

13 Week 13

13.1 Homogeneous coordinates

The affine transformation

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, L(x, y) = (ax + by + c, dx + ey + f)$$

can be written by using the matrix language and by equations:

1. (a) identifying the vectors $(x, y) \in \mathbb{R}^2$ with the line matrices $[x \ y] \in \mathbb{R}^{1 \times 2}$ and implicitly \mathbb{R}^2 with $\mathbb{R}^{1 \times 2}$:

$$L[x \ y] = [x \ y] \begin{bmatrix} a & d \\ b & e \end{bmatrix} + [c \ f].$$

- (b) identifying the vectors $(x, y) \in \mathbb{R}^2$ with the column matrices $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2 \times 1}$ and implicitly \mathbb{R}^2 with $\mathbb{R}^{2 \times 1}$:

$$L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}.$$

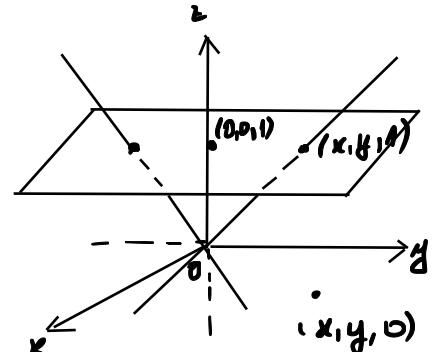
2. $\begin{cases} x' = ax + by + c \\ y' = dx + ey + f. \end{cases} \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$

Observe that the representation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$



In this lesson we identify the points $(x, y) \in \mathbb{R}^2$ with the points $(x, y, 1) \in \mathbb{R}^3$ and even with the punctured lines of \mathbb{R}^3 , (rx, ry, r) , $r \in \mathbb{R}^*$. Due to technical reasons we shall actually identify the points $(x, y) \in \mathbb{R}^2$ with the punctured lines of \mathbb{R}^3 represented in the form

$$\mathbb{R}^3 - \{(0, 0, 0)\} \quad \begin{bmatrix} rx \\ ry \\ r \end{bmatrix}, r \in \mathbb{R}^*, \quad \mathbb{R}^3 \setminus \{(0, 0, 0)\}$$

and the latter ones we shall call *homogeneous coordinates* of the point $(x, y) \in \mathbb{R}^2$. The set of homogeneous coordinates (x, y, w) will be denoted by \mathbb{RP}^2 and call it the real *projective plane*. The homogeneous coordinates $(x, y, w) \in \mathbb{RP}^2$, $w \neq 0$ și $(\frac{x}{w}, \frac{y}{w}, 1)$ represent the same element of \mathbb{RP}^2 .

Remark 13.1. The projective plane \mathbb{RP}^2 is actually the quotient set $(\mathbb{R}^3 \setminus \{(0, 0, 0)\}) / \sim'$, where ' \sim' ' is the following equivalence relation on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$:

$$(x, y, w) \sim (\alpha, \beta, \gamma) \Leftrightarrow \exists r \in \mathbb{R}^* \text{ a.s.t. } (x, y, w) = r(\alpha, \beta, \gamma).$$

$$[(x, y, z)] = \{(rx, ry, rz) / r \in \mathbb{R}^*\}$$



Observe that the equivalence classes of the equivalence relation \sim' are the punctured lines of \mathbb{R}^3 through the origin without the origin itself, i.e. the elements of the real projective plane \mathbb{RP}^2 . By the column matrix

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

we also denote the equivalence class of $(x, y, w) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. The meaning of this notation will be understood, each time, from the context.

Definition 13.1. A projective transformation of the projective plane \mathbb{RP}^2 is a transformation

$$L\left(\begin{matrix} x \\ y \\ z \end{matrix}\right) = \lambda L\left(\begin{matrix} x \\ y \\ z \end{matrix}\right) \quad L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$L: \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2, \quad L \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} ax + by + cw \\ dx + ey + fw \\ gx + hy + kw \end{bmatrix}, \quad (13.1)$$

where $a, b, c, d, e, f, g, h, k \in \mathbb{R}$. Note that

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

is called the *homogeneous transformation matrix* of L . Note that every nonsingular 3×3 matrix defines a projective transformation. Also, if λ is a nonzero real scalar, then the nonsingular matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \text{ and } \lambda \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

define the same projective transformation \mathbb{RP}^2 . Therefore, the homogeneous transformation matrix of a projective transformation is an entire class, appearing as an element, of a projective space rather than one single matrix.

Observe that a projective transformation (14.2) is well defined since

$$L \begin{bmatrix} rx \\ ry \\ rw \end{bmatrix} = \begin{bmatrix} arx + bry + crw \\ drx + ery + frw \\ grx + hry + krw \end{bmatrix} = \begin{bmatrix} r(ax + by + cw) \\ r(dx + ey + fw) \\ r(gx + hy + kw) \end{bmatrix}.$$

If $g = h = 0$ and $k \neq 0$, then the projective transformation (14.2) is said to be *affine*. The restriction of the affine transformation (14.2), which corresponds to the situation $g = h = 0$ and $k = 1$, to the subspace $w = 1$, has the form

$$L \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + cw \\ dx + ey + fw \\ 1 \end{bmatrix}, \quad (13.2)$$

i.e.

$$L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix} \quad \begin{cases} x' = ax + by + c \\ y' = dx + ey + f. \end{cases} \quad (13.3)$$

Remark 13.2. If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two projective applications, then their product (concatenation) transformation $L_1 \circ L_2$ is also a projective transformation and its homogeneous transformation matrix is the product of the homogeneous transformation matrices of L_1 and L_2 .

Indeed, if

$$L_1 \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

and

$$L_2 \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

then

$$(L_1 \circ L_2) \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \left(\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Remark 13.3. If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two affine applications, then their product $L_1 \circ L_2$ is also an affine transformation.

13.2 Transformations of the plane in homogeneous coordinates

In this section we shall identify an affine transformation of \mathbb{RP}^2 with its homogeneous transformation matrix

13.3 Translations and scalings

- The homogeneous transformation matrix of the translation $T(h, k)$ is

$$T(h, k) = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}.$$

- The homogeneous transformation matrix of the scaling $S(s_x, s_y)$ is

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Scaling about $P_0(x_0, y_0)$

$$\begin{aligned} T(x_0, y_0) \cdot S(s_x, s_y) \cdot T(-x_0, -y_0) &= \\ \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix} &= \\ \begin{pmatrix} s_x & 0 & x_0 \\ 0 & s_y & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix} &= \\ \begin{pmatrix} s_x & 0 & (1-s_x)x_0 \\ 0 & s_y & (1-s_y)y_0 \\ 0 & 0 & 1 \end{pmatrix} & \end{aligned}$$

13.4 Reflections

- The homogeneous transformation matrix of reflection r_x about the x -axis is

$$r_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- The homogeneous transformation matrix of reflection r_y about the y -axis is

$$r_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- The homogeneous transformation matrix of reflection r_l about the line $l : ax + by + c = 0$ is

$$r_l = \begin{bmatrix} \frac{b^2 - a^2}{a^2 + b^2} & -\frac{2ab}{a^2 + b^2} & -\frac{2ac}{a^2 + b^2} \\ \frac{2ab}{a^2 + b^2} & \frac{a^2 - b^2}{a^2 + b^2} & -\frac{2bc}{a^2 + b^2} \\ -\frac{a^2 + b^2}{a^2 + b^2} & \frac{a^2 + b^2}{a^2 + b^2} & \frac{a^2 + b^2}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since in homogeneous coordinates multiplication by a factor does not affect the result, the above matrix can be multiplied by a factor $a^2 + b^2$ to give the homogeneous matrix of a general reflection

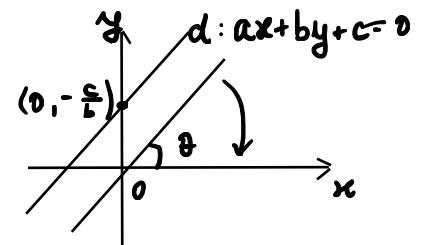
$$\begin{bmatrix} b^2 - a^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 & -2bc \\ 0 & 0 & a^2 + b^2 \end{bmatrix}.$$

Example 13.1. Consider a line (d) $ax + by + c = 0$ whose slope is $\tan \theta = -\frac{a}{b}$. By using the observation that the reflection r_d in the line d is the following concatenation (product)

$$r_d = T(0, -c/b) \circ R_\theta \circ r_x \circ R_{-\theta} \circ T(0, c/b),$$

one can show that the homogeneous transformation matrix of r_d is

$$\begin{bmatrix} \frac{b^2 - a^2}{a^2 + b^2} & -\frac{2ab}{a^2 + b^2} & -\frac{2ac}{a^2 + b^2} \\ \frac{2ab}{a^2 + b^2} & \frac{a^2 - b^2}{a^2 + b^2} & -\frac{2bc}{a^2 + b^2} \\ -\frac{a^2 + b^2}{a^2 + b^2} & \frac{a^2 + b^2}{a^2 + b^2} & \frac{a^2 + b^2}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$



Solution. The homogeneous matrix of the concatenation

$$T(0, -c/b) \circ R_\theta \circ r_x \circ R_{-\theta} \circ T(0, c/b)$$

is

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c/b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c/b \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & 2 \sin \theta \cos \theta & 2 \frac{c}{b} \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta & \sin^2 \theta - \cos^2 \theta & \frac{c}{b} (\sin^2 \theta - \cos^2 \theta - 1) \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (13.4)$$

Since $\operatorname{tg}\theta = -\frac{a}{b}$, it follows that $\frac{a^2}{b^2} = \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{\sin^2 \theta}{1 - \sin^2 \theta} = \frac{1 - \cos^2 \theta}{\cos^2 \theta}$, namely

$$\sin^2 \theta = \frac{a^2}{a^2 + b^2} \text{ and } \cos^2 \theta = \frac{b^2}{a^2 + b^2}.$$

Thus

$$\sin \theta = \pm \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \cos \theta = \mp \frac{b}{\sqrt{a^2 + b^2}}, \text{ as } \frac{\sin \theta}{\cos \theta} = \operatorname{tg}\theta = -\frac{a}{b}.$$

Therefore $\sin \theta \cos \theta = -\frac{ab}{a^2 + b^2}$ and the matrix (13.4) becomes

$$\begin{bmatrix} \frac{b^2 - a^2}{a^2 + b^2} & -\frac{2ab}{a^2 + b^2} & -\frac{c}{b} \frac{2ab}{a^2 + b^2} \\ -\frac{2ab}{a^2 + b^2} & \frac{a^2 - b^2}{a^2 + b^2} & \frac{c}{b} \left(\frac{a^2 - b^2}{a^2 + b^2} - 1 \right) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{b^2 - a^2}{a^2 + b^2} & -\frac{2ab}{a^2 + b^2} & -\frac{2ac}{a^2 + b^2} \\ -\frac{2ab}{a^2 + b^2} & \frac{a^2 - b^2}{a^2 + b^2} & -\frac{2bc}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

13.5 Rotations

The homogeneous transformation matrix of the rotation R_θ about the origin through an angle θ is

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 13.2. The homogeneous transformation matrix of the product (concatenation) $T(h, k) \circ R_\theta$ is the product

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & h \\ \sin \theta & \cos \theta & k \\ 0 & 0 & 1 \end{bmatrix}.$$

In order to find the homogeneous transformation matrix of the inverse transformation

$$(T(h, k) \circ R_\theta)^{-1} = R_\theta^{-1} \circ T(h, k)^{-1} = R_{-\theta} \circ T(-h, -k)$$

of the product (concatenation) homogeneous transformation $T(h, k) \circ R_\theta$ we can either multiply the homogeneous transformation matrices of the inverse transformations $R_\theta^{-1} = R_\theta$ and $T(h, k)^{-1} =$

$T(-h, -k)$ or use the next proposition. The product of the homogeneous transformation matrices of the inverse transformations $R_\theta^{-1} = R_\theta$ and $T(h, k)^{-1} = T(-h, -k)$ is

$$\begin{aligned} & \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} = \\ & = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} = \\ & = \begin{bmatrix} \cos \theta & \sin \theta & -h \cos \theta - k \sin \theta \\ -\sin \theta & \cos \theta & h \sin \theta - k \cos \theta \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Proposition 13.1. A homogeneous transformation L is invertible if and only if its homogeneous transformation matrix, say T , is invertible and T^{-1} is the transformation matrix of L^{-1} .

Proof. Suppose that L has an inverse L^{-1} with transformation matrix T_1 . The product transformation $L \circ L^{-1} = id$ has the transformation matrix $TT_1 = I_3$. Similarly, $L^{-1} \circ L = I_3$ has the transformation matrix $T_1T = I_3$. Thus $T_1 = T^{-1}$. Conversely, assume that T has an inverse T^{-1} , and let L_1 be the homogeneous transformation defined by T^{-1} . Since $TT^{-1} = I_3$ and $T^{-1}T = I_3$, it follows that $L \circ L_1 = I$ and $L_1 \circ L = I$. Hence L_1 is the inverse transformation of L .

Example 13.3. The homogeneous transformation matrix of inverse

$$(T(h, k) \circ R_\theta)^{-1} = R_\theta^{-1} \circ T(h, k)^{-1} = R_{-\theta} \circ T(-h, -k)$$

of the product (concatenation) homogeneous transformation $T(h, k) \circ R_\theta$ is the matrix

$$\begin{bmatrix} \cos \theta & -\sin \theta & h \\ \sin \theta & \cos \theta & k \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos \theta & \sin \theta & -h \cos \theta - k \sin \theta \\ -\sin \theta & \cos \theta & h \sin \theta - k \cos \theta \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \text{The shear about the point} \\ P_0(x_0, y_0) \end{array}$$

$\begin{array}{c} T(x_0, y_0) \circ S_{P_0}(v, r) \circ T(-x_0, y_0) \\ \left(\begin{array}{ccc} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{array} \right) \begin{array}{c} \left(\begin{array}{ccc} 1 - rv_1 v_2 & rv_1^2 & 0 \\ -rv_2^2 & 1 + rv_1 v_2 & 0 \\ 0 & 0 & 1 \end{array} \right) \left(\begin{array}{ccc} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{array} \right) \\ = \end{array} \end{array}$

13.6 Shears

The homogeneous transformation matrix of the shear is

$$[Sh(v, r)] = \begin{bmatrix} 1 - rv_1 v_2 & rv_1^2 & 0 \\ -rv_2^2 & 1 + rv_1 v_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - rv_1 v_2 & rv_1^2 & -x_0(1 - rv_1 v_2) - y_0(rv_1 v_2) \\ -rv_2^2 & 1 + rv_1 v_2 & -x_0(rv_2^2) - y_0(1 + rv_1 v_2) \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - rv_1 v_2 & rv_1^2 & x_0(1 - rv_1 v_2) - y_0(rv_1 v_2) \\ -rv_2^2 & 1 + rv_1 v_2 & x_0(rv_2^2) - y_0(1 + rv_1 v_2) \\ 0 & 0 & 1 \end{pmatrix}$$

13.7 Problems

- Find the concatenation (product) of an anticlockwise rotation about the origin through an angle of $\frac{3\pi}{2}$ followed by a scaling by a factor of 3 units in the x -direction and 2 units in the y -direction.
(Hint: $S(3, 2)R_{3\pi/2}$)

Solution $[R_\theta] = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow [R_{\frac{3\pi}{2}}] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$[S(3, 2)] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow [S(3, 2) \circ R_{\frac{3\pi}{2}}] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$$

$$\sim [S(3, 2) \circ R_{\frac{3\pi}{2}}] \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 3y \\ -2x \\ 1 \end{pmatrix} \Rightarrow (S(3, 2) \circ R_{\frac{3\pi}{2}}) \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 3y \\ -2x \\ 1 \end{pmatrix}$$

2. Find the homogeneous matrix of the product (concatenation) $S(3, 2) \circ R_{\frac{3\pi}{2}}$.

Solution

3. Find the equations of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ .

Solution The homogeneous transformation matrix of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ is

$$\begin{aligned} R_\theta(x_0, y_0) &= T(x_0, y_0)R_\theta T(-x_0, -y_0) \\ &= \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta & -x_0 \cos \theta + y_0 \sin \theta + x_0 \\ \sin \theta & \cos \theta & -x_0 \sin \theta - y_0 \cos \theta + y_0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Thus, the equations of the required rotation are:

$$\begin{cases} x' = x \cos \theta - y \sin \theta - x_0 \cos \theta + y_0 \sin \theta + x_0 \\ y' = x \sin \theta + y \cos \theta - x_0 \sin \theta - y_0 \cos \theta + y_0. \end{cases}$$

4. Show that the concatenation (product) of two rotations, the first through an angle θ about a point $P(x_0, y_0)$ and the second about a point $Q(x_1, y_1)$ (distinct from P) through an angle $-\theta$ is a translation.

Solution $[R_{-\theta}(x_1, y_1)] = \begin{pmatrix} \cos \theta & -\sin \theta & d_1 \\ \sin \theta & \cos \theta & p_1 \\ 0 & 0 & 1 \end{pmatrix}_{-1}$

$$[R_\theta(x_0, y_0)] = \begin{pmatrix} \cos \theta & -\sin \theta & d_0 \\ \sin \theta & \cos \theta & p_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$^{-1}[R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0)] = \begin{pmatrix} \cos \theta & -\sin \theta & d_1 \\ -\sin \theta & \cos \theta & p_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & d_0 \\ \sin \theta & \cos \theta & p_0 \\ 0 & 0 & 1 \end{pmatrix} = M$$

$$m_{11} = \cos^2 \theta + \sin^2 \theta = 1$$

$$m_{21} = -\sin \theta \cos \theta + \sin \theta \cdot \cos \theta = 0$$

$$m_{12} = \sin \theta \cdot \cos \theta - \sin \theta \cdot \cos \theta = 0$$

$$m_{22} = \sin^2 \theta + \cos^2 \theta = 1$$

$$m_{13} = d_0 \cdot \cos \theta + p_0 \sin \theta + d_1$$

$$m_{23} = -d_0 \cdot \sin \theta + p_0 \cos \theta + p_1$$

$$m_{31} = 0, m_{32} = 0, m_{33} = 1$$

$$\Rightarrow [R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0)] = \begin{pmatrix} 1 & 0 & m_{13} \\ 0 & 1 & m_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

(matrix of translation \Rightarrow)

$$^{-1}[R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0)] = T(m_{13}, m_{23})$$

$$m_{13} = d_0 \cdot \cos \theta + p_0 \sin \theta + d_1$$

$$\begin{aligned} m_{13} &= \cos \theta \cdot (-x_0 \cos \theta + y_0 \sin \theta + x_0) + \\ &\quad \sin \theta \cdot (-x_0 \sin \theta - y_0 \cos \theta + y_0) - x_1 \cos \theta - y_1 \sin \theta + x_1 = \\ &= -x_0 \cos^2 \theta + \cancel{y_0 \sin \theta \cos \theta} + x_0 \cos \theta - x_0 \sin^2 \theta - \cancel{y_0 \sin \theta \cos \theta} + \\ &\quad + \sin \theta \cdot y_0 - x_1 \cos \theta - y_1 \sin \theta + x_1 \end{aligned}$$

$$= (x_1 - x_0) + \cos \theta \cdot (x_0 - x_1) + \sin \theta \cdot (y_0 - y_1)$$

$$= (x_0 - x_1) \cos \theta - (y_0 - y_1) \sin \theta$$

$$m_{23} = -d_0 \sin \theta \cdot (-x_0 \cos \theta + y_0 \sin \theta + x_0) + \cos \theta \cdot (-x_0 \sin \theta - y_0 \cos \theta + y_0) -$$

$$+ x_1 \sin \theta - y_1 \cos \theta + y_1 = \dots$$

$$= (x_0 - x_1) \sin \theta + (y_0 - y_1) (\cos \theta - 1)$$

Projective plane

$$\mathbb{RP}^2 = \mathbb{P}(\mathbb{R}^3) = \mathbb{P}^2(\mathbb{R})$$

$$\mathbb{RP}^2 = \{[x:y:z] / x, y, z \in \mathbb{R}, (x, y, z) \neq (0, 0, 0)\}$$

homogeneous
coordinates/vectors

$$[x:y:z] = [\lambda x : \lambda y : \lambda z], \forall \lambda \in \mathbb{R} \setminus \{0\}$$

$$\mathbb{RP}^2 = \mathbb{R}^3 \setminus \{0\}$$

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \sim \exists \lambda \in \mathbb{R} \setminus \{0\}: (x_2, y_2, z_2) = \lambda(x_1, y_1, z_1)$$

\mathbb{RP}^2 = set of lines in \mathbb{R}^3 that contain the origin $(0, 0, 0)$

$\mathbb{RP}^2 = \mathbb{RA}^2 \cup \mathbb{R}\infty$ line at infinity

projective plane affine plane

$$\mathbb{RA}^2 = \{[x:y:z] \in \mathbb{RP}^2 / z \neq 0\} = \left\{ \left[\frac{x}{z} : \frac{y}{z} : 1 \right] \in \mathbb{RP}^2 / z \neq 0 \right\} = \{[x:y:1] \in \mathbb{RP}^2 / x, y \in \mathbb{R}\}$$

$\mathbb{RA}^2 \rightarrow \mathbb{R}^2$ bijective map

$$[x:y:z] \mapsto \left(\frac{x}{z}, \frac{y}{z} \right)$$

This is how we embed $\mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$

$$\mathbb{R}\infty = \{[x:y:z] / z = 0\} = \{[x:y:0] / x, y \in \mathbb{R}, (x, y) \neq (0, 0)\}$$

Every projective point $[x:y:0]$ corresponds to all the parallel lines in \mathbb{R}^2 whose direction vector is (x, y) .

Why we care:

φ_1, φ_2 affine transformations.

$$\varphi_1 \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = M_1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_1$$

$$\Rightarrow (\varphi_1 \circ \varphi_2) \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \varphi_1 \left(\varphi_2 \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) \right) = \varphi_1 \left(M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_2 \right) =$$

$$\varphi_2 \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_2$$

$$= M_1 \left(M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_2 \right) + v_1 = M_1 M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + M_1 v_2 + v_1$$

Instead of defining $\psi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (affine) as $\psi \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = M \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) + v_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ we see it temporarily as a projective transf: $\psi: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$

$$\Rightarrow \psi \left[\begin{pmatrix} x \\ y \end{pmatrix} \right] = \begin{pmatrix} M & | & v_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax+bx+x_0 \\ cx+dy+y_0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} ax+bx+x_0 \\ cx+dy+y_0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

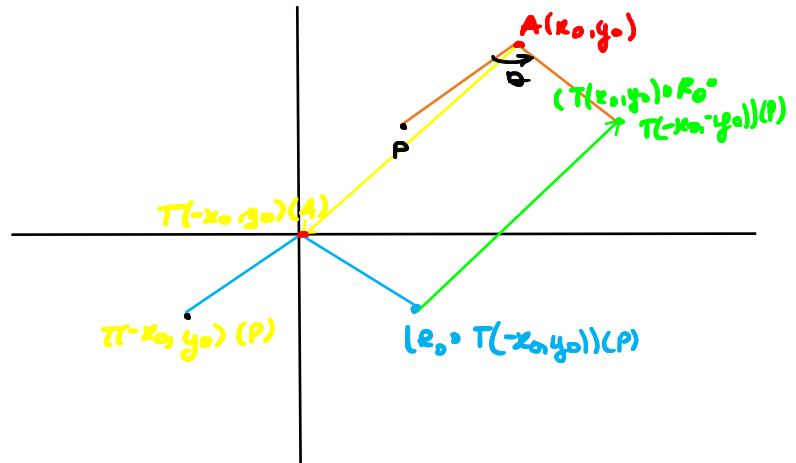
From here we deduce that: $\psi \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} ax+bx+x_0 \\ cx+dy+y_0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

A projective transformation is a function:

$$\Psi: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2, \Psi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix}$$

A projective transf Ψ is affine if $a_{31} = a_{32} = 0$ and $a_{33} \neq 0$.

13.3. $R_\theta(x_0, y_0) = T(x_0, y_0) \circ R_0 \circ T(-x_0, -y_0)$



$$[R_\theta(x_0, y_0)] = \begin{pmatrix} \cos \theta & -\sin \theta & x_0 \\ \sin \theta & \cos \theta & y_0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow x_0 = -x_0 \cos \theta + y_0 \sin \theta + x_0 \\ y_0 = -x_0 \sin \theta - y_0 \cos \theta + y_0$$

13.* Consider two parallel lines l_1 and l_2 show that $\kappa_{l_1} \circ \kappa_{l_2}$ is a translation.

$$l_1: ax + by + c_1 = 0$$

$$l_2: ax + by + c_2 = 0$$

$$[\kappa_{l_1}] = \begin{pmatrix} b^2 - a^2 & -2ab & -2ac_1 \\ -2ab & a^2 - b^2 & -2bc_1 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} \Rightarrow$$

$$[\kappa_{l_2}] = \begin{pmatrix} b^2 - a^2 & -2ab & -2ac_2 \\ -2ab & a^2 - b^2 & -2bc_2 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}$$

$$[\kappa_{l_1} \circ \kappa_{l_2}] = \begin{pmatrix} b^2 - a^2 & -2ab & -2ac_1 \\ -2ab & a^2 - b^2 & -2bc_1 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} \begin{pmatrix} b^2 - a^2 & -2ab & -2ac_2 \\ -2ab & a^2 - b^2 & -2bc_2 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} = M$$

$$m_{11} = (b^2 - a^2)^2 + 4a^2b^2 - (a^2 + b^2)^2$$

$$m_{12} = (b^2 - a^2) \cdot (-2ab) - (b^2 - a^2) \cdot (-2ab) = 0$$

$$m_{13} = (b^2 - a^2) \cdot (-2ac_2) + 1ab^2c_2 - 2ac_1(a^2 + b^2) = 2a(c_2 - c_1)(a^2 + b^2)$$

$$m_{21} = -2ab(b^2 - a^2) - 2ab(a^2 - b^2) = 0$$

$$m_{22} = 4a^2b^2 + (a^2 - b^2)^2 - (a^2 + b^2)^2$$

$$m_{23} = 4a^2bc_2 + (a^2 - b^2) \cdot (-2bc_2) - 2bc_1(a^2 + b^2) = 2b(c_2 - c_1)(a^2 + b^2)$$

$$m_{31} = 0, m_{32} = 0, m_{33} = (a^2 + b^2)^2$$

$$[\kappa_{l_1} \circ \kappa_{l_2}] = \begin{pmatrix} (a^2 + b^2)^2 & 0 & m_{13} \\ 0 & (a^2 + b^2)^2 & m_{23} \\ 0 & 0 & (a^2 + b^2)^2 \end{pmatrix}$$

$$[\kappa_{l_1} \circ \kappa_{l_2}] = \begin{pmatrix} 1 & 0 & \frac{m_{13}}{(a^2 + b^2)^2} \\ 0 & 1 & \frac{m_{23}}{(a^2 + b^2)^2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2a(c_2 - c_1) \\ 0 & 1 & 2b(c_2 - c_1) \\ 0 & 0 & 1 \end{pmatrix}$$