Exercise 11.1 - Study the continuity and the partial differentiability of f at 0_2

$$f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq 0_2\\ 0, & (x,y)=0_2\end{cases}$$

f continuous at $0_2 \Leftrightarrow \lim\limits_{(x,y)\to 0_2} f(x,y)=f(0,0)=0$

$$\lim_{(x,y)\to 0_2} f(x,y)=\lim_{(x,y)\to 0_2}\frac{xy}{\sqrt{x^2+y^2}}=\lim_{r\to 0}\frac{r^2\sin\alpha\cos\alpha}{\sqrt{r^2\sin^2\alpha+r^2\cos\alpha}}=\lim_{r\to 0}\frac{r^2\sin\alpha\cos\alpha}{\sqrt{r^2}}=$$

$x=r\cos\alpha$, $y=r\sin\alpha$, $r>0$

$$=\lim_{r\to 0}\frac{r^{\cancel{2}}\sin\alpha\cos\alpha}{\cancel{r}}=\lim_{r\to 0} r\sin\alpha\cos\alpha=0 \Rightarrow$$

$(r>0\Rightarrow\sqrt{r^2}=r)$

$\Rightarrow f$ continuous at 0_2

$$\left.\begin{aligned}\frac{\partial f}{\partial x}(0,0)&=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{\overset{0}{\frac{\cdot}{\sqrt{x^2}}}-0}{x-0}=\lim_{x\to 0}\frac{0-0}{x}=0\\ \frac{\partial f}{\partial y}(0,0)&=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{\overset{0}{\frac{\cdot}{\sqrt{y^2}}}-0}{y-0}=\lim_{y\to 0}\frac{0-0}{y}=0\end{aligned}\right\}\Rightarrow$$

$\Rightarrow f$ is partially differentiable w.r.t x and y at 0_2

Exercise 11.2 - The gradient and the Hessian matrix of f at $c=(0,0,\pi/6)$

$f:\mathbb{R}^3\to\mathbb{R},\ f(x,y,z)=e^{2x+y}\cdot\cos(3z)$

$\frac{\partial f}{\partial x}(x,y,z)=2e^{2x+y}\cos(3z)$ $\quad\frac{\partial f}{\partial y}(x,y,z)=e^{2x+y}\cdot\cos(3z)$ $\quad\frac{\partial f}{\partial z}(x,y,z)=-3e^{2x+y}\sin(3z)$

$$\left.\begin{aligned}\frac{\partial f}{\partial x}(0,0,\pi/6)&=2e^0\cos\pi/2=2\cos\pi/2=0\\ \frac{\partial f}{\partial y}(0,0,\pi/6)&=e^0\cos\pi/2=\cos\pi/2=0\\ \frac{\partial f}{\partial z}(0,0,\pi/6)&=-3e^0\sin\pi/2=-3\sin\pi/2=-3\end{aligned}\right\}\Rightarrow \nabla f(0,0,\pi/6)=(0,0,-3)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 4e^{2x+y}\cdot\cos(3z) \qquad \frac{\partial^2 f}{\partial y^2}(x,y,z) = e^{2x+y}\cdot\cos(3z) \qquad \frac{\partial^2 f}{\partial z^2}(x,y,z) = -9e^{2x+y}\cdot\cos(3z)$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y,z) = 2e^{2x+y}\cdot\cos(3z) \qquad \frac{\partial^2 f}{\partial z\partial x}(x,y,z) = -6e^{2x+y}\cdot\sin(3z)$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y,z) = 2e^{2x+y}\cdot\cos(3z) \qquad \frac{\partial^2 f}{\partial z\partial y}(x,y,z) = -3e^{2x+y}\cdot\sin(3z)$$

$$\frac{\partial^2 f}{\partial x\partial z}(x,y,z) = -6e^{2x+y}\sin(3z) \qquad \frac{\partial^2 f}{\partial y\partial z}(x,y,z) = -3e^{2x+y}\sin(3z)$$

$$\frac{\partial^2 f}{\partial x^2}(0,0,\pi/6) = 4e^0\cdot\cos\pi/2 = 0$$

$$\frac{\partial^2 f}{\partial y^2}(0,0,\pi/6) = e^0\cos\pi/2 = 0$$

$$\frac{\partial^2 f}{\partial z^2}(0,0,\pi/6) = -9e^0\cos\pi/2 = 0$$

$$\frac{\partial^2 f}{\partial x\partial y}(0,0,\pi/6) = \frac{\partial^2 f}{\partial y\partial x}(0,0,\pi/6) = 2e^0\cos\pi/2 = 0$$

$$\frac{\partial^2 f}{\partial x\partial z}(0,0,\pi/6) = \frac{\partial^2 f}{\partial z\partial x}(0,0,\pi/6) = -6e^0\sin\pi/2 = -6$$

$$\frac{\partial^2 f}{\partial z\partial y}(0,0,\pi/6) = \frac{\partial^2 f}{\partial y\partial z}(0,0,\pi/6) = -3e^0\sin\pi/2 = -3$$

$$H_f\underbrace{(0,0,\pi/6)}_{c} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(c) & \frac{\partial^2 f}{\partial x\partial y}(c) & \frac{\partial^2 f}{\partial x\partial z}(c) \\ \frac{\partial^2 f}{\partial y\partial x}(c) & \frac{\partial^2 f}{\partial y^2}(c) & \frac{\partial^2 f}{\partial y\partial z}(c) \\ \frac{\partial^2 f}{\partial z\partial x}(c) & \frac{\partial^2 f}{\partial z\partial y}(c) & \frac{\partial^2 f}{\partial z^2}(c) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -6 \\ 0 & 0 & -3 \\ -6 & -3 & 0 \end{pmatrix}$$

(12.1) $f:\mathbb{R}^2 \to \mathbb{R}, f(x,y) = x^3+y^3-3xy$. find the local extremum points and specify their type

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) = (3x^2-3y, 3y^2-3x)$$

$$\nabla f(x,y) = (0,0) \Leftrightarrow (3x^2-3y, 3y^2-3x) = (0,0) \Rightarrow \begin{cases} 3x^2-3y=0 & |:3 \\ 3y^2-3x=0 & |:3 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x^2-y=0 \\ y^2-x=0 \end{cases} \Rightarrow \begin{cases} x^2=y \\ y^2-x=0 \end{cases} \Rightarrow x^4-x=0$$

$$x^4-x=0 \Rightarrow x(x^3-1)=0 \Rightarrow x(x-1)(x^2+x+1)=0 \Rightarrow x\in\{0,1\}$$

$$\left.\begin{array}{l} x=0 \Rightarrow y=0 \\ x=1 \Rightarrow y=1 \end{array}\right\} \Rightarrow S=\{(0,0),(1,1)\}$$ - the set of stationary points of f

$$\nabla f(x,y) = (3x^2-3y, 3y^2-3x)$$

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial y\partial x}(x,y) \\ \frac{\partial^2 f}{\partial x\partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

$$H_f(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} \quad \Delta_1 = 0,\ \Delta_2 = -9 < 0 \Rightarrow \text{inconclusive}$$

$$H_f(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \quad \Delta_1 = 6>0,\ \Delta_2 = 36-9=27>0 \Rightarrow H_f(1,1) \text{ positive definite} \Rightarrow$$

$\Rightarrow (1,1)$ local minimum point

$H_f(0,0)$ - inconclusive (using Sylvester's method)

$$\phi(h) = (h_1\ h_2)\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = (-3h_2\ \ -3h_1)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = -3h_1h_2-3h_1h_2 = -6h_1h_2$$

let $v_1=(-1,1)$ and $v_2=(1,1)$

$$\left.\begin{array}{l} \phi(v_1) = \phi(-1,1) = -6\cdot(-1)\cdot 1 = 6>0 \\ \phi(v_2) = \phi(1,1) = -6\cdot 1\cdot 1 = -6<0 \end{array}\right\} \Rightarrow H_f(0,0) \text{ is indefinite} \Rightarrow$$

$\Rightarrow (0,0)$ is not a local extremum point

Exercise 13.1 $f:[1;+\infty)\to\mathbb{R}$, $f(x)=\dfrac{x^{\alpha}\arctan x}{1+x^{\beta}}$, $\alpha,\beta\in\mathbb{R}$

- study the improper integrability

Using Theorem 5, Lecture 11: $\exists L=\lim\limits_{x\to\infty} x^{p}\cdot f(x)\in[0;+\infty)\cup\{+\infty\}$

$$L=\lim_{x\to\infty} x^{p}\cdot\frac{x^{\alpha}\cdot\arctan x}{1+x^{\beta}}=\lim_{x\to\infty}\frac{x^{p+\alpha}\cdot\overset{\nearrow \pi/2}{\arctan x}}{1+x^{\beta}}=\lim_{x\to\infty}\frac{\pi}{2}\cdot\frac{x^{p+\alpha}}{1+x^{\beta}}$$

$\boxed{I}$ $\beta>0\Rightarrow\lim\limits_{x\to\infty}x^{\beta}=+\infty$,

we choose $p=\beta-\alpha$

$$L=\lim_{x\to\infty}\frac{\pi}{2}\cdot\frac{x^{\beta-\alpha+\alpha}}{1+x^{\beta}}=\lim_{x\to\infty}\frac{\pi}{2}\cdot\frac{x^{\beta}}{1+x^{\beta}}=\frac{\pi}{2}$$

- $p=\beta-\alpha>1$, $L=\frac{\pi}{2}<+\infty\Rightarrow f$ is improperly integrable on $[1;+\infty)$; $\int_1^{+\infty}f(x)\,dx=\frac{\pi}{2}$
- $p=\beta-\alpha\le 1$, $L=\frac{\pi}{2}>0\Rightarrow f$ is not improperly integrable on $[1;+\infty)$

$\boxed{II}$ $\beta=0\Rightarrow\lim\limits_{x\to\infty}x^{\beta}=1$

we choose $p=-\alpha$

$$L=\lim_{x\to\infty}\frac{\pi}{2}\cdot\frac{x^{-\alpha+\alpha}}{1+1}=\lim_{x\to\infty}\frac{\pi}{2}\cdot\frac{\overset{\nearrow 1}{x^{0}}}{2}=\frac{\pi}{4}$$

- $p=-\alpha>1$, $L=\frac{\pi}{4}<+\infty\Rightarrow f$ is improperly integrable on $[1;+\infty)$; $\int_1^{+\infty}f(x)=\frac{\pi}{4}$
- $p=-\alpha\le 1$, $L=\frac{\pi}{4}>0\Rightarrow f$ is not improperly integrable on $[1;+\infty)$

$\boxed{III}$ $\beta<0\Rightarrow\lim\limits_{x\to\infty}x^{\beta}=0$

we choose $p=-\alpha$

$$L=\lim_{x\to\infty}\frac{\pi}{2}\cdot\frac{x^{-\alpha+\alpha}}{1+\underbrace{x^{\beta}}_{\to 0}}=\lim_{x\to\infty}\frac{\pi}{2}\cdot\overset{\nearrow 1}{x^{0}}=\frac{\pi}{2}$$

- $p=-\alpha>1$, $L=\frac{\pi}{2}<+\infty\Rightarrow f$ is improper integrable on $[1;+\infty)$; $\int_1^{+\infty}f(x)=\frac{\pi}{2}$
- $p=-\alpha\le 1$, $L=\frac{\pi}{2}>0\Rightarrow f$ is not improper integrable on $[1;+\infty)$