$A \subseteq \mathbb{R}$ (subset of $\mathbb{R}$) $\to$ $ub(A) = \{x \in \mathbb{R} : x \geq a, \forall a \in A\}$ - set of upper bounds of A

$\to$ $lb(A) = \{x \in \mathbb{R} : x \leq a, \forall a \in A\}$ - set of lower bounds of A

$x \in \mathbb{R}$ (number) $\to$ upper (lower) bound of A if $x \in ub(A)$ ($x \in lb(A)$)

$\to$ maximum (greatest el.) of A if $x \in A \cap ub(A)$ - noted $\max A$

$\to$ minimum (lowest el.) of A if $x \in A \cap lb(A)$ - noted $\min A$

$A \subseteq \mathbb{R}$ $\to$ bounded above (below) if $ub(A) \neq \emptyset$ ($lb(A) \neq \emptyset$)

$\to$ bounded if bounded above and below

$\to$ unbounded if not bounded

$A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ $\to$ $ub(A) \neq \emptyset \Rightarrow$ x - supremum of A (sup A) if $x = \min(ub(A))$ - least upper bound

$\to$ $lb(A) \neq \emptyset \Rightarrow$ x - infimum of A (inf A) if $x = \max(lb(A))$ - greatest lower bound

SUPREMUM PROPERTY (SP): Every nonempty subset of $\mathbb{R}$ which is bd. above has a supremum in $\mathbb{R}$

Nested Interval Property (NIP): For $n \in \mathbb{N}$, $I_n = [a_n, b_n]$, $a_n \leq b_n$

If $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, i.e. $I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq I_{n+1} \supseteq \dots$ is a nested sequence of closed intervals, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ ($\exists\, x \in \mathbb{R}$ s.t. for all $n \in \mathbb{N}, x \in I_n$)

Archimedean Property (AP): $x \in \mathbb{R} \Rightarrow \exists\, n \in \mathbb{N}$ s.t. $n > x$

Density Property of $\mathbb{Q}$ in $\mathbb{R}$: $x, y \in \mathbb{R}, x < y \Rightarrow \exists\, q \in \mathbb{Q}$ s.t. $x < q < y$

V subset of $\mathbb{R}$ $\to$ neighborhood of $x \in \mathbb{R}$ if $\exists\, \varepsilon \in \mathbb{R}, \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq V$

$\to$ neighborhood of $+\infty$ if $\exists\, a \in \mathbb{R}$ s.t. $(a, +\infty) \subseteq V$

$\to$ neighborhood of $-\infty$ if $\exists\, a \in \mathbb{R}$ s.t. $(-\infty, a) \subseteq V$

$\mathcal{V}(x)$ - family of all neighborhoods of x, $x \in \overline{\mathbb{R}}$

$x \in \overline{\mathbb{R}} \Rightarrow$ $\to$ if $x \in \mathbb{R}$ and $V \in \mathcal{V}(x) \Rightarrow x \in V$

$\to$ if $V \in \mathcal{V}(x)$ and $U \subseteq \mathbb{R}$ s.t. $V \subseteq U \Rightarrow U \in \mathcal{V}(x)$

$\to$ if $U, V \in \mathcal{V}(x) \Rightarrow U \cap V \in \mathcal{V}(x)$

Real numbers - basic concepts

~ Lecture I ~

Lecture 2 – Sequences of real numbers

(x_n) - ~~sequen~~ sequence | $\{x_n \mid n \in \mathbb{N}\}$ - set of its values

(x_n)
- bounded below $\Leftrightarrow \exists a \in \mathbb{R}$ s.t. $x_n \geq a, \forall n \in \mathbb{N}$
- bounded above $\Leftrightarrow \exists a \in \mathbb{R}$ s.t. $x_n \leq a, \forall n \in \mathbb{N}$
- bounded $\Leftrightarrow \exists a \in \mathbb{R}$ s.t. $|x_n| \leq a, \forall n \in \mathbb{N}$
- unbounded $\Leftrightarrow \forall a \in \mathbb{R}$. $\exists n \in \mathbb{N}$. s.t. $|x_n| > a$

(x_n) sequence in $\mathbb{R}$
- increasing (decreasing) if $\forall n \in \mathbb{N}$, $x_n \leq x_{n+1}$ ($x_n \geq x_{n+1}$)
- strictly increasing (strictly decreasing) if $\forall n \in \mathbb{N}$, $x_n < x_{n+1}$ ($x_n > x_{n+1}$)
- monotone (strictly monotone) if either increasing or decreasing (if either strictly increasing or strictly decreasing)

(x_n) sequence in $\mathbb{R}$
- convergent if it has a finite limit ((x_n) converges to $\lim_{n\to\infty} x_n \in \mathbb{R}$)
- divergent if it is not convergent (i.e. has no limit or is infinite)

(x_n), $x_n = \alpha^n$
- convergent for $\alpha \in [-1;1]$
- divergent for $\alpha \leq -1$ or $\alpha > 1$

$\lim_{n\to\infty} x_n$
- $\nexists$ for $\alpha \leq -1$
- 0 for $\alpha \in (-1;1)$
- 1 for $\alpha = 1$
- $+\infty$ for $\alpha > 1$

Every increasing (decreasing) sequence is bounded below (above)

Every convergent sequence (x_n) in $\mathbb{R}$ is bounded

! Bounded sequences are not always convergent

(x_n) monotone sequence in $\mathbb{R}$: (i) (x_n) has a limit in $\overline{\mathbb{R}}$

(ii) (x_n) increasing $\Rightarrow \lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} x_n$; (x_n) convergent $\Leftrightarrow (x_n)$ bounded above

(iii) (x_n) decreasing $\Rightarrow \lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n$; (x_n) convergent $\Leftrightarrow (x_n)$ bounded below

$(x_n), (y_n)$ sequences, $\forall n \in \mathbb{N}, x_n \leq y_n$
- if (x_n) and (y_n) convergent $\Rightarrow \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$
- $\lim_{n\to\infty} x_n = +\infty \Rightarrow \lim_{n\to\infty} y_n = +\infty$
- $\lim_{n\to\infty} y_n = -\infty \Rightarrow \lim_{n\to\infty} x_n = -\infty$

Squeeze Theorem: $(x_n), (y_n), (z_n)$ sequences in $\mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $x_n \leq y_n \leq z_n$; (x_n) & (z_n) are convergent, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l \Rightarrow (y_n)$ also convergent, $\lim_{n\to\infty} y_n = l$

Stolz-Césaro: $(x_n), (y_n)$ sequences in $\mathbb{R}$ s.t. $\to$ (y_n) strictly increasing & $\lim_{n\to\infty} y_n = +\infty$

$\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L \in \overline{\mathbb{R}}$

$\boxed{\text{Then } \lim_{n\to\infty} \frac{x_n}{y_n} = L}$

Consequences of Stolz-Césaro Th.:

1. if $\lim_{n\to\infty} x_n = x \in \overline{\mathbb{R}} \Rightarrow \lim_{n\to\infty} \frac{x_1+x_2+\dots+x_n}{n} = x$

2. if $\forall n \in \mathbb{N}, x_n > 0$ and $\lim_{n\to\infty} = x \in [0;+\infty) \cup \{+\infty\} \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n} = x$

3. if $\forall n \in \mathbb{N}, x_n > 0$ and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L \in [0;+\infty) \cup \{+\infty\} \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = L$

Limit laws: $x + \infty = \infty + x = \infty, \forall x \in \mathbb{R}$ $\qquad \infty + \infty = \infty$

$x + (-\infty) = (-\infty) + x = (-\infty), \forall x \in \mathbb{R}$ $\qquad (-\infty) + (-\infty) = -\infty$

$x \cdot \infty = \infty \cdot x = \begin{cases} \infty, & x > 0 \\ -\infty, & x < 0 \end{cases}$ $\qquad x \cdot (-\infty) = (-\infty) \cdot x = \begin{cases} -\infty, & x > 0 \\ \infty, & x < 0 \end{cases}$

$\infty \cdot \infty = \infty, \quad (-\infty) \cdot (-\infty) = \infty, \quad \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$

$\frac{x}{\infty} = \frac{x}{-\infty} = 0, \forall x \in \mathbb{R} \qquad \frac{1}{0^+} = +\infty, \quad \frac{1}{0^-} = -\infty$

$x^{\infty} = \begin{cases} \infty \text{ if } x > 1 \\ 0, x \in [0;1) \end{cases} \qquad x^{-\infty} = \begin{cases} 0, x > 1 \\ \infty, x \in [0;1) \end{cases} \qquad (\infty)^x = \begin{cases} \infty, x > 0 \\ 0, x < 0 \end{cases}$

$\infty^{\infty} = \infty, \quad \infty^{-\infty} = 0$

Not defined: $\infty + (-\infty), \ (-\infty) + \infty$

$0 \cdot \infty, \ \infty \cdot 0, \ 0 \cdot (-\infty), \ (-\infty) \cdot 0$

$\frac{\infty}{\infty}, \ \frac{-\infty}{-\infty}, \ \frac{\infty}{-\infty}, \ \frac{-\infty}{\infty}$

$1^{\infty}, \ 0^0, \ \infty^0, \ 1^{-\infty}$

Lecture 3- Subsequences

(y_k) is a subsequence of (x_n) in $\mathbb{R}$ given by $y_k = x_{n_k}$, $k \in \mathbb{N}$, where (n_k) is a strictly increasing subsequence in $\mathbb{N}$

(x_n) sequence in $\mathbb{R}$ w/ a limit (in $\overline{\mathbb{R}}$). Then any subsequence (x_{n_k}) has the same lim.

If a sequence has two subsequences that have different limits, then the sequence has no limit.

Bolzano - Weierstross Th.: A bounded sequence in $\mathbb{R}$ has a convergent subsequence

- $f, g : \mathbb{N} \to [0; +\infty)$; f is big-O of g if: $\exists c, n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq n_0, f(n) \leq c \cdot g(n)$

  not.: $f(n) = O(g(n))$

  $f : \mathbb{N} \to [0; +\infty)$, $g : \mathbb{N} \to (0; +\infty)$, $\exists L = \lim\limits_{n \to \infty} \frac{f(n)}{g(n)} \in [0; +\infty) \cup \{+\infty\}$

  $f(n) = O(g(n)) \Leftrightarrow L \in [0; +\infty)$

- $f : \mathbb{N} \to [0; +\infty)$, $g : \mathbb{N} \to (0; +\infty)$; f is little-o of g if $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = 0$

  not.: $f(n) = o(g(n))$

  (i) $f(n) = o(g(n)) \Leftrightarrow \forall c > 0, \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq n_0, f(n) < c \cdot g(n)$

  (ii) $f(n) = o(g(n)) \Rightarrow f(n) = O(g(n))$

  (iii) $f(n) \neq o(f(n))$

- $f, g : \mathbb{N} \to [0; +\infty)$; f is big-Theta of g if $f(n) = O(g(n))$ and $g(n) = O(f(n))$

  not.: $f(n) = \Theta(g(n))$

  $f, g : \mathbb{N} \to (0; +\infty)$; $\exists L = \lim\limits_{n \to \infty} \frac{f(n)}{g(n)} \in [0; +\infty) \cup \{+\infty\}$

  (i) $L = 0 \Rightarrow f(n) = o(g(n))$, hence ~~$g(n) = o$~~ $f(n) = O(g(n))$.

  (ii) $L \in (0; +\infty) \Rightarrow f(n) = \Theta(g(n))$

  (iii) $L = +\infty \Rightarrow g(n) = o(f(n))$, hence $g(n) = O(f(n))$

Lecture 3 – Series of real numbers

(x_n) sequence in $\mathbb{R}$. (s_n) sequence given by $s_n = x_1 + x_2 + \dots + x_n$, $n \in \mathbb{N}$

The pair $((x_n),(s_n))$ – the series w/ terms x_n (not.: $\sum_{n\geq 1} x_n$ or $\sum x_n$)

s_n – the n^{th} partial sum of the series

(s_n) converges (diverges) $\rightarrow$ $\sum x_n$ is convergent (divergent)

(s_n) has a limit $\rightarrow$ $\sum x_n$ has a sum $\rightarrow$ $\boxed{\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n}$

- $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ convergent series, $c \in \mathbb{R}$.

(i) $\sum_{n\geq 1}(x_n + y_n)$ convergent and $\sum_{n=1}^{\infty}(x_n+y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$

(ii) $\sum_{n\geq 1}(c x_n)$ convergent and $\sum_{n=1}^{\infty}(c x_n) = c\sum_{n=1}^{\infty} x_n$

Telescoping series: (x_n) sequence in $\mathbb{R}$. $\sum_{n\geq 1}(x_n - x_{n+1})$ is called a telescoping series

This series is convergent $\Leftrightarrow$ (x_n) convergent $\rightarrow$

$$\rightarrow \boxed{\sum_{n=1}^{\infty}(x_n - x_{n+1}) = x_1 - \lim_{n\to\infty} x_n}$$

Lecture 4

The harmonic series $\boxed{\sum_{n\geq 1} \frac{1}{n} \text{ is divergent}}$ with $\boxed{\text{sum } +\infty}$

The generalized harmonic series: $\sum_{n\geq 1} \frac{1}{n^\alpha}$ → convergent if $\alpha > 1$; → divergent if $\alpha \leq 1$

In particular $\boxed{\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}}$

$\boxed{\sum_{n\geq 0}^{\infty} \frac{1}{n!} \text{ convergent with sum } e}$

n-th term test: if the series $\sum_{n\geq 1} x_n$ converges $\rightarrow \lim_{n\to\infty} x_n = 0$

Series w/ nonnegative terms

(x_n) sequence in $\mathbb{R}$. $\sum_{n\geq 1} x_n$ series, (S_n) sequence of partial sums

A series $\sum_{n\geq 1} x_n$ is w/ nonnegative (positive) terms if $\forall n \in \mathbb{N}$, $x_n \geq 0$ ($x_n > 0$)

Series w/ nonnegative terms always have a sum in $[0;+\infty) \cup \{+\infty\}$.

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} S_n = \sup_{n\in\mathbb{N}} S_n$$

First Comparison Test: $\sum_{n\geq 1} x_n$, $\sum_{n\geq 1} y_n$ series w/ nonnegative terms, $\exists n_0 \in \mathbb{N}$ s.t.

$\forall n \geq n_0$, $x_n \leq y_n$

$\sum_{n\geq 1} y_n$ convergent $\Rightarrow \sum_{n\geq 1} x_n$ convergent

$\sum_{n\geq 1} x_n$ divergent $\Rightarrow \sum_{n\geq 1} y_n$ divergent

Second Comparison Test: $\sum_{n\geq 1} x_n$ w/ nonnegative terms, $\sum_{n\geq 1} y_n$ w/ positive terms

$\exists L = \lim_{n\to\infty} \frac{x_n}{y_n} \in [0;+\infty) \cup \{+\infty\}$

$L \in (0;+\infty)$: $\sum_{n\geq 1} x_n$ conv. $\Leftrightarrow \sum_{n\geq 1} y_n$ conv. ($\sum_{n\geq 1} x_n$ div. $\Leftrightarrow \sum_{n\geq 1} y_n$ div.)

$L = 0$: $\sum_{n\geq 1} y_n$ conv. $\Rightarrow \sum_{n\geq 1} x_n$ conv. ($\sum_{n\geq 1} x_n$ div. $\Rightarrow \sum_{n\geq 1} y_n$ div.)

$L = +\infty$: $\sum_{n\geq 1} x_n$ conv. $\Rightarrow \sum_{n\geq 1} y_n$ conv. ($\sum_{n\geq 1} y_n$ div. $\Rightarrow \sum_{n\geq 1} x_n$ div.)

Ratio Test / D'Alembert : $\sum_{n\geq 1} x_n$ w/ positive terms ; $L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} \in [0;+\infty) \cup \{+\infty\}$

$L<1 \Rightarrow \sum_{n\geq 1} x_n$ convergent

$L>1 \Rightarrow \sum_{n\geq 1} x_n$ divergent

$L=1$ gives no information

Root Test / Cauchy : $\sum_{n\geq 1} x_n$ w/ nonnegative terms ; $L = \lim_{n\to\infty} \sqrt[n]{x_n} \in [0;+\infty) \cup \{+\infty\}$

$L<1 \Rightarrow \sum_{n\geq 1} x_n$ convergent

$L>1 \Rightarrow \sum_{n\geq 1} x_n$ divergent

$L=1$ gives no information

Raabe's Test : $\sum_{n\geq 1} x_n$ w/ positive terms ; $L = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) \in \overline{\mathbb{R}}$

$L<1 \Rightarrow \sum_{n\geq 1} x_n$ divergent

$L>1 \Rightarrow \sum_{n\geq 1} x_n$ convergent

$L=1$ gives no information

Series with arbitrary terms

$\sum_{n\geq 1} x_n$ alternating if :
- $x_n = (-1)^{n+1}|x_n|, \forall n\in\mathbb{N}$: $x_1 \geq 0, x_2 \leq 0, x_3 \geq 0, \dots$
- $x_n = (-1)^{n}|x_n|, \forall n\in\mathbb{N}$: $x_1 \leq 0, x_2 \geq 0, x_3 \leq 0, \dots$

Alternating Series Test / Leibniz : $\sum_{n\geq 1} x_n$ alternating

if $(|x_n|)$ decreasing then $\boxed{\sum_{n\geq 1} x_n \text{ convergent} \Leftrightarrow \lim_{n\to\infty} x_n = 0}$

$\sum_{n\geq 1} x_n$
- absolutely convergent if $\sum_{n\geq 1} |x_n|$ is convergent
- semi-convergent if it is only convergent (not abs. convergent)

$\boxed{\sum_{n\geq 1} x_n \text{ absolutely convergent} \Rightarrow \sum_{n\geq 1} x_n \text{ convergent}}$

The Alternating generalized harmonic series :

$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^\alpha}$
- divergent if $\alpha \leq 0$
- semi-convergent if $\alpha \in (0;1]$
- absolutely convergent if $\alpha > 1$

Lecture 5 - Real-valued functions of one real variable

Limits of functions

$A \subseteq \mathbb{R}$ and $c \in \overline{\mathbb{R}}$: c - accumulation point of A if $\forall V \in \mathcal{V}(c)$, $V \cap (A \setminus \{c\}) \neq \emptyset$

the set of all accumulation points of A is called the derived set of A (noted A')

$\boxed{\text{c - isolated point of A if } c \in A \setminus A'}$

accumulation points of A may or may not belong to A

$a \in A \setminus A' \Leftrightarrow \exists V \in \mathcal{V}(a)$ s.t. $V \cap A = \{a\}$

Sequential characterization of acc. points : $A \subseteq \mathbb{R}$, $c \in \overline{\mathbb{R}}$. $c \in A' \Leftrightarrow \exists (x_n)$ in $A \setminus \{c\}$ s.t. $\boxed{\lim_{n\to\infty} x_n = c}$

Let $f : A \to \mathbb{R}$, $c \in A'$. f has a limit at c if $\exists L \in \overline{\mathbb{R}}$ s.t.

$\forall V \in \mathcal{V}(L)$, $\exists U \in \mathcal{V}(c)$ s.t. $\forall x \in U \cap (A \setminus \{c\})$ we have $f(x) \in V$ (1)

f cannot have two distinct limits at c

If $f : A \to \mathbb{R}$ has a limit at $c \in A'$, then $L \in \overline{\mathbb{R}}$ (satisfying (1)) is called the limit of f at c (noted $\boxed{\lim_{x \to c} f(x)}$ or $\boxed{f(x) \to L \text{ as } x \to c}$)

Sequential characterization of limits : let $f : A \to \mathbb{R}$, $c \in A'$, $L \in \overline{\mathbb{R}}$

$\lim_{x\to c} f(x) = L \Leftrightarrow \forall (x_n)$ in $A \setminus \{c\}$ w/ $\lim_{n\to\infty} x_n = c$ we have $\lim_{n\to\infty} f(x_n) = L$

$f, g : A \to \mathbb{R}$, $c \in A'$. Suppose $\exists U \in \mathcal{V}(c)$ s.t. $f(x) \leq g(x)$, $\forall x \in U \cap (A \setminus \{c\})$

→ if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then $\boxed{\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)}$

→ $\lim_{x\to c} f(x) = +\infty \Rightarrow \lim_{x\to c} g(x) = +\infty$

→ $\lim_{x\to c} g(x) = -\infty \Rightarrow \lim_{x\to c} f(x) = -\infty$

Squeeze th. for functions : $f, g, h : A \to \mathbb{R}$, $c \in A'$. Suppose $\exists U \in \mathcal{V}(c)$ s.t. $f(x) \leq g(x) \leq h(x)$ $\forall x \in U(A \setminus \{c\})$; if $L \in \mathbb{R}$ is the limit of f and h ~~and~~ at c, then L is also the limit of g at c.

One sided limits

$f: A \to \mathbb{R}$, $c \in \mathbb{R}$. If c is an accumulation point of $A \cap (-\infty; c)$ and $f|_{A \cap (-\infty; c)}$ has a limit at c, then we call this the left-hand limit of f at c (noted $\boxed{\lim\limits_{\substack{x \to c \\ x < c}} f(x) ; \lim\limits_{x \nearrow c} f(x)}$)

right-hand limit of f at c - in a similar way, considering the set $A \cap (c; +\infty)$

noted $\left(\boxed{\lim\limits_{\substack{x \to c \\ x > c}} f(x) ; \lim\limits_{x \searrow c} f(x)}\right)$

$\boxed{\lim\limits_{x \to c} f(x) = L \Leftrightarrow \lim\limits_{\substack{x \to c \\ x < c}} f(x) = L = \lim\limits_{\substack{x \to c \\ x > c}} f(x)}$, c - acc. point of both $A \cap (-\infty; c)$ & $A \cap (c; +\infty)$

→ $\lim\limits_{x \to c} f(x)$ is also called the two-sided limit of f at c

Continuous functions

$f: A \to \mathbb{R}$, $c \in A$; f is continuous at c if $\forall V \in \mathcal{V}(f(c)), \exists U \in \mathcal{V}(c)$ s.t. $\forall x \in U \cap A$, $f(x) \in V$

f is continuous at $c \Leftrightarrow \forall (x_n)$ in A w/ $\lim\limits_{n \to \infty} x_n = c$ we have $\boxed{\lim\limits_{n \to \infty} f(x_n) = f(c)}$

$f: A \to \mathbb{R}$, $c \in A$ discontinuity point of f (f not continuous at c)

- discontinuity point of the first kind of f if both one-sided limits exist and are finite
- –//– of the second kind of f if not of the first kind
- first kind → jump discontinuity if the one-sided limits are not equal
  → removable disc. if equal but not equal to $f(c)$

$f: A \to \mathbb{R}$ is bounded if the set $f(A)$ is bounded

f attains its maximum if $\exists \overline{x} \in A$ s.t. $\forall x \in A$, $f(x) \le f(\overline{x})$

f attains its minimum if $\exists \underline{x} \in A$ s.t. $\forall x \in A$, $f(\underline{x}) \le f(x)$

$a, b \in \mathbb{R}$, $a < b$, $f: [a,b] \to \mathbb{R}$, continuous $\Rightarrow f$ bounded & attains its maximum and minimum

$f: [a,b] \to \mathbb{R}$ ~~cont~~ continuous if $f(a) < \gamma < f(b)$ or $f(b) < \gamma < f(a) \Rightarrow \exists c \in (a,b)$, $f(c) = \gamma$

Differentiation of functions

$f: A \to \mathbb{R}$, $c \in A \cap A$; f has a derivative at c if $\boxed{\exists \lim_{x \to c} \frac{f(x)-f(c)}{x-c}}$ (denoted by $f'(c)$)

f differentiable at $c \Rightarrow f$ continuous at c

c - acc. point of $A \cap (-\infty; c)$; f has a left-hand derivative at c if $\boxed{\exists \lim_{\substack{x \to c \\ x < c}} \frac{f(x)-f(c)}{x-c} \in \overline{\mathbb{R}}}$ (noted $f'_l(c)$)

similarly for right-hand derivative (noted $f'_r(c)$)

Chain Rule: $f: I \to J$, $g: J \to \mathbb{R}$, $c \in I$; f diff. at c & g diff. at $f(c) \Rightarrow$

$\Rightarrow g \circ f: I \to \mathbb{R}$ diff. at c & $(g \circ f)'(x) = g'(f(c)) \cdot f'(c)$

Inverse Function Theorem: $f: I \to J$ invertible ; if diff. at c, $f'(c) \neq 0$, $f^{-1}: J \to I$ cont. at $f(c)$

$\Rightarrow f^{-1}$ diff. at $f(c)$: $(f^{-1})'(f(c)) = \frac{1}{f'(c)}$

Lecture 6 - Local extrema & derivatives

$A \subseteq \mathbb{R}$; $f: A \to \mathbb{R}$ — attains a local maximum (minimum) at $c \in A$ if $\exists V \in \mathcal{V}(c)$ s.t. c is a maximum (minimum) point for $f|_{A \cap V}$. In this case, c is called a local maximum (minimum) point for f

— attains a local extremum at $c \in A$ if it attains a local minimum / maximum point at c. (c – local extremum point for f)

Fermat Th.: $f:(a,b) \to \mathbb{R}$, $c \in (a;b)$; if f has a derivative at c & f attains a local extremum at c, then $\boxed{f'(c) = 0}$!!! ($f'(c) = 0 \not\Rightarrow f$ attains local extremum at c)

Darboux Th.: $f:[a,b] \to \mathbb{R}$ differentiable; if $\gamma \in \mathbb{R}$ s.t. $f'(a) < \gamma < f'(b)$ or $f'(b) < \gamma < f'(a) \Rightarrow$
$\Rightarrow \exists c \in (a,b)$ s.t. $\boxed{f'(c) = \gamma}$

a function is called continuously differentiable if diff. and its derivative is continuous

Rolle Th.: $f:[a,b] \to \mathbb{R}$: if f continuous on $[a;b]$, diff. on (a,b) and $f(a) = f(b) \Rightarrow \exists c \in (a,b)$ s.t. $\boxed{f'(c) = 0}$

Mean Val. Th / Lagrange: $f:[a,b] \to \mathbb{R}$, continuous on $[a;b]$, diff on $(a;b) \Rightarrow \exists c \in (a,b)$ s.t. $\boxed{f(b) - f(a) = f'(c)(b-a)}$

Generalized M.V.T / Cauchy: $f, g: [a,b] \to \mathbb{R}$ cont. on $[a;b]$, diff. on $(a;b) \Rightarrow \exists c \in (a;b)$ s.t.

$$\boxed{(f(b) - f(a)) \cdot g'(c) = (g(b) - g(a)) \cdot f'(c)}$$

Second Derivative Test. $f:(a,b) \to \mathbb{R}$, $c \in (a;b)$ if f is twice diff. at c, $f'(c) = 0$, $f''(c) \neq 0 \Rightarrow$

$$\Rightarrow \begin{cases} f''(c) > 0 \Rightarrow f \text{ attains a local minimum at } c \\ f''(c) < 0 \Rightarrow f \text{ attains a local maximum at } c \end{cases}$$

$f''(c) = 0$ gives no information

Taylor Polynomials

$$\boxed{T_n(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x - x_0)^k}$$

(remainder) $\boxed{R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}}$, $c \in (\min\{x, x_0\}, \max\{x, x_0\})$

$$\boxed{f(x) = T_n(x) + R_n(x)}$$

Taylor Series: $\boxed{\sum_{n \geq 0} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n}$ — taylor series of f around x_0

Taylor Series expansion of $f(x)$ around x_0: $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n}$

f can be expanded as a T.S. around x_0 on $J \Leftrightarrow \lim_{n \to \infty} R_n(x) = 0, \forall x \in J \Rightarrow$

$$\Rightarrow f(x) = T_n(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Lecture 7 - Functions of several variables

Euclidean space $\mathbb{R}^m$

- $\boxed{\langle x,y\rangle} = \sum_{i=1}^{m} x_i y_i = x_1y_1 + \dots + x_m y_m$ - scalar product of x and y
- $\boxed{\|x\|} = \sqrt{\langle x,x\rangle} = \sqrt{x_1^2 + x_2^2 + \dots + x_m^2}$ - Euclidean norm of x
- $\boxed{\|x-y\|} = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_m-y_m)^2}$ - Euclidean dist. between x and y

$\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$ $\quad$ $\langle x,y\rangle = \langle y,x\rangle$ $\quad$ $\langle 0_m, x\rangle = 0, \forall x\in\mathbb{R}^m$

$\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$ $\quad$ $\langle x,x\rangle > 0, \forall x\in \mathbb{R}^m\setminus\{0_m\}$ $\quad$ $\langle x,x\rangle = 0 \iff x = 0_m$

Cauchy-Schwarz inequality: $|\langle x,y\rangle| \le \|x\|\cdot\|y\|, \forall x,y\in\mathbb{R}^m$

Euclidean norm properties:
- $\|x\| = 0 \iff x = 0_m$
- $\|\alpha x\| = |\alpha|\cdot\|x\|, \forall \alpha\in\mathbb{R}, x\in\mathbb{R}^m$
- $\|x+y\| \le \|x\| + \|y\|, \forall x,y\in\mathbb{R}^m$ (triangle inequality)

$\rightarrow B(x_0,r) = \{x\in\mathbb{R}^m \mid \|x-x_0\| < r\}$ - open ball of radius r centered at x_0

$\rightarrow \overline{B}(x_0,r) = \{x\in\mathbb{R}^m \mid \|x-x_0\| \le r\}$ - closed ball of radius r cetered at x_0

$$x_0\in\mathbb{R}^m,\ r_1 > r > 0 \rightarrow \begin{cases} x_0 \in B(x_0,r) \\ B(x_0,r) \subseteq \overline{B}(x_0,r) \subseteq B(x_0,r_1) \subseteq \overline{B}(x_0,r_1) \\ \forall x\in B(x_0,r),\ B(x, r-\|x_0-x\|) \subseteq B(x_0,r) \end{cases}$$

$V\subseteq\mathbb{R}^m$ for which $\exists r>0$ s.t. $B(x,r)\subseteq V$ - neighborhood of x ($\mathcal{V}(x)$ - family of all neighborhoods)

Sequences in $\mathbb{R}^m$

$x^1 = (x_1^1, x_2^1, \dots, x_m^1), \dots, x^k = (x_1^k, x_2^k, \dots, x_m^k)$

a sequence (x^k) in $\mathbb{R}^m$ is said to converge (tend) to $x \in \mathbb{R}^m$ if

$\rightarrow \forall V\in\mathcal{V}(x), \exists k_V\in\mathbb{N}$ s.t. $\forall k\in\mathbb{N}, k\ge k_V, x^k\in V$

sequence (x^k) converges to $x\in\mathbb{R}^m$ $\rightarrow$ (x^k) convergent, x - the limit of (x^k)

(x^k) sequence, $x = (x_1, \dots, x_m)$

$$\begin{cases} \lim\limits_{k\to\infty} x^k = x \\ \forall \varepsilon>0, \exists k_\varepsilon\in\mathbb{N} \text{ s.t. } \forall k\in\mathbb{N}, k\ge k_\varepsilon, \|x^k - x\| < \varepsilon \\ \forall i\in\{1,\dots,m\}, \lim\limits_{k\to\infty} x_i^k = x_i \end{cases}$$

$A\subseteq\mathbb{R}^m$ nonempty $\Rightarrow$ $f: A\to\mathbb{R}$ is a real-valued function of m variables

$c\in\mathbb{R}^m$ - acc. point of A if $\forall V\in\mathcal{V}(c), V\cap(A\setminus\{c\}) \ne \emptyset$ (all acc. points - A', derived set of A)

f has a limit at c if $\exists L\in\overline{\mathbb{R}}$ s.t. $\forall V\in\mathcal{V}(L), \exists U\in\mathcal{V}(c)$ s.t. $\forall x\in U\cap(A\setminus\{c\})$ we have $f(x)\in V$

not.: $\lim\limits_{x\to c} f(x) = L$ - limit of f at c

... sequence (x^k) in $A\setminus\{c\}$ with $\lim x^k = c$ we have $\lim f(x^k) = L$

Lecture 8 - Continuous functions of several variables

$A \subseteq \mathbb{R}^m$, $A \neq \emptyset$

$f: A \to \mathbb{R}$, $c \in A$; f continuous at c if $\forall V \in \mathcal{V}(f(c)), \exists U \in \mathcal{V}(c)$ s.t. $\forall x \in U \cap A, f(x) \in V$

if $c \in A \cap A'$ $\Rightarrow$ f continuous at $c \Leftrightarrow \lim_{x \to c} f(x) = f(c)$

$A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}$, $a \in A$; $f: A \to B$, $g: B \to \mathbb{R}$

f cont. at a, g cont. at $f(a)$ $\Rightarrow$ $g \circ f: A \to \mathbb{R}$ cont. at a

Partial derivatives

$A \subseteq \mathbb{R}^m$, $c \in A$ - interior point of A if $\exists r > 0$ s.t. $B(c, r) \subseteq A$

int A - set of all interior points of A

$f: A \to \mathbb{R}$, $c = (c_1, \ldots, c_m) \in \text{int } A$, $j \in \{1, \ldots, m\}$

f is partially differentiable w.r.t x_j at c if: $\boxed{\exists \lim_{x_j \to c_j} \dfrac{f(c_1, \ldots, c_{j-1}, x_j, c_{j+1}, \ldots c_m) - f(c_1, \ldots, c_m)}{x_j - c_j} \in \mathbb{R}}$

(first order) partial derivative of f w.r.t. x_j at c ($\boxed{\dfrac{\partial f}{\partial x_j}(c) \text{ or } f'_{x_j}(c)}$)

if f partial diff. w.r.t. all x_j at c, then f is p. diff. at c $\Rightarrow$

$\Rightarrow \boxed{\left(\dfrac{\partial f}{\partial x_1}(c), \ldots, \dfrac{\partial f}{\partial x_m}(c)\right)} \in \mathbb{R}^m$ - the gradient of f at c ($\nabla f(c)$)

f partially diff. at c $\not\Rightarrow$ f cont. at c ; partial deriv. of f are cont. $\Rightarrow$ f cont.

Higher-order partial derivatives

f is twice partially diff. w.r.t. (x_i, x_j) at c if $\exists V \in \mathcal{V}(c)$, $V \subseteq A$ s.t. f p. diff. w.r.t x_i on V and

$\dfrac{\partial f}{\partial x_i}: V \to \mathbb{R}$, $x \in V \longmapsto \dfrac{\partial f}{\partial x_i}(x) \in \mathbb{R}$ is p. diff. w.r.t x_j at c

not.: $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ - second order partial derivative

$f: A \to \mathbb{R}$ - twice continuously partially differentiable if it is twice p. diff. and all first & second order partial derivatives are continuous ($f \in C^2(A)$)

A open, $f \in C^2(A) \Rightarrow \dfrac{\partial^2 f}{\partial x_j \partial x_i} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$

$$\begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(c) & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_m}(c) \\ \vdots & & \\ \dfrac{\partial^2 f}{\partial x_m \partial x_1}(c) & \cdots & \dfrac{\partial^2 f}{\partial x_m^2}(c) \end{pmatrix}$$ - Hessian matrix of f at c ($H_f(c)$)

Vector-valued functions of several variables - Lecture 9

$pr_j : \mathbb{R}^m \to \mathbb{R}$, $pr_j(y) = y_j$, $\forall y = (y_1, ..., y_m) \in \mathbb{R}^m$

$A \subseteq \mathbb{R}^n$ nonempty ; $f : A \to \mathbb{R}^m$ - vector-valued function of n variables

components of f are the real-valued functions $f_1, ..., f_m : A \to \mathbb{R}$, $f_j = pr_j \circ f$, $\forall j \in \{1, ..., m\}$

not.: $f = (f_1, ..., f_m)$

The chain rule

$J \subseteq \mathbb{R}$ interval, $A \subseteq \mathbb{R}^n$, $g = (g_1, ..., g_n) : J \to A$ s.t. $\forall i \in \{1, ..., n\}$, g_i diff. at c and $g(c) \in \operatorname{int} A$

if $f = f(x_1, ..., x_n) : A \to \mathbb{R}$ is C^1 near $g(c)$, then $\begin{cases} f \circ g : J \to \mathbb{R} \text{ is diff. at } c \\ (f \circ g)'(c) = \boxed{\displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(g(c))\, g_i'(c)} \end{cases}$

Local extrema and partial derivatives

$A \subseteq \mathbb{R}^n$ nonempty, $f : A \to \mathbb{R}$, $c \in A$

c
- local maximum (min.) point for f if $\exists V \in \mathcal{V}(c)$ s.t. $\forall x \in V \cap A$ $\boxed{f(c) \geq f(x) \ \ (f(c) \leq f(x))}$ (1)
- local extremum point for f if is either a local max. or min. point
- global max. (min.) point if (1) holds $\forall x \in A$
- global extremum point for f if is either global max. or min. for f

f attains a global max./min./extremum at c / f attains its max/min if it has at least one global max./min. point

f attains a local ~~glo~~ max./min./extremum at c

Maximum-Minimum Th (Weierstrass) : $A \subseteq \mathbb{R}^n$ nonempty, closed and bounded, $f : A \to \mathbb{R}$, f cont. $\to$ f attains both its max. and min.

Fermat Th.: $A \subseteq \mathbb{R}^n$ nonempty and open, $f : A \to \mathbb{R}$; if $c \in A$, f p.diff at c and f attains a local extremum at c, then $\nabla f(c) = 0_n$

$c \in A$ at which f is p.diff. is called a stationary point (or critical) for f if $\nabla f(c) = 0_n$

$C = (c_{ij})_{1 \leq i,j \leq n}$ $n \times n$ matrix $\Phi_C : \mathbb{R}^n \to \mathbb{R}$, $\Phi_C(h) = \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} h_i h_j$, $\forall h = (h_1, ..., h_n) \in \mathbb{R}^n$ - quadratic form associated to C

Φ_C (or C)
- positive (negative) definite if $\forall h \in \mathbb{R}^n \setminus \{0_n\}$, $\Phi_C(h) > 0$ ($\Phi_C(h) < 0$)
- positive (negative) semidefinite — // — $\Phi_C(h) \geq 0$ ($\Phi_C(h) \leq 0$)
- indefinite if $\exists a, b \in \mathbb{R}^n$ s.t. $\Phi_C(a) < 0 < \Phi_C(b)$

Sylvester Th.: $\forall k \in \{1, ..., n\}$, $\boxed{\Delta_k = \det(c_{ij})_{1 \leq i,j \leq k}}$

C is pos. definite $\Leftrightarrow$ $\boxed{\Delta_k > 0}$, $\forall k \in \{1, ..., n\}$

neg. definite $\Leftrightarrow$ $\boxed{(-1)^k \Delta_k > 0}$, $\forall k \in \{1, ..., n\}$

$f \in C^2(A)$ $\to$ c local min/max. point of f, then $\boxed{\nabla f(c) = 0}$ $\Rightarrow$ $H_f(c)$ pos. (neg.) semidef.

Lecture 10 - Directional derivatives

$A \subseteq \mathbb{R}^m, A \neq \emptyset$

$f: A \to \mathbb{R}$, $c \in \operatorname{int} A$, $v \in \mathbb{R}^m$ unit vector ($\|v\| = 1$); f diff. in the direction v at c if

$$\boxed{\exists \lim_{t \to 0} \frac{f(c+tv) - f(c)}{t}}$$ - directional derivative of f in the direction v at c ($f'(c; v)$)

suppose that f is C^1 near c. $\forall v \in \mathbb{R}^m, \|v\| = 1$, $f'(c; v) = \langle \nabla f(c), v \rangle$

f diff. in every direction at $c \not\Rightarrow f$ continuous at c

Riemann integrals

a partition of an interval $[a,b]$ is a finite ordered set $P = (x_0, ..., x_m)$ s.t. $a = x_0 < x_1 < ... < x_m = b$

$[x_{i-1}, x_i]$ - subintervals of the partition P

norm of P, $\|P\| = \max\{x_1 - x_0, x_2 - x_1, ..., x_m - x_{m-1}\}$

$\forall i \in \{1, ..., m\}$, ξ_i chosen in $[x_{i-1}, x_i]$; $\xi = (\xi_1, ..., \xi_m) \to (P, \xi)$ - tagged partition of $[a;b]$

$$\boxed{\sigma(f, P, \xi) = \sum_{i=1}^{m} f(\xi_i)(x_i - x_{i-1})}$$ - Riemann sum of f w.r.t. the tagged partition (P, ξ)

$f: [a,b] \to \mathbb{R}$ - Riemann integrable on $[a;b]$ if $\exists J \in \mathbb{R}$ s.t.

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t. ~~$\forall$~~ (P, ξ) w/ $\|P\| < \delta$, $|\sigma(f, P, \xi) - J| < \varepsilon$ (1)

$\boxed{\mathcal{R}[a;b]}$ - family of all Riemann integrable functions on $[a;b]$

if $f \in \mathcal{R}[a;b]$, then $J \in \mathbb{R}$ is unique (Riemann integral / definite) on f on $[a;b]$ $\boxed{\int_a^b f(x)dx = \int_a^b f = J}$

$f: [a,b] \to \mathbb{R}$; constantly equal to $M \to f \in \mathcal{R}[a;b]$, $\boxed{\int_a^b f = M(b-a)}$

continuous $\to f \in \mathcal{R}[a;b]$

monotone $\to f \in \mathcal{R}[a;b]$

$f \in \mathcal{R}[a;b] \to f$ bounded

$a, b \in \mathbb{R}, a < b, f, g \in \mathcal{R}[a;b], \alpha \in \mathbb{R}$: $f+g \in \mathcal{R}[a,b]$; $\boxed{\int_a^b (f+g) = \int_a^b f + \int_a^b g}$

$(\alpha f) \in \mathcal{R}[a;b] \to \int_a^b \alpha f = \alpha \int_a^b f$; $f \cdot g \in \mathcal{R}[a;b]$, $|f| \in \mathcal{R}[a;b]$; $\boxed{f \leq g \Rightarrow \int_a^b f \leq \int_a^b g}$

$f: [a,b] \to \mathbb{R}$, $c \in (a;b)$; $f \in \mathcal{R}[a;b] \Leftrightarrow f|_{[a,c]} \in \mathcal{R}[a;c]$ and $f|_{[c;b]} \in \mathcal{R}[c;b]$

in this case $\boxed{\int_a^b f = \int_a^c f + \int_c^b f}$

First fundam. Th. of Calculus: $F: [a,b] \to \mathbb{R}$, $F(t) = \int_a^t f$ - F cont.; if f cont. at $c \in [a;b] \Rightarrow F$ diff. at c $\boxed{(F'(c) = f(c))}$

Second fundam. Th. of Calculus: $f \in \mathcal{R}[a;b]$; ~~$F(x)$~~ $F: [a;b] \to \mathbb{R}$ antiderivative of f ($F'(x) = f(x)$ for all $x \in [a;b]$), then $\boxed{\int_a^b f = F(b) - F(a)}$

Lecture 11 - Improper integrals

$a\in\mathbb{R}$, $b\in\mathbb{R}\cup\{+\infty\}$, $a<b$, $f:[a,b)\to\mathbb{R}$ continuous. f imp. int. on $[a;b)$ if $\exists \lim\limits_{\substack{t\to b\\ t<b}} \int_a^t f(x)\,dx \in\mathbb{R}$

$a\in\mathbb{R}\cup\{-\infty\}$, $b\in\mathbb{R}$, $a<b$, $f:(a;b]\to\mathbb{R}$ cont. —//— $\exists \lim\limits_{\substack{t\to a\\ t>a}} \int_t^b f(x)\,dx\in\mathbb{R}$

$a,b\in\overline{\mathbb{R}}$, $a<b$, $f:(a;b)\to\mathbb{R}$ cont. —//— if $\exists c\in(a;b)$ s.t. f imp. int. on $(a;c]$ and $[c;b)$

$$\boxed{\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx}$$

Integral test for Convergence of Series: $m\in\mathbb{N}$, $f:[m;\infty)\to[0;\infty)$ cont. and decreasing

$$\boxed{f \text{ impr. int. on } [m;+\infty) \Leftrightarrow \sum_{n\ge m} f(n) \text{ convergent}}$$

Generalized harmonic series: $\displaystyle\sum_{n\ge 1}\frac{1}{n^\alpha}$ conv. if $\alpha>1$, div. if $\alpha\le 1$, $\alpha\in\mathbb{R}$

Comparison Test for Improper Integrals: $a\in\mathbb{R}$, $b\in\mathbb{R}\cup\{+\infty\}$, $a<b$, $f,g:[a,b)\in\mathbb{R}$ continuous and $\exists c\in(a,b)$ s.t. $\forall x\in[c;b)$, $\boxed{0\le f(x)\le g(x)}$

$$\begin{cases} g \text{ imp. int. on } [a,b) \Rightarrow f \text{ imp. int. on } [a;b) \\ f \text{ not imp. int on } [a,b) \Rightarrow g \text{ not imp. int. on } [a;b) \end{cases}$$

$a,b\in\mathbb{R}$, $a<b$, $f:[a,b)\to[0;+\infty)$ cont., $p\in\mathbb{R}$ s.t. $\boxed{\exists L=\lim\limits_{\substack{x\to b\\ x<b}}(b-x)^p f(x)}\in[0;+\infty)\cup\{+\infty\}$

$$\begin{cases} p<1,\ L<\infty \Rightarrow f \text{ imp. int. on } [a,b) \\ p\ge 1,\ L>0 \Rightarrow f \text{ not imp. int. on } [a;b) \end{cases}$$

$a,b\in\mathbb{R}$, $a<b$, $f:(a,b]\to[0;+\infty)$, $p\in\mathbb{R}$ s.t. $\boxed{\exists L=\lim\limits_{\substack{x\to a\\ x>a}}(x-a)^p f(x)}\in[0;+\infty)\cup\{+\infty\}$

$$\begin{cases} p<1,\ L<\infty \Rightarrow f \text{ imp. int. on } [a,b) \\ p\ge 1,\ L>0 \Rightarrow f \text{ not imp. int. on } [a,b) \end{cases}$$

$a\in\mathbb{R}$, $f:[a,+\infty)\to[0;+\infty)$ continuous, $p\in\mathbb{R}$ s.t. $\boxed{\exists L=\lim\limits_{x\to\infty} x^p f(x)}\in[0;+\infty)\cup\{+\infty\}$

$$\begin{cases} p>1,\ L<\infty \Rightarrow f \text{ imp. int. on } [a;+\infty) \\ p\le 1,\ L>0 \Rightarrow f \text{ not imp. int. on } [a;+\infty) \end{cases}$$

Lecture 12 - Multiple integrals

$\iint_A f(x,y)\,dxdy = I$ - double integral of f on A

$f,g \in \mathcal{R}(A), \alpha \in \mathbb{R}$: $f+g \in \mathcal{R}(A)$, $\iint_A (f(x,y)+g(x,y))\,dxdy = \iint_A f(x,y)\,dxdy + \iint_A g(x,y)\,dxdy$

$\alpha f \in \mathcal{R}(A)$, $\iint_A (\alpha f(x,y))\,dxdy = \alpha \iint_A f(x,y)\,dxdy$

$(f\cdot g) \in \mathcal{R}(A)$; $|f| \in \mathcal{R}(A)$ $\quad f(x,y) \le g(x,y) \Rightarrow \iint_A f(x,y)\,dxdy \le \iint_A g(x,y)\,dxdy$

a set $M \subseteq \mathbb{R}^2$ – simple w.r.t. y-axis if $\exists a,b \in \mathbb{R}$ ~~s.t.~~ $a<b$, $\varphi_1,\varphi_2 : [a,b] \to \mathbb{R}$ cont. w/ $\varphi_1 \le \varphi_2$

s.t. $M = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$

– simple w.r.t. x-axis if $\exists c,d \in \mathbb{R}$, $c<d$, $\varphi_3,\varphi_4 : [a;b] \to \mathbb{R}$ cont, $\varphi_3 \le \varphi_4$

s.t. $M = \{(x,y) \in \mathbb{R}^2 \mid c \le y \le d,\ \varphi_3(y) \le x \le \varphi_4(y)\}$

$M \subseteq \mathbb{R}^2$, $f : M \to \mathbb{R}$ continuous

M simple w.r.t y-axis $\to$ f Riemann integrable on M, $\iint_M f(x,y)\,dxdy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dydx$

M simple w.r.t x-axis $\to$ f Riemann int. on M, $\iint_M f(x,y)\,dxdy = \int_c^d \int_{\varphi_3(y)}^{\varphi_4(y)} f(x,y)\,dx\,dy$