The Commutation of the Γ Operators

An Algebraic Proof for One-Part Partitions

Let λ and μ be one-part partitions — i.e. integers. We wish to show the identity

$$\langle\mu|\,\Gamma_-(b)\Gamma_+(a)\,|\lambda\rangle = \frac{1}{1-ab}\,\langle\mu|\,\Gamma_+(a)\Gamma_-(b)\,|\lambda\rangle\,.$$

We begin with the right side. We have that

$$\langle\mu|\,\Gamma_+(a)\Gamma_-(b)\,|\lambda\rangle = \sum_{\mu\succ\eta\prec\lambda} b^{|\lambda|-|\eta|}a^{|\mu|-|\eta|}.$$

Let $M = \max(\lambda,\mu)$ and $m = \min(\lambda,\mu)$. Since η can be at most a one-part partition, we can write the previous equation as

$$\sum_{\mu\geq\eta\leq\lambda} b^{\lambda-\eta}a^{\mu-\eta} = \sum_{0\leq\eta\leq m} b^{\lambda-\eta}a^{\mu-\eta}.$$

Now let's consider the left side of the identity. Since the first operator is Γ_+, we will be summing over η that are at most two-partitions, which we can write as $\eta = (\eta_1,\eta_2)$. Then we have

$$\begin{aligned}
\langle\mu|\,\Gamma_-(b)\Gamma_+(a)\,|\lambda\rangle &= \sum_{\mu\prec\eta\succ\lambda} b^{|\eta|-|\lambda|}a^{|\eta|-|\mu|} \\
&= \sum_{\eta_1\geq M}\sum_{0\leq\eta_2\leq m} a^{\eta_1+\eta_2-\lambda}b^{\eta_1+\eta_2-\mu} \\
&= \sum_{0\leq\eta_2\leq m}\sum_{\eta_1\geq M} a^{\eta_1+\eta_2-\lambda}b^{\eta_1+\eta_2-\mu} \\
&= \sum_{0\leq\eta_2\leq m}\sum_{\eta_1\geq 0} a^{\eta_1+\eta_2-\lambda+M}b^{\eta_1+\eta_2-\mu+M}.
\end{aligned}$$

Now $\lambda+\mu = m+M$, so $M-\lambda = \mu-m$ and $M-\mu = \lambda-m$. Since $0\leq\eta_2\leq m$, we have that $-m\leq\eta_2-m\leq 0$. Therefore, setting $i = -(\eta_2-m)$, we have

$$
\begin{aligned}
\langle \mu | \, \Gamma_-(b)\Gamma_+(a) \, | \lambda \rangle &= \sum_{0 \le \eta_2 \le m} \sum_{\eta_1 \ge 0} a^{\eta_1 + \eta_2 + \mu - m} b^{\eta_1 + \eta_2 + \lambda - m} \\
&= \sum_{0 \le i \le m} \sum_{\eta_1 \ge 0} a^{\eta_1 + \mu - i} b^{\eta_1 + \lambda - i} \\
&= \sum_{0 \le i \le m} \sum_{\eta_1 \ge 0} (ab)^{\eta_1} a^{\mu - i} b^{\lambda - i} \\
&= \frac{1}{1 - ab} \sum_{0 \le i \le m} a^{\mu - i} b^{\lambda - i} \\
&= \frac{1}{1 - ab} \langle \mu | \, \Gamma_+(a)\Gamma_-(b) \, | \lambda \rangle ,
\end{aligned}
$$

as required. Thus Γ_+ and Γ_- commute for one-part partitions.

A Combinatorial Proof for Arbitrary Partitions

We now give a combinatorial proof of the commutation for any partitions μ and λ with an equal number of parts, say m. This amounts to a bijection sending η with $\mu \prec \eta \succ \lambda$ to another partition η' with $\mu \succ \eta' \prec \lambda$ and an integer n, such that

$$
a^{|\eta| - |\lambda|} b^{|\eta| - |\mu|} = (ab)^n b^{|\lambda| - |\eta'|} a^{|\mu| - |\eta'|}.
$$

We accomplish this using toggles. Toggling η with respect to λ and μ produces a partition η' as follows: we set $\eta'_i = \min(\lambda_i, \mu_i) + \max(\lambda_{i+1}, \mu_{i+1}) - \eta_{i+1}$ for $1 \le i \le m$, where we set $\lambda_{m+1} = \mu_{m+1} = 0$. We then take $n = \eta_1 - \max(\lambda_1, \mu_1)$.

We have $\mu \prec \eta \succ \lambda$ if and only if $\min(\lambda_i, \mu_i) \ge \eta_{i+1} \ge \max(\lambda_{i+1}, \mu_{i+1})$ for $1 \le i \le n$, and $\eta_1 \ge \max(\lambda_1, \mu_1)$. But the first criterion is equivalent to

$$
-\min(\lambda_i, \mu_i) \le -\eta_{i+1} \le -\max(\lambda_{i+1}, \mu_{i+1}),
$$

which is equivalent to

$$
\max(\lambda_{i+1}, \mu_{i+1}) \le \min(\lambda_i, \mu_i) + \max(\lambda_{i+1}, \mu_{i+1}) - \eta_{i+1} \le \min(\lambda_i, \mu_i),
$$

and therefore

$$
\max(\lambda_{i+1}, \mu_{i+1}) \le \eta'_i \le \min(\lambda_i, \mu_i).
$$

But this is exactly the statement that $\lambda \succ \eta' \prec \mu$. And the second criterion is equivalent to $n \geq 0$. Now we can recover every η_i for $i > 1$ by $\eta_i = \min(\lambda_{i-1}, \mu_{i-1}) + \max(\lambda_i, \mu_i) - \eta'_{i-1}$, and we can recover η_1 by $\eta_1 = \max(\lambda_1, \mu_1) + n$. Thus the operation is invertible.

It remains to show that the weights match — specifically, that $|\eta| - |\lambda| = |\mu| - |\eta'|$. We have that $|\eta'| = |\lambda| + |\mu| - |\eta|$ since each entry of λ and μ is added to η' exactly once. Thus the weights are as required, and together with invertibility, our modified toggle is therefore a bijection. Combinatorially, this means a partition interlacing both λ and μ can be identified with its toggle relative to λ and μ, along with the amount that its first entry is larger than the minimum possible value.