American Options and Early Exercise

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Contents

American options are contracts that may be exercised early, prior to expiry. These options are contrasted with European options, which can only be exercised at expiry. Due to this early exercise feature, an American option is at least as valuable as its European counterpart. In the United States, most traded stock and futures options are of American style, while most index options are European.

When pricing an American option, we face two problems: determining its fair price and, more interestingly, identifying its optimal exercise strategy. Later, we will see that these two problems can be solved simultaneously.

1 Pricing Models for American Options

1.1 No Arbitrage Argument

Using the no-arbitrage argument, we can obtain some properties of American option prices. For example, we can show the following results:

(i) Early exercise is never optimal for an American call option provided that the underlying stock does not pay a dividend.

(ii) An American option price must be higher than or equal to its early exercise payoff, i.e.,

$$V_t \geq \varphi(S_t), \tag{1}$$

where $\varphi(S_t)$ is the early exercise payoff:

$$\varphi(S) = \begin{cases} S - X & \text{for a call} \\ X - S & \text{for a put.} \end{cases}$$

However, to price an (American) option, we must assume that the underlying stock price follows a certain stochastic process.

1.2 *A Continuous-Time Model

Consider the Black-Scholes model, where the stock price follows a geometric Brownian motion in the real world:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

Here W_t is a standard Brownian motion, and μ and σ are the expected return rate and volatility of the stock.

We attempt to derive the pricing model for American options. Without loss of generality, we take into account a put as an example. Let $V_t = V(S_t, t)$ be the option value. At expiry, we have

$$V(S,T) = (X - S)^+. \tag{2}$$

The constraint (1) can be rewritten as

$$V(S,t) \geq X - S. \tag{3}$$

We now construct a portfolio of one long American option position and a short position in some quantity Δ, of the underlying.

$$\Pi_t = V(S_t, t) - \Delta_t S_t.$$

With the choice $\Delta_t = \frac{\partial V}{\partial S}(S_t, t)$, the value of this portfolio changes by the amount

$$d\Pi_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})|_{(S_t,t)} dt - q\Delta_t S_t dt,$$

where q is the constant dividend yield. In the Black-Scholes argument for European options, we set this expression equal to the riskless return to preclude arbitrage. However, when the option in the portfolio is of American style, all we can say is that we can earn no more than the risk-free rate on our portfolio, that is,

$$d\Pi_t \leq r\Pi_t dt = r(V - S\frac{\partial V}{\partial S})|_{(S_t,t)} dt.$$

The reason is the option's holder determines early exercise. When he fails to optimally exercise the option, the change in the portfolio value would be less than the riskless return. Thus we arrive at an inequality

$$(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - qS\frac{\partial V}{\partial S})|_{(S_t,t)} dt \leq r(V - S\frac{\partial V}{\partial S})|_{(S_t,t)} dt.$$

It follows

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S\frac{\partial V}{\partial S} - rV \leq 0. \tag{4}$$

Remark 1 *For American options, the long/short position is asymmetrical. The holder of an American option is given more rights, as well as more headaches: when should he exercise?*

By contrast, the writer of the option can do no more than sit back and enjoy the view. The writer can make more than the risk-free rate if the holder does not exercise optimally.

It is worth pointing out that the conditions given in (2)-(4) do not suffice to determine the option price, as there are many solutions satisfying these conditions. We need to exploit more information. Note that if $V(S,t) > X - S$, which implies that the option should not be exercised at the moment, then the equality must hold in the inequality (4), namely,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0 \quad \text{if } V(S,t) > X - S. \tag{5}$$

Denote

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV.$$

Combining (3)-(5), we obtain a variational inequality problem:

$$-\mathcal{L}V(S,t) \geq 0, \qquad V(S,t) \geq X - S, \tag{6}$$

$$-\mathcal{L}V(S,t)[V(S,t) - (X-S)] = 0, \tag{7}$$

in $(S,t) \in D \equiv \{(S,t) : S > 0, t \in [0,T)\}$. It can be shown that there exists a unique solution to the variational inequality (6)-(8) with the terminal condition (2) and the following boundary conditions:[1]

$$V(0,t) = K \quad \text{for any } t \in [0,T), \tag{8}$$

$$V(S,t) \to 0 \quad \text{for any } t \in [0,T). \tag{9}$$

A succinct expression of the variational inequality (6)-(8) is

$$\min\{-\mathcal{L}V, V(S,t) - (X-S)\} \;=\; 0$$

in $(S,t) \in D$.

Alternatively, we rewrite the variational inequality as

$$-\mathcal{L}V(S,t) = 0 \quad \text{if } \; V(S,t) > X - S$$
$$-\mathcal{L}V(S,t) \geq 0 \quad \text{if } \; V(S,t) = X - S$$

in $(S,t) \in D$.

[1] Note that the boundary conditions (8)-(9) are needed when the implicit finite difference method is employed to solve the problem numerically.

Remark 2 *From the perspective of stochastic analysis, the American (put) option price is governed by an optimal stopping time problem:*

$$V(S,t) = \max_{t' \geq t} \widehat{\mathbb{E}}_t \left[e^{-r(t'-t)}(X - S_{t'})^+ | S_t = S \right], \tag{10}$$

where t' is a stopping time. Intuitively, a stopping time t' can be thought of as an exercise strategy, and the option price corresponds to the optimal exercise strategy. Mathematically we can show the equivalence between (10) and the above variational inequality model.

1.3 A Binomial Model

Let T be the expiration date, and $[0,T]$ be the lifetime of the option. If N is the number of discrete time points, we have time points $n\Delta t$, where $n = 0, 1, ..., N$, with $\Delta t = T/N$. Let V_j^n be the option price at time point $n\Delta t$ with stock price S_j. Suppose the stock price S_j will move either up to $S_{j+1} = S_j u$ or down to $S_{j-1} = S_j d$ after the next timestep. Similar to the arguments in the continuous time case, we are able to derive the binomial model:

$$\begin{cases} V_j^n = \max \left\{ e^{-r\Delta t}[pV_{j+1}^{n+1} + (1-p)V_{j-1}^{n+1}], \varphi_j \right\}, \\ \qquad\qquad\qquad \text{for } j = -n, -n+2, ..., n \text{ and } n = 0, 1, ...N-1 \\ V_j^N = \varphi_j^+, \quad \text{for } j = -N, -N+2, ..., N \end{cases}$$

where $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, $p = \frac{e^{(r-q)\Delta t} - d}{u-d}$, and

$$\varphi_j = \begin{cases} S_0 u^j - X & \text{for a call} \\ X - S_0 u^j & \text{for a put.} \end{cases}$$

1.4 From the Binomial Model to the Continuous-Time Model

The binomial model for American options can be written as

$$V(S, t - \Delta t) = \max\{\frac{1}{\rho}[pV(Su,t) + (1-p)V(Sd,t)], \varphi(S)\}$$

or

$$\min\left\{ V(S,t-\Delta t) - \frac{1}{\rho}[pV(Su,t) + (1-p)V(Sd,t)], V(S,t-\Delta t) - \varphi(S) \right\} = 0.$$

For a sufficiently smooth function $V(S,t)$, we have shown using Taylor expansions:

$$
\begin{aligned}
& V(S,t-\Delta t) - \frac{1}{\rho}[pV(Su,t) + (1-p)V(Sd,t)] \\
= & \left[rV - \frac{\partial V}{\partial t} - (r-q)S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right]_{(S,t)} \Delta t + O(\Delta t^2).
\end{aligned}
$$

So for American options, we have

$$
\min\left\{ \left[rV - \frac{\partial V}{\partial t} - (r-q)S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right]_{(S,t)} \Delta t + O(\Delta t^2), V(S,t-\Delta t) - \varphi(S) \right\} = 0.
$$

Note that

$$
\min\{a\Delta t, b\} = 0 \quad \text{for } \Delta t > 0
$$

is equivalent to

$$
\min\{a, b\} = 0.
$$

It follows that

$$
\min\left\{ \left[rV - \frac{\partial V}{\partial t} - (r-q)S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right]_{(S,t)} + O(\Delta t), V(S,t-\Delta t) - \varphi(S) \right\} = 0.
$$

Sending $\Delta t \to 0$, one obtains the continuous-time model:

$$
\min\left\{ -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV, V(S,t) - \varphi(S) \right\} = 0.
$$

Exercise region and holding region. We give definitions of Stopping Region E (or Exercise Region) and Continuation Region H (or Holding Region) :

$$
\begin{aligned}
E &= \{(S,t) \in D : V(S,t) = X - S\}, \\
H &= D \backslash E = \{(S,t) \in D : V(S,t) > X - S\}.
\end{aligned}
$$

It can be shown that there exists a boundary $S^*(t)$, called the optimal exercise boundary hereafter, such that

$$
\begin{aligned}
E &= \{(S,t) \in D : S \le S^*(t)\}, \\
H &= \{(S,t) \in D : S > S^*(t)\}.
\end{aligned}
$$

Assignment 3. Use the binomial model to find the price of an American put option and the optimal exercise boundary, where $r = 0.03$, $q = 0$, $X = 1$, $\sigma = 0.3$, $S_0 = 1$, and $T = 1$ (you can use any program language).

2 Replicating American Options

Without loss of generality, we focus on an American put, assuming that the underlying does not pay a dividend.

Assertion: The value of American put option $V = V(S_t, t)$ satisfies

$$\begin{aligned} \min\left\{-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV, V - \varphi(S)\right\} &= 0, \ S > 0, \ t \in [0, T) \\ V(S, T) &= \varphi(S)^+. \end{aligned}$$

Moreover, the optimal exercise time

$$\begin{aligned} t^* &= \inf\{t \in [0, T) | V(S_t, t) = \varphi(S_t)\} \\ &= \inf\{0 \leq t < T | S_t \leq S^*(t)\}. \end{aligned} \tag{11}$$

In other words, we have the following:

I. If, at time 0, the option is sold at a price $P_0 > V(S_0, 0)$, then there is an arbitrage opportunity for the seller.

II. If at time 0, the option is sold at a price $P_0 < V(S_0, 0)$, then there is an arbitrage opportunity for the buyer.

III. If the option is sold at the fair price, $P_0 = V(S_0, 0)$, but the buyer does not exercise it at the optimal exercise time t^*, then there is an arbitrage opportunity for the seller.

2.1 Proof of Part I

Consider the option seller who sold the option at $t_0 = 0$ at price $P_0 > V(S_0, 0)$. She forms the following portfolio up to time τ, at which the option is exercised.

1. Sell the option at the option price P_0, which is used as the initial endowment for investment.

2. In any investment period $(t, t + dt]$, hold $\Delta_t = \frac{\partial V(S_t, t)}{\partial S}$ shares of stock with the rest invested in the bond.

3. At time τ, cash the portfolio and pay the buyer, so

$$\begin{aligned}
P_0 &> V(S_0, 0) \\
dP_t &= \Delta_t dS_t + (P_t - \Delta_t S_t) r dt, \ t \in (0, \tau) \\
PL &= P_\tau - \varphi(S_\tau),
\end{aligned}$$

where PL is the profit or loss at time τ.

We now show that $PL > 0$. By Ito's lemma, we have

$$\begin{aligned}
&d(P_t - V(S_t, t)) \\
&= \Delta_t dS_t + (P_t - \Delta_t S_t) r dt - \left(\frac{\partial V(S_t, t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S_t, t)}{\partial S^2} \right) |_{S=S_t} dt - \frac{\partial V(S_t, t)}{\partial S} dS_t \\
&= [P_t - V(S_t, t)] r dt - \left\{ \frac{\partial V(S_t, t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S_t, t)}{\partial S^2} + rS \frac{\partial V(S_t, t)}{\partial S} - rV(S_t, t) \right\} dt \\
&= [P_t - V(S_t, t)] r dt + \xi(S_t, t) dt,
\end{aligned}$$

where

$$\xi = - \left\{ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \right\}.$$

It follows

$$d(P_t - V(S_t, t)) - [P_t - V(S_t, t)] r dt = \xi(S_t, t) dt$$

or

$$d\left[e^{-rt}(P_t - V(S_t, t))\right] = \xi(S_t, t) e^{-rt} dt.$$

Integrating both sides, we have

$$\int_0^t d\left[e^{-ru}(P_u - V(S_u, u))\right] = \int_0^t \xi(S_u, u) e^{-ru} du,$$

that is

$$e^{-rt}[P_t - V(S_t, t)] - [P_0 - V(S_0, 0)] = \int_0^t \xi(S_u, u) e^{-ru} du.$$

Therefore,

$$P_t - V(S_t, t) = [P_0 - V(S_0, 0)] e^{rt} + \int_0^t \xi(S_u, u) e^{r(t-u)} du, \ 0 \le t \le \tau.$$

So,

$$\begin{aligned}PL &= P_\tau - \varphi(S_\tau) = P_\tau - V(S_\tau,\tau) + [V(S_\tau,\tau) - \varphi(S_\tau)] \\ &= [P_0 - V(S_0,0)]e^{r\tau} + \int_0^\tau \xi(S_u,u)e^{r(\tau-u)}du + [V(S_\tau,\tau) - \varphi(S_\tau)] \\ &\geq [P_0 - V(S_0,0)]e^{r\tau} > 0,\end{aligned}$$

where the second and third terms on the right-hand side of the last equality are nonnegative because of the variational inequality for V. Then the seller has an arbitrage opportunity.

2.2 *Proof of Part II

Omitted.

2.3 *Proof of Part III

The seller can construct a portfolio as in the proof of the first assertion. Just note

$$\begin{aligned}PL &= [P_0 - V(S_0,0)]e^{r\tau} + \int_0^\tau \xi(S_u,u)e^{(\tau-u)}du + [V(S_\tau,\tau) - \varphi(S_\tau)] \\ &= \int_0^\tau \xi(S_u,u)e^{(\tau-u)}du + [V(S_\tau,\tau) - \varphi(S_\tau)]\end{aligned}$$

and the fact that if the option buyer chooses to exercise in the holding region $\{V > \varphi\}$, then the second term is positive, and if he chooses to exercise in the interior of the exercise region (not on the boundary) where $\xi > 0$, then the first term is positive since he must have stayed in the exercise region for a positive length of time.

3 *Free Boundary Formulation

3.1 Formulation

We still take a put for example. In the continuation region $H = \{S > S^*(t)\}$, the price function of an American put satisfies the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \tag{12}$$

for $S > S^*(t)$, $t \in [0, T)$. On $S = \infty$ and $S = S^*(t)$, we have

$$\begin{aligned} V(S,t) &\rightarrow 0 \quad \text{as } S \rightarrow \infty, && (13) \\ V(S^*(t),t) &= X - S^*(t), && (14) \end{aligned}$$

respectively. The terminal condition is

$$V(S,T) = (X - S)^+. \tag{15}$$

However, these conditions are not sufficient to determine $V(S,t)$ because $S^*(t)$ is unknown. It is worth highlighting that $S^*(t)$ and $V(S,t)$ must be solved simultaneously. Therefore, we need an additional boundary condition

$$\frac{\partial V}{\partial S}(S^*(t),t) = -1. \tag{16}$$

The condition is often called the *high-contact* condition, which means that the hedging ratio Δ is continuous across the optimal exercise boundary. Equations (12)-(16) form a complete model known as a free boundary problem.

3.2 Perpetual American options

Pricing perpetual American options can provide insights into understanding free boundary problems. A perpetual American put can be exercised at any time and does not have an expiry date. Note that the price function of the option is independent of time and is denoted by $P_\infty(S)$. Then $P_\infty(S)$ satisfies

$$-\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_\infty}{\partial S^2} - (r-q)S\frac{\partial P_\infty}{\partial S} + rP_\infty = 0, \text{ for } S > S_\infty \tag{17}$$

$$P_\infty(S_\infty) = X - S_\infty, \qquad \frac{\partial P_\infty}{\partial S}(S_\infty) = -1. \tag{18}$$

This is an ordinary differential equation with a free boundary. We now seek a solution of the form S^α to Equation (17), where α satisfies

$$-\frac{1}{2}\sigma^2\alpha(\alpha-1) - (r-q)\alpha + r = 0$$

or

$$\frac{1}{2}\sigma^2\alpha^2 + (r - q - \frac{\sigma^2}{2})\alpha - r = 0.$$

The two solutions of the above equation are

$$\alpha_{\pm} = \frac{-(r - q - \frac{\sigma^2}{2}) \pm \sqrt{(r - q - \frac{\sigma^2}{2})^2 + 2r\sigma^2}}{\sigma^2}. \tag{19}$$

So the general solution of Equation (17) is

$$AS^{\alpha_+} + BS^{\alpha_-}$$

where A and B are arbitrary constants.

For the perpetual American put, the coefficient A must be zero; as $S \to \infty$, the value of the option must tend to zero. What about B? Here we need to use the condition (18)

$$\begin{aligned} BS_\infty^{\alpha_-} &= X - S_\infty \\ \alpha_- BS_\infty^{\alpha_- - 1} &= -1 \end{aligned}$$

to get

$$\begin{aligned} S_\infty &= \frac{\alpha_-}{\alpha_- - 1} X, \\ B &= \frac{X - S_\infty}{S_\infty^{\alpha_-}}. \end{aligned}$$

Thus, we obtain the perpetual American option price function:

$$\begin{aligned} P_\infty(S) &= (X - S_\infty) \left(\frac{S}{S_\infty} \right)^{\alpha_-} \quad \text{for } S > S_\infty, \\ P_\infty(S) &= X - S \qquad \text{for } S \le S_\infty. \end{aligned}$$