4.1. (x_n) sequence, $x_1 \in (0;1)$, $x_{n+1} = x_n - x_n^2$, $n \in \mathbb{N}$

prove that (x_n) converges, $\lim_{n\to\infty} x_n = ?$

study convergence for $(n \cdot x_n)$

$x_{n+1} - x_n = x_n - x_n^2 - x_n = -x_n^2 < 0, \forall n \in \mathbb{N} \Rightarrow (x_n)$ is decreasing (1)

$P(n): x_n \in (0;1), \forall n \in \mathbb{N}$

$P(1)$ is true, $x_1 \in (0;1)$

we assume $P(k)$ is true, and we prove that $P(k+1)$ is also true

$P(k): x_k \in (0;1)$ true

$P(k+1): x_{k+1} \in (0;1)$

$x_k \in (0;1) \Rightarrow 0 < x_k < 1 \;|-1$

$-1 < x_k - 1 < 0 \;|\cdot(-1)$

$0 < 1 - x_k < 1$

$\left.\begin{array}{l} 0 < x_k < 1 \\ 0 < 1 - x_k < 1 \end{array}\right\} \Rightarrow 0 < x_k(1-x_k) < 1$

$0 < x_k - x_k^2 < 1 \Rightarrow 0 < x_{k+1} < 1 \Rightarrow x_{k+1} \in (0;1) \Rightarrow P(k+1)$ true $\Rightarrow$

$\Rightarrow P(n)$ true, $\forall n \in \mathbb{N} \Rightarrow x_n \in (0;1), \forall n \in \mathbb{N} \Rightarrow (x_n)$ bounded (2)

(1), (2) $\Rightarrow (x_n)$ convergent

$\left.\begin{array}{l} \text{Let } \lim_{n\to\infty} x_n = L \\ (x_n) \text{ convergent} \end{array}\right\} \Rightarrow \lim_{n\to\infty} x_{n+1} = L$

$x_{n+1} = x_n - x_n^2 \;\Big|\cdot \lim_{n\to\infty}$

$L = L - L^2$

$L^2 = 0 \Rightarrow L = 0 \Rightarrow \lim_{n\to\infty} x_n = 0$

$(n \cdot x_n)$ - converges

let $(a_n), (b_n), n \in \mathbb{N}$ sequences, $a_n = n$, $b_n = \frac{1}{x_n}$

(x_n) strictly decreasing $\Rightarrow \left(\frac{1}{x_n}\right)$ strictly increasing $\Rightarrow (b_n)$ strictly increasing

$\lim\limits_{n\to\infty} x_n = 0 \Rightarrow \lim\limits_{n\to\infty} \frac{1}{x_n} = +\infty \Rightarrow \lim\limits_{n\to\infty} b_n = +\infty$

$$L = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{\cancel{n}+1-\cancel{n}}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \lim_{n\to\infty} \frac{x_{n+1} \cdot x_n}{x_n - x_{n+1}} = \lim_{n\to\infty} \frac{(x_n - x_n^2)\, x_n}{x_n - x_n + x_n^2} =$$

$$= \lim_{n\to\infty} \frac{x_n^2 - x_n^3}{x_n^2} = \lim_{n\to\infty} \frac{\cancel{x_n^2}(1-x_n)}{\cancel{x_n^2}} = \lim_{n\to\infty} 1 - \underset{\downarrow\atop 0}{x_n} = 1 \Rightarrow$$

$$\overset{St.}{\underset{Cez.}{\Rightarrow}} \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n \cdot x_n = 1$$

$(n \cdot x_n)$ has a finite limit $\Rightarrow (n \cdot x_n)$ convergent

4.3. a) $\sum\limits_{n\geq 1} \left(\frac{-\pi}{4}\right)^n$

$$S_n = \sum_{k=1}^{n} \left(\frac{-\pi}{4}\right)^k = -\frac{\pi}{4} \cdot \frac{(-\pi/4)^n - 1}{-\pi/4 - 1} = -\frac{\pi}{\cancel{4}} \cdot \frac{(-\pi/4)^n - 1}{\frac{-\pi-4}{\cancel{4}}} = \pi \cdot \frac{(-\pi/4)^n - 1}{\pi + 4}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \pi \cdot \frac{\overset{\to 0}{\left(\frac{-\pi}{4}\right)^n} - 1}{\pi + 4} = \pi \cdot \frac{-1}{\pi+4} = \frac{-\pi}{\pi+4} \Rightarrow$$

$$\Rightarrow \sum_{n\geq 1} \left(\frac{-\pi}{4}\right)^n = \frac{-\pi}{\pi+4}$$

b) $\sum\limits_{n\geq 0} \frac{2^{3n}}{5^{n-1}}$

$$S_n = \sum_{k=0}^{n} \frac{2^{3k}}{5^{k-1}} = \sum_{k=0}^{n} \frac{(2^3)^k}{\frac{5^k}{5}} = 5 \sum_{k=0}^{n} \frac{8^k}{5^k} = 5 \sum_{k=0}^{n} \left(\frac{8}{5}\right)^k =$$

$$= \cancel{5} \cdot \frac{8}{\cancel{5}} \cdot \frac{(8/5)^n - 1}{8/5 - 1} = 8 \cdot \frac{(8/5)^n - 1}{3/5} = \frac{40}{3}\left[\left(\frac{8}{5}\right)^n - 1\right]$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{40}{3}\left[\overset{\to +\infty}{\left(\frac{8}{5}\right)^n} - 1\right] = +\infty \Rightarrow \sum_{n\geq 0} \frac{2^{3n}}{5^{n-1}} = +\infty$$

~~$\frac{8}{5} > 1 \Rightarrow \lim \left(\frac{8}{5}\right)^n = +\infty \Rightarrow \lim S_n = +\infty \Rightarrow \sum \frac{2^{3n}}{\ldots} = +\infty$~~

c) $\sum_{n\geq 1} \frac{1}{4n^2-1}$

$$S_n = \sum_{k=1}^{n} \frac{1}{4k^2-1} = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}$$

$$\frac{1}{(2k-1)(2k+1)} = \frac{\overset{2k+1}{A}}{2k-1} + \frac{\overset{2k-1}{B}}{2k+1} = \frac{(2A+2B)k+A-B}{(2k-1)(2k+1)} \Rightarrow \left.\begin{matrix} A+B=0 \\ A-B=1 \end{matrix}\right\} \Rightarrow \begin{matrix} A=\frac{1}{2} \\ B=\frac{-1}{2} \end{matrix}$$

$$\Rightarrow S_n = \sum_{k=1}^{n} \frac{1}{2(2k-1)} - \frac{1}{2(2k+1)} = \frac{1}{2} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{6}} - \cancel{\frac{1}{10}} + \dots + \cancel{\frac{1}{2(2n-1)}} - \frac{1}{2(2n+1)} =$$

$$= \frac{\overset{2n+1}{1}}{2} - \frac{1}{2(2n+1)} = \frac{\cancel{2}n}{\cancel{2}(2n+1)} = \frac{n}{2n+1} \Rightarrow$$

$$\Rightarrow \sum_{n\geq 1} \frac{1}{4n^2-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

d) $\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right)$

$$S_n = \sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n} [\ln(k+1) - \ln k] = \cancel{\ln 2} - \ln 1 + \cancel{\ln 3} - \cancel{\ln 2} +$$

$$+ \dots + \ln(n+1) - \cancel{\ln n} = \ln(n+1)$$

$$\sum_{n\geq 1} \ln(1+1/n) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = +\infty$$

e) $\sum_{n\geq 1} \frac{3n-2}{2^n}$

$$S_n = \sum_{k=1}^{n} \frac{3k-2}{2^k} = \sum_{k=1}^{n} \frac{4k-8-k+6}{2^k} = \sum_{k=1}^{n} \frac{4(k-2)}{2^k} - \frac{k}{2^k} + \frac{6}{2^k} = \underbrace{\sum_{k=1}^{n} \frac{k-2}{2^{k-2}} - \frac{k}{2^k}}_{S_1} + \underbrace{6\sum_{k=1}^{n} \frac{1}{2^k}}_{S_2}$$

$$S_1 = \sum_{k=1}^{n} \frac{k-2}{2^{k-2}} - \frac{k}{2^k} = \frac{-1}{2^{-1}} - \cancel{\frac{1}{2}} + 0 - \cancel{\frac{2}{2^2}} + \cancel{\frac{1}{2}} - \cancel{\frac{3}{2^3}} + \cancel{\frac{2}{2^2}} - \cancel{\frac{4}{2^4}} + \dots + \cancel{\frac{n-3}{2^{n-3}}} - \frac{n-1}{2^{n-1}} +$$

$$+ \cancel{\frac{n-2}{2^{n-2}}} - \frac{n}{2^n} = \frac{-1}{2^{-1}} - \frac{\overset{2}{}n-1}{2^{n-1}} - \frac{n}{2^n} = -2 - \frac{n-2}{2^n}$$

$$S_2 = 6\sum_{k=1}^{n} \frac{1}{2^k} = 6 \cdot \frac{1}{2} \cdot \frac{(1/2)^n - 1}{-1/2} = -6\left[\left(\frac{1}{2}\right)^n - 1\right]$$

$$S_n = -2 - \frac{n-2}{2^n} - \frac{6}{2^n} + 6 = 4 - \frac{n+4}{2^n}$$

$$\lim_{n\to\infty} \frac{n+4}{2^n} = 0 \Rightarrow \lim_{n\to\infty} S_n = 4 \Rightarrow \sum_{n\geq 1} \frac{3n-2}{2^n} = 4$$

(5.1) a) $\sum_{n\geq 1}\left(1-\frac{1}{n}\right)^n$

$x_n=\left(1-\frac{1}{n}\right)^n$; $\lim_{n\to\infty} x_n=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n=\lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{-n}\right]^{\frac{-1}{n}\cdot n}=e^{-1}=\frac{1}{e}\neq 0 \Rightarrow$

$\Rightarrow \sum_{n\geq 1} x_n$ divergent

b) $\sum_{n\geq 1}\sin\frac{1}{n^{5/4}}$ $\qquad x_n=\frac{1}{n^{5/4}}$

$\lim_{n\to\infty} x_n=0 \Rightarrow \sin x_n < x_n \Rightarrow \sum_{n\geq 1}\sin\frac{1}{n^{5/4}} < \sum_{n\geq 1}\frac{1}{n^{5/4}} \Rightarrow$

convergent

$\stackrel{\text{F.C.T.}}{\Rightarrow} \sum_{n\geq 1}\sin\frac{1}{n^{5/4}}$ convergent

c) $\sum_{n\geq 1}\frac{\sqrt{n}}{n\sqrt[3]{n}+2}$

$n\sqrt[3]{n}+2 < n\sqrt{n} \Rightarrow \frac{1}{n\sqrt[3]{n}+2} > \frac{1}{n\sqrt{n}} \Big| \cdot(\sqrt{n})$;

$\frac{\sqrt{n}}{n\sqrt[3]{n}+2} > \frac{1}{n} \Rightarrow \sum_{n\geq 1}\frac{\sqrt{n}}{n\sqrt[3]{n}+2} > \sum_{n\geq 1}\frac{1}{n}$

$\sum_{n\geq 1}\frac{1}{n}$ divergent

$\stackrel{\text{F.C.T}}{\Rightarrow} \sum_{n\geq 1}\frac{\sqrt{n}}{n\sqrt[3]{n}+2}$ divergent

d) $\sum_{n\geq 1}\frac{n!}{3\cdot 5\cdot \ldots \cdot(2n+1)}=\sum_{n\geq 1} x_n$

$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{(n+1)!}{\cancel{3\cdot 5\cdot\ldots\cdot(2n+1)}(2n+3)}\cdot\frac{\cancel{3\cdot 5\cdot\ldots\cdot(2n+1)}}{\cancel{n!}}=\lim_{n\to\infty}\frac{n+1}{2n+3}=$

$=\frac{1}{2}<1 \underset{\text{Ratio Test}}{\Rightarrow} \sum_{n\geq 1} x_n$ convergent

e) $\sum_{n\geq 1}\frac{n^3\cdot 5^n}{2^{3n+1}}=\sum_{n\geq 1} x_n$

$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{(n+1)^3\cdot 5^{n+1}}{\cancel{2^{3n+1}}\,8}\cdot\frac{\cancel{2^{3n+1}}}{n^3\cdot 5^n}=\lim_{n\to\infty}\frac{5\cdot(n+1)^3}{8\cdot n^3}=\frac{5}{8}<1 \underset{\text{Ratio Test}}{\Rightarrow} \sum_{n\geq 1} x_n$ conv.

f) $\sum_{n\geq 1}\frac{2\cdot 5\cdot\ldots\cdot(3n-1)}{3\cdot 6\cdot\ldots\cdot 3n}=\sum_{n\geq 1} x_n$

$x_{n+1}=\frac{2\cdot 5\cdot\ldots\cdot(3n-1)(3n+2)}{3\cdot 6\cdot\ldots\cdot 3n(3n+3)}=x_n\cdot\frac{3n+2}{3n+3}$

(Raabe's Test)

$\lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{3n+3}{3n+2}-\frac{3n+2}{1}\right)=\lim_{n\to\infty}\frac{n}{3n+2}=\frac{1}{3}<1 \Rightarrow \sum_{n\geq 1} x_n$ conv.

(5.2) $(x_n), (y_n)$ sequences of positive numbers, $\sum_{n\geq 1} \frac{x_n}{y_n}$, $\sum_{n\geq 1} y_n$ convergent

is $\sum_{n\geq 1} \sqrt{x_n}$ conv. as well?

$$\left.\begin{array}{l}\sum_{n\geq 1} \frac{x_n}{y_n} \text{ conv.} \xrightarrow{F.C.T} \lim_{n\to\infty} \frac{x_n}{y_n} = 0 \\ \\ \sum_{n\geq 1} y_n \text{ conv.}\end{array}\right\} \xrightarrow{S.C.T.} \sum_{n\geq 1} x_n \text{ convergent}$$

if $\sum_{n\geq 1} x_n$ is convergent, it doesn't necessarily mean that $\sum_{n\geq 1} \sqrt{x_n}$ is also convergent, as it can also be divergent (i.e. for $x_n = \frac{1}{n^2}$, $\sum_{n\geq 1} \frac{1}{n^2}$ is convergent, but $\sum_{n\geq 1} \frac{1}{n}$ is divergent)

(6.1) a) $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n\sqrt{n+1}}$

$$\left.\begin{array}{l} x_n = \frac{(-1)^{n+1}}{n\sqrt{n+1}} ; \lim_{n\to\infty} x_n = 0 \\ |x_n| = \frac{1}{n\sqrt{n+1}} \\ \frac{|x_{n+1}|}{|x_n|} = \frac{n\sqrt{n+1}}{(n+1)\sqrt{n+2}} < 1 \Rightarrow (|x_n|) \text{ decreasing}\end{array}\right\} \Rightarrow \sum_{n\geq 1} x_n \text{ convergent}$$

$$\left.\begin{array}{l} \frac{1}{n\sqrt{n+1}} < \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}} \\ \sum_{n\geq 1} \frac{1}{n^{3/2}} \text{ ~~divergent~~ convergent}\end{array}\right\} \xrightarrow{F.C.T.} \sum_{n\geq 1} \frac{1}{n\sqrt{n+1}} \text{ convergent} \Rightarrow \sum_{n\geq 1} |x_n| \text{ convergent} \Rightarrow$$

$$\Rightarrow \sum_{n\geq 1} x_n \text{ absolutely convergent}$$

b) $\sum_{n\geq 1} \frac{n}{n^2+1} \cdot \cos(n\pi)$ let $x_n = \frac{n}{n^2+1} \cdot \cos(n\pi)$

$$|x_n| = \left|\frac{n \cdot \cos(n\pi)}{n^2+1}\right| = \frac{n}{n^2+1}$$

$$\left.\begin{array}{l} \frac{|x_{n+1}|}{|x_n|} = \frac{n+1}{(n+1)^2+1} \cdot \frac{n^2+1}{n} = \frac{n^3+n^2+n+1}{n^3+2n^2+2n} < 1 \Rightarrow |x_n| \text{ decreasing} \\ \lim_{n\to\infty} \frac{n\cdot\cos(n\pi)}{n^2+1} = 0\end{array}\right\} \Rightarrow \sum_{n\geq 1} x_n \text{ convergent}$$

let $b_n = \frac{1}{n}$, $a_n = \frac{n}{n^2+1}$; $L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

$$\left.\begin{array}{l}\sum_{n\geq 1} b_n \text{ divergent} \\ L = 1\end{array}\right\} \xrightarrow{S.C.T} \sum_{n\geq 1} \frac{n}{n^2+1} \text{ divergent} \Rightarrow \sum_{n\geq 1} |x_n| \text{ divergent} \Rightarrow \sum_{n\geq 1} x_n \text{ semi-convergent}$$

6.2 $f,g:[0,1]\to\mathbb{R}$, f,g continuous s.t. $f(x)=g(x)$, $\forall x\in[0;1]\cap\mathbb{Q}$

f,g continuous on $[0;1]$ $\Rightarrow$ f,g are also continuous in $\alpha\in[0;1]\cap(\mathbb{R}\setminus\mathbb{Q})$ $\Rightarrow$

$\Rightarrow$ $\forall (x_n)\in[0;1]\cap\mathbb{Q}$, $\lim\limits_{n\to\infty} x_n=\alpha$ $\Rightarrow$

$$\Rightarrow \begin{cases} \lim\limits_{n\to\infty} f(x_n)=f(\alpha) \\ \lim\limits_{n\to\infty} g(x_n)=g(\alpha) \end{cases}$$

$f(x_n)=g(x_n)$, $\forall n\in\mathbb{N}$ $\Rightarrow$ $f(\alpha)=g(\alpha)$, $\forall \alpha\in[0;1]\cap(\mathbb{R}\setminus\mathbb{Q})$

$$\left.\begin{array}{l} f(x)=g(x),\ \forall x\in[0;1]\cap\mathbb{Q} \\ f(x)=g(x),\ \forall x\in[0;1]\cap(\mathbb{R}\setminus\mathbb{Q}) \end{array}\right\} \Rightarrow f(x)=g(x),\ \forall x\in[0;1]$$