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① $\begin{cases} x' = 5x - 7y \\ y' = 7x + 5y \end{cases}$ - fundam. matrix sol.

$\dot{X} = AX$, $\dot{X} = \begin{pmatrix} x' \\ y' \end{pmatrix}$, $A = \begin{pmatrix} 5 & -7 \\ 7 & 5 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$

from $x' = 5x - 7y$, we get that $y = \frac{x' - 5x}{-7} = \frac{-x' + 5x}{7}$

$y' = 7x + 5y \Leftrightarrow \left(\frac{-x'+5x}{7}\right)' = 7x + 5\left(\frac{-x'+5x}{7}\right)$

$\frac{-x'' + 5x'}{7} = \frac{49x - 5x' + 25x}{7} \Leftrightarrow x'' - 10x' + 74x = 0$

char. eq: $r^2 - 10r + 74 = 0 \Rightarrow r_{1,2} = 5 \pm 7i \longmapsto e^{5t}\cos 7t$, $\cancel{e^{5t}\cos}$ $e^{5t}\sin 7t \Rightarrow$

$\Rightarrow x = c_1 \cdot e^{5t}\cos 7t + c_2 e^{5t}\sin 7t$, $c_1, c_2 \in \mathbb{R}$

$x' = (c_1 e^{5t}\cos 7t + c_2 e^{5t}\sin 7t)' = (c_1 e^{5t}\cos 7t)' + (c_2 e^{5t}\sin 7t)' =$

$= c_1(e^{5t}\cos 7t)' + c_2(e^{5t}\sin 7t)' =$

$= -7c_1 e^{5t}\sin 7t + 5c_1 e^{5t}\cos 7t + 7c_2 e^{5t}\cos 7t + 5c_2 e^{5t}\sin 7t$

$\Rightarrow y = \Big(-7c_1 e^{5t}\sin 7t + \cancel{5c_1 e^{5t}\cos 7t} + 7c_2 e^{5t}\cos 7t + \cancel{5c_2 e^{5t}\sin 7t} -$

$- \cancel{5(c_1 e^{5t}\cos 7t + c_2 e^{5t}\sin 7t)}\Big) \cdot \frac{1}{7} =$

$= \frac{-7c_1 e^{5t}\sin 7t + 7c_2 e^{5t}\cos 7t}{7} = -c_1 e^{5t}\sin 7t + c_2 e^{5t}\cos 7t$

$\Rightarrow$ the gen. sol. of the system is:

$\begin{cases} x = c_1 e^{5t}\cos 7t + c_2 e^{5t}\sin 7t \\ y = -c_1 e^{5t}\sin 7t + c_2 e^{5t}\cos 7t \end{cases}$, $c_1, c_2 \in \mathbb{R}$

Let $U = \begin{pmatrix} e^{5t}\cos 7t & e^{5t}\sin 7t \\ -e^{5t}\sin 7t & e^{5t}\cos 7t \end{pmatrix}$; $\det(U) = e^{10t}\cos^2 7t + e^{10t}\sin^2 7t =$

$= e^{10t}(\cos^2 7t + \sin^2 7t) =$

$= e^{10t} \neq 0 \Rightarrow U$ is a fundam. matrix solution

(2.) $x' - 3\lambda x = e^{3t} \Rightarrow a(t) = -3\lambda,\ f(t) = e^{3t}$

$A(t) = -\int_{t_0}^{t} a(s)\,ds = -\int_0^t (-3\lambda)\,ds = 3\lambda t \Rightarrow \mu(t) = e^{-3\lambda t}$ is the integrating factor

$x' - 3\lambda x = e^{3t} \,\Big|\cdot e^{-3\lambda t} \Rightarrow x' \cdot e^{-3\lambda t} - 3\lambda x \cdot e^{-3\lambda t} = e^{3t} \cdot e^{-3\lambda t} \Rightarrow$

$\Rightarrow (x \cdot e^{-3\lambda t})' = e^{3t} \cdot e^{-3\lambda t} \underset{\text{we integrate}}{\Rightarrow} x e^{-3\lambda t} = \int_0^t e^{3s(1-\lambda)}\,ds + C \Rightarrow$

$\Rightarrow x e^{-3\lambda t} = \dfrac{e^{3s(1-\lambda)}}{3-3\lambda}\Big|_0^t + c,\ c \in \mathbb{R}$; we multiply with $e^{3\lambda t} \Rightarrow$

$\Rightarrow x = \dfrac{e^{3t(1-\lambda)} \cdot e^{3\lambda t}}{3-3\lambda} + c \cdot e^{3\lambda t} \Rightarrow x = \dfrac{e^{3t}}{3-3\lambda} + c \cdot e^{3\lambda t},\ c \in \mathbb{R}$

case 1: $\lambda \neq 1 \Rightarrow x = \dfrac{e^{3t}}{3-3\lambda} + c e^{3\lambda t},\ c \in \mathbb{R}$

case 2: $\lambda = 1 \Rightarrow x' - 3x = e^{3t}$

$x' - 3x = e^{3t} \,\Big|\cdot e^{-3t} \Leftrightarrow x' \cdot e^{-3t} - 3e^{-3t} x = 1 \Rightarrow (x \cdot e^{-3t})' = 1 \underset{\text{we integrate}}{\Rightarrow}$

$\Rightarrow x e^{-3t} = \int_0^t 1\,ds + c \Rightarrow x e^{-3t} = t + c \,\Big|\cdot e^{3t} \Rightarrow$

$\Rightarrow x = t e^{3t} + c e^{3t},\ c \in \mathbb{R}$

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(3.) $f:\mathbb{R}\to\mathbb{R}$, C^1, injective, $f(2)=-1$, $f(5)=2$

a) $\dot{x}=f(x)$ - global repeller eq. point

f injective $\Rightarrow$ f monotone

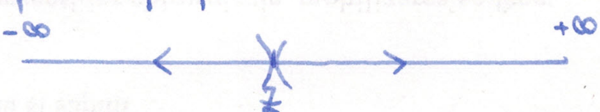
$\left.\begin{array}{l} f(2)<f(5) \\ (2<5) \end{array}\right\} \Rightarrow f$ is strictly increasing

$\left.\begin{array}{l} f(2)=-1<0 \\ f(5)=2>0 \\ f \text{ increasing} \end{array}\right\} \Rightarrow \exists$ a point 'z' between 2 and 5 such ~~tat~~ that $f(z)=0$; f being strictly increasing, means that this point is unique

x	$-\infty$	2	z	5	$+\infty$
$f(x)$	$- - -$	-1	$- - 0$	$+ \; 2$	$+ + +$

$f(z)=0 \Rightarrow z$ is an equilibrium point of f

we draw the phase portrait :



$\left.\begin{array}{l} \text{on the interval } (-\infty; z),\ f(x)<0 \\ \text{on the interval } (z;+\infty),\ f(x)>0 \end{array}\right\} \Rightarrow z$ is a repeller

because z is the only eq. point of the function, z is a global repeller

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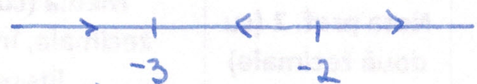
(4) we find the equilibrium point(s) of $\dot{x} = x^2+5x+6$

$$x^2+5x+6=0$$

$$\Delta = 25-24 = 1 \Rightarrow x_{1,2} = \frac{-5\pm 1}{2} \Rightarrow x_1=-2, \quad x_2=-3$$

x	$-\infty$		-3		-2		$+\infty$
$\dot{x}$		$+ + + +$	0	$- - -$	0	$+ + + +$	

phase portrait: 

from the phase portrait we have that $x=-3$ is the attractor eq. point

Euler's numerical formula:

$$x_{n+1} = x_n + h(x_n^2+5x_n+6) = hx_n^2 + x_n(1+5h) + 6h$$

$$hx_m^2 + x_m(1+5h) + 6h = 0$$

$$\Delta = h^2+10h+1 \Rightarrow x_{m_{1,2}} = \frac{-(1+5h)\pm\sqrt{h^2+10h+1}}{2h}$$

$$\cancel{x_m=-3 \Rightarrow \frac{-(1+5h)+\sqrt{h^2+10h+1}}{2h}=-3 \Rightarrow -(1+5h)+\sqrt{h^2+10h+1}=-6h}$$

$$\cancel{\sqrt{h^2+10h+1} = -h+1 = 1-h \;\uparrow^2 \Rightarrow h^2+10h+1 = 1-2h+h^2}$$

$$\cancel{12h=0 \Rightarrow h=0}$$

$$x_n=-3 \Rightarrow \frac{-(1+5h)\pm\sqrt{h^2+10h+1}}{2h} = 3$$

$$\sqrt{h^2+10h+1} = \underbrace{-6h+5h+1}_{=1-h} \;\uparrow^2 \Rightarrow h^2+10h+1 = 1-2h+h^2$$

$$12h=0 \Rightarrow h=0 \Rightarrow$$

$\Rightarrow \nexists\, h$ that satisfies the condition