Linear Hard Margin SVM for Classification and Pattern Recognition

André Ambrósio Boechat

Departamento de Automação e Sistemas
Universidade Federal de Santa Catarina

Florianópolis, October 2012

Contents

Data-Driven Models

- Regression, pattern recognition, classification

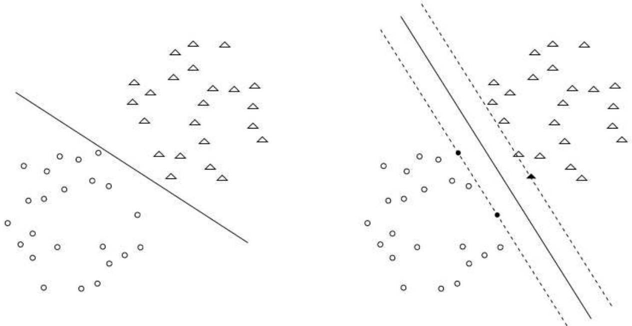
Classification

Separation of samples in different classes, trying to find a possible decision boundary.

- Well-known classifiers
  - Naive Bayes classifier
  - Logistic regression
  - Artificial Neural networks (ANN)
  - Support Vector Machines (SVM)

Support Vector Machines

- The foundations were recently developed (Vapnik, 1995)
- Greater ability to generalize than ANN
- Many possible formulations
  - linear
  - nonlinear
  - regression
- Large margin classifier



Geometrical Interpretation [Burges, 1998]

Training data composed of m samples

$$\left\{\mathbf{x}^{(i)}, y^{(i)}\right\}, \quad i = 1, \cdots, m$$
$$\mathbf{x}^{(i)} \in \mathbb{R}^n$$
$$y^{(i)} \in \{-1, 1\}$$

We want to find a $f(\mathbf{x})$ which determines a separating hyperplane

$$\mathbf{w}^T\mathbf{x} + b = 0 \tag{1}$$

For the linearly separable case, a hyperplane with the largest margin must obey the following constraints

$$\begin{aligned} \mathbf{w}^T\mathbf{x}^{(i)} + b \geq 1, \qquad & \text{for} \quad y^{(i)} = 1 \\ \mathbf{w}^T\mathbf{x}^{(i)} + b \leq -1, \qquad & \text{for} \quad y^{(i)} = -1 \end{aligned} \tag{2}$$

Hyperplane with the Largest Margin

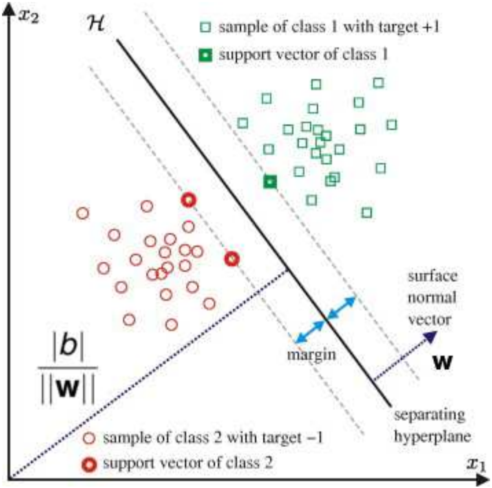


Figure: Adapted from [Fisch et al., 2010]

The support hyperplanes given by the constraints of Eq. (2) are

$$\begin{aligned} \mathcal{H}_1 : \quad & \mathbf{w}^T \mathbf{x}^{(i)} + b = 1 \\ \mathcal{H}_2 : \quad & \mathbf{w}^T \mathbf{x}^{(i)} + b = -1 \end{aligned} \tag{3}$$

With simple geometry we could find that the sum of $\mathcal{H}_1$ and $\mathcal{H}_2$ margins is given by

$$\frac{2}{||\mathbf{w}||}$$

Rewritten the constraints of Eq. (2), we have

SVM problem — Hard margin

$$\begin{aligned} & \text{Minimize} \quad ||\mathbf{w}||^2 \\ & \text{s.t.} \quad -y^{(i)} \left(\mathbf{w}^T \mathbf{x}^{(i)} + b \right) + 1 \leq 0, \quad \forall i \end{aligned} \tag{4}$$

Lagrangian Formulation

Primal problem

$$\max_{\lambda} \min_{\mathbf{w},b} \quad \mathcal{L}(\mathbf{w}, b, \lambda) = \frac{1}{2}||\mathbf{w}||^2 - \sum_{i=1}^{m} \lambda_i y^{(i)} \left(\mathbf{w}^T \mathbf{x}^{(i)} + b\right) + \sum_{i=1}^{m} \lambda_i$$
$$\lambda_i \geq 0 \tag{5}$$

Dual problem

$$\max_{\lambda} \quad g(\lambda) = \inf_{\mathbf{w},b} \mathcal{L}(\mathbf{w}, b, \lambda)$$
$$\lambda_i \geq 0 \tag{6}$$

Lagrangian Dual Problem

To find $\inf_{\mathbf{w},b} \mathcal{L}(\mathbf{w}, b, \lambda)$ the gradient with respect to $\mathbf{w}$ and b must vanish, that gives the conditions

$$\mathbf{w} = \sum_{i=1}^{m} \lambda_i y^{(i)} \mathbf{x}^{(i)} \tag{7}$$

$$\sum_{i=1}^{m} \lambda_i y^{(i)} = 0 \tag{8}$$

Dual problem

$$\begin{aligned} \max_{\lambda} \quad & g(\lambda) = \sum_{i=1}^{m} \lambda_i - \frac{1}{2} \sum_{i,j=1}^{m} \lambda_i \lambda_j y^{(i)} y^{(j)} \left\langle \mathbf{x}^{(i)}, \mathbf{x}^{(j)} \right\rangle \\ & \sum_{i=1}^{m} \lambda_i y^{(i)} = 0 \\ & \lambda_i \geq 0 \end{aligned} \tag{9}$$

Considering the solution [Gunn, 1998]

$$\mathbf{w}^* = \sum_{i=1}^{m} \lambda_i y^{(i)} \mathbf{x}^{(i)}$$
$$b^* = -\frac{1}{2} \left\langle \mathbf{w}^*, \mathbf{x}^{(s_1)} + \mathbf{x}^{(s_2)} \right\rangle$$

- Those training samples for which $\lambda_i > 0$ are called support vectors and lie on $\mathcal{H}_1$ or $\mathcal{H}_2$.
- All other training samples have $\lambda_i = 0$ and lie either on $\mathcal{H}_1$ or $\mathcal{H}_2$, or on the half space determined by $\mathcal{H}_1$ or $\mathcal{H}_2$.
- The support vectors are the critical elements of the training set.

Karush-Kuhn-Tucker Conditions

Solving the SVM problem is equivalent to finding a solution to the KKT conditions. [Burges, 1998]

$$
\begin{aligned}
f_i(\mathbf{w}) = -y^{(i)}(\mathbf{w}^T\mathbf{x}^{(i)} + b) + 1 \leq \quad & 0 \\
\lambda_i \geq \quad & 0 \\
\lambda_i f_i(\mathbf{w}) = \lambda_i \left[-y^{(i)}(\mathbf{w}^T\mathbf{x}^{(i)} + b) + 1 \right] = \quad & 0 \\
\frac{\partial}{\partial \mathbf{w}}\mathcal{L} = \mathbf{w} - \sum_{i=1}^{m} \lambda_i y^{(i)}\mathbf{x}^{(i)} = \quad & 0 \\
\frac{\partial}{\partial b}\mathcal{L} = \sum_{i=1}^{m} \lambda_i y^{(i)} = \quad & 0
\end{aligned}
\tag{10}
$$

Equation (10) is used to determine the *b* value.

"If $\tilde{\mathbf{w}}, \tilde{b}, \tilde{\lambda}$ satisfy KKT for a convex problem, then they are optimal"

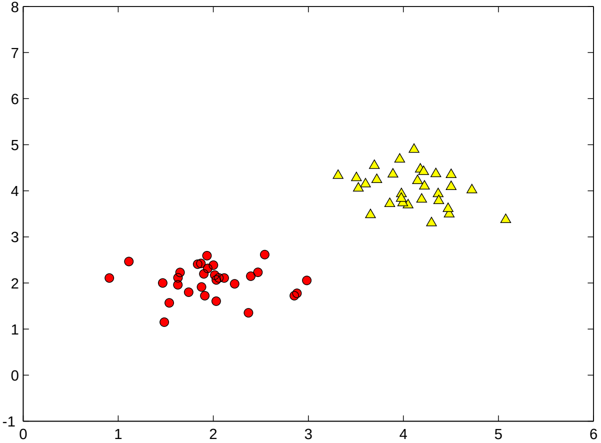
Tools

- SVM solvers/packages
  - LIBSVM
  - SVM light
  - Matlab SVM toolbox
  - List of softwares: www.support-vector-machines.org/SVM_soft.html
- CVX, a convex modeling framework for Matlab
  - problem description very similar to the mathematical formulation

Dataset Example

```
%% Linear separable data samples generation for training.
% Features dimension
n = 2;
% Number of samples
m = 2*30;
% Center of the classes
c1 = [2 2];
c2 = [4 4];
% Standard deviation from center
stdc = [.4 .4];
% Data samples -> X is MxN
X1 = repmat(c1, m/2, 1) + repmat(stdc, m/2, 1) .* randn(m/2, n);
X2 = repmat(c2, m/2, 1) + repmat(stdc, m/2, 1) .* randn(m/2, n);
X = [X1; X2];
% Labels -> Y is Mx1
Y = [ones(m/2, 1); -1*ones(m/2,1)];
```

Linear Separable Dataset



SVM Primal Problem

```
function [w, b] = svm_primal(X, Y)

[m, n] = size(X);
%% SVM formulation
cvx_begin
    variables w(1,n) b(1)
    minimize(pow_pos(norm(w, 2), 2))
    - Y .* (X * w' + ones(m,1)*b) + ones(m,1) <= zeros(m,1);
cvx_end
```

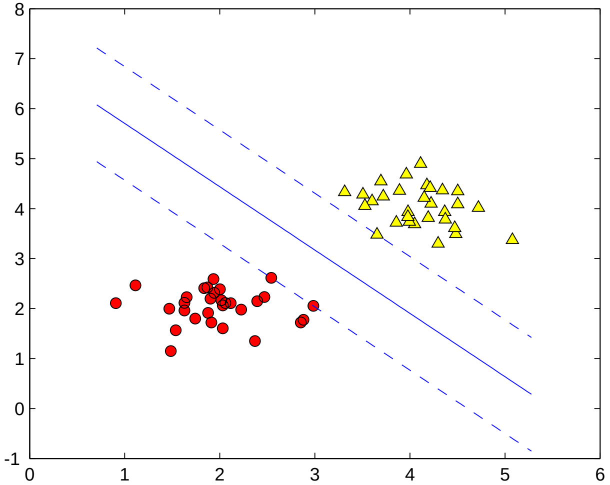
SVM Dual Problem

```
function [w, b] = svm_lagrangian(X, Y)

[m, n] = size(X);
%% Dual problem of the Lagrangian formulation.
Z = repmat(Y, 1, n) .* X;
H = Z * Z'; % MxM matrix
cvx_begin
    variables lambda(m, 1)
    %maximize ( sum(lambda) - .5 * lambda' * H * lambda )
    maximize ( sum(lambda) - .5 * quad_form(lambda, H) )
    lambda'*Y == 0;
    lambda >= zeros(m, 1);
cvx_end

% w is 1xN
w = (lambda .* Y)' * X;
% Non-zero lagrangian multipliers.
l1 = find(lambda > 1e-6 & Y == 1);
l2 = find(lambda > 1e-6 & Y == -1);
% b calculation as Gunn1998 suggested.
b = - .5 * w * (X(l1(1), :) + X(l2(1), :))';
```

Problem Solution



```
-------------------------------------------------------------------
 number of iterations   = 22
 primal objective value = -2.01515477e+00
 dual   objective value = -2.01515479e+00
 gap := trace(XZ)       = 1.58e-08
 relative gap           = 3.13e-09
 actual relative gap    = 3.13e-09
 rel. primal infeas     = 2.76e-12
 rel. dual   infeas     = 1.00e-12
 Total CPU time (secs)  = 0.61
 CPU time per iteration = 0.03
 termination code       =  0
-------------------------------------------------------------------
 number of iterations   = 14
 primal objective value = -1.00757737e+00
 dual   objective value = -1.00757740e+00
 gap := trace(XZ)       = 3.09e-08
 relative gap           = 1.02e-08
 actual relative gap    = 1.02e-08
 rel. primal infeas     = 1.56e-12
 rel. dual   infeas     = 1.01e-12
 Total CPU time (secs)  = 0.20
 CPU time per iteration = 0.01
 termination code       =  0
-------------------------------------------------------------------
```

Conclusions

- SVM is a famous technique for classification problems
- The SVM problem consists in looking for separating hyperplane with the largest margins
- It leads to a convex problem
- There are advantages in using the Lagrangian formulation
- The solution of the dual problem is optimal (KKT conditions)
- With a little modification, the presented formulation can handle more complex problems

Further Reading

Burges, C. (1998).
A tutorial on support vector machines for pattern recognition.
Data mining and knowledge discovery, 43:1–43.

Fisch, D., Kühbeck, B., Sick, B., and Ovaska, S. J. (2010).
So near and yet so far: New insight into properties of some well-known classifier paradigms.
Information Sciences, 180(18):3381 – 3401.

Gunn, S. (1998).
Support vector machines for classification and regression.
Technical Report 2.