7 Week 7: The triple scalar product

The *triple scalar product* $(\vec{a}, \vec{b}, \vec{c})$ of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the real number $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

Proposition 7.1. *If* $[\vec{i}, \vec{j}, \vec{k}]$ *is a direct orthonormal basis and*

$$\begin{aligned} \vec{a} &= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \\ \vec{b} &= b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k} \\ \vec{c} &= c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k} \end{aligned}$$

then

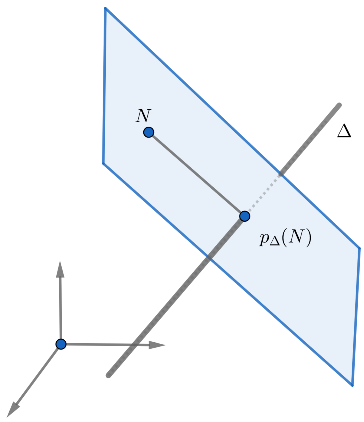
$$(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{7.1}$$

Proof. Indeed, we have successively:

$$\begin{aligned} (\vec{a}, \vec{b}, \vec{c}) &= (\vec{a} \times \vec{b}) \cdot \vec{c} \\ &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \right) \cdot (c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}) \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \end{aligned}$$

□

Remark 7.1. Taking into account the formula (7.2) for the distance $\delta(N, \Delta)$ from the point $N(x_N, y_N, z_N)$ to the straight line $\Delta : \dfrac{x - x_0}{p} = \dfrac{y - y_0}{q} = \dfrac{z - z_0}{r}$ as well as Proposition 6.3 we deduce that



$$\begin{aligned} \delta(N, \Delta) &= \| \overrightarrow{N p_{\Delta}(N)} \| \\ &= \| \overrightarrow{NO} + \overrightarrow{O p_{\Delta}(N)} \| = \left\| \overrightarrow{NA_0} - \frac{\vec{d}_{\Delta} \cdot \overrightarrow{NA_0}}{\| \vec{d}_{\Delta} \|^2} \vec{d}_{\Delta} \right\| \end{aligned} \tag{7.2}$$

$$= \frac{\| \left(\vec{d}_\Delta \cdot \vec{d}_\Delta\right) \overrightarrow{NA_0} - \left(\vec{d}_\Delta \cdot \overrightarrow{NA_0}\right) \vec{d}_\Delta \|}{\| \vec{d}_\Delta \|^2}$$
$$= \frac{\| \vec{d}_\Delta \times (\overrightarrow{NA_0} \times \vec{d}_\Delta) \|}{\| \vec{d}_\Delta \|^2} = \frac{\| \overrightarrow{NA_0} \times \vec{d}_\Delta \|}{\| \vec{d}_\Delta \|}.$$

Thus, we recovered the distance formula from one point to one straight line (see formula 6.4) by using different arguments.

Corollary 7.2. 1. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent (collinear) iff $(\vec{a}, \vec{b}, \vec{c}) = 0$*

2. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly independent (noncollinear) if and only if $(\vec{a}, \vec{b}, \vec{c}) \neq 0$*

3. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ form a basis of the space $\mathcal{V}$ if and only if $(\vec{a}, \vec{b}, \vec{c}) \neq 0$.*

4. *The correspondence $F : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$, $F(\vec{a}, \vec{b}, \vec{c}) = (\vec{a}, \vec{b}, \vec{c})$ is trilinear and skew-symmetric, i.e.*

$$\begin{array}{l} (\alpha \vec{a} + \alpha' \vec{a}\,', \vec{b}, \vec{c}) = \alpha(\vec{a}, \vec{b}, \vec{c}) + \alpha'(\vec{a}\,', \vec{b}, \vec{c}) \\ (\vec{a}, \beta \vec{b} + \beta' \vec{b}\,', \vec{c}) = \beta(\vec{a}, \vec{b}, \vec{c}) + \beta'(\vec{a}, \vec{b}\,', \vec{c}) \\ (\vec{a}, \vec{b}, \gamma \vec{c} + \gamma' \vec{c}\,') = \gamma(\vec{a}, \vec{b}, \vec{c}) + \gamma'(\vec{a}, \vec{b}, \vec{c}\,') \end{array} \tag{7.3}$$

$\forall\, \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{R}$, $\forall\ \vec{a}, \vec{b}, \vec{c}, \vec{a}\,', \vec{b}\,', \vec{c}\,' \in \mathcal{V}$ şi

$$(\vec{a}_1, \vec{a}_2, \vec{a}_3) = sgn(\sigma)(\vec{a}_{\sigma(1)}, \vec{a}_{\sigma(2)}, \vec{a}_{\sigma(3)}),\ \forall\ \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V} \text{ şi } \forall\, \sigma \in S_3 \tag{7.4}$$

Remark 7.2. *One can rewrite the relations (7.4) as follows:*

$$\begin{array}{rcl} (\vec{a}_1, \vec{a}_2, \vec{a}_3) & = & (\vec{a}_2, \vec{a}_3, \vec{a}_1) = (\vec{a}_3, \vec{a}_1, \vec{a}_2) \\ & = & -(\vec{a}_2, \vec{a}_1, \vec{a}_3) = -(\vec{a}_1, \vec{a}_3, \vec{a}_2) = -(\vec{a}_3, \vec{a}_2, \vec{a}_1), \end{array}$$

$\forall\ \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V}$

Corollary 7.3. 1. *$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})\ \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.*

2. *For every $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathcal{V}$ the* Laplace formula holds:

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}.$$

Proof. While the first identity is obvious, for the Laplace formula we have successively:

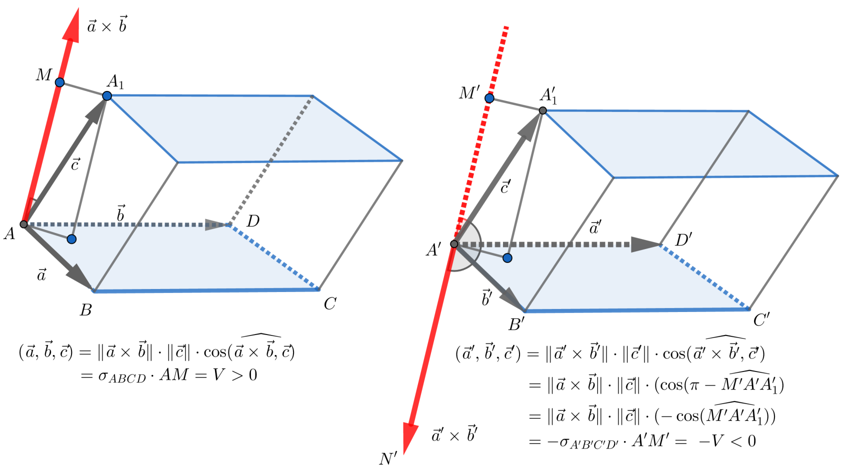
$$\begin{aligned} (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= \left(\vec{a}, \vec{b}, \vec{c} \times \vec{d}\right) = \left(\vec{c} \times \vec{d}, \vec{a}, \vec{b}\right) \\ &= \left[(\vec{c} \times \vec{d}) \times \vec{a}\right] \cdot \vec{b} = -\left[(\vec{a} \cdot \vec{d})\, \vec{c} - (\vec{a} \cdot \vec{c})\, \vec{d}\right] \cdot \vec{b} \\ &= -(\vec{a} \cdot \vec{d})(\vec{c} \cdot \vec{b}) + (\vec{a} \cdot \vec{c})(\vec{d} \cdot \vec{b}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}. \end{aligned}$$

□

Definition 7.1. The basis $[\vec{a}, \vec{b}, \vec{c}]$ of the space $\mathcal{V}$ is said to be *directe* if $(\vec{a}, \vec{b}, \vec{c}) > 0$. If, on the contrary, $(\vec{a}, \vec{b}, \vec{c}) < 0$, we say that the basis $[\vec{a}, \vec{b}, \vec{c}]$ is *inverse*

Definition 7.2. The *oriented* volume of the parallelepiped constructed on the noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ is $\varepsilon \cdot V$, where V is the volume of this parallelepiped and $\varepsilon = +1$ or -1 insomuch as the basis $[\vec{a}, \vec{b}, \vec{c}]$ is directe or inverse respectively.

Proposition 7.4. *The triple scalar product $(\vec{a}, \vec{b}, \vec{c})$ of the noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ is equal with the oriented volume of the parallelepiped constructed on these vectors.*



7.1 Applications of the triple scalar product

7.1.1 The distance between two straight lines

If $d_1,\ d_2$ are two straight lines, then the distance between them, denoted by $\delta(d_1, d_2)$, is being defined as

$$\min\{||\ \overrightarrow{M_1M_2}\ ||\ |\ M_1 \in d_1,\ M_2 \in d_2\}.$$

1. If $d_1 \cap d_2 \neq \emptyset$, then $\delta(d_1, d_2) = 0$.

2. If $d_1 || d_2$, then $\delta(d_1, d_2) = ||\ \overrightarrow{MN}\ ||$ where $\{M\} = d \cap d_1,\ \ \{N\} = d \cap d_2$ and d is a straight line perpendicular to the lines d_1 and d_2. Obviously $||\ \overrightarrow{MN}\ ||$ is independent on the choice of the line d.

3. We now assume that the straight lines d_1, d_2 are noncoplanar (skew lines). In this case there exits a unique straight line d such that $d \perp d_1, d_2$ and $d \cap d_1 = \{M_1\},\ \ d \cap d_2 = \{M_2\}$. The straight line d is called the *common perpendicular* of the lines d_1, d_2 and obviously $\delta(d_1, d_2) = ||\ \overrightarrow{M_1M_2}\ ||$.

Assume that the straight lines d_1, d_2 are given by their points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$ and their vectors şi au vectorii directori $\vec{d}_1\ (p_1, q_1, r_1)\ \vec{d}_2\ (p_2, q_2, r_2)$, that is, thei equations are

$$d_1 : \frac{x - x_1}{p_1} = \frac{y - y_1}{q_1} = \frac{z - z_1}{r_1}$$

$$d_2 : \frac{x - x_2}{p_2} = \frac{y - y_2}{q_2} = \frac{z - z_2}{r_2}.$$

The common perpendicular of the lines d_1, d_2 is the intersection line between the plane containing the line d_1 which is parallel to the vector $\vec{d}_1 \times \vec{d}_2$, and the plane containing the line d_2 which is parallel to $\vec{d}_1 \times \vec{d}_2$. Since

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} \vec{i} + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} \vec{j} + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \vec{k}$$

it follows that the equations of the common perpendicular are

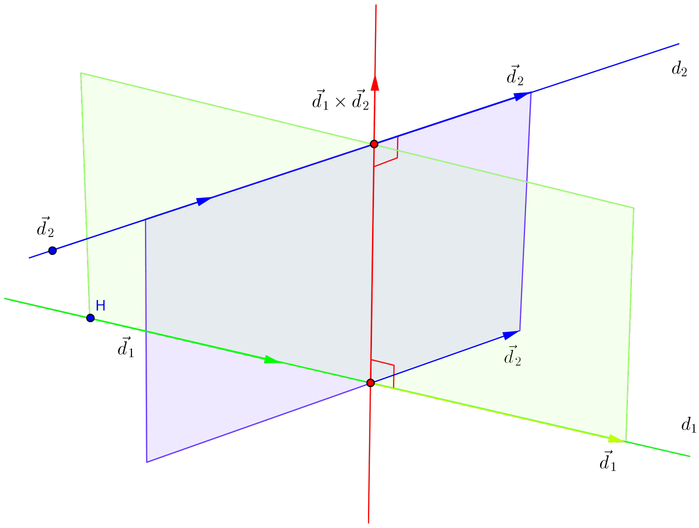


Figure 8: Perpendiculara comună a dreptelor d_1 şi d_2

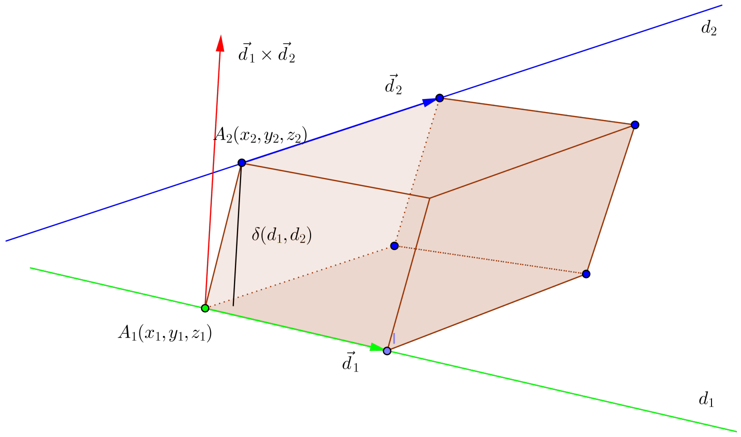
$$\begin{cases} \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ p_1 & q_1 & r_1 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0 \\ \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ p_2 & q_2 & r_2 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0. \end{cases} \tag{7.5}$$

The distance between the straight lines d_1, d_2 can be also regarded as the height of the parallelogram constructed on the vectors $\vec{d}_1, \vec{d}_2, \vec{d}_1 \times \vec{d}_2$. Thus

$$\delta(d_1, d_2) = \frac{|(\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2)|}{|| \vec{d}_1 \times \vec{d}_2 ||}. \tag{7.6}$$

Therefore we obtain

$$\delta(d_1, d_2) = \frac{\left| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} \right|}{\sqrt{\begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}^2 + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}^2 + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}^2}} \tag{7.7}$$



7.1.2 The coplanarity condition of two straight lines

Using the notations of the previous section, observe that the straight lines d_1, d_2 are coplanar if and only if the vectors $\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2$ are linearly dependent (coplanar), or equivalently $(\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2) = 0$. Consequently the stright lines d_1, d_2 are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0 \tag{7.8}$$

7.2 Problems

1. **(2p)** Show that

   (a) $|(\vec{a}, \vec{b}, \vec{c})| \leq \| \vec{a} \| \cdot \| \vec{b} \| \cdot \| \vec{c} \|$;

   Solution.

$$|(\vec{a}, \vec{b}, \vec{c})| = |(\vec{a} \times \vec{b}) \cdot \vec{c}| = \|\vec{a} \times \vec{b}\| \cdot \|\vec{c}\| \cdot \underbrace{|\cos(\widehat{\vec{a} \times \vec{b}, \vec{c}})|}_{\leq 1} \leq \|\vec{a} \times \vec{b}\| \cdot \|\vec{c}\| =$$
$$= \|\vec{a}\| \cdot \|\vec{b}\| \cdot \|\vec{c}\| \cdot \underbrace{\sin(\vec{a}, \vec{b})}_{\leq 1} \leq \|\vec{a}\| \cdot \|\vec{b}\| \cdot \|\vec{c}\|$$

(b) **(2p)** $(\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}) = 2(\vec{a},\vec{b},\vec{c})$.

Solution.

$$(\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}) = (\vec{a}+\vec{b})\left[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\right] =$$

$$= (\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\underbrace{\vec{c}\times\vec{c}}_{\vec{0}}+\vec{c}\times\vec{a}) =$$

$$= \vec{a}\cdot(\vec{b}\times\vec{c}) + \underbrace{\vec{b}\cdot(\vec{b}\times\vec{c})}_{0} + \underbrace{\vec{a}\cdot(\vec{b}\times\vec{a})}_{0} + \underbrace{\vec{b}(\vec{b}\times\vec{a})}_{0} + \underbrace{\vec{a}\cdot(\vec{c}\times\vec{a})}_{0} + \vec{b}(\vec{c}\times\vec{a})$$

$$= \vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{b}(\vec{c}\times\vec{a}) = 2\cdot(\vec{a},\vec{b},\vec{c})$$

2. **(3p)** Prove the following identity:

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a}, \vec{c}, \vec{d})\,\vec{b} - (\vec{b}, \vec{c}, \vec{d})\,\vec{a} = (\vec{a}, \vec{b}, \vec{d})\,\vec{c} - (\vec{a}, \vec{b}, \vec{c})\,\vec{d}\,.$$

Solution. By using the identity $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\,\vec{v} - (\vec{u} \cdot \vec{v})\,\vec{v}$ for $\vec{u} = \vec{a} \times \vec{b}$, $\vec{v} = \vec{c}$ and $\vec{w} = \vec{d}$ we obtain

$$\begin{aligned}(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= \vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\,\vec{v} - (\vec{u} \cdot \vec{v})\,\vec{v} \\ &= \left[(\vec{a} \times \vec{b}) \cdot \vec{d}\right]\vec{c} - \left[(\vec{a} \times \vec{b}) \cdot \vec{c}\right]\vec{d} \\ &= (\vec{a}, \vec{b}, \vec{d})\,\vec{c} - (\vec{a}, \vec{b}, \vec{c})\,\vec{d}\,.\end{aligned}$$

By using the identity $(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\,\vec{v} - (\vec{v} \cdot \vec{w})\,\vec{u}$ for $\vec{u} = \vec{a}$, $\vec{v} = \vec{b}$ and $\vec{w} = \vec{c} \times \vec{d}$ we obtain

$$\begin{aligned}(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= (\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\,\vec{v} - (\vec{v} \cdot \vec{w})\,\vec{u} \\ &= \left[\vec{a} \cdot (\vec{c} \times \vec{d})\right]\vec{b} - \left[\vec{b} \cdot (\vec{c} \times \vec{d})\right]\vec{a} \\ &= (\vec{a}, \vec{c}, \vec{d})\,\vec{b} - (\vec{b}, \vec{c}, \vec{d})\,\vec{a}\,.\end{aligned}$$

3. **(3p)** Prove the following identity: $(\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}) = (\vec{u}, \vec{v}, \vec{w})^2$.

Solution. We have successively:

$$\begin{aligned}(\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}) &= \left[(\vec{u} \times \vec{v}) \times (\vec{v} \times \vec{w})\right] \cdot (\vec{w} \times \vec{u}) \\ &= \left[(\vec{u}, \vec{v}, \vec{w})\,\vec{v} - (\vec{u}, \vec{v}, \vec{v})\,\vec{w}\right] \cdot (\vec{w} \times \vec{u}) \\ &= (\vec{u}, \vec{v}, \vec{w})\left[\vec{v} \cdot (\vec{w} \times \vec{u})\right] = (\vec{u}, \vec{v}, \vec{w})(\vec{v}, \vec{w}, \vec{u}) = (\vec{u}, \vec{v}, \vec{w})^2.\end{aligned}$$

4. **(3p)** The *reciprocal vectors* of the noncoplanar vectors $\vec{u}, \vec{v}, \vec{w}$ are defined by

$$\vec{u}' = \frac{\vec{v} \times \vec{w}}{(\vec{u}, \vec{v}, \vec{w})}, \quad \vec{v}' = \frac{\vec{w} \times \vec{u}}{(\vec{u}, \vec{v}, \vec{w})}, \quad \vec{w}' = \frac{\vec{u} \times \vec{v}}{(\vec{u}, \vec{v}, \vec{w})}.$$

Show that:

(a)

$$\begin{aligned}\vec{a} &= (\vec{a} \cdot \vec{u}')\,\vec{u} + (\vec{a} \cdot \vec{v}')\,\vec{v} + (\vec{a} \cdot \vec{w}')\,\vec{w} \\ &= \frac{(\vec{a}, \vec{v}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})}\,\vec{u} + \frac{(\vec{u}, \vec{a}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})}\,\vec{v} + \frac{(\vec{u}, \vec{v}, \vec{a})}{(\vec{u}, \vec{v}, \vec{w})}\,\vec{w}\,.\end{aligned}$$

(b) the reciprocal vectors of $\vec{u}', \vec{v}', \vec{w}'$ are the vectors $\vec{u}, \vec{v}, \vec{w}$.

Solution. (4a) Obviously $\vec{a} = \alpha\,\vec{u} + \beta\,\vec{v} + \gamma\,\vec{c}$, as $\vec{u}, \vec{v}, \vec{w}$ are three linearly independent vectors of the three dimensional vector space $\mathcal{V}$, i.e. $\vec{u}, \vec{v}, \vec{w}$ form a basis of $\mathcal{V}$. Moreover we have

$$\begin{aligned}\vec{a} \cdot \vec{u}' &= \frac{\vec{a} \cdot (\vec{v} \times \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} = \frac{(\vec{a}, \vec{v}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} = \frac{(\alpha\,\vec{u} + \beta\,\vec{v} + \gamma\,\vec{c}, \vec{v}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} \\ &= \frac{\alpha(\vec{u}, \vec{v}, \vec{w}) + \beta(\vec{v}, \vec{v}, \vec{w}) + \gamma(\vec{w}, \vec{v}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} = \alpha.\end{aligned}$$

One can similalry show that

$$\vec{a} \cdot \vec{v}' = \frac{(\vec{u}, \vec{a}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})} = \beta \text{ and } \vec{a} \cdot \vec{w}' = \frac{(\vec{u}, \vec{v}, \vec{a})}{(\vec{u}, \vec{v}, \vec{w})} = \gamma.$$

(4b) Let us first observe that

$$(\vec{u}', \vec{v}', \vec{w}') = (\vec{w}', \vec{u}', \vec{v}') = \frac{(\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u})}{(\vec{u}, \vec{v}, \vec{w})^3} = \frac{(\vec{u}, \vec{v}, \vec{w})^2}{(\vec{u}, \vec{v}, \vec{w})^3} = \frac{1}{(\vec{u}, \vec{v}, \vec{w})}.$$

On the other hand we have:

$$\frac{\vec{v}' \times \vec{w}'}{(\vec{u}', \vec{v}', \vec{w}')} = (\vec{u}, \vec{v}, \vec{w})(\vec{v}' \times \vec{w}') = (\vec{u}, \vec{v}, \vec{w})\frac{(\vec{w} \times \vec{u}) \times (\vec{u} \times \vec{v})}{(\vec{u}, \vec{v}, \vec{w})^2} = \frac{(\vec{w}, \vec{u}, \vec{v})\,\vec{u} - (\vec{w}, \vec{u}, \vec{u})\,\vec{v}}{(\vec{u}, \vec{v}, \vec{w})} = \vec{u}.$$

One can similarly show that

$$\frac{\vec{w}' \times \vec{u}'}{(\vec{u}', \vec{v}', \vec{w}')} = \vec{v} \text{ and } \frac{\vec{u}' \times \vec{v}'}{(\vec{u}', \vec{v}', \vec{w}')} = \vec{w}.$$

5. **(2p)** Find the value of the parameter α for which the pencil of planes through the straight line AB has a common plane with the pencil of planes through the straight line CD, where $A(1, 2\alpha, \alpha)$, $B(3, 2, 1)$, $C(-\alpha, 0, \alpha)$ and $D(-1, 3, -3)$.

Solution. Condition: AB, CD - coplanar

Let $P_1 \in AB$, $P_2 \in CD$, $\vec{d_1} = \vec{AB}$, $\vec{d_2} = \vec{CD}$

AB, CD - coplanar $\Rightarrow (\vec{P_1P_2}, \vec{d_1}, \vec{d_2}) = 0$

$\vec{d_1} = \vec{AB} = (x_B - x_A, y_B - y_A, z_B - z_A) = (2, 2-2\alpha, 1-\alpha)$

$\vec{d_2} = \vec{CD} = (-1+\alpha, 3, -3-\alpha)$

We choose $P_1 = B \in AB$, $P_1(3,2,1)$ $\Rightarrow \vec{P_1P_2}(-4,1,-4)$

$P_2 = D \in CD$, $P_2(-1,3,-3)$

$$(\vec{P_1P_2}, \vec{d_1}, \vec{d_2}) = 0 \Leftrightarrow \begin{vmatrix} -4 & 1 & -4 \\ 2 & 2-2\alpha & 1-\alpha \\ -1+\alpha & 3 & -3-\alpha \end{vmatrix} = 0 \Leftrightarrow 8(1-\alpha)(3+\alpha) + (1-\alpha)(\alpha-1) - 24 + 8(1-\alpha)(\alpha-1) + 12(1-\alpha) + 2(3+\alpha) = 0 \Leftrightarrow$$

$$\Leftrightarrow (8-8\alpha)(3+\alpha) - (\alpha-1)^2 - 24 - 8(\alpha-1)^2 + 12 - 12\alpha + 6 + 2\alpha = 0 \underset{\dots}{\Leftrightarrow} -17\alpha^2 - 8\alpha + 9 = 0$$

$\Delta = 64 + 4 \cdot 9 \cdot 17 = 64 + 612 = 676$

$$\Rightarrow \alpha_1 = \frac{8 + \sqrt{676}}{-34} = \frac{8+26}{-34} = -1$$

$$\alpha_2 = \frac{8-26}{-34} = \frac{-18}{-34} = \frac{9}{17}$$

$\Rightarrow \alpha \in \{-1, \frac{9}{17}\}$

6. **(2p)** Find the value of the parameter λ for which the straight lines

$$(d_1)\ \frac{x-1}{3}=\frac{y+2}{-2}=\frac{z}{1},\ (d_2)\ \frac{x+1}{4}=\frac{y-3}{1}=\frac{z}{\lambda}$$

are coplanar. Find the coordinates of their intersection point in that case.

Solution.

$\begin{cases} A_1(1,-2,0)\in d_1 \\ A_2(-1,3,0)\in d_2 \end{cases} \Rightarrow \overrightarrow{A_1A_2}(-2,5,0)$

$\overrightarrow{d_1}(3,-2,1)$

$\overrightarrow{d_2}(4,1,\lambda)$

$(\overrightarrow{A_1A_2}, \overrightarrow{d_1}, \overrightarrow{d_2})$

d_1, d_2 are coplanar iff $\begin{vmatrix} -1-1 & 3-(-2) & 0-0 \\ 3 & -2 & 1 \\ 4 & 1 & \lambda \end{vmatrix}=0$

$\Leftrightarrow \begin{vmatrix} -2 & 5 & 0 \\ 3 & -2 & 1 \\ 4 & 1 & \lambda \end{vmatrix}=0 \Leftrightarrow 4\lambda+20+2-15\lambda=0 \Leftrightarrow -11\lambda=-22 \Rightarrow \lambda=2$

$d_1: \frac{x-1}{3}=\frac{y+2}{-2}=\frac{z}{1}$; $d_2: \frac{x+1}{4}=\frac{y-3}{1}=\frac{z}{2}$ coplanar, not parallel because the coordinates are not proportional

$(d_1)\begin{cases} x=1+3t \\ y=-2-2t \\ z=t \end{cases}, t\in\mathbb{R}$ $(d_2)\begin{cases} x=-1+4s \\ y=3+s \\ z=2s \end{cases}, s\in\mathbb{R}$

$\{M\}=d_1\cap d_2 \Leftrightarrow \begin{cases} M\in d_1 \\ M\in d_2 \end{cases} \Leftrightarrow \begin{cases} 1+3t=-1+4s \\ -2-2t=3+s \\ t=2s \end{cases} \Rightarrow \begin{cases} 3t-4s=-2 \\ t=2s \end{cases} \Leftrightarrow \begin{cases} 6s-4s=-2 \\ t=2s \end{cases} \Leftrightarrow \begin{cases} 2s=-2 \\ t=2s \end{cases} \Leftrightarrow$

We are looking for s, t

$\Leftrightarrow \begin{cases} s=-1 \\ t=-2 \end{cases}$. $-2-2\cdot(-2)=3+(-1) \Leftrightarrow -2+4=2 \Leftrightarrow 2=2$ true $\Rightarrow$ sol: $\begin{cases} s=-1 \\ t=-2 \end{cases}$

$M(-5,2,-2)$

7. **(2p)** Find the distance between the straight lines

$$(d_1)\ \frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{1},\ (d_2)\ \frac{x+1}{3}=\frac{y}{4}=\frac{z-1}{3}$$

as well as the equations of the common perpendicular.

Solution. $\vec{d_1}(2,3,1)$, $\vec{d_2}(3,4,3)$

$$\vec{d_1}\times\vec{d_2}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & 1\\ 3 & 4 & 3\end{vmatrix}=9\vec{i}+8\vec{k}+3\vec{j}-9\vec{k}-4\vec{i}-6\vec{j}=5\vec{i}-3\vec{j}-\vec{k}\Rightarrow \vec{d_1}\times\vec{d_2}(5,-3,-1)$$

$$\begin{cases}\Pi_1: \begin{vmatrix}x-1 & y+1 & z\\ 2 & 3 & 1\\ 5 & -3 & -1\end{vmatrix}=0\\ \Pi_2: \begin{vmatrix}x+1 & y & z-1\\ 3 & 4 & 3\\ 5 & -3 & -1\end{vmatrix}=0\end{cases} \Leftrightarrow \begin{cases}7y+7-21z=0\\ 5x+5+18y-29z+29=0\end{cases}\Leftrightarrow$$

$$\Leftrightarrow \begin{cases}y-3z+1=0\\ 5x+18y-29z+34=0\end{cases}$$ Eq. of the common perpendicular

$$d(d_1,d_2)=\frac{|(\overrightarrow{A_1A_2},\vec{d_1},\vec{d_2})|}{\|\vec{d_1}\times\vec{d_2}\|}=\frac{\left|\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\end{vmatrix}\right|}{\sqrt{5^2+(-3)^2+(-1)^2}}=\ldots=\frac{14}{\sqrt{35}}$$

8. **(2p)** Find the distance between the straight lines M_1M_2 and d, where $M_1(-1,0,1)$, $M_2(-2,1,0)$ and

$$(d)\ \begin{cases} x + y + z = 1 \\ 2x - y - 5z = 0. \end{cases}$$

as well as the equations of the common perpendicular.

Solution.

$d_1 = M_1M_2$

$\overrightarrow{M_1M_2} = (-2-(-1))\vec{i} + (1-0)\vec{j} + (0-1)\vec{k} = -\vec{i} + \vec{j} - \vec{k} \Rightarrow p_1 = -1,\ q_1 = 1,\ r_1 = -1$

$M_1 \in d_1 \Rightarrow x_1 = -1,\ y_1 = 0,\ z_1 = 1$

$(d_1): \frac{x-(-1)}{-1} = \frac{y-0}{1} = \frac{z-1}{-1} \Leftrightarrow \frac{x+1}{-1} = \frac{y}{1} = \frac{z-1}{-1}$

$d_2 = d: \begin{cases} x+y+z=1 \\ 2x-y-5z=0 \end{cases}$ $\oplus$

$3x - 4z = 1 \Rightarrow x = \frac{1+4z}{3} \Rightarrow \frac{1+4z}{3} + y + z = 1 \Rightarrow y = \frac{2-7z}{3} \Rightarrow \begin{cases} x = \frac{1}{3} + \frac{4}{3}\lambda \\ y = \frac{2}{3} - \frac{7}{3}\lambda \\ z = \lambda \end{cases} \Rightarrow$

$\Rightarrow d_2: \frac{x-\frac{1}{3}}{\frac{4}{3}} = \frac{y-\frac{2}{3}}{-\frac{7}{3}} = z$

$x_2 = \frac{1}{3},\ y_2 = \frac{2}{3},\ z_2 = 0$

$p_2 = \frac{4}{3},\ q_2 = -\frac{7}{3},\ r_2 = 1$

The common perpendicular of the lines d_1, d_2 is the intersection line between the plane containing the line d_1 which is parallel to the vector $\vec{d}_1 \times \vec{d}_2$, and the plane containing the line d_2 which is parallel to $\vec{d}_1 \times \vec{d}_2$. Since

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}\vec{i} + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}\vec{j} + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}\vec{k}$$

$\begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -\frac{7}{3} & 1 \end{vmatrix}$

$\begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & \frac{4}{3} \end{vmatrix}$

$\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ \frac{4}{3} & -\frac{7}{3} \end{vmatrix}$

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & -1 \\ \frac{4}{3} & -\frac{7}{3} & 1 \end{vmatrix} = \left(1 - \frac{7}{3}\right)\vec{i} + \left(-\frac{4}{3} + 1\right)\vec{j} + \left(\frac{7}{3} - \frac{4}{3}\right)\vec{k} = -\frac{4}{3}\vec{i} - \frac{1}{3}\vec{j} + \vec{k} \Rightarrow p = -\frac{4}{3},\ q = -\frac{1}{3},\ r = 1$$

Perpendiculara comună a dreptelor d_1 şi d_2

$$\begin{cases} \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ p_1 & q_1 & r_1 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0 \\ \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ p_2 & q_2 & r_2 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0. \end{cases}$$

The eq of the common perpendicular

$$l: \begin{cases} \begin{vmatrix} x+1 & y & z-1 \\ -1 & 1 & -1 \\ -\frac{4}{3} & -\frac{1}{3} & 1 \end{vmatrix} = 0\ (\pi_1) \\ \Leftrightarrow \\ \begin{vmatrix} x-\frac{1}{3} & y-\frac{2}{3} & z \\ \frac{4}{3} & -\frac{7}{3} & 1 \\ -\frac{4}{3} & -\frac{1}{3} & 1 \end{vmatrix} = 0\ (\pi_2) \end{cases}$$

$$\Leftrightarrow \; e_1 \begin{cases} \left(1-\frac{1}{3}\right)(x+1)+\left(\frac{4}{3}+1\right)y+\left(\frac{1}{3}+\frac{4}{3}\right)(z-1)=0 \\ \quad \rightarrow 2x+7y+5z-3=0 \; (\pi_1) \\ \left(-\frac{7}{3}+\frac{1}{3}\right)\left(x-\frac{1}{3}\right)+\left(-\frac{4}{3}-\frac{4}{3}\right)\left(y-\frac{2}{3}\right)+\left(-\frac{4}{9}-\frac{28}{9}\right)z=0 \\ \quad \rightarrow 18x+24y+z-22=0 \; (\pi_2) \end{cases}$$

The equations of the common perpendicular

The distance between the straight lines d_1, d_2 can be also regarded as the height of the parallelogram constructed on the vectors $\vec{d}_1, \vec{d}_2, \vec{d}_1 \times \vec{d}_2$. Thus

$$\delta(d_1,d_2)=\frac{|(\overrightarrow{A_1A_2},\vec{d}_1,\vec{d}_2)|}{\|\vec{d}_1\times\vec{d}_2\|}. \tag{7.6}$$

Therefore we obtain

$$\delta(d_1,d_2)=\frac{\left|\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}\right|}{\sqrt{\begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}^2+\begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}^2+\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}^2}} \tag{7.7}$$

$$d(d_1,d_2)=d(d,M_1M_2)=\frac{|(\overrightarrow{A_1A_2},\vec{d_1},\vec{d_2})|}{\|\vec{d_1}\times\vec{d_2}\|}, \quad A_1(x_1,y_1,z_1) \Rightarrow A_1(-1,0,1), \quad A_2(x_2,y_2,z_2) \Rightarrow A_2\left(\frac{1}{3},\frac{2}{3},0\right)$$

$$(\overrightarrow{A_1A_2},\vec{d_1},\vec{d_2})=\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}=\begin{vmatrix} \frac{4}{3} & \frac{2}{3} & -1 \\ -1 & 1 & -1 \\ \frac{4}{3} & -\frac{7}{3} & 1 \end{vmatrix}=\frac{1}{9}\begin{vmatrix} 4 & 2 & -3 \\ -1 & 1 & -1 \\ 4 & -7 & 3 \end{vmatrix}=$$

$$=\frac{1}{9}\begin{vmatrix} -4 & -2 & 3 \\ 1 & -1 & 1 \\ 4 & -7 & 3 \end{vmatrix}=\frac{1}{9}(12-21-8+12-28+6)=\frac{-27}{9}=-3$$

$$\|\vec{d_1}\times\vec{d_2}\|=\sqrt{p^2+q^2+r^2}=\sqrt{\frac{16}{9}+\frac{1}{9}+1}=\sqrt{\frac{26}{9}}=\frac{\sqrt{26}}{3} \Rightarrow$$

$$\Rightarrow d(d,M_1M_2)=\frac{|-3|}{\frac{\sqrt{26}}{3}}=\frac{3\cdot 3}{\sqrt{26}}=\frac{9}{\sqrt{26}}\simeq 1{,}76$$

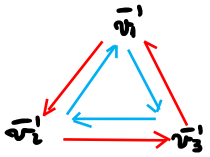
Seminar 916 - 7/04/2021

The mixed product (the triple scalar product)

$\vec{a}, \vec{b}, \vec{c} \in \mathcal{V} \Rightarrow (\vec{a}, \vec{b}, \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

If the reference system $(O, [\vec{i}, \vec{j}, \vec{k}])$ is orthonormal and direct, then:

$\vec{v_1}(a_1, b_1, c_1), \vec{v_2}(a_2, b_2, c_2), \vec{v_3}(a_3, b_3, c_3) \Rightarrow (\vec{v_1}, \vec{v_2}, \vec{v_3}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

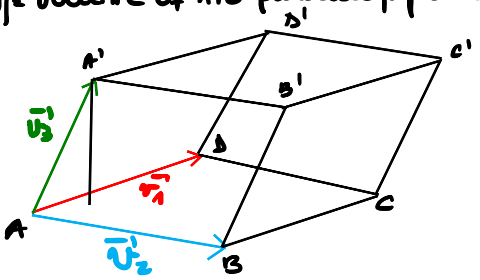


$(\vec{v_1}, \vec{v_2}, \vec{v_3}) = (\vec{v_2}, \vec{v_3}, \vec{v_1}) = (\vec{v_3}, \vec{v_1}, \vec{v_2}) =$

$= -(\vec{v_1}, \vec{v_3}, \vec{v_2}) = -(\vec{v_2}, \vec{v_1}, \vec{v_3}) = -(\vec{v_3}, \vec{v_2}, \vec{v_1})$

$|(\vec{v_1}, \vec{v_2}, \vec{v_3})|$ = volume of the parallelipiped built on $\vec{v_1}, \vec{v_2}, \vec{v_3}$

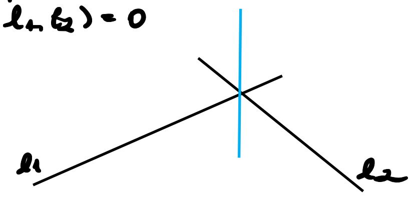
$dist(A', (ABCD)) = \dfrac{|(\vec{v_1}, \vec{v_2}, \vec{v_3})|}{\|\vec{v_1} \times \vec{v_2}\|}$



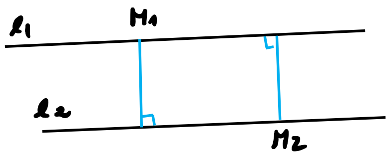
The distance between two lines (and the common perpendicular)

→ ℓ_1, ℓ_2 - lines in space

$\ell_1 \cap \ell_2 \neq \emptyset \Rightarrow dist(\ell_1, \ell_2) = 0$



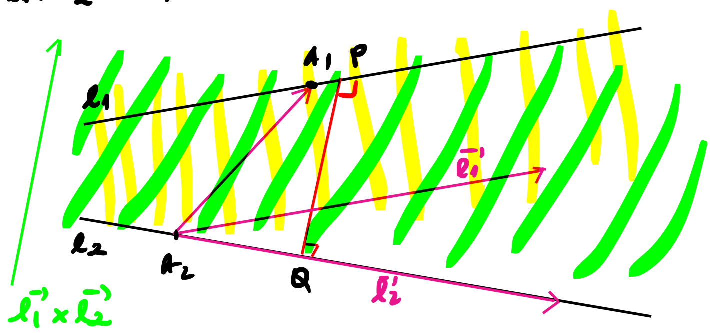
→ $\ell_1 \cap \ell_2 = \emptyset$, $\ell_1 \parallel \ell_2$



$dist(\ell_1, \ell_2) = dist(M_1, \ell_2) = dist(M_2, \ell_1), \forall M_1 \in \ell_1$
$\forall M_2 \in \ell_2$

common perp = perp from $\forall M_1 \in \ell_1$ onto ℓ_2

→ $\ell_1 \cap \ell_2 = \emptyset$, $\ell_1 \nparallel \ell_2$, ℓ_1 and ℓ_2 are skew (noncoplanar), $A_1 \in \ell_1, A_2 \in \ell_2$



π_1: plane that contains ℓ_1 and is parallel to $\vec{\ell_1} \times \vec{\ell_2}$

π_2: plane that contains ℓ_2 and is parallel to $\vec{\ell_1} \times \vec{\ell_2}$

⇒ the common perpendicular is: $\ell_1 = \pi_1 \cap \pi_2$

$dist(\ell_1, \ell_2) = \dfrac{Vol(\overrightarrow{A_1A_2}, \vec{\ell_1}, \vec{\ell_2})}{Area(\vec{\ell_1}, \vec{\ell_2})} = \dfrac{|(\overrightarrow{A_1A_2}, \vec{\ell_1}, \vec{\ell_2})|}{\|\vec{\ell_1} \times \vec{\ell_2}\|}$

Checking for coplanarity

l_1, l_2 - lines, $A_1 \in l_1$, $A_2 \in l_2$

l_1, l_2 coplanar $\Leftrightarrow$ $\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}$ are linearly dependent $\Leftrightarrow$ $(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}) = 0$ $\Leftrightarrow$

$\Leftrightarrow$ the volume of the parallelipiped built on $\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}$ is 0