# 14 Week 14

## 14.1 Transformations of the space

**Definition 14.1.** An affine transformation of the spacee is a perturbation bya atranslation of a linear transformation, i.e.

$$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ T(x,y,z) = (ax+by+cz+d, ex+fy+gz+h, kx+ly+mz+n), \tag{14.1}$$

for some constant real numbers $a,b,c,d,e,f,g,h,k,l,m,n$.

By using the matrix language, the action of the map $L$ can be written in the form

$$L(x,y,z) = [x\ y\ z]\begin{bmatrix} a & e & k \\ b & f & l \\ c & g & m \end{bmatrix} + [d\ h\ n].$$

The affine transformation $L$ can be also identified with the map $L^c : \mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}^{3\times 1}$ given by

$$\begin{aligned} L^c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) &= \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix} \\ &= [L]\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix},\ \text{where } [L] = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}. \end{aligned}$$

n

$(x,y,z)$

**Definition 14.2.** An affine transformation (14.1) is said to be *singular* if

$$\begin{vmatrix} a & b & c \\ e & f & g \\ k & l & m \end{vmatrix} = 0.$$

and non-singular otherwise.

### 14.1.1 Translations

The *translation of* $\mathbb{R}^3$ *of vector* $(h,k,l) \in \mathbb{R}^3$ is the affine transformation

$$T(h,k,l) : \mathbb{R}^3 \to \mathbb{R}^3,\ T(h,k,l)(x_1,x_2,x_3) = (x_1+h, x_2+k, x_3+l).$$

Its associated transformation is

$$T(h,k,l)^c : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ T(h,k,l)^c\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \cancel{\begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} h \\ k \\ l \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[T(h,k,l)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 + h \\ w_2 = x_2 + k \\ w_3 = x_3 + l \end{cases}.$$

$(T(h,k,l))^{-1} = T(-h,-k,-l)$

### 14.1.2 Scaling about the origin

The *scaling about the origin* by non-zero scaling factors $(s_x, s_y, s_z) \in \mathbb{R}^3$ is the affine transformation

$$S(s_x, s_y, s_z) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ \left[S(s_x, s_y, s_z)\right](x, y, z) = (s_x \cdot x, s_y \cdot y, s_z \cdot z).$$

Thus

$$\left[S(s_x, s_y, s_z)^c\right]\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \\ s_z \cdot z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

i.e.

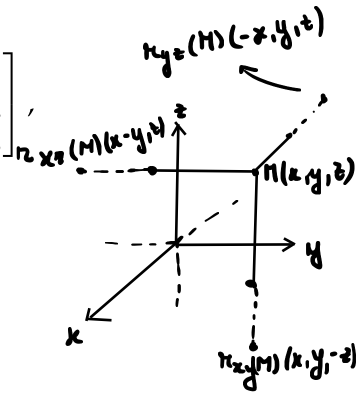
$$[S(s_x, s_y, s_z)] = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}.$$

Note that the scaling about the origin by non-zero scaling factors $(s_x, s_y, s_z) \in \mathbb{R}^3$ is non-singular (invertible) and $(S(s_x, s_y, s_z))^{-1} = S(s_x^{-1}, s_y^{-1}, s_z^{-1})$.

### 14.1.3 Reflections about planes

1. The *reflection of* $\mathbb{R}^3$ *through the xy-plane is* $r_{xy} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{xy}(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. Its associated transformation is

$$r_{xy}^c : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r_{xy}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$




which shows that its standard matrix and equations are:

$$[r_{xy}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 \\ w_2 = x_2 \\ w_3 = -x_3 \end{cases}.$$

2. The *reflection of* $\mathbb{R}^3$ *through the xz-plane is* $r_{xz} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{xz}(x_1, x_2, x_3) = (x_1, -x_2, x_3)$. Its associated transformation is

$$r_{xz} : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r_{xz}^c\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[r_{xz}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 \\ w_2 = -x_2 \\ w_3 = x_3 \end{cases}.$$

3. The *reflection of* $\mathbb{R}^3$ *through the yz-plane is* $r_{yz} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{yz}(x_1, x_2, x_3) = (-x_1, x_2, x_3)$. Its associated transformation is

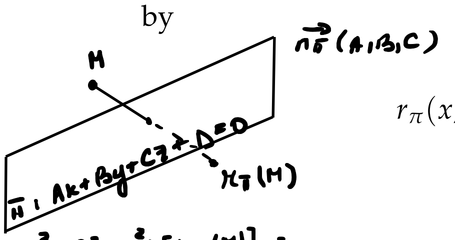
$$r_{yz}^c : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r_{yz}^c\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[r_{yz}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = -x_1 \\ w_2 = x_2 \\ w_3 = x_3 \end{cases}.$$

4. The *reflection of* $\mathbb{R}^3$ *through an arbitrary plane* $\pi : ax_1 + bx_2 + cx_3 + d = 0$ *is* $r_\pi : \mathbb{R}^3 \to \mathbb{R}^3$, given by




$$r_\pi(x,y,z) = \left(\frac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2},\right.$$
$$\frac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2},$$
$$\left.\frac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2}\right).$$

$(A^2+B^2+C^2)[r_\pi(M)]_R = \begin{pmatrix} -A^2+B^2+C^2 & -2AB & -2AC \\ -2AB & A^2-B^2+C^2 & -2BC \\ -2AC & -2BC & A^2+B^2-C^2 \end{pmatrix}[M]_R - 2D[n_\pi]_B \quad R=(O,\vec{i},\vec{j},\vec{k})$

Its associated transformation $r_\pi : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1}$ is given by

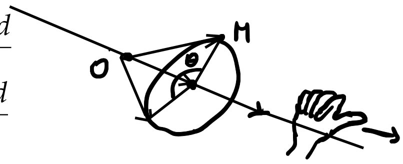
$$r_\pi^c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \dfrac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2} \\ \dfrac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2} \\ \dfrac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2} \end{bmatrix}$$
$$= \frac{1}{a^2+b^2+c^2}\left(\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} - 2d\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right).$$

which shows that its standard matrix and equations are:

$$[r_\pi] = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

and

$$\begin{cases} w_1 = \dfrac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2} \\ w_2 = \dfrac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2} \\ w_3 = \dfrac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2} \end{cases}$$



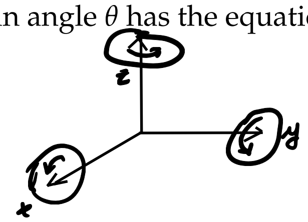

### 14.1.4 Rotations

The *rotation operator of* $\mathbb{R}^3$ *through a fixed angle* $\theta$ *about an oriented axis*, rotates about the axis of rotation each point of $\mathbb{R}^3$ in such a way that its associated vector sweeps out some portion of the cone determine by the vector itself and by a vector which gives the direction and the orientation of the considered oriented axis. The angle of the rotation is measured at the base of the cone and it is measured clockwise or counterclockwise in relation with a viewpoint along the axis looking toward the origin. As in $\mathbb{R}^2$, the positives angles generates counterclockwise rotations and negative angles generates clockwise roattions. The counterclockwise sense of rotaion can be determined by the right-hand rule: If the thumb of the right hand points the direction of the direction of the oriented axis, then the cupped fingers points in a counterclockwise direction. The rotation operators in $\mathbb{R}^3$ are linear.

For example

1. The counterclockwise rotation about the positive $x$-axis through an angle $\theta$ has the equations

$$\begin{aligned} w_1 &= x \\ w_2 &= y\cos\theta - z\sin\theta \\ w_3 &= y\sin\theta + z\cos\theta \end{aligned},$$



its standard matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

2. The counterclockwise rotation about the positive $y$-axis through an angle $\theta$ has the equations

$$\begin{array}{lcl} w_1 & = & x\cos\theta + z\sin\theta \\ w_2 & = & y \\ w_3 & = & -x\sin\theta + z\cos\theta \end{array},$$

its standard matrix is

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

3. The counterclockwise rotation about the positive $z$-axis through an angle $\theta$ has the equations

$$\begin{array}{lcl} w_1 & = & x\cos\theta - y\sin\theta \\ w_2 & = & x\sin\theta + y\cos\theta \\ w_3 & = & z \end{array},$$

its standard matrix is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

## 14.2 Homogeneous coordinates

The affine transformation

$$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ T(x,y,z) = (ax+by+cz+d, ex+fy+gz+h, kx+ly+mz+n),$$

can be written by using the matrix language and by equations:

1. (a) indentifying the vectors $(x,y,z) \in \mathbb{R}^3$ with the line matrices $[x\ y\ z] \in \mathbb{R}^{1\times 3}$ and implicitely $\mathbb{R}^3$ with $\mathbb{R}^{1\times 3}$. With this identification, the action of $L$ is given by

$$L[x\ y\ z] = [x\ y\ z] \begin{bmatrix} a & e & k \\ b & f & l \\ c & g & m \end{bmatrix} + [d\ h\ n].$$

   (b) indentifying the vectors $(x,y,z) \in \mathbb{R}^3$ with the column matrices $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^{3\times 1}$ and implicitely $\mathbb{R}^3$ with $\mathbb{R}^{3\times 1}$. We de note by $L^c : \mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}^{3\times 1}$ the associated map via this identification, and its action is given by

$$\begin{aligned} L^c\left(\underset{\substack{\uparrow \\ (x,y,z)}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}\right) &= \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix} \\ &= [L] \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}, \text{ where } [L] = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}. \end{aligned}$$

2. $$\begin{cases} x' &= ax+by+cz+d \\ y' &= ex+fy+gz+h \\ z' &= kx+ly+mz+n \end{cases} \Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}$$

Observe that the representation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

In this section we identify the points $(x,y,z) \in \mathbb{R}^3$ with the points $(x,y,z,1) \in \mathbb{R}^4$ and even with the punctured lines of $\mathbb{R}^4$, $(rx,ry,rz,r)$, $r \in \mathbb{R}^*$. Due to technical reasons we shall actually identify the points $(x,y,z) \in \mathbb{R}^3$ with the punctured lines of $\mathbb{R}^4$ represented in the form

$$\begin{bmatrix} rx \\ ry \\ rz \\ r \end{bmatrix}, \ r \in \mathbb{R}^*,$$

and the latter ones we shall call *homogeneous coordinates* of the point $(x,y,z) \in \mathbb{R}^3$. The set of homogeneous coordinates $(x,y,z,w)$ will be denoted by $\mathbb{RP}^3$ and call it the real *projective space*. The homogeneous coordinates $(x,y,z,w) \in \mathbb{RP}^3$, $w \neq 0$ and $\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, 1\right)$ represent the same element of $\mathbb{RP}^3$.

**Remark 14.1.** The projective space $\mathbb{RP}^3$ is actually the quotient set $(\mathbb{R}^4 \setminus \{0\})/\sim$, where $'\sim'$ is the following equivalence relation on $\mathbb{R}^4 \setminus \{0\}$:

$$(x,y,z,w) \sim (\alpha,\beta,\gamma,\delta) \Leftrightarrow \exists r \in \mathbb{R}^* \text{ a.î. } (x,y,z,w) = r(\alpha,\beta,\gamma,\delta).$$

$[(x,y,z,w)] = \mathbb{R}^* \cdot [(\alpha,\beta,\gamma,\delta)]$

Observe that the equivalence classes of the equivalence relation $\sim'$ are the punctured lines of $\mathbb{R}^3$ through the origin without the origin itself, i.e. the elements of the real projective plane $\mathbb{RP}^3$. By the column matrix

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

we also denote the equivalence class of $(x,y,z,w) \in \mathbb{R}^3 \setminus \{0\}$. The meaning of this notation will be understood, each time, from the context.

**Definition 14.3.** A *projective transformation* of the projective space $\mathbb{RP}^3$ is a transformation

$$L : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3, \ L \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} ax+by+cz+dw \\ ex+fy+gz+hw \\ kx+ly+mz+nw \\ px+qy+rz+sw \end{bmatrix}, \tag{14.2}$$

where $a,b,c,d,e,f,g,h,k,l,m,n,p,q,r,s \in \mathbb{R}$. Note that

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{bmatrix}$$

is called the *homogeneous transformation matrix* of $L$. Note that every nonsingular $4 \times 4$ matrix defines a projective transformation. Also, if $\lambda$ is a nonzero real scalar, then the nonsingular matrices

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{bmatrix} \text{ and } \lambda \cdot \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{bmatrix}$$

define the same projective transformation of $\mathbb{RP}^3$. Therefore, the homogeneous transformation matrix of a projective transformation is an entire class, appearing as an element, of a projective space rather than one single matrix.

Observe that a projective transformation (14.2) is well defined since

$$L\begin{bmatrix} tx \\ ty \\ tz \\ tw \end{bmatrix} = \begin{bmatrix} atx+bty+ctz+dtw \\ etx+fty+gtz+htw \\ ktx+lty+mtz+ntw \\ ptx+qty+rtz+tsw \end{bmatrix} = \begin{bmatrix} t(ax+by+cz+dw) \\ t(ex+fy+gz+hw) \\ t(kx+ly+mz+nw) \\ t(px+qy+rz+sw) \end{bmatrix}.$$

If $p = q = r = 0$ and $s \neq 0$, then the projective transformation (14.2) is said to be *affine*. The restriction of the affine transformation (14.2), which corresponds to the situation $p = q = r = 0$ and $s = 1$, to the subspace $w = 1$, has the form

$$L\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \\ 1 \end{bmatrix}, \tag{14.3}$$

i.e.

$$\begin{cases} x' &= ax+by+cz+d \\ y' &= ex+fy+gz+h \\ z' &= kx+ly+mz+n. \end{cases} \tag{14.4}$$

**Remark 14.2.** If $L_1, L_2 : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3$ are two projective applications, then their product (concatenation) transformation $L_1 \circ L_2$ is also a projective transformation and its homogeneous transformation matrix is the product of the homogeneous transformation matrices of $L_1$ and $L_2$ .

Indeed, if

$$L_1\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ e_1 & f_1 & g_1 & h_1 \\ k_1 & l_1 & m_1 & n_1 \\ p_1 & q_1 & r_1 & s_1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

and

$$L_2\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ e_2 & f_2 & g_2 & h_2 \\ k_2 & l_2 & m_2 & n_2 \\ p_2 & q_2 & r_2 & s_2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

then

$$(L_1 \circ L_2)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \left(\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ e_1 & f_1 & g_1 & h_1 \\ k_1 & l_1 & m_1 & n_1 \\ p_1 & q_1 & r_1 & s_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ e_2 & f_2 & g_2 & h_2 \\ k_2 & l_2 & m_2 & n_2 \\ p_2 & q_2 & r_2 & s_2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

**Remark 14.3.** If $L_1, L_2 : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3$ are two affine applications, then their product $L_1 \circ L_2$ is also an affine transformation.

## 14.3 Transformations of the space in homogeneous coordinates

### 14.3.1 Translations

The homogeneous transformation matrix of the translation

$$T(h,k,l) : \mathbb{R}^3 \to \mathbb{R}^3,\ T(h,k,l)(x_1,x_2,x_3) = (x_1+h, x_2+k, x_3+l)$$

is

$$\begin{bmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Scaling about the point $Q_0(x_0, y_0, z_0)$

$$T(x_0,y_0,z_0) \circ S(s_x,s_y,s_z) \circ T(-x_0,-y_0,-z_0)$$

$$\begin{pmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} s_x & 0 & 0 & x_0 \\ 0 & s_y & 0 & y_0 \\ 0 & 0 & s_z & z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} s_x & 0 & 0 & (1-s_x)x_0 \\ 0 & s_y & 0 & (1-s_y)y_0 \\ 0 & 0 & s_z & (1-s_z)z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

### 14.3.2 Scaling about the origin

The homogeneous transformation matrix of the scaling

$$S(s_x,s_y,s_z) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ \left[S(s_x,s_y,s_z)\right](x,y,z) = (s_x \cdot x, s_y \cdot y, s_z \cdot z)$$

is

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

### 14.3.3 Reflections about planes

1. The homogeneous transformation matrix of the reflection

$$r_{xy} : \mathbb{R}^3 \to \mathbb{R}^3,\ r_{xy}(x_1,x_2,x_3) = (x_1,x_2,-x_3)$$

of $\mathbb{R}^3$ through the $xy$-plane is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. The homogeneous transformation matrix of the reflection

$$r_{yz} : \mathbb{R}^3 \to \mathbb{R}^3,\ r_{yz}(x_1,x_2,x_3) = (-x_1,x_2,x_3)$$

of $\mathbb{R}^3$ through the $yz$-plane is

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. The homogeneous transformation matrix of the reflection

$$r_{xz} : \mathbb{R}^3 \to \mathbb{R}^3,\ r_{yz}(x_1,x_2,x_3) = (x_1,-x_2,x_3)$$

of $\mathbb{R}^3$ through the $xz$-plane is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. The homogeneous transformation matrix of the reflection $r_\pi : \mathbb{R}^3 \to \mathbb{R}^3$

$$r_\pi(x,y,z) = \left(\frac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2}, \frac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2}, \frac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2}\right).$$

through an arbitrary plane $\pi : ax_1 + bx_2 + cx_3 + d = 0$ is

$$\begin{bmatrix} \frac{-a^2+b^2+c^2}{a^2+b^2+c^2} & \frac{-2ab}{a^2+b^2+c^2} & \frac{-2ac}{a^2+b^2+c^2} & \frac{-2ad}{a^2+b^2+c^2} \\ \frac{-2ab}{a^2+b^2+c^2} & \frac{a^2-b^2+c^2}{a^2+b^2+c^2} & \frac{-2bc}{a^2+b^2+c^2} & \frac{-2bd}{a^2+b^2+c^2} \\ \frac{-2ac}{a^2+b^2+c^2} & \frac{-2bc}{a^2+b^2+c^2} & \frac{a^2+b^2-c^2}{a^2+b^2+c^2} & \frac{-2cd}{a^2+b^2+c^2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since in homogeneous coordinates multiplication by a factor does not affect the result, the above matrix can be multiplied by a factor $a^2+b^2+c^2$ to give the homogeneous matrix of a general reflection

$$\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac & -2ad \\ -2ab & a^2-b^2+c^2 & -2bc & -2bd \\ -2ac & -2bc & a^2+b^2-c^2 & -2cd \\ 0 & 0 & 0 & a^2+b^2+c^2 \end{bmatrix}.$$

### 14.3.4 Rotations

1. The homogeneous transformation matrix of the counterclockwise rotation $\mathrm{Rot}_x(\theta_x)$ about the positive $x$-axis through an angle $\theta_x$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. The homogeneous transformation matrix of the counterclockwise rotation $\mathrm{Rot}_x(\theta_y)$ about the positive $y$-axis through an angle $\theta_y$ is

$$\begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. The homogeneous transformation matrix of the counterclockwise rotation $\mathrm{Rot}_x(\theta_z)$ about the positive $z$-axis through an angle $\theta_z$ is

$$\begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

## 14.4 Problems

1. Find the homogeneous transformation matrix of the product (concatenation)

$$T(1,1,-2) \circ Rot_y(\pi/6),$$

where $Rot_y(\pi/6)$ stands for the rotation about the positive $y$-axis through an angle $\theta$.

*Solution.*

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\frac{\pi}{6} & 0 & \sin\frac{\pi}{6} & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\frac{\pi}{6} & 0 & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 1 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

2. Find the homogeneous transformation matrix of the rotation through an angle $\theta$, of the space, about an arbitrary line.

   *Solution.* Consider two points $P(p_1, p_2, p_3)$, $Q(q_1, q_2, q_3)$ on the given line such that it becomes the line $PQ$ as well as the versor $\overrightarrow{OR} = \frac{1}{\|\overrightarrow{PQ}\|}\overrightarrow{PQ}$ whose coordinates, which are also the coordinates of the point $R$, are denoted by $(r_1, r_2, r_3)$. The required rotation is the concatenation of the following transformations:

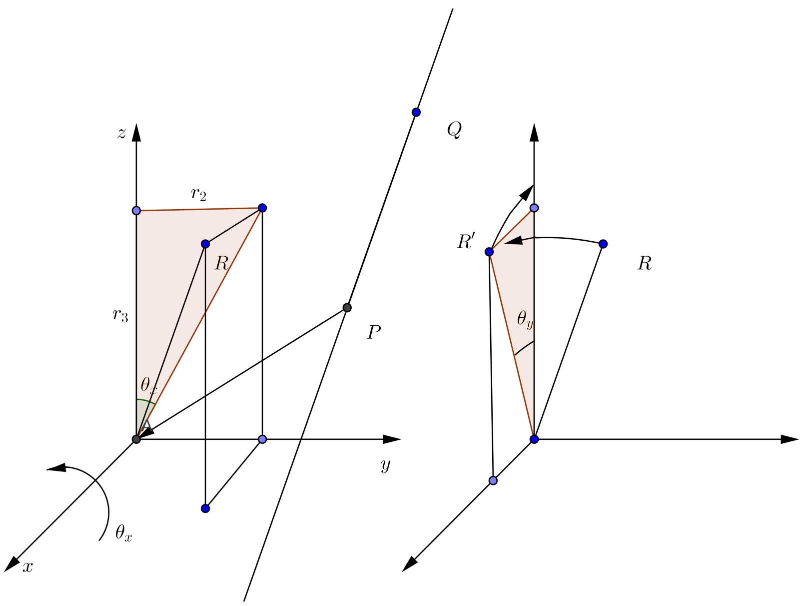
   (a) The translation $T(-p_1, -p_2, -p_3)$, which maps the point $P$ to the origin and the rotation axis to the line $OR$. If the unit vector $\overrightarrow{OR}$ is parallel to one of the coordinates axes $Ox$ (i.e. $r_2 = r_3 = 0$), $Oy$ (i.e. $r_1 = r_3 = 0$), $Oz$ (i.e. $r_1 = r_2 = 0$), then the required rotation is $T(p_1, p_2, p_3)\mathrm{Rot}_x(\theta)T(-p_1, -p_2, -p_3)$ or $T(p_1, p_2, p_3)\mathrm{Rot}_y(\theta)T(-p_1, -p_2, -p_3)$ or $T(p_1, p_2, p_3)\mathrm{Rot}_z(\theta)T(-p_1, -p_2, -p_3)$ respectively.

   (b) Assume now that $r_2 \neq 0$ and $r_3 \neq 0$. Apply the rotation about the $x$-axis through the angle $\theta_x$ given by
   $$\sin\theta_x = \frac{r_2}{\sqrt{r_2^2 + r_3^2}} \text{ and } \cos\theta_x = \frac{r_3}{\sqrt{r_2^2 + r_3^2}},$$
   which maps the point $R$ in $R'(r_1, 0, \sqrt{r_2^2 + r_3^2})$ and the line $OR$ into the $xOz$ plane.

   (c) Apply the rotation about the $y$-axis through the angle $\theta_y$ given by
   $$\sin\theta_y = r_1 \text{ and } \cos\theta_y = \frac{r_3}{\sqrt{r_2^2 + r_3^2}},$$
   which maps the line $OR'$ in the $z$-axis.

   (d) Apply the rotation through an angle $\theta$ about the $z$-axis.

   (e) Apply the inverse of the rtansformations 2a-2c in reverse order.




   Thus, the general rotation through an angle $\theta$ about the line $PQ$, determined by the points $P(p_1, p_2, p_3)$ and $Q(q_1, q_2, q_3)$ is
   $$T(p_1, p_2, p_3) \circ \mathrm{Rot}_x(-\theta_x) \circ \mathrm{Rot}_x(\theta_y) \circ \mathrm{Rot}_z(\theta) \circ \mathrm{Rot}_x(-\theta_y) \circ \mathrm{Rot}_x(\theta_x) \circ T(-p_1, -p_2, -p_3).$$

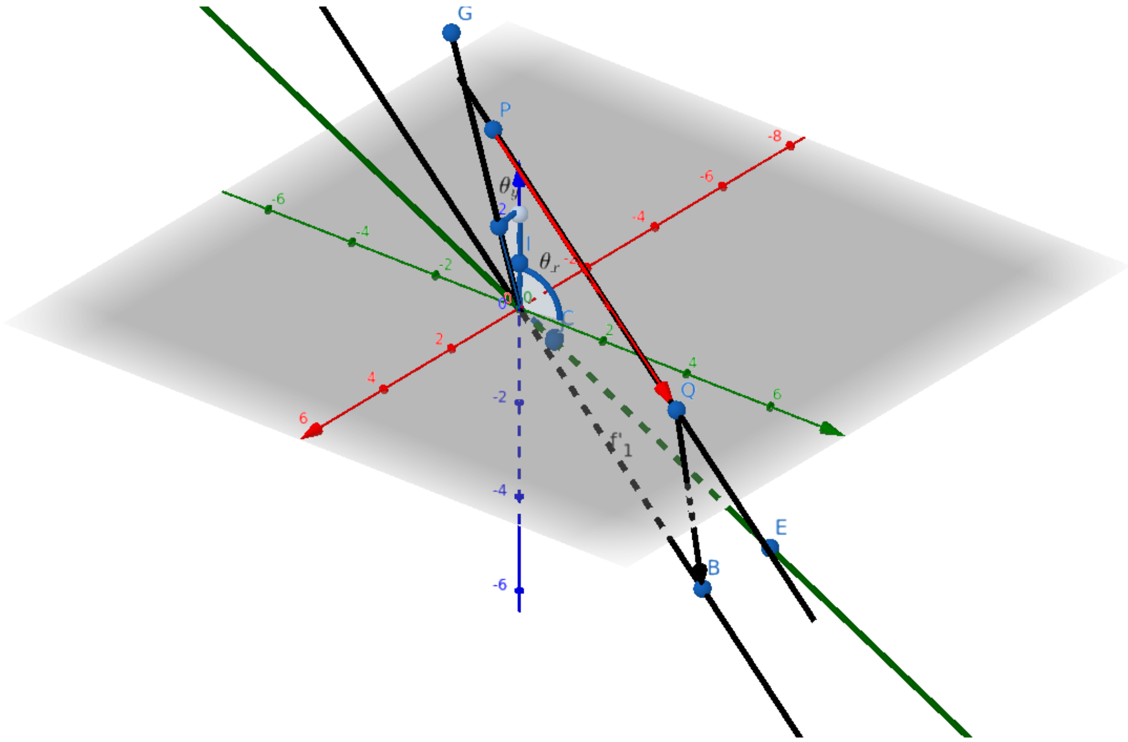
3. Find the homogeneous transformation matrix of the rotation through an angle $\theta$ about the line $PQ$, where $P(2, 1, 5)$ and $Q(4, 7, 2)$.

*Solution.* The unit vector $\overrightarrow{OR} = \frac{1}{\|\overrightarrow{PQ}\|}\overrightarrow{PQ}$, along with its extreme point $R$, has the coordinates $(2/7, 6/7. -3/7)$, as the magnitude of $\overrightarrow{PQ}$ is $\sqrt{2^2+6^2+(-3)^2}=7$, and $\sqrt{r_2^2+r_3^3}=\frac{3}{7}\sqrt{5}$. Also $\sin\theta_x = \frac{2}{\sqrt{5}}$, $\cos\theta_x = -\frac{1}{\sqrt{5}}$, $\sin\theta_y = \frac{2}{7}$, $\cos\theta_x = \frac{3}{7}\sqrt{5}$. Thus, the homogeneous transformation of the required rotation is

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{7\sqrt{5}} & 0 & \frac{2}{7} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{2}{7} & 0 & \frac{3}{7\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} \frac{3}{7\sqrt{5}} & 0 & -\frac{2}{7} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{7} & 0 & \frac{3}{7\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

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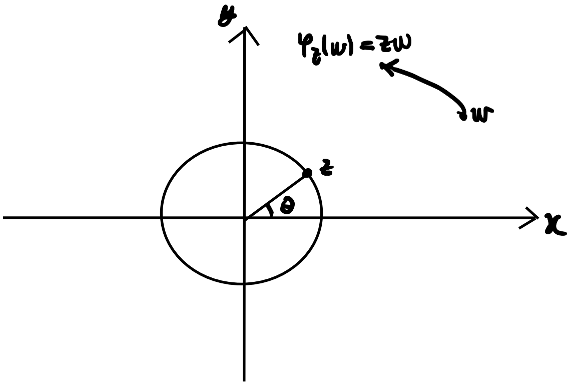
fixed
$z \in \mathbb{C}, |z| = 1, z = e^{i\theta} = \cos\theta + i\sin\theta$

$w \in \mathbb{C}, w = |w| \cdot e^{i\varphi} = |w|(\cos\varphi + i\sin\varphi)$

$z \cdot w = |w| \cdot (\cos(\varphi+\theta) + i\sin(\varphi+\theta)) = |w| e^{i(\varphi+\theta)}$

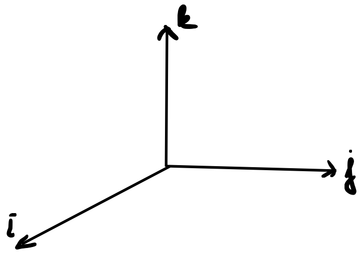
$\varphi_z : \mathbb{C} \to \mathbb{C}, \varphi_z(w) = zw$




QUATERNIONS

$\mathbb{R}^4 \ni (x,y,z,w) = x\underbrace{(1,0,0,0)}_{e_1} + y\underbrace{(0,1,0,0)}_{e_2} + z\underbrace{(0,0,1,0)}_{e_3} + w\underbrace{(0,0,0,1)}_{e_4}$

$= x\underset{1}{\underset{\sim}{e_1}} + y\underset{i}{\underset{\sim}{e_2}} + z\underset{j}{\underset{\sim}{e_3}} + w\underset{k}{\underset{\sim}{e_4}} = x \cdot 1 + y \cdot i + z \cdot j + wk$




* $ij = -ji = k$
* $jk = -kj = i$
* $ik = -ki = -j$
* $i^2 = j^2 = k^2 = -1$
* $1 \cdot 1 = 1$

$(x_1 \cdot 1 + y_1 \cdot i + z_1 j + w_1 k) + (x_2 \cdot 1 + y_2 i + z_2 j + w_2 k) = (x_1 + x_2) 1 + (y_1 + y_2) i + (z_1 + z_2) j + (w_1 + w_2) k$

$(s_1 1 + x_1 i + y_1 j + z_1 k)(s_2 1 + x_2 i + y_2 j + z_2 k) =$

$= s_1 s_2 \cdot 1 + s_1 x_2 i + s_1 y_2 j + s_1 z_2 k + x_1 s_2 i + x_1 x_2 i^2 + x_1 y_2 ij + x_1 z_2 ik +$

$+ y_1 s_2 j + y_1 x_2 ji + y_1 y_2 j^2 + y_1 z_2 jk + z_1 s_2 k + z_1 x_2 ki + z_1 y_2 kj + z_1 z_2 k^2 =$

$= (s_1 s_2 - x_1 x_2 - y_1 y_2 - z_1 z_2) \cdot 1 + s_1 x_2 i + s_1 y_2 j + s_1 z_2 k + x_1 s_2 \cdot i + x_1 y_2 k - x_1 z_2 j + y_1 s_2 j - y_1 x_2 k + y_1 z_2 i + z_1 s_2 k +$

$+ z_1 x_2 j - z_1 y_2 i = (s_1 s_2 - x_1 x_2 - y_1 y_2 - z_1 z_2) \cdot 1 + (s_1 x_2 + x_1 s_2 + y_1 z_2 - z_1 y_2) i + (s_1 y_2 - x_1 z_2 + y_1 s_2 + z_1 x_2) j +$

$+ (s_1 z_2 + x_1 y_2 - y_1 x_2 + z_1 s_2) k$

$s \cdot 1 + x i + yj + zk \rightsquigarrow (s, v) \quad v = (x, y, z)$

$(s_1, v_1) + (s_2, v_2) = (s_1 + s_2, v_1 + v_2)$

$(s_1, v_1) \cdot (s_2, v_2) = (s_1 s_2 - v_1 v_2, s_1 v_2 + s_2 v_1 + v_1 \times v_2)$

→ NOT COMMUTATIVE

$v_1 \times v_2 = -v_2 \times v_1$

PROPERTIES

1) $p + 0 = 0 + p = p, \forall p$

2) $1 \cdot p = p \cdot 1 = p, \forall p$

3) $p + q = q + p, \forall p, q$

4) $(p+q)+r = p + (q+r), \forall p, q, r$

5) $(pq)r = p(qr), \forall p, q, r$

6) $p(q+r) = pq + pr$ and $(p+q)r = pr + qr$

*) $s1 + xi + yj + zk \rightsquigarrow s + xi + yj + zk$

$q = s + xi + yj + zk$ $\quad q^{-1} = \dfrac{\bar{q}}{|q|^2}, \ |q| = (q \cdot \bar{q})^{1/2} = (s^2 + |v|^2)^{1/2}$

$\bar{q} = s - xi - yj - zk$ $\quad \underline{v} = xi + yj + zk$ $\quad q \neq 0$

Any unit quaternion $q = (s, v)$ has the form $q(\cos\theta, \sin\theta\, I)$ for some angle $\theta$ and some $v(p_1, p_2, p_3), p_1^2 + p_2^2 + p_3^2 = 1$

If $q = s + xi + yj + zk, p = w + p_1 i + p_2 j + p_3 k \Rightarrow qp = pL_q = (w, p_1, p_2, p_3) = \begin{pmatrix} s & x & y & z \\ -x & s & z & -y \\ -y & -z & s & x \\ -z & y & -x & s \end{pmatrix}$

$$pq = pR_q = (w, p_1 p_2 p_3) = \begin{pmatrix} s & x & y & z \\ -x & s & -z & y \\ -y & z & s & -x \\ -z & -y & x & s \end{pmatrix}$$

$C_q(p) = qpq^{-1} (= qp\bar{q}$, when $|q| = 1)$

$$C_q = L_q R_q^{-1} = R_q^{-1} L_q = \begin{pmatrix} s^2+x^2+y^2+z^2 & 0 & 0 & 0 \\ 0 & s^2+x^2-y^2-z^2 & 2xy+2sz & 2xz-2sy \\ 0 & 2xy-2sz & s^2-x^2+y^2-z^2 & 2yz+2sx \\ 0 & 2xz+2sy & 2yz-2sx & s^2-x^2-y^2+z^2 \end{pmatrix}$$

THE MATRIX REPRESENTATION OF A ROTATION

Th: Let $q = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2} I)$ be a unit quaternion. Then $C_q(p) = qpq^{-1} = qp\bar{q}$ yields a rotation of $p$ about the axis $I$ through the angle $\theta$. Conversely, any rotation is given by $C_q$ for some unit quaternion $q$.