12 Week 12. Transformations

12.1 Transformations of the plane

Definition 12.1. An *affine transformation* of the plane is a perturbation by a translation of a linear transformation, i.e.

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ L(x, y) = (ax + by + c, dx + ey + f), \tag{12.1}$$

for some constant real numbers a, b, c, d, e, f.

By using the matrix language, the action of the map L can be written in the form

$(x,y) \in \mathbb{R}^2, [x\ y] \in \mathbb{R}^{1\times 2}$

$$L(x, y) = [x\ y]\begin{bmatrix} a & d \\ b & e \end{bmatrix} + [c\ f].$$

The affine transformation L can be also identified with the map $L^c : \mathbb{R}^{2\times1} \longrightarrow \mathbb{R}^{2\times1}$ given by

$(x,y) \in \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1}$

$$L^c\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax + by + c \\ dx + ey + f \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix} \rightsquigarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$
$$= [L]\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}, \text{ where } [L] = \begin{bmatrix} a & b \\ d & e \end{bmatrix}.$$

$B \neq 0, y = -\frac{A}{B}x - \frac{C}{B}$

Lemma 12.1. *If $(aB - bA)^2 + (dB - eA)^2 > 0$, then the affine transformation (14.1) maps the line*

$$= \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ -\frac{A}{B}x - \frac{C}{B} \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

$aB - bA \neq 0$ or $dB - eA \neq 0$

$$(d)\ Ax + By + C = 0$$

$A \neq 0$ or $B \neq 0$

$$= \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ -\frac{A}{B}x \end{bmatrix} - \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} 0 \\ \frac{C}{B} \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

$= \ldots$

to the line

$$(eA - dB)x + (aB - bA)y + (bf - ce)A - (af - cd)B + (ae - bd)C = 0.$$

$$= \begin{bmatrix} \frac{aB - bA}{B}x + \frac{cB - bC}{B} \\ \frac{dB - eA}{B}x + \frac{fB - eC}{B} \end{bmatrix}$$

$= \ldots$

If $aB - bA = dB - eA = 0$, then $ae - bd = 0$ and $L\big|_d$ is the constant map $\left(\frac{cB - bC}{B}, \frac{fB - eC}{B}\right)$.

Definition 12.2. An affine transformation (14.1) is said to be *singular* if

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0 \text{ i.e. } ae - bd = 0.$$

and non-singular otherwise.

12.1.1 Translations

Note that the affine transformation L is nonsingular if and only if it is invertible. In such a case the inverse L^{-1} is a non-singular affine transformation and $[L^{-1}] = [L]^{-1}$.

Definition 12.3. The *translation* of vector $(h, k) \in \mathbb{R}^2$ is the affine transformation

$$T(h, k) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ [T(h, k)](x, y) = (x + h, y + k).$$

$\begin{cases} x' = x + h \\ y' = y + k \end{cases}$

Thus

$$[T(h, k)^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + h \\ y + k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix},$$

i.e.

$$[T(h, k)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$T(h,k) \circ T(l,m) = T(h+l, k+m)$

$T(0,0) = Id_{\mathbb{R}^2}$

Note that the translation $T(h, k)$ is non-singular (invertible) and $(T(h, k))^{-1} = T(-h, -k)$.

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12.1.2 Scaling about the origin

Definition 12.4. The *scaling about the origin* by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ is the affine transformation

$$S(s_x, s_y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ \left[S(s_x, s_y)\right](x, y) = (s_x \cdot x, s_y \cdot y).$$

Thus

$$\left[S(s_x,s_y)^c\right]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$S(s_x,s_y)\circ S(s'_x,s'_y) = S(s_x s'_x, s_y s'_y)$
$S(1,1) = id_{\mathbb{R}^2}$

$$[S(s_x, s_y)] = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}.$$

$\begin{cases} x' = s_x \cdot x \\ y' = s_y \cdot y \end{cases}$

Note that the scaling about the origin by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ is non-singular (invertible) and $(S(s_x, s_y))^{-1} = S(s_x^{-1}, s_y^{-1})$.

12.1.3 Reflections

Definition 12.5. The *reflections about the x-axis* and the y-axis respectively are the affine transformation

$$r_x, r_y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ r_x(x, y) = (x, -y),\ r_y = (-x, y).$$

$r_x : \begin{cases} x' = x \\ y' = -y \end{cases}$

Thus

$r_x \circ r_y = id_{\mathbb{R}^2}$
$r_y \circ r_x = id_{\mathbb{R}^2}$

$$[r_x^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

$r_y : \begin{cases} x' = -x \\ y' = y \end{cases}$

i.e.

$$[r_x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \text{ Similarly } [r_y] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $r_x = S(-1, 1)$ and $r_y = S(1, -1)$. Thus the two reflections are non-singular (invertible) and $r_x^{-1} = r_x$, $r_y^{-1} = r_y$.

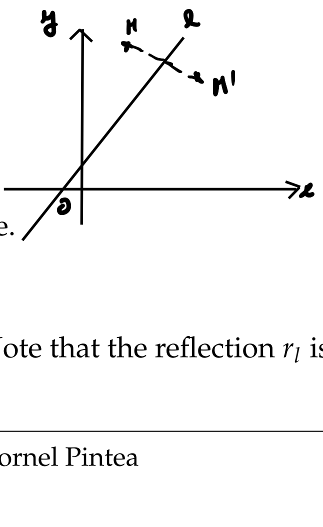
Definition 12.6. The *reflection* $r_l : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *about the line* l maps a given point M to the point M' defined by the property that l is the perpendicular bisector of the segment MM'. One can show that the action of the reflection about the line $l : ax + by + c = 0$ is

$$r_l(x,y) = \left(\frac{b^2-a^2}{a^2+b^2}x - \frac{2ab}{a^2+b^2}y - \frac{2ac}{a^2+b^2}, -\frac{2ab}{a^2+b^2}x + \frac{a^2-b^2}{a^2+b^2}y - \frac{2bc}{a^2+b^2}\right).$$

Thus

$$[r_l^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2}x - \frac{2ab}{a^2+b^2}y - \frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2}x + \frac{a^2-b^2}{a^2+b^2}y - \frac{2bc}{a^2+b^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{2ac}{a^2+b^2} \\ \frac{2bc}{a^2+b^2} \end{bmatrix},$$



i.e.

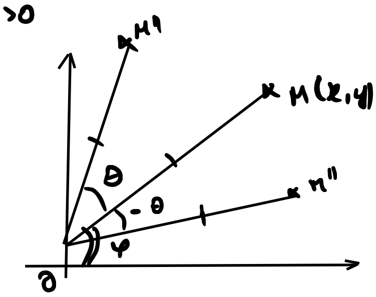
$$[r_l] = \frac{1}{a^2+b^2}\begin{bmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}.$$

Note that the reflection r_l is non-singular (invertible) and $r_l^{-1} = r_l$.

12.1.4 Rotations

Definition 12.7. The *rotation* $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the origin through an angle θ maps a point $M(x,y)$ into a point $M'(x',y')$ with the properties that the segments $[OM]$ and $[OM']$ are congruent and the $m(\widehat{MOM'}) = \theta$. If $\theta > 0$ the rotation is supposed to be *anticlockwise* and for $\theta < 0$ the rotation is *clockwise*. If $(x,y) = (r\cos\varphi, r\sin\varphi)$, then the coordinates of the rotated point are $\big(r\cos(\theta+\varphi), r\sin(\theta+\varphi)\big) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, i.e.

$$R_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$ $\theta > 0$



Thus

$$[R_\theta^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ $r = \|OM\|$

Note that the rotation R_θ is non-singular (invertible) and $R_\theta^{-1} = R_{-\theta}$.

$\begin{cases} x' = r\cos(\theta+\varphi) \\ y' = r\sin(\theta+\varphi) \end{cases}$

$\Rightarrow \begin{cases} x' = r\cos\varphi\cdot\cos\theta - r\sin\varphi\sin\theta \\ y' = r\cos\varphi\sin\theta + r\sin\varphi\cdot\cos\theta \end{cases} \Rightarrow \begin{cases} x' = x\cos\theta - x\sin\theta \\ y' = y\sin\theta + y\cos\theta \end{cases}$

12.1.5 Shears

Definition 12.8. Given a fixed direction in the plane specified by a unit vector $v = (v_1, v_2)$, consider the lines Δ with direction v and the oriented distance d from the origin. The *shear* about the origin of factor r in the direction v is defined to be the transformation which maps a point $M(x,y)$ on d to the point $M' = M + rdv$. The equation of the line through M of direction v is $v_2X - v_1Y + (v_1y - v_2x) = 0$. The oriented distacnce from the origin to this line is $v_1y - v_2x$. Thus the action of the shear $Sh(v,r) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the origin of factor r in the direction v is

$$\begin{aligned} Sh(v,r)(x,y) &= (x,y) + rd(v_1,v_2) \\ &= (x,y) + (r(v_1y - v_2x)v_1, r(v_1y - v_2x)v_2) \\ &= (x,y) + \big(-rv_1v_2x + rv_1^2y, -rv_2^2x + rv_1v_2y\big) \\ &= \big((1 - rv_1v_2)x + rv_1^2y, -rv_2^2x + (1 + rv_1v_2)y\big) \end{aligned}$$

Thus

$$[Sh(v,r)^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} (1 - rv_1v_2)x + rv_1^2y \\ -rv_2^2x + (1 + rv_1v_2)y \end{bmatrix}$$

$$= \begin{bmatrix} 1 - rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1 + rv_1v_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

$\Delta : \frac{X - x}{v_1} = \frac{Y - y}{v_2}$

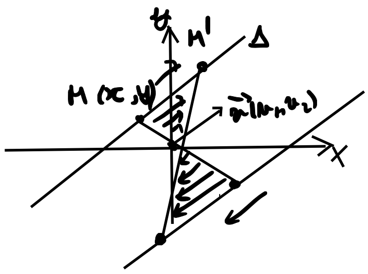
$\Delta : v_2X - v_1Y + v_1y - v_2x = 0$

$dist(O(0,0), \Delta) = \frac{|v_2 0 - v_1 0 + v_1y - v_2x|}{\sqrt{v_1^2 + v_2^2}} = |v_1y - v_2x| = \left\| \begin{matrix} v_1 & v_2 \\ x & y \end{matrix} \right\|$

i.e. $[Sh(v,r)] = \begin{bmatrix} 1 - rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1 + rv_1v_2 \end{bmatrix}$. $v_1y - v_2x$

the oriented distance from $O(0,0)$ to Δ

$M(x,y) \mapsto M' = M + rdv$



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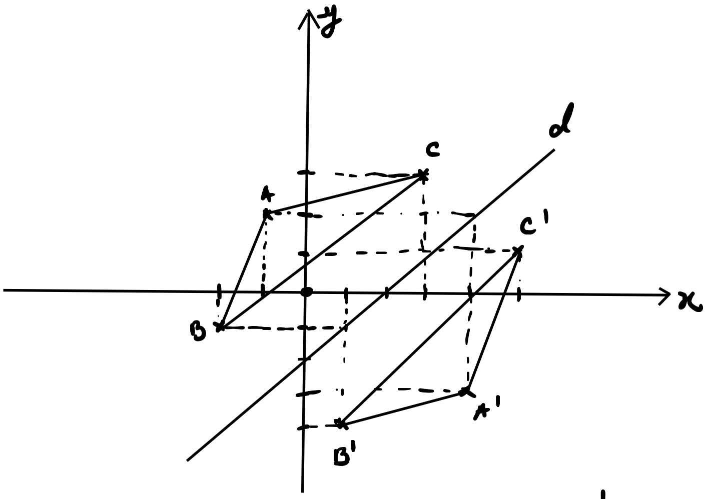
12.2 Problems

1. Find the image of the triangle ABC through the reflection in the line $(d)\ x - y = 2$, where $A(-1, 2)$, $B(-2, -1)$ and $C(3, 3)$.

   Solution.



$$(d): x - y = 2 \Rightarrow x = y + 2 \Rightarrow x - y - 2 = 0 \Rightarrow \begin{cases} a = 1 \\ b = -1 \\ c = -2 \end{cases}$$

$$r_d(x, y) = \frac{1}{a^2 + b^2}\left((b^2 - a^2)x - 2aby - 2ac, (a^2 - b^2)y - 2abx - 2bc\right) \Rightarrow$$

$$\Rightarrow r_d(x, y) = \frac{1}{1+1}\left((1 - 1)x - 2 \cdot 1 \cdot (-1)y - 2 \cdot 1 \cdot (-2), (1 - 1)y - 2 \cdot 1 \cdot (-1)x - 2 \cdot (-1) \cdot (-2)\right) =$$

$$= \frac{1}{2}(2y + 4, 2x - 4) = (y + 2, x - 2)$$

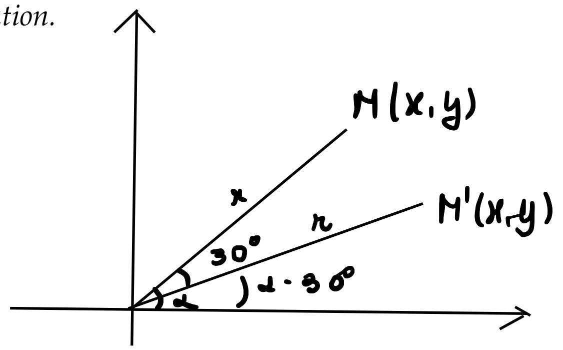
For $A(-1, 2)$: $A' : r_d(-1, 2) = (2 + 2, -1 - 2) = (4, -3) \Rightarrow A'(4, -3)$

$B(-2, -1)$: $B' : r_d(-2, -1) = (-1 + 2, -2 - 2) = (1, -4) \Rightarrow B'(1, -4)$

$C(3, 3)$: $C' : r_d(3, 3) = (3 + 2, 3 - 2) = (5, 1) \Rightarrow C'(5, 1)$

2. Find the image of the triangle ABC through the clockwise rotation of angle $30°$, where $A(6,4)$, $B(6,2)$ and $C(10,6)$.

Solution.



$\cos 30° = \frac{\sqrt{3}}{2}$

$\sin 30° = \frac{1}{2}$

$$(x,y) = (r\cos\alpha, r\sin\alpha) \Rightarrow (x',y') = (r\cos(\alpha-30°), r\sin(\alpha-30°)) =$$

$$= (r\cos\alpha\cdot\cos 30° + r\sin\alpha\cdot\sin 30°, r\sin\alpha\cdot\cos 30° - r\sin 30°\cdot\cos\alpha) =$$

$$= \left(x\cdot\frac{\sqrt{3}}{2} + y\cdot\frac{1}{2}, y\cdot\frac{\sqrt{3}}{2} - x\cdot\frac{1}{2}\right) = \frac{1}{2}(x\sqrt{3}+y, y\sqrt{3}-x)$$

For $A(6,4) \Rightarrow A'(x',y') = \frac{1}{2}(6\sqrt{3}+4, 4\sqrt{3}-6) = (3\sqrt{3}+2, 2\sqrt{3}-3)$

$B(6,2) \Rightarrow B'(x',y') = \frac{1}{2}(6\sqrt{3}+2, 2\sqrt{3}-6) = (3\sqrt{3}+1, \sqrt{3}-3)$

$C(10,6) \Rightarrow C'(x,y) = \frac{1}{2}(10\sqrt{3}+6, 6\sqrt{3}-10) = (5\sqrt{3}+3, 3\sqrt{3}-5)$

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3. Consider a quadrilateral with vertices $A(1,1)$, $B(3,1)$, $C(2,2)$, and $D(1.5,3)$. Find the image quadrilaterals through the translation $T(1,2)$, the scaling $S(2,2.5)$, the reflections about the x and y-axes, the clockwise and anticlockwise rotations through the angle $\pi/2$ and the shear $Sh\left(\left(2/\sqrt{5},1/\sqrt{5}\right),1.5\right)$.

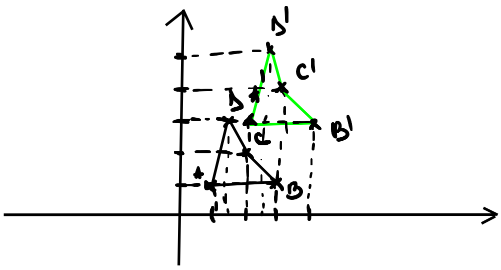
Solution.

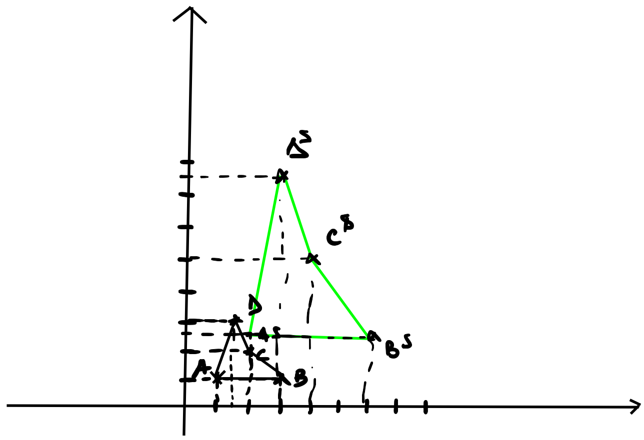
Translation $T(1,2)$: $[T(1,2)](1,1) = (1+1,1+2) = (2,3) \Rightarrow A^T(2,3)$

$[T(1,2)](3,1) = (1+3,2+1) = (4,3) \Rightarrow B^T(4,3)$

$[T(1,2)](2,2) = (1+2,2+2) = (3,4) \Rightarrow C^T(3,4)$

$[T(1,2)](1{,}5,3) = (1+1{,}5,3+2) = (2{,}5,5) \Rightarrow D^T(2{,}5,5)$





Scaling $(2,2{,}5)$:

$S(2,2{,}5)(1,1) = (2,2{,}5) \Rightarrow A^S(2,2{,}5)$

$S(2,2{,}5)(3,1) = (6,2{,}5) \Rightarrow B^S(6,2{,}5)$

$S(2,2{,}5)(2,2) = (4,5) \Rightarrow C^S(4,5)$

$S(2,2{,}5)(1{,}5,3) = (3,7{,}5) \Rightarrow D^S(3,7{,}5)$

Reflection about the x and y axes:

$r_x(1,1) = (1,-1) \Rightarrow A^x(1,-1)$

$r_x(3,1) = (3,-1) \Rightarrow B^x(3,-1)$

$r_x(2,2) = (2,-2) \Rightarrow C^x(2,-2)$

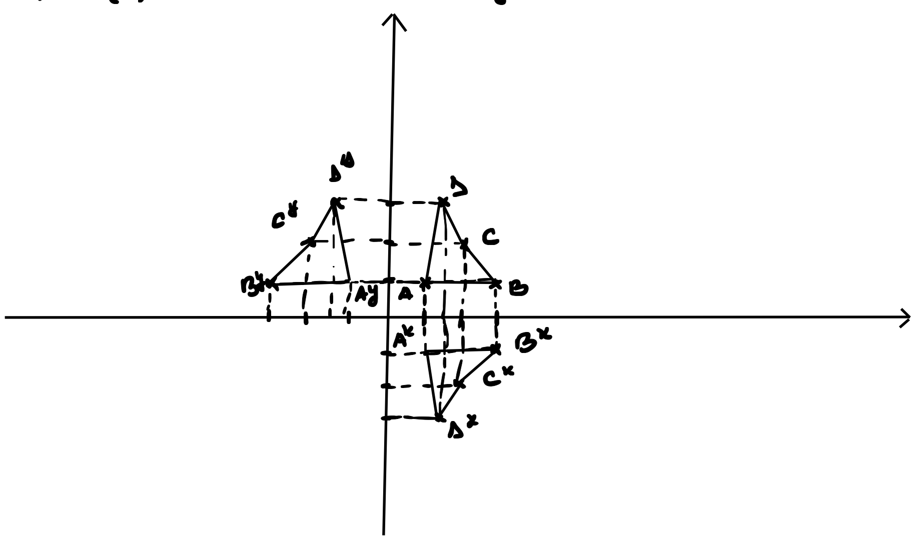
$r_x(1{,}5,3) = (1{,}5,-3) \Rightarrow D^x(1{,}5,-3)$

$r_y(1,1) = (-1,1) \Rightarrow A^y(-1,1)$

$r_y(3,1) = (-3,1) \Rightarrow B^y(-3,1)$

$r_y(2,2) = (-2,2) \Rightarrow C^y(-2,2)$

$r_y(1{,}5,3) = (-1{,}5,3) \Rightarrow D^y(-1{,}5,3)$



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4. Let $M(x, y)$ be a mobile point on the ellipse $(\mathcal{E})$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the locus of centroids of the triangles MFF', where F and F' are the foci of the ellipse, is the image through a scalling of equal factors (a homothety) of the given ellipse $\mathcal{E}$. Find the equation of the locus..

Solution.

The foci: $F(-c,0)$ and $F(c,0)$, where $c = \sqrt{a^2-b^2}$

$\mathcal{E}: \begin{cases} x = a\cdot\cos(t) \\ y = b\cdot\sin(t) \end{cases}$. Let $M(t) = (x(t), y(t)) \in \mathcal{E}$, and $G(t)$ - centroid of $\Delta M(t)FF'$ $\Rightarrow$

$\Rightarrow \begin{cases} x_{G(t)} = \dfrac{a\cos(t) + (-c) + c}{3} = \dfrac{1}{3}a\cos(t) = \dfrac{1}{3}x_{M(t)} \\ y_{G(t)} = \dfrac{b\sin(t) + 0 + 0}{3} = \dfrac{1}{3}b\sin(t) = \dfrac{1}{3}y_{M(t)} \end{cases}$ $\Rightarrow$ Locus of the centroids is $s(1/3, 1/3)(\mathcal{E}) = \mathcal{E}'$, with:

$\mathcal{E}': \dfrac{x^2}{\left(\frac{a}{3}\right)^2} + \dfrac{y^2}{\left(\frac{b}{3}\right)^2} = 1$

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5. Consider the line (d) $ax + by + c = 0$ and the points $A, B \notin d$. Find the coordinates of the point $M \in d$ such that $\text{dist}(A, B) + \text{dist}(M, B)$ is minimal.

   Solution.

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Affine transformations

$y = mx + n$, $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto mx + n$ affine function

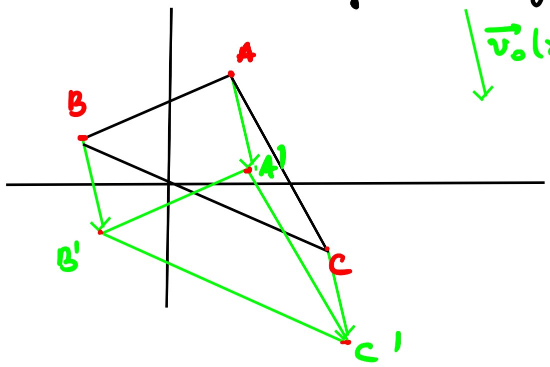
$f(x_1 + x_2) \neq f(x_1) + f(x_2)$

$y = mx$ linear function

$\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transformation if $\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{M}_{\in \mathcal{M}_2(\mathbb{R})} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}_{= \vec{v_0}}$

→ they preserve lines and parallelism (but not necessarily distances and angles)

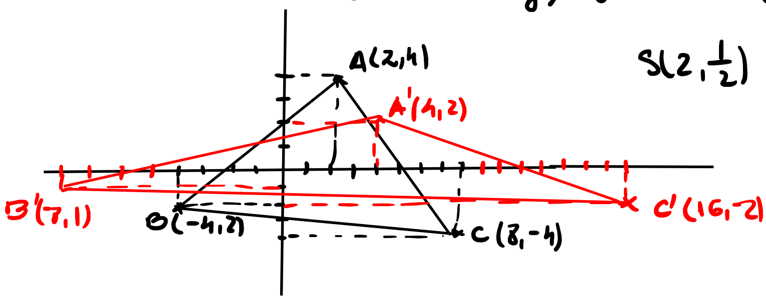
Translations: $T(x_0, y_0)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}_{\vec{v_0}(x_0, y_0)} = \begin{pmatrix} x + x_0 \\ y + y_0 \end{pmatrix}$



If we have $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ an affine transformation, how do we tell if φ is a translation?

- Pick a point A, let $A' = \varphi(A)$
- If φ is a translation, then $\varphi = T(\overrightarrow{AA'})$
- Check this against all other points

Scalings: $S(s_x, s_y)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} s_x \cdot x \\ s_y \cdot y \end{pmatrix}$

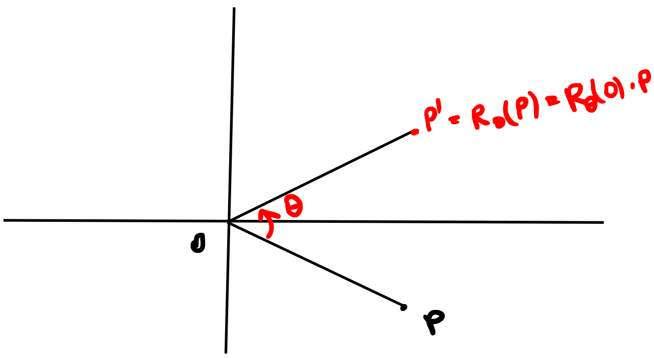


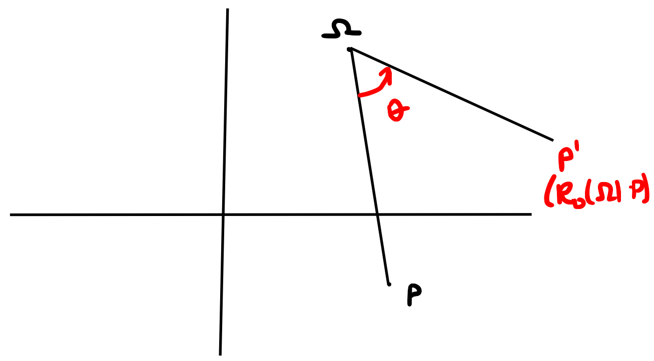
If $s_x = s_y \to S(s_x, s_y)$ "homothety"

If we have $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ an affine transformation, how do we tell if φ is a scaling ?

- Pick a point A, let $A' = \varphi(A)$
- If φ is a scaling $\Rightarrow s_x = \frac{x_{A'}}{x_A}$, $s_y = \frac{y_{A'}}{y_A}$
- Check this against all other points

Rotations (around the origin): R_θ





$$R_\theta\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}$$

$f: A\to B$, $Fix(f) = \{x\in A \,/\, f(x) = x\}$

If we have $\varphi:\mathbb{R}^2\to\mathbb{R}^2$ an affine transformation, how do we tell if φ is a rotation?

- Check if $Fix(\varphi) = \{P_0\}$
- If so, then $\varphi = R_\theta(P_0)$, we don't know θ

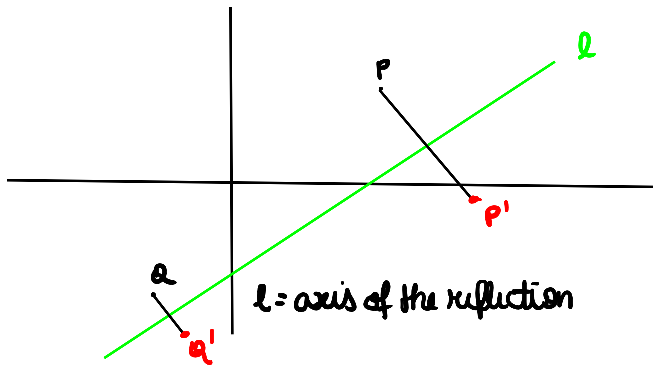
(- check if $\forall P: P_0P = P_0\varphi(P)$)

- Check if $\forall P$: $m(\widehat{PP_0P'})$ is the same. If so, $\theta = m(\widehat{PP_0P'})$

Reflections (orthogonal): r_ℓ

(w.r.t a line)

$\ell: ax+by+c=0$



$$r_\ell\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{\begin{pmatrix}\frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2}\\ \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2}\end{pmatrix}}_{= M}\begin{pmatrix}x\\y\end{pmatrix} + \underbrace{\begin{pmatrix}\frac{-2ac}{a^2+b^2}\\ \frac{-2bc}{a^2+b^2}\end{pmatrix}}_{= \vec{v_0}}$$

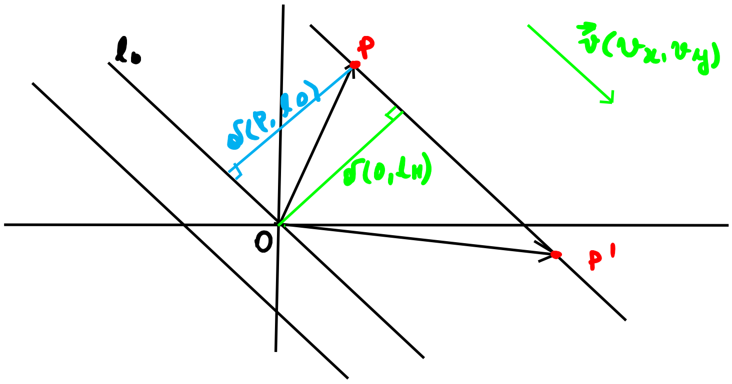
If $c=0$ (i.e. $\ell \ni O$), then $r_\ell\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2}\\ \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

If we have $\varphi:\mathbb{R}^2\to\mathbb{R}^2$ an affine transformation, how do we tell if φ is a reflection?

- Check that $Fix(\varphi) = \ell$, ℓ line
- Check that $\forall A$, $A' = \varphi(A) \to \ell$ is the perp. bisector of AA'

Shearing: $sh(\vec{v}, r)$

$\vec{v} \in \mathbb{R}^2, \|\vec{v}\| = 1, r \in \mathbb{R}$



$sh(\vec{v}, r)(P) = sh(\vec{v}, r)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + r \cdot \delta(O, \ell_H)\,\vec{v} = P'$

ℓ_H = line through H with direction $\vec{v}$

$\ell : ax + by + c = 0$

$P(x_P, y_P) \to \delta(P, \ell) = \dfrac{ax_P + by_P + c}{\sqrt{a^2 + b^2}}$

└ oriented distance

$sh(\vec{v}, r)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - r v_x v_y & r v_x^2 \\ -r v_y^2 & 1 + r v_x v_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

If we have $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ an affine transformation, how do we tell if φ is a shear ?

- Check if $Fix(\varphi) = \ell$, line

Check if φ is a reflection — or — Check if $\forall A, A' = \varphi(A), \overrightarrow{AA'} \parallel \ell$

- If we know that φ is a shear with axis ℓ, we choose $\vec{v} = \dfrac{1}{\|\vec{\ell}\|} \cdot \vec{\ell}$
- Find r using the definition