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① $\begin{cases} \dot{x} = 2y \\ \dot{y} = -5x \end{cases}$ - find a first integral

$x' = \frac{dx}{dt}, y' = \frac{dy}{dt} \Rightarrow \cancel{\frac{x'}{y'}} = \quad \frac{y'}{x'} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-5x}{2y} \Leftrightarrow \frac{dy}{dx} = \frac{-5x}{2y} \Leftrightarrow$

$\Leftrightarrow 2y\,dy = -5x\,dx \,\Big|\int \Leftrightarrow 2\int y\,dy = -5\int x\,dx \Leftrightarrow \frac{2y^2}{2} = \frac{-5x^2}{2} + c,\ c \in \mathbb{R} \Rightarrow$

$\Rightarrow \frac{2y^2+5x^2}{2} = c \Rightarrow 2y^2+5x^2 = c_1,\ c_1 \in \mathbb{R}$

We define $H(x,y) = 2y^2+5x^2,\ \forall (x,y) \in \mathbb{R}^2$

$H: \mathbb{R}^2 \to \mathbb{R},\ H \in C^1(\mathbb{R})$ is not locally constant

We check if H is a global first integral

$\frac{\partial H}{\partial x}\cdot f_1 + \frac{\partial H}{\partial y}\cdot f_2 = 0$, where $\begin{cases} f_1 = 2y \\ f_2 = -5x \end{cases}$

$10x\cdot 2y + 4y\cdot(-5x) = 0 \Leftrightarrow 20xy - 20xy = 0 \Leftrightarrow 0 = 0$, true $\Rightarrow H(x,y) = 2y^2+5x^2$ is a global first integral

② phase portrait of the scalar dynamical system

$\dot{x} = \frac{-1}{5}x + x^2 - x^3$; find $\varphi(t,0)$ and det. the propr. of $\varphi(t,1/4), \varphi(t,1/2), \varphi(t,1)$

$\dot{x} = \underbrace{\frac{-1}{5}x + x^2 - x^3}_{f(x)}$ $\qquad f(\eta^*) = 0 \Leftrightarrow \cancel{\frac{-1}{5}x + x^2 - x^3 = 0}$

$\cancel{x\left(-x^2+x-\frac{1}{5}\right) = 0} \to x-$

$f(\eta^*) = 0 \Leftrightarrow \cancel{-x^3} + \eta^{*3} + \eta^{*2} - \frac{1}{5}\eta^* = 0 \Leftrightarrow \eta^*\left(-\eta^{*2} + \eta^* - \frac{1}{5}\right) = 0 \quad \begin{cases} \eta_1^* = 0 \\ -\eta^{*2} + \eta^* - \frac{1}{5} = 0 \end{cases}$

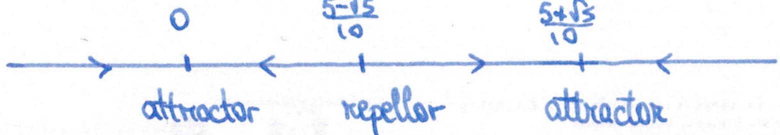
$-\eta^{*2} + \eta^* - \frac{1}{5} = 0 \Leftrightarrow \eta^{*2} - \eta^* + \frac{1}{5} = 0 \Rightarrow \eta^*_{2,3} = \frac{5 \pm \sqrt{5}}{10}$

$\cancel{\Delta = \frac{-1}{5}}$

$\Rightarrow \eta_1^* = 0,\ \eta_2^* = \frac{5+\sqrt{5}}{10},\ \eta_3^* = \frac{5-\sqrt{5}}{10}$ - equilibrium points

x	$-\infty$		0		$\frac{5-\sqrt{5}}{10}$		$\frac{5+\sqrt{5}}{10}$		$+\infty$
$f(x)$	+ + + + +		0	− −	0	+ + + +	0	− − − −	

③ the phase portrait :



the orbits : $(-\infty;0)$; $\{0\}$; $\left(0;\frac{5-\sqrt{5}}{10}\right)$; $\left\{\frac{5-\sqrt{5}}{10}\right\}$; $\left(\frac{5-\sqrt{3}}{10};\frac{5+\sqrt{5}}{10}\right)$;

$\left\{\frac{5+\sqrt{3}}{10}\right\}$; $\left(\frac{5+\sqrt{5}}{10};+\infty\right)$

• find $\varphi(t,0)$

$\eta\in(-\infty;0)\Rightarrow f(\eta)>0$ $\qquad$ $\eta\in\left(\frac{5-\sqrt{3}}{10};\frac{5+\sqrt{3}}{10}\right)\Rightarrow f(\eta)>0$

$\eta\in\left(0;\frac{5-\sqrt{3}}{10}\right)\Rightarrow f(\eta)<0$ $\qquad$ $\eta\in\left(\frac{5+\sqrt{3}}{10};+\infty\right)\Rightarrow f(\eta)<0$

$\eta_1^*=0$ equilibrium point $\xrightarrow[\text{theory}]{}$ $\varphi(t,0)=0$

$\varphi(t;1/4)$; $\frac{1}{4}\in\left(0;\frac{5-\sqrt{3}}{10}\right)\Rightarrow$ the orbit of $\varphi(t;\frac{1}{4})$ is $\left(0;\frac{5-\sqrt{3}}{10}\right)$

$$\left.\begin{array}{l} f(x)<0 \text{ on } 0;\left(\frac{5-\sqrt{3}}{10}\right) \\ \varphi(t;\frac{1}{4}) \text{ is a sol. of IVP} \begin{cases} \dot{x}=f(x),<0 \\ x(0)=\frac{1}{4} \end{cases} \end{array}\right\} \Rightarrow \varphi(t;1/4) \text{ is strictly decreasing}$$

$\lim\limits_{t\to\infty}\varphi(t;1/4)=0$, $\lim\limits_{t\to-\infty}\varphi(t;1/4)=\frac{5-\sqrt{3}}{10}$

$\varphi(t;1/2)$; $\frac{1}{2}\in\left(\frac{5-\sqrt{3}}{10};\frac{5+\sqrt{3}}{10}\right)\Rightarrow$ the orbit of $\varphi(t;\frac{1}{2})$ is $\left(\frac{5-\sqrt{3}}{10};\frac{5+\sqrt{3}}{10}\right)$

$$\left.\begin{array}{l} f(x)>0 \text{ on } \left(\frac{5-\sqrt{3}}{10};\frac{5+\sqrt{3}}{10}\right) \\ \varphi(t;1/2) \text{ is a sol. of IVP} \begin{cases} \dot{x}=f(x)>0 \\ x(0)=\frac{1}{2} \end{cases} \end{array}\right\} \Rightarrow \varphi(t;1/2) \text{ strictly increasing}$$

$\lim\limits_{t\to\infty}\varphi(t;\frac{1}{2})=\frac{5-\sqrt{3}}{10}$, $\lim\limits_{t\to-\infty}\varphi(t;1/2)=\frac{5+\sqrt{5}}{10}$

$\varphi(t;1)$; $1\in\left(\frac{5+\sqrt{5}}{10};+\infty\right)\Rightarrow$ the orbit of $\varphi(t;1)$ is $\left(\frac{5+\sqrt{5}}{10};+\infty\right)$

$$\left.\begin{array}{l} \text{on } \left(\frac{5+\sqrt{5}}{10};+\infty\right), f(x)<0 \\ \varphi(t,1) \text{ is a sol. of IVP} \begin{cases} \dot{x}=f(x),<0 \\ x(0)=1 \end{cases} \end{array}\right\} \Rightarrow \varphi(t,1) \text{ is strictly decreasing}$$

$\lim\limits_{t\to\infty}\varphi(t,1)=\frac{5+\sqrt{5}}{10}$, $\lim\limits_{t\to-\infty}\varphi(t,1)=+\infty$

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(3) planar system $\begin{cases} \dot{x} = x - 2xy \\ \dot{y} = x - y \end{cases}$

a) eq. points

$\dot{x} = f(x,y)$, $x = \binom{x}{y} \in \mathbb{R}^2$, $f(x,y) = \begin{pmatrix} x-2xy \\ x-y \end{pmatrix}$

η^* is an eq. $\Leftrightarrow f(\eta^*) = 0$

$Jf(x,y) = \begin{pmatrix} 1-2y & -2x \\ 1 & -1 \end{pmatrix}$ - the Jacobian matrix of f

$f(x,y) = 0 \Rightarrow \begin{cases} x-2xy=0 \\ x-y=0 \end{cases} \Leftrightarrow \begin{cases} x-2xy=0 \\ x=y \end{cases} \Leftrightarrow \begin{cases} y-2y^2=0 \\ x=y \end{cases} \Leftrightarrow \begin{cases} y(1-2y)=0 \\ x=y \end{cases} \Rightarrow$

$\Rightarrow \eta_1^* = (0,0)$ and $\eta_2^* = (\frac{1}{2}, \frac{1}{2})$ the equilibrium points of the system

b) $\eta_1^* = (0,0) \Rightarrow J(0;0) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

eigenvalues : $\lambda_{1,2} = \pm 1$

$\lambda_1, \lambda_2 \in \mathbb{R}$, $Re(\lambda_1) \neq 0$, $Re(\lambda_2) \neq 0 \Rightarrow \eta_1^*$ is hyperbolic

$\lambda_1 = -1 < 0 < \lambda_2 = 1 \Rightarrow \eta_1^*$ is a saddle (unstable)

$\eta_2^* = (\frac{1}{2}, \frac{1}{2}) \Rightarrow J(\frac{1}{2}; \frac{1}{2}) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

eigenvalues: $\lambda_1, \lambda_2 = \frac{-1 \pm \sqrt{3}i}{2}$

$Re(\lambda_1) \neq 0$, $Re(\lambda_2) \neq 0 \Rightarrow \eta_2^*$ is hyperbolic

η_2^* is a focus attractor ($Re(\lambda_1) < 0$, $Re(\lambda_2) < 0$)