Digital Signal Processing Lab
Paper Report

Wiener Filters for Audio and Image Processing

1 About Wiener Filters

Wiener filters approximate a signal from an input signal using a statistical approach. This statistical approach is often termed as a Least Squares Optimisation Problem or the MSE use-case, i.e. minimising the mean square error in the predicted signal and predicting the correct signal.

In Figure 1, we infer that for the input signal $x[n]$ we predict a signal $d[n]$ by reducing the error in the signal, $e[n]$.

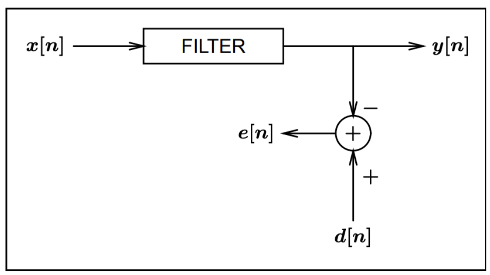

Figure 1: Wiener Filter System, Source: Stanford EE264:Lec12

This error in a signal is also termed as the blurriness or noise in various applications. Wiener filters are very common and are used everywhere for noise reduction and for suppressing them in audio and images. This report will discuss both the applications.

2 Math around Wiener Filters

For any least squares problem, we need a cost function or loss function which is to be minimized. For us the squared loss here is simply the average of the difference between the predicted signal and the actual signal which we can write as

$$M.S.E. = E[(d[n] - y[n])^2] \tag{1}$$

We know that for a given input signal $x[n]$, and where $h[n]$ is the impulse response for any arbitrary filter used to produce the signal, then our predicted signal can be written as,

$$d[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \tag{2}$$

Substituting back $d[n]$ in the Equation (1), we get M.S.E as

$$M.S.E. = E[(\sum_{m=-\infty}^{\infty} h[m]x[n-m] - y[n])^2] \tag{3}$$

Assumptions before we move ahead,
1) We assume a Gaussian error, i.e., our error in the signal is equally distributed, i.e. has zero mean.
2) The M.S.E. used to solve for the value of $h[n]$, is convex and is differentiable. 3) Our input signal has a constant mean and variance.

Now we differentiate the M.S.E. w.r.t h[m] as we need to find the optimal h[m] which is called the Wiener Filter.

$$\frac{\partial M.S.E.}{\partial h[m]} = E[2(\sum_{m=-\infty}^{\infty} h[m]x[n-m] - y[n])x[n-m]]$$

Equating $\frac{\partial M.S.E.}{\partial h[m]}$ with 0 and substituting back $\sum_{m=-\infty}^{\infty} h[m]x[n-m] - y[n]$ as $e[n]$,

$$E[e[n]x[n-m]] = 0 \tag{4}$$

To define the correlation of two signals we use the term

$$\phi_{xy} = E[x[n]y[n+k]]$$

Using the above relation, Eq. (4) is basically,

$$E(d[n]{-}y[n])x[n{-}m] = 0$$

$$\phi_{dx} - \phi_{yx} = 0$$

$$\phi_{dx} = \phi_{yx} \tag{5}$$

Now we know that,

$$\phi_{dx} = H^* \phi_{xx} \tag{6}$$

Substituting ϕ_{dx} from (6) in (5), we arrive at the following relation for the optimal filter,

$$H^* = \frac{\phi_{yx}}{\phi_{xx}} \tag{7}$$

This relation tells us that the impulse response of the optimal filter depends on the cross-correlation of the input x[n] and y[n] and the autocorrelation of x[n]. This is also often called the transfer function for the Wiener filter.

3 Application in Real World Data

Signals are usually not always zero mean and the images, audio samples we obtain in real world might have noise added to it from before.
Hence, Wiener Filters have applications in both the scenarios:

- If signals need to be estimated from a given impulse response and input sample we call that Signal Estimation and,
- Deblurring or Noise Smoothing: Adding noise to a blurred or filtered signal from a stable system, which will eventually deblur the original signal.

3.1 Estimation

If we have an input signal which is a function of the output signal and a added noise,

$$x[n] = y[n] + v[n]$$

In this case if we use the equation (7), our impulse response for the optimal filter will be,

$$H^* = \frac{\phi_{yx}}{\phi_{yx} + \phi_{vv}}$$

Thus we can conclude that if the noise v[n] is too high we our impulse predicted will be very close to 0 and very close to 1 if the noise is almost 0.

3.2 Denoising

While there are various ways to reduce the noise from a given signal using Wiener Filters, this paper [3] discusses a technique which is a modification of the same. Going by the fact that images do not always have noise which is globally stationary, this paper [3] discussed here concentrates at a local area of the image, the technique called as locally adaptive Wiener filters. This report interprets and discusses local adaptive wiener filtering using the concepts and equations derived in Section 2.

Considering that we divide the images into rows and columns of these local samples (spatial data), we can write every element as $x(i, j)$ and the noisy image as a summation of the input image and the noise signal as a linear equation.

Assumptions to be made here,
1. Noise in image is spread not globally but locally at different statistical levels.
2. Noise is additive and ϕ_{nn} is not 0, i.e. the noise is of zero mean but uncorrelated.
3. We are dealing with a Non Stationary Mean, $E(x(i, j))$ and a Non Stationary Variance ($\sigma^2_{x(i,j)}$) at any spatial position (i,j) in the image, called as the NMNV image model which as described in the paper means that any image with noise can be decomposed as a statistical formulation of these factors and the stationary mean. The non stationary mean and the stationary mean both describe the overall structure of the image thus it makes sense to

formulate the linear adaptive filters in terms of these factors.

Mathematically the noisy image here i.e. y,

$$y(i,j) = Bx(i,j) + e(i,j)$$

where B is a blurring matrix, x is the input and e is the additive noise in the signal and we formulate $\hat{x}$, which is the denoised image.

3.2.1 Statistical Inference

We know that variance in terms of mean can be written as $V = E[(x - E(x))^2]$, similarly since these are matrices we can write,

$$\sigma^2 = E[(x - E(x))(x - E(x))^T] \tag{8}$$

From (3) we know that M.S.E. is $E[(y - Hx)^2]$, so the min. M.S.E. when derived in terms of y and $H^* = \frac{\phi_{yx}}{\phi_{xx}}$ gives us the equation,

$$\phi_e = \phi_y - \phi_{\hat{x}} = \phi_y - \phi_{yx}^T {\phi_x}^{-1} \phi_{yx} = 0 \tag{9}$$

Now using, $Cov_{xy} = \frac{\phi_{xy}}{\sigma_x \sigma_y}$ and (8) in (9), the formulation of the above equation as a detailed linear minimum mean square error estimator is written as the sum of means and the covariances.

$$\hat{x}(i,j) = E(x(i,j)) + Cov_{x(i,j)} Cov_y(i,j)^{-1}(y(i,j) - E(y(i,j)))$$

3.2.2 Conclusion

As assumed in the NMNV model that the images carry statistical information at local (spatial) level, the formula to predict this likelihood comes down to the summation of all such local spaces. Further proofs in this paper tell us that eventually the linear local MMSE is a weighted sum of spatial mean $E(x(i,j))$ and the additive term which is called the signal to noise ratio(SNR). The lower the SNR the image has more noise and thus the weights are more towards the values of $E(x(i,j))$ and for a higher SNR the image is sharper towards the edges.

$$\hat{x}(i,j) = E(x(i,j)) + \frac{\sigma_x^2(i,j)}{\sigma_x^2(i,j) + \sigma_e^2(i,j)}(y(i,j) - E(y(i,j)))$$

References

[1] MIT Open Course Ware, Signals, Systems and Inference, Chapter 11: Wiener Filtering
[2] Rice University, ELEC431: Wiener Filtering and Image Processing
[3] D. T. Kuan, A. A. Sawchuk, T. C. Strand and P. Chavel, "Adaptive Noise Smoothing Filter for Images with Signal-Dependent Noise," in IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. PAMI-7, no. 2, pp. 165-177, March 1985.