2 Week 2: Straight lines and planes

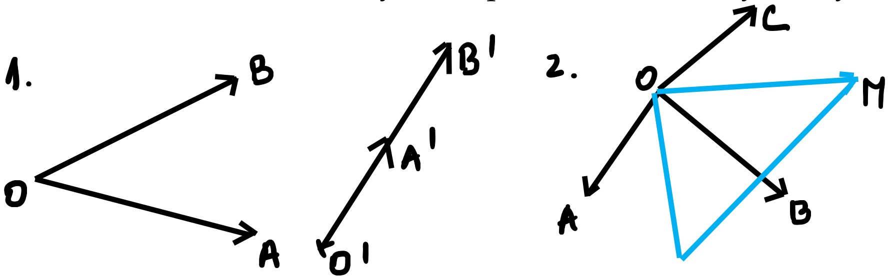
2.1 Linear dependence and linear independence of vectors

Definition 2.1. 1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *collinear* if the points O, A, B are collinear. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *noncollinear*.

2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are said to be *coplanar* if the points O, A, B, C are coplanar. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are *noncoplanar*.

Remark 2.1. 1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are linearly (in)dependent if and only if they are (non)collinear.

2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are linearly (in)dependent if and only if they are (non)coplanar.



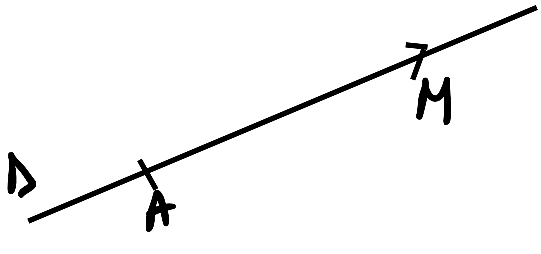
Proposition 2.1. *The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ form a basis of $\mathcal{V}$ if and only if they are noncoplanar.*

Corollary 2.2. *The dimension of the vector space of free vectors $\mathcal{V}$ is three.*

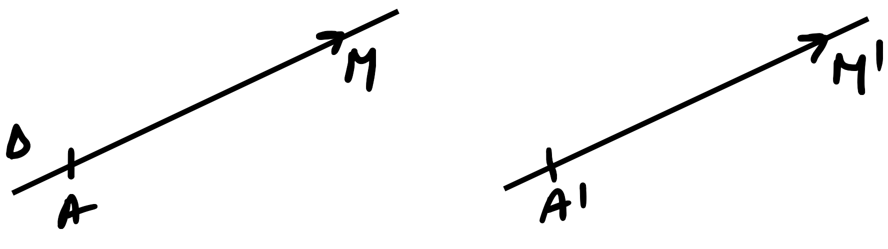
Proposition 2.3. *Let Δ be a straight line and let $A \in \Delta$ be a given point. The set*

$$\overrightarrow{\Delta} = \{\overrightarrow{AM} \mid M \in \Delta\}$$

is an one dimensional subspace of $\mathcal{V}$. It is independent on the choice of $A \in \Delta$ and is called the director subspace *of Δ or the* direction *of Δ.*



Remark 2.2. The straight lines Δ, Δ' are parallel if and only if $\overrightarrow{\Delta} = \overrightarrow{\Delta}'$

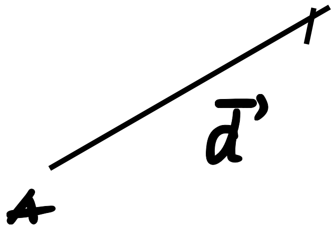


Definition 2.2. *We call* director vector *of the straigh line Δ every nonzero vector $\overrightarrow{d} \in \overrightarrow{\Delta}$.*

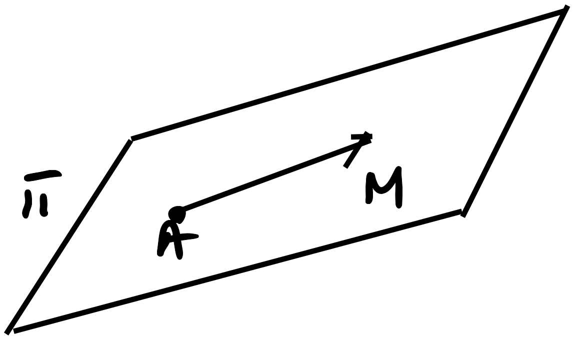
If $\overrightarrow{d} \in \mathcal{V}$ is a nonzero vector and $A \in \mathcal{P}$ is a given point, then there exits a unique straight line which passes through A and has the direction $\langle \overrightarrow{d} \rangle$. This stright line is

$$\Delta = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d} \rangle\}.$$

Δ is called the straight line which passes through O and is parallel to the vector $\overrightarrow{d}$.



Proposition 2.4. *Let π be a plane and let $A \in \pi$ be a given point. The set $\overrightarrow{\pi} = \{\overrightarrow{AM} \in \mathcal{V} \mid M \in \pi\}$ is a two dimensional subspace of $\mathcal{V}$. It is independent on the position of A inside π and is called the* director subspace, the director plane *or the* direction *of the plane π.*

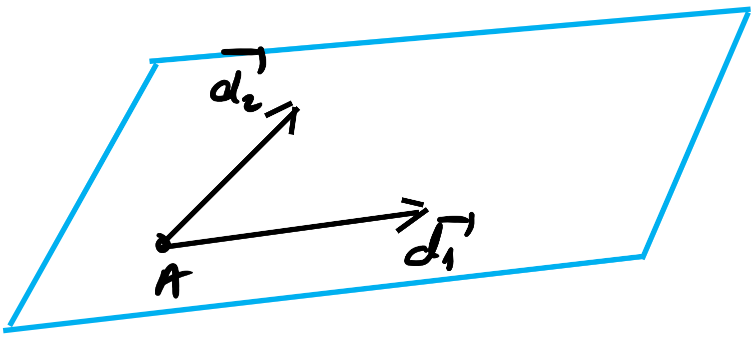


Remark 2.3. • *The planes π, π' are parallel if and only if $\overrightarrow{\pi} = \overrightarrow{\pi}'$.*

• If $\overrightarrow{d}_1, \overrightarrow{d}_2$ are two linearly independent vectors and $A \in \mathcal{P}$ is a fixed point, then there exists a unique plane through A whose direction is $\langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle$. This plane is

$$\pi = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle\}.$$

We say that π is the plane which passes through the point A and is parallel to the vectors $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$.



Remark 2.4. Let $\Delta \subset \mathcal{P}$ be a straight line and $\pi \subset \mathcal{P}$ be given plane.

1. If $A \in \Delta$ is a given point, then $\varphi_O(\Delta) = \overrightarrow{r}_A + \overrightarrow{\Delta}$.

2. If $B \in \Delta$ is a given point, then $\varphi_O(\pi) = \vec{r}_B + \vec{\pi}$.

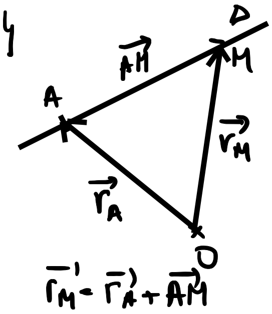
$(O \in \mathcal{P}),\ \varphi_O : \mathcal{P} \to \mathcal{V}$

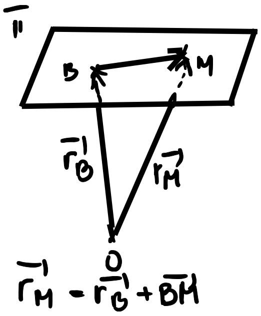
2) $\varphi_O(\Delta) = \{\varphi_O(M) \,/\, M \in \Delta\} = \{\vec{OM} \,/\, M \in \Delta\}$

$= \{\vec{OA} + \vec{AM} \,|\, M \in \Delta\}$

$= \vec{r_A} + \{\vec{AM} \,|\, M \in \Delta\} = \vec{r_A} + \vec{\Delta}$

Similarly, $\varphi_O(\pi) = \vec{r_B} + \vec{\pi}$





Generally speaking, a subset X of a vector space is called *linear variety* if either $X = \emptyset$ or there exists $a \in V$ and a vector subspace U of V, such that $X = a + U$.

$$\dim(X) = \begin{cases} -1 & \text{dacă } X = \emptyset \\ \dim(U) & \text{dacă } X = a + U, \end{cases}$$

Proposition 2.5. *The bijection φ_O transforms the straight lines and the planes of the affine space $\mathcal{P}$ into the one and two dimnensional linear varieties of the vector space $\mathcal{V}$ respectively.*

2.2 The vector ecuations of the straight lines and planes

Proposition 2.6. *Let Δ be a straight line, let π be a plane, $\{\vec{d}\}$ be a basis of $\vec{\Delta}$ and let $[\vec{d}_1, \vec{d}_2]$ be an ordered basis of $\vec{\pi}$.*

1. *The points $M \in \Delta$ are characterized by the* vector equation of Δ

$$\vec{r}_M = \vec{r}_A + \lambda \vec{d},\ \lambda \in \mathbb{R} \tag{2.1}$$

where $A \in \Delta$ is a given point.

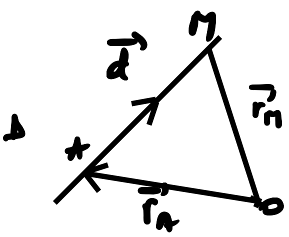
2. *The points $M \in \pi$ are characterized by the* vector equation of π

$$\vec{r}_M = \vec{r}_A + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}, \tag{2.2}$$

where $A \in \pi$ is a given point.

PROOF. $\varphi_O(\Delta) = \vec{r_A} + \vec{\Delta}$ $\qquad \vec{\Delta} = \langle \vec{d} \rangle$

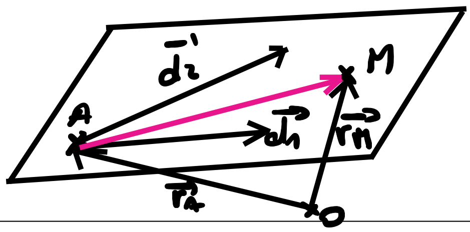
$M \in \Delta \Leftrightarrow \varphi_O(M) \in \vec{r_A} + \vec{\Delta} \Leftrightarrow \vec{r_M} \in \vec{r_A} + \langle \vec{d} \rangle \Leftrightarrow \exists\, \lambda \in \mathbb{R}$ s.t. $\underline{\vec{r_M} = \vec{r_A} + \lambda \vec{d}}$



$\varphi_O(\pi) = \vec{r_A} + \vec{\pi},\ A \in \pi$

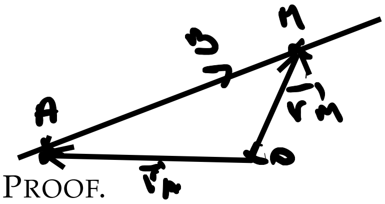
$M \in \pi \Leftrightarrow \varphi_O(M) \in \varphi_O(\pi) = \vec{r_A} + \vec{\pi} \Leftrightarrow \vec{r_M} \in \vec{r_A} + \langle \vec{d_1}, \vec{d_2} \rangle \Leftrightarrow$

$\Leftrightarrow \exists\, \lambda_1, \lambda_2 \in \mathbb{R}$ s.t. $\underline{\vec{r_M} = \vec{r_A} + \lambda_1 \vec{d_1} + \lambda_2 \vec{d_2}}$



□

Corollary 2.7. *If $A, B \in \mathcal{P}$ are different points, then the vector equation of the line AB is*



$$\vec{r}_M = (1-\lambda)\,\vec{r}_A + \lambda\,\vec{r}_B,\ \lambda \in \mathbb{R}. \tag{2.3}$$

PROOF.

$\vec{r_M} = \vec{r_A} + \lambda \vec{d}$

We can choose $\vec{d} = \vec{AB} = \vec{OB} - \vec{OA} = \vec{r_B} - \vec{r_A}$

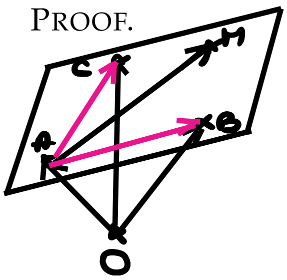
$\vec{r_M} = \vec{r_A} + \lambda(\vec{r_B} - \vec{r_A}) \Leftrightarrow \vec{r_M} = \vec{r_A} + \lambda\vec{r_B} - \lambda\vec{r_A} \Leftrightarrow \vec{r_M} = (1-\lambda)\vec{r_A} + \lambda\vec{r_B}$

□

Corollary 2.8. *If A, B, $C \in \mathcal{P}$ are three noncolinear points, then the vector equation of the plane (ABC) is*

$$\vec{r}_M = (1-\lambda_1-\lambda_2)\,\vec{r}_A + \lambda_1\,\vec{r}_B + \lambda_2\,\vec{r}_C,\ \lambda_1,\ \lambda_2 \in \mathbb{R}. \tag{2.4}$$

PROOF.



$\vec{r_M} = \vec{r_A} + \lambda_1\vec{d_1} + \lambda_2\vec{d_2}$ (1)

$\vec{d_1}, \vec{d_2}$ are lin. indep and $\vec{\pi} = \langle \vec{d_1}, \vec{d_2} \rangle$

We can choose $\vec{d_1} = \vec{AB} = \vec{r_B} - \vec{r_A}$

$\vec{d_2} = \vec{AC} = \vec{r_C} - \vec{r_A}$

(1) $\Leftrightarrow \vec{r_M} = (1-\lambda_1-\lambda_2)\vec{r_A} + \lambda_1\vec{r_B} + \lambda_2\vec{r_C}$

□

Example 2.1. Consider the points C' and B' on the sides AB and AC of the triangle ABC such that $\vec{AC'} = \lambda\,\vec{BC'}$, $\vec{AB'} = \mu\,\vec{CB'}$. The lines BB' and CC' meet at M. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \vec{PA}$, $\vec{r}_B = \vec{PB}$, $\vec{r}_C = \vec{PC}$ are the position vectors, with respect to P, of the vertices A, B, C respectively, show that

$$\vec{r}_M = \frac{\vec{r}_A - \lambda\,\vec{r}_B - \mu\,\vec{r}_C}{1-\lambda-\mu}. \tag{2.5}$$

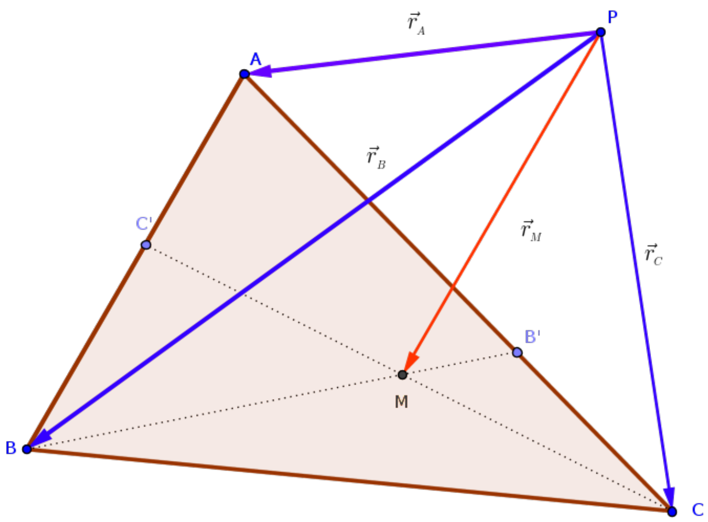
SOLUTION: $BB' \cap CC' = \{M\}$

BB': $\vec{r_X} = (1-s)\vec{r_B} + s\vec{r_{B'}}$, $s \in \mathbb{R}$

$\vec{AB'} = \mu\vec{CB'} \Leftrightarrow \vec{r_{B'}} - \vec{r_A} = \mu(\vec{r_{B'}} - \vec{r_C}) \Leftrightarrow (1-\mu)\vec{r_{B'}} = \vec{r_A} - \mu\vec{r_C}$

$\Leftrightarrow \vec{r_{B'}} = \dfrac{\vec{r_A} - \mu\vec{r_C}}{1-\mu}$

BB': $\vec{r_X} = (1-s)\vec{r_B} + s\vec{r_{B'}} = (1-s)\vec{r_B} + s \cdot \dfrac{\vec{r_A} - \mu\vec{r_C}}{1-\mu}$, $s \in \mathbb{R}$



$* \; BB': \vec{r_x} = \frac{s}{1-\mu}\vec{r_A} + (1-s)\vec{r_B} - \frac{\mu s}{1-\mu}\vec{r_C}$

$CC': \vec{r_y} = (1-t)\vec{r_C} + t\cdot\vec{r_{C'}}, \; t\in\mathbb{R}$

$\vec{AC'} = \lambda\vec{BC'} \Leftrightarrow \vec{r_{C'}} - \vec{r_A} = \lambda(\vec{r_{C'}} - \vec{r_B}) \Leftrightarrow (1-\lambda)\vec{r_{C'}} = \vec{r_A} - \lambda\vec{r_B} \Leftrightarrow$

$\vec{r_{C'}} = \frac{\vec{r_A} - \lambda\vec{r_B}}{1-\lambda} \Rightarrow \vec{r_y} = (1-t)\vec{r_C} + \frac{t\vec{r_A}}{1-\lambda} - \frac{t\lambda\vec{r_B}}{1-\lambda} \Leftrightarrow$

$\Leftrightarrow \vec{r_y} = \frac{t}{1-\lambda}\vec{r_A} - \frac{t\lambda}{1-\lambda}\vec{r_B} + (1-t)\vec{r_C}, \; t\in\mathbb{R} \; *$

$\{M\} = BB'\cap CC' \Rightarrow \vec{r_M} = \frac{s}{1-\mu}\vec{r_A} + (1-s)\vec{r_B} - \frac{\mu s}{1-\mu}\vec{r_C}, \; s,t\in\mathbb{R}$

$= \frac{t}{1-\lambda}\vec{r_A} - \frac{t\lambda}{1-\lambda}\vec{r_B} + (1-t)\vec{r_C}$

We are looking for $s,t\in\mathbb{R}$ s.t.

$$\begin{cases} \frac{s}{1-\mu} = \frac{t}{1-\lambda} \Leftrightarrow t = \frac{(1-\lambda)s}{1-\mu} \\ 1-s = -\frac{\lambda t}{1-\lambda} \Rightarrow 1-s = -\frac{\lambda s}{1-\mu} \\ \frac{-\mu s}{1-\mu} = 1-t \quad (*) \end{cases}$$

check $(*)$ $-\frac{\mu}{\cancel{1-\mu}}\cdot\frac{\cancel{1-\mu}}{1-\mu-\lambda} = 1 - \frac{1-\lambda}{1-\mu-\lambda}$

$\Rightarrow \frac{-\mu}{1-\mu-\lambda} = \frac{-\mu}{1-\mu-\lambda}$, true

$s\left(\frac{\lambda}{1-\mu} - 1\right) = -1 \Rightarrow s = \frac{1}{1-\frac{\lambda}{1-\mu}} = \frac{1-\mu}{1-\mu-\lambda} \Rightarrow t = \frac{1-\lambda}{1-\mu-\lambda} \Rightarrow$

$\Rightarrow \vec{r_M} = \frac{1}{1-\mu-\lambda}\vec{r_A} - \frac{\lambda}{1-\mu-\lambda}\vec{r_B} - \frac{\mu}{1-\mu-\lambda}\vec{r_C} \Rightarrow$ □

$\Rightarrow \vec{r_M} = \frac{\vec{r_A} - \lambda\vec{r_B} - \mu\vec{r_C}}{1-\lambda-\mu}$

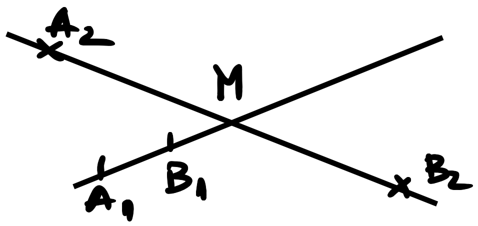
l - line in the Euclidian plane

We fix a reference system

Then $\forall M \in l \ \exists! \lambda \in \mathbb{R}$ s.t.: $\vec{r}_M = \lambda \vec{r}_A + (1-\lambda)\vec{r}_B$

If $M \in [AB]$ and $\frac{AM}{MB} = \alpha \Rightarrow \vec{r}_M = \frac{\alpha}{\alpha+1}\vec{r}_B + \frac{1}{\alpha+1}\vec{r}_A$

Let $\{M\} = l_1 \cap l_2$, $A_1, B_1 \in l_1$, $A_2, B_2 \in l_2$



Template for proofs

Step 1: We write the fact that $M \in l_1$ and $M \in l_2$ by using the vector eq. $\Rightarrow \exists\, \lambda, \mu \in \mathbb{R}$: $\vec{r}_M = \lambda \vec{r}_{A_1} + (1-\lambda)\vec{r}_{B_1}$ (1)

$\vec{r}_M = \mu \vec{r}_{A_2} + (1-\mu)\vec{r}_{B_2}$ (2)

Step 2: We find two vectors $\vec{v}, \vec{w}$ that are always linearly independent.

Step 3: We write $\vec{r}_{A_1}, \vec{r}_{B_1}, \vec{r}_{A_2}, \vec{r}_{B_2}$ in terms of $\vec{v}, \vec{w}$.

Step 4: You have obtained from (1) and (2) that:

$\alpha(\lambda,\mu)\vec{v} + \beta(\lambda,\mu)\vec{w} = \vec{0}$

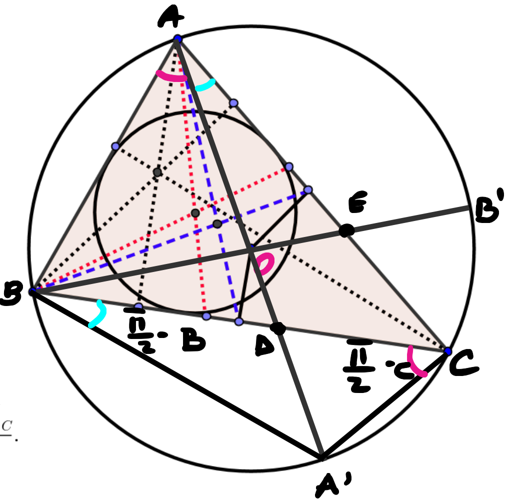
Step 5: $\vec{v}, \vec{w}$ lin. indep $\Rightarrow \begin{cases} \alpha(\lambda,\mu) = 0 \\ \beta(\lambda,\mu) = 0 \end{cases}$. Solve the system to get λ (and μ).

Step 6: Replace λ (or μ) in (1) or (2). $\Rightarrow$ Find $\vec{r}_M$ in terms of $\vec{v}$ and $\vec{w}$.

2.3 Problems

1. ([4, Problem 17, p. 5]) Consider the triangle ABC, its centroid G, its orthocenter H, its incenter I and its circumcenter O. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$, $\vec{r}_C = \overrightarrow{PC}$ are the position vectors with respect to P of the vertices A, B, C respectively, show that:

$a = BC,\ b = AC,\ c = AB$



a) $\vec{r}_G := \overrightarrow{PG} = \dfrac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}$.

b) $\vec{r}_I := \overrightarrow{PI} = \dfrac{a\,\vec{r}_A + b\,\vec{r}_B + c\,\vec{r}_C}{a+b+c}$.

c) $\vec{r}_H := \overrightarrow{PH} = \dfrac{(\tan A)\,\vec{r}_A + (\tan B)\,\vec{r}_B + (\tan C)\,\vec{r}_C}{\tan A + \tan B + \tan C}$.

d) $\vec{r}_O := \overrightarrow{PO} = \dfrac{(\sin 2A)\,\vec{r}_A + (\sin 2B)\,\vec{r}_B + (\sin 2C)\,\vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}$.

We fix a reference system.

Solution. $AA' \cap BC = \{D\}$, $BB' \cap AC = \{E\}$. Show that $\dfrac{BD}{DC} = \dfrac{\sin 2C}{\sin 2B}$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$, R - radius of the circum. circle

Using the sin th in $\triangle ABD$: $\dfrac{BD}{\sin \widehat{BAD}} = \dfrac{AD}{\sin B} = \dfrac{c}{\sin \widehat{ADB}}$ $\Rightarrow$

$\triangle ADC$: $\dfrac{CD}{\sin \widehat{CAD}} = \dfrac{b}{\sin \widehat{ADC}} = \dfrac{AD}{\sin C}$

$\Rightarrow \dfrac{BD}{CD} \cdot \dfrac{\sin \widehat{CAD}}{\sin \widehat{BAD}} = \dfrac{\frac{AD}{\sin B}}{\frac{AD}{\sin C}} = \dfrac{\sin C}{\sin B} \Rightarrow \dfrac{BD}{CD} \cdot \dfrac{\cos B}{\cos C} = \dfrac{\sin C}{\sin B}$ (1)

$\widehat{BAA'} \equiv \widehat{BCA'}$ (they corresp. to $\widehat{BA'}$) and $\widehat{CAA'} \equiv \widehat{CBA'}$ (they both corresp to $\widehat{CA'}$)

$m(\widehat{ACA'}) = \frac{\pi}{2} \Rightarrow m(\widehat{BCA'}) = \frac{\pi}{2} - m(\hat{C})$

$m(\widehat{ABA'}) = \frac{\pi}{2} \Rightarrow m(\widehat{CBA'}) = \frac{\pi}{2} - m(\hat{B})$

$\Rightarrow \sin(\widehat{CAD}) = \sin(\frac{\pi}{2} - B) = \cos B$

$\sin(\widehat{BAD}) = \sin(\frac{\pi}{2} - C) = \cos C$

(1) $\Rightarrow \dfrac{BD}{CD} = \dfrac{\cos C \cdot \sin C}{\cos B \cdot \sin B} = \dfrac{\sin 2C}{\sin 2B}$, $\dfrac{AE}{EC} = \dfrac{\sin 2C}{\sin 2A}$

d) $\sin 2A =: \alpha$, $\sin 2B =: \beta$, $\sin 2C =: \gamma$

$\vec{r}_O = \lambda \vec{r}_D + (1-\lambda)\vec{r}_A$
$= \mu \vec{r}_E + (1-\mu)\vec{r}_B$, $\lambda, \mu \in \mathbb{R}$ and $\dfrac{BD}{DC} = \dfrac{\gamma}{\beta} \Rightarrow \vec{r}_D = \dfrac{\vec{r}_B + \frac{\gamma}{\beta}\vec{r}_C}{1 + \frac{\gamma}{\beta}} = \dfrac{\beta \vec{r}_B + \gamma \vec{r}_C}{\beta + \gamma}$

$\frac{AE}{EC} = \frac{\gamma}{\alpha} \Rightarrow \vec{r_E} = \frac{\alpha \vec{r_A} + \gamma \vec{r_C}}{\alpha + \gamma}$

$\Rightarrow (1-\lambda)\vec{r_A} + \frac{\lambda\beta}{\beta+\gamma}\vec{r_B} + \frac{\lambda\gamma}{\beta+\gamma}\vec{r_C} = (1-\mu)\vec{r_B} + \frac{\mu\alpha}{\alpha+\gamma}\vec{r_A} + \frac{\mu\gamma}{\alpha+\gamma}\vec{r_C}$

$\Rightarrow \left(1-\lambda-\frac{\mu\alpha}{\alpha+\gamma}\right)\vec{r_A} + \left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)\vec{r_B} + \left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)\vec{r_C} = \vec{0}$ (1)

$\vec{AB}$ and $\vec{AC}$ are linearly indep., because if not, then ΔABC would be degenerate.

We choose $\vec{v} = \vec{AB}$ and $\vec{w} = \vec{AC}$

$\vec{r_B} = \vec{r_A} + \vec{v}$, $\vec{r_C} = \vec{r_A} + \vec{w}$

(1) $\Rightarrow \left(1-\lambda-\frac{\mu\alpha}{\alpha+\gamma}\right)\vec{r_A} + \left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)(\vec{r_A}+\vec{v}) + \left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)(\vec{r_A}+\vec{w}) = \vec{0}$

$\underbrace{\left(\cancel{1}-\lambda-\frac{\alpha\mu}{\alpha+\gamma}+\frac{\lambda\beta}{\beta+\gamma}\cancel{-1}+\mu+\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)}_{0}\vec{r_A} + \left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)\vec{v} + \left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)\vec{w} = \vec{0}$

$\left.\begin{array}{l} \Rightarrow \left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)\vec{v} + \left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)\vec{w} = \vec{0} \\ \vec{v}, \vec{w} \text{ - lin. indep} \end{array}\right\} \Rightarrow \begin{cases} \frac{\lambda\beta}{\beta+\gamma} - 1 + \mu = 0 \\ \frac{\lambda\gamma}{\beta+\gamma} - \frac{\mu\gamma}{\alpha+\gamma} = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \mu = 1 - \frac{\lambda\beta}{\beta+\gamma} \\ \frac{\lambda\gamma}{\beta+\gamma} + \frac{\lambda\beta\gamma}{(\beta+\gamma)(\alpha+\gamma)} - \frac{\gamma}{\alpha+\gamma} = 0 \end{cases}$

$(\alpha+\gamma)(\beta+\gamma)$, $\backslash\gamma$

$\Rightarrow \lambda = \frac{\frac{\gamma}{\alpha+\gamma}}{\frac{\gamma}{\beta+\gamma} + \frac{\gamma\beta}{(\beta+\gamma)(\alpha+\gamma)}} = \frac{\beta+\gamma}{(\alpha+\gamma)+\beta} = \frac{\beta+\gamma}{\beta+\gamma+\alpha}$

$\Rightarrow \vec{r_O} = \frac{\alpha}{\alpha+\beta+\gamma}\vec{r_A} + \frac{\beta}{\alpha+\beta+\gamma}\vec{r_B} + \frac{\gamma}{\alpha+\beta+\gamma}\vec{r_C}$, replace α, β, γ with $\sin 2A, \sin 2B, \sin 2C \Rightarrow$

$\Rightarrow$ conclusion

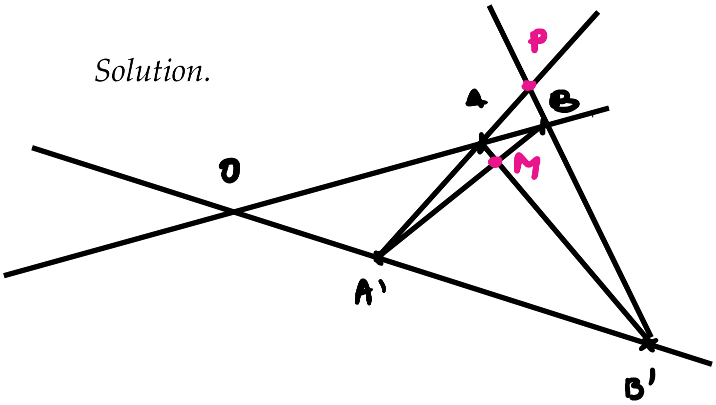
2. Consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\,\overrightarrow{OA} \;+\; n\frac{1-m}{1-mn}\,\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\,\overrightarrow{OA} \;+\; n\frac{m-1}{m-n}\,\overrightarrow{OA'}.$$

where $\{M\} = AB' \cap A'B$, $\{N\} = AA' \cap BB'$, $\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\,\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\,\overrightarrow{OA'}$.

Solution.



$$\overrightarrow{OM} = \lambda \overrightarrow{OA} + (1-\lambda)\overrightarrow{OB'} = \lambda \cdot \vec{u} + (1-\lambda) n \vec{v} \Rightarrow$$

$$= \mu \overrightarrow{OB} + (1-\mu)\overrightarrow{OA'} = \mu m \cdot \vec{u} + (1-\mu)\vec{v}$$

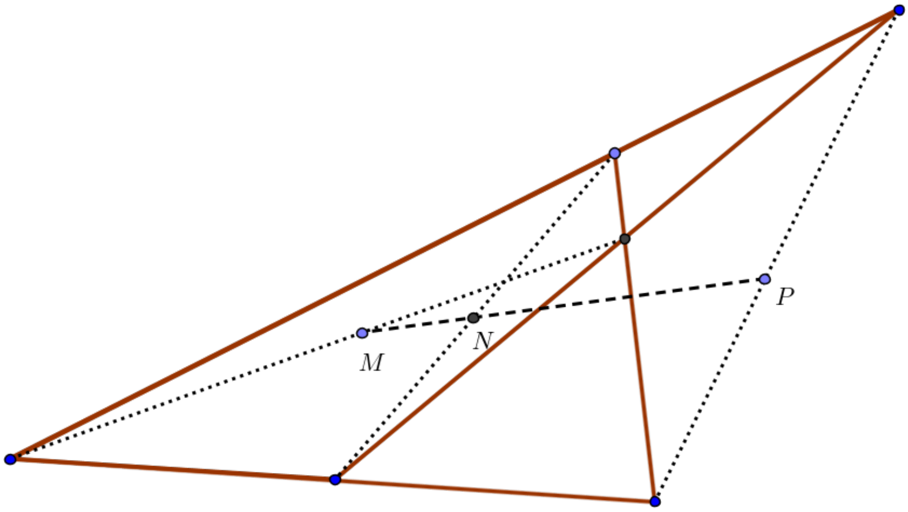
$$\Rightarrow \left.\begin{array}{l}(\lambda - \mu m)\vec{u} + ((1-\lambda)n - 1 + \mu)\vec{v} = \vec{0} \\ \vec{u}, \vec{v} \text{ - lin indep}\end{array}\right\} \Rightarrow \begin{cases} \lambda - \mu m = 0 \\ (1-\lambda)n - 1 + \mu = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \mu = \frac{n-1}{1-nm} \Rightarrow \overrightarrow{OM} = \frac{n-1}{1-nm}\cdot m \cdot \vec{u} + \left(1 - \frac{n-1}{1-nm}\right)\vec{v} =$$

$$= \frac{n-1}{1-nm}\cdot m \cdot \vec{u} - \frac{n(m+1)}{1-nm}\vec{v}$$

Same thing for $\overrightarrow{ON}$.

3. Show that the midpoints of the diagonals of a complet quadrilateral are collinear (Newton’s theorem).



Solution.

4. Let d, d' be concurrent straight lines and $A, B, C \in d$, $A', B', C' \in d'$. If the following relations $AB' \nparallel A'B$, $AC' \nparallel A'C$, $BC' \nparallel B'C$ hold, show that the points $\{M\} := AB' \cap A'B$, $\{N\} := AC' \cap A'C$, $\{P\} := BC' \cap B'C$ are collinear (Pappus' theorem).

   SOLUTION.

5. Let d, d' be two straight lines and $A, B, C \in d$, $A', B', C' \in d'$ three points on each line such that $AB' \| BA'$, $AC' \| CA'$. Show that $BC' \| CB'$ (the affine Pappus' theorem).

   SOLUTION.

6. Let us consider two triangles ABC and $A'B'C'$ such that the lines AA', BB', CC' are concurrent at a point O and $AB \nparallel A'B'$, $BC \nparallel B'C'$ and $CA \nparallel C'A'$. Show that the points $\{M\} = AB \cap A'B'$, $\{N\} = BC \cap B'C'$ and $\{P\} = CA \cap C'A'$ are collinear (Desargues).

   SOLUTION.

