The Commutation of the Γ Operators

An Algebraic Proof for One-Part Partitions

Let λ and μ be one-part partitions — i.e. integers. We wish to show the identity

$$\langle\mu|\, \Gamma_-(b)\Gamma_+(a)\, |\lambda\rangle = \frac{1}{1-ab}\, \langle\mu|\, \Gamma_+(a)\Gamma_-(b)\, |\lambda\rangle\,.$$

We begin with the right side. We have that

$$\langle\mu|\, \Gamma_+(a)\Gamma_-(b)\, |\lambda\rangle = \sum_{\mu \succ \eta \prec \lambda} b^{|\lambda|-|\eta|} a^{|\mu|-|\eta|}.$$

Let $M = \max(\lambda, \mu)$ and $m = \min(\lambda, \mu)$. Since η can be at most a one-part partition, we can write the previous equation as

$$\sum_{\mu \geq \eta \leq \lambda} b^{\lambda-\eta} a^{\mu-\eta} = \sum_{0 \leq \eta \leq m} b^{\lambda-\eta} a^{\mu-\eta}.$$

Now let's consider the left side of the identity. Since the first operator is Γ_+, we will be summing over η that are at most two-partitions, which we can write as $\eta = (\eta_1, \eta_2)$. Then we have

$$\begin{aligned}
\langle\mu|\, \Gamma_-(b)\Gamma_+(a)\, |\lambda\rangle &= \sum_{\mu \prec \eta \succ \lambda} b^{|\eta|-|\lambda|} a^{|\eta|-|\mu|} \\
&= \sum_{\eta_1 \geq M} \sum_{0 \leq \eta_2 \leq m} a^{\eta_1+\eta_2-\lambda} b^{\eta_1+\eta_2-\mu} \\
&= \sum_{0 \leq \eta_2 \leq m} \sum_{\eta_1 \geq M} a^{\eta_1+\eta_2-\lambda} b^{\eta_1+\eta_2-\mu} \\
&= \sum_{0 \leq \eta_2 \leq m} \sum_{\eta_1 \geq 0} a^{\eta_1+\eta_2-\lambda+M} b^{\eta_1+\eta_2-\mu+M}.
\end{aligned}$$

Now $\lambda + \mu = m + M$, so $M - \lambda = \mu - m$ and $M - \mu = \lambda - m$. Since $0 \leq \eta_2 \leq m$, we have that $-m \leq \eta_2 - m \leq 0$. Therefore, setting $i = -(\eta_2 - m)$, we have

$$
\begin{aligned}
\langle\mu|\,\Gamma_-(b)\Gamma_+(a)\,|\lambda\rangle &= \sum_{0\le\eta_2\le m}\sum_{\eta_1\ge 0} a^{\eta_1+\eta_2+\mu-m} b^{\eta_1+\eta_2+\lambda-m} \\
&= \sum_{0\le i\le m}\sum_{\eta_1\ge 0} a^{\eta_1+\mu-i} b^{\eta_1+\lambda-i} \\
&= \sum_{0\le i\le m}\sum_{\eta_1\ge 0} (ab)^{\eta_1} a^{\mu-i} b^{\lambda-i} \\
&= \frac{1}{1-ab}\sum_{0\le i\le m} a^{\mu-i} b^{\lambda-i} \\
&= \frac{1}{1-ab}\,\langle\mu|\,\Gamma_+(a)\Gamma_-(b)\,|\lambda\rangle\,,
\end{aligned}
$$

as required. Thus Γ_+ and Γ_- commute for one-part partitions.

A Combinatorial Proof for Arbitrary Partitions

Γ_+ commutes with Γ_-

We now give a combinatorial proof of the commutation for any partitions μ and λ with an equal number of parts, say m. This amounts to a bijection sending η with $\mu \prec \eta \succ \lambda$ to another partition η' with $\mu \succ \eta' \prec \lambda$ and an integer n, such that

$$a^{|\eta|-|\lambda|} b^{|\eta|-|\mu|} = (ab)^n b^{|\lambda|-|\eta'|} a^{|\mu|-|\eta'|}.$$

We accomplish this using toggles. Toggling η with respect to λ and μ produces a partition η' as follows: we set $\eta'_i = \min(\lambda_i,\mu_i)+\max(\lambda_{i+1},\mu_{i+1})-\eta_{i+1}$ for $1\le i\le m$, where we set $\lambda_{m+1}=\mu_{m+1}=0$. We then take $n=\eta_1-\max(\lambda_1,\mu_1)$.

We have $\mu \prec \eta \succ \lambda$ if and only if $\min(\lambda_i,\mu_i)\ge\eta_{i+1}\ge\max(\lambda_{i+1},\mu_{i+1})$ for $1\le i\le n$, and $\eta_1\ge\max(\lambda_1,\mu_1)$. But the first criterion is equivalent to

$$-\min(\lambda_i,\mu_i)\le-\eta_{i+1}\le-\max(\lambda_{i+1},\mu_{i+1}),$$

which is equivalent to

$$\max(\lambda_{i+1},\mu_{i+1})\le\min(\lambda_i,\mu_i)+\max(\lambda_{i+1},\mu_{i+1})-\eta_{i+1}\le\min(\lambda_i,\mu_i),$$

and therefore

$$\max(\lambda_{i+1}, \mu_{i+1}) \leq \eta'_i \leq \min(\lambda_i, \mu_i).$$

But this is exactly the statement that $\lambda \succ \eta' \prec \mu$. And the second criterion is equivalent to $n \geq 0$. Now we can recover every η_i for $i > 1$ by $\eta_i = \min(\lambda_{i-1}, \mu_{i-1}) + \max(\lambda_i, \mu_i) - \eta'_{i-1}$, and we can recover η_1 by $\eta_1 = \max(\lambda_1, \mu_1) + n$. Thus the operation is invertible.

It remains to show that the weights match — specifically, that $|\eta| - |\lambda| = |\mu| - |\eta'|$. We have that $|\eta'| = |\lambda| + |\mu| - |\eta|$ since each entry of λ and μ is added to η' exactly once. Thus the weights are as required, and together with invertibility, our modified toggle is therefore a bijection. Combinatorially, this means a partition interlacing both λ and μ can be identified with its toggle relative to λ and μ, along with the amount that its first entry is larger than the minimum possible value.

Γ_+ and Γ_- commute with themselves

The logic is identical for both Γ_+ and Γ_- — we will discuss the former. We wish to show

$$\sum_{\lambda \prec \eta \prec \mu} a^{|\eta|-|\lambda|} b^{|\mu|-|\eta|} = \sum_{\lambda \prec \eta \prec \mu} a^{|\mu|-|\eta|} b^{|\eta|-|\lambda|},$$

so in order to give a combinatorial proof, we will need to identify a partition η with $\lambda \prec \eta \prec \mu$ with another partition η' also satisfying $\lambda \prec \eta' \prec \mu$, such that $|\mu| - |\eta| = |\eta'| - |\lambda|$ and $|\eta| - |\lambda| = |\mu| - |\eta'|$. Once again, we can use toggles. Since $\mu \succ \eta$, the toggle of η relative to λ and μ no longer destroys its first entry, removing the free variable we labeled n in the previous section. We have

$$\eta'_i = \min(\lambda_{i-1}, \mu_i) + \max(\lambda_i, \mu_{i+1}) - \eta_i, i > 1,$$

$$\eta'_1 = \mu_1 + \max(\lambda_1, \mu_2) - \eta_1.$$

As before, this ensures that $\lambda \prec \eta' \prec \mu$, and we also have that $|\eta'| = |\lambda| + |\mu| - |\eta|$. Rearranging, we see that $|\mu| - |\eta| = |\eta'| - |\lambda|$ and $|\eta| - |\lambda| = |\mu| - |\eta'|$, as required.