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MECHANICS

Dynamics of Maxwell's Pendulum

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Abstract—The stability of motion of Maxwell's pendulum is investigated in a uniform gravity field. By means of several canonic transforms of the equations of pendulum motion and the method of the surfaces of Poincaré sections, the problem is reduced to investigation of the immobile-point stability retaining the area of mapping of the plane into itself. In the space of dimensionless parameters, the stability and instability regions are singled out.

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FORMULATION OF PROBLEM

Maxwell's pendulum is a physical device that represents a massive disk rigidly fixed on the axis perpendicular to the disk plane and passing through its center of gravity. The axis by means of two long threads is suspended so that the disk-equilibrium state occupies the horizontal position. This device is used for demonstration of the basic theorems of dynamics by the example of plane motion of a solid [1, 2]. When carrying out an experiment, the thread is wound on the axis accurately coil-to-coil due to which the disk rises to a certain height. If the disk is then released, it starts to fall downwards under the action of gravity rotating around its axis. In this case, the threads are stretched and the axis remains horizontal. Such a motion proceeds up to the moment of the full unwinding of the threads. After that moment, the disk causes a "jerk" on the threads and starts to rise upwards continuing to rotate, and the threads remaining stretched wind onto the axis. If the threads are inextensible and the resistance of the external medium is negligible, the disk rises to its initial height, and, subsequently, the process is repeated. The periodicity of this process gives grounds to call the device under consideration a pendulum.

In the motion described, the angle which is made by the threads with the vertical is zero. It is of interest to consider the problem about the stability of the pendulum motion with respect to small deviations of threads from the vertical.

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EQUATIONS OF MOTION

Let m be the pendulum mass (equal to the sum of masses of the disk and the axis on which it is fixed). We consider that the axis is a cylinder of radius r, and the pendulum center of masses C coincides with the center of masses of the disk. We designate the moment of inertia of the pendulum with respect to the axis perpendicular to the disk plane and passing through the point C as J_c. Let ℓ be the length of each thread in the unwound state, and let OA be the distance from the point O of the suspension of a certain thread to the point A of its descent from the pendulum axis (Fig. 1). We set the position of the pendulum with two generalized coordinates: the angle θ between the vertical and the direction of threads, and the value of $\xi = \ell - OA$. Because of the inextensibility of threads, we have a unilateral constraint $\xi \geq 0$.

At $\xi = 0$ (the stressed constraint), the motion of the pendulum has the character of an absolutely elastic impact: the preimpact ($\dot{\xi}^-$) and after-impact ($\dot{\xi}^+$) values of the generalized velocity are related as $\dot{\xi}^+ = -\dot{\xi}^-$.

At $\xi > 0$ (the weakened constraint), the motion of the pendulum is described by canonical equations with the Hamiltonian function

$$\Gamma = \frac{r^2}{2(J_c + mr^2)} p_\xi^2 + \frac{(p_\theta - rp_\xi)^2}{2m(\ell - \xi)^2} + mg[\xi\cos\theta + r\sin\theta + \ell(1 - \cos\theta)], \tag{1}$$

where p_θ and p_ξ are the generalized momenta corresponding to the coordinates θ and ξ and g is acceleration due to gravity.

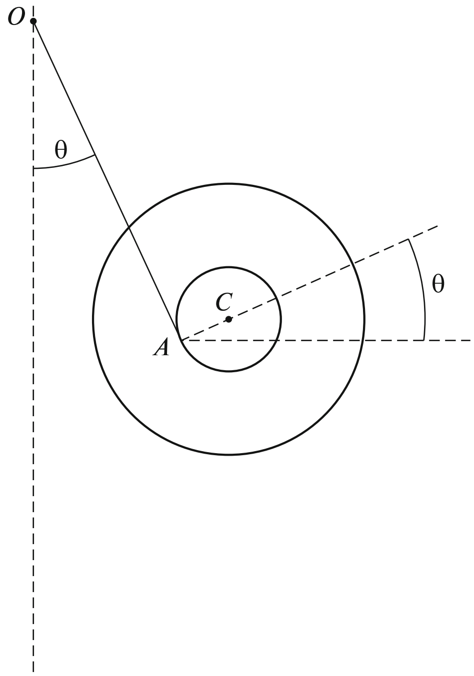

Fig. 1. Maxwell's pendulum deviated from the vertical.

During the impact, the preimpact (p_ξ^-) and afterimpact (p_ξ^+) values of the momentum p_ξ are related as [3]

$$p_\xi^+ = -p_\xi^- + 2\frac{J_c + mr^2}{r[J_c + m(r^2 + \ell^2)]}p_\theta. \tag{2}$$

The equations of motion admit a periodic solution with period $T = 2\sqrt{2\alpha\beta\ell/g}$, where α and β are dimensionless parameters:

$$\alpha = \frac{h}{\ell} \quad (0 < \alpha < 1), \quad \beta = 1 + \frac{J_c}{mr^2} \quad (\beta > 1).$$

This solution is set by the equalities

$$\xi = h\varphi_2(\tau), \quad p_\xi = m\sqrt{2\alpha\beta g\ell}\,\varphi_1(\tau), \quad \theta = 0, \\ p_\theta = rp_\xi, \quad \tau = 2\pi t/T. \tag{3}$$

Here, φ_1 and $\varphi_2 - 2\pi$ are the periodic functions of τ. At $0 \le \tau < 2\pi$, they are set by the equalities

$$\varphi_1 = 1 - \frac{\tau}{\pi}, \quad \varphi_2 = \frac{\tau}{\pi}\left(2 - \frac{\tau}{\pi}\right). \tag{4}$$

For solution (3), the pendulum threads are vertical and the disk accomplishes the motion described above. For the initial moment of time $t = 0$, we accepted the moment when $\xi = 0$. The center of masses of the disk begins its motion upwards with the velocity $\sqrt{\frac{2\alpha g\ell}{\beta}}$; the reverse motion downwards begins at the moment of time $t = T/2$, when ξ achieves the highest value equal to h.

TRANSFORM OF THE HAMILTONIAN FUNCTION

For investigating the orbital stability of periodic motion (3), we preliminarily perform several canonical transformations following [4, 5]. First, we replace the variables of the form

$$\xi = \ell x, \quad \theta = y + \sqrt{\beta}\left(\arctan\frac{1-x}{\mu} - \arctan\frac{1}{\mu}\right),$$
$$p_\xi = \frac{2\pi m\ell}{T}\left(p_x + \frac{\mu\sqrt{\beta}}{\mu^2 + (1-x)^2}p_y\right),$$
$$p_\theta = \frac{2\pi m\ell^2}{T}p_y, \quad \mu = \sqrt{\frac{J_c + mr^2}{m\ell^2}}. \tag{5}$$

The dimensionless parameter μ can be considered as small ($0 < \mu \ll 1$) because of the large length of threads.

If we accept the value of τ from (3) for a new independent variable, the following Hamiltonian function corresponds to the new variables

$$G = G_0 + \frac{p_y^2}{2(1-x)^2}$$
$$+ \frac{2\alpha\beta}{\pi^2}(1-x)(1-\cos\theta) + \mu\frac{2\alpha\sqrt{\beta}}{\pi^2}\sin\theta \tag{6}$$
$$+ \frac{\mu^2}{2(1-x)^2}\left[\frac{p_x^2}{\beta} - \frac{p_y^2}{\mu^2 + (1-x)^2}\right], \quad G_0 = \frac{p_x^2}{2\beta} + \frac{2\alpha\beta}{\pi^2}x.$$

The value of θ in Eq. (6) is determined by the second equality of set (5). During the entire time of motion, the inequality $x \ge 0$ is fulfilled. At the stage of impact motion, when $x = 0$, the values of y, p_y are constant and

$$p_x^+ = -p_x^-. \tag{7}$$

In set (6) with the Hamiltonian function, we introduce the variables q_i, p_i $i = 1, 2$, by the formulas

$$x = \left(\frac{9\pi^4}{16\alpha\beta^2}\right)^{1/3} p_1^{2/3}\varphi_2(q_1),$$
$$p_x = \left(\frac{6\alpha\beta^2}{\pi}\right)^{1/3} p_1^{1/3}\varphi_1(q_1), \quad y = q_2, \quad p_y = p_2.$$

The values of p_1, q_1 are the action–angle variables in the set with the Hamiltonian function G_0 (with taking into account relation (7)), and φ_1, φ_2 are functions (4).

In the variables q_i, p_i, periodic motion (3) of the pendulum is written in the form $q_i = f_i(\tau)$, $p_i = g_i(\tau)$,