3. $f(x) = \sqrt{1+x^2} = (1+x^2)^{1/2}$ $\quad f(0) = 1$

$f'(x) = 1/2(1+x^2)^{-1/2}(2x) = x(1+x^2)^{-1/2}$ $\quad f'(0) = 0$

$f''(x) = x(-1/2)(1+x^2)^{-3/2}\,2x + (1+x^2)^{-1/2}$ $\quad f''(0) = 1$

$= -x^2(1+x^2)^{-3/2} + (1+x^2)^{-1/2}$

$f'''(x) = -x^2(-3/2)(1+x^2)^{-5/2}(2x) + (1+x^2)^{-3/2}(-2x) +$ $\quad f'''(0) = 0$

$-1/2(1+x^2)^{-3/2}(2x)$

$= 3x^3(1+x^2)^{-5/2} - 2x(1+x^2)^{-3/2} - x(1+x^2)^{-3/2}$

$f^4(x) = 3x^3(-5/2)(1+x^2)^{-7/2}(2x) + (1+x^2)^{-5/2}(9x^2)$ $\quad f^4(0) = 3$

$- 2x(-3/2)(1+x^2)^{-5/2}(2x) + (1+x^2)^{-3/2}(2)$

$- x(-3/2)(1+x^2)^{-5/2}(2x) + (1+x^2)^{-3/2}$

$= -15x^4(1+x^2)^{-7/2} + 9x^2(1+x^2)^{-5/2}$

$+ 6x^2(1+x^2)^{-5/2} + 2(1+x^2)^{-3/2}$

$+ 3x^2(1+x^2)^{-5/2} + (1+x^2)^{-3/2}$

$f^5(x) = -15x^4(-7/2)(1+x^2)^{-9/2}(2x) + (1+x^2)^{-7/2}(-60x^3)$ $\quad f^5(0) = 0$

$+ 9x^2(-5/2)(1+x^2)^{-7/2}(2x) + (1+x^2)^{-5/2}(18x)$

$+ 6x^2(-5/2)(1+x^2)^{-7/2}(2x) + (1+x^2)^{-5/2}(12x)$

$+ 2(-3/2)(1+x^2)^{-5/2}(2x)$

$+ 3x^2(-5/2)(1+x^2)^{-7/2}(2x) + (1+x^2)^{-5/2}(6x)$

$+ -3/2(1+x^2)^{-5/2}(2x)$

$= 105x^5(1+x^2)^{-9/2} - 60x^3(1+x^2)^{-7/2}$

$-45x^3(1+x^2)^{-7/2} + 18x(1+x^2)^{-5/2}$

$-30x^3(1+x^2)^{-7/2} + 12x(1+x^2)^{-5/2}$

$-6x(1+x^2)^{-5/2} - 15x^3(1+x^2)^{-7/2}$

$+ 6x(1+x^2)^{-5/2} - 3x(1+x^2)^{-5/2}$

$f^6(x) = 105x^5(-9/2)(1+x^2)^{-11/2}(2x) + (1+x^2)^{-9/2}(525x^4)$ $\quad f^6(0) = 45$

$- 60x^3(-7/2)(1+x^2)^{-9/2}(2x) + (1+x^2)^{-7/2}(180x^2)$

$-45x^3(-7/2)(1+x^2)^{-9/2}(2x) + (1+x^2)^{-7/2}\,135x^2$

$+ 18x(-5/2)(1+x^2)^{-7/2}(2x) + (1+x^2)^{-5/2}(18)$

$-30x^3(-7/2)(1+x^2)^{-9/2} + (1+x^2)^{-7/2}(90x^2)$

$+ 12x(-5/2)(1+x^2)^{-7/2}(2x) + (1+x^2)^{-5/2}(12)$

$- 6x(-5/2)(1+x^2)^{-7/2}(2x) + (1+x^2)^{-5/2}(6)$

$-15x^3(-7/2)(1+x^2)^{-9/2}(2x) + (1+x^2)^{-7/2}(45x^2)$

$+ 6x(-5/2)(1+x^2)^{-7/2}(2x) + (1+x^2)^{-5/2}(6)$

$- 3x(-5/2)(1+x^2)^{-7/2}(2x) + (1+x^2)^{-5/2}(3)$

$P_6(x) = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6$

Answers 1.1

5. $R(x) = \dfrac{|x|^6 e^{\xi}}{6!}$ $\quad x \in [-1/2, 1/2]$

$$e^{-1/2} \le e^{\xi} \le e^{1/2}$$

$$\frac{e^{1/2}}{64 \cdot 720}$$

7. $R_n(x) = \dfrac{(x-x_0)^{n+1} f^{(n+1)}(\xi_x)}{(n+1)!}$

$$\frac{x^{n+1} f^{(n+1)}(\xi_x)}{(n+1)!}$$

$$\frac{x^{n+1} e^{\xi}}{(n+1)!}$$

$$|R_n(x)| = \frac{(1/2)^{n+1} e^{1/2}}{(n+1)!} \le 10^{-3}$$

$n = 4$ $\quad \dfrac{(1/2)^5 e^{1/2}}{5!} = 4.29 \times 10^{-4}$

11. a) $f(x) = e^{-x}, \quad x \in [0,1]$

$f'(x) = -e^{-x}$

$f''(x) = e^{-x}$

$f'''(x) = -e^{-x}$

$f(0) = 1$

$f'(0) = -1$

$f''(0) = 1$

$f'''(0) = -1$

$$P_3(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

$$R_3(x) = \frac{x^4}{24} e^{-\xi}$$

$$|R_3(x)| \leq \frac{1}{24} e$$

b) $f(x) = \ln(1+x), \quad x \in [-1,1]$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f(0) = 0$

$f'(0) = 1$

$f''(x) = -(1+x)^{-2}$

$f'''(x) = 2(1+x)^{-3}$

$f''''(x) = -6(1+x)^{-4}$

$f''(0) = -1$

$f'''(0) = 2$

$$P_3(x) = x - \frac{x^2}{2} + \frac{2 \cdot x^3}{3!} = x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$R_3(x) = \frac{-x^4}{4(1+\xi_x)^4}$$

$x = -1$ is the upper bound but causes division by zero

c) $f(x) = \sin x$, $x \in [0, \pi]$

$f(0) = 0$

$f'(x) = \cos x$

$f'(0) = 1$

$f''(x) = -\sin x$

$f''(0) = 0$

$f'''(x) = -\cos x$

$f'''(0) = -1$

$f''''(x) = \sin x$

$f^5(x) = \cos x$

$P_3(x) = x - \frac{x^3}{6}$

$R_3(x) = \frac{x^5}{120} \cos(\xi_x)$

$|R_3(x)| = \left| \frac{x^5}{120} \cos(\xi_x) \right|$

$\frac{x^5}{120} |\cos(\xi_x)|$

$0 \le \cos(\xi_x) \le \frac{\pi^5}{120}$

d) $f(x) = \ln(1+x)$, $x \in [-1/2, 1/2]$

$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$

$R_3(x) = \frac{-x^4}{4} \frac{1}{(1+\xi_x)^4}$

$|R_3(x)| = \left| \frac{-x^4}{4} \frac{1}{(1+\xi_x)^4} \right|$

$\left| \frac{-x^4}{4} \right| \frac{1}{(1+\xi_x)^4}$

$\frac{(1/2)^4}{4} \frac{1}{(3/2)^4} \le \frac{1}{(1+\xi_x)^4} \le \frac{(1/2)^4}{4} \frac{1}{(1/2)^4} = \frac{1}{4}$

23. $R_k(x) = \frac{f^{(k+1)}(\xi_x)(x-x_0)^{k+1}}{(k+1)!}$

$$R_k(x) = \frac{1}{k!}\int_{x_0}^{x}(x-t)^k f^{(k+1)}(t)\,dt$$

$$R_k(x) = \frac{f^{(k+1)}(z)}{k!}\int_{x_0}^{x}(x-t)^k\,dt$$

$$\frac{f^{(k+1)}(z)}{k!}\cdot\left[\frac{(x-t)^{k+1}}{k+1}\right]_{x_0}^{x}$$

$$-\frac{\cancel{(x-x)^{k+1}}}{k+1} - \left(-\frac{(x-x_0)^{k+1}}{k+1}\right)$$

$$\frac{f^{(k+1)}(z)}{k!}\frac{(x-x_0)^{k+1}}{k+1} \qquad k!\cdot(k+1)=(k+1)!$$

$$\frac{f^{(k+1)}(z)(x-x_0)^{k+1}}{(k+1)!}$$

27.

$$S = \sum_{k=1}^{n} a_k f(x_k) \le f_M \sum_{k=1}^{n} a_k = f_M \qquad \text{since } \sum_{k=1}^{n} a_k = 1$$

And

$$S = \sum_{k=1}^{n} a_k f(x_k) \ge f_m \sum_{k=1}^{n} a_k = f_m$$

Therefore $f_m \le S \le f_M$

Let $w = S$

Since w is always greater than or equal to the function minimum and greater than or equal to the function maximum then if $\eta \in [a,b]$ there must be some $f(\eta)$ such that $f(\eta) = w$ because the function is continuous