Exercise 1.1.

$$\sum_{m=1}^{n} \frac{1}{\sqrt{m}} > \sqrt{n}, \forall n \in \mathbb{N}, n \geq 2$$

$$P(n): \sum_{m=1}^{n} \frac{1}{\sqrt{m}} > \sqrt{n}, \forall n \in \mathbb{N}, n \geq 2$$

I $P(2): \sum_{m=1}^{2} \frac{1}{\sqrt{m}} > \sqrt{2}$; $\frac{1}{1}^{\sqrt{2})} + \frac{1}{\sqrt{2}} > \sqrt{2}$; $\frac{1+\sqrt{2}}{\sqrt{2}} > \sqrt{2} \Rightarrow 1+\sqrt{2} > 2 \mid -1$

$\sqrt{2} > 1$ true $\Rightarrow P(2)$ true

II We assume that $P(k)$ is true and prove $P(k+1)$ is true

$$P(k): \sum_{m=1}^{k} \frac{1}{\sqrt{m}} > \sqrt{k}$$

$$P(k+1): \sum_{m=1}^{k+1} \frac{1}{\sqrt{m}} > \sqrt{k+1}$$

$$\left.\begin{array}{l} \sum_{m=1}^{k+1} \frac{1}{\sqrt{m}} = \sum_{m=1}^{k} \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{k+1}} \\ \sum_{m=1}^{k} \frac{1}{\sqrt{m}} > \sqrt{k} \end{array}\right\} \Rightarrow \sum_{m=1}^{k} \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1} \mid \cdot (\sqrt{k+1})$$

$$\sqrt{k(k+1)} + \cancel{1} > k + \cancel{1} \quad \uparrow^2$$

$$k^2 + k > k^2 \mid -(k^2)$$

$k > 0$, true $\Rightarrow P(k+1)$ true

$(I, II) \Rightarrow P(n)$ true $\forall n \in \mathbb{N}, n \geq 2$

Exercise 1.2.

$G(x_1, x_2, \ldots, x_n) \leq A(x_1, x_2, \ldots, x_n)$, $x_1, x_2, \ldots, x_n > 0$, $x > 0$, $n \in \mathbb{N}$

a) $$\frac{x^n}{1+x+\ldots+x^{2n}} \leq \frac{1}{2n+1}$$

$$G(1, x, \ldots, x^{2n}) \leq A(1, x, \ldots, x^{2n}) \Rightarrow \sqrt[2n+1]{1 \cdot x \cdot \ldots \cdot x^{2n}} \leq \frac{1+x+\ldots+x^{2n}}{2n+1}$$

$$\sqrt[2n+1]{x^{1+2+\ldots+2n}} \leq \frac{1+x+\ldots+x^{2n}}{2n+1} ; \left[x^{\frac{\cancel{(2n+1)} \cdot \cancel{2}n}{\cancel{2}}}\right]^{\frac{1}{\cancel{2n+1}}} \leq \frac{1+x+\ldots+x^{2n}}{2n+1} \Rightarrow$$

$$\Rightarrow x^n \leq \frac{1+x+\ldots+x^{2n}}{2n+1} \mid : (1+x+\ldots+x^{2n}) \Rightarrow \frac{x^n}{1+x+\ldots+x^{2n}} \leq \frac{1}{2n+1}$$

b) $1+(m+1)x \le (1+x)^{m+1}$, $x>0$, $m\in\mathbb{N}$

$P(m): 1+(m+1)x \le (1+x)^{m+1}, \forall x>0, m\in\mathbb{N}$

I $P(1): 1+2x \le (1+x)^2$

$1+2x \le 1+2x+x^2 \Rightarrow x^2 \ge 0$; True, $\forall x>0$

II We assume $P(k)$ is true and prove $P(k+1)$ is true

$P(k): 1+(k+1)x \le (1+x)^{k+1}$

$P(k+1): 1+(k+2)x \le (1+x)^{k+2}$

$(1+x)^{k+2} = (1+x)(1+x)^{k+1} \Rightarrow [1+(k+1)x](1+x) \le (1+x)(1+x)^{k+1}$

$[1+(k+1)x](x+1) \ge 1+(k+2)x$

$(1+xk+x)(x+1) \ge 1+xk+2x$

$\cancel{x}+x^2k+x^2+\cancel{1}+\cancel{xk}+\cancel{x} \ge \cancel{1}+\cancel{xk}+\cancel{2x}$

$x^2k+x^2 \ge 0$

$$\left.\begin{array}{l} x^2(k+1)\ge 0 \\ x^2 \ge 0, \forall x>0 \\ k+1>0, \forall k\in\mathbb{N} \end{array}\right\} \Rightarrow x^2(k+1)\ge 0 \Rightarrow P(k+1) \text{ true}$$

$(\mathrm{I},\mathrm{II}) \Rightarrow P(m)$ true, $\forall x>0, m\in\mathbb{N}$

$\boxed{\text{Exercise 2.1}}$

$A_1 = [-8;\pi)\cap\mathbb{Z}$; $A_1 = \{-8,-7,\dots,3\}$

$lb(A_1) = (-\infty;8]$ $\quad$ $\min(A_1) = -8 = \inf(A_1)$

$ub(A_1) = [3;+\infty)$ $\quad$ $\max(A_1) = 3 = \sup(A_1)$

$A_2 = \{2^m+n! \mid m,n\in\mathbb{N}\}$

$lb(A_2) = (-\infty;2]$ $\quad$ $\min(A_2) = 2 = \inf(A_2)$

$ub(A_2) = \emptyset$ $\quad$ no maximum, $\sup(A_2) = +\infty$

$A_3 = \{x+\frac{1}{x} \mid x\in\mathbb{R}, x<0\}$ $\quad$ $a+\frac{1}{a} \ge 2$, $\forall a>0 \Rightarrow$

$lb(A_3) = \emptyset$ $\quad$ $\Rightarrow -a+\frac{1}{-a} \le 2$ $(\forall)\, a>0$

$ub(A_3) = [-2;+\infty)$

$\inf(A_3) = -\infty$, no minimum $\quad$ $-a = x \Rightarrow x+\frac{1}{x} \le 2, (\forall)\, x<0 \Rightarrow A_3 = (-\infty;-2]$

$\sup(A_3) = -2 = \max(A_3)$

$A_4 = \left\{ \frac{n}{1-n^2} \mid n \in \mathbb{N}, n \geq 2 \right\}$ $\quad A_4 \subseteq \left[\frac{-2}{3}; 0 \right]$

$\lim\limits_{n \to 0} \frac{n}{1-n^2} = 0 \Rightarrow \frac{n}{1-n^2} < 0, \forall n \in \mathbb{N}, n \geq 2$

$n = 2 \Rightarrow \frac{n}{1-n^2} = -\frac{2}{3}$

$lb(A_4) = \left(-\infty; \frac{-2}{3}\right]$ $\quad \inf(A_4) = \frac{-2}{3} = \max(A_4)$

$ub(A_4) = [0; +\infty)$ $\quad \sup(A_4) = 0$, no minimum

$\boxed{\text{Exercise 2.3}}$

$A_1 = (-1; 0] \cup \{1\}$, $A_1 \notin V(0)$

$A_2 = \left[1 - \frac{3}{2}; 1 + \frac{3}{2}\right] \cup (3; 4)$

$A_2 = \left[\frac{-1}{2}; \frac{5}{2}\right] \cup (3; 4)$, $A_2 \in V(0)$, because $\left(\frac{-1}{2}; \frac{1}{2}\right) \in A_2$

$A_3 = \mathbb{R}$, $A_3 \in V(0)$, because $(-\varepsilon, \varepsilon) \in A_3, \forall \varepsilon > 0$

$A_4 = \mathbb{R} \setminus \mathbb{Q}$, $A_4 \notin V(0)$, because it doesn't contain intervals

$\boxed{\text{Exercise 3.1.}}$

a) $\lim\limits_{n \to \infty} \frac{n + \sin^2 n}{\cos n - 3n} = \lim\limits_{n \to \infty} \frac{\cancel{n}\left(1 + \overbrace{\frac{\sin^2 n}{n}}^{\to 0}\right)}{\cancel{n}\left(\underbrace{\frac{\cos n}{n}}_{\to 0} - 3\right)} = -\frac{1}{3}$

b) $\lim\limits_{n \to \infty} (n^2 + n)^{\frac{-n}{n+1}} = \infty^{-1} = \frac{1}{\infty} = 0$

$\lim\limits_{n \to \infty} (n^2 + n) = \infty$; $\lim\limits_{n \to \infty} -\frac{n}{n+1} = -1$

c) $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n^3 + 2n^2}\right)^{n - n^3} \underset{(1^\infty)}{=} \lim\limits_{n \to \infty} \left[\left(1 + \frac{1}{n^3 + 2n^2}\right)^{n^3 + 2n^2}\right]^{\frac{-n^3 + n}{n^3 + 2n^2}} = \lim\limits_{n \to \infty} e^{\frac{-n^3 + n}{n^3 + 2n^2}} =$

$= e^{\lim\limits_{n \to \infty} \frac{-n^3 + n}{n^3 + 2n^2}} = e^{-1} = \frac{1}{e}$

d) $\lim\limits_{n \to \infty} \frac{1 \cdot 1! + 2 \cdot 2! + \dots + n \cdot n!}{(n+1)!} = \lim\limits_{n \to \infty} \frac{(n+1)! - 1}{(n+1)!} = \lim\limits_{n \to \infty} \left[\frac{(n+1)!}{(n+1)!} - \overbrace{\frac{1}{(n+1)!}}^{\to 0}\right] = 1$

$1 \cdot 1! + 2 \cdot 2! + \dots + n \cdot n! = \sum\limits_{k=1}^{n} k \cdot k! = \sum\limits_{k=1}^{n} (k+1)k! - k! = \sum\limits_{k=1}^{n} (k+1)! - k! =$

$= \cancel{2!} - 1! + \cancel{3!} - \cancel{2!} + \cancel{4!} - \cancel{3!} + \dots + (n+1)! - \cancel{n!} = (n+1)! - 1$

e) $\lim\limits_{n\to\infty}\sqrt[n]{1+2+\dots+n} = \lim\limits_{n\to\infty}\left[\frac{n(n+1)}{2}\right]^{\frac{1}{n}} = \lim\limits_{n\to\infty} e^{\left(\frac{\ln\frac{n(n+1)}{2}}{n}\right)} =$

$= \lim\limits_{n\to\infty} e^{\left[\frac{\ln\frac{n^2(1+\overset{\to 0}{1/n})}{2}}{n}\right]} = \lim\limits_{n\to\infty} e^{\frac{\ln n^2 - \ln 2}{n}} = e^{\lim\limits_{n\to\infty}\left(\overset{\to 0}{\frac{2\ln n}{n}} - \overset{\to 0}{\frac{\ln 2}{n}}\right)} = e^0 = 1$

f) $\lim\limits_{n\to\infty} n\left[(1+1/n)^{1+\frac{1}{n}} - 1\right] = \lim\limits_{n\to\infty}\frac{(1+1/n)^{1+1/n}-1}{1/n}$

$\left.\begin{array}{l} 1+\frac{1}{n} = m \\ n\to\infty \Rightarrow m\to 1 \end{array}\right\} \Rightarrow \lim\limits_{n\to\infty}\frac{(1+\frac{1}{n})^{1+\frac{1}{n}}-1}{\frac{1}{n}} = \lim\limits_{m\to 1}\frac{m^m-1}{m-1} \underset{(l'H)}{\overset{(\frac{0}{0})}{=}} \lim\limits_{m\to 1}(m^m)(\ln m+1) = 1$

$(m^m)' = (e^{m\ln m})' = e^{m\cdot\ln m}\cdot(m\cdot\ln m)' = m^m(\ln m+1)$

Exercise 3.3.

(x_n) sequence in $\mathbb{Z}$, (x_n) convergent

is (x_n) eventually constant? (i.e. $\exists\, n_0\in\mathbb{N}$ s.t. $\forall\, m,n\in\mathbb{N}$, $m,n\ge n_0$, $x_m = x_n$)

$\left.\begin{array}{l}(x_n) \text{ convergent} \Rightarrow \lim\limits_{n\to\infty} x_n = L \\ x_n\in\mathbb{Z}\end{array}\right\} \Rightarrow L\in\mathbb{Z}$

$\left.\begin{array}{l}\lim\limits_{n\to\infty} x_n = L \Leftrightarrow (\forall)\,\varepsilon>0,\ \exists\, n_\varepsilon\in\mathbb{N} \text{ s.t. } x_n\in(L-\varepsilon, L+\varepsilon), (\forall)\, n\ge n_\varepsilon \\ \exists\,\varepsilon>0 \text{ s.t. } (L-\varepsilon, L+\varepsilon)\cap\mathbb{Z} = \{L\}\end{array}\right\} \Rightarrow$

$\Rightarrow x_n = L, (\forall)\, n\ge n_\varepsilon \Rightarrow (x_n)$ is eventually constant