Preliminaries

Min Dai

https://sites.google.com/view/mindai

The Hong Kong Polytechnic University

Contents

1 Cox-Ross-Rubinstein Binomial Model

1.1 Single-period model

Consider an option with a current value denoted by V_0 at time $t = 0$, which depends on the underlying stock price S_0. Let the expiration date of the option be T. Assume that during the life of the option, the stock price S_0 can either move up to $S_0 u$ with probability p', or move down to $S_0 d$ with probability $1 - p'$. The no arbitrage condition requires that $u > e^{rT} > d$ and $0 < p' < 1$. Correspondingly, the payoff from the option will be either V_u (for an upward movement in the stock price) or V_d (for a downward movement).

$\Delta-$hedging analysis. We construct a portfolio consisting of a long position in the option and a short position in Δ shares. At time $t = 0$, the portfolio has a value of

$$V - \Delta S_0.$$

If there is an upward movement in the stock price, the value of the portfolio at $t = T$ is

$$V_u - \Delta S_0 u.$$

If there is a downward movement in the stock price, the value becomes

$$V_d - \Delta S_0 d.$$

To make the portfolio risk-free, we set these two values equal:

$$V_u - \Delta S_0 u = V_d - \Delta S_0 d,$$

which gives

$$\Delta = \frac{V_u - V_d}{S_0 \left(u - d\right)}. \tag{1}$$

According to the no-arbitrage principle, a risk-free portfolio must earn the risk-free interest rate. Therefore,

$$V_u - \Delta S_0 u = e^{rT}(V - \Delta S).$$

Substituting equation (1) into the above formula, we obtain

$$V = e^{-rT}\left[pV_u + (1-p)V_d\right],$$

where

$$p = \frac{e^{rT} - d}{u - d}.$$

This is the single-period binomial model. Here, p is called the risk-neutral probability. Note that the objective probability p' does not appear in the binomial model. This is known as the risk-neutral pricing principle.

An alternative derivative approach: option replication. Given the initial option premium V, we invest ΔS_0 in stock and the remainder $V - \Delta S_0$ in a bank account at $t = 0$. At maturity $t = T$,

$$\begin{aligned}
&\Delta S_0 u + e^{rT}\left(V - \Delta S_0\right) \text{ for an up movement;} \\
&\Delta S_0 d + e^{rT}\left(V - \Delta S_0\right) \text{ for a down movement.}
\end{aligned}$$

The fair value V is such that the option's payoff can be exactly replicated:

$$\begin{aligned}
\Delta S_0 u + e^{rT}\left(V - \Delta S_0\right) &= V_u, \\
\Delta S_0 d + e^{rT}\left(V - \Delta S_0\right) &= V_d.
\end{aligned}$$

Solving these equations leads to the desired result.

1.2 Multi-period model

Let T be the expiration date, and $[0, T]$ be the lifetime of the option. If N is the number of discrete time points, we have time points $n\Delta t$, where $n = 0, 1, ..., N$, with $\Delta t = \frac{T}{N}$. At time $t = 0$, the underlying stock price is known and denoted by S_0. At time Δt, there are two possible stock prices: $S_0 u$ and $S_0 d$. Without loss of generality, we assume

$$ud = 1.$$

At time $2\Delta t$, there are three possible stock prices, $S_0 u^2$, S_0, and $S_0 d^2 = S_0 u^{-2}$; and so on. In general, at time $n\Delta t$, $n+1$ stock prices are involved: $S_0 u^{-n}, S_0 u^{-n+2}, ..., S_0 u^n$. A complete tree is then constructed. Let V_j^n be the option price at time point $n\Delta t$ with stock price $S_j = S_0 u^j$. Note that S_j will jump either up to S_{j+1} or down to S_{j-1} at time $(n+1)\Delta t$, and the value of the option at $(n+1)\Delta t$ will become either V_{j+1}^{n+1} or V_{j-1}^{n+1}. Since the length of the time period is Δt, the discounting factor is $e^{-r\Delta t}$. Then, similar to the arguments in the single-period case, we have

$$V_j^n = e^{-r\Delta t}\left[pV_{j+1}^{n+1} + (1-p)V_{j-1}\right] \tag{2}$$

for $j = -n, -n+2, ..., n$ and $n = 0, 1, ..., N-1$, where

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

At expiry, the terminal condition is $V_j^N = \max(\varphi_j, 0) =: \varphi_j^+$ for $j = -N, -N+2, ..., N$, where

$$\varphi_j = \begin{cases} (S_0 u^j - K)^+ & \text{for a call,} \\ (K - S_0 u^j)^+ & \text{for a put,} \end{cases}$$

This is the multi-period binomial model for European options.

Suppose that the stock volatility is σ. Then we can choose

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}.$$

2 From the Binomial Model to the Black-Sholes Model

The binomial scheme (2) for European options can be rewritten as

$$V(S, t-\Delta t) = e^{-r\Delta t}[pV(Su, t) + (1-p)V(Sd, t)].$$

Here, for convenience, we take the current time to be $t - \Delta t$. Assuming sufficient smoothness of $V(S,t)$, we perform the Taylor series expansion of the binomial scheme at (S,t) as follows:

$$\begin{aligned}
0 &= -V(S, t-\Delta t) + e^{-r\Delta t}[pV(Su, t) + (1-p)V(Sd, t)] \\
&= -V(S,t) + \frac{\partial V(S,t)}{\partial t}\Delta t + O(\Delta t^2) \\
&\quad + e^{-r\Delta t}V(S,t) + \frac{\partial V(S,t)}{\partial S} S e^{-r\Delta t}[p(u-1) + (1-p)(d-1)] \\
&\quad + \frac{1}{2}\frac{\partial^2 V(S,t)}{\partial S^2} S^2 e^{-r\Delta t}[p(u-1)^2 + (1-p)(d-1)^2] \\
&\quad + \frac{1}{6}\frac{\partial^3 V(S,t)}{\partial S^3} S^3 e^{-r\Delta t}[p(u-1)^3 + (1-p)(d-1)^3] + O(\Delta t^2).
\end{aligned}$$

Observe that

$$\begin{aligned}
e^{-r\Delta t}[p(u-1)+(1-p)(d-1)] &= r\Delta t + O(\Delta t^2), \\
e^{-r\Delta t}[p(u-1)^2+(1-p)(d-1)^2] &= \sigma^2\Delta t + O(\Delta t^2), \\
e^{-r\Delta t}[p(u-1)^3+(1-p)(d-1)^3] &= O(\Delta t^2).
\end{aligned}$$

We then get

$$\begin{aligned}
0 &= -V(S,t-\Delta t) + e^{-r\Delta t}[pV(Su,t)+(1-p)V(Sd,t)] \\
&= [-rV(S,t) + \frac{\partial V(S,t)}{\partial t} + rS\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2}]\Delta t + O(\Delta t^2),
\end{aligned}$$

namely,

$$-rV(S,t) + \frac{\partial V(S,t)}{\partial t} + rS\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} = O(\Delta t).$$

Sending Δt to zero leads to the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

3 Continuous-time Black-Scholes Model

3.1 Brownian Motion, Ito Integral, and Ito's Lemma

Brownian motion. A process W_t is a Brownian motion if W_t is continuous and:

(i) The change ΔW during any period of time $[t, t+\Delta t]$ is a random variable drawn from a normal distribution with zero mean and variance Δt, i.e.,

$$\Delta W = \phi\sqrt{\Delta t}.$$

where ϕ is a random variable drawn from a standard normal distribution, which has zero mean, unit variance and a density function given by

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\ x \in (-\infty, \infty).$$

(ii) The values of ΔW for any two non-overlapping intervals are independent.

Ito process and Ito integral. In most cases, we assume that the underlying stock price follows an Ito process,

$$dS_t = a_t dt + b_t dW_t, \tag{3}$$

where W_t is a standard Brownian motion, and a_t and b_t are adapted with respect to the filtration generated by W_t.

A precise expression of the stock price process is

$$S_t = S_0 + \int_0^t a_\tau d\tau + \int_0^t b_\tau dW_\tau,$$

where the first integral is the Lebesgue integral, and the second is the Ito integral.

For a rigorous definition of the Ito integral, see Oksendal (2003). Here, we only provide an intuitive interpretation. For any partition $0 = t_1 < t_2 < ... < t_n = t$, the integral $\int_0^t f_\tau dW_\tau$ is the limit of

$$\sum_{i=0}^{n-1} f_{t_i} \left(W_{t_{i+1}} - W_{t_i} \right)$$

as $\max_i (t_{i+1} - t_i) \to 0$ (f is adapted with respect to the filtration generated by W_t). We emphasize that f is evaluated at the left-hand side point of the interval (t_i, t_{i+1}). In contrast, the Riemann integral $\int_0^t f(\tau)\, d\tau$ is defined as the limit of

$$\sum_{i=0}^{n-1} f(\xi_i)(t_{i+1} - t_i),$$

as $\max_i (t_{i+1} - t_i) \to 0$, where $\xi_i \in [t_i, t_{i+1}]$. Next, we show that the definition of the Ito integral aligns with investment decisions.

A self-financing weath process. Consider a self-financing procedure, which means that there is no withdrawal or infusion of funds during the investment period. Let Z_t be the value of a self-financing wealth process at time t. Consider an investment strategy for which transactions take place at discrete times t_i, $i = 0, ..., t_{n-1}$. Assume that at time t_i, based on all information up to time t_i, the investor decides to hold Δ_{t_i} number of shares, and the remainder $Z_{t_i} - \Delta_{t_i} S_{t_i}$ in a bank account. During (t_i, t_{i+1}), the number of shares remains fixed. Then the profit/loss during $[t_i, t_{i+1}]$ is $r(Z_{t_i} - \Delta_{t_i} S_{t_i})(t_{i+1} - t_i) + \Delta_{t_i}(S_{t_{i+1}} - S_{t_i})$. The accumulative profit or loss during $[0,t]$, $Z_t - Z_0$, equals

$$\begin{aligned}
& \sum_{i=0}^{n-1}\Big(r(Z_{t_i}-\Delta_{t_i}S_{t_i})(t_{i+1}-t_i)+\Delta_{t_i}(S_{t_{i+1}}-S_{t_i})\Big) \\
= & \sum_{i=0}^{n-1}\Big(r(Z_{t_i}-\Delta_{t_i}S_{t_i})(t_{i+1}-t_i)+\Delta_{t_i}a_{t_i}(t_{i+1}-t_i)+\Delta_{t_i}b_{t_i}(W_{t_{i+1}}-W_{t_i})\Big) \\
= & \sum_{i=0}^{n-1}\Big([r(Z_{t_i}-\Delta_{t_i}S_{t_i})+\Delta_{t_i}a_{t_i}](t_{i+1}-t_i)+\Delta_{t_i}b_{t_i}(W_{t_{i+1}}-W_{t_i})\Big)
\end{aligned}$$

Sending $\max_i (t_{i+1}-t_i) \to 0$, we obtain

$$Z_t - Z_0 = \int_0^t [r(Z_\tau - \Delta_\tau S_\tau)+\Delta_\tau a_\tau]d\tau + \int_0^t \Delta_\tau b_\tau dW_\tau. \tag{4}$$

Ito's lemma. Ito's Lemma is essentially the differential chain rule of a function involving random variables. First let us recall the standard differential chain rule of a function of deterministic variables. Let $V(.,.)$ be a deterministic function of two state variables. Consider $V(S_t,t)$, where

$$dS_t = a_t dt.$$

Then,

$$\begin{aligned}
dV(S_t,t) &= \frac{\partial V(S_t,t)}{\partial t}dt + \frac{\partial V(S_t,t)}{\partial S}dS_t = \frac{\partial V(S_t,t)}{\partial t}dt + \frac{\partial V(S_t,t)}{\partial S}a_t dt \\
&= \left[\frac{\partial V(S_t,t)}{\partial t} + a_t\frac{\partial V(S_t,t)}{\partial S}\right]dt.
\end{aligned}$$

Now let us return to the stochastic process (3). Keep in mind that $dW_t = \phi\sqrt{dt}$ and $(dW_t)^2 = dt$. So, formally we have

$$(dS_t)^2 = (a_t dt + b_t dW_t)^2 = a_t^2 dt^2 + 2a_t b_t dt dW_t + b_t^2 (dW_t)^2 = b_t^2 dt + \cdots.$$

As a result, when applying the Taylor series expansion to $V(S_t,t)$, we need to retain the second order term of dS. Thus,

$$\begin{aligned}
dV(S_t,t) &= \frac{\partial V(S_t,t)}{\partial t}dt + \frac{\partial V(S_t,t)}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 V(S_t,t)}{\partial S^2}(dS_t)^2 \\
&= \frac{\partial V(S_t,t)}{\partial t}dt + \frac{\partial V(S_t,t)}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 V(S_t,t)}{\partial S^2}b_t^2 dt \\
&= \left[\frac{\partial V(S_t,t)}{\partial t} + \frac{1}{2}b_t^2\frac{\partial^2 V(S_t,t)}{\partial S^2}\right]dt + \frac{\partial V(S_t,t)}{\partial S}dS_t \\
&= \left[\frac{\partial V(S_t,t)}{\partial t} + a_t\frac{\partial V(S_t,t)}{\partial S} + \frac{1}{2}b_t^2\frac{\partial^2 V(S_t,t)}{\partial S^2}\right]dt + b_t\frac{\partial V(S_t,t)}{\partial S}dW.
\end{aligned}$$

This is the Ito formula, the chain rule of stochastic calculus.

3.2 Black-Scholes Model

Consider a market where only two basic assets are traded. One is a bond (money-market account), whose price process is given by

$$dP_t = rP_t dt,$$

where r is the risk-free rate. The other asset is a stock whose price process is governed by geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ and σ are the expected return rate and volatility of the underlying stock, and W_t is the Brownian motion. Assume that the risk-free rate r and the stock volatility σ are known constants over the life of the option.

While one can derive the Black-Scholes model using Δ-hedging, we instead adopt an option replication argument.

Continuous-time option replication. Without loss of generality, we focus on a European call option whose payoff is

$$V_T = (S_T - K)^+ .$$

Our target is to replcate the option through a self-financing wealth process. According to equation (4), a self-financing wealth process Z_t is described by

$$dZ_t = [rZ_t + (\mu - r)\Delta_t S_t]\,dt + \sigma\Delta_t S_t dW_t, \tag{5}$$

where Δ_t is the number of shares held at time t.

Consider the replication problem from the perspective of the option's writer who sells the option at the price V_0 at time $t = 0$ and attempts to replicate the option's payoff V_T at expiry by selecting an appropriate strategy Δ_t. More generally, suppose the option value $V = V(S_t, t)$. We aim to have $Z_t = V(S_t, t)$ for all $t \in [0, T]$.

Applying Ito lemma,

$$dV_t = \left(\frac{\partial V(S_t, t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S_t, t)}{\partial S^2} + \mu S_t \frac{\partial V(S_t, t)}{\partial S} \right) dt + \sigma S_t \frac{\partial V(S_t, t)}{\partial S} dW_t. \tag{6}$$

Comparing this with the self-financing process given in (5), we get

$$\Delta_t = \frac{\partial V(S_t, t)}{\partial S}$$

and

$$\frac{\partial V(S_t, t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S_t, t)}{\partial S^2} + \mu S_t \frac{\partial V(S_t, t)}{\partial S} = rV(S_t, t) + (\mu - r)\, \Delta_t S_t.$$

In the end, we derive the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \tag{7}$$

in $S > 0$ and $t \in [0, T)$. At maturity,

$$V(S, T) = (S - K)^+ \tag{8}$$

The above derivation indicates the replication procedure is as follows:

- Solve the Black-Scholes equation (7) with the boundary condition (8) to obtain $V(S, t)$.
- At time 0, sell the option at the price $V(S_0, 0)$.
- At any time $t \in [0, T)$, hold $\Delta_t = \frac{\partial V(S_t, t)}{\partial S}$ shares of stock.

3.3 Risk-neutral pricing and theoretical basis of Monte-Carlo simulation

The expected return rate μ of the underlying stock, which clearly depends on risk preference, does not appear in the Black-Scholes equation. All of the variables and parameters appearing in the equation are independent of risk preference. Therefore, risk references do not affect the solution to the Black-Scholes equation.

In a risk-neutral world, all investors are risk-neutral, meaning the expected return on all securities is the risk-free rate of interest r. Thus, the present value of any cash flow in this world can be obtained by discounting its expected value at the risk-free rate. Specifically, the price of an option (a European call, for example) can be represented by

$$V(S,t) = \widehat{\mathbb{E}}_t \left[e^{-r(T-t)}(S_T - X)^+ | S_t = S \right]. \tag{9}$$

Here, $\widehat{\mathbb{E}}_t$ denotes the conditional expectation in a risk-neutral world under which the underlying stock price S_t follows

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t. \tag{10}$$

Note that in this situation, the expected return rate of the underlying stock is the risk-free rate of interest rr (assuming the underlying pays no income).

The equivalence between (7)-(8) and (9) follows from the Feynman-Kac formula; see Oksendal (2003).

Equation (9) is the theoretical basis of Monte Carlo simulation for derivative pricing. The simulation can be carried out by the following procedure:

(i) Simulate the price movement of the underlying stock in a risk-neutral world according to (10);

(ii) Calculate the expected terminal payoff of the derivative;

(iii) Discount the expected payoff at the risk-free interest rate.

3.4 Continuous Dividend Payment

Let q be the continuous dividend yield. This means that in a time period dt, the underlying stock pays a dividend of $qS_t dt$.

Assignment 1: Show that for the binomial model, the risk neutral probability is adjusted as

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}.$$

Assignment 2: Assume that the underlying stock pays a continuous dividend with yield q. Use the option replication argument to derive the corresponding the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0.$$

Exercise: Derive the Black-Scholes equation from the binomial model when the underlying stock pays a continuous dividend.