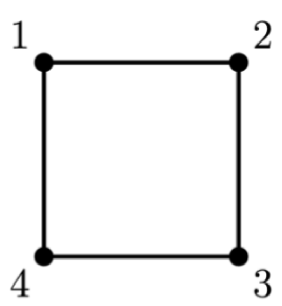
GINZBURG–LANDAU ON LARGE GRAPH LIMITS

Edith Zhang[1], James Scott[1], Qiang Du[1], Mason Porter[2]

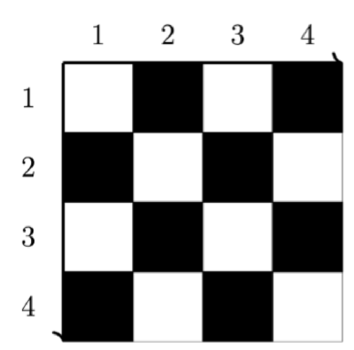
[1]Columbia University Department of Applied Mathematics
[2]University of California, Los Angeles Department of Mathematics

Graphons: large graph limits

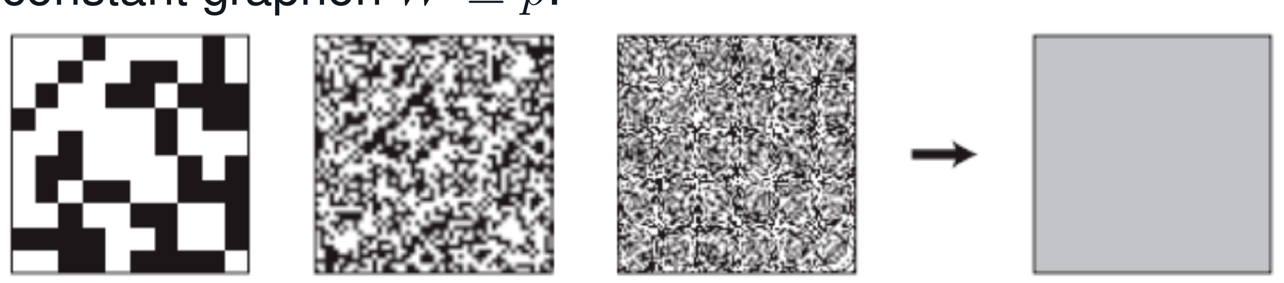
- Large networks are increasingly popular in datasets, e.g. social networks. One studies optimization problems on graphs, e.g. the min-cut problem.
- Idea: describe a continuum limit of large graphs: number of nodes $n \to \infty$. Graphon is a notion of infinite-node graphs: a function
$$W : [0,1]^2 \to \mathbb{R}$$
that generalizes the adjacency matrix (a linear operator $\mathbb{R}^n \to \mathbb{R}^n$) to a function (a linear operator $L^p \to \mathbb{R}$) [3].
- Example: 4-cycle graph, its adjacency matrix, and its graphon.



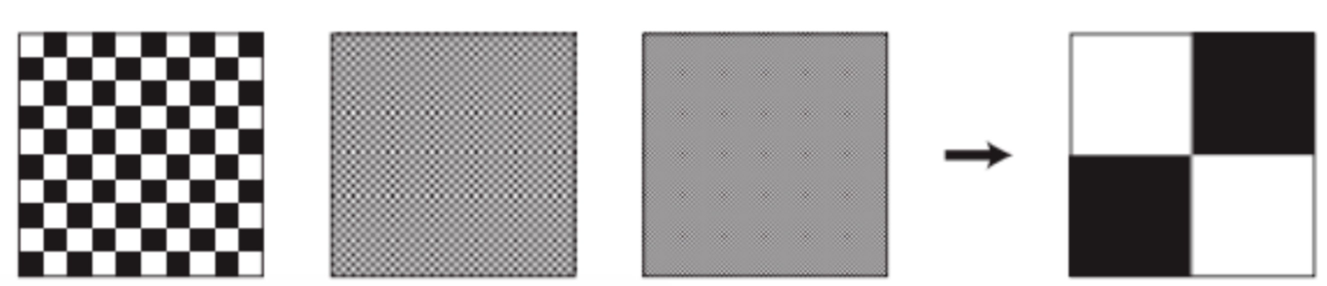
$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$



- Notion of graph convergence is defined by "cut convergence", which is convergence with respect to the following cut norm:
$$\|W\|_\square = \sup_{S,T \subseteq [0,1]} \left| \int_{S\times T} W(x,y)\,dx\,dy \right|.$$
- Example: the Erdos-Renyi random graph ER(p) converges to the constant graphon $W \equiv p$:



- Note however that not all graphs that appear random converge to a constant graphon, since graphons are invariant up to permutations.



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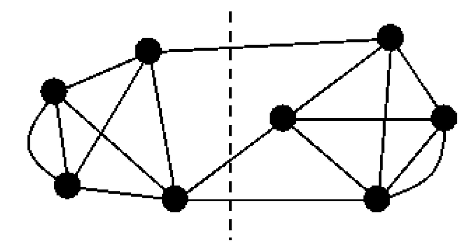
Ginzburg–Landau on graphs

- Ginzburg–Landau theory is a rich field, but we study one application: a relaxation of the minimum cut problem.



- The graph min-cut problem stated in terms of graph functions:
$$\min_{S\subset\{1,\dots,n\}} \sum_{i\in S, j\in S^c} A_{ij} = \min_{u:\{1,\dots,n\}\to\{-1,1\}} \sum_{i,j=1}^{n} A_{ij}|u(i)-u(j)|^2$$
- This is a problem that optimizes over $u : [0,1]^2 \to \{-1,1\}^n$! This is a huge combinatorial optimization problem. Ginzburg–Landau theory relaxes this problem to smooth functions $u : [0,1]^2 \to [-1,1]$.

Contributions

- Γ-convergence is a notion of convergence of functionals used in calculus of variations that guarantees that if functionals $F_n \xrightarrow{\Gamma} F$, then $\arg\min F_n \to \arg\min F$ [1]. We demonstrate four Γ-convergences as shown in the figure.

$$\begin{tikzcd}
 & \mathrm{TV}^{W_n}\ (8) \arrow[dr, "\substack{n\to\infty \\ \text{limit (2)}}"] & \\
\mathrm{GL}_\epsilon^{W_n}\ (7) \arrow[ur, "\substack{\epsilon\to 0 \\ \text{limit (1)}}"] \arrow[dr, "\substack{n\to\infty \\ \text{limit (3)}}"'] & & \mathrm{TV}_\epsilon^{W}\ (15) \\
 & \mathrm{GL}_\epsilon^{W}\ (13) \arrow[ur, "\substack{\epsilon\to 0 \\ \text{limit (4)}}"'] &
\end{tikzcd}$$

- A similar diagram of Γ-convergences was shown in [4], but restricted to the case of the square grid graph, which has the Euclidean plane as its large-graph limit, and worked with classical GL and TV functionals.
- We show more general, nonlocal, limits.
- Limit (4) is our main technical result, and extends the main theoretical result of [2].

Graphons and functional analysis

- Graphons can be viewed as a nonlocal kernel.
- For example, the Dirichlet energy functional $D(u) = \int_\Omega |\nabla^2 u(x)|dx$. Whereas, the graph Dirichlet energy is $D^A(u) = \frac{1}{n}\sum_{i,j=1}^n A_{ij}|u(i)-u(j)|^2$. The graphon Dirichlet energy $\int_0^1\int_0^1 W(x,y)|u(x)-u(y)|^2 dx dy$ is continuous and nonlocal.
- Graphons can be associated with the kernel operator $T_W(f)(x) = \int_0^1\int_0^1 W(x,y)f(y)dy$. The cut norm $\|W\|_\square$ is equivalent to operator norm of this kernel operator induced by W.
- Graphons are well-suited for calculus of variations, because we can continuously deform graphon-related objects.

Limitations and future work

- Limitation and future study: understanding the appearance of Young measures.
- Applications such as image segmentation, and developing an algorithm to compute minimizers.
- Ask about the rate of Γ-convergence.

Acknowledgements

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