

The Commutation of the Γ Operators

An Algebraic Proof for One-Part Partitions

Let λ and μ be one-part partitions — i.e. integers. We wish to show the identity

$$\langle \mu | \Gamma_-(b) \Gamma_+(a) | \lambda \rangle = \frac{1}{1-ab} \langle \mu | \Gamma_+(a) \Gamma_-(b) | \lambda \rangle.$$

We begin with the right side. We have that

$$\langle \mu | \Gamma_+(a) \Gamma_-(b) | \lambda \rangle = \sum_{\mu > \eta < \lambda} b^{|\lambda| - |\eta|} a^{|\mu| - |\eta|}.$$

Let $M = \max(\lambda, \mu)$ and $m = \min(\lambda, \mu)$. Since η can be at most a one-part partition, we can write the previous equation as

$$\sum_{\mu \geq \eta \leq \lambda} b^{\lambda - \eta} a^{\mu - \eta} = \sum_{0 \leq \eta \leq m} b^{\lambda - \eta} a^{\mu - \eta}.$$

Now let's consider the left side of the identity. Since the first operator is Γ_+ , we will be summing over η that are at most two-partitions, which we can write as $\eta = (\eta_1, \eta_2)$. Then we have

$$\begin{aligned} \langle \mu | \Gamma_-(b) \Gamma_+(a) | \lambda \rangle &= \sum_{\mu < \eta > \lambda} b^{|\eta| - |\lambda|} a^{|\eta| - |\mu|} \\ &= \sum_{\eta_1 \geq M} \sum_{0 \leq \eta_2 \leq m} a^{\eta_1 + \eta_2 - \lambda} b^{\eta_1 + \eta_2 - \mu} \\ &= \sum_{0 \leq \eta_2 \leq m} \sum_{\eta_1 \geq M} a^{\eta_1 + \eta_2 - \lambda} b^{\eta_1 + \eta_2 - \mu} \\ &= \sum_{0 \leq \eta_2 \leq m} \sum_{\eta_1 \geq 0} a^{\eta_1 + \eta_2 - \lambda + M} b^{\eta_1 + \eta_2 - \mu + M}. \end{aligned}$$

Now $\lambda + \mu = m + M$, so $M - \lambda = \mu - m$ and $M - \mu = \lambda - m$. Since $0 \leq \eta_2 \leq m$, we have that $-m \leq \eta_2 - m \leq 0$. Therefore, setting $i = -(\eta_2 - m)$, we have

$$\begin{aligned}
\langle \mu | \Gamma_-(b) \Gamma_+(a) | \lambda \rangle &= \sum_{0 \leq \eta_2 \leq m} \sum_{\eta_1 \geq 0} a^{\eta_1 + \eta_2 + \mu - m} b^{\eta_1 + \eta_2 + \lambda - m} \\
&= \sum_{0 \leq i \leq m} \sum_{\eta_1 \geq 0} a^{\eta_1 + \mu - i} b^{\eta_1 + \lambda - i} \\
&= \sum_{0 \leq i \leq m} \sum_{\eta_1 \geq 0} (ab)^{\eta_1} a^{\mu - i} b^{\lambda - i} \\
&= \frac{1}{1 - ab} \sum_{0 \leq i \leq m} a^{\mu - i} b^{\lambda - i} \\
&= \frac{1}{1 - ab} \langle \mu | \Gamma_+(a) \Gamma_-(b) | \lambda \rangle,
\end{aligned}$$

as required. Thus Γ_+ and Γ_- commute for one-part partitions.

A Combinatorial Proof for Arbitrary Partitions

We now give a combinatorial proof of the commutation for any partitions μ and λ with an equal number of parts, say m . This amounts to a bijection sending η with $\mu < \eta > \lambda$ to another partition η' with $\mu > \eta' < \lambda$ and an integer n , such that

$$a^{|\eta| - |\lambda|} b^{|\eta| - |\mu|} = (ab)^n b^{|\lambda| - |\eta|} a^{|\mu| - |\eta|}.$$

We accomplish this using toggles. Toggling η with respect to λ and μ produces a partition η' as follows: we set $\eta'_i = \min(\lambda_i, \mu_i) + \max(\lambda_{i+1}, \mu_{i+1}) - \eta_{i+1}$ for $1 \leq i \leq m$, where we set $\lambda_{m+1} = \mu_{m+1} = 0$. We then take $n = \eta_1 - \max(\lambda_1, \mu_1)$.

We have $\mu < \eta > \lambda$ if and only if $\min(\lambda_i, \mu_i) \geq \eta_{i+1} \geq \max(\lambda_{i+1}, \mu_{i+1})$ for $1 \leq i \leq n$, and $\eta_1 \geq \max(\lambda_1, \mu_1)$. But the first criterion is equivalent to

$$-\min(\lambda_i, \mu_i) \leq -\eta_{i+1} \leq -\max(\lambda_{i+1}, \mu_{i+1}),$$

which is equivalent to

$$\max(\lambda_{i+1}, \mu_{i+1}) \leq \min(\lambda_i, \mu_i) + \max(\lambda_{i+1}, \mu_{i+1}) - \eta_{i+1} \leq \min(\lambda_i, \mu_i),$$

and therefore

$$\max(\lambda_{i+1}, \mu_{i+1}) \leq \eta'_i \leq \min(\lambda_i, \mu_i).$$

But this is exactly the statement that $\lambda > \eta' < \mu$. And the second criterion is equivalent to $n \geq 0$. Now we can recover every η_i for $i > 1$ by $\eta_i = \min(\lambda_{i-1}, \mu_{i-1}) + \max(\lambda_i, \mu_i) - \eta'_{i-1}$, and we can recover η_1 by $\eta_1 = \max(\lambda_1, \mu_1) + n$. Thus the operation is invertible, and therefore a bijection. Combinatorially, this means a partition interlacing both λ and μ can be identified with its toggle relative to λ and μ , along with the amount that its first entry is larger than the minimum possible value.