Path-dependent Options

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Contents

So far, the options we have discussed are vanilla options. We will explore some exotic options, such as barrier options, Asian options, and lookback options, whose payoffs are path-dependent.

1 Barrier Options

Barrier (or knock-in, knock-out) options are activated when the underlying asset hits a prescribed value at some time before expiry. For example, as long as the asset remains below a pre-determined barrier price throughout the option's life, the contract will have a call payoff at expiry. However, should the asset reach this level before expiry, then the option becomes worthless because it has "knocked out". Barrier options are clearly path-dependent. A barrier option is cheaper than a similar vanilla option since it provides the holder with fewer rights.

1.1 Types of barrier options

There are two main types of barrier options:

- An out option only pays off if its barrier level is not reached. If the barrier is reached, then the option is said to have knocked out;
- An in option pays off as long as its barrier level is reached before expiry. If the barrier is reached, then the option is said to have knocked in.

Barrier options can be further characterized by the position of the barrier relative to the initial value of the underlying:

- If the barrier is above the initial asset value, the option is an up option.
- If the barrier is below the initial asset value, the option is a down option.

Finally, barrier options can also be classified as calls or puts according to the payoffs. In addition, if early exercise is permitted, the option is known as the *American-style* barrier option. In what follows, barrier options always refer to European-style options unless otherwise stated.

1.2 In-out parity

For the European style barrier option, the relationship between an 'in' barrier option and an 'out' barrier option (with the same payoff and the same barrier level) is very simple:

$$\text{in} + \text{out} = \text{vanilla}.$$

If the 'in' barrier is triggered, so is the 'out' barrier. Thus, no matter whether or not the barrier is triggered, the portfolio of 'in' and 'out' options has the vanilla payoff at expiry.

However, the above in-out parity does not hold for American-style barrier options.

1.3 Pricing by Monte-Carlo simulation

For illustration, consider an up-out-call option whose terminal payoff can be written as

$$(S_T - X)^+ \mathrm{I}_{\{S_t < H,\ t\in[0,T]\}}$$

where H is the barrier level, I is the indicator function, i.e.

$$\mathrm{I}_{\{S_t < H,\ t\in[0,T]\}} = \begin{cases} 1, & \text{if } S_t < H \text{ for all } t \in [0,T] \\ 0, & \text{otherwise} \end{cases} .$$

By the risk-neutral pricing principle, the option value is

$$e^{-rT}\widehat{\mathbb{E}}[(S_T - X)^+ \mathrm{I}_{\{S_t < H,\ t\in[0,T]\}}]$$

We can then use the Monte-Carlo simulation to get an approximate value of the option.

It should be noted that the simulation can only apply to European-style options.

1.4 Pricing in the PDE framework

Barrier options are weakly path-dependent. We only have to know whether or not the barrier has been triggered, and we do not need any other information about the path. This contrasts with some contracts we will discuss shortly, such as the Asian and lookback options, that are strongly path-dependent.

We can use $V(S,t)$ to denote the value of the barrier contract *before* the barrier has been triggered. This value still satisfies the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0. \tag{1}$$

The details of the barrier feature are specified through the boundary conditions and solution domains.

1.4.1 'Out' Barriers

The contract becomes worthless if the underlying asset reaches the barrier in an 'out' barrier option. This leads to the following boundary condition:

$$V(H, t) = 0 \text{ for } t \in [0, T].$$

The final condition is the same as that of a vanilla option:

$$V(S, T) = \begin{cases} (S - X)^+ \text{ for call} \\ (X - S)^+ \text{ for put.} \end{cases}$$

The solution domain is $\{0 < S < H\} \times [0, T)$ for an up-out option, and $\{H < S < \infty\} \times [0, T)$ for a down-out option.

1.4.2 'In' Barriers

An 'in' option only pays off if the barrier is triggered. Remember that $V(S, t)$ stands for the option value before the barrier has been triggered. If the barrier is not triggered during the option's life, the option expires worthless so that we have the final condition:

$$V(S, T) = 0.$$

Once the barrier is triggered, the barrier-in option becomes a vanilla option. As such, on the barrier, the contract must have the same value as a vanilla contract:

$$V(H, t) = \begin{cases} C_E(H, t), \text{ for call} \\ P_E(H, t), \text{ for put,} \end{cases} \text{ for } t \in [0, T],$$

where $C_E(S, t)$ and $P_E(S, t)$ represent the values of the (European-style) vanilla call and put, respectively.

The solution domain is $\{0 < S < H\} \times [0, T)$ for an up-in option, and $\{H < S < \infty\} \times [0, T)$ for a down-in option.

1.4.3 *Explicit solutions

Closed-form solutions for European-style barrier options are available. We refer interested readers to those references for these formulas. Here we only present the formula for the down-and-out call option with $H \leq X$:

$$C_{do}(S,t) = C_E(S,t) - \left(\frac{S}{H}\right)^{1-2(r-q)/\sigma^2} C_E(\frac{H^2}{S},t), \tag{2}$$

where $C_E(S,t)$ is the value of the European vanilla call.

Let us confirm that this is indeed the solution. Clearly, on the barrier $C_{do}(H,t) = C_E(H,t) - C_E(H,t) = 0$. At expiry

$$\begin{aligned} C_{do}(S,T) &= (S-X)^+ - \left(\frac{S}{H}\right)^{1-2(r-q)/\sigma^2} (\frac{H^2}{S} - X)^+ \\ &= (S-X)^+, \text{ for } S > H. \end{aligned}$$

So the remaining task is to show $C_{do}(S,t)$ satisfies the Black-Scholes equation. The first term on the right-hand side of (2) does as well. The second term does also. Actually, if we have any solution V_{BS} of the Black-Scholes equation, it is easy to show that (exercise)

$$S^{1-2(r-q)/\sigma^2} V_{BS}(\frac{A}{S},t)$$

is also a solution for any constant A.

1.5 American early exercise

For American knock-out options, the pricing model is a free boundary problem:

$$\begin{aligned} \min\left\{-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV, V - \varphi\right\} &= 0, \ (S,t) \in D \\ V(H,t) &= 0 \\ V(S,T) &= \varphi^+ \end{aligned}$$

where $D = (H,\infty) \times [0,T)$ for a down-out, and $D = (0,H) \times [0,T)$ for an up-out.

However, an American knock-in option price is governed still by the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \ (S,t) \in D \tag{3}$$

The reason is $V(S,t)$ represents the option value *before* the barrier is triggered. Thus for $(S,t) \in D$, early exercise is not permitted because it has not been activated. In fact, the option should be referred to as a *knock-into American option*. As in the European case, at expiry we have

$$V(S,T) = 0. \tag{4}$$

Once the barrier is triggered, the option becomes an American vanilla option then on the barrier. So,

$$V(H,t) = \begin{cases} C_A(H,t), & \text{for call} \\ P_A(H,t), & \text{for put} \end{cases}, \text{ for } t \in [0,T], \tag{5}$$

where $C_A(S,t)$ and $P_A(S,t)$ represent the values of the American vanilla call and put, respectively. (3-5) forms a complete model for American knock-in options.

1.6 BTM

The binomial tree method (BTM) can be readily extended to handle barrier options. Recall that the Black-Scholes equation still holds. So the BTM must hold, too, that is

$$V(S, t-\Delta t) = e^{-r\Delta t}[pV(Su,t) + (1-p)V(Sd,t)]$$

with appropriate final and boundary conditions. For instance, for a down-out call option,

$$\begin{aligned} V(H,t) &= 0 \\ V(S,T) &= (S-X)^+, \end{aligned}$$

and the backward procedure is conducted in the region $\{S > H\} \times [0,T)$.

It is easy to see that the BTM can handle early exercise in American barrier options.

1.7 Hedging

Barrier options usually have a discontinuous delta at the barrier. Such a discontinuity makes hedging a barrier option difficult. We refer interested students to Wilmott (1998) and references therein.

1.8 Other features

The barrier options discussed above are regular. In practice, the barrier level in the contract can be time-dependent. The level may begin at one level and then rise, for example. Usually the level is a piecewise-constant function of time.

Another style of barrier option is the double barrier option, which has both an upper and a lower barrier, the first above and the second below the current asset price. A double 'out' option becomes worthless if either of the barriers is reached. A double 'in' option pays off when either of the barriers is reached before expiry. Other possibilities can be imagined: one barrier is an 'in' and the other 'out', such that at expiry the contract could have either and 'in' or an 'out' payoff.

Sometimes, a rebate is paid if the barrier level is reached. This is often the case for 'out' barriers, in which the rebate can be thought of as cushioning the blow of losing the rest of the payoff. The rebate may be paid as soon as the barrier is triggered or at expiry.

2 Asian Options

An Asian option provides the holder with a payoff that depends on the average price of the underlying during the option's life.

2.1 Payoff types

The payoff from a fixed strike Asian call (or average call) is $(A_T - X)^+$, and that from a fixed strike Asian put (or average price put) is $(X - A_T)^+$, where A_T is the average value of the underlying asset calculated over a predetermined average period. Fixed strike Asian options are less expensive than vanilla options and are more suitable than vanilla options for meeting some of the needs of corporate treasurers. Suppose that a U.S. corporate treasurer expects to receive a cash flow of 100 million Australian dollars spread evenly over the next year from the company's Australian subsidiary. The treasurer might be interested in an option that guarantees that the average exchange rate realized during the year is above some level. A fixed strike Asian put can achieve this more effectively than vanilla put options.

Another type of Asian option is a floating strike option (or average strike option). A floating strike Asian call pays off $(S_T - A_T)^+$ and a floating strike Asian put pays off $(A_T - S_T)^+$. Floating strike options can ensure that the average price paid for an asset in frequent trading over a period of time is not greater than the final price. Alternatively, it can guarantee that the average price received for an asset in frequent trading over a period of time is not less than the final price.

2.2 Types of averaging

The two simplest types of averages are arithmetic and geometric averages. The arithmetic average of the price is the sum of all the constituent prices, equally weighted, divided by the total number of prices used. The geometric average is the exponential of the sum of all the logarithms of the constituent prices, equally weighted, divided by the total number of prices used. Furthermore, the average may be based on discretely sampled prices or on continuously sampled prices. Then, we have

$$A_T = \begin{cases} \frac{1}{n}\sum_{i=1}^{n} S_{t_i}, \text{ discretely sampled arithmetic} \\ \frac{1}{T}\int_0^T S_\tau d\tau, \text{ continuously sampled arithmetic} \\ \exp\left(\frac{1}{n}\sum_{i=1}^{n} \ln S_{t_i}\right) = (S_{t_1}S_{t_2}...S_{t_n})^{1/n}, \text{ discretely sampled geometric} \\ \exp\left(\frac{1}{T}\int_0^T \ln S_\tau d\tau\right), \text{ continuously sampled geometric} \end{cases}$$

2.3 Extending the Black-Scholes equation

Monte Carlo simulation is straightforward for pricing all kinds of European-style Asian options. However, for American-style Asian options, we must rely on the PDE framework or BTM. In the following we only consider the continuously sampled Asian options. We refer interested readers to Wilmott et al. (1995), Wilmott (1998) for the PDE formulation of discretely sampled Asian options.

We start by assuming that the underlying asset follows the lognormal random walk

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

The value of an Asian option is not only a function of S and t, but also a function of the historical average A, that is $V = V(S_t, A_t, t)$. Here A_t will be a new independent variable. Let us focus on the arithmetic average for which

$$A_t = \frac{1}{t}\int_0^t S_\tau d\tau.$$

When using Ito lemma, we need to know the stochastic differential equation satisfied by A_t. It is not hard to verify that

$$dA_t = \frac{tS_t - \int_0^t S_t d\tau}{t^2} dt = \frac{S_t - A_t}{t} dt.$$

We can see that its stochastic differential equation contains no stochastic terms. So, the Ito lemma for $V = V(S, A, t)$ is

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial A} dA + \frac{\partial V}{\partial S} dS \\ &= \left(\frac{\partial V}{\partial t} + \frac{S - A}{t}\frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial S} dS. \end{aligned}$$

To derive the pricing PDE, we set up a portfolio containing one of the path-dependent option and short a number Δ of the underlying asset:

$$\Pi = V(S, A, t) - \Delta S$$

The change in the value of this portfolio is given by

$$\begin{aligned} d\Pi &= dV - \Delta dS \\ &= \left(\frac{\partial V}{\partial t} + \frac{S-A}{t}\frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left(\frac{\partial V}{\partial S} - \Delta \right) dS. \end{aligned}$$

Choosing

$$\Delta = \frac{\partial V}{\partial S}$$

to hedge the risk, we find that

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{S-A}{t}\frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

This change is risk-free, thus earns the risk-free rate of interest r, leading to the pricing equation

$$\frac{\partial V}{\partial t} + \frac{S-A}{t}\frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{6}$$

The solution domain is $\{S > 0,\ A > 0,\ t \in [0, T)\}$.

This is to be solved subject to

$$V(S, A, T) = \begin{cases} (A-S)^+, & \text{floating put} \\ (S-A)^+, & \text{floating call} \\ (A-X)^+, & \text{fixed call} \\ (X-A)^+, & \text{fixed put} \end{cases}$$

The obvious changes can be made to accommodate dividends on the underlying. This completes the formulation of the valuation problem.

For the geometric average,

$$\begin{aligned} dA &= d\exp\left(\frac{1}{t}\int_0^t \ln S_\tau d\tau\right) = \exp\left(\frac{1}{t}\int_0^t \ln S_\tau d\tau\right) \frac{t\ln S - \int_0^t \ln S_\tau d\tau}{t^2} dt \\ &= \frac{A\ln\frac{S}{A}}{t}. \end{aligned}$$

So the PDE satisfied by geometric Asian options is

$$\frac{\partial V}{\partial t}+\frac{A\ln\frac{S}{A}}{t}\frac{\partial V}{\partial A}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0.$$

The solution domain is $\{S>0,\ A>0,\ t\in[0,T)\}$.

2.4 Early exercise

The only point to mention is that the details of the payoff on early exercise must be well defined. The payoff at expiry depends on the value of the average up to expiry; this will, of course, not be known until expiry. Typically, on early exercise it is the average to date that is used. For example, in an American floating strike arithmetic put the early payoff would be

$$\left(\frac{1}{t}\int_0^t S_\tau d\tau - S_t\right)^+.$$

In general, we denote the exercise payoff at time t by $\Lambda(S_t, A_t)$, where

$$\Lambda(S_t,A_t)=\begin{cases}(A_t-S_t)^+, & \text{floating put}\\(S_t-A_t)^+, & \text{floating call}\\(A_t-X)^+, & \text{fixed call}\\(X-A_t)^+, & \text{fixed put}\end{cases}.$$

Such representation is consistent with the terminal payoff. Then the pricing model is formulated by

$$\begin{aligned}\min\left\{-\frac{\partial V}{\partial t}-LV, V-\Lambda(S,A)\right\} &= 0\\ V(S,A,T) &= \Lambda(S,A)\end{aligned}$$

where

$$L=\begin{cases}\frac{S-A}{t}\frac{\partial}{\partial A}+\frac{1}{2}\sigma^2S^2\frac{\partial^2}{\partial S^2}+(r-q)S\frac{\partial}{\partial S}-r, & \text{for arithmetic}\\ \frac{A\ln(S/A)}{t}\frac{\partial}{\partial A}+\frac{1}{2}\sigma^2S^2\frac{\partial^2}{\partial S^2}+(r-q)S\frac{\partial}{\partial S}-r, & \text{for geometric}\end{cases}$$

Remark 1 *All European-style Asian geometric options have explicit price formulas. However, that is generally not true for Asian arithmetic options and all American-style Asian options, which must be solved by numerical approaches.*

2.5 Reductions in dimensionality

The price function of Asian options depends on three variables. For some cases, a reduction in dimensionality of the problem is permitted. We leave it as an exercise.

Remark 2 *For any floating strike Asian option (arithmetic or geometric, European or American), one can find an appropriate transformation to reduce the model to a lower-dimensional problem. As for fixed strike Asian options, it is true only for European-style.*

2.6 Parity relation

We will provide one example. Consider European-style floating strike arithmetic call options. The payoff at expiry for a portfolio of one call held long and one put held short is

$$(S-A)^+ - (A-S)^+,$$

where is simply

$$S-A.$$

The value of the portfolio then satisfies the equation (6) with the final condition

$$V(S,A,T) = S-A.$$

It can be verified that the solution is

$$V(S,A,t) = S - \frac{S}{rT}(1-e^{-r(T-t)}) - \frac{t}{T}e^{-r(T-t)}A.$$

Therefore, for floating strike arithmetic Asian options, we have the put-call parity relation

$$V_{fc}(S,A,t) - V_{fp}(S,A,t) = S - \frac{S}{rT}(1-e^{-r(T-t)}) - \frac{t}{T}e^{-r(T-t)}A.$$

We have similar results for other European Asian options.

2.7 Model-dependent and model-independent results

It is important to note that some results depend on assumptions of the model. For example, the above Asian put-call parity holds for the Black-Scholes model, but it might not be true in general provided that the geometric Brownian assumption is given up.

Recall that some results are model-independent. For example:

(1) the put-call parity for vanilla options

(2) the in-out parity for barrier options

(3) American call options should never be exercised early if there is no dividend payment.

To derive these results, one only needs the no-arbitrage principle.

But some are true only under the Black-Scholes framework. For instance:

(1) the above Asian put-call parity

(2) the put-call symmetry relation

2.8 *BTM

The BTM for European-style Asian options is as follows:

$$\begin{aligned} V(S, A, t - \Delta t) &= e^{-r\Delta t} \left[pV(Su, A_u, t) + (1-p)V(Sd, A_d, t)\right] \\ V(S, A, T) &= \Lambda(S, A) \end{aligned}$$

in $t < T$, $S > 0$, $A > 0$, where

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

and

$$A_u = \frac{tA + \Delta t Su}{t + \Delta t}, \text{ and } A_d = \frac{tA + \Delta t Sd}{t + \Delta t}.$$

Early exercise can be easily incorporated into the above algorithm to handle American style options.

3 Lookback Options

The ideal contract for a speculator would be one that pays the difference between the highest and the lowest asset prices realized by an asset over some period. Such a contract is an example of a lookback option. The contract that pays this is an example of a lookback option, an option that pays off some function of the realized maximum and/or minimum of the underlying asset over some prescribed period. Due to their extreme payoff structure, lookback options tend to be expensive

3.1 Types of payoff

For basic lookback contracts, the payoff comes in two varieties, similar to Asian options: the fixed strike and the floating strike, respectively. In a floating strike option, the vanilla exercise price is replaced by the maximum or minimum price. In a fixed strike option, the asset value in the vanilla payoff is replaced by the maximum or minimum price. That is

$$\Lambda(S_T, M_T) = \begin{cases} M_T - S_T, \text{ floating put} \\ S_T - m_T, \text{ floating call} \\ (M_T - X)^+, \text{ fixed call} \\ (X - m_T)^+, \text{ fixed put} \end{cases}$$

Here

$$M_T = \max_{0 \le \tau \le T} S_\tau \text{ and } m_T = \min_{0 \le \tau \le T} S_\tau.$$

3.2 Extending the Black-Scholes equations

Let $V = V(S_t, M_t, t)$ (or. $V = V(S_t, m_t, t)$) be the lookback option value, where

$$M_t = \max_{0 \le \tau \le t} S_\tau \text{ and } m_t = \min_{0 \le \tau \le t} S_\tau.$$

If $S < M$, then $dM_t = 0$. Using the $\Delta-$hedging argument and Ito lemma, we infer

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \text{ for } S < M$$

At expiry we have

$$V(S, M, T) = \Lambda(S, M)$$

The solution domain is $\{S \le M\} \times [0, T]$. An auxiliary condition is required on $S = M$,

$$\left.\frac{\partial V}{\partial M}\right|_{M=S} = 0,$$

whichcan be derived using the BTM.

For a floating call or fixed put, we can establish its governing equation in a similar way. And the solution domain is $\{S \ge m\} \times [0, T]$.

3.3 BTM

Consider a floating put or fixed call. We have

$$V(S, M, t - \Delta t) = e^{-r\Delta t}\left[pV(Su, M_u, t) + (1 - p)V(Sd, M_d, t)\right]$$

$t < T$, $S \le M$, where

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

and

$$M_u = \max(M, Su), \text{ and } M_d = \max(M, Sd).$$

Since $M \ge S$ and $d < 1$, we have

$$M_d = M.$$

Thus, the BTM is given by

$$\begin{aligned} V(S, M, t - \Delta t) &= e^{-r\Delta t}\left[pV(Su, M_u, t) + (1-p)V(Sd, M, t)\right], \text{ for } S \le M, \ t < T \\ V(S, M, T) &= \Lambda(S, A) \end{aligned}$$

3.4 Consistency of the BTM and the continuous-time model:

Note that $M \ge S$. We only need to consider two cases:

(1) $M \ge Su$: In this case $M_u = M_d = M$. Then the BTM can be rewritten as

$$V(S, M, t - \Delta t) = e^{-r\Delta t}\left[pV(Su, M, t) + (1-p)V(Sd, M, t)\right].$$

Using the Taylor expansion, we obtain

$$-rV(S, M, t) + \frac{\partial V}{\partial t}(S, M, t) + (r-q)S\frac{\partial V}{\partial S}(S, M, t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S, M, t) = O(\Delta t).$$

(2) $M = S$: Then $M_u = Su$ and $M_d = S$, and the BTM becomes

$$V(S, S, t - \Delta t) = e^{-r\Delta t}\left[pV(Su, Su, t) + (1-p)V(Sd, S, t)\right].$$

Using the Taylor expansion, we have

$$\begin{aligned} & V(S, S, t - \Delta t) - e^{-r\Delta t}\left[pV(Su, Su, t) + (1-p)V(Sd, S, t)\right] \\ = & -[\frac{\partial V}{\partial t}(S, S, t) + (r-q)S\frac{\partial V}{\partial S}(S, S, t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S, S, t) - rV(S, S, t)]\Delta t \\ & -e^{-r\Delta t}p(Su - S)\frac{\partial V}{\partial M}(S, S, t) + O(\Delta t) \\ = & -\Delta t^{1/2}\frac{\partial V}{\partial M}(S, S, t) + O(\Delta t), \end{aligned}$$

which implies

$$\frac{\partial V}{\partial M} = 0 \text{ at } M = S.$$

Remark 3 *For Asian options, the Taylor expansion also provides consistency.*

3.5 Similarity reduction

The similarity reduction also applies to some lookback options. For example, for floating strike lookback put options, it follows from the transformations $\frac{V(S,M,t)}{S} = W(x,t)$ and $x = \frac{M}{S}$,

$$\begin{aligned} \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} + (q-r)x\frac{\partial W}{\partial x} - qW &= 0,\ t \in [0,T),\ x \in (1,\infty) \\ \left.\frac{\partial W}{\partial x}\right|_{x=1} &= 0 \\ W(x,T) &= x-1. \end{aligned}$$

The reduction can be extended to the BTM. Taking the same transformations, we have for the floating strike lookback call

$$\begin{aligned} SW(\frac{M}{S}, t-\Delta t) &= e^{-r\Delta t}\left[pSuW(\frac{\max(M,Su)}{Su}, t) + (1-p)SdW(\frac{M}{Sd}, t)\right] \\ SW\left(\frac{M}{S}, T\right) &= M - S,\ \text{for } M \geq S,\ t < T \end{aligned}$$

or

$$\begin{aligned} W(x,t-\Delta t) &= e^{-r\Delta t}\left[puW(\max(xd,1),t) + (1-p)dW(xu,t)\right],\ x \geq 1,\ t<T \\ W(x,T) &= x-1,\ \text{for } x \geq 1. \end{aligned} \tag{7}$$

(7) can be rewritten as

$$\begin{aligned} W(x,t-\Delta t) &= e^{-r\Delta t}\left[puW(xd,t) + (1-p)dW(xu,t)\right],\ \text{for } x \geq u,\ t<T \\ W(1,t-\Delta t) &= e^{-r\Delta t}\left[puW(1,t) + (1-p)dW(u,t)\right], \\ W(x,T) &= x-1. \end{aligned}$$

3.6 Russian options

The similarity reduction can also be applied to the American-style lookback option. Here, we consider the Russian option, a special perpetual option, whose payoff is M. Since it is a perpetual option, the option value is independent of time. Let $V = V(S, M)$ denote the option value. The pricing model for the Russian option is given by

$$\begin{aligned}
\min\left\{-\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV, V - M\right\} &= 0, \text{ for } S < M \\
\left.\frac{\partial V}{\partial M}\right|_{S=M} &= 0.
\end{aligned}$$

Using the transformations

$$W(x) = \frac{V(S, M)}{S} \text{ and } x = \frac{M}{S},$$

we have

$$\begin{aligned}
\min\left\{-\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - (q-r)x\frac{\partial W}{\partial x} + qW, W - x\right\} &= 0, \text{ for } x > 1 \\
\left.\frac{\partial V}{\partial x}\right|_{x=1} &= 0.
\end{aligned}$$

For a fixed S, it becomes more attractive to exercise the Russian option if M is sufficiently large. Then, the above model can be rewritten as

$$\begin{aligned}
-\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - (q-r)x\frac{\partial W}{\partial x} + qW &= 0, \text{ for } 1 < x < x^* \\
W(x^*) &= x^* \\
\frac{\partial W}{\partial x}(x^*) &= 1 \\
\frac{\partial W}{\partial x}(1) &= 0.
\end{aligned}$$

Solving the equations gives the option value. A question: what about the option value if $q = 0$.

4 Appendix

4.1 Double barrier options

Consider an option with two barrier levels H_u and H_d, above and below the initial asset price, respectively. First let us consider the case that if the asset touches either barrier before expiry then the option knocks out. Otherwise, at expiry the option has a payoff of $(S - X)^+$.

Let $V(S,t;H_u,H_d)$ be the price function of the double barrier option before either barrier is hit. It is easy to see

$$L_{BS}V = 0,\ S \in (H_u, H_d)\,,\ t \in [0,T).$$

When $S = H_u$, the option knocks out, so

$$V\,(H_u, t) = 0,\ t \in [0,T).$$

Similarly

$$V\,(H_d, t) = 0,\ t \in [0,T).$$

At maturity

$$V\,(S,T) = (S-X)^+\,,\ S \in [H_u, H_d]\,.$$

Now let us consider another case. If the asset touches both barriers before expiry, then the option has a payoff of $(S-X)^+$. Otherwise the option is worthless. Again we let $V(S,t;H_u,H_d)$ be the price function of the double barrier option before either barrier is hit. Still we have

$$L_{BS}V = 0,\ S \in (H_u, H_d)\,,\ t \in [0,T).$$

The terminal condition is

$$V\,(S,T) = 0,\ S \in [H_u, H_d]\,.$$

We need to prescribe the boundary conditions at $S = H_u, H_d$. The key point is to examine what the option becomes. At $S = H_u$, the option reduces to a down-in call option,

$$V\,(H_u, t) = C_{di}(H_u, t; H_d),\ t \in [0,T).$$

Similarly the option becomes an up-in call option at $S = H_d$,

$$V\,(H_d, t) = C_{ui}\,(H_d, t; H_u)\,,\ t \in [0,T).$$

4.2 *Why we can write $V = V\,(S_t, A_t, t)$ for Asian options

By the martingale approach, we conclude that a European-style option value is the discounted expectation of terminal payoff.

For a vanilla call,

$$\begin{aligned}
V_t &= e^{-r(T-t)}\widehat{E}_t\left[(S_T-K)^+\right] \\
&= e^{-r(T-t)}\widehat{E}_t\left[\left(S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\widehat{W}_T-\widehat{W}_t)}-K\right)^+\right] \\
&\equiv V(S_t,t). \qquad (8)
\end{aligned}$$

For an Asian option, define

$$A_t=\frac{1}{t}\int_0^t S_\tau d\tau.$$

Then

$$\begin{aligned}
V_t &= e^{-r(T-t)}\widehat{E}_t\left[f\left(S_T,A_T\right)\right] \\
&= e^{-r(T-t)}\widehat{E}_t\left[f\left(S_T,\frac{1}{T}\int_0^T S_\tau d\tau\right)\right] \\
&= e^{-r(T-t)}\widehat{E}_t\left[f\left(S_T,\frac{1}{T}\int_0^t S_\tau d\tau+\frac{1}{T}\int_t^T S_\tau d\tau\right)\right] \\
&= e^{-r(T-t)}\widehat{E}_t\left[f\left(S_T,\frac{t}{T}A_t+\frac{1}{T}\int_t^T S_\tau d\tau\right)\right] \\
&= e^{-r(T-t)}\widehat{E}_t\left[f\left(S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\widehat{W}_T-\widehat{W}_t)},\frac{t}{T}A_t+\frac{S_t}{T}\int_t^T e^{(r-\frac{\sigma^2}{2})(\tau-t)+\sigma(\widehat{W}_\tau-\widehat{W}_t)}d\tau\right)\right] \\
&\equiv V(S_t,A_t,t).
\end{aligned}$$

4.3 *How to make transformations?

Let us consider the fixed strike Asian arithmetic call option:

$$\frac{\partial V}{\partial t}+\frac{S-A}{t}\frac{\partial V}{\partial A}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+(r-q)\,S\frac{\partial V}{\partial S}-rV=0 \qquad (9)$$

in $S>0$, $A>0$, $t\in[0,T)$ with

$$V(S,A,T)=(A-X)^+\,.$$

The transformation is

$$V\left(S,A,t\right)=\frac{SU\left(x,t\right)}{T},\ x=\frac{TX-tA}{S}.$$

We aim to derive the equation for $U\left(x,t\right)$. Note that what we have is the equation for $V\left(S,A,t\right)$ where the partial derivatives of V are involved. So we need to express these partial derivatives by using those of $U(x,t)$:

$$\begin{aligned}
\frac{\partial x}{\partial t} &= -\frac{A}{S}, \ \frac{\partial x}{\partial S} = -\frac{TX-tA}{S^2} = -\frac{x}{S}, \ \frac{\partial x}{\partial A} = -\frac{t}{S} \\
\frac{\partial V}{\partial t} &= \frac{S}{T}\left(\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}\frac{\partial x}{\partial t}\right) = \frac{S}{T}\left(\frac{\partial U}{\partial t} - \frac{A}{S}\frac{\partial U}{\partial x}\right) \\
\frac{\partial V}{\partial S} &= \frac{1}{T}U + \frac{S}{T}\frac{\partial U}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{T}U - \frac{x}{T}\frac{\partial U}{\partial x} \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{T}U - \frac{x}{T}\frac{\partial U}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{1}{T}U - \frac{x}{T}\frac{\partial U}{\partial x}\right)\frac{\partial x}{\partial S} \\
&= \left(\frac{1}{T}\frac{\partial U}{\partial x} - \frac{1}{T}\frac{\partial U}{\partial x} - \frac{x}{T}\frac{\partial^2 U}{\partial x^2}\right)\left(-\frac{x}{S}\right) \\
&= \frac{x^2}{TS}\frac{\partial^2 U}{\partial x^2} \\
\frac{\partial V}{\partial A} &= \frac{S}{T}\frac{\partial U}{\partial x}\frac{\partial x}{\partial A} = -\frac{t}{T}\frac{\partial U}{\partial x}
\end{aligned}$$

Substituting into (9), we have

$$\begin{aligned}
&\frac{S}{T}\left(\frac{\partial U}{\partial t} - \frac{A}{S}\frac{\partial U}{\partial x}\right) + \frac{1}{2}\sigma^2 S^2 \frac{x^2}{TS}\frac{\partial^2 U}{\partial x^2} + (r-q)\,S\left(\frac{1}{T}U - \frac{x}{T}\frac{\partial U}{\partial x}\right) \\
&-r\frac{SU(x,t)}{T} + \frac{S-A}{t}\left(-\frac{t}{T}\frac{\partial U}{\partial x}\right) = 0,
\end{aligned}$$

that is,

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 U}{\partial x^2} - \left[(r-q)\,x + 1\right]\frac{\partial U}{\partial x} - qU = 0$$

in $x \in (-\infty, +\infty)$, $t \in [0,T)$, with the terminal condition

$$U(x,T) = (-x)^+ .$$

The above reduction also implies

$$U(x,t) = e^{-q(T-t)} E\left[(-x_T)^+\right], \tag{10}$$

where

$$dx_t = -\left[(r-q)\,x + 1\right]dt + \sigma x d\widetilde{W}_t.$$