(5.1) Using the semantic tableaux method, prove '$\exists$' (the existential quantifier) is semi-distributive over '$\wedge$' (conjunction):

$$\begin{cases} \models (\exists x)(A(x) \wedge B(x)) \rightarrow (\exists x)A(x) \wedge (\exists x)B(x) \text{ and} \\ \not\models (\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\exists x)(A(x) \wedge B(x)) \end{cases}$$

We have to prove that $\boxed{U_1 \text{ is a tautology}}$ and $\boxed{U_2 \text{ is not valid}}$, where:

- $U_1 = (\exists x)(A(x) \wedge B(x)) \rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$
- $U_2 = (\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\exists x)(A(x) \wedge B(x))$

$\models U \Leftrightarrow \neg U$ has a closed semantic tableau, so we build the semantic tableaux of $\neg U_1$ and $\neg U_2$

$\neg U_1 = \neg((\exists x)(A(x) \wedge B(x)) \rightarrow (\exists x)A(x) \wedge (\exists x)B(x))$ (1)

| α-rule for (1)

(2) $(\exists x)(A(x) \wedge B(x))$

|

(3) $\neg((\exists x)A(x) \wedge (\exists x)B(x))$

| δ-rule for (2) - c a new ~~variable~~ constant

(4) $A(c) \wedge B(c)$

| α-rule for (4)

$A(c)$

|

$B(c)$

| β-rule for (3)

(5) $\neg(\exists x)A(x)$ $\neg(\exists x)B(x)$ (6)

γ-rule for (5) | | γ-rule for (6)

$\neg A(c)$ $\neg B(c)$

⊗ closed branch   ⊗ closed branch

- All the branches are closed, so $\neg U_1$ has a closed semantic tableau $\Rightarrow$
  $\Rightarrow U_1$ is a tautology

$\neg U_2 = \neg((\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\exists x)(A(x) \wedge B(x)))$ (1)

| α-rule for (1)

(2) $(\exists x)A(x) \wedge (\exists x)B(x)$

|

(3) $\neg((\exists x)(A(x) \wedge B(x)))$

| α-rule for (2)

(4) $(\exists x)A(x)$

|

(5) $(\exists x)B(x)$

| δ-rule for (4), c_1 new constant

$A(c_1)$

| δ-rule for (5), c_2 new constant

$B(c_2)$

| γ-rule for (3)

(6) $\neg(A(c_1) \wedge B(c_1))$

|

(7) $\neg(A(c_2) \wedge B(c_2))$

|

(8) $\neg((\exists x)(A(x) \wedge B(x)))$ - copy of (3)

| β-rule for (6)

$\neg A(c_1)$ ⊗ closed branch

$\neg B(c_1)$

| β-rule for (7)

$\neg A(c_2)$ ⊗ ~~closed~~ open branch

$\neg B(c_2)$ ⊗ closed branch

- There exists an open branch, so $\neg U_2$ doesn't have a closed semantic tableau and it is not a tautology ($\nvDash U_2$)
- Conclusion: we proved $\vDash U_1$ and $\nvDash U_2$, so the existential quantifier is only semi-distributive over conjunction