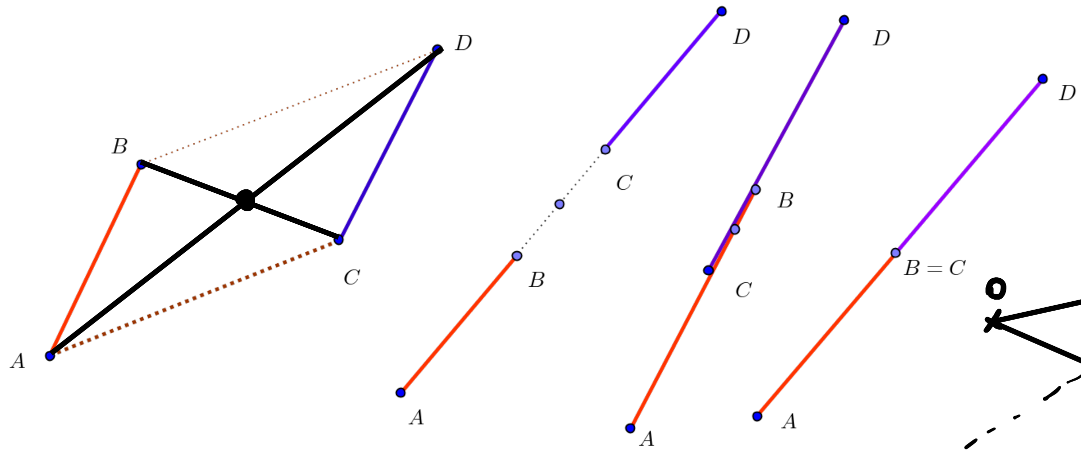
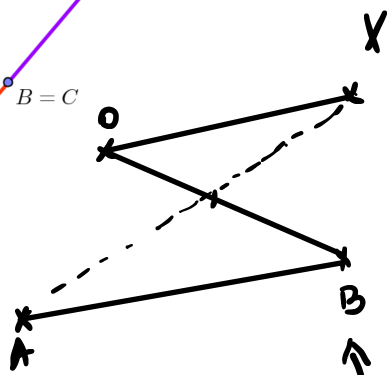
1 Week 1: Vector algebra

1.1 Free vectors

Vectors Let $\mathcal{P}$ be the three dimensional physical space in which we can talk about points, lines, planes and various relations among them. If $(A, B) \in \mathcal{P} \times \mathcal{P}$ is an ordered pair, then A is called the *original point* or the *origin* and B is called the *terminal point* or the *extremity* of (A, B).

Definition 1.1. The ordered pairs (A, B), (C, D) are said to be equipollent, written $(A, B) \sim (C, D)$, if the segments $[AD]$ and $[BC]$ have the same midpoint.





Pairs of equipollent points $(A, B) \sim (C, D)$

Remark 1.1. If the points A, B, C, $D \in \mathcal{P}$ are not collinear, then $(A, B) \sim (C, D)$ if and only if $ABDC$ is a parallelogram. In fact the length of the segments $[AB]$ and $[CD]$ is the same whenever $(A, B) \sim (C, D)$.

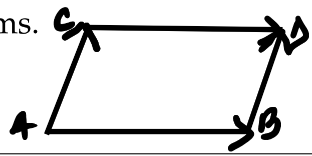
Proposition 1.1. *If (A, B) is an ordered pair and $O \in \mathcal{P}$ is a given point, then there exists a unique point X such that $(A,B) \sim (O, X)$.*

Proposition 1.2. *The equipollence relation is an equivalence relation on $\mathcal{P} \times \mathcal{P}$.*

Definition 1.2. The equivalence classes with respect to the equipollence relation are called *(free) vectors*.

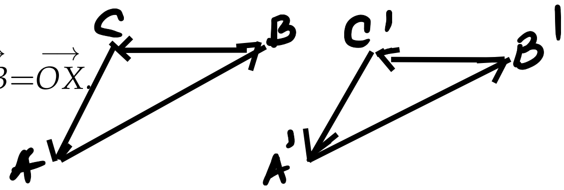
Denote by $\overrightarrow{AB}$ the equivalence class of the ordered pair (A, B), that is $\overrightarrow{AB} = \{(X, Y) \in \mathcal{P} \times \mathcal{P} \mid (X, Y) \sim (A, B)\}$ and let $\mathcal{V} = \mathcal{P} \times \mathcal{P}/_{\sim} = \{\overrightarrow{AB} \mid (A, B) \in \mathcal{P} \times \mathcal{P}\}$ be the set of (free) vectors. The *length* or the *magnitude* of the vector $\overrightarrow{AB}$, denoted by $\| \overrightarrow{AB} \|$ or by $| \overrightarrow{AB} |$, is the length of the segment $[AB]$.

Remark 1.2. If two ordered pairs (A, B) and (C, D) are equippllent, i.e. the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal, then they have the same length, the same direction and the same sense. In fact a vector is determined by these three items.



Proposition 1.3. *1. $\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \overrightarrow{AC} = \overrightarrow{BD}$.*

2. $\forall A, B, O \in \mathcal{P}$, $\exists! X \in \mathcal{P}$ *such that* $\overrightarrow{AB} = \overrightarrow{OX}$.
3. $\overrightarrow{AB} = \overrightarrow{A'B'}$, $\overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$.



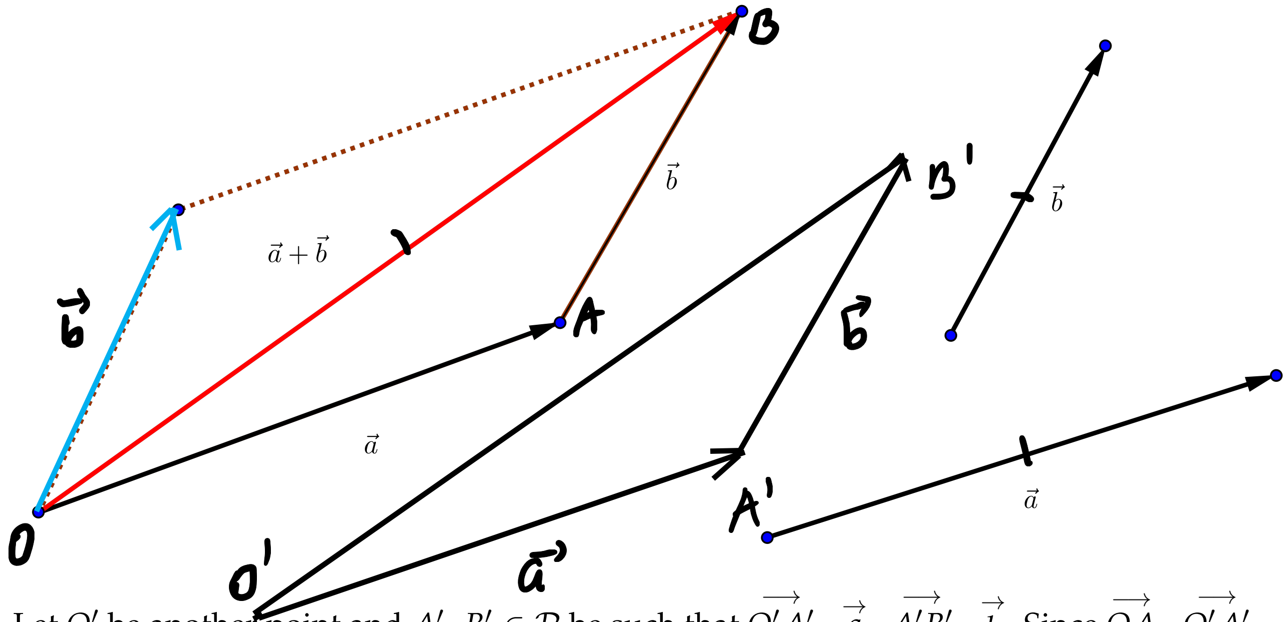
Definition 1.3. If O, $M \in \mathcal{P}$, the the vector $\overrightarrow{OM}$ is denoted by $\vec{r}_M$ and is called the *position vector* of M with respect to O.

Corollary 1.4. *The map* $\varphi_O : \mathcal{P} \to \mathcal{V}$, $\varphi_O(M) = \vec{r}_M$ *is one-to-one and onto, i.e. bijective.*

1.1.1 Operations with vectors

$\vec{a} + \vec{b} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ (def)

• **The addition of vectors** Let $\vec{a}, \vec{b} \in \mathcal{V}$ and $O \in \mathcal{P}$ be such that $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{AB}$. The vector $\overrightarrow{OB}$ is called the *sum* of the vectors $\vec{a}$ and $\vec{b}$ and is written $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$.



Let O' be another point and A', $B' \in \mathcal{P}$ be such that $\overrightarrow{O'A'} = \vec{a}$, $\overrightarrow{A'B'} = \vec{b}$. Since $\overrightarrow{OA} = \overrightarrow{O'A'}$ and $\overrightarrow{AB} = \overrightarrow{A'B'}$ it follows, according to Proposition 1.3(3), that $\overrightarrow{OB} = \overrightarrow{O'B'}$. Therefore the vector $\vec{a} + \vec{b}$ is independent on the choice of the point O.

Proposition 1.5. *The set* $\mathcal{V}$ *endowed to the binary operation* $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, $(\vec{a}, \vec{b}) \mapsto \vec{a} + \vec{b}$, *is an abelian group whose zero element is the vector* $\overrightarrow{AA} = \overrightarrow{BB} = \vec{0}$ *and the opposite of* $\overrightarrow{AB}$, *denoted by* $-\overrightarrow{AB}$, *is the vector* $\overrightarrow{BA}$. $\overrightarrow{BA} = -\overrightarrow{AB}$

In particular the addition operation is associative and the vector

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

is usually denoted by $\vec{a} + \vec{b} + \vec{c}$. Moreover the expression

$$((\cdots(\vec{a}_1 + \vec{a}_2) + \vec{a}_3 + \cdots + \vec{a}_n)\cdots), \tag{1.1}$$

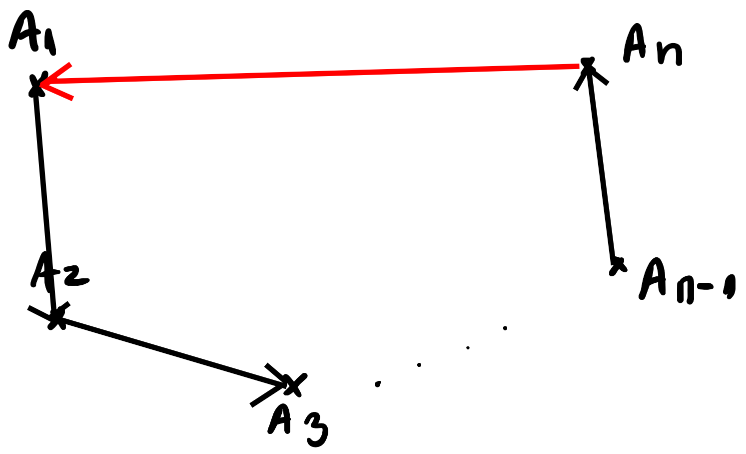
is independent of the distribution of paranthesis and it is usually denoted by

$$\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n .$$

Example 1.1. If $A_1, A_2, A_3, \ldots, A_n \in \mathcal{P}$ are some given points, then

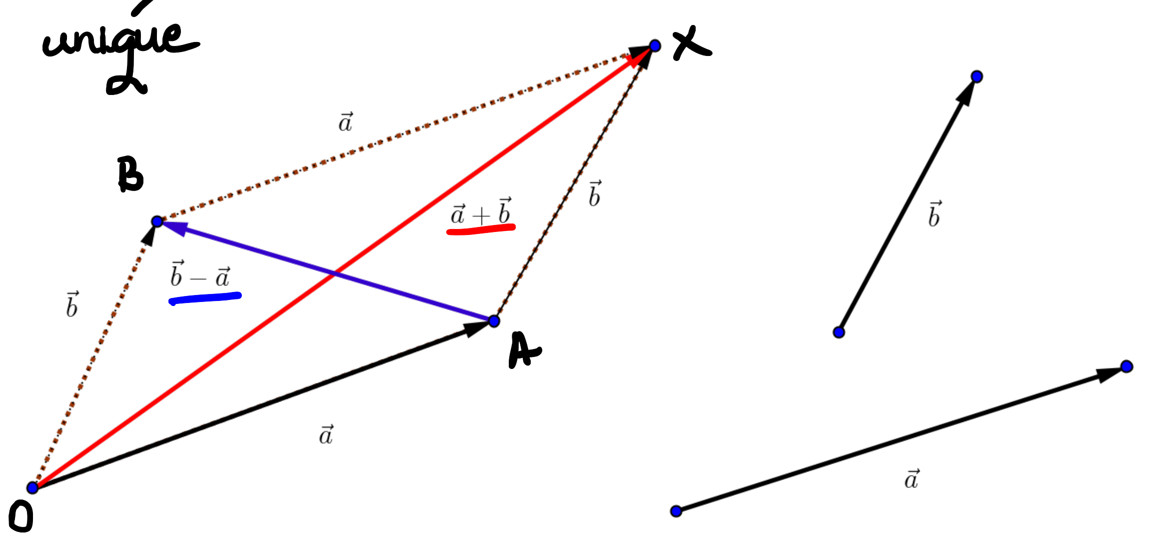
$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{A_1A_n} .$$

This shows that $\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1} = \overrightarrow{0}$, namely the sum of vectors constructed on the edges of a closed broken line is zero.



Corollary 1.6. *If $\overrightarrow{a} = \overrightarrow{OA}$, $\overrightarrow{b} = \overrightarrow{OB}$ are given vectors, there exists a unique vector $\overrightarrow{x} \in \mathcal{V}$ such that $\overrightarrow{a} + \overrightarrow{x} = \overrightarrow{b}$. In fact $\overrightarrow{x} = \overrightarrow{b} + (-\overrightarrow{a}) = \overrightarrow{AB}$ and is denoted by $\overrightarrow{b} - \overrightarrow{a}$.* $\vec{b} - \vec{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

unique



• The multiplication of vectors with scalars

Let $\alpha \in \mathbb{R}$ be a scalar and $\overrightarrow{a} = \overrightarrow{OA} \in \mathcal{V}$ be a vector. We define the vector $\alpha \cdot \overrightarrow{a}$ as follows: $\alpha \cdot \overrightarrow{a} = \overrightarrow{0}$ if $\alpha = 0$ or $\overrightarrow{a} = \overrightarrow{0}$; if $\overrightarrow{a} \neq \overrightarrow{0}$ and $\alpha > 0$, there exists a unique point on the half line $]OA$ such that $||OB|| = \alpha \cdot ||OA||$ and define $\alpha \cdot \overrightarrow{a} = \overrightarrow{OB}$; if $\alpha < 0$ we define $\alpha \cdot \overrightarrow{a} = -(|\alpha| \cdot \overrightarrow{a})$. The external binary operation

$$\mathbb{R} \times \mathcal{V} \to \mathcal{V},\ (\alpha, \overrightarrow{a}) \mapsto \alpha \cdot \overrightarrow{a}$$

is called the *multiplication of vectors with scalars*.

Proposition 1.7. *The following properties hold:*

$(v1)$ $(\alpha+\beta)\cdot \vec{a} = \alpha\cdot \vec{a} + \beta\cdot \vec{a},\ \forall\, \alpha,\beta \in \mathbb{R},\ \vec{a} \in \mathcal{V}.$

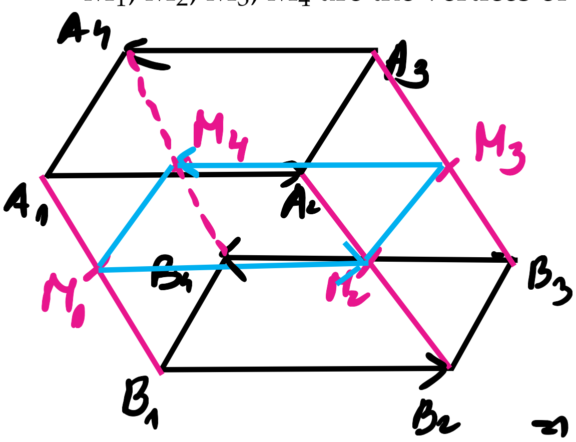
$(v2)$ $\alpha\cdot(\vec{a} + \vec{b}) = \alpha\cdot \vec{a} + \alpha\cdot \vec{b},\ \forall\, \alpha \in \mathbb{R},\ \vec{a}, \vec{b} \in \mathcal{V}.$

$(v3)$ $\alpha\cdot(\beta\cdot \vec{a}) = (\alpha\beta)\cdot \vec{a}, \forall\, \alpha,\beta \in \mathbb{R}.$

$(v4)$ $1\cdot \vec{a} = \vec{a},\ \forall\ \vec{a} \in \mathcal{V}.$

Application 1.1. Consider two parallelograms, $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ in $\mathcal{P}$, and M_1, M_2, M_3, M_4 the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$, $[A_4B_4]$ respectively. Then:

- $2\,\overrightarrow{M_1M_2} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2}$ and $2\,\overrightarrow{M_3M_4} = \overrightarrow{A_3A_4} + \overrightarrow{B_3B_4}$.
- M_1, M_2, M_3, M_4 are the vertices of a parallelogram.



$\left.\begin{array}{l}\overrightarrow{A_1A_2} + \overrightarrow{A_2M_2} + \overrightarrow{M_2M_1} + \overrightarrow{M_1A_1} = \vec{0} \\ \overrightarrow{B_1B_2} + \overrightarrow{B_2M_2} + \overrightarrow{M_2M_1} + \overrightarrow{M_1B_1} = \vec{0}\end{array}\right| \Rightarrow$

$\vec{0}$ $\vec{0}$

$\Rightarrow \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + 2\overrightarrow{M_2M_1} = \vec{0} \Rightarrow$

$\Rightarrow -2\overrightarrow{M_2M_1} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} \Rightarrow$

$\Rightarrow 2\overrightarrow{M_1M_2} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2}$

$\left.\begin{array}{l}2\overrightarrow{M_1M_2} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} \\ 2\overrightarrow{M_3M_4} = \overrightarrow{A_3A_4} + \overrightarrow{B_3B_4} \Rightarrow 2\overrightarrow{M_4M_3} = \overrightarrow{A_4A_3} + \overrightarrow{B_4B_3}\end{array}\right\} \Rightarrow$

$\Rightarrow \overrightarrow{M_1M_2} = \overrightarrow{M_4M_3} \Rightarrow M_1M_2M_3M_4$ - parallelogram

1.1.2 The vector structure on the set of vectors

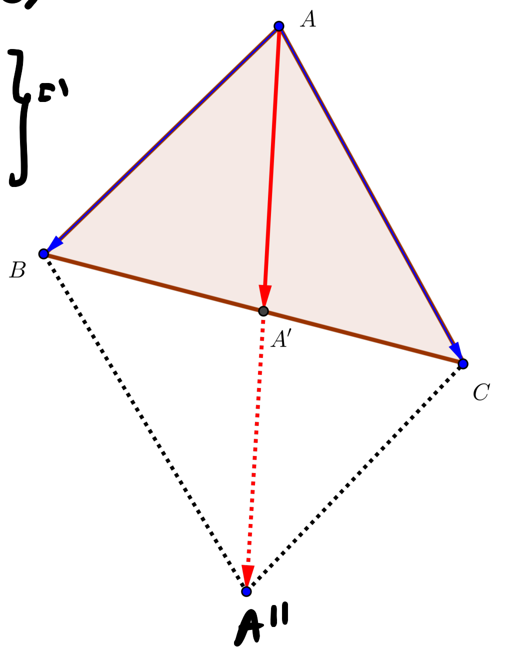
Theorem 1.8. *The set of (free) vectors endowed with the addition binary operation of vectors and the external binary operation of multiplication of vectors with scalars is a real vector space.*

Example 1.2. If A' is the midpoint of the egde $[BC]$ of the triangle ABC, then

$$\overrightarrow{AA'} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}).$$

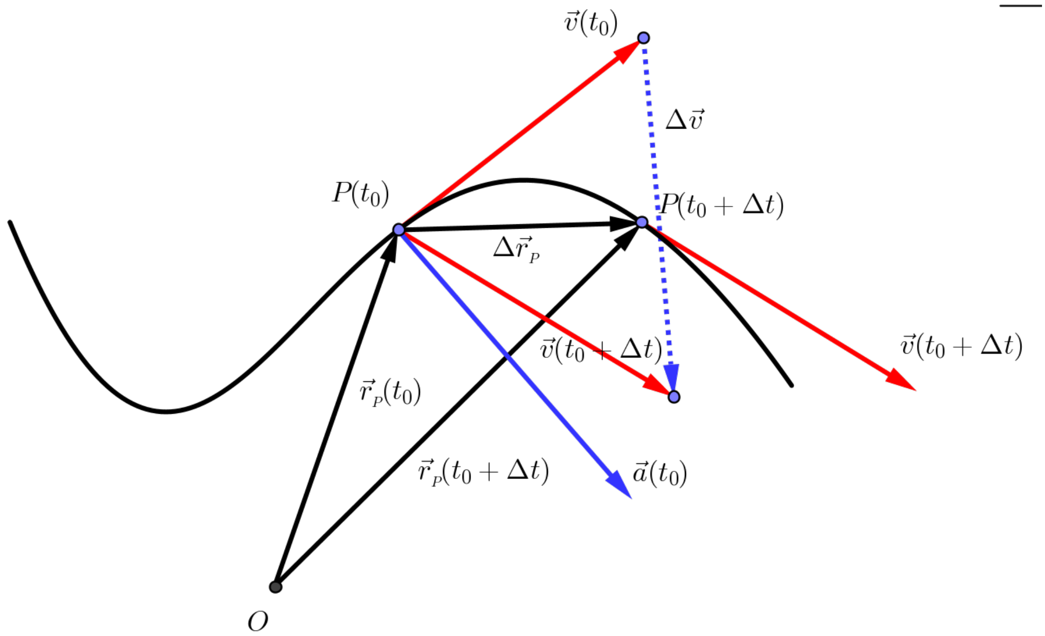
$\overrightarrow{AA'} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$

$\left.\begin{array}{l}\overrightarrow{AA''} = \overrightarrow{AB} + \overrightarrow{AC} \\ 2\overrightarrow{AA'} = \overrightarrow{AB} + \overrightarrow{AC}\end{array}\right\} \Rightarrow 2\overrightarrow{AA'} = \overrightarrow{AA''} \Rightarrow \overrightarrow{AA'} = \frac{1}{2}\overrightarrow{AA''}$



A few vector quantities:

1. The force, usually denoted by $\overrightarrow{F}$.
2. The velocity $\frac{d\overrightarrow{r}_P}{dt}$ of a moving particle P, is usually denoted by $\overrightarrow{v}_P$ or simply by $\overrightarrow{v}$.
3. The acceleration $\frac{d\overrightarrow{v}_P}{dt}$ of a moving particle P, is usually denoted by $\overrightarrow{a}_P$ or simply by $\overrightarrow{a}$.



• **Newton’s law of gravitation**, statement that any particle of matter in the universe attracts any other with a force varying directly as the product of the masses and inversely as the square of the distance between them. In symbols, the magnitude of the attractive force F is equal to G (the gravitational constant, a number the size of which depends on the system of units used and which is a universal constant) multiplied by the product of the masses (m_1 and m_2) and divided by the square of the distance R: $F = G(m_1 m_2)/R^2$. (Encyclopdia

Britannica)
• **Newton's second law** is a quantitative description of the changes that a force can produce on the motion of a body. It states that the time rate of change of the momentum of a body is equal in both magnitude and direction to the force imposed on it. The momentum of a body is equal to the product of its mass and its velocity. Momentum, like velocity, is a vector quantity, having both magnitude and direction. A force applied to a body can change the magnitude of the momentum, or its direction, or both. Newtons second law is one of the most important in all of physics. For a body whose mass m is constant, it can be written in the form $F = ma$, where F (force) and a (acceleration) are both vector quantities. If a body has a net force acting on it, it is accelerated in accordance with the equation. Conversely, if a body is not accelerated, there is no net force acting on it. (Encyclopdia Britannica)

1.2 Problems

1. Consider a tetrahedron $ABCD$. Find the the following sums of vectors:

   (a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$.

   (b) $\overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{DC}$.

   (c) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{CD}$.

   Solution.

a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

b) $\underline{\overrightarrow{AD}} + \overrightarrow{CB} + \underline{\overrightarrow{DC}} = \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$

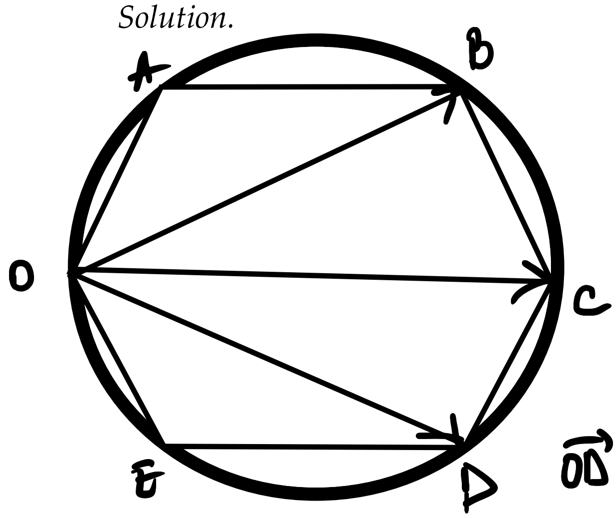
$\overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB}$

c) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{CD} = \overrightarrow{AC} + \overrightarrow{DA} + \overrightarrow{CB} = \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DA} = \overrightarrow{AD} + \overrightarrow{DA} =$

$= \overrightarrow{0}$

2. ([4, Problem 3, p. 1]) Let $OABCDE$ be a regular hexagon in which $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OE}=\vec{b}$. Express the vectors $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of the vectors $\vec{a}$ and $\vec{b}$. Show that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}+\overrightarrow{OE}=3\,\overrightarrow{OC}$.

Solution.



$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\vec{a}+\vec{a}+\vec{b}=2\vec{a}+\vec{b}$

$\overrightarrow{OC}=2\overrightarrow{OO'}=2\overrightarrow{O'C}=2\overrightarrow{O'B}+2\overrightarrow{BC}=$

$=2\vec{a}+2\vec{b}$

$\overrightarrow{OB}=\overrightarrow{OO'}+\overrightarrow{O'B}=\vec{a}+\vec{b}+\vec{a}=2\vec{a}+\vec{b}$

$\overrightarrow{OD}=\overrightarrow{OO'}+\overrightarrow{O'D}=\vec{a}+\vec{b}+\vec{b}=\vec{a}+2\vec{b}$

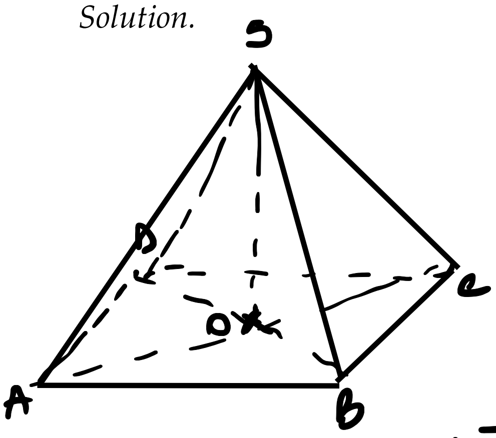
$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}+\overrightarrow{OE}=\vec{a}+2\vec{a}+\vec{b}+2\vec{a}+2\vec{b}+\vec{a}+2\vec{b}+\vec{b}=$

$=6\vec{a}+6\vec{b}=3(2\vec{a}+2\vec{b})=3\overrightarrow{OC}$

$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OE}+\overrightarrow{EB}=-\vec{a}+\vec{b}+2\vec{a}=\vec{a}+\vec{b}$

3. Consider a pyramid with the vertex at S and the basis a parallelogram $ABCD$ whose diagonals are concurrent at O. Show the equality $\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\,\overrightarrow{SO}$.

Solution.



$$\left.\begin{array}{l}\overrightarrow{SA}+\overrightarrow{AO}=\overrightarrow{SO}\\ \overrightarrow{SB}+\overrightarrow{BO}=\overrightarrow{SO}\\ \overrightarrow{SC}+\overrightarrow{CO}=\overrightarrow{SO}\\ \overrightarrow{SD}+\overrightarrow{DO}=\overrightarrow{SO}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \overrightarrow{SA}+\overrightarrow{AO}+\overrightarrow{SB}+\overrightarrow{BO}+\overrightarrow{SC}+\overrightarrow{CO}+\overrightarrow{SD}+\overrightarrow{DO}=4\overrightarrow{SO}$$

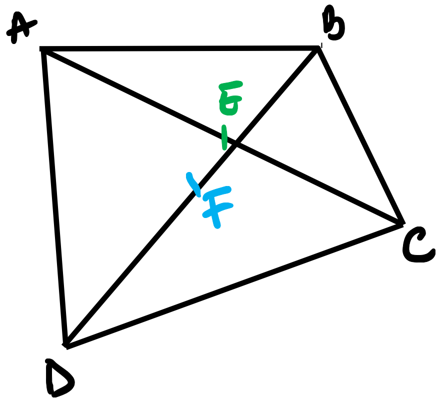
$$\Rightarrow \overrightarrow{SA}+\overrightarrow{SB}+\overrightarrow{SC}+\overrightarrow{SD}+\underbrace{\overrightarrow{AO}-\overrightarrow{OC}}_{\overrightarrow{0}}+\underbrace{\overrightarrow{BO}-\overrightarrow{OD}}_{\overrightarrow{0}}=4\overrightarrow{SO} \Rightarrow$$

$$\Rightarrow \overrightarrow{SA}+\overrightarrow{SB}+\overrightarrow{SC}+\overrightarrow{SD}=4\overrightarrow{SO}$$

4. Let E and F be the midpoints of the diagonals of a quadrilateral $ABCD$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right) = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{CB}\right).$$

Solution.



$$\left.\begin{aligned}\overrightarrow{EF} &= \overrightarrow{EA} + \overrightarrow{AB} + \overrightarrow{BF}\\ \overrightarrow{EF} &= \overrightarrow{EC} + \overrightarrow{CD} + \overrightarrow{DF}\end{aligned}\right\} \Rightarrow 2\overrightarrow{EF} = \underbrace{\overrightarrow{EA} + \overrightarrow{EC}}_{\overrightarrow{0}} + \overrightarrow{AB} + \overrightarrow{CD} + \underbrace{\overrightarrow{BF} + \overrightarrow{DF}}_{\overrightarrow{0}} \Rightarrow$$

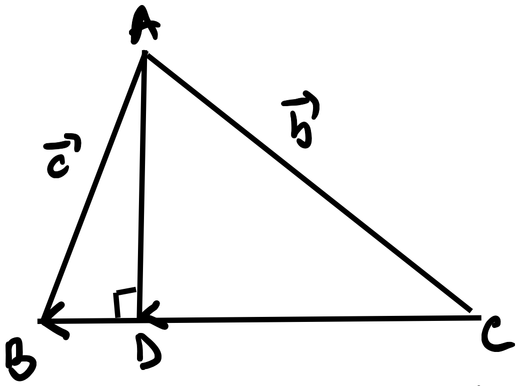
$$\Rightarrow \overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right)$$

$$\left.\begin{aligned}\overrightarrow{EF} &= \overrightarrow{EA} + \overrightarrow{AD} + \overrightarrow{DF}\\ \overrightarrow{EF} &= \overrightarrow{EC} + \overrightarrow{CB} + \overrightarrow{BF}\end{aligned}\right\} \Rightarrow 2\overrightarrow{EF} = \underbrace{\overrightarrow{EA} + \overrightarrow{EC}}_{\overrightarrow{0}} + \overrightarrow{AD} + \overrightarrow{CB} + \underbrace{\overrightarrow{BF} + \overrightarrow{DF}}_{\overrightarrow{0}} \Rightarrow$$

$$\Rightarrow \overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{CB}\right)$$

5. In a triangle ABC we consider the height AD from the vertex A ($D \in BC$). Find the decomposition of the vector AD in terms of the vectors $\vec{c} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AC}$.

   Solution.



In ΔADC, $\tan \hat{C} = \dfrac{|AD|}{|DC|}$

In ΔADB, $\tan \hat{B} = \dfrac{|AD|}{|DB|}$

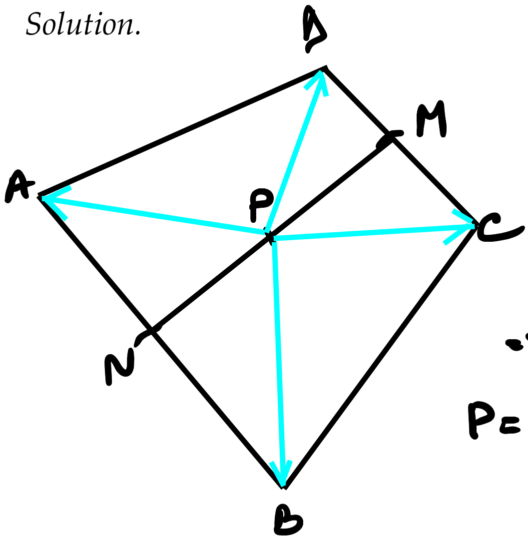
$\exists\, t \in \mathbb{R}$ s.t. $\overrightarrow{DC} = t \cdot \overrightarrow{DB}$

$\overrightarrow{AC} - \overrightarrow{AD} = t(\overrightarrow{AB} - \overrightarrow{AD}) \Rightarrow (t-1)\overrightarrow{AD} = t\overrightarrow{AB} - \overrightarrow{AC} \Rightarrow \overrightarrow{AD} = \dfrac{\overrightarrow{AC} - t\overrightarrow{AB}}{1-t}$

6. ([4, Problem 12, p. 3]) Let M, N be the midpoints of two opposite edges of a given quadrilateral $ABCD$ and P be the midpoint of $[MN]$. Show that

$$\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}+\overrightarrow{PD}=0$$

Solution.



In ΔAPB : $N = \mathrm{mid}(AB) \Rightarrow$

$\Rightarrow \overrightarrow{PN} = \frac{1}{2}(\overrightarrow{PA}+\overrightarrow{PB}) \Rightarrow 2\overrightarrow{PN} = \overrightarrow{PA}+\overrightarrow{PB}$

In ΔPDC : $M = \mathrm{mid}(DC) \Rightarrow$

$\Rightarrow \overrightarrow{PM} = \frac{1}{2}(\overrightarrow{PD}+\overrightarrow{PC}) \Rightarrow 2\overrightarrow{PM} = \overrightarrow{PC}+\overrightarrow{PD}$

$P = \mathrm{mid}(MN) \Rightarrow \overrightarrow{NP} = \overrightarrow{PM}$

$\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}+\overrightarrow{PD} = 2\overrightarrow{PN}+2\overrightarrow{PM} = 2(\overrightarrow{PN}+\overrightarrow{PM}) =$

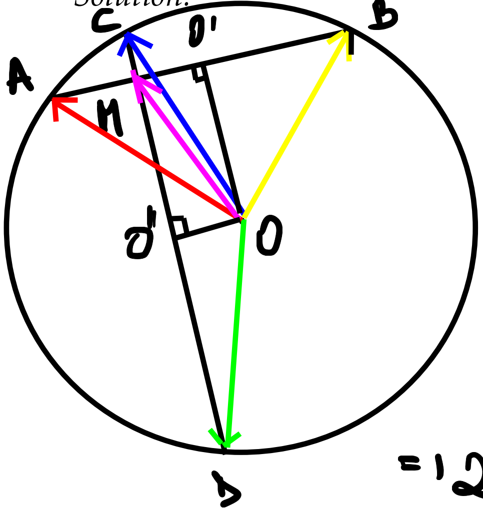
$= 2(\overrightarrow{PN}+\overrightarrow{NP}) = 2\vec{0} = \vec{0} \Rightarrow$

$\Rightarrow \overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}+\overrightarrow{PD} = 0$

7. ([4, Problem 12, p. 7]) Consider two perpendicular chords AB and CD of a given circle and $\{M\} = AB \cap CD$. Show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2\,\overrightarrow{OM}\,.$$

Solution.



In $\triangle OAB$, $2\overrightarrow{OO'} = \overrightarrow{OB} + \overrightarrow{OA}$

In $\triangle OCD$, $2\overrightarrow{OO''} = \overrightarrow{OC} + \overrightarrow{OD}$ $\Big\} \Rightarrow$

$\Rightarrow 2\overrightarrow{OO'} + 2\overrightarrow{OO''} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} \Rightarrow$

$\Rightarrow 2(\overrightarrow{OO'} + \overrightarrow{OO''}) = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$

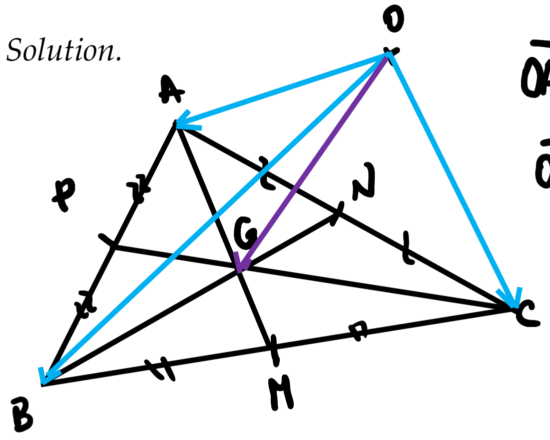
In $OO'MO''$, using parallelogram rule $\Rightarrow \overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{OO''}$ $\Big\} \Rightarrow$

$\Rightarrow 2\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$

8. ([4, Problem 13, p. 3]) If G is the centroid of a tringle ABC and O is a given point, show that

$$\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}.$$

Solution.



$$\left.\begin{aligned} \overrightarrow{OA} &= \overrightarrow{OG} + \overrightarrow{GA} \\ \overrightarrow{OB} &= \overrightarrow{OG} + \overrightarrow{GB} \\ \overrightarrow{OC} &= \overrightarrow{OG} + \overrightarrow{GC} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 3\overrightarrow{OG} + \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = 3\overrightarrow{OG} + \vec{0} = 3\overrightarrow{OG} \quad (1)$$

$$\left.\begin{aligned} \overrightarrow{GA} &= \tfrac{2}{3}\overrightarrow{MA} \\ \overrightarrow{GB} &= \tfrac{2}{3}\overrightarrow{NB} \\ \overrightarrow{GC} &= \tfrac{2}{3}\overrightarrow{PC} \end{aligned}\right\} \Rightarrow \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \frac{2}{3}\left(\overrightarrow{MA} + \overrightarrow{NB} + \overrightarrow{PC}\right) =$$

$$= -\frac{2}{3}\left(\overrightarrow{AM} + \overrightarrow{BN} + \overrightarrow{CP}\right) = -\frac{2}{3}\cdot\left(\frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{CB}\right)\right) =$$

$$= -\frac{1}{3}\cdot\left(\vec{0} + \vec{0} + \vec{0}\right) = \vec{0}$$

$$(1)\,/{:}3 \Rightarrow \overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}$$

9. ([4, Problem 14, p. 4]) Consider the triangle ABC alongside its orthocenter H, its circumcenter O and the diametrically opposed point A' of A on the latter circle. Show that:

   (a) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$.

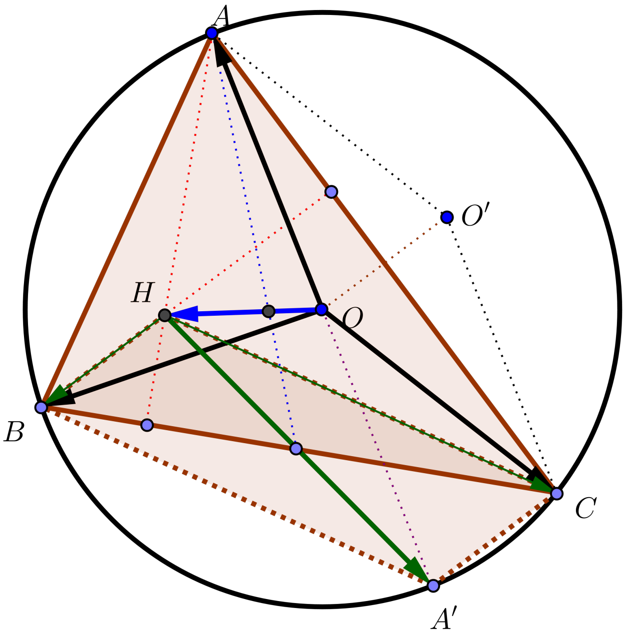
   (b) $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$.

   (c) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\,\overrightarrow{HO}$.

c) $\vec{HA}+\vec{HB}+\vec{HC} = (\vec{OA}-\vec{OH})+(\vec{OB}-\vec{OH})+$
$+(\vec{OC}-\vec{OH})$
$= \vec{OA}+\vec{OB}+\vec{OC}-3\vec{OH}$
$= \vec{OH}-3\vec{OH} = -2\vec{OH} =$
$= 2\vec{HO}$

b) $\vec{HB}+\vec{HC} = 2\vec{HO}-\vec{HA} =$
$= \vec{HO}+\vec{AH}+\vec{HO} =$
$= \vec{HO}+\vec{AO} =$
$= \vec{HO}+\vec{OA'} =$
$= \vec{HA'}$

A' and A - diametrically opposed ⇒ O = mid (AA') ⇒
⇒ $\vec{AO} = \vec{OA'}$

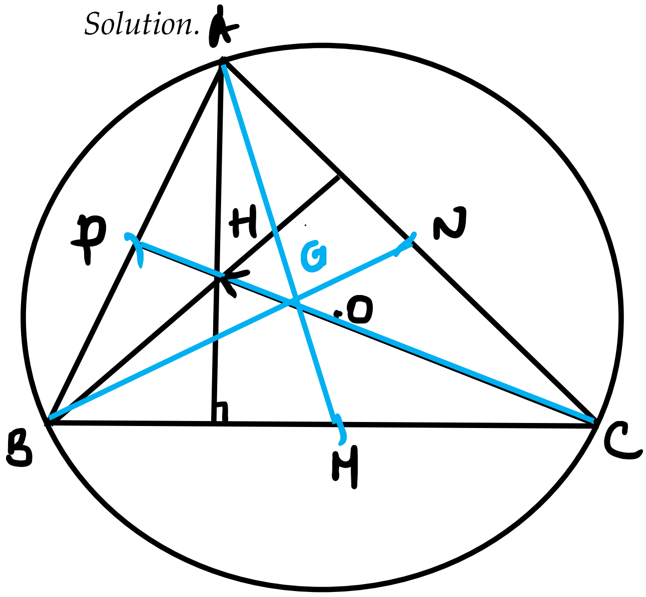


Solution.

a) Let $M \in \mathcal{P}$ s.t. $\vec{OA}+\vec{OB}+\vec{OC} = \vec{OM} \Rightarrow \vec{AM} \perp \vec{BC}$?
$\vec{OB}+\vec{OC} = \vec{OM}-\vec{OA}$
$\vec{OB}+\vec{OC} = \vec{AM}$ ⇒ $\vec{OB}+\vec{OC}$ is the diagonal of the ◊ det. by O, B, C and the symmetric of O wrt BC ⇒
$OB = OC$
⇒ $\vec{OB}+\vec{OC} \perp \vec{BC} \Rightarrow \vec{AM} \perp \vec{BC}$ } ⇒ M - orthocentre of ΔABC ⇒ M = H ⇒
Analogously, $\vec{BM} \perp \vec{AC}$
⇒ $\vec{OA}+\vec{OB}+\vec{OC} = \vec{OH}$

10. ([4, Problem 15, p. 4]) Consider the triangle ABC alongside its centroid G, its orthocenter H and its circumcenter O. Show that O, G, H are collinear and $3\overrightarrow{HG} = 2\overrightarrow{HO}$.

Solution.



$$\left.\begin{aligned} \overrightarrow{HG} &= \overrightarrow{HA} + \overrightarrow{AG} \\ \overrightarrow{HG} &= \overrightarrow{HB} + \overrightarrow{BG} \\ \overrightarrow{HG} &= \overrightarrow{HC} + \overrightarrow{CG} \end{aligned}\right\} \Rightarrow 3\overrightarrow{HG} = (\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC}) + (\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG}) \Rightarrow$$

$$\Rightarrow 3\overrightarrow{HG} = 2\overrightarrow{HO} + \overrightarrow{0} \Rightarrow 3\overrightarrow{HG} = 2\overrightarrow{HO} \quad (1)$$

$$\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = \frac{2}{3}\overrightarrow{AM} + \frac{2}{3}\overrightarrow{BN} + \frac{2}{3}\overrightarrow{CP} =$$

$$= \frac{2}{3}\left(\frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}) + \frac{1}{2}\cdot(\overrightarrow{BA} + \overrightarrow{BC}) + \frac{1}{2}(\overrightarrow{CB} + \overrightarrow{CA})\right) =$$

$$= \frac{2}{3}\cdot\frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BA} + \overrightarrow{AC} + \overrightarrow{CA} + \overrightarrow{BC} + \overrightarrow{CB}\right) = \overrightarrow{0}$$

$$(1) \Rightarrow \overrightarrow{HG} = \frac{2}{3}\overrightarrow{HO} \Rightarrow H, G, O \text{ - collinear}$$

11. ([4, Problem 27, p. 13]) Consider a tetrahedron $A_1A_2A_3A_4$ and the midpoints A_{ij} of the edges A_iA_j, $i \neq j$. Show that:

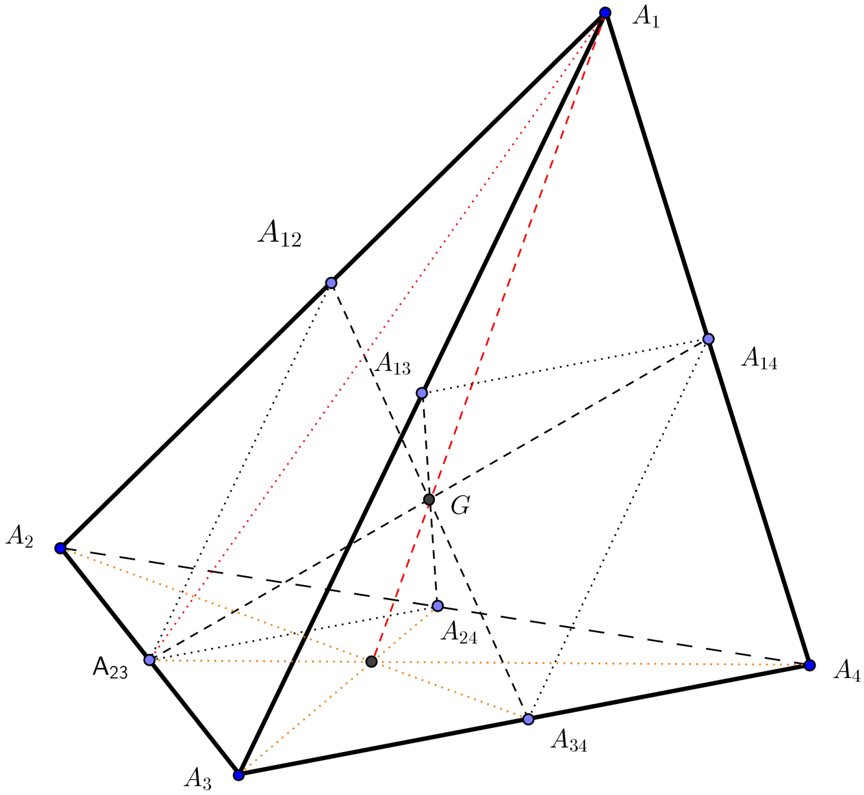
   (a) The lines $A_{12}A_{34}$, $A_{13}A_{24}$ and $A_{14}A_{23}$ are concurrent in a point G.

   (b) The medians of the tetrahedron (the lines passing through the vertices and the centroids of the opposite faces) are also concurrent at G.

   (c) Determine the ratio in which the point G divides each median.

   (d) Show that $\overrightarrow{GA_1} + \overrightarrow{GA_2} + \overrightarrow{GA_3} + \overrightarrow{GA_4} = \overrightarrow{0}$.

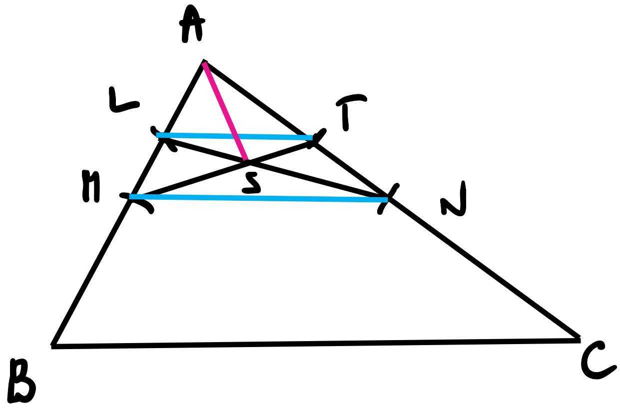
   (e) If M is an arbitrary point, show that $\overrightarrow{MA_1} + \overrightarrow{MA_2} + \overrightarrow{MA_3} + \overrightarrow{MA_4} = 4\,\overrightarrow{MG}$.



Solution.

12. In a triangle ABC consider the points M, L on the side AB and N, T on the side AC such that $3\overrightarrow{AL} = 2\overrightarrow{AM} = \overrightarrow{AB}$ and $3\overrightarrow{AT} = 2\overrightarrow{AN} = \overrightarrow{AC}$. Show that $\overrightarrow{AB} + \overrightarrow{AC} = 5\overrightarrow{AS}$, where $\{S\} = MT \cap LN$.

Solution.



$$\frac{AL}{AB} = \frac{AT}{AC} = \frac{1}{3} \Rightarrow \Delta ALT \sim \Delta ABC$$

$$\Rightarrow LT \parallel BC$$

$$\frac{AM}{AB} = \frac{AN}{AC} = \frac{1}{2} \Rightarrow \Delta AMN \sim \Delta ABC \Rightarrow$$

$$\Rightarrow MN \parallel BC$$

$$\Rightarrow LT \parallel MN$$

$$\Rightarrow \left.\begin{array}{l} \sphericalangle NMT \equiv \sphericalangle LTM \\ \sphericalangle LNM \equiv \sphericalangle TLN \end{array}\right\} \Rightarrow \Delta LTS \sim \Delta NMS \Rightarrow \frac{LS}{NS} = \frac{TS}{MS} = k$$

$$\left.\begin{array}{l} \overrightarrow{AS} = \frac{1}{1+k}\left(\overrightarrow{AL} + k\overrightarrow{AN}\right) \\ \overrightarrow{AS} = \frac{1}{1+k}\left(\overrightarrow{AT} + k\overrightarrow{AM}\right) \end{array}\right\} \Rightarrow \overrightarrow{AL} + k\overrightarrow{AN} = \overrightarrow{AT} + k\overrightarrow{AM} \Rightarrow$$

$$\Rightarrow \frac{1}{3}\overrightarrow{AB} + \frac{1}{2}k\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AC} + \frac{1}{2}k\overrightarrow{AB} \Big/ \cdot 6 \Rightarrow$$

$$\Rightarrow 2\overrightarrow{AB} + 3k\overrightarrow{AC} = 2\overrightarrow{AC} + 3k\overrightarrow{AB} \Rightarrow (3k-2)(\overrightarrow{AB} - \overrightarrow{AC}) = \overrightarrow{0} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} (3k-2)\cdot\overrightarrow{CB} = \overrightarrow{0} \\ \overrightarrow{CB} \neq \overrightarrow{0} \end{array}\right| \Rightarrow 3k-2 = 0 \Rightarrow k = \frac{2}{3} \Rightarrow$$

$$\Rightarrow \overrightarrow{AS} = \frac{1}{1+\frac{2}{3}}\left(\overrightarrow{AL} + \frac{2}{3}\overrightarrow{AN}\right) = \frac{1}{\frac{5}{3}}\left(\frac{1}{3}\overrightarrow{AB} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\cdot\overrightarrow{AC}\right) =$$

$$= \frac{3}{5}\cdot\frac{1}{3}\cdot\left(\overrightarrow{AB} + \overrightarrow{AC}\right) = \frac{1}{5}\left(\overrightarrow{AB} + \overrightarrow{AC}\right) \Rightarrow$$

$$\Rightarrow 5\overrightarrow{AS} = \overrightarrow{AB} + \overrightarrow{AC}$$

$$\vec{AS} = \vec{AL} + \vec{LS} = \vec{AT} + \vec{TS} = \vec{AM} + \vec{MS} = \vec{AN} + \vec{NS}$$

$$\vec{AS} = \vec{AL} + \alpha\,\vec{LN},\ \alpha \in \mathbb{R}$$
$$= \vec{AM} + \beta\,\vec{MT},\ \beta \in \mathbb{R}$$

$$\vec{AS} = \vec{AL} + \alpha \cdot (\vec{AN} - \vec{AL}) \Rightarrow \vec{AS} = \frac{1}{3}\vec{AB} + \alpha \cdot \left(\frac{1}{2}\vec{AC} - \frac{1}{3}\vec{AB}\right)$$
$$= \vec{AM} + \beta \cdot (\vec{AT} - \vec{AM}) \qquad = \frac{1}{2}\vec{AB} + \beta \cdot \left(\frac{1}{3}\vec{AC} - \frac{1}{2}\vec{AB}\right)$$

$$\Rightarrow \vec{AS} = \frac{1}{3}(1-\alpha)\vec{AB} + \frac{\alpha}{2}\vec{AC}$$
$$= \frac{1}{2}(1-\beta)\vec{AB} + \frac{\beta}{3}\vec{AC} \Rightarrow \begin{cases} 2(1-\alpha) = 3(1-\beta) \\ 2\beta = 3\alpha \end{cases} \Rightarrow$$

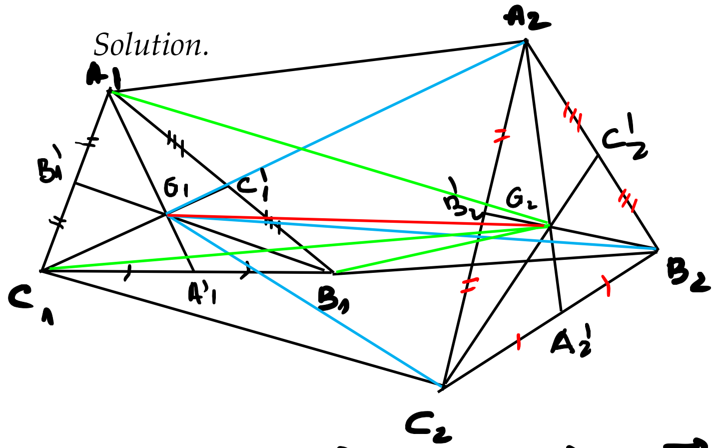
$$\Rightarrow \begin{cases} 2 - 2\alpha = 3 - 3\beta \\ \beta = \frac{3\alpha}{2} \end{cases} \Rightarrow 3\beta - 2\alpha = 1 \Rightarrow \frac{9\alpha}{2} - 2\alpha = 1 \Rightarrow 2{,}5\alpha = 1 \Rightarrow$$

$$\Rightarrow \alpha = \frac{10}{25} = \frac{2}{5} = 0{,}4 \Rightarrow \beta = 0{,}6 \Rightarrow$$

$$\Rightarrow \vec{AS} = \frac{1}{2} \cdot \overset{0{,}2}{\cancel{0{,}4}}\,\vec{AB} + 0{,}2\,\vec{AC} \;/\; 5 \Rightarrow 5\vec{AS} = \vec{AB} + \vec{AC}$$

13. Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$, not necesarily in the same plane, alongside their centroids G_1, G_2. Show that $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3\,\overrightarrow{G_1G_2}$.

Solution.



$$\left.\begin{aligned}
\overrightarrow{G_1G_2} &= \overrightarrow{G_1A_1} + \overrightarrow{A_1G_2}\\
\overrightarrow{G_1G_2} &= \overrightarrow{G_1B_1} + \overrightarrow{B_1G_2}\\
\overrightarrow{G_1G_2} &= \overrightarrow{G_1C_1} + \overrightarrow{C_1G_2}\\
\overrightarrow{A_1G_2} &= \overrightarrow{A_1A_2} + \overrightarrow{A_2G_2}\\
\overrightarrow{B_1G_2} &= \overrightarrow{B_1B_2} + \overrightarrow{B_2G_2}\\
\overrightarrow{C_1G_2} &= \overrightarrow{C_1C_2} + \overrightarrow{C_2G_2}
\end{aligned}\right| \Rightarrow$$

$$\Rightarrow 3\overrightarrow{G_1G_2} = \overrightarrow{G_1A_1} + \overrightarrow{A_1A_2} + \overrightarrow{A_2G_2} + \overrightarrow{G_1B_1} + \overrightarrow{B_1B_2} + \overrightarrow{B_2G_2} + \overrightarrow{G_1C_1} + \overrightarrow{C_1C_2} + \overrightarrow{C_2G_2}$$

$$\Rightarrow 3\overrightarrow{G_1G_2} = (\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2}) + (\overrightarrow{G_1A_1} + \overrightarrow{G_1B_1} + \overrightarrow{G_1C_1}) + (\overrightarrow{A_2G_2} + \overrightarrow{B_2G_2} + \overrightarrow{C_2G_2}) \quad (*)$$

$$\overrightarrow{G_1A_1} + \overrightarrow{G_1B_1} + \overrightarrow{G_1C_1} = -\frac{2}{3}(\overrightarrow{A_1A_1'} + \overrightarrow{B_1B_1'} + \overrightarrow{C_1C_1'}) = -\frac{2}{3}\cdot\frac{1}{2}(\overrightarrow{A_1B_1} + \overrightarrow{A_1C_1} + \overrightarrow{B_1C_1} + \overrightarrow{B_1A_1} + \overrightarrow{C_1B_1} + \overrightarrow{C_1A_1}) = -\frac{1}{3}\vec{0} = \vec{0} \quad (1)$$

$$\overrightarrow{A_2G_2} + \overrightarrow{B_2G_2} + \overrightarrow{C_2G_2} = \frac{2}{3}(\overrightarrow{A_2A_2'} + \overrightarrow{B_2B_2'} + \overrightarrow{C_2C_2'}) = \frac{2}{3}\cdot\frac{1}{2}(\overrightarrow{A_2B_2} + \overrightarrow{A_2C_2} + \overrightarrow{B_2C_2} + \overrightarrow{B_2A_2} + \overrightarrow{C_2B_2} + \overrightarrow{C_2A_2}) = \frac{1}{3}\vec{0} = \vec{0} \quad (2)$$

$$(*) \overset{(1)}{\underset{(2)}{\Rightarrow}} 3\overrightarrow{G_1G_2} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2}$$