5 Week 5: Products of vectors

5.1 The dot product

Definition 5.1. The real number

$$\vec{a}\cdot\vec{b}=\begin{cases} 0 \text{ if } \vec{a}=0 \text{ or } \vec{b}=0 \\ \|\vec{a}\|\cdot\|\vec{b}\|cos(\widehat{\vec{a},\vec{b}}) \text{ if } \vec{a}\neq 0 \text{ and } \vec{b}\neq 0 \end{cases} \tag{5.1}$$

is called the *dot product* of the vectors $\vec{a}$, $\vec{b}$.

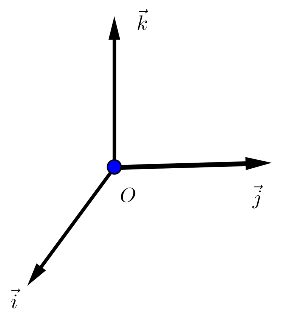
Remark 5.1.
1. $\vec{a}\perp\vec{b}\Leftrightarrow\vec{a}\cdot\vec{b}=0$.
2. $\vec{a}\cdot\vec{a}=\|\vec{a}\|\cdot\|\vec{a}\|\cos 0=\|\vec{a}\|^2$.

Proposition 5.1. *The dot product has the following properties:*

1. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a},\ \forall\ \vec{a},\vec{b}\in\mathcal{V}$.
2. $\vec{a}\cdot(\lambda\vec{b})=\lambda(\vec{a}\cdot\vec{b}),\ \forall\lambda\in\mathbb{R},\vec{a},\vec{b}\in\mathcal{V}$.
3. $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c},\ \forall\ \vec{a},\vec{b},\vec{c}\in\mathcal{V}$.
4. $\vec{a}\cdot\vec{a}\geq 0,\ \forall\ \vec{a}\in\mathcal{V}$.
5. $\vec{a}\cdot\vec{a}=0\Leftrightarrow\vec{a}=\vec{0}$.

$[\vec{i},\vec{j},\vec{k}]$

Definition 5.2. A basis of the vector space $\mathcal{V}$ is said to be *orthonormal*, if $\|\vec{i}\|=\|\vec{j}\|=\|\vec{k}\|=1$, $\vec{i}\perp\vec{j},\ \vec{j}\perp\vec{k},\ \vec{k}\perp\vec{i}$ ($\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1,\ \vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=0$). A Cartesian reference system $R=(O,\vec{i},\vec{j},\vec{k})$ is said to be *orthonormal* if the basis $[\vec{i},\vec{j},\vec{k}]$ is orthonormal.



orthonormal basis

Proposition 5.2. *Let* $[\vec{i},\vec{j},\vec{k}]$ *be an orthonormal basis and* $\vec{a},\vec{b}\in\mathcal{V}$. *If* $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$, *then*

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3 \tag{5.2}$$

Proof. Indeed,

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\cdot(b_1\vec{i}+b_2\vec{j}+b_3\vec{k}) \\ &= a_1b_1\vec{i}\cdot\vec{i}+a_1b_2\vec{i}\cdot\vec{j}+a_1b_3\vec{i}\cdot\vec{k} \\ &+a_2b_1\vec{j}\cdot\vec{i}+a_2b_2\vec{j}\cdot\vec{j}+a_2b_3\vec{j}\cdot\vec{k} \\ &+a_3b_1\vec{k}\cdot\vec{i}+a_3b_2\vec{k}\cdot\vec{j}+a_3b_3\vec{k}\cdot\vec{k} \\ &= a_1b_1+a_2b_2+a_3b_3.\end{aligned}$$

□

Remark 5.2. Let $[\vec{i},\vec{j},\vec{k}]$ be an orthonormal basis and $\vec{a},\vec{b}\in\mathcal{V}$. If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$, then

1. $\vec{a}\cdot\vec{a}=a_1^2+a_2^2+a_3^2$ and we conclude that $\|\vec{a}\|=\sqrt{\vec{a}\cdot\vec{a}}=\sqrt{a_1^2+a_2^2+a_3^2}$.

2.
$$\begin{aligned}\cos(\widehat{\vec{a},\vec{b}}) &= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}\\ &= \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}.\end{aligned}\tag{5.3}$$

   In particular
$$\begin{aligned}\cos(\widehat{\vec{a},\vec{i}}) &= \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}};\\ \cos(\widehat{\vec{a},\vec{j}}) &= \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}};\\ \cos(\widehat{\vec{a},\vec{k}}) &= \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}.\end{aligned}$$

3. $\vec{a}\perp\vec{b}\Leftrightarrow a_1b_1+a_2b_2+a_3b_3=0$

5.1.1 Applications of the dot product

◊ The two dimensional setting

- **The distance between two points** Consider two points $A(x_A,y_A),\ B(x_B,y_B)\in\pi$. The norm of the vector $\overrightarrow{AB}\ (x_B-x_A,y_B-y_A)$ is
$$\|\overrightarrow{AB}\|=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}.$$

- **The equation of the circle**
  Recall that the circle $\mathcal{C}(O,r)$ is the locus of points M in the plane such that $\text{dist}(O,M)=r\Longleftrightarrow\|\overrightarrow{OM}\|=r$. If (a,b) are the coordinates of O and (x,y) are the coordinates of M, then
$$\begin{aligned}\|\overrightarrow{OM}\|=r \Longleftrightarrow \sqrt{(x-a)^2+(y-b)^2}=r \Longleftrightarrow (x-a)^2+(y-b)^2=r^2\\ \Longleftrightarrow x^2+y^2-2ax-2by+c=0,\end{aligned}\tag{5.4}$$
  where $c=a^2+b^2-r^2$. Conversely, every equation of the form $x^2+y^2+2ex+2fy+g=0$ is the equation of the circle centered at $(-e,-f)$ and having the radius $r=\sqrt{e^2+f^2-g}$, whenever $e^2+f^2\geq g$. One can find the equation of the circle circumscribed to the triangle ABC by imposing the requirement on the coordinates (x_A,y_A), (x_B,y_B) and (x_C,y_C) of its vertices A,B,C to verify the equation $x^2+y^2+2ex+2fy+g=0$. A point $M(x,y)$ belongs to this circumcircle if and only if
$$\begin{cases}x^2+y^2+2ex+2fy+g=0\\ x_A^2+y_A^2+2ex_A+2fy_A+g=0\\ x_B^2+y_B^2+2ex_B+2fy_B+g=0\\ x_C^2+y_C^2+2ex_C+2fy_C+g=0\end{cases}\tag{5.5}$$

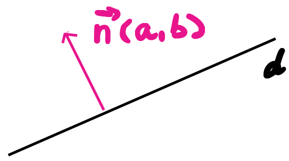
On can regard the system (5.5) as linear with the unknowns e, g, f, whose compatibility is given, via the Kronecker-Capelli theorem, by

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_A^2+y_A^2 & x_A & y_A & 1 \\ x_B^2+y_B^2 & x_B & y_B & 1 \\ x_C^2+y_C^2 & x_C & y_C & 1 \end{vmatrix} = 0,$$

which is the equation of the circumcircle of the triangle ABC.

- **The normal vector of a line** If $R = (O, b)$ is the orthonormal Cartesian reference sytem behind the equation of a line $(d)\ ax + by + c = 0$, then $\overrightarrow{n}\ (a, b)$ is a normal vector to the direction $\overrightarrow{d}$ of d. Indeed, every vector of the direction $\overrightarrow{d}$ of d has the form $\overrightarrow{PM}$, where $P(x_P, y_P)$ and $M(x, y)$ are two points on the line d. Thus, $ax_P + by_P + c = 0 = ax_M + by_M + c$, which shows that

$$a(x_M - x_P) + b(y_M - y_P) = 0,$$



namely

$$\overrightarrow{n} \cdot \overrightarrow{PM} = 0 \Longleftrightarrow \overrightarrow{n} \perp \overrightarrow{PM}.$$

- **The distance from a point to a line** If $(d)\ ax + by + c = 0$ is a line and $M(x_M, y_M) \in \pi$ a given point, then the distance from M to d is

$$\delta(M, d) = \frac{|ax_M + by_M + c|}{\sqrt{a^2 + b^2}}. \tag{5.6}$$

Indeed, $\delta(M, d) = |\delta|$, where δ is the real scalar with the property $\overrightarrow{PM} = \delta \frac{\overrightarrow{n}}{\|\overrightarrow{n}\|}$ and $P(x_P, y_P)$ is the orthogonal projection of $M(x_M, y_M)$ on d. Thus $\overrightarrow{PM}\ (x_M - x_P, y_M - y_P)$ andul

$$\begin{aligned} \delta(M, d) &= |\delta| = \left| \overrightarrow{PM} \cdot \frac{\overrightarrow{n}}{\| \overrightarrow{n} \|} \right| = \frac{|\overrightarrow{PM} \cdot \overrightarrow{n}|}{\|\overrightarrow{n}\|} = \frac{|a(x_M - x_P) + b(y_M - y_P)|}{\sqrt{a^2 + b^2}} \\ &= \frac{|ax_M + by_M - ax_P - by_P)|}{\sqrt{a^2 + b^2}} = \frac{|ax_M + by_M + c|}{\sqrt{a^2 + b^2}}. \end{aligned}$$

◊ **The three dimensional setting**

- **The distance between two points** Consider two points $A(x_A, y_A, z_A),\ B(x_B, y_B, z_B) \in \mathcal{P}$. The norm of the vector $\overrightarrow{AB}\ (x_B - x_A, y_B - y_A, z_B - z_A)$ is

$$\|\ \overrightarrow{AB}\ \| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

- **The equation of the sphere**
Recall that the sphere $\mathcal{S}(O, r)$ is the locus of points M in space such that $\text{dist}(O, M) = r \Longleftrightarrow \|\ \overrightarrow{OM}\ \| = r$. If (a, b, c) are the coordinates of O and (x, y, z) are the coordinates of M, then

$$\begin{aligned} \|\ \overrightarrow{OM}\ \| = r &\Longleftrightarrow \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r \Longleftrightarrow (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \\ &\Longleftrightarrow x^2 + y^2 + z^2 - 2ax - 2by - 2cz + d = 0, \end{aligned}$$

where $d = a^2 + b^2 + c^2 - r^2$. Conversely, every equation of the form

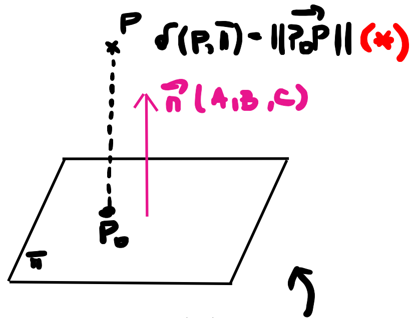
$$x^2 + y^2 + z^2 + 2ex + 2fy + 2gz + h = 0$$

is the equation of the sphere centered at $(-e,-g,-f)$ and having the radius $r=\sqrt{e^2+f^2+g^2-h}$, whenever $e^2+f^2+g^2\geq h$. One can find the equation of the sphere circumscribed to the tetrahedron $ABCD$ by imposing the requirement on the coordinates (x_A,y_A,z_A), (x_B,y_B,z_B) and (x_C,y_C,z_C) and (x_D,y_D,z_D) of its vertices A,B,C,D to verify the equation $x^2+y^2+z^2-2ax-2by-2cz+d=0$. A point $M(x,y,z)$ belongs to this circumcircle if and only if

$$\begin{cases} x^2+y^2+z^2+2ex+2fy+2gz+h=0\\ x_A^2+y_A^2+z_A^2+2ex_A+2fy_A+2gz_A+h=0\\ x_B^2+y_B^2+z_B^2+2ex_B+2fy_B+2gz_B+h=0\\ x_C^2+y_C^2+z_C^2+2ex_C+2fy_C+2gz_C+h=0\\ x_D^2+y_D^2+z_D^2+2ex_D+2fy_D+2gz_A+h=0 \end{cases} \tag{5.7}$$

On can regard the system (5.7) as linear with the unknowns e,g,f,h, whose compatibility is given, via the Kronecker-Capelli theorem, by

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1\\ x_A^2+y_A^2+z_A^2 & x_A & y_A & z_A & 1\\ x_B^2+y_B^2+z_B^2 & x_B & y_B & z_B & 1\\ x_C^2+y_C^2+z_C^2 & x_C & y_C & z_C & 1\\ x_D^2+y_D^2+z_D^2 & x_D & y_D & z_A & 1 \end{vmatrix}=0,$$



which is the equation of the circumsphere of the tetrahedron $ABCD$.

- **The normal vector of a plane.** Consider the plane $\pi : Ax+By+Cz+D=0$ and the point $P(x_0,y_0,z_0)\in\pi$. The equation of π becomes

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0. \tag{5.8}$$

If $M(x,y,z)\in\pi$, the coordinates of $\overrightarrow{PM}$ are $(x-x_0,y-y_0,z-z_0)$ and the equation (5.8) tells us that $\overrightarrow{n}\cdot\overrightarrow{PM}=0$, for every $M\in\pi$, that is $\overrightarrow{n}\perp\overrightarrow{PM}=0$, for every $M\in\pi$, which is equivalent to $\overrightarrow{n}\perp\overrightarrow{\pi}$, where $\overrightarrow{n}\,(A,B,C)$. This is the reason to call $\overrightarrow{n}\,(A,B,C)$ *the normal vector* of the plane π.

- **The distance from a point to a plane.** Consider the plane $\pi : Ax+By+Cz+D=0$, a point $P(x_P,y_P,z_P)\in\mathcal{P}$ and M the orthogonal projection of P on π. The real number δ given by $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$ is called the *oriented distance* from P to the plane π, where $\overrightarrow{n}_0=\frac{1}{\|\overrightarrow{n}\|}\overrightarrow{n}$ is the versor of the normal vector $\overrightarrow{n}\,(A,B,C)$. Since $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$, it follows that $\delta(P,M)=\|\overrightarrow{MP}\|=|\delta|$, where $\delta(P,M)$ stands for the distance from P to π. We shall show that

$$\delta=\frac{Ax_P+By_P+Cz_P+D}{\sqrt{A^2+B^2+C^2}}.$$

Indeed, since $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$, we get successively:

$$\begin{aligned}\delta &= \overrightarrow{n}_0\cdot\overrightarrow{MP}=\left(\frac{1}{\|\overrightarrow{n}\|}\overrightarrow{n}\right)\cdot\overrightarrow{MP}=\frac{\overrightarrow{n}\cdot\overrightarrow{MP}}{\|\overrightarrow{n}\|}\\ &= \frac{A(x_P-x_M)+B(y_P-y_M)+C(z_P-z_M)}{\sqrt{A^2+B^2+C^2}}\\ &= \frac{Ax_P+By_P+Cz_P-(Ax_M+By_M+Cz_M)}{\sqrt{A^2+B^2+C^2}}\\ &= \frac{Ax_P+By_P+Cz_P+D}{\sqrt{A^2+B^2+C^2}}.\end{aligned}$$

Consequently, the distance from P to the plane π is

$$\delta(P,\pi)=\|\overrightarrow{MP}\|=|\delta|=\frac{|Ax_P+By_P+Cz_P+D|}{\sqrt{A^2+B^2+C^2}}. \quad (*)$$

Example 5.1. Compute the distance from the point $A(3,1,-1)$ to the plane

$$\pi : 22x + 4y - 20z - 45 = 0.$$

SOLUTION.

$$\delta(A,\pi) = \frac{|22\cdot 3 + 4\cdot 1 - 20\cdot(-1) - 45|}{\sqrt{22^2+4^2+(-20)^2}} = \frac{45}{\sqrt{900}} = \frac{45}{30} = \frac{3}{2}.$$

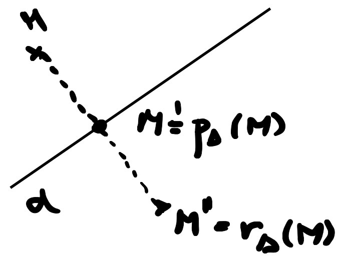
5.2 Appendix: Orthogonal projections and reflections

5.2.1 The two dimensional setting

Asssume that $R = (O, \overrightarrow{i}, \overrightarrow{j})$ is the orthonormal Cartesian system of a plane π behind the equation of the line $\Delta : ax + by + c = 0$.

• **The orthogonal projection of a point on a line.** We define the projection of the ambient plane $p_\Delta : \pi \to \Delta$ on Δ, whose value p_Δ at $M \in \pi$ is the intersection point between Δ and the line through M perpendicular to Δ. Due to relations (4.24), the coordinates of $p_\Delta(M)$, in terms of the coordinates of M are:

$$x_M - p\frac{F(x_M,y_M)}{a^2+b^2}$$
$$y_M - q\frac{F(x_M,y_M)}{a^2+b^2},$$



where $F(x,y) = ax + by + c$. Consequently, the position vector of $p_\Delta(M)$ is

$$\overrightarrow{Op_\Delta(M)} = \overrightarrow{OM} - \frac{F(M)}{a^2+b^2}\overrightarrow{n}_\Delta,$$

where $\overrightarrow{n}_\Delta = a\overrightarrow{i} + b\overrightarrow{j}$.

Proposition 5.3. *If $R = (O, \overrightarrow{i}, \overrightarrow{j})$ is the orthonormal Cartesian reference system of the plane π behind the equations of the line*

$$\Delta : ax + by + c = 0,$$

then

$$[p_\Delta(M)]_R = \frac{1}{a^2+b^2}\begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix}[M]_R - \frac{c}{a^2+b^2}[\overrightarrow{n}_\Delta]_b, \tag{5.9}$$

where b stands for the orthonormal basis $[\overrightarrow{i}, \overrightarrow{j}]$ of $\overrightarrow{\pi}$.

• **The reflection of the plane about a line.** We call the function $r_\Delta : \pi \to \pi$, whose value r_Δ at $M \in \pi$ is the symmetric point of M with respect to $p_\Delta(M)$, the *reflection of π about Δ*. For the position vector of $r_\Delta(M)$ we have

$$\overrightarrow{Op_\Delta(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Or_\Delta(M)}}{2}, \text{ i.e.}$$

$$\overrightarrow{Or_\Delta(M)} = 2\overrightarrow{Op_\Delta(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{a^2+b^2}\overrightarrow{n}_\Delta,$$

where $F(x,y) = ax + by + c$ and $\overrightarrow{n}_\Delta = a\overrightarrow{i} + b\overrightarrow{j}$. Thus, the coordinates of $s_{\Delta,d}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_M - 2p\dfrac{F(x_M,y_M)}{a^2+b^2} \\ y_M - 2q\dfrac{F(x_M,y_M)}{a^2+b^2}. \end{cases}$$

Proposition 5.4. *If $R = (O, \overrightarrow{i}, \overrightarrow{j})$ is the orthonormal Cartesian reference system of the plane π behind the equations of the line*

$$\Delta : ax + by + c = 0,$$

then

$$[r_\Delta(M)]_R = \frac{1}{a^2+b^2}\begin{pmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix}[M]_R - \frac{2c}{a^2+b^2}\left[\overrightarrow{n}_\Delta\right]_b, \tag{5.10}$$

where b stands for the orthonormal basis $[\overrightarrow{i}, \overrightarrow{j}]$ of $\overrightarrow{\pi}$.

Example 5.2. Find the coordinates of the reflected point of $P(-5, 13)$ with respect to the line

$$d : 2x - 3y - 3 = 0,$$

knowing that the Cartesian reference system R behind the coordinates of A and the equation of (d) is orthonormal.

HINT. According to 5.11 it follows that

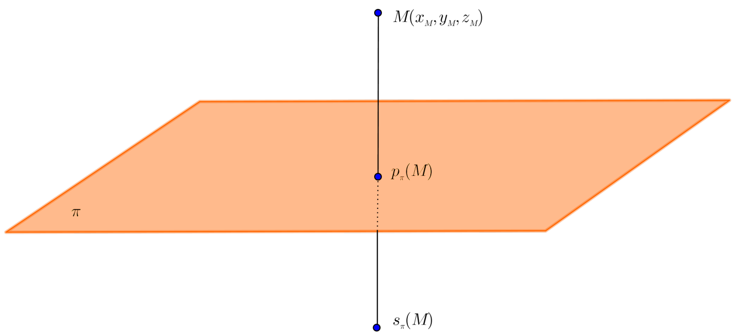
$$[r_d(P)]_R = \frac{1}{2^2+(-3)^2}\begin{pmatrix} -2^2+(-3)^2 & -2\cdot 2\cdot(-3) \\ -2\cdot 2\cdot(-3) & 2^2-(-3)^2 \end{pmatrix}\begin{bmatrix} -5 \\ 13 \end{bmatrix} - \frac{2\cdot(-3)}{2^2+(-3)^2}\begin{bmatrix} 2 \\ -3 \end{bmatrix}. \tag{5.11}$$

5.2.2 The three dimensional setting

• **The orthogonal projection of a point on a plane.** For a given plane

$$\pi : Ax + By + Cz + D = 0$$

and a given point $M(x_M, y_M, z_M)$, we shall determine the coordinates of its orthogonal projection on the plane π, as well as the coordinates of its (orthogonal) symmetric with respect to π. The equation of the plane and the coordinates of M are considered with respect to some cartezian coordinate system $R = (O, \vec{i}, \vec{j}, \vec{k})$. In this respect we consider the orthogonal line on π which passes through M.



Its parametric equations are

$$\begin{cases} x = x_M + At \\ y = y_M + Bt \\ z = z_M + Ct \end{cases}, \ t \in \mathbb{R}. \tag{5.12}$$

The orthogonal projection $p_\pi(M)$ of M on the plane π is at its intersection point with the orthogonal line (5.12) and the value of $t \in \mathbb{R}$ for which this orthogonal line (5.12) puncture the plane π can be determined by imposing the condition on the point of coordinates $(x_M + At, y_M + Bt, z_M + Ct)$ to verify the equation of the plane, namely $A(x_M + At) + B(y_M + Bt) + C(z_M + Ct) + D = 0$. Thus

$$t = -\frac{Ax_M + By_M + Cz_M + D}{A^2 + B^2 + C^2} = -\frac{F(x_M, y_M, z_M)}{\| \vec{n}_\pi \|^2},$$

where $F(x, y, z) = Ax + By + Cz + D$ şi $\vec{n}_\pi = A\vec{i} + B\vec{j} + C\vec{k}$ is the normal vector of the plasne π.

• **The orthogonal projection of the space on a plane.**

The coordinates of the orthogonal projection $p_\pi(M)$ of M on th eplane π are

$$\begin{cases} x_M - A\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ y_M - B\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ z_M - C\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2}. \end{cases}$$

Therefore, the position vector of the orthogonal projection $p_\pi(M)$ is

$$\overrightarrow{Op_\pi(M)} = \overrightarrow{OM} - \frac{F(M)}{\| \vec{n}_\pi \|^2} \vec{n}_\pi . \tag{5.13}$$

Proposition 5.5. *If $R = (O, b)$ is the orthonormal Cartesian reference system behind the equation of the plane $(\pi)\ Ax + By + Cz + D = 0$, then*

$$(A^2 + B^2 + C^2)[p_\pi(M)]_R = \begin{pmatrix} B^2 + C^2 & -AB & -AC \\ -AB & A^2 + C^2 & -BC \\ -AC & -BC & A^2 + B^2 \end{pmatrix} [M]_R - D[\vec{n}_\pi]_b. \tag{5.14}$$

Remark 5.3. The distance from the point $M(x_M, y_M, z_M)$ to the plane $\pi : Ax + By + Cz + D = 0$ can be equally computed by means of (5.13). Indeed,

$$\begin{aligned} \delta(M, \pi) &= \| \overrightarrow{Mp_\pi(M)} \| = \| \overrightarrow{Op_\pi(M)} - \overrightarrow{OM} \| \\ &= \left| -\tfrac{F(M)}{\|\vec{n}_\pi\|^2} \right| \cdot \| \vec{n}_\pi \| = \frac{|F(M)|}{\| \vec{n}_\pi \|}. \end{aligned}$$

• **The reflection of the space about a plane.** In order to find the position vector of the orthogonally symmetric point $r_\pi(M)$ of M w.r.t. π, we use the relation

$x' = x_M - 2A\dfrac{Ax+By+Cz+D}{A^2+B^2+C^2}$

$$\overrightarrow{Op_\pi(M)} = \frac{1}{2}\left(\overrightarrow{OM} + \overrightarrow{Or_\pi(M)} \right),$$

$y' = y_M - 2B\dfrac{Ax+By+Cz+D}{A^2+B^2+C^2}$

namely

$z' = z_M - 2C\dfrac{Ax+By+Cz+D}{A^2+B^2+C^2}$

$$\overrightarrow{Or_\pi(M)} = 2\,\overrightarrow{Op_\pi(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{\| \vec{n}_\pi \|^2} \vec{n}_\pi .$$

The correspondence which associate to some point M its orthogonally symmetric point w.r.t. π, is called the *reflection* in the plane π and is denoted by r_π.

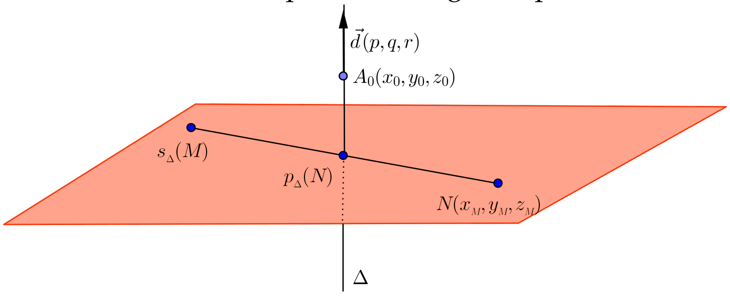
Proposition 5.6. *If $R = (O, b)$ is the orthonormal Cartesian reference system behind the equation of the plane $(\pi)\ Ax + By + Cz + D = 0$, then*

$$(A^2+B^2+C^2)[r_\pi(M)]_R = \begin{pmatrix} -A^2+B^2+C^2 & -2AB & -2AC \\ -2AB & A^2-B^2+C^2 & -2BC \\ -2AC & -2BC & A^2+B^2-C^2 \end{pmatrix} [M]_R - 2D[\vec{n}_\pi]_b. \tag{5.15}$$

• **The orthogonal projection of the space on a line.** For a given line

$$\Delta : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

and a point $N(x_N, y_N, z_N)$, we shall find the coordinates of its orthogonal projection on the line Δ, as well as the coordinates of the orthogonally symmetric point M with respect to Δ. The equations of the line and the coordinates of the point N are considered with respect to an orthonormal coordinate system $R = (O, \vec{i}, \vec{j}, \vec{k})$. In this respect we consider the plane $p(x-x_N) + q(y-y_N) + r(z-z_N) = 0$ orthogonal on the line Δ which passes through the point N.



The parametric equations of the line Δ are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \ t \in \mathbb{R}. \tag{5.16}$$

The orthogonal projection of N on the line Δ is at its intersection point with the plane

$$p(x-x_N) + q(y-y_N) + r(z-z_N) = 0,$$

and the value of $t \in \mathbb{R}$ for which the line Δ puncture the orthogonal plane $p(x-x_N) + q(y-y_N) + r(z-z_N) = 0$ can be found by imposing the condition on the point of coordinate (x_0+pt, y_0+qt, z_0+rt) to verify the equation of the plane, namely $p(x_0+pt-x_N) + q(y_0+qt-y_N) + r(z_0+rt-z_N) = 0$. Thus

$$t = -\frac{p(x_0-x_N) + q(y_0-y_N) + r(z_0-z_N)}{p^2+q^2+r^2} = -\frac{G(x_0,y_0,z_0)}{\| \vec{d}_\Delta \|^2},$$

where $G(x,y,z) = p(x-x_N) + q(y-y_N) + r(z-z_N)$ and $\vec{d}_\pi = p\,\vec{i} + q\,\vec{j} + r\,\vec{k}$ is the director vectoir of the line Δ. Ths coordinates of the orthogonal projection $p_\Delta(N)$ of N on the line Δ are therefore

$$\begin{cases} x_0 - p\dfrac{G(x_0,y_0,z_0)}{p^2+q^2+r^2} \\ y_0 - q\dfrac{G(x_0,y_0,z_0)}{p^2+q^2+r^2} \\ z_0 - r\dfrac{G(x_0,y_0,z_0)}{p^2+q^2+r^2}. \end{cases}$$

Thus, the position vector of the orthogonal projection $p_\Delta(N)$ is

$$\overrightarrow{Op_\Delta(N)} = \overrightarrow{OA_0} - \frac{G(A_0)}{\| \vec{d}_\Delta \|^2}\vec{d}_\Delta, \tag{5.17}$$

where $A_0(x_0, y_0, z_0) \in \Delta$.

• **The reflection of the space about a line.** In order to find the position vector of the orthogonally symmetric point $r_\Delta(N)$ of N with respect to the line Δ we use the relation

$$\overrightarrow{Op_\Delta(N)} = \frac{1}{2}\left(\overrightarrow{ON} + \overrightarrow{Or_\Delta(N)}\right)$$

i.e.

$$\overrightarrow{Os_{\Delta}(N)} = 2\,\overrightarrow{Op_{\Delta}(N)} - \overrightarrow{ON} = 2\,\overrightarrow{OA_0} - 2\frac{G(A_0)}{\| \overrightarrow{d}_{\Delta} \|^2}\,\overrightarrow{d}_{\Delta} - \overrightarrow{ON}\,.$$

The correspondence which associate to some point M its orthogonally symmetric point w.r.t. δ, is called the *reflection* in the line δ and is denoted by r_{δ}.

5.3 Problems

1. (**2p**) Consider the triangle ABC and the midpoint A' of the side $[BC]$. Show that

$$4\,\overrightarrow{AA'}^{\,2} - \overrightarrow{BC}^{\,2} = 4\,\overrightarrow{AB}\cdot\overrightarrow{AC}\,.$$

Solution.

A' - midpoint of $[BC] \Rightarrow \overrightarrow{AA'} = \dfrac{\overrightarrow{AB}+\overrightarrow{AC}}{2} \Big/^{2} \Rightarrow \overrightarrow{AA'}^2 = \dfrac{(\overrightarrow{AB}+\overrightarrow{AC})^2}{4}$ $\Big\}\Rightarrow$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} \Rightarrow \overrightarrow{BC}\cdot\overrightarrow{BC} = (\overrightarrow{AC}-\overrightarrow{AB})\cdot(\overrightarrow{AC}-\overrightarrow{AB}) \Rightarrow \overrightarrow{BC}^2 = (\overrightarrow{AC}-\overrightarrow{AB})^2$

$\Rightarrow 4\overrightarrow{AA'}^2 - \overrightarrow{BC}^2 = \cancel{4}\dfrac{(\overrightarrow{AB}+\overrightarrow{AC})^2}{\cancel{4}} - (\overrightarrow{AC}-\overrightarrow{AB})^2 =$

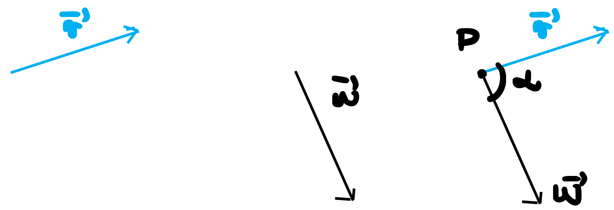
$= \cancel{\overrightarrow{AB}^2} + 2\overrightarrow{AB}\cdot\overrightarrow{AC} + \cancel{\overrightarrow{AC}^2} - \cancel{\overrightarrow{AC}^2} + 2\overrightarrow{AC}\cdot\overrightarrow{AB} - \cancel{\overrightarrow{AB}^2} =$

$= 4\overrightarrow{AB}\cdot\overrightarrow{AC}$

Seminar Micu - 24/03/2021

Dot product (Scalar product)

$\vec{v}, \vec{w} \in \mathcal{V} \Rightarrow \vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\vec{v}, \vec{w})$



$\alpha \in [0, \pi]$

If we fix a ref system that is orthonormal, then:

$\vec{v}(a_1, b_1, c_1), \vec{w}(a_2, b_2, c_2) \Rightarrow \vec{v} \cdot \vec{w} = a_1 a_2 + b_1 b_2 + c_1 c_2$

$\mathcal{R} = (O, [\vec{i}, \vec{j}, \vec{k}])$, orthonormal = orthogonal + normed

$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$ $\qquad$ $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$

The distance from a point to a plane

$\pi : Ax + By + Cz + D = 0$
$P(x_0, y_0, z_0)$
$\Rightarrow dist(P, \pi) = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

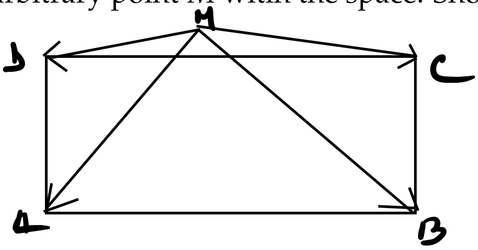
The distance from a point to a line in 2D

$\ell : Ax + By + C = 0$
$P(x_0, y_0, z_0)$
$\Rightarrow dist(P, \ell) = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2}}$

2. (**2p**) Consider the rectangle $ABCD$ and the arbitrary point M witin the space. Show that

   (a) $\overrightarrow{MA} \cdot \overrightarrow{MC} = \overrightarrow{MB} \cdot \overrightarrow{MD}$.

   (b) $\overrightarrow{MA}^2 + \overrightarrow{MC}^2 = \overrightarrow{MB}^2 + \overrightarrow{MD}^2$.



Solution.

ABCD - rectangle

a) $\overrightarrow{MB} = \overrightarrow{MA} + \overrightarrow{AB}$
$\overrightarrow{MD} = \overrightarrow{MC} + \overrightarrow{CD}$ $\Rightarrow \overrightarrow{MB} \cdot \overrightarrow{MD} = (\overrightarrow{MA} + \overrightarrow{AB})(\overrightarrow{MC} + \overrightarrow{CD}) =$

$= \overrightarrow{MA} \cdot \overrightarrow{MC} + \overrightarrow{MA} \cdot \overrightarrow{CD} + \overrightarrow{AB} \cdot \overrightarrow{MC} + \cancel{\overrightarrow{AB} \cdot \overrightarrow{CD}}$ $\Rightarrow$

$\overrightarrow{AB} = -\overrightarrow{CD}$

$\Rightarrow \overrightarrow{MB} \cdot \overrightarrow{MD} = \overrightarrow{MA} \cdot \overrightarrow{MC} + \overrightarrow{CD}(\underbrace{\overrightarrow{MA} - \overrightarrow{MC} - \overrightarrow{CD}}_{\underbrace{\overrightarrow{CM} + \overrightarrow{MA}}_{\overrightarrow{CA}} + \overrightarrow{DC}}) = \overrightarrow{MA} \cdot \overrightarrow{MC} + \overrightarrow{CD} \cdot \overrightarrow{DA} =$

$= \overrightarrow{MA} \cdot \overrightarrow{MC} + \underbrace{\overrightarrow{AB} \cdot \overrightarrow{AD}}_{0,\ \overrightarrow{AB} \perp \overrightarrow{AD}} \Rightarrow$

$\Rightarrow \overrightarrow{MB} \cdot \overrightarrow{MD} = \overrightarrow{MA} \cdot \overrightarrow{MC}$

b) $\overrightarrow{MB}^2 + \overrightarrow{MD}^2 = (\overrightarrow{MA} + \overrightarrow{AB})^2 + (\overrightarrow{MC} + \overrightarrow{CD})^2 = \overrightarrow{MA}^2 + 2\overrightarrow{MA} \cdot \overrightarrow{AB} + \overrightarrow{AB}^2 + \overrightarrow{MC}^2 + 2\overrightarrow{MC} \cdot \overrightarrow{CD} + \overrightarrow{CD}^2 =$

$= \overrightarrow{MA}^2 + \overrightarrow{MC}^2 + 2\overrightarrow{MA} \cdot \overrightarrow{AB} + 2\overrightarrow{MC} \cdot \overrightarrow{CD} + \overrightarrow{AB}^2 + \overrightarrow{CD}^2 =$

$= \overrightarrow{MA}^2 + \overrightarrow{MC}^2 + 2\overrightarrow{MA} \cdot \overrightarrow{AB} - 2\overrightarrow{MC} \cdot \overrightarrow{AB} + 2\overrightarrow{AB}^2 =$

$= \overrightarrow{MA}^2 + \overrightarrow{MC}^2 + 2\overrightarrow{AB}(\overrightarrow{MA} - \overrightarrow{MC} + \overrightarrow{AB}) =$

$= \overrightarrow{MA}^2 + \overrightarrow{MC}^2 + 2\overrightarrow{AB}\,(\underbrace{\overrightarrow{CA} + \overrightarrow{AB}}_{\overrightarrow{CB}}) = \overrightarrow{MA}^2 + \overrightarrow{MC}^2 + \vec{0} = \overrightarrow{MA}^2 + \overrightarrow{MC}^2$

$\overrightarrow{AB} \perp \overrightarrow{CB}$

3. (**3p**) Find the angle between:

   (a) the straight lines

$$(d_1)\ \begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases} \qquad (d_2)\ \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0. \end{cases}$$

   (b) the planes

$$\pi_1 :\ x + 3y + 2z + 1 = 0 \text{ and } \pi_2 : 3x + 2y - z = 6.$$

   (c) the plane xOy and the straight line M_1M_2, where $M_1(1, 2, 3)$ and $M_2(-2, 1, 4)$.

Solution.

(a) $d_1 \Leftrightarrow \begin{cases} 2x+2z=0 \\ x-2y+z+1=0 \end{cases} \Leftrightarrow \begin{cases} x=-z \\ -z-2y+z+1=0 \end{cases} \Leftrightarrow \begin{cases} x=-z \\ 2y=1 \end{cases} \Leftrightarrow \begin{cases} y=\frac{1}{2} \\ x=-z \end{cases} \Leftrightarrow$

$\Leftrightarrow d_1 : \begin{cases} x=t \\ y=\frac{1}{2}+0\cdot t \\ z=-t \end{cases} \Rightarrow \vec{d_1}(1,0,-1)$

$d_2 \Leftrightarrow \begin{cases} x-y-z-1=0 \\ x-y+2z+1=0 \end{cases} \Leftrightarrow \begin{cases} -3z-2=0 \\ x-y+2z+1=0 \end{cases} \Leftrightarrow \begin{cases} z=-\frac{2}{3} \\ x-y-\frac{4}{3}+1=0 \end{cases} \Leftrightarrow \begin{cases} z=-\frac{2}{3} \\ x-y=\frac{1}{3} \end{cases} \Leftrightarrow$

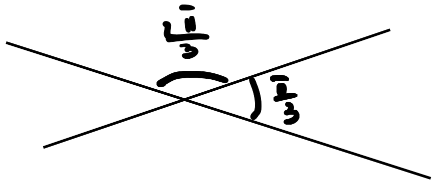
$\Leftrightarrow \begin{cases} x=y+\frac{1}{3} \\ z=-\frac{2}{3} \end{cases} \Leftrightarrow d_2 : \begin{cases} x=t \\ y=t-\frac{1}{3} \\ z=-\frac{2}{3}+t\cdot 0 \end{cases} \Rightarrow \vec{d_2}(1,1,0)$

$\vec{d_1}\cdot\vec{d_2} = (1,0,-1)\cdot(1,1,0) = 1\cdot 1+0\cdot 1+(-1)\cdot 0 = 1$

$\|\vec{d_1}\| = \sqrt{1+0+1} = \sqrt{2}$

$\|\vec{d_2}\| = \sqrt{1+1+0} = \sqrt{2}$

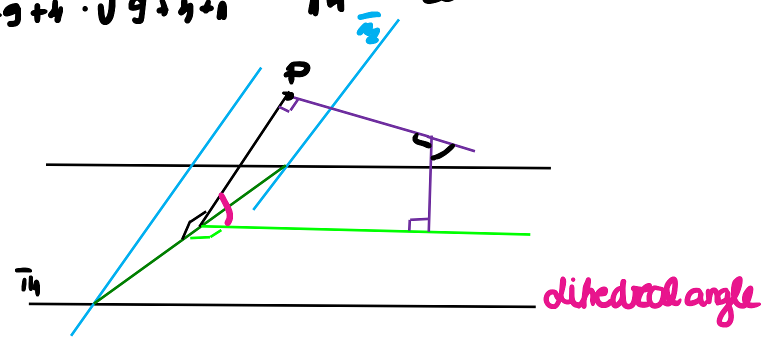
$\Rightarrow \cos(\vec{d_1},\vec{d_2}) = \dfrac{\vec{d_1}\cdot\vec{d_2}}{\|\vec{d_1}\|\cdot\|\vec{d_2}\|} = \dfrac{1}{2} \Rightarrow m(\widehat{d_1,d_2}) = \dfrac{\pi}{3}$



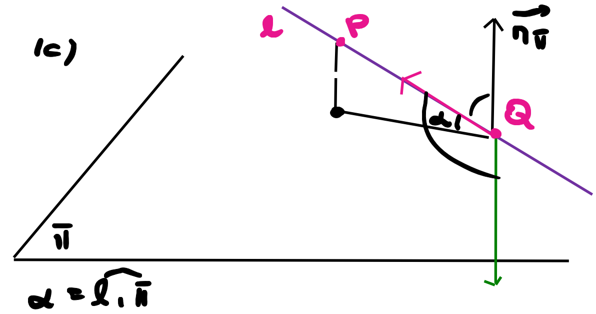
(b) $\pi_1 : x+3y+2z+1=0$ $\Rightarrow$ $\vec{n_{\pi_1}}(1,3,2)$ $\Rightarrow \cos(\widehat{\vec{n_{\pi_1}},\vec{n_{\pi_2}}}) = \dfrac{\vec{n_{\pi_1}}\cdot\vec{n_{\pi_2}}}{\|\vec{n_{\pi_1}}\|\cdot\|\vec{n_{\pi_2}}\|} =$

$\pi_2 : 3x+2y-z=0$ $\qquad \vec{n_{\pi_2}}(3,2,-1)$

$= \dfrac{1\cdot 3+3\cdot 2+2\cdot(-1)}{\sqrt{1+9+4}\cdot\sqrt{9+4+1}} = \dfrac{7}{14} = \dfrac{1}{2} \Rightarrow m(\widehat{\pi_1,\pi_2}) = \dfrac{\pi}{3}$



1c)



$\alpha = \widehat{l, \pi}$

$$\cos(\widehat{\vec{n_{xOy}}, \vec{M_1M_2}}) = \frac{0\cdot(-3)+0\cdot(-1)+1\cdot 1}{\sqrt{9+1+1}\cdot\sqrt{1}} = \frac{1}{\sqrt{11}}$$

$$\arccos\left(\frac{1}{\sqrt{11}}\right) \in \left[0; \frac{\pi}{2}\right] \Bigg\} \Rightarrow$$

$\Rightarrow m(M_1M_2, xOy) = \frac{\pi}{2} - \arccos\left(\frac{1}{\sqrt{11}}\right)$

$M_1M_2: \frac{x-1}{-2-1} = \frac{y-2}{1-2} = \frac{z-3}{4-3} \Leftrightarrow \frac{x-1}{-3} = \frac{y-2}{-1} = \frac{z-3}{1} \Rightarrow \vec{M_1M_2}(-3,-1,1) \Rightarrow$

$\Rightarrow$ plane $xOy \Rightarrow z=0 \Rightarrow \vec{n_{xOy}}(0,0,1)$

4. (**3p**) Consider the noncoplanar vectors $\overrightarrow{OA}\ (1,-1,-2)$, $\overrightarrow{OB}\ (1,0,-1)$, $\overrightarrow{OC}\ (2,2,-1)$ related to an orthonormal basis $\vec{i}, \vec{j}, \vec{k}$. Let H be the foot of the perpendicular through O on the plane ABC. Determine the components of the vectors $\overrightarrow{OH}$.

   Solution.

We know that $O(0,0,0) \Rightarrow x_0=0, y_0=0, z_0=0 \Rightarrow A(1,-1,-2), B(1,0,-1), C(2,2,-1)$

A, B, C - noncollinear points $\Rightarrow$ eq of the plane (ABC):

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{vmatrix} x & y & z & 1 \\ 1 & -1 & -2 & 1 \\ 1 & 0 & -1 & 1 \\ 2 & 2 & -1 & 1 \end{vmatrix} = 0 \Leftrightarrow x \cdot \begin{vmatrix} -1 & -2 & 1 \\ 0 & -1 & 1 \\ 2 & -1 & 1 \end{vmatrix} - y \begin{vmatrix} 1 & -2 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 1 \end{vmatrix} + z \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 2 & -1 \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow x(\cancel{1}-4+2\cancel{-1}) - y(\cancel{-1}\cancel{-1}-4+\cancel{2}+1+2) + z(\cancel{2}\cancel{-2}-2+1) - (\cancel{-4}\cancel{+2}\cancel{+2}-1) = 0 \Leftrightarrow$$

$$\Leftrightarrow 2x + y - z + 1 = 0 \Rightarrow \text{the eq of the plane } (ABC)$$

For the plane (ABC) and point $O(0,0,0)$ we shall det. the coord. of its orthogonal projection on the plane. The eq. of the plane and the coord. of O are considered with respect to the cartezian coordinate system $R = (O, \vec{i}, \vec{j}, \vec{k})$. In this respect we consider the orthogonal line on (ABC) which passes through O $(\overline{OH})$

Param. eq of $\overrightarrow{OH}$: $\begin{cases} x = x_0 + At \\ y = y_0 + Bt, \ t \in \mathbb{R} \\ z = z_0 + Ct \end{cases}$

$$\Rightarrow t = -\frac{Ax_0 + By_0 + Cz_0 + D}{A^2 + B^2 + C^2} = \frac{-2\cdot 0 + 1\cdot 0 - 1\cdot 0 + 1}{4+1+1} = \frac{-1}{6} \Rightarrow \begin{cases} x = 0 + (-2)\cdot\frac{-1}{6} \\ y = 0 + 1 \cdot \frac{-1}{6} \\ z = 0 - 1 \cdot \frac{-1}{6} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = \frac{1}{3} \\ y = -\frac{1}{6} \\ z = \frac{1}{6} \end{cases} \Rightarrow \overrightarrow{OH}\left(\frac{1}{3}, \frac{-1}{6}, \frac{1}{6}\right)$$

• **The orthogonal projection of a point on a plane.** For a given plane

$$\pi : Ax + By + Cz + D = 0$$

and a given point $M(x_M, y_M, z_M)$, we shall determine the coordinates of its orthogonal projection on the plane π, as well as the coordinates of its (orthogonal) symmetric with respect to π. The equation of the plane and the coordinates of M are considered with respect to some cartezian coordinate system $R = (O, \vec{i}, \vec{j}, \vec{k})$. In this respect we consider the orthogonal line on π which passes through M.

Its parametric equations are

$$\begin{cases} x = x_M + At \\ y = y_M + Bt \\ z = z_M + Ct \end{cases}, \ t \in \mathbb{R}. \tag{5.12}$$

The orthogonal projection $p_\pi(M)$ of M on the plane π is at its intersection point with the orthogonal line (5.12) and the value of $t \in \mathbb{R}$ for which this orthogonal line (5.12) puncture the plane π can be determined by imposing the condition on the point of coordinates $(x_M + At, y_M + Bt, z_M + Ct)$ to verify the equation of the plane, namely $A(x_M + At) + B(y_M + Bt) + C(z_M + Ct) + D = 0$. Thus

$$t = -\frac{Ax_M + By_M + Cz_M + D}{A^2 + B^2 + C^2} = -\frac{F(x_M, y_M, z_M)}{\| \vec{n}_\pi \|^2},$$

where $F(x,y,z) = Ax + By + Cz + D$ şi $\vec{n}_\pi = A\vec{i} + B\vec{j} + C\vec{k}$ is the normal vector of the plasne π.

5. (**2p**) Find the points on the z-axis which are equidistant with respect to the planes

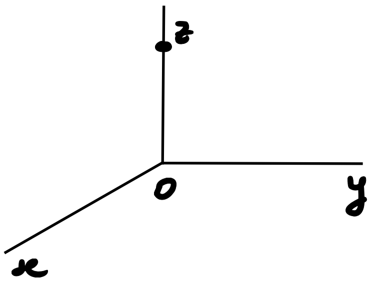
$$\pi_1 : \ 12x + 9y - 20z - 19 = 0 \text{ and } \pi_2 : \ 16x + 12y + 15z - 9 = 0.$$

Solution.

$P(x_0, y_0, z_0)$

$$\text{dist}(P, \pi_1) = \frac{|12x_0 + 9y_0 - 20z_0 - 19|}{\sqrt{12^2 + 9^2 + 20^2}} = \frac{|12x_0 + 9y_0 - 20z_0 - 19|}{25}$$

$$\text{dist}(P, \pi_2) = \frac{|16x_0 + 12y_0 + 15z_0 - 9|}{\sqrt{16^2 + 12^2 + 15^2}} = \frac{|16x_0 + 12y_0 + 15z_0 - 9|}{25}$$



$$P \in Oz \Rightarrow x_0 = y_0 = 0 \Rightarrow \text{dist}(P, \pi_1) = \frac{|-20z_0 - 19|}{25}$$

$$\text{dist}(P, \pi_2) = \frac{|15z_0 - 9|}{25}$$

$$\text{dist}(P, \pi_1) = \text{dist}(P, \pi_2) \Rightarrow |-20z_0 - 19| = |15z_0 - 9|$$

Case I: $-20z_0 - 19 = 15z_0 - 9$

$\Rightarrow z_0 = \frac{-10}{35} = \frac{-2}{7}$

$\Rightarrow P_1\left(0, 0, -\frac{2}{7}\right)$

loc geometric

Case II: $-20z_0 - 19 = -15z_0 + 9$

$\Rightarrow z_0 = \frac{-28}{5}$

$\Rightarrow P_2\left(0, 0, -\frac{28}{5}\right)$

The locus of points equidistant to two non-parallel planes consists of two perpendicular planes called the bisector planes.

6. (**2p**) Consider two planes

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0$$
$$(\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0$$

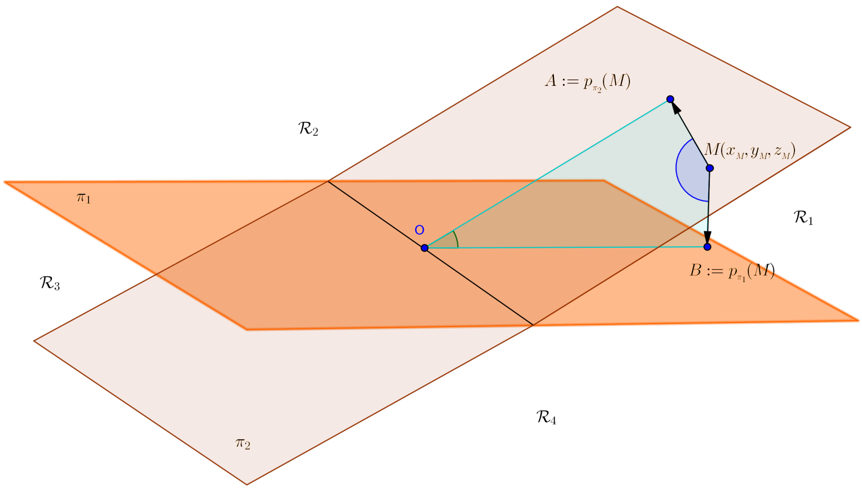
which are not parallel and not perpendicular as well. The two planes π_1, π_2 devide the space into four regions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$, two of which, say $\mathcal{R}_1$ and $\mathcal{R}_3$, correspond to the acute dihedral angle of the two planes. Show that $M(x,y,z) \in \underbrace{\mathcal{R}_1 \cup \mathcal{R}_3}_{\text{acute region}}$, if and only if

$$F_1(x,y,z) \cdot F_2(x,y,z)\underbrace{(A_1A_2 + B_1B_2 + C_1C_2)}_{\vec{n}_{\pi_1}\cdot\vec{n}_{\pi_2}} < 0,$$

where $F_1(x,y,z) = A_1x + B_1y + C_1z + D_1$ and $F_2(x,y,z) = A_2x + B_2y + C_2z + D_2$.

Hint. The non-parallellism relation between the two planes is equivalent with the condition

$$\text{rank}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2.$$



The point M belongs to the union $\mathcal{R}_1 \cup \mathcal{R}_3$ if and only if the angle of the vectors $\overrightarrow{Mp_{\pi_1}(M)}$ and $\overrightarrow{Mp_{\pi_2}(M)}$ is at leat $90°$, as the quadrilateral $OAMB$ is inscriptible. More formally

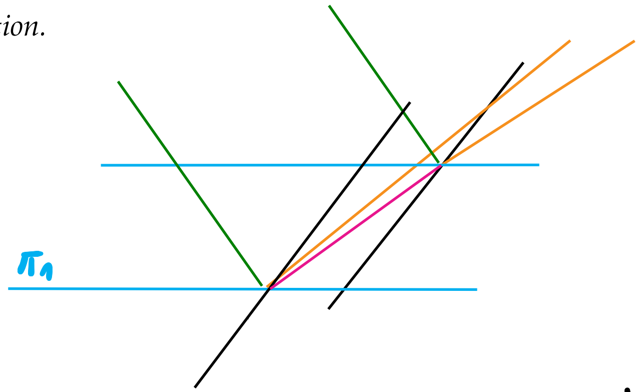
$$M(x,y,z) \in \mathcal{R}_1 \cup \mathcal{R}_3 \Leftrightarrow m(\widehat{\overrightarrow{Mp_{\pi_1}(M)}, \overrightarrow{Mp_{\pi_2}(M)}}) > 90°$$
$$\Leftrightarrow \overrightarrow{Mp_{\pi_1}(M)} \cdot \overrightarrow{Mp_{\pi_2}(M)} < 0,$$

where $p_{\pi_1}(M), p_{\pi_2}(M)$ are the orthogonal projections of M on the planes π_1 and π_2 respectively.

Solution.

7. (**3p**) Consider the planes $(\pi_1)\ \overset{F_1}{2x + y - 3z - 5} = 0$, $(\pi_2)\ \overset{F_2}{x + 3y + 2z + 1} = 0$. Find the equations of the bisector planes of the dihedral angles formed by the planes π_1 and π_2 and select the one contained into the acute regions of the dihedral angles formed by the two planes.

Solution.



$$\left.\begin{aligned} \text{Let } P(x,y,z) \Rightarrow \operatorname{dist}(P,\pi_1) &= \frac{|2x+y-3z-5|}{\sqrt{14}} \\ \operatorname{dist}(P,\pi_2) &= \frac{|x+3y+2z+1|}{\sqrt{14}} \\ \operatorname{dist}(P,\pi_1) &= \operatorname{dist}(P,\pi_2) \end{aligned}\right\} \Rightarrow |2x+y-3z-5| = |x+3y+2z+1| \Rightarrow$$

$\Rightarrow$ case I: $2x+y-3z-5 = x+3y+2z+1 \Rightarrow \pi_3:\ x-2y-5z-6=0$

case II: $2x+y-3z-5 = -x-3y-2z-1 \Rightarrow \pi_4:\ 3x+4y-z-4=0$

Let $M(0,-3,0) \in \pi_4$. We check if M belongs to the acute region.

$F_1(M) = F_1(0,-3,0) = -8$

$F_2(M) = F_2(0,-3,0) = -8$

$A_1A_2 + B_1B_2 + C_1C_2 = \overrightarrow{n_{\pi_1}} \cdot \overrightarrow{n_{\pi_2}} = (2,1,-3)\cdot(1,3,2) = 2+3-6 = -1$

$\Rightarrow F_1(M)\cdot F_2(M)\cdot(\overrightarrow{n_{\pi_1}}\cdot\overrightarrow{n_{\pi_2}}) = -64 < 0 \Rightarrow M \in$ acute region $\Rightarrow \pi_4 \subset$ acute regions

8. (**3p**) Let a, b be two real numbers such that $a^2 \neq b^2$. Consider the planes:

$$(\alpha_1) ax + by - (a+b)z = 0$$

$$(\alpha_2) ax - by - (a-b)z = 0$$

and the quadric $(\mathcal{C}) : a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 - 2a^2xz + 2b^2yz - a^2b^2 = 0$. If $a^2 < b^2$, show that the quadric $\mathcal{C}$ is contained in the acute regions of the dihedral angles formed by the two planes. If, on the contrary, $a^2 > b^2$, show that the quadric $\mathcal{C}$ is contained in the obtuse regions of the dihedral angles formed by the two planes.

Solution.

We need to check the conditions for a point of the quadric (any point)

that is: $P(x,y,z)$ s.t. $a^2x^2 - b^2y^2 + (a^2-b^2)z^2 - 2a^2xz + 2b^2yz - a^2b^2 = 0$

$\Leftrightarrow a^2x^2 - b^2y^2 + (a^2-b^2)z^2 - 2a^2xz + 2b^2yz = a^2b^2 \quad (1) \qquad \forall (x,y,z) \in \mathbb{R}^3$

* for being contained in the acute / obtuse regions

For this we use what we proved at exercise (6):

$P(x,y,z) \in R_1 \cup R_3 \Leftrightarrow F_1(x,y,z) \cdot F_2(x,y,z) \cdot (A_1A_2 + B_1B_2 + C_1C_2) < 0$ (acute regions)

so $P(x,y,z) \in R_2 \cup R_4 \Leftrightarrow F_1(x,y,z) \cdot F_2(x,y,z) \cdot (A_1A_2 + B_1B_2 + C_1C_2) > 0$ (obtuse regions)

where $F_1(x,y,z) = A_1x + B_1y + C_1z + D_1$

$F_2(x,y,z) = A_2x + B_2y + C_2z + D_2$

For our case

$P(x,y,z) \in R_1 \cup R_3 \Leftrightarrow [ax + by - (a+b)z]\cdot[ax - by - (a-b)z]\cdot[a^2 - b^2 + (a^2 - b^2)] < 0$

$\Leftrightarrow [(ax - az) + (by - bz)]\cdot[(ax - az) - (by - bz)]\cdot[a^2 - b^2 + a^2 - b^2] < 0$

$\Leftrightarrow [(ax - az)^2 - (by - bz)^2]\cdot(2a^2 - 2b^2) < 0$

$\Leftrightarrow (a^2x^2 + a^2z^2 - 2a^2xz - b^2y^2 - b^2z^2 + 2b^2yz)\cdot 2(a^2 - b^2) < 0$

$\Leftrightarrow (\underbrace{a^2x^2 - b^2y^2 + (a^2 - b^2)z - 2a^2xz + 2b^2yz}) \cdot 2(a^2 - b^2) < 0$

because we check these for a point $P(x,y,z) \in \mathcal{C}$ $\overset{(1)}{\Leftrightarrow}$ $a^2b^2 \cdot 2(a^2 - b^2) < 0$

but $a^2b^2 > 0, \forall a, b \in \mathbb{R}$

$2 > 0$

$\Leftrightarrow a^2 - b^2 < 0$

Therefore:

- $a^2 < b^2$: $F_1(x,y,z) \cdot F_2(x,y,z) \cdot (A_1A_2 + B_1B_2 + C_1C_2) < 0$

$\Rightarrow \forall P(x,y,z) \in \mathcal{C} \to P(x,y,z) \in R_1 \cup R_3$

$\Rightarrow$ the quadric $\mathcal{C}$ is contained in the acute regions

- otherwise $a^2 > b^2$: $F_1(x,y,z) \cdot F_2(x,y,z) \cdot (A_1A_2 + B_1B_2 + C_1C_2) > 0$

$\Rightarrow \forall P(x,y,z) \in \mathcal{C} \to P(x,y,z) \in R_2 \cup R_4$

$\Rightarrow$ the quadric $\mathcal{C}$ is contained in the obtuse regions

9. If two pairs of opposite edges of the tetrahedron $ABCD$ are perpendicular ($AB \perp CD$, $AD \perp BC$), show that

   (a) The third pair of opposite edges are perpendicular too ($AC \perp BD$).

   (b) $AB^2 + CD^2 = AC^2 + BD^2 = BC^2 + AD^2$.

   (c) The heights of the tetrahedron are concurrent.
   (Such a tetrahedron is said to be orthocentric)

Solution. Denote by $\overrightarrow{AB}=\vec{b}$, $\overrightarrow{AC}=\vec{c}$ and $\overrightarrow{AD}=\vec{d}$.

(a) $AB \perp CD \implies \vec{b}\,(\vec{d}-\vec{c}) = 0 \implies \vec{b}\vec{d}=\vec{b}\vec{c}=k$
$AD \perp BC \implies \vec{d}\,(\vec{c}-\vec{b}) = 0 \implies \vec{c}\vec{d}=\vec{b}\vec{d}=k,$
deci
$\vec{c}\vec{b}=\vec{c}\vec{d} \implies \vec{c}\,(\vec{b}-\vec{d}) = 0 \implies AC \perp BD.$

(b) $AB^2 + CD^2 = \vec{b}^{\,2} + (\vec{d}-\vec{c})^2 = \vec{b}^{\,2} + \vec{d}^{\,2} + \vec{c}^{\,2} - 2k;$
$AC^2 + BD^2 = \vec{c}^{\,2} + (\vec{d}-\vec{b})^2 = \vec{b}^{\,2} + \vec{c}^{\,2} + \vec{d}^{\,2} - 2k;$
$BC^2 + AD^2 = \vec{d}^{\,2} + (\vec{c}-\vec{b})^2 = \vec{b}^{\,2} + \vec{c}^{\,2} + \vec{d}^{\,2} - 2k.$

(c) We shall show that there exists a point H such that $AH \perp (DBC)$, $BH \perp (ACD)$, $CH \perp (ABD)$, $DH \perp (ABC)$. Let $\overrightarrow{h} = \overrightarrow{AH} = m\,\overrightarrow{a} + n\,\vec{b} + p\,\vec{c}$. Writing the conditions $\overrightarrow{AH} \perp \overrightarrow{BC}$, $\overrightarrow{CD}$; $\overrightarrow{BH} \perp \overrightarrow{AC}, \overrightarrow{AD}$; $\overrightarrow{CH} \perp \overrightarrow{AB}, \overrightarrow{AD}$; $\overrightarrow{DH} \perp \overrightarrow{AB}, \overrightarrow{AC}$ we obtain a consistent system with one single solution:

$$\begin{cases} b^2 m + kn + kp = k \\ km + c^2 n + kp = k \\ km + kn + d^2 p = k. \end{cases} \tag{5.18}$$

Indeed the the matrix of the system is

$$A = \begin{pmatrix} b^2 & k & k \\ k & c^2 & k \\ k & k & d^2 \end{pmatrix}$$

and for its determinant we have successively

$$\det(A) = \begin{vmatrix} b^2 & k & k \\ k & c^2 & k \\ k & k & d^2 \end{vmatrix} = \begin{vmatrix} b\cdot b & b\cdot c & b\cdot c \\ c\cdot b & c\cdot c & c\cdot d \\ d\cdot b & d\cdot c & d\cdot d \end{vmatrix}$$

$$= \begin{vmatrix} b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 & b_1d_1 + b_2d_2 + b_3d_3 \\ c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 & c_1d_1 + c_2d_2 + c_3d_3 \\ d_1b_1 + d_2b_2 + d_3b_3 & d_1c_1 + d_2c_2 + d_3c_3 & d_1^2 + d_2^2 + d_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 & d_1 \\ b_1 & c_2 & d_2 \\ b_1 & c_3 & d_3 \end{vmatrix} = (\vec{b}, \vec{c}, \vec{d}) \cdot \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = (\vec{b}, \vec{c}, \vec{d})^2.$$

The linear independence of the vectors $\vec{b}, \vec{c}, \vec{d}$ ensure that $(\vec{b}, \vec{c}, \vec{d}) \neq 0$ and shows that the linear system (5.18) is consistent and has one single solution.

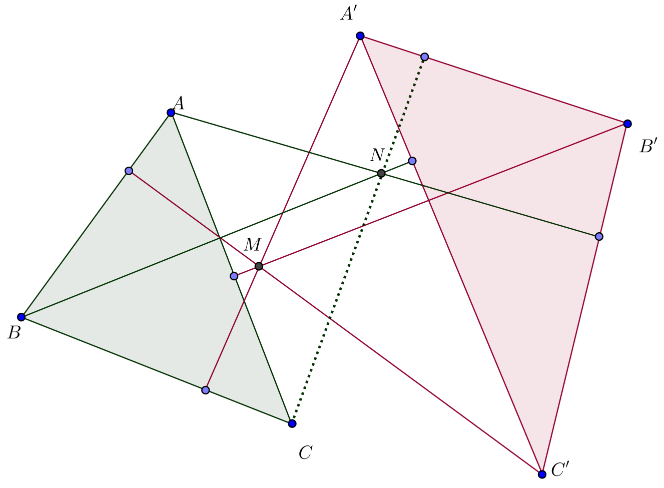
10. Two triangles ABC şi $A'B'C'$ are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices A', B', C' on the sides BC, CA, AB are concurrent. Show

that, in this case, the perpendicular lines from the vertices A, B, C on the sides $B'C', C'A', A'B'$ are concurrent too.

Solution Due to the given hypothesis, we have

$$\overrightarrow{MA'} \cdot \overrightarrow{BC} = \overrightarrow{MB'} \cdot \overrightarrow{CA} = \overrightarrow{MC'} \cdot \overrightarrow{AB} = 0 \tag{5.19}$$

We now consider the perpendicular lines from the vertices A and B on the edges $B'C'$ and $C'A'$ and denote by N their intersection point.



Thus

$$\overrightarrow{NA} \cdot \overrightarrow{B'C'} = \overrightarrow{NB} \cdot \overrightarrow{C'A'} = 0.$$

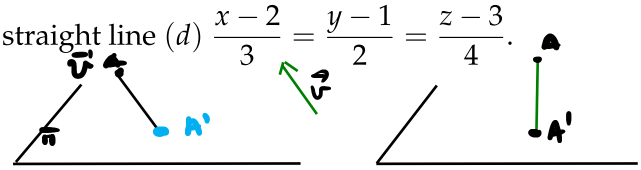
By using the relations (5.19) we obtain

$$\begin{aligned}
& \overrightarrow{MA'} \cdot \overrightarrow{BC} + \overrightarrow{MB'} \cdot \overrightarrow{CA} + \overrightarrow{MC'} \cdot \overrightarrow{AB} = 0 \\
\Leftrightarrow\ & \overrightarrow{MA'} \cdot (\overrightarrow{NC} - \overrightarrow{NB}) + \overrightarrow{MB'} \cdot (\overrightarrow{NA} - \overrightarrow{NC}) + \overrightarrow{MC'} \cdot (\overrightarrow{NB} - \overrightarrow{NA}) = 0 \\
\Leftrightarrow\ & (\overrightarrow{MB'} - \overrightarrow{MC'}) \cdot \overrightarrow{NA} + (\overrightarrow{MC'} - \overrightarrow{MA'}) \cdot \overrightarrow{NB} + (\overrightarrow{MA'} - \overrightarrow{MB'}) \cdot \overrightarrow{NC} = 0 \\
\Leftrightarrow\ & \overrightarrow{C'B'} \cdot \overrightarrow{NA} + \overrightarrow{A'C'} \cdot \overrightarrow{NB} + \overrightarrow{B'A'} \cdot \overrightarrow{NC} = 0 \\
\Leftrightarrow\ & \overrightarrow{B'A'} \cdot \overrightarrow{NC} = 0 \Leftrightarrow NC \perp A'B'.
\end{aligned}$$

11. (**2p**) Find the orthogonal projection

    (a) of the point $A(1,2,1)$ on the plane $\pi: \ x + y + 3z + 5 = 0$.

    (b) of the point $B(5,0,-2)$ on the straight line $(d)\ \dfrac{x-2}{3} = \dfrac{y-1}{2} = \dfrac{z-3}{4}$.



Solution.

(a) $\overrightarrow{r_{P_{\pi,v}(A)}} = \vec{r_A} - \dfrac{F(A)}{\vec{n_\pi} \cdot \vec{v}} \vec{v}$

Since the projection is orthogonal, we have $\vec{v} = \vec{n_\pi}$ ⇒

⇒ $\overrightarrow{r_{P_\pi(A)}} = \vec{r_A} - \dfrac{F(A)}{\|\vec{n_\pi}\|^2} \vec{n_\pi}$ ⇒ $\overrightarrow{r_{P_\pi(A)}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \dfrac{1+2+3+5}{1^2+1^2+3^2} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} =$

$= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \dfrac{11}{11} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ ⇒ $P_\pi(A)$ has the coords $(0,1,-2)$

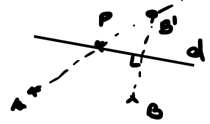
b) orthogonal projection of $B(5,0,-2)$ on straight line (d) : $\frac{x-2}{\underset{p}{3}} = \frac{y-1}{\underset{q}{2}} = \frac{z-3}{\underset{r}{4}}$ $\begin{cases} x_0 = 2 \\ y_0 = 1 \\ z_0 = 3 \end{cases}$

$G(x,y,z) = p(x-x_B) + q(y-y_B) + r(z-z_B)$
$= p(x-5) + q \cdot y + r(z+2)$

$\Rightarrow G(x_0,y_0,z_0) = 3(2-5) + 2 \cdot 1 + 4(3+2) =$
$= 3 \cdot (-3) + 2 + 4 \cdot 5 =$
$= -9 + 2 + 20 = 13$

$\Rightarrow$ the coordinates of the projection are $\begin{cases} x_0 - p\frac{G(x_0,y_0,z_0)}{p^2+q^2+r^2} \\ y_0 - q\frac{G(x_0,y_0,z_0)}{p^2+q^2+r^2} \\ z_0 - r\frac{G(x_0,y_0,z_0)}{p^2+q^2+r^2} \end{cases} \Rightarrow \begin{cases} 2 - \frac{13}{29} \cdot 3 \\ 1 - \frac{13}{29} \cdot 2 \\ 3 - \frac{13}{29} \cdot 4 \end{cases} \Rightarrow \begin{cases} \frac{58-39}{29} \\ \frac{29-26}{29} \\ \frac{87-52}{29} \end{cases} \Rightarrow \begin{cases} \frac{19}{29} \\ \frac{3}{29} \\ \frac{35}{29} \end{cases}$

$\Rightarrow P_d(B)\left(\frac{19}{29}, \frac{3}{29}, \frac{35}{29}\right)$



A few questions in the two dimensional setting

12. (**3p**) Find the coordinates of the point P on the line $d : 2x - y - 5 = 0$ for which the sum $AP + PB$ is minimum, when $A(-7,1)$ and $B(-5,5)$.

$P(x_P,y_P) \in d \Rightarrow 2x_P - y_P - 5 = 0$, $AP = \sqrt{(x_P+7)^2+(y_P-1)^2}$, $PB = \sqrt{(-5-x_P)^2+(5-y_P)^2} = \sqrt{(x_P+5)^2+(5-y_P)^2}$

The eq of AB: $m_{AB} = \frac{y_B-y_A}{x_B-x_A} = \frac{5-1}{-5+7} = \frac{4}{2} = 2 \Rightarrow d_{AB}: y - y_B = m_{AB}(x-x_B) \Leftrightarrow 2x - y + 15 = 0$

$d \cap AB = \{M(x_M,y_M)\} \Leftrightarrow \begin{cases} 2x_M - y_M - 5 = 0 \\ 2x_M - y_M + 15 = 0 \end{cases}$, $\Rightarrow$ d and d_{AB} do not intersect $\Rightarrow A, B$ are on the same side of D

The refl. of B w.r.t d: $[R_d(B)] = \frac{1}{2^2+(-1)^2} \cdot \begin{pmatrix} -2^2+(-1)^2 & -2\cdot 2(-1) \\ -2\cdot 2\cdot(-1) & 2^2-(-1)^2 \end{pmatrix} \begin{bmatrix} -5 \\ 5 \end{bmatrix} - \frac{2\cdot(-5)}{2^2+(-1)^2} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \ldots = \begin{pmatrix} 11 \\ -3 \end{pmatrix} \Rightarrow$

$\Rightarrow B'(11,-3)$ is the refl of B w.r.t d

$d \cap AB' = \{P(x_P,y_P)\}$, $AB': y - 1 = \frac{-3-1}{11+7} \cdot (x+7) \Leftrightarrow y - 1 = \frac{-4}{18}(x+7) /\cdot 9 \Leftrightarrow 2x + 9y + 5 = 0$

$\Rightarrow \begin{cases} 2x_P - y_P - 5 = 0 \\ 2x_P + 9y_P + 5 = 0 \end{cases} \Rightarrow 10y_P + 10 = 0 \Rightarrow \begin{matrix} y_P = -1 \\ x_P = 2 \end{matrix} \Rightarrow P(2,-1)$

13. (**2p**) Find the coordinates of the circumcenter (the center of the circumscribed circle) of the triangle determined by the lines $\underbrace{4x - y + 2 = 0}_{d_1}$, $\underbrace{x - 4y - 8 = 0}_{d_2}$ and $\underbrace{x + 4y - 8 = 0}_{d_3}$.

Solution.

$d_1 \cap d_2 = \{A\} \Rightarrow \begin{cases} 4x - y + 2 = 0 \\ x - 4y - 8 = 0 /\cdot(-4) \end{cases} \Leftrightarrow \begin{cases} 4x - y + 2 = 0 \\ -4x + 16y + 32 = 0 \end{cases} \Rightarrow 15y = -34 \Rightarrow y = \frac{-34}{15} \Rightarrow$

$\Rightarrow 4x + \frac{34}{15} + 2 = 0 \Leftrightarrow 4x = \frac{-64}{15} /:4 \Leftrightarrow x = \frac{-16}{15} \Rightarrow A\left(-\frac{16}{15}, -\frac{34}{15}\right)$

$d_2 \cap d_3 = \{B\} \Rightarrow \begin{cases} x - 4y - 8 = 0 \\ x + 4y - 8 = 0 \end{cases} \Leftrightarrow 2x = 16 /:2 \Leftrightarrow x = 8 \Rightarrow 4y = 8 - x \Rightarrow$

$\Rightarrow 4y = 0 \Rightarrow y = 0 \Rightarrow B(8,0)$

$d_1 \cap d_3 = \{C\} \Rightarrow \begin{cases} 4x - y + 2 = 0 /\cdot 4 \\ x + 4y - 8 = 0 \end{cases} \Leftrightarrow \begin{cases} 16x - 4y + 8 = 0 \\ x + 4y - 8 = 0 \end{cases} \Rightarrow 17x = 0 \Rightarrow x = 0 \Rightarrow$

$\Rightarrow +y = +2 \Rightarrow C(0,+2)$

$O(x,y) \Rightarrow |OA| = |OB| = |OC|$

$d_1^2 = \left(x + \frac{16}{15}\right)^2 + \left(y + \frac{34}{15}\right)^2$; $d_2^2 = (x-8)^2 + (y-0)^2 = (x-8)^2 + y^2$

$d_1^2 = d_2^2 \Leftrightarrow \left(x + \frac{16}{15}\right)^2 + \left(y + \frac{34}{15}\right)^2 = (x-8)^2 + y^2 \Leftrightarrow x = \frac{191}{60} - \frac{1}{4}y /\cdot 4 \Rightarrow -y = 4x - \frac{191}{15}$

$d_3^2 = (x-0)^2 + (y-2)^2 = x^2 + (y-2)^2$

$d_2^2 = d_3^2 \Leftrightarrow (x-8)^2 + y^2 = x^2 + (y-2)^2 \Leftrightarrow 16x - 4y - 60 = 0 \Leftrightarrow 4x - y - 15 = 0 \Rightarrow$

$\Rightarrow x = \frac{52}{15}, y = \frac{-17}{15}$

14. (**3p**) Given the bundle of lines of equations $(1-t)x+(2-t)y+t-3=0$, $t\in\mathbb{R}$ and $x+y-1=0$, find:

   (a) the coordinates of the vertex of the bundle;

   (b) the equation of the line in the bundle which cuts Ox and Oy in M respectively N, such that $OM^2\cdot ON^2=4(OM^2+ON^2)$.

Solution. $A(x_A, y_A)$ - vertex of the bundle

a) $\begin{cases}(1-t)x_A+(2-t)y_A+t-3=0, \forall t\in\mathbb{R}\\ x_A+y_A-1=0\end{cases} \Leftrightarrow \begin{cases}(-x_A-y_A+1)t+x_A+2y_A-3=0, \forall t\in\mathbb{R}\\ -x_A-y_A+1=0\end{cases} \Leftrightarrow \begin{cases}-x_A-y_A+1=0\\ x_A+2y_A-3=0\end{cases} \oplus \Rightarrow y_A=2 \Rightarrow x_A=-1 \Rightarrow$

$\Rightarrow A(-1,2)$

b) Check if $d_1: x+y-1=0$ is a solution $\Rightarrow d_1\cap Ox: y=0 \Rightarrow x+0-1=0 \Rightarrow x=1 \Rightarrow d_1\cap Ox=\{M_1(1,0)\}$

$\Rightarrow d_1\cap Oy: x=0 \Rightarrow 0+y-1=0 \Rightarrow y=1 \Rightarrow d_1\cap Oy=\{N_1(0,1)\}$

$\left.\begin{array}{l}OM_1^2=1, ON_1^2=1 \Rightarrow OM_1^2\cdot ON_1^2=1\\ 4(OM_1^2+ON_1^2)=4(1+1)=8\end{array}\right\} \Rightarrow d_1$ is not a solution

We search for $d: (1-t)x+(2-t)y+t-3=0, t\in\mathbb{R}$

If $t=1 \Rightarrow d\parallel Ox$, $t=2 \Rightarrow d\parallel Oy$, So we put the cond: $t\neq 1, t\neq 2$

Let $d\cap Ox=\{M(x_M,0)\}$, $d\cap Oy=\{N(0,y_N)\} \Rightarrow OM=|x_M|, ON=|y_N| \Rightarrow OM^2=x_M^2, ON^2=y_N^2 \Rightarrow x_M^2 y_N^2=4(x_M^2+y_N^2)$ (1)

$M(x_M,0)\in d \Rightarrow (1-t)x_M+t-3=0$ (2)

$N(0,y_N)\in d \Rightarrow (2-t)y_N+t-3=0$ (3)

From (1),(2),(3) $\Rightarrow \begin{cases}(1-t)x_M+t-3=0\\ (2-t)y_N+t-3=0\\ x_M^2 y_N^2=4(x_M^2+y_N^2)\end{cases} \Rightarrow \begin{cases}x_M=\frac{3-t}{1-t}\\ y_N=\frac{3-t}{2-t}\\ x_M^2 y_N^2=4(x_M^2+y_N^2)\end{cases} \Rightarrow \left(\frac{3-t}{1-t}\right)^2\cdot\left(\frac{3-t}{2-t}\right)^2=4\cdot(3-t)^2\cdot\left(\frac{1}{(1-t)^2}+\frac{1}{(2-t)^2}\right) \Leftrightarrow$

$\Leftrightarrow \frac{(3-t)^4}{(1-t)^2(2-t)^2}=4(3-t)^2\cdot\frac{(2-t)^2+(1-t)^2}{(1-t)^2(2-t)^2} \Leftrightarrow (3-t)^4-4(3-t)^2[(2-t)^2+(1-t)^2]=0 \Leftrightarrow$

$\Leftrightarrow (3-t)^2[(3-t)^2-4(2-t)^2-4(1-t)^2]=0$

I. $(3-t)^2=0 \Rightarrow 3-t=0 \Rightarrow t_1=3 \Rightarrow x_M=y_N=0 \Rightarrow d\cap Ox=d\cap Oy=O(0,0) \Rightarrow OM=ON=0$ and $d: -2x-y=0$

II. $9-6t+t^2-4(4-4t+t^2)-4(1-2t+t^2)=0 \Leftrightarrow 9-6t+t^2-16+16t-4t^2-4+8t-4t^2=0 \Leftrightarrow$

$\Leftrightarrow -7t^2+18t-11=0 \rightarrow t_2=1$, False

$\searrow t_3=\frac{11}{7} \Rightarrow \begin{cases}x_M=\frac{3-\frac{11}{7}}{1-\frac{11}{7}}=\frac{21-11}{7-11}=\frac{10}{-4}=-\frac{5}{2}\\ y_N=\frac{3-\frac{11}{7}}{2-\frac{11}{7}}=\frac{21-11}{14-11}=\frac{10}{3}\end{cases} \Rightarrow d': -\frac{4}{7}x+\frac{3}{7}y-\frac{10}{7}=0 /\cdot 7 \Rightarrow$

$\Rightarrow -4x+3y-10=0$

From I, II $\Rightarrow d: -2x-y=0$ and $d': -4x+3y-10=0$ are solutions

15. (**2p**) Let $\mathcal{B}$ be the bundle of lines of vertex $M_0(5,0)$. An arbitrary line from $\mathcal{B}$ intersects the lines $d_1 : y-2=0$ and $d_2 : y-3=0$ in M_1 respectively M_2. Prove that the line passing through M_1 and parallel to OM_2 passes through a fixed point.

On the next page.

16. (**3p**) The vertices of the quadrilateral $ABCD$ are $A(4,3)$, $B(5,-4)$, $C(-1,-3)$ and $D((-3,-1)$.

    (a) Find the coordinates of the intersection points $\{E\} = AB \cap CD$ and $\{F\} = BC \cap AD$;

    (b) Prove that the midpoints of the segments $[AC]$, $[BD]$ and $[EF]$ are collinear.

Solution. $E(x_E, y_E) \in AB \cap CD$; $F(x_F, y_F) \in BC \cap AD$

a) $AB: \dfrac{x-x_A}{x_B-x_A} = \dfrac{y-y_A}{y_B-y_A} \Leftrightarrow AB: \dfrac{x-4}{1} = \dfrac{y-3}{-7} \Rightarrow AB: 7x+y-31=0$

$CD: \dfrac{x-x_C}{x_D-x_C} = \dfrac{y-y_C}{y_D-y_C} \Leftrightarrow CD: \dfrac{x+1}{-2} = \dfrac{y+3}{2} \Rightarrow CD: x+y+4=0$

$\Rightarrow \begin{cases} 7x_E+y_E-31=0 \\ x_E+y_E+4=0 \end{cases} \Rightarrow 6x_E-35 \Rightarrow x_E = \dfrac{35}{6}, \quad y_E = -\dfrac{59}{6}$

$\Rightarrow E\left(\dfrac{35}{6}, -\dfrac{59}{6}\right)$

$BC: \dfrac{x-x_B}{x_C-x_B} = \dfrac{y-y_B}{y_C-y_B} \Leftrightarrow BC: \dfrac{x-5}{-6} = \dfrac{y+4}{1} \Rightarrow BC: x+6y+19=0$

$AD: \dfrac{x-x_A}{x_D-x_A} = \dfrac{y-y_A}{y_D-y_A} \Leftrightarrow AD: \dfrac{x-4}{-7} = \dfrac{y-3}{-4} \Rightarrow AD: 4x-7y+5=0$

$\Rightarrow \begin{cases} x_F+6y_F+19=0 \\ 4x_F-7y_F+5=0 \end{cases} \Leftrightarrow 31y_F = -71 \Rightarrow y_F = \dfrac{-71}{31} \Rightarrow x_F = \dfrac{-163}{31}$

$\Rightarrow F\left(\dfrac{-163}{31}, -\dfrac{71}{31}\right)$

b) Let M - midpoint of $[AC]$, N - midpoint of $[BD]$, P - midpoint of $[EF]$ $\Rightarrow \begin{cases} M\left(\dfrac{3}{2}, 0\right) \\ N\left(1, -\dfrac{5}{2}\right) \\ P\left(\dfrac{107}{372}, -\dfrac{2255}{372}\right) \end{cases}$

$MN: \dfrac{x-x_M}{x_N-x_M} = \dfrac{y-y_M}{y_N-y_M} \Leftrightarrow MN: \dfrac{x-\frac{3}{2}}{1-\frac{3}{2}} = \dfrac{y}{-\frac{5}{2}} \Leftrightarrow MN: \dfrac{2x-3}{-1} = \dfrac{2y}{-5} \Rightarrow MN: 10x-2y-15=0$

V: $10x_P - 2y_P - 15 = 10 \cdot \dfrac{107}{372} + 2 \cdot \dfrac{2255}{372} - 15 = \dfrac{1070+4510-5580}{372} = 0 \Rightarrow P \in MN \Rightarrow M, N, P$ - collinear

Exercise 15.

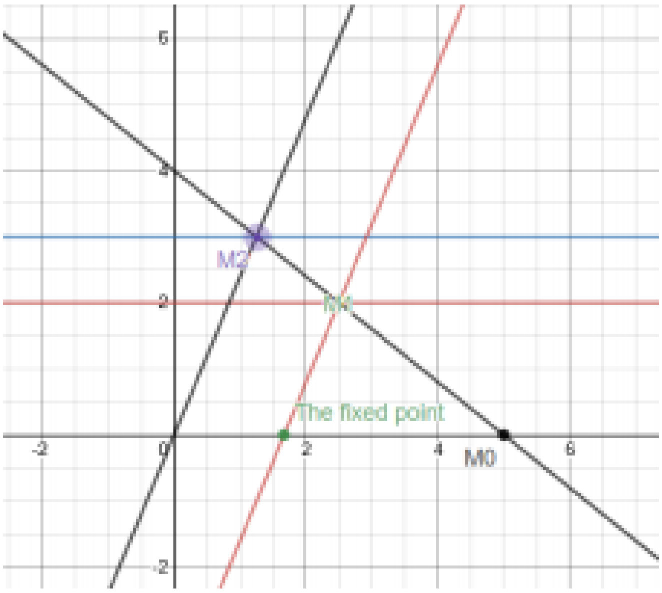
Let β be the bundle of vertex $M_0(5,0)$. An arbitrary line from β intersects the lines $d_1: y-2=0$, $d_2: y-3=0$ in M_1 respectively M_2. Prove that the line passing through M_1 and parallel to OM_2 passes through a fixed point.

Let $l: r(x-x_0)+s(y-y_0)=0, \forall(r,s)\in\mathbb{R}^2\{(0,0)\}$

$$l: r(x-5)+sy=0, \forall(r,s)\in\mathbb{R}^2\{(0,0)\}$$

$$\begin{cases} y-2=0 \\ r(x-5)+sy=0 \end{cases} \Rightarrow \begin{cases} y=2 \\ r(x-5)+2s=0 \end{cases} \Rightarrow \begin{cases} y=2 \\ x=5-\dfrac{2s}{r} \end{cases} \Rightarrow M_1(5-\frac{2s}{r},2)$$

$$\begin{cases} y-3=0 \\ r(x-5)+sy=0 \end{cases} \Rightarrow \begin{cases} y=3 \\ r(x-5)+3s=0 \end{cases} \Rightarrow \begin{cases} y=3 \\ x=5-\dfrac{3s}{r} \end{cases} \Rightarrow M_2(5-\frac{3s}{r},3)$$



$$OM_2: \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 5-\dfrac{3s}{r} & 3 & 1 \end{vmatrix}=0$$

$$OM_2: \left(5-\frac{3s}{r}\right)y-3x=0 \Rightarrow m=\frac{3}{5-\frac{3s}{r}}=\frac{3r}{5r-3s}$$

$d: y-y_0=m(x-x_0)$, d – parallel with OM_2 going through M_1.

$$d: y-2=\frac{3r}{5r-3s}\left(x-5+\frac{2s}{r}\right)$$

$$d: y-2=\frac{3r}{5r-3s}x-\frac{15r}{5r-3s}+\frac{6s}{5r-3s} \mid \cdot(5r-3s)$$

$$d: (5r-3s)y-10r+6s=3rx-15r+6s$$

$$d: 3rx+(3s-5r)y-5r=0$$

Let A be a possible fixed point. If it does exist, we can find it at the intersection of 2 different lines d, generated by the bundle.

$$r=1, s=0: 3x-5y-5=0$$

$$r=0, s=1: 3y=0$$

$$\begin{cases} 3x-5y-5=0 \\ 3y=0 \end{cases} \Rightarrow \begin{cases} 3x-5=0 \\ y=0 \end{cases} \Rightarrow \begin{cases} x=\dfrac{5}{3} \\ y=0 \end{cases} \Rightarrow A\left(\frac{5}{3},0\right)$$

If A is a fixed point then $A\in d, \forall(r,s)\in\mathbb{R}^2\{(0,0)\}$.

$3r\cdot\frac{5}{3}+(3s-5r)\cdot 0-5r=5r-5r=0 \Rightarrow A$ is a fixed point on d.

17. (**3p**) Let M be a point whose coordinates satisfy

$$\frac{4x+2y+8}{3x-y+1} = \frac{5}{2}.$$

(a) Prove that M belongs to a fixed line (d);

(b) Find the minimum of x^2+y^2, when $M \in d \setminus \{M_0(-1,-2)\}$.

Solution.

(a) $2(4x+2y+8) = 5(3x-y+1) \Rightarrow 8x+4y+16 = 15x-5y+5 \Rightarrow$

$\Rightarrow 7x-9y-11=0$ - line l

(b) $x^2+y^2 \geq 2xy$ with equality if $x=y$

We will look at the point on the line that satisfies this, which is $Q(a,a) \Rightarrow$

$\Rightarrow 7a-9a-11=0 \Rightarrow a=-\frac{11}{2} \Rightarrow \min(x^2+y^2) = 2\cdot\frac{121}{4} = \frac{121}{2}$

$(x,y) \neq (-1,-2)$

18. (**3p**) Find the locus of the points whose distances to two orthogonal lines have a constant ratio.
Solution.

Let $R = (O, \vec{i}, \vec{j})$ the orthonormal reference of the plane and (d_1, d_2) two orthogonal lines in the plane, and consider $\{E\} = d_1 \cap d_2$, and let S be the locus of the points whose distances to d_1 and d_2 have the same ratio.

Denote $T = \{S, d_1, d_2, E\}$ the system determined by the orthogonal lines, their intersection and S and denote $\theta = (\widehat{\vec{d_1}, \vec{i}})$ the oriented angle determined by the oriented directions $d_1$and i(which orientation we choose for d_1 does not influence the result.). Applying to the system T the translation by the vector $\overrightarrow{EO}$ and the rotation of center O and angle $-\theta$ we overlap the lines d_1, d_2 and the point E over the axes of the plane. Now because those 2 trasformations preserved all the distances in the system T, it is enough to determine the locus S' that resulted after the transformations.

That being said, we are going to determine the locus S' of points whose distances to the axes Ox and Oy have the same ratio.

Let $P(x, y) \in S'$ and deote $k \in \mathbb{R}$ the ratio. $\Rightarrow \left|\frac{x}{y}\right| = k \Rightarrow x = ky$,and therefore we obtain the equation of 2 lines:

$$l_1 : x - ky = 0 \tag{1}$$

$$l_2 : x + ky = 0 \tag{2}$$

We do not take into consideration the ratio $\frac{y}{x}$; that would make no sense since it would inverse the order of the lines. However, all that it would have changed is an extra line belonging to S.

Because $\arctan\left(\frac{1}{k}\right)$ (k = 0 treated separately) is the oriented angle of lines $\vec{l_1}$ and $\vec{i}$ by reversing the rotation we get the oriented angle $\arctan\left(\frac{1}{k}\right) + \theta$ for the original direction and the Ox axis, so $m_1 = \tan\left(\arctan\left(\frac{1}{k}\right) + \theta\right)$ is the original slope of one of the lines in S and $m_2 = \tan\left(\arctan\left(-\frac{1}{k}\right) + \theta\right)$ Now because the original lines go through point $E(e_1, e_2)$ and therefore we can compute theor original equations:

$$l_1' : (y - e_1) = m_1(x - e_2) \tag{3}$$

$$l_2' : (y - e_1) = m_2(x - e_2) \tag{4}$$

In conclusion, the set S is constituted by 2 straight lines passing through E that are symmetrical with respect to the first line of the orthogonal lines that we consider the distance to.