective and all constraint function are convex:

$$\begin{array}{llll} \underset{x}{\text{minimize}} & f_0(x) & & \\ \text{subject to:} & f_i(x) \leq 0, & i = 1, \dots, m & \\ & h_i(x) = 0, & i = 1, \dots, p, & \\ & f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y), & \alpha + \beta = 1, & 0 \leq \alpha, 0 \leq \beta, \quad 0 \leq i \leq m \\ & h_i(x) = a'_i x + b_i & & 0 \leq i \leq p \end{array}$$

- The LP and LS are special cases; however, only LS has closed-form solution.
- Very robust, reliable, and effective set of methods, including *interior point methods*.
- Complexity is almost: $O(\max(n^3, n^2 m, F))$, where F is the cost of evaluating 1st and 2nd derivatives of f_i and h_i .
- Similar to LS and LP, we can solve a problem of thousands of variables.
- However, it is not as very mature technology as the LP and LS yet.
- There are many practical problems that can be re-formulated as convex problem **BUT** requires mathematical skills; but once done the problem is solved. **Hint:** realizing that the problem is convex requires more mathematical maturity than those required for LP and LS.

1.4 Nonlinear Optimization

A *non-linear optimization* problem is an optimization problem with objective and constraint functions are non-linear **BUT** not known to be convex (**so far**). Even simple-looking problems in 10 variables can be extremely challenging. Several approaches for solutions:

Local Optimization : starting at initial point in space, using differentiability, then navigate

- does not guarantee global optimal.
- affected heavily by initial point.
- depends heavily on numerical algorithm and their parameters.
- More art than technology.
- In contrast to convex optimization, where a lot of art and mathematical skills are required to formulate the problem as convex; then numerical solution is straightforward.

Global Optimization : the true global solution is found; the compromise is complexity.

- The complexity goes exponential with dimensions.
- Sometimes it is worth it when: the cost is huge, not in real time, and dimensionality is low.

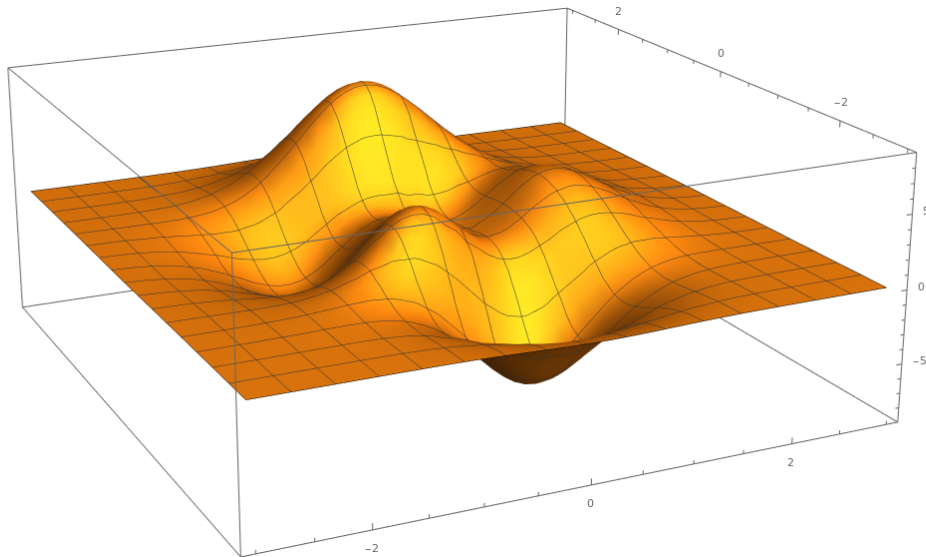
Role of Convex Optimization :

- Approximate the non-linear function to a convex one, finding the exact solution, then using it as a starting point for the original problem. (Also does not guarantee optimality)
- Setting bounds on the global solution.

Evolutionary Computations : Genetic Algorithm (GA), Simulated Annealing (SA), Particle Swarm Optimization (PSO), etc.

Example 5 (Nonlinear Objective Function) : (*Chong and Zak, 2013, Ex. 14.3, P.290*)

$$f(x, y) = 3(1 - x)^2 e^{-x^2 - (y+1)^2} - 10e^{-x^2 - y^2} \left(-x^3 + \frac{x}{5} - y^5 \right) - \frac{1}{3} e^{-(x+1)^2 - y^2}$$



Part I

Theory

Chapter 2

Convex sets

2.1 Affine and convex sets

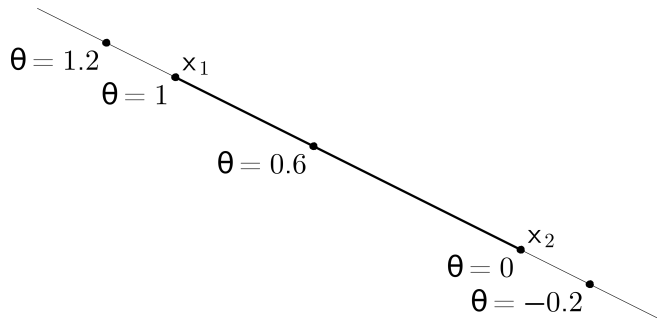
2.1.1 Lines and line segments

Definition 6 (line and line segment) Suppose $x_1 \neq x_2 \in \mathbf{R}^n$. Points of the form

$$\begin{aligned} y &= \theta x_1 + (1 - \theta)x_2 \\ &= x_2 + \theta(x_1 - x_2), \end{aligned}$$

where $\theta \in \mathbf{R}$, form the line passing through x_1 and x_2 .

- As usual, this is a definition for high dimensions taken from a proof for $n \leq 3$.
- We have done it many times: angle, norm, cardinality of sets, etc.
- if $0 \leq \theta \leq 1$ this forms a line segment.



2.1.2 Affine sets

Definition 7 (Affine sets) A set $C \subset \mathbf{R}^n$ is affine if the line through any two distinct points in C lies in C . I.e., $\forall x_1, x_2 \in C$ and $\theta \in \mathbf{R}$, we have $\theta x_1 + (1 - \theta)x_2 \in C$. In other words, C contains any linear combination (summing to one) of any two points in C .

Examples: what about line, line segment, circle, disk, strip, first quadrant?

Corollary 8 Suppose C is an affine set, and $x_1, \dots, x_k \in C$, then C contains every general affine combination of the form $\theta_1 x_1 + \dots + \theta_k x_k$, where $\theta_1 + \dots + \theta_k = 1$.

Wrong Proof. Suppose $y_1, y_2 \in C$, then

$$x = \sum_{i=1}^k \theta_i x_i = \sum_{i=1}^k \theta_i (\alpha_i y_1 + (1 - \alpha_i) y_2);$$

and the summation of the coefficients will be

$$\sum_{i=1}^k \theta_i \alpha_i + \sum_{i=1}^k \theta_i (1 - \alpha_i) = \sum_{i=1}^k \theta_i (\alpha_i + 1 - \alpha_i) = \sum_{i=1}^k \theta_i = 1.$$

Where is the bug?

Correct Proof. base: $k = 3$.

$$\begin{aligned} x &= \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \\ &= (1 - \theta_3) \left(\frac{\theta_1}{1 - \theta_3} x_1 + \frac{\theta_2}{1 - \theta_3} x_2 \right) + \theta_3 x_3. \\ &= (1 - \theta_3)(\cdot \in C) + \theta_3(\cdot \in C). \end{aligned}$$

induction: suppose it is true for some $k \geq 3$; i.e., $\sum_{i=1}^k \theta_i x_i \in C$ when $\sum_{i=1}^k \theta_i = 1$. Then

$$\begin{aligned} x &= \sum_{i=1}^{k+1} \theta_i x_i \\ &= \sum_{i=1}^k \theta_i x_i + \theta_{k+1} x_{k+1} \\ &= (1 - \theta_{k+1}) \sum_{i=1}^k \theta_i / (1 - \theta_{k+1}) x_i + \theta_{k+1} x_{k+1} \\ &= (1 - \theta_{k+1})(\cdot \in C) + \theta_{k+1}(\cdot \in C), \\ &\quad \text{(from the induction hypothesis)} \end{aligned}$$

■ which completes the proof. ■

Definition 9 (Subspace from Linear Algebra) A set $V \subset \mathbf{R}^n$ of vector (here points) is a subspace if it is closed under sums and scalar multiplication. I.e., $\forall v_1, v_2 \in V$ and $\forall \alpha, \beta \in \mathbf{R}$ we have $\alpha v_1 + \beta v_2 \in V$.

Remember:

- $\alpha + \beta$ not necessarily equals 1
- $\alpha = 0, \beta = 0 \rightarrow \mathbf{0} \in V$.
- Any subspace V is the solution set of $A_{m \times n} x_{n \times 1} = \mathbf{0}$, which is $\mathcal{N}(A)$ (the null space of A). Geometry? I.e., $V = \{x \mid Ax = \mathbf{0}\}$
- **rank**(A) = $n - \dim(V)$.

Corollary 10 .

1. If C is affine, then $V = C - x_0 = \{x - x_0 \mid x, x_0 \in C\}$ is a subspace.
2. If V is a subspace, then $C = V + x_0 = \{x + x_0 \mid x \in V\}$ is affine $\forall x_0$.
3. An affine set C can be represented as the solution set of a nonhomogeneous linear system $Ax = b$, where $V = C - x_0$ is $\mathcal{N}(A)$.
4. The solution set of any nonhomogeneous system is an affine set. (Ex. 2.1)

Proof.

1. Suppose $x_1, x_2, x_0 \in C$, an affine set. Both $x_1 - x_0$ and $x_2 - x_0$, by construction, $\in V$; then

$$\begin{aligned} x &= \alpha(x_1 - x_0) + \beta(x_2 - x_0) + x_0 \\ &= \alpha x_1 + \beta x_2 + (1 - \alpha - \beta)x_0 \in C \end{aligned}$$

Then $x - x_0 = \alpha(x_1 - x_0) + \beta(x_2 - x_0) \in V$; hence V is a subspace.

2. Suppose $x_1, x_2 \in V$, a subspace. Both $x_1 + x_0$ and $x_2 + x_0$, by construction, $\in C$; then

$$\begin{aligned} x &= \theta(x_1 + x_0) + (1 - \theta)(x_2 + x_0) \\ &= \theta x_1 + (1 - \theta)x_2 + x_0 = (\cdot \in V) + x_0 \in C \end{aligned}$$

3. If C is affine and $x_0 \in C$, then

$$\begin{aligned} C - x_0 &= \{x \mid Ax = \mathbf{0}\} \quad (\text{since it is a subspace}) \\ C &= \{x + x_0 \mid A(x + x_0) = Ax_0\} \\ C &= \{c \mid Ac = b\}. \end{aligned}$$

4. $C = \{x \mid Ax = b\}$; if $x_0 \in C$ then $Ax_0 = b$ and

$$C - x_0 = \{x - x_0 \mid A(x - x_0) = b - Ax_0 = \mathbf{0}\}.$$

Hence, $C - x_0$ is a subspace and C is affine. ■

Proof of the book. Suppose $x_1, x_2 \in C$, where $C = \{x \mid Ax = b\}$. Then

$$A(\theta x_1 + (1 - \theta)x_2) = \theta Ax_1 + (1 - \theta)Ax_2 = \theta b + (1 - \theta)b = b,$$

which means $\theta x_1 + (1 - \theta)x_2 \in C$ as well. ■

Remark 1 :

- *The dimension of affine is defined to be the dimension of the associate subspace.*
- *affine is a subspace plus offset.*
- *every subspace is affine but not the vice versa; i.e., subspace is a special case of affine.*

Definition 11 (affine hull) The “smallest” set of all affine combinations of some set C (not necessarily affine) is called the affine hull (**aff** C):

$$\mathbf{aff} C = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in C, \sum_i \theta_i = 1 \right\}.$$

Corollary 12 **aff** C is affine.

Proof. For $x_1 = \sum_i \alpha_i x_i$, $\sum_i \alpha_i = 1$, and $x_2 = \sum_i \beta_i x_i$, $\sum_i \beta_i = 1$, we have

$$\theta x_1 + (1 - \theta)x_2 = \theta \sum_i \alpha_i x_i + (1 - \theta) \sum_i \beta_i x_i = \sum_i (\theta \alpha_i + (1 - \theta) \beta_i) x_i$$

$$\sum_i (\theta \alpha_i + (1 - \theta) \beta_i) = \theta \sum_i \alpha_i + (1 - \theta) \sum_i \beta_i = \theta + (1 - \theta) = 1.$$

Hence, **aff** C is affine as well. ■

Example 13 Construct the affine hull of the set $C = \{(-1, 0), (1, 0), (0, 1)\}$

$$\begin{aligned} \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 &= (1 - \theta_3) \left(\frac{\theta_1}{1 - \theta_3} x_1 + \frac{\theta_2}{1 - \theta_3} x_2 \right) + \theta_3 x_3 \\ &= (1 - \alpha_3) \left((1 - \alpha_2) x_1 + \alpha_2 x_2 \right) + \alpha_3 x_3 = (1 - \alpha_2)(1 - \alpha_3) x_1 + \alpha_2(1 - \alpha_3) x_2 + \alpha_3 x_3, \end{aligned}$$

$$\begin{array}{lll} \theta_3 = \alpha_3 & \theta_2 = \alpha_2(1 - \alpha_3) & \theta_1 = 1 - \theta_2 - \theta_3 = (1 - \alpha_2)(1 - \alpha_3) \\ \alpha_3 = \theta_3 & \alpha_2 = \theta_2 / (1 - \theta_3) & \alpha_1 = 1 - \alpha_2 = \theta_1 / (1 - \theta_3). \end{array}$$

HW: Derive expressions for α_i and θ_i for n -point combination.

2.1.3 Affine dimension and relative interior

Definition 14 (some basic topology in \mathbf{R}^n) :

1. The ball of radius r and center x in the norm $\|\cdot\|$.

$$B(x, r) = \{y \mid \|y - x\| \leq r\}.$$

2. An element $x \in C \subseteq \mathbf{R}^n$ is called an interior point of C if $\exists \varepsilon > 0$ for which

$$B(x, \varepsilon) = \{y \mid \|y - x\|_2 \leq \varepsilon\} \subseteq C.$$

I.e., \exists a ball centered at x that lies entirely in C .

3. The set of all points interior to C is called the interior of C and is denoted $\text{int } C$.
4. A set C is open if $\text{int } C = C$. I.e., every point in C is an interior point.
5. A set C is closed if its complement is open

$$\mathbf{R}^n \setminus C = \{x \in \mathbf{R}^n \mid x \notin C\}$$

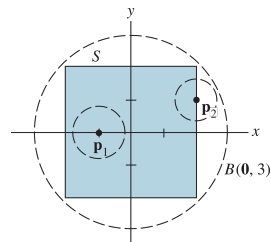
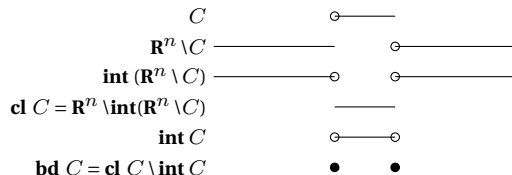
6. The closure of a set C is defined as

$$\text{cl } C = \mathbf{R}^n \setminus \text{int}(\mathbf{R}^n \setminus C).$$

7. The boundary C is defined as

$$\text{bd } C = \text{cl } C \setminus \text{int } C.$$

Corollary 15 A boundary point (a point $x \in \text{bd } C$) satisfies: $\forall \varepsilon > 0, \exists y \in C$ and $z \notin C$ s.t. $y, z \in B(x, \varepsilon)$.



Definition 16 (alter. equiv. def.) :

- $\text{int } C$ and $\text{bd } C$ are defined as 2,3, corollary. (It is obvious that: $\text{int } C \cap \text{bd } C = \phi$.)
- C is open if $\text{int } C = C \iff C \cap \text{bd } C = \phi$.
- C is closed if $\text{bd } C \subseteq C$.
- $\text{cl } C = \text{bd } C \cup \text{int } C$.

Definition 17 We define the affine dimension of a set C as the dimension of its affine hull.

Example 18 The unit circle in \mathbf{R}^2 , i.e., $\{x \mid x_1^2 + x_2^2 = 1\}$ has affine hull of whole \mathbf{R}^2 . So its affine dimension is 2. However, it has a dimensionality of 1 since it is parametric in just one parameter (manifold).

Definition 19 We define the relative interior of the set C , denoted **relint** C , as its interior relative to **aff** C

$$\mathbf{relint} \ C = \{x \in C \mid B(x, r) \cap \mathbf{aff} \ C \subseteq C \text{ for some } r > 0\},$$

and its relative boundary, denoted **relbd** C is defined as

$$\mathbf{relbd} \ C = \mathbf{cl} \ C \setminus \mathbf{relint} \ C.$$

Example 20 Consider a square in the (x_1, x_2) -plane in \mathbf{R}^3 , defined as:

$$C = \{x \in \mathbf{R}^3 \mid -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1, x_3 = 0\}.$$

Then:

$$\mathbf{int} \ C = \Phi$$

$$\mathbf{cl} \ C = \mathbf{R}^n \setminus \mathbf{int}(\mathbf{R}^n \setminus C) = C$$

$$\mathbf{bd} \ C = \mathbf{cl} \ C \setminus \mathbf{int} \ C = C$$

$$\mathbf{aff} \ C = \{x \in \mathbf{R}^3 \mid x_3 = 0\}$$

$$\mathbf{relint} \ C = \{x \in \mathbf{R}^3 \mid -1 < x_1 < 1, -1 < x_2 < 1, x_3 = 0\}$$

$$\mathbf{relbd} \ C = \{x \in \mathbf{R}^3 \mid \max\{|x_1|, |x_2|\} = 1, x_3 = 0\}$$

2.1.4 Convex sets

Definition 21 (convex set) A set C is convex if the line segment between any two points in C lies in C ; i.e., if for any $x_1, x_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we have

$$\theta x_1 + (1 - \theta)x_2 \in C.$$

Corollary 22 Suppose C is convex set, and $x_1, \dots, x_k \in C$, then C contains every general convex combination (also called mixture); i.e.,

$$\sum_i \theta_i x_i \in C, \sum_i \theta_i = 1, \theta_i \geq 0.$$

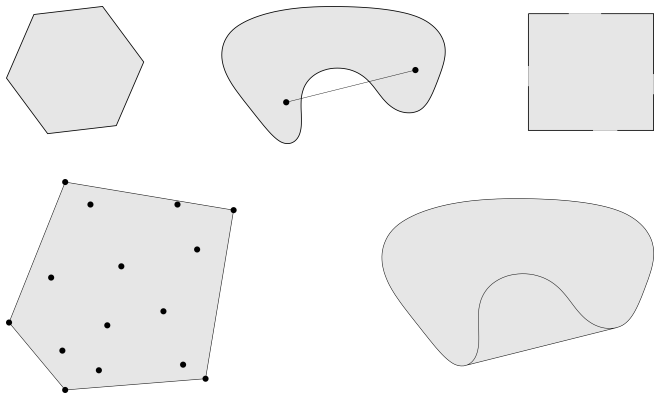
Proof. identical to proof of corollary 8.

Definition 23 (convex hull) The “smallest” set of all convex combinations of some set C (not necessarily convex) is called the convex hull (**conv** C)

$$\text{conv } C = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in C, \sum_i \theta_i = 1, \theta_i \geq 0 \right\}.$$

Corollary 24 **conv** C is convex.

Proof. identical to proof of corollary 12.



■ **Example 25** Revisit example 13.

Example 26 (Applications) : Suppose $X \in C$ is a r.v., C is convex. Then $\mathbb{E}X \in C$ if it exists:

$$\mathbb{E}X = \sum_{i=1}^n p_i x_i$$

$$\mathbb{E}X = \sum_{i=1}^{\infty} p_i x_i$$

$$\mathbb{E}X = \int_C f_X(x) x \, dx \quad (\text{Riemann sum})$$

2.1.5 Cones

Definition 27 A set C is called a cone (or nonnegative homogeneous) if $\forall x \in C, \theta \geq 0$ we have $\theta x \in C$; and it is a convex cone if it is convex in addition to being a cone.

Definition 28 A point of the form $\sum_{i=1}^k \theta_i x_i, \theta_i \geq 0$ is called a conic combination.

Corollary 29 A set C is a convex cone if and only if it contains all conic combinations of its elements; i.e.,

$$\sum_i \theta_i x_i \in C \quad \forall x_i \in C \text{ and } \theta_i \geq 0.$$

Proof.

Sufficiency: is obvious. Choosing $\sum_i \theta_i = 1$ implies C is convex; and setting $\theta_i = 0 \quad \forall i > 1$ implies C is cone.

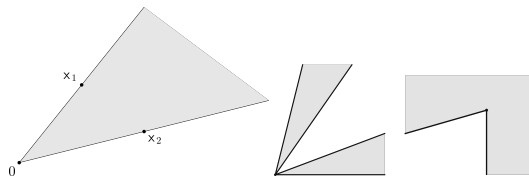
Necessity: Since C is convex cone, then $\forall x_i \in C, \theta_i \geq 0$ we have:

$$\theta_i x_i \in C \quad (\text{cone})$$

$$\sum_i (1/n)(\theta_i x_i) \in C \quad (\text{convex})$$

$$n \sum_i (1/n)(\theta_i x_i) = \sum_i \theta_i x_i \in C \quad (\text{cone})$$

■

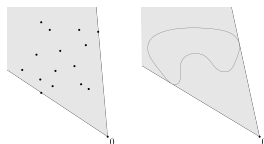


Definition 30 A conic hull of a set C is the minimum set of all conic combination:

$$\text{cone } C = \left\{ \sum_i \theta_i x_i \mid x_i \in C, \theta_i \geq 0, i = 1, \dots, n \right\}.$$

Corollary 31 cone C is convex cone.

Proof. If $y \in \text{cone } C, \alpha \geq 0$, then $\alpha y = \alpha \sum_i \theta_i x_i = \sum_i (\alpha \theta_i) x_i \in \text{cone } C$. And if $y_1, y_2 \in \text{cone } C$ then $\alpha y_1 + (1 - \alpha) y_2 = \alpha \sum_i \theta_i x_i + (1 - \alpha) \sum_i \mu_i x_i = \sum_i (\alpha \theta_i + (1 - \alpha) \mu_i) x_i \in \text{cone } C$ ■



2.2 Some important examples

Fast Revision

- Each of the sets: ϕ , x_0 (a singleton), \mathbf{R}^n are affine and convex.
- Any line is affine. If it passes through zero, it is a subspace and a convex cone.
- Any subspace is convex cone but not vice versa.
- A line segment is convex, but not affine (unless it reduces to a singleton).
- A ray, $\{x_0 + \theta v \mid \theta \geq 0, v \neq 0\}$ is convex but not affine. It is convex cone if $x_0 = 0$.

2.2.1 Hyperplanes and halfspaces

Definition 32 A hyperplane is a set of the form

$$\begin{aligned} \mathcal{S} &= \{x \mid a^T x = b\}, & a, b \in \mathbf{R}^n, a \neq 0 \\ &\equiv \{x \mid a^T(x - x_0) = 0\}, & a^T x_0 = b. \end{aligned}$$

- Vectors with inner product with a is b : $\frac{a^T}{\|a\|}x = \frac{b}{\|a\|}$.
I.e., from $\mathbf{0}$, walk a distance $\frac{b}{\|a\|}$ (either + or -) in the direction of a , then draw perpendicular line.

Definition 33 A closed halfspace is the region generated by the hyperplane and defined as:

$$\begin{aligned} \mathcal{H} &= \{x \mid a^T x \leq b\}, & a, b \in \mathbf{R}^n, a \neq 0 \\ &\equiv \{x \mid a^T(x - x_0) \leq 0\}, & a^T x_0 = b. \end{aligned}$$

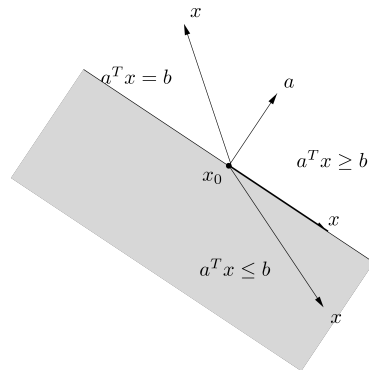
- region of all vectors with projection $< b/\|a\|$.
- Vectors with obtuse angle with a : ($\cos \theta = \frac{a^T x}{\|a\|\|x\|}$).
- Line passing with p_0 and \perp on \mathcal{S} :

$$x = p_0 + \theta \bar{a} \quad (\text{parametric eq.})$$

$$a^T x_0 = a^T p_0 + \theta_0 \|a\|$$

$$\theta_0 = (b - a^T p_0) / \|a\| \quad (x_0 \text{ pt. of intersection.})$$

$$x_0 - p_0 = \frac{(b - a^T p_0)}{\|a\|} \bar{a}.$$



Corollary 34 \mathcal{S} is affine, \mathcal{H} is convex and not affine, $\text{int } \mathcal{H} = \mathcal{H} \setminus \mathcal{S}$, and $\text{bd } \mathcal{H} = \mathcal{S}$.

Proof. \mathcal{S} is affine done. \mathcal{H} is convex: take $0 \leq \theta \leq 1$
 $\theta a^T x_1 + (1 - \theta) a^T x_2 \leq \theta b + (1 - \theta)b = b$. (why not affine?!)

$$y = x + ru, \quad 0 \leq \|u\| \leq 1 \quad (y \in B(x, r))$$

$$a^T y = a^T x + r a^T u = b - (b - a^T x) + r \|a\| \|u\| \cos(a, u)$$

If $b = a^T x$, i.e., $x \in \mathcal{S}$, $a^T u > 0$ or < 0 (depending on the angle) and hence $a^T y > b$ or $< b$. Then $\mathcal{S} \subseteq \text{bd } \mathcal{H}$.

If $a^T x < b$, i.e., $x \in \mathcal{H} \setminus \mathcal{S}$, $\exists r < \frac{b - a^T x}{\|a\|}$, s.t. $a^T y < b$. Hence:
 $\text{int } \mathcal{H} = \mathcal{H} \setminus \mathcal{S}$ and $\text{bd } \mathcal{H} = \mathcal{S}$. ■

2.2.2 Euclidean balls and ellipsoids

Definition 35 A Euclidean ball in \mathbf{R}^n is the set:

$$\begin{aligned} B(x_c, r) &= \{x = x_c + ru \mid \|u\|_2 \leq 1\} \\ &= \{x \mid \|x - x_c\|_2 / r \leq 1\} \\ &= \{x \mid (x - x_c)'(x - x_c) / r^2 \leq 1\}. \end{aligned}$$

Definition 36 Ellipsoid in \mathbf{R}^n is the set:

$$\begin{aligned} \mathcal{E} &= \{x = x_c + Au \mid \|u\|_2 \leq 1, A > 0\} \\ &= \{x \mid \|A^{-1}(x - x_c)\|_2 \leq 1, A > 0\} \\ &= \{x \mid (x - x_c)'(A^{-1})'A^{-1}(x - x_c) \leq 1\} \end{aligned}$$

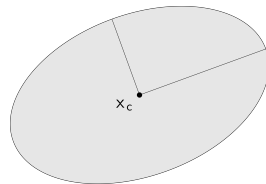
Spectral decomposition for $A = A'$.

$$\begin{aligned} Au &= (\lambda_1 v_1 v_1' + \lambda_2 v_2 v_2' + \cdots + \lambda_n v_n v_n')u \\ &= \lambda_1 v_1 (v_1' u) + \lambda_2 v_2 (v_2' u) + \cdots + \lambda_n v_n (v_n' u), \end{aligned}$$

which reduces to a Ball when $\lambda_i = r$.

Remark 2 A does not have to be symmetric; put:

$$\begin{aligned} P^{-1} &= (A^{-1})'A^{-1} = V\Sigma^{-1/2}\Sigma^{-1/2}V' && \text{symmetric} \\ P^{1/2}u_2 &= Au_1 && \text{is bijection} \\ \|u_2\|^2 &= u_1' A' P^{-1/2} P^{-1/2} A u_1 = \|u_1\|^2 \end{aligned}$$



Remark 3 (Contours of $\mathcal{N}(\mu, \Sigma)$) :

$$f_X(x) = \frac{1}{((2\pi)^p |\Sigma|)^{1/2}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

Corollary 37 An ellipsoid, hence a ball, is convex

Proof. For $x_1, x_2 \in \mathcal{E}, 0 \leq \theta \leq 1$,

$$\begin{aligned} x_1 &= x_c + Au_1, \|u_1\| \leq 1 \\ x_2 &= x_c + Au_2, \|u_2\| \leq 1 \\ x &= \theta(x_c + Au_1) + (1 - \theta)(x_c + Au_2) \\ &= x_c + A(\theta u_1 + (1 - \theta)u_2) \\ \|u\| &= \|\theta u_1 + (1 - \theta)u_2\| \\ &\leq \theta\|u_1\| + (1 - \theta)\|u_2\| \\ &\leq \theta + (1 - \theta) = 1. \end{aligned}$$

2.2.3 Norm balls and norm cones

Definition 38 (Norm) Let $x, y \in \mathbf{R}^n$, $t \in \mathbf{R}$; a function $f: \mathbf{R}^n \mapsto \mathbf{R}_+$ with **dom** $f = \mathbf{R}^n$ is called a norm if

1. $f(x) = 0 \rightarrow x = 0$ (definite)
 2. $f(tx) = |t|f(x)$ (homogeneous)
 3. $f(x+y) \leq f(x) + f(y)$ (triangle inequality)
- $f(0) = 0$ is implied from (2) (positive definite)

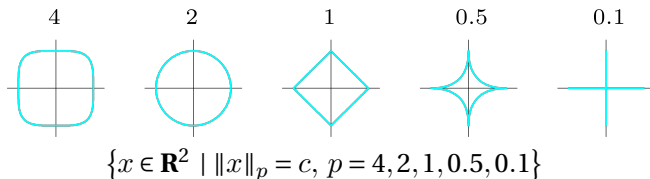
Definition 39 (L^p -norm ($\|\cdot\|_p$)) is defined as

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} = (|x_1|^p + \dots + |x_n|^p)^{1/p}.$$

Proof of $\|\cdot\|_p$ is a norm. :

1. $\left(\sum_{i=1}^n |x_i|^p \right)^{1/p} = 0 \rightarrow \sum_{i=1}^n |x_i|^p = 0 \rightarrow x_i = 0.$
2. $\|tx\|_p = \left(\sum_{i=1}^n |tx_i|^p \right)^{1/p} = |t| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} = |t| \|x\|_p$
3. $\|x+y\|_p \leq \|x\|_p + \|y\|_p$:
(Minkowski's inequality proof (Kreyszig, 1978))

counter example for $p < 1$: $\|(0, 1)\|_{1/2} + \|(1, 0)\|_{1/2} = 1 + 1 = 2$, whereas $\|(1, 1)\|_{1/2} = (1 + 1)^{1/(1/2)} = 4$. ■



- L_1 -norm, Manhattan dist., Taxicab, abs. value

$$\|x\|_1 = \left(\sum_{i=1}^n |x_i| \right).$$

- L_2 -norm, Euclidean distance (most meaningful)

$$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}.$$

- L_∞ -norm

$$\|x\|_\infty = \lim_{p \rightarrow \infty} \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} = \max_i |x_i|$$

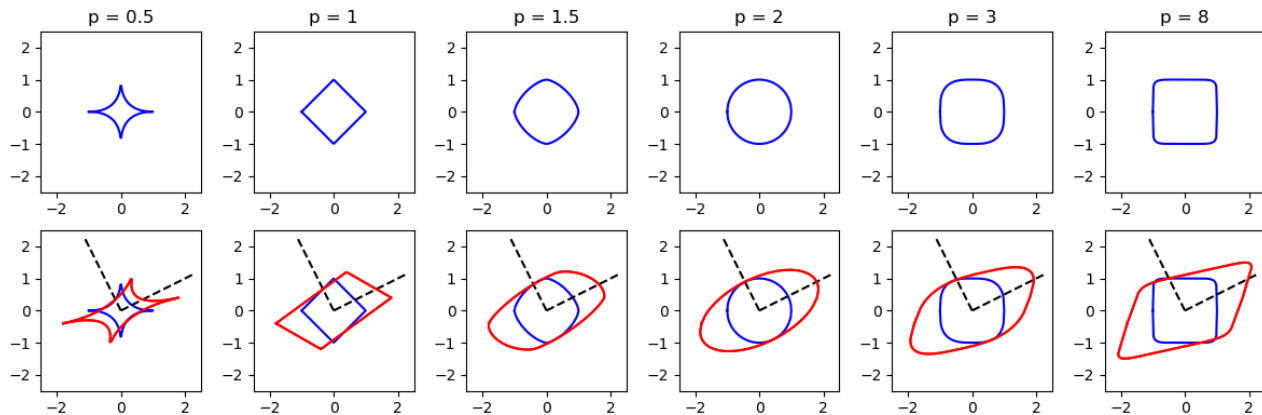
Corollary 40 (properties of $\|\cdot\|_p$) :

1. $|x_i| < \|x\|_p \quad \forall p < \infty.$
2. L_P -norm is monotonic in p .

■ **Proof.** is HW. ■

Definition 41 (A general norm ellipsoid in \mathbb{R}^n) is the set generated by a norm ball, for any norm $\|\cdot\|$, of radius r , centered at x_c , and transformed by any symmetric matrix $A > 0$:

$$\mathcal{E} = \{x = x_c + Au \mid \|u\|_p \leq 1, A > 0\} \equiv \{x \mid \|A^{-1}(x - x_c)\|_p \leq 1, A > 0\}. \quad (2.1)$$



- $A = \lambda_1 v_1 v_1' + \lambda_2 v_2 v_2'$, $v_1 = (2, 1)' / \sqrt{5}$, $v_2 = (-1, 2)' / \sqrt{5}$, $\lambda_1 = 2$, $\lambda_2 = 1$.
- The unit ball intersect with the ellipsoid at v_2 ; why? The ellipsoids, of course, no longer have unit L_p -norm.

Corollary 42 The general ellipsoid (2.1) is convex.

Definition 43 (Metric) Let $x, y, z \in S$, a function $\delta : S \times S \mapsto \mathbf{R}_+$ is called a metric on S if:

1. $\delta(x, y) = 0 \leftrightarrow x = y$ (positive definite)
2. $\delta(x, y) = \delta(y, x)$ (symmetric)
3. $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$ (triangle inequality)

Lemma 44 : $\delta(x, y) = f(x - y)$ is a metric ($\delta(x, 0) = f(x)$):

$$\delta(x, y) = f(x - y) = 0 \leftrightarrow x - y = 0 \leftrightarrow x = y$$

$$\delta(x, y) = f(x - y) = f(-1(y - x)) = f(y - x) = \delta(y, x)$$

$$\begin{aligned} \delta(x, y) &= f(x - y) = f((x - z) + (z - y)) \\ &\leq f(x - z) + f(z - y) = \delta(x, z) + \delta(y, z) \end{aligned}$$

Definition 45 (Loss) Let $x \in S_1$ (called set of nature), and $y \in S_2$ (called set of actions); then a function $L : S_1 \times S_2 \mapsto \mathbf{R}_+$ is called a loss incurred from assigning the action y based on the truth of nature x . For details on loss and utility theory see [Berger \(1993\)](#).

Remark 4 :

- Not any metric defines a norm; e.g., $\delta(x, y) = I_{x \neq y}$: First, prove it is a metric (HW). Then: $\delta(x, 0) = f(x) = 1 \neq 10 = f(10x) = \delta(10x, 0) \quad \forall x \neq 0$.
- Why? metric suites any set even categorical.
- Loss do not have to follow metric properties at all; e.g., $L(P, N) \neq L(N, P)$ in medical classification prediction.

Norm balls and norm cones

Definition 46 The norm cone associated with any norm $\|x\|$, $x \in \mathbf{R}^n$ is the set

$$C = \{(x, t)' \in \mathbf{R}^{n+1} \mid \|x\| \leq t, t > 0\}$$

Example 47 The second-order cone is the norm cone for the Euclidean norm; i.e.,

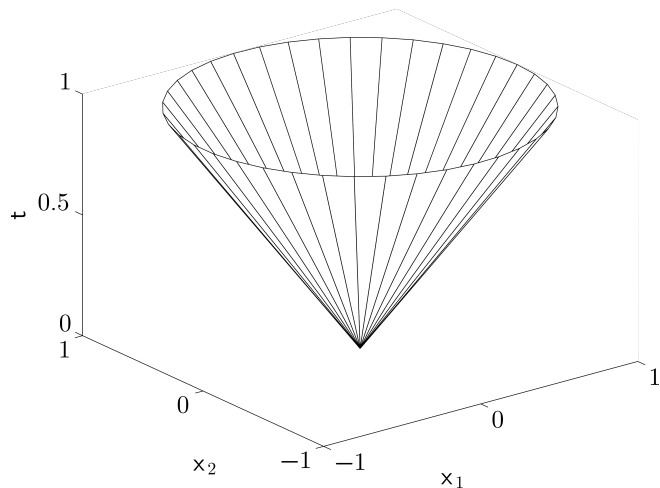
$$\begin{aligned} C &= \{(x, t)' \in \mathbf{R}^{n+1} \mid \|x\|_2 \leq t\} \\ &= \{(x, t)' \mid x'x \leq t, t > 0\} \end{aligned}$$

Corollary 48 The norm cone is convex.

Proof. : given $p_i = (x_i, t_i)$, $\|x_i\| \leq t_i$, $i = 1, 2$, then

$$\begin{aligned} p &= \theta(x_1, t_1) + (1 - \theta)(x_2, t_2) = (x, t) \\ &= (\theta x_1 + (1 - \theta)x_2, \theta t_1 + (1 - \theta)t_2) \\ \|x\| &= \|\theta x_1 + (1 - \theta)x_2\| \\ &\leq \theta \|x_1\| + (1 - \theta)\|x_2\| \\ &\leq \theta t_1 + (1 - \theta)t_2 = t. \end{aligned}$$

■



To imagine it: pay attention to that the radius is the same as the height t . Therefore, the cross section is convex and in the z -direction is convex as well. E.g.,

$$C = \{(x, t^2) \in \mathbf{R}^{n+1} \mid \|x\| \leq t, t > 0\}$$

is convex; however:

$$C = \{(x, \sqrt{t}) \in \mathbf{R}^{n+1} \mid \|x\| \leq t, t > 0\}$$

is not.

2.2.4 Polyhedra

(Remember the early definition 1.2.2).

Definition 49 A polyhedron is defined as the solution set of a finite number of linear qualities and inequalities:

$$\mathcal{P} = \{x \mid a'_j x \leq b_j, j = 1, \dots, m, \quad c'_j x = d_j, j = 1, \dots, p\}.$$

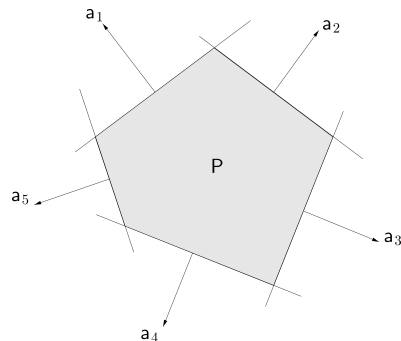
For short notation, we write

$$\mathcal{P} = \{x \mid Ax \leq b, Cx = d\}, \quad A = \begin{pmatrix} a'_1 \\ \vdots \\ a'_m \end{pmatrix}, \quad C = \begin{pmatrix} c'_1 \\ \vdots \\ c'_m \end{pmatrix}.$$

The polyhedron is called polytope if it is bounded.

Example 50 The nonnegative orthant

$$\mathbf{R}_+^n = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \dots, n\} = \{x \in \mathbf{R}^n \mid x \geq 0\}$$



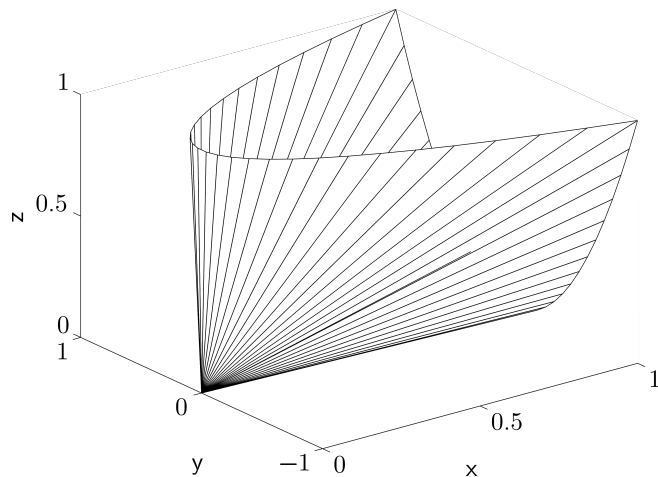
Corollary 51 The polyhedron is convex.

Proof. $x_1, x_2 \in \mathcal{P}$, $x = \theta x_1 + (1 - \theta)x_2$, $0 \leq \theta \leq 1$. Then:

$$\begin{aligned} a'_j x &= \theta a'_j x_1 + (1 - \theta)a'_j x_2 \\ &\leq \theta b_j + (1 - \theta)b_j = b_j \\ c'_j x &= \theta c'_j x_1 + (1 - \theta)c'_j x_2 \\ &= \theta d_j + (1 - \theta)d_j = d_j. \end{aligned}$$

Hence, all conditions are satisfied; the proof is complete. ■

2.2.5 The positive semidefinite cone



2.3 Operations that preserve convexity

2.4 Generalized inequalities

2.5 Separating and supporting hyperplanes

2.6 Dual cones and generalized inequalities

Part II

Applications

Part III

Algorithms

Bibliography

Bagirov, A. (2014), *Introduction to nonsmooth optimization : theory, practice and software*, Cham: Springer.

Berger, J. O. (1993), *Statistical decision theory and Bayesian analysis*, New York: Springer-Verlag, 2nd ed.

Bertsekas, D. (2003), *Convex analysis and optimization*, Belmont, Mass: Athena Scientific.

Borwein, J. (2006), *Convex analysis and nonlinenar optimization : theory and examples*, New York: Springer.

Boyd, S. and Vandenberghe, L. (2004), *Convex Optimization*, Cambridge: Cambridge University Press.

Chong, E. K. and Zak, Stanislaw, H. (2013), *An Introduction to Optimization*, Wiley-Interscience, 4th ed.

Dattorro, J. (2009), *Convex Optimization & Euclidean Distance Geometry*, California: $\mathcal{M}\epsilon\beta\mathcal{O}\mathcal{O}$ Publishing.

Hiriart (2001), *Fundamentals of convex analysis*, Berlin New York: Springer.

Kreyszig, E. (1978), *Introductory functional analysis with applications*, New York: Wiley.

Luenberger, D. (1968), *Optimization by vector space methods*, New York: Wiley.

Rockafellar, R. (1997), *Convex analysis*, Princeton, N.J: Princeton University Press.

Rudin, W. (1976), *Principles of mathematical analysis*, New York: McGraw-Hill, 3rd ed.

Wade, W. R. (2000), *An introduction to analysis*, Upper Saddle River, NJ: Prentice Hall, 2nd ed.