Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză, Curs: Dynamical Systems, An: Primăvara 2021

Laboratory 4.

1. Introduce the matrix $A = [[0, -2, 0], [1, -2, 0], [0, 0, -2]]$.

(i) Compute its determinant, inverse, characteristic polynomial, eigenvalues and eigenvectors.

(ii) Check that the column vector $u_1 = (0, 0, 1)$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = -2$, i.e. check that $Au_1 = \lambda_1 u_1$.

(iii) Check that the column vector $u_2 = (1 + i, 1, 0)$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = -1 + i$, i.e. check that $Au_2 = \lambda_2 u_2$.

(iv) Check that the column vector $u_3 = (1 - i, 1, 0)$ is an eigenvector corresponding to the eigenvalue $\lambda_3 = -1 - i$, i.e. check that $Au_3 = \lambda_3 u_3$.

(v) Introduce the matrix P whose columns are u_1, u_2, u_3 in this order.

(vi) Introduce the diagonal matrix J whose elements on the main diagonal are $\lambda_1, \lambda_2, \lambda_3$ in this order.

(vii) Check that $A = PJP^{-1}$. Of course, A is not diagonalizable over $\mathbb{R}$, but, due to this relation, it is said that A is diagonalizable over $\mathbb{C}$.

(viii) Compute e^{tJ} and e^{tA}. The matrix e^{tJ} has complex entries with nonzero imaginary part because J has complex entries with nonzero imaginary part. The matrix A has real entries and we know that, by definition, e^{tA} has also real entries. In Maple this is evident, but in Sage it is not so evident.

(ix) Compute the limit as $t \to \infty$ for each entry of e^{tA}.

(x) True/False:

"Each solution of the differential system $X' = AX$ satisfies $\lim_{t\to\infty} X(t) = 0_3$."

Here 0_3 is the null column vector in $\mathbb{R}^3$.

2. Find a 4×4 real matrix A with the eigenvalues $2, 2, -1, 0$ which is not diagonal but it is diagonalizable over $\mathbb{R}$.

Hint: First find an invertible 4×4 real matrix P (there are many!), then introduce the diagonal matrix J with $2, 2, -1, 0$ on the main diagonal. Take $A = PJP^{-1}$. From this relation we deduce that A is similar to the diagonal matrix J, thus, by definition, it is diagonalizable.

3. We consider $x' = 1 - x^2$.

(i) Find its equilibrium points. Find the expression of $\phi(t, -1)$, $\phi(t, 1)$.

I think this is faster without Maple or Sage. In fact, Sage will not solve the IVPs $x' = 1 - x^2, \quad x(0) = -1$ and $x' = 1 - x^2, \quad x(0) = 1$.

(ii) Find the expression of each of the solutions $\varphi(t, -2)$, $\varphi(t, 0)$, $\varphi(t, 2)$.

Sage will integrate the differential equation (like we explained in Lecture 7), giving an expression which defines implicitly the solution. So, we have to solve the (algebraic) equation obtained with respect to the unknown $x(t)$. To succeed, we have first to multiply with 2, then to use $log_simplify$. Pay attention, that the output of **solve** is a list with one equation. The first element in a list is taken with **[0]**, while the right-hand side of an equation is taken with **rhs**. Of course, if this looks too difficult for you, solve it by hand.

Note that $\varphi(t, -2) = \frac{e^{2t}+3}{e^{2t}-3}, \quad \varphi(t, 0) = \frac{e^{2t}-1}{e^{2t}+1}, \quad \varphi(t, 2) = \frac{e^{2t}+1/3}{e^{2t}-1/3}$.

In Maple these expressions have other forms. Use

convert(convert(tanh(t-arctanh(2)),exp),exp) to obtain the expression $\frac{e^{2t}+3}{e^{2t}-3}$.

(iii) Represent the graph of $\varphi(t, -2)$, $\varphi(t, 0)$, $\varphi(t, 2)$. Note that their maximal interval of definition has the form $I_{-2} = (-\infty, \beta_{-2})$, $I_0 = \mathbb{R}$ and $I_2 = (\alpha_2, +\infty)$, where $\beta_{-2}, \alpha_2 \in \mathbb{R}$. Pay attention that we talk about the interval of definition! And $\mathbb{R} \setminus \{a\}$ is not an interval, it is a union of two intervals.

If you want, find the exact values $\beta_{-2} = \ln \sqrt{3}$ and $\alpha_2 = -\ln \sqrt{3}$.

In Sage, plot the graph of $\varphi(t, -2)$ also on the interval $(-2, 1/2)$, the graph of $\varphi(t, 0)$ also on the interval $(-10, 10)$, the graph of $\varphi(t, 2)$ on the interval $(-1/2, 2)$.

(iv) Find $\lim\limits_{t \to -\infty} \varphi(t, -2)$, $\lim\limits_{t \to -\infty} \varphi(t, 0)$, $\lim\limits_{t \to +\infty} \varphi(t, 0)$, $\lim\limits_{t \to +\infty} \varphi(t, 2)$.

(v) Specify the monotonicity of the functions found at (ii) looking at their graph. Find the image of each function. More exactly, prove that $\gamma_{-2} = (-\infty, -1)$, $\gamma_0 = (-1, 1)$, $\gamma_2 = (1, \infty)$.

(vi) Finally, in your notebooks represent the phase portrait of $x' = 1 - x^2$ and confirm (using the theory presented in the lecture) the properties you found.