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1 a) $x_n = \ln(n+1) - \ln n,\ n \in \mathbb{N}$

i) (x_n) monotone if it is either increasing or decreasing

$$x_n = \ln(n+1) - \ln n = \ln\left(\frac{n+1}{n}\right)$$

$$x_{n+1} = \ln\left(\frac{n+2}{n+1}\right)$$

$$x_{n+1} - x_n = \ln\left(\frac{n+2}{n+1}\right) - \ln\left(\frac{n+1}{n}\right) = \ln\left(\frac{n+2}{n+1}\cdot\frac{n}{n+1}\right) = \ln\Big(\underbrace{\frac{n^2+2n}{n^2+2n+1}}_{<1}\Big) \Rightarrow$$

$$\Rightarrow \ln\left(\frac{n^2+2n}{n^2+2n+1}\right) < 0 \Rightarrow x_{n+1} < x_n \Rightarrow (x_n) \text{ decreasing} \Rightarrow (x_n) \text{ monotone}$$

(x_n) bounded if $\exists a \in \mathbb{R}$ s.t. $|x_n| \le a,\ \forall n \in \mathbb{N}$

$\cancel{a_1 = \ln 2}$

$\cancel{a_n = \ln\left(\frac{n+1}{n}\right)} \Rightarrow$

$\cancel{1 < \frac{n+1}{n} \le 2}$

(x_n) decreasing $\Rightarrow x_1 \ge x_n,\ \forall n \in \mathbb{N}$

$x_1 = \ln 2 \Rightarrow x_n \le \ln 2,\ \forall n \in \mathbb{N}$ (1)

$\ln\left(\frac{n+1}{n}\right) > 0 \Rightarrow x_n > 0,\ \forall n \in \mathbb{N}$ (2)

$\overset{(1)}{\underset{(2)}{\Rightarrow}}\ 0 < x_n \le \ln 2 \Rightarrow (x_n)$ bounded

(x_n) convergent $\Rightarrow \lim\limits_{n\to\infty} x_n \in \mathbb{R}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \ln\left(\lim_{n\to\infty}\frac{n+1}{n}\right) = \ln 1 = 0 \in \mathbb{R} \Rightarrow (x_n) \text{ convergent}$$

ii) $$\lim_{n\to\infty}((2n+1)\cdot x_n) = \lim_{n\to\infty}\left[(2n+1)\ln\left(\frac{n+1}{n}\right)\right] = \lim_{n\to\infty}\left[\ln\left(\frac{n+1}{n}\right)^{(2n+1)}\right] =$$

$$= \ln\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n+1}\right] = \ln\left(\lim_{n\to\infty}\Big[\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{e}\Big]^{\frac{2n+1}{n}}\right) = \ln e^{\lim\limits_{n\to\infty}\frac{2n+1}{n}} =$$

$$= \ln e^2 = 2$$

$$\lim_{n\to\infty}\left(\frac{1+\frac{1}{3}+\dots+\frac{1}{2n-1}}{\ln n}\right) \overset{\text{Stolz-Césaro}}{=} \lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{\frac{1}{2n+1}}{\ln\left(\frac{n+1}{n}\right)} =$$

$1+\frac{1}{3}+\dots+\frac{1}{2n-1} = (a_n)$

$\ln n = (b_n)$

(b_n) strictly increasing

$\lim\limits_{n\to\infty} b_n = +\infty$

$$= \lim_{n\to\infty}\frac{1}{\underbrace{(2n+1)\cdot\ln\left(\frac{n+1}{n}\right)}_{2}} = \frac{1}{2}$$

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iii) $\sum\limits_{n\geq 1} x_n$ convergent or divergent

$\sum\limits_{n\geq 1} x_n$ convergent (divergent) $\Rightarrow (S_n)$ convergent (divergent), $S_n = x_1 + x_2 + \dots + x_n, n \in \mathbb{N}$

$$S_n = x_1 + x_2 + \dots + x_n = \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \dots + \ln\left(\frac{n+1}{n}\right) =$$

$$= \ln\left(\frac{\cancel{2}}{1} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \ldots \cdot \frac{n+1}{\cancel{n}}\right) = \ln(n+1)$$

$\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \ln(n+1) = +\infty \Rightarrow (S_n)$ divergent $\Rightarrow \sum\limits_{n\geq 1} x_n$ also divergent

b) (x_n) sequence in $[0;+\infty)$; $\sum\limits_{n\geq 1} \underbrace{\frac{x_n}{1+n^3 x_n}}_{a_n}$ conv. / div. (?)

let $\sum\limits_{n\geq 1} b_n = \sum\limits_{n\geq 1} \frac{x_n}{n^3 x_n} = \sum\limits_{n\geq 1} \frac{1}{n^3}$

$$\frac{x_n}{1+n^3 x_n} \leq \frac{x_n}{n^3 x_n}, \forall n \in [0;+\infty) \Rightarrow a_n \leq b_n \quad (1)$$

$\sum\limits_{n\geq 1} \frac{1}{n^\alpha}$ convergent when $\alpha > 1 \Rightarrow \sum\limits_{n\geq 1} b_n$ convergent (2)

$(1), (2) \overset{F.C.T.}{\Longrightarrow} \sum\limits_{n\geq 1} a_n = \sum\limits_{n\geq 1} \frac{x_n}{1+n^3 x_n}$ is convergent

F.C.T : $\sum\limits_{n\geq 1} b_n$ conv. $\Rightarrow \sum\limits_{n\geq 1} a_n$ conv.

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② $f: \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}, f(x,y) = \frac{\sqrt[3]{3xy+8}-2}{x^2+y^2}$; does f have a lim. at 0_2?

$$\lim_{\substack{x\to 0\\ x<0}} \left(\lim_{y\to 0} \frac{\sqrt[3]{3xy+8}-2}{x^2+y^2} \right) = \lim_{\substack{x\to 0\\ x<0}} \frac{\sqrt[3]{3x+8}-2}{x^2} = \lim_{\substack{x\to 0\\ x<0}} \frac{3x+8-8}{x^2((\sqrt[3]{3x+8})^2+2\sqrt[3]{3x+8}+4)} =$$

$$= \lim_{\substack{x\to 0\\ x<0}} \frac{3}{x((\sqrt[3]{3x+8})^2+2\sqrt[3]{3x+8}+4)} = \frac{3}{0_-} = -\infty \quad (1)$$

$$\lim_{\substack{x\to 0\\ x>0}} \left(\lim_{y\to 0} \frac{\sqrt[3]{3yx+8}-2}{x^2+y^2} \right) = \frac{3}{0^+} = +\infty \quad (2)$$

(1),(2) $\Rightarrow \nexists$ lim at 0_2

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③ $f:\mathbb{R}^2 \to \mathbb{R},\ f(x,y) = x^3 + 3xy^2 + 6xy$

a) $\nabla f(x,y),\ H_f(x,y)$

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$$

$$\left.\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= 3x^2 + 3y^2 + 6y \\ \frac{\partial f}{\partial y}(x,y) &= \cancel{6xy+6x=12x}\ 6xy + 6x = 6x(y+1)\end{aligned}\right\} \Rightarrow \nabla f(x,y) = (3x^2+3y^2+6y,\ 6xy+6x)$$

$$H_f(x,y) = \begin{matrix}\frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y)\end{matrix}$$

$$\left.\begin{aligned}\frac{\partial^2 f}{\partial x^2}(x,y) &= \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial f}{\partial x}(3x^2+3y^2+6y) = 6x \\ \frac{\partial^2 f}{\partial y^2}(x,y) &= \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial f}{\partial y}(6xy+6x) = 6x \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) &= \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial f}{\partial x}(6xy+6x) = 6y+6 \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) &= \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial f}{\partial y}(3x^2+3y^2+6y) = 6y+6\end{aligned}\right\} \Rightarrow H_f(x,y) = \begin{pmatrix}6x & 6y+6 \\ 6y+6 & 6x\end{pmatrix}$$

b) stationary points of f

c - stationary point of $f \Leftrightarrow \nabla f(c) = 0_2$

$$\nabla f(x,y) = (0,0) \Rightarrow \begin{cases}3x^2+3y^2+6y = 0 \\ 6x(y+1) = 0\end{cases} \Rightarrow x = 0\ /\ y = -1$$

$\boxed{\text{I}}$ $x = 0 \Rightarrow 3y^2 + 6y = 0$

$3y(y+2) = 0 \Rightarrow y \in \{0,-2\}$ $\Rightarrow$ for $x=0$ we have $(0,0), (0,-2)$ - stationary points

$\boxed{\text{II}}$ $y = -1 \Rightarrow 3x^2 + 3 - 6 = 0;\ 3x^2 = 3 \Rightarrow x = \pm 1$ $\Rightarrow$ for $y=-1$ we have $(-1,-1), (1,-1)$ - stationary points

I, II $\Rightarrow S = \{(0,0), (0,-2), (1,-1), (-1,-1)\}$ - set of stationary points

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1) $(x,y)=(0,0) \Rightarrow H_f(0,0)=\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$

$\Phi_c=(h_1 \; h_2)\cdot\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = (6h_2+6h_1)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 6h_1h_2+6h_2h_1 = 12h_1h_2$

let $a_1=(1,-1)$, $b_1=(1,1)$

$\left.\begin{array}{l} \Phi_c(a_1)=-12<0 \\ \Phi_c(b_1)=12>0 \end{array}\right\} \Rightarrow \Phi_c(a_1)<0<\Phi_c(b_1) \Rightarrow H_f(0,0)$ indefinite

2. $(x,y)=(0,-2) \Rightarrow H_f(0,-2)=\begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$

$\Phi_c=(h_1 \; h_2)\begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = (-6h_2+-6h_1)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = -12h_1h_2$

let $a_2=(1,-1)$, $b_2=(1,1)$

$\left.\begin{array}{l} \Phi_c(a_2)=12>0 \\ \Phi_c(b_2)=-12<0 \end{array}\right\} \Rightarrow \Phi_c(b_2)<0<\Phi_c(a_2) \Rightarrow H_f(0,-2)$ indefinite

3. $(x,y)=(1,-1) \Rightarrow H_f(+1,-1)=\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

$\Delta_1=6>0$, $\Delta_2=\begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix}=36>0 \Rightarrow H_f$ positive definite $\Rightarrow (1,-1)$ local min. point

4. $(x,y)=(-1,-1) \Rightarrow H_f(-1,-1)=\begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$

$\Delta_1=-6<0$, $\Delta_2=\begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix}=36>0 \Rightarrow H_f$ negative definite $\Rightarrow (-1,-1)$ local max. point

c) $\left.\begin{array}{l} (1,-1) \text{ the only local min. point (unique)} \\ (-1,-1) \text{ the only local max. point (unique)} \end{array}\right\} \Rightarrow (-1,-1),(1,-1)$ are global extremum points

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4) a) $f:[0,2)\to\mathbb{R},\ f(x)=\ln(2-x)$

$f:[a,b)\to\mathbb{R}$ improperly integrable on $[a;b)$ if $\exists \lim\limits_{t\nearrow b}\int_a^t f(x)\,dx\in\mathbb{R}$

$\int_0^t \ln(2-x)\,dx = \cancel{\int_0^t \ln(x-2)\,dx}$

$\left.\begin{array}{l} f(x)=\ln(2-x)\\ g'(x)=1\end{array}\right\} \Rightarrow \int_0^t \ln(2-x)\,dx = \ln|2-x|\cdot x\Big|_0^t - \int_0^t \frac{x}{x-2}\,dx =$

$=(\ln|2-t|\cdot t - \underbrace{\ln 2\cdot 0}_{0}) - \underbrace{\int_0^t \frac{x}{x-2}\,dx}_{J}$

$f'(x)=\frac{-1}{2-x},\ g(x)=x$

$f'(x)=\frac{1}{x-2}$

$J=\int_0^t\frac{x-2+2}{x-2}\,dx=\int_0^t dx+2\int_0^t\frac{1}{x-2}\,dx = x\Big|_0^t+2\ln|x-2|\Big|_0^t = \cancel{t+2\ln|t|}=$

$=t+2\ln|t-2|-2\ln 2$

$\Rightarrow \int_0^t \ln(2-x)\,dx = t\ln|2-t|-t-2\ln|t-2|+2\ln 2$

$L=\lim\limits_{\substack{t\to2\\ t<2}} t\ln|2-t|-t-2\ln|t-2|+2\ln2 = \lim\limits_{\substack{t\to2\\ t-2<0}} t\ln|2-t|-t-2\ln|t-2|+2\ln2=$

$=\lim\limits_{\substack{t\to2\\ t-2<0}} t\underbrace{\ln 0^+}_{\to-\infty}-t\underbrace{-2\ln 0^+}_{\to-\infty}+2\ln2=-\infty$

$L=\lim\limits_{\substack{t\to2\\ t<2}}\ln|t-2|(t-2)+2\ln2-\cancel{t}=\lim\limits_{\substack{t\to2\\ t<2}}(t-2)\ln|t-2|=\lim\limits_{\substack{t\to2\\ t<2}}\frac{\ln(2-t)}{\frac{1}{t-2}} \overset{\frac{\infty}{\infty}}{=}$ (l'Hopital)

$=\lim\limits_{\substack{t\to2\\ t<2}} \frac{\frac{-1}{2-t}}{\frac{-1}{(t-2)^2}} = \lim\limits_{\substack{t\to2\\ t<2}} \frac{1}{2-t}\cdot(t-2)^2=\lim\limits_{\substack{t\to2\\ t<2}}(2-t)=0 \Rightarrow$ f is improperly integrable on $[0;2)$

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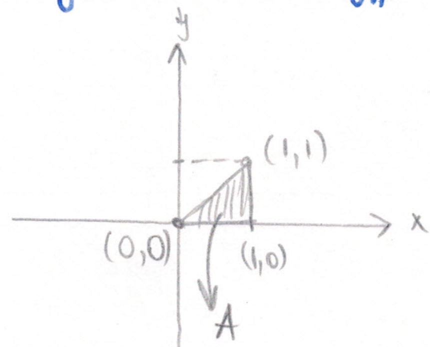
b) $M \subseteq \mathbb{R}^2$ w/ vertices $(0,0), (1,0), (1,1)$; $J=\iint_M \cos\frac{\pi x^2}{2}\,dxdy$

$A=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ 0\le y\le x\}$



$$J=\underbrace{\int_0^x\int_0^1 \cos\frac{\pi x^2}{2}\,dx\,dy}_{J_1} = \int_0^1\int_0^x \cos\frac{\pi x^2}{2}\,dy\,dx =$$

$$\cancel{J_1=\int_0^1 \cos\frac{\pi x^2}{2}\,dx}$$

$$= \int_0^1 \left(\cos\frac{\pi x^2}{2}\cdot y\Big|_0^x\right)dx = \int_0^1 \left(x\cos\left(\frac{x^2\pi}{2}\right)\right)dx = \int_0^{\pi/2} \cos t\,dt = \sin t\Big|_0^{\pi/2} =$$

$\frac{x^2\pi}{2}=t$ $\quad x=0 \Rightarrow t=0$

$\quad\quad\quad\quad x=1 \to t=\pi/2$

$x\,dx=dt$

$$= \sin \pi/2 - \sin 0 = 1$$