Exercise 7.1 - find the n^{th} derivative of $f:\mathbb{R}\to\mathbb{R}$, $f(x) = e^x \sin x$

$$f(x) = e^x \sin x$$

$$f'(x) = (e^x)' \cdot \sin x + e^x \cdot (\sin x)' = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$$

$$f''(x) = (f'(x))' = \left[e^x(\sin x + \cos x)\right]' = (e^x)' \cdot (\sin x + \cos x) + e^x(\sin x + \cos x)' =$$

$$= e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = e^x \cdot 2\cos x = 2e^x \cos x$$

$$f^{(3)}(x) = (f''(x))' = (2e^x \cos x)' = 2(e^x \cos x)' = 2(e^x)' \cos x + 2e^x(\cos x)' =$$

$$= 2e^x \cos x - 2e^x \sin x = 2e^x(\cos x - \sin x)$$

$$f^{(4)}(x) = (f^{(3)}(x))' = \left[2e^x(\cos x - \sin x)\right]' = 2\left[(e^x)'(\cos x - \sin x) + e^x(\cos x - \sin x)'\right] =$$

$$= 2e^x(\cos x - \sin x) + 2e^x(-\cos x - \sin x) = 2e^x \cdot -2\sin x = -4e^x \sin x$$

$$f^{(5)}(x) = -4e^x(\sin x + \cos x)$$

$$f^{(6)}(x) = -8e^x \cos x$$

$$f^{(7)}(x) = -8e^x(\cos x - \sin x)$$

$$f^{(8)}(x) = 16 e^x \sin x$$

...

$$f^{(n)}(x) = \begin{cases} (-1)^k \cdot 2^{2k} \cdot e^x \sin x, & n = 4k \\ (-1)^k \cdot 2^{2k} \cdot e^x(\sin x + \cos x), & n = 4k+1 \\ (-1)^k \cdot 2^{2k+1} \cdot e^x \cos x, & n = 4k+2 \\ (-1)^k \cdot 2^{2k+1} \cdot e^x(\cos x - \sin x), & n = 4k+3 \end{cases} = \begin{cases} (-4)^k \cdot e^x \sin x, & n = 4k \\ (-4)^k e^x(\sin x + \cos x), & n = 4k+1 \\ (-4)^k \cdot 2e^x \cos x, & n = 4k+2 \\ (-4)^k \cdot 2e^x(\cos x - \sin x), & n = 4k+3 \end{cases}$$

we prove $P(n): f^{(n)}(x)$ true, $\forall n \in \mathbb{N}$

$\boxed{I}$ $P(0)$: $$f^{(n)}(x) = \begin{cases} (-1)^0 \cdot 2^0 \cdot e^x \sin x, & n = 0 \\ (-1)^0 \cdot 2^0 \cdot e^x(\sin x + \cos x), & n = 1 \\ (-1)^0 \cdot 2 \cdot e^x \cos x, & n = 2 \\ (-1)^0 \cdot 2 \cdot e^x(\cos x - \sin x), & n = 3 \end{cases} = \begin{cases} e^x \sin x, & n = 0 \\ e^x(\sin x + \cos x), & n = 1 \\ 2e^x \cos x, & n = 2 \\ 2e^x(\cos x - \sin x), & n = 3 \end{cases}$$

$\Rightarrow P(0)$ true (1)

$\boxed{II}$ we assume $P($~~k~~$n)$ is true and prove $P(n+1)$ also true

$$P(n): f^{(m)}(x) = \begin{cases} (-4)^n \cdot e^x \sin x, & m = 4n \\ (-4)^n \cdot e^x(\sin x + \cos x), & m = 4n+1 \\ (-4)^n \cdot 2e^x \cos x, & m = 4n+2 \\ (-4)^n \cdot 2e^x(\cos x - \sin x), & m = 4n+3 \end{cases}$$

$$P(n+1): f^{(m)}(x) = \begin{cases} (-4)^{n+1} \cdot e^x \sin x, & m = 4n+4 \\ (-4)^{n+1} \cdot e^x(\sin x + \cos x), & m = 4n+5 \\ (-4)^{n+1} \cdot 2e^x \cos x, & m = 4n+6 \\ (-4)^{n+1} \cdot 2e^x(\cos x - \sin x), & m = 4n+7 \end{cases}$$

To prove $P(n+1)$, we have to calculate $f^{(m)}(x)$, $m = \overline{4n+4, 4n+7}$

$$f^{(4n+4)}(x) = \left(f^{(4n+3)}(x)\right)' = \left[(-4)^n \cdot 2e^x(\cos x - \sin x)\right]' = (-4)^n \cdot 2\left[e^x(\cos x - \sin x) + \right.$$
$$\left. + e^x(-\sin x - \cos x)\right] = (-4)^n \cdot 2e^x(\cos x - \sin x - \sin x - \cos x) =$$
$$= (-4)^n \cdot (-4) e^x \sin x = (-4)^{n+1} \cdot e^x \sin x$$

$$f^{(4n+5)}(x) = \left(f^{(4n+4)}(x)\right)' = \left[(-4)^{n+1} \cdot e^x \sin x\right]' = (-4)^{n+1}(e^x \sin x + e^x \cos x) =$$
$$= (-4)^{n+1} \cdot e^x(\sin x + \cos x)$$

$$f^{(4n+6)}(x) = \left(f^{(4n+5)}(x)\right)' = \left[(-4)^{n+1} \cdot e^x(\sin x + \cos x)\right]' = (-4)^{n+1}\left[e^x(\sin x + \cos x) + \right.$$
$$\left. + e^x(\cos x - \sin x)\right] = (-4)^{n+1} \cdot e^x(\cancel{\sin x} + \cos x + \cos x - \cancel{\sin x}) =$$
$$= (-4)^{n+1} \cdot 2e^x \cos x$$

$$f^{(4n+7)}(x) = \left(f^{(4n+6)}(x)\right)' = \left[(-4)^{n+1} \cdot 2e^x \cos x\right]' = (-4)^{n+1} \cdot 2(e^x \cos x - e^x \sin x) =$$
$$= (-4)^{n+1} \cdot 2e^x(\cos x - \sin x)$$

$$\Rightarrow f^{(m)}(x) = \begin{cases} (-4)^{n+1} \cdot e^x \sin x, & m = 4n+4 \\ (-4)^{n+1} \cdot e^x(\sin x + \cos x), & m = 4n+5 \\ (-4)^{n+1} \cdot 2e^x \cos x, & m = 4n+6 \\ (-4)^{n+1} \cdot 2e^x(\cos x - \sin x), & m = 4n+7 \end{cases} \Rightarrow$$

$\Rightarrow$ ~~P(k+1)~~ $P(n+1)$ true (2)

$$(1), (2) \Rightarrow P(m) \text{ true}, \forall m \in \mathbb{N} \Rightarrow f^{(m)}(x) = \begin{cases} (-4)^k \cdot e^x \sin x, & m = 4k \\ (-4)^k e^x(\sin x + \cos x), & m = 4k+1 \\ (-4)^k \cdot 2e^x \cos x, & m = 4k+2 \\ (-4)^k \cdot 2e^x(\cos x - \sin x), & m = 4k+3 \end{cases}$$

Exercise 7.2. - Compute the following limits

a) $\lim_{x\to\infty} \frac{x+\ln x}{x\cdot\ln x} \overset{\left(\frac{\infty}{\infty}\right)}{\underset{\text{l'Hopital}}{=\!=}} \lim_{x\to\infty} \frac{1+\overbrace{1/x}^{\to 0}}{\underbrace{\ln x+1}_{\to\infty}} = \frac{1}{\infty} = 0$

b) $\lim_{\substack{x\to 0\\ x>0}} x\cdot\ln(\sin x) \overset{0\cdot\infty}{=} \lim_{\substack{x\to 0\\ x>0}} \frac{\ln\sin x}{\frac{1}{x}} \overset{\frac{\infty}{\infty}}{\underset{(l'H)}{=}} \lim_{\substack{x\to 0\\ x>0}} \frac{\frac{1}{\sin x}\cdot\cos x}{\frac{-1}{x^2}} =$

$= \lim_{\substack{x\to 0\\ x>0}} \frac{-x^2\cos x}{\sin x} \overset{\left(\frac{0}{0}\right)}{\underset{l'H}{=}} \lim_{\substack{x\to 0\\ x>0}} \frac{\overbrace{-2x\cos x}^{\to 0}+\overbrace{x^2\sin x}^{\to 0}}{\underbrace{\cos x}_{\to 0}} = 0$

c) $\lim_{\substack{x\to 0\\ x>0}} (\sin x)^x = \lim_{\substack{x\to 0\\ x>0}} e^{\ln(\sin x)^x} = \lim_{\substack{x\to 0\\ x>0}} e^{x\ln\sin x} = e^{\lim_{\substack{x\to 0\\ x>0}} x\ln\sin x} = e^0 = 1$

(from b) we have that $\lim_{\substack{x\to 0\\ x>0}} x\ln\sin x = 0$)

Exercise 7.3. - Find the 3^{rd} Taylor polynomial $T_3(x)$ ~~at~~ of f at 1.

$f:\mathbb{R}\to\mathbb{R}, f(x) = x^3-3x^2+5x+1$

$T_n(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

$n=3,\ x_0=1 \Rightarrow T_3(x) = f(1) + \sum_{k=1}^{3} \frac{f^{(k)}(1)}{k!}(x-1)^k$

$T_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$

$f'(x) = 3x^2-6x+5$

$f''(x) = 6x-6$

$f^{(3)}(x) = 6$

$T_3(x) = 1-3+5+1 + \frac{3-6+5}{1!}(x-1) + \cancel{6+6}\ \overbrace{\frac{6-6}{2!}(x-1)^2}^{0} + \frac{6}{3!}(x-1)^3 =$

$= 4 + 2(x-1) + (x-1)^3$

$= x^3-3x^2-x+2$

Exercise 8.1 - Prove that f can be expanded as a T.S. around 1 on $[1,2)$ and find the corresponding T.S. expansion

$f:(0;+\infty)\to\mathbb{R}, f(x)=\frac{1}{x^2}$

f can be expanded as a T.S. $\Leftrightarrow \lim\limits_{n\to\infty} R_n(x)=0$, where $R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$,

c between x and x_0

$f'(x)=\left(\frac{1}{x^2}\right)'=\frac{-2x}{x^4}=\frac{-2}{x^3}=\frac{(-1)^1\cdot 2!}{x^3}$

$f''(x)=\left(\frac{-2}{x^3}\right)'=-2\left(\frac{1}{x^3}\right)'=-2\cdot\frac{-3}{x^4}=\frac{6}{x^4}=\frac{(-1)^2\cdot 3!}{x^4}$

...

$f^{(n)}(x)=\frac{(-1)^n(n+1)!}{x^{n+2}}=(-1)^n(n+1)!\cdot x^{-n-2}$

$P(n): f^{(n)}(x)=(-1)^n(n+1)!\cdot x^{-n-2},\ n\in\mathbb{N}^*$

$P(1): f'(x)=(-1)^1\cdot 2!\cdot x^{-1-2}=-2\cdot x^{-3}=\frac{-2}{x^3}$, true (1)

assume $P(k)$ true and prove $P(k+1)$ also true

$P(k): f^{(k)}(x)=(-1)^k(k+1)!\cdot x^{-k-2}$, true

$P(k+1): f^{(k+1)}(x)=(-1)^{k+1}\cdot(k+2)!\cdot x^{-k-3}$

$f^{(k+1)}(x)=\left[f^{(k)}(x)\right]'=\left[(-1)^k\cdot(k+1)!\cdot x^{-k-2}\right]'=(-1)^k\cdot(k+1)!\cdot(-k-2)\cdot x^{-k-2-1}=$

$=(-1)^k\cdot(-1)\cdot(k+1)!(k+2)\cdot x^{-k-3}=(-1)^{k+1}\cdot(k+2)!\cdot x^{-k-3}$, true (2)

(1),(2) $\Rightarrow P(n)$ true, $\forall n\in\mathbb{N}^*$

$\Rightarrow f^{(n+1)}(x)=\frac{(-1)^{n+1}\cdot(n+2)!}{x^{n+3}}$

$R_n(x)=\frac{(-1)^{n+1}\cdot(n+2)\cancel{!}}{c^{n+3}}\cdot\frac{1}{\cancel{(n+1)!}}\cdot(x-1)^{n+1}=\frac{(-1)^{n+1}\cdot(n+2)\cdot(x-1)^{n+1}}{c^{n+3}}$

$|R_n(x)|=\frac{(n+2)(x-1)^{n+1}}{c^{n+3}}$ $\qquad \lim\limits_{n\to\infty}\frac{(n+2)(x-1)^{n+1}}{c^{n+3}}\le\lim\limits_{n\to\infty}\underbrace{\frac{n+2}{c^{n+3}}}_{a_n}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{n+3}{c^{\cancel{n+4}}}\cdot\frac{c^{\cancel{n+3}}}{n+2}=\lim_{n\to\infty}\frac{n+3}{c(n+2)}=\frac{1}{c}<1\Rightarrow$$

$$\Rightarrow\lim_{n\to\infty}a_n=0$$

$$\lim_{n\to\infty}\frac{-(n+2)}{c^{n+3}}\leq\lim_{n\to\infty}\frac{(n+2)(x-1)^{n+1}}{c^{n+3}}\leq\lim_{n\to\infty}\frac{n+2}{c^{n+3}}\Rightarrow$$

(with the outer limits $\to 0$)

$$\xRightarrow{\text{Squeeze Th.}}\lim_{n\to\infty}R_n(x)=0\Rightarrow f \text{ can be expanded as a T.S. around } 1 \text{ on } [1,2)\Rightarrow$$

$$\Rightarrow f(x)=T_n(x)=f(x_0)+\sum_{k=1}^{\infty}\frac{f^{(k)}(x)}{k!}(x-x_0)^k$$

$$f(x)=f(1)+\sum_{k=1}^{\infty}\frac{(-1)^k(k+1)!}{x^{k+2}\cdot k!}\cdot(x-1)^k=1+\sum_{k=1}^{\infty}\frac{(-1)^k(k+1)}{x^{k+2}}\cdot(x-1)^k=$$

$$=1+\frac{-2(x-1)}{x^3}+\frac{3(x-1)^2}{x^4}-\frac{4(x-1)^3}{x^5}+\ldots=\frac{1}{x^2},\ x\in[1;2)$$

Exercise 8.2 - Prove that if $x,y\in\overline{B}(z,r)$, $z\in\mathbb{R}^n$, $r>0$ s.t. $\|x-y\|\geq\varepsilon r$, $\varepsilon\in(0;2]$, then $\left\|z-\frac{x+y}{2}\right\|\leq r\sqrt{1-\frac{\varepsilon^2}{4}}$

$$x,y\in\overline{B}(z,r)\Rightarrow\begin{cases}\|x-z\|\leq r & \uparrow^2\\ \|y-z\|\leq r & \uparrow^2\end{cases}\Rightarrow\begin{cases}\|x-z\|^2\leq r^2 & |\cdot(\tfrac{1}{2})\\ \|y-z\|^2\leq r^2 & |\cdot(\tfrac{1}{2})\end{cases}$$

$$\begin{cases}\frac{1}{2}\|x-z\|^2\leq r^2/2\\ \frac{1}{2}\|y-z\|^2\leq r^2/2\end{cases}\xRightarrow{(+)}\frac{1}{2}\left(\|x-z\|^2+\|y-z\|^2\right)\leq r^2\quad(1)$$

$$\|x-y\|\geq\varepsilon r\,|\cdot(\tfrac{1}{2})\ ;\quad\frac{1}{2}\|x-y\|\geq\frac{\varepsilon r}{2}\Rightarrow\left\|\frac{x-y}{2}\right\|\geq\frac{\varepsilon r}{2}\ \uparrow^2;$$

$$\left\|\frac{x-y}{2}\right\|^2\geq\frac{\varepsilon^2r^2}{4}\Rightarrow\frac{1}{4}\|x-y\|^2\geq\frac{\varepsilon^2r^2}{4}\Rightarrow\frac{-1}{4}\|x-y\|^2\leq-\frac{\varepsilon^2r^2}{4}$$

$$(1)+(2)\Rightarrow\frac{1}{2}\left(\|x-z\|^2+\|y-z\|^2\right)-\frac{1}{4}\|x-y\|^2\leq r^2-\frac{\varepsilon^2r^2}{4}$$

$$\left\|z-\frac{x+y}{2}\right\|^2 = \sum_{i=1}^{m}\left(z_i-\frac{x_i}{2}-\frac{y_i}{2}\right)^2 = \sum_{i=1}^{m}\left(z_i^2+\frac{x_i^2}{4}+\frac{y_i^2}{4}-z_ix_i-z_iy_i+\frac{x_iy_i}{2}\right) =$$

$$= \sum_{i=1}^{m}\left(\frac{4z_i^2+2x_i^2-x_i^2+2y_i^2-y_i^2-4z_ix_i-4z_iy_i+2x_iy_i}{4}\right) = \frac{1}{4}\sum_{i=1}^{m}\Big[(2x_i^2-4z_ix_i+2z_i^2)+$$

$$+(2y_i^2-4z_iy_i+2z_i^2)-(x_i^2-2x_iy_i+y_i^2)\Big] = \frac{1}{4}\sum_{i=1}^{m}\left[2(x_i-z_i)^2+2(y_i-z_i)^2-(x_i-y_i)^2\right] =$$

$$= \frac{2}{4}\sum_{i=1}^{m}(x_i-z_i)^2+\frac{2}{4}\sum_{i=1}^{m}(y_i-z_i)^2-\frac{1}{4}\sum_{i=1}^{m}(x_i-y_i)^2 = \frac{1}{2}\|x-z\|^2+\frac{1}{2}\|y-z\|^2-\frac{1}{4}\|x-y\|^2 \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow \left\|z-\frac{x+y}{2}\right\|^2 = \frac{1}{2}\|x-z\|^2+\frac{1}{2}\|y-z\|^2-\frac{1}{4}\|x-y\|^2 \\ \frac{1}{2}\|x-z\|^2+\frac{1}{2}\|y-z\|-\frac{1}{4}\|x-y\|^2 \le r^2\left(1-\frac{\varepsilon^2}{4}\right) \end{array}\right\} \Rightarrow \left\|z-\frac{x+y}{2}\right\|^2 \le r^2\left(1-\frac{\varepsilon^2}{4}\right) \Rightarrow \left\|z-\frac{x+y}{2}\right\| \le r\sqrt{1-\frac{\varepsilon^2}{4}}$$

Exercise 9.1 - Study continuity at O_2

a) $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=\begin{cases}\dfrac{xy+x^2y\ln(x^2+y^2)}{x^2+y^2}, & (x,y)\neq O_2\\ 0, & (x,y)=O_2\end{cases}$

$f(x,y)$ continuous at $O_2 \Leftrightarrow \lim_{(x,y)\to O_2} f(x,y) = f(0,0) = 0$

$$\lim_{(x,y)\to O_2} f(x,y) = \lim_{(x,y)\to O_2}\frac{xy+x^2y\ln(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0}\frac{r^2\cos\alpha\sin\alpha + r^2\cos^2\alpha\cdot r\sin\alpha\cdot\ln(r^2(\cos^2\alpha+\sin^2\alpha))}{r^2(\cos^2\alpha+\sin^2\alpha)}$$

$x = r\cos\alpha$

$y = r\sin\alpha$

$$= \lim_{r\to 0}\frac{r^2\cos\alpha\sin\alpha + r^3\cos^2\alpha\sin\alpha\cdot\ln r^2}{r^2} =$$

$$= \lim_{r\to 0}\cos\alpha\sin\alpha + r\cos^2\alpha\sin\alpha\cdot\ln r^2 =$$

$$= \cos\alpha\sin\alpha + \lim_{r\to 0} r\cos^2\alpha\sin\alpha\cdot\ln r^2 \overset{0\cdot\infty}{=} \cos\alpha\sin\alpha + \lim_{r\to 0}\frac{\cos^2\alpha\sin\alpha\cdot\ln r^2}{\frac{1}{r}} \underset{\text{l'Hopital}}{\overset{\frac{\infty}{\infty}}{=}}$$

$$= \cos\alpha\sin\alpha + \lim_{r\to 0}\frac{\cos^2\alpha\sin\alpha\cdot\frac{2r}{r^2}}{-1/r^2} = \cos\alpha\sin\alpha - 2\lim_{r\to 0} \underbrace{r\cos^2\alpha\sin\alpha}_{\to 0} =$$

$$= \cos\alpha\sin\alpha \neq 0 \Rightarrow$$

$\Rightarrow f$ is not continuous at O_2

b) $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=\begin{cases}\dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}, & (x,y)\neq O_2\\ 0, & (x,y)=O_2\end{cases}$

f continuous at $O_2 \Leftrightarrow \lim\limits_{(x,y)\to O_2} f(x,y)=f(0,0)=0$

$x^4+y^4\geq\dfrac{(x^2+y^2)^2}{2};\quad x^4+y^4\geq\dfrac{x^4+2x^2y^2+y^4}{2};\quad 2x^4+2y^4\geq x^4+2x^2y^2+y^4 \Rightarrow$

$\Rightarrow x^4-2x^2y^2+y^4\geq 0$

$(x^2-y^2)^2\geq 0$ true $\forall\, x,y\in\mathbb{R}$

$x^4+y^4\geq\dfrac{(x^2+y^2)^2}{2}\Rightarrow\dfrac{1}{x^4+y^4}\leq\dfrac{2}{(x^2+y^2)^2}$

$\lim\limits_{(x,y)\to O_2}\dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}\leq\lim\limits_{(x,y)\to O_2}\dfrac{2\cdot e^{-\frac{1}{x^2+y^2}}}{(x^2+y^2)^2}=\lim\limits_{r\to 0}\dfrac{2\cdot e^{-\frac{1}{r^2(\cos^2\alpha+\sin^2\alpha)}}}{[r^2(\cos^2\alpha+\sin^2\alpha)]^2}=$

$x=r\cos\alpha$
$y=r\sin\alpha$

$=\lim\limits_{r\to 0}\dfrac{2e^{-1/r^2}}{r^4}=2\lim\limits_{r\to 0}\dfrac{1}{r^4}\cdot\dfrac{1}{e^{1/r^2}}=2\cdot\dfrac{1}{\underbrace{\lim\limits_{r\to 0}r^4\cdot e^{1/r^2}}_{L_1}}$ (1)

$L_1=\lim\limits_{r\to 0}r^4\cdot e^{1/r^2}\overset{(0\cdot\infty)}{=}\lim\limits_{r\to 0}\dfrac{e^{1/r^2}}{\frac{1}{r^4}}\overset{\left(\frac{\infty}{\infty}\right)}{\underset{l'H}{=}}\lim\limits_{r\to 0}\dfrac{e^{1/r^2}\cdot\frac{-2}{r^3}}{\frac{-4}{r^5}}=\lim\limits_{r\to 0}\dfrac{e^{1/r^2}\cdot r^2}{2}\overset{(\infty\cdot 0)}{=}\dfrac{1}{2}\lim\limits_{r\to 0}\dfrac{e^{1/r^2}}{\frac{1}{r^2}}\overset{\frac{\infty}{\infty}}{\underset{l'H}{=}}$

$=\dfrac{1}{2}\lim\limits_{r\to 0}\dfrac{e^{1/r^2}\cdot\cancel{-2/r^3}}{\cancel{-2/r^3}}=\dfrac{1}{2}\cdot\lim\limits_{r\to 0}e^{1/r^2}=+\infty$ (2)

(1),(2) $\Rightarrow\lim\limits_{(x,y)\to O_2}f(x,y)\leq 2\cdot\dfrac{1}{\infty}=0\Rightarrow f$ is continuous at O_2

Exercise 9.2 – Find the second order derivatives

a) $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=\sin(x\sin y)$

$\dfrac{\partial f}{\partial x}(x,y)=[\sin(x\sin y)]'=\cos(x\sin y)\cdot(x\sin y)'=\cos(x\sin y)\cdot\sin y$

$\dfrac{\partial f}{\partial y}(x,y)=[\sin(x\sin y)]'=\cos(x\sin y)\cdot(x\sin y)'=\cos(x\sin y)\cdot x\cos y$

$\dfrac{\partial^2 f}{\partial x^2}=\dfrac{\partial^2 f}{\partial x\partial x}(x,y)=[\cos(x\sin y)\cdot\sin y]'=[\cos(x\sin y)]'\sin y+\cos(x\sin y)\cdot(\sin y)'=$
$=-\sin(x\sin y)\cdot\sin^2 y$

$$\frac{\partial^2 f}{\partial y^2}(x,y)=\frac{\partial^2 f}{\partial y \partial y}(x,y)=\left[\cos(x\sin y)\cdot x\cos y\right]'=\left[\cos(x\sin y)\right]'\cdot x\cos y+$$

$$+\cos(x\sin y)\cdot(x\cos y)'=-\sin(x\sin y)\cdot x^2\cos^2 y+\cos(x\sin y)\cdot(-x\sin y)$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y)=\frac{\partial}{\partial x}\left(\cos(x\sin y)\cdot x\cos y\right)=$$

$$=\left[\cos(x\sin y)\right]'\cdot x\cos y+\cos(x\sin y)\cdot(x\cos y)'=$$

$$=-\sin(x\sin y)\cdot\sin y\cdot x\cos y+\cos(x\sin y)\cdot\cos y$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)=\frac{\partial}{\partial y}\left(\cos(x\sin y)\cdot\sin y\right)=$$

$$=\left[\cos(x\sin y)\right]'\cdot\sin y+\cos(x\sin y)\cdot(\sin y)'=$$

$$=-\sin(x\sin y)\cdot x\cos y\cdot\sin y+\cos(x\sin y)\cdot\cos y$$

b) $f:\mathbb{R}^3\to\mathbb{R},\ f(x,y,z)=(1+x^2)ye^z$

$$\frac{\partial f}{\partial x}(x,y,z)=2x\cdot ye^z;\quad \frac{\partial f}{\partial y}(x,y,z)=(1+x^2)e^z;\quad \frac{\partial f}{\partial z}(x,y,z)=(1+x^2)ye^z$$

$$\frac{\partial^2 f}{\partial x^2}(x,y,z)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y,z)=\frac{\partial}{\partial x}(2x\cdot ye^z)=2\cdot ye^z$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y,z)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y,z)=\frac{\partial}{\partial y}(2x\cdot ye^z)=2x\cdot e^z$$

$$\frac{\partial^2 f}{\partial z\partial x}(x,y,z)=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)(x,y,z)=\frac{\partial}{\partial z}(2xye^z)=2xye^z$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y,z)=\frac{\partial}{\partial y}\left((1+x^2)\cdot e^z\right)=0$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y,z)=\frac{\partial}{\partial x}\left((1+x^2)\cdot e^z\right)=2x\cdot e^z$$

$$\frac{\partial^2 f}{\partial z\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)(x,y,z)=\frac{\partial}{\partial z}\left((1+x^2)\cdot e^z\right)=(1+x^2)\cdot e^z$$

$$\frac{\partial^2 f}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)(x,y,z)=\frac{\partial}{\partial z}\left((1+x^2)\cdot ye^z\right)=(1+x^2)\cdot ye^z$$

$$\frac{\partial^2 f}{\partial x\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)(x,y,z)=\frac{\partial}{\partial x}\left((1+x^2)ye^z\right)=2xye^z$$

$$\frac{\partial^2 f}{\partial y\partial z}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)(x,y,z)=\frac{\partial}{\partial y}\left((1+x^2)ye^z\right)=(1+x^2)e^z$$