6 Week 6:

6.1 The vector product

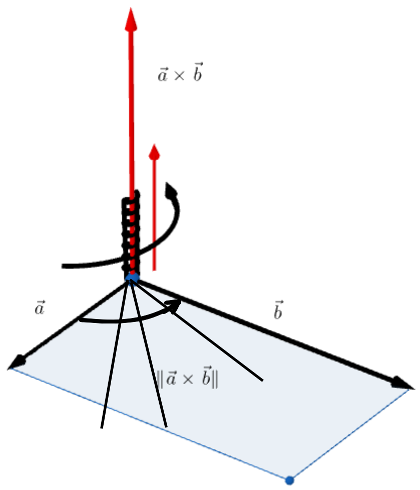
Definition 6.1. The *vector product* or the *cross product* of the vectors $\vec{a}, \vec{b} \in \mathcal{V}$ is a vector, denoted by $\vec{a} \times \vec{b}$, which is defined to be zero if $\vec{a}, \vec{b}$ are linearly dependent (collinear), and if $\vec{a}, \vec{b}$ are linearly independent (noncollinear), then it is defined by the following data:

1. $\vec{a} \times \vec{b}$ is a vector orthogonal on the two-dimensional subspace $\langle \vec{a}, \vec{b} \rangle$ of $\mathcal{V}$;

2. if $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, then the sense of $\vec{a} \times \vec{b}$ is the one in which a right-handed screw, placed along the line passing through O orthogonal to the vectors $\vec{a}$ and $\vec{b}$, advances when it is being rotated simultaneously with the vector $\vec{a}$ from $\vec{a}$ towards $\vec{b}$ within the vector subspace $\langle \vec{a}, \vec{b} \rangle$ and the support half line of $\vec{a}$ sweeps the interior of the angle $\widehat{AOB}$ (Screw rule).

3. the *norm* (*magnitude* or *length*) of $\vec{a} \times \vec{b}$ is defined by

$$|| \vec{a} \times \vec{b} || = || \vec{a} || \cdot || \vec{b} || \sin(\widehat{\vec{a}, \vec{b}}). \geq 0$$



Remark 6.1. 1. *The* norm *(*magnitude *or* length*) of the vector $\vec{a} \times \vec{b}$ is actually the area of the parallelogram constructed on the vectors $\vec{a}, \vec{b}$.*

2. *The vectors $\vec{a}, \vec{b} \in \mathcal{V}$ are linearly dependent (collinear) if and only if $\vec{a} \times \vec{b} = \vec{0}$.*

Proposition 6.1. *The vector product has the following properties:*

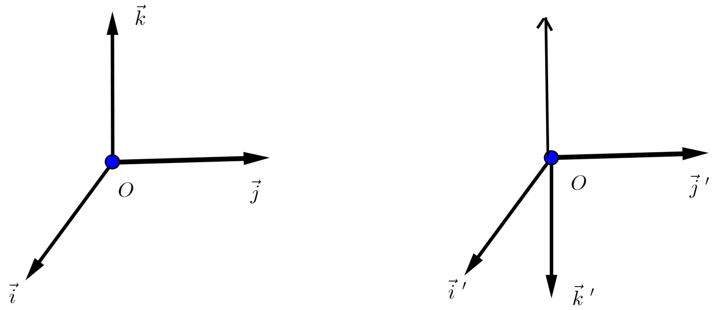
1. $\vec{a} \times \vec{b} = - \vec{b} \times \vec{a}, \forall \vec{a}, \vec{b} \in \mathcal{V}$;

2. $(\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b}) = \lambda(\vec{a} \times \vec{b}), \forall \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}$;

3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}, \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.

6.2 The vector product in terms of coordinates

If $[\vec{i}, \vec{j}, \vec{k}]$ is an orthonormal basis, observe that $\vec{i} \times \vec{j} \in \{-\vec{k}, \vec{k}\}$. We say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *direct* if $\vec{i} \times \vec{j} = \vec{k}$. If, on the contrary, $\vec{i} \times \vec{j} = -\vec{k}$, we say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *inverse*.



direct orthonormal basis inverse orthonormal basis

Therefore, if $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis, then $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$ and obviously $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

Proposition 6.2. *If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, then*

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}, \tag{6.1}$$

or, equivalently,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \tag{6.2}$$

Proof. Indeed,

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \times (b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}) \\ &= a_1 b_1 \vec{i} \times \vec{i} + a_1 b_2 \vec{i} \times \vec{j} + a_1 b_3 \vec{i} \times \vec{k} \\ &+ a_2 b_1 \vec{j} \times \vec{i} + a_2 b_2 \vec{j} \times \vec{j} + a_2 b_3 \vec{i} \times \vec{k} \\ &+ a_3 b_1 \vec{k} \times \vec{i} + a_3 b_2 \vec{k} \times \vec{j} + a_3 b_3 \vec{k} \times \vec{k} \\ &= a_1 b_2 \vec{k} - a_1 b_3 \vec{j} - a_2 b_1 \vec{k} + a_2 b_3 \vec{i} + a_3 b_1 \vec{j} - a_3 b_2 \vec{i} \\ &= (a_2 b_3 - a_3 b_2)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k} \end{aligned}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

□

One can rewrite formula (6.1) in the form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{6.3}$$

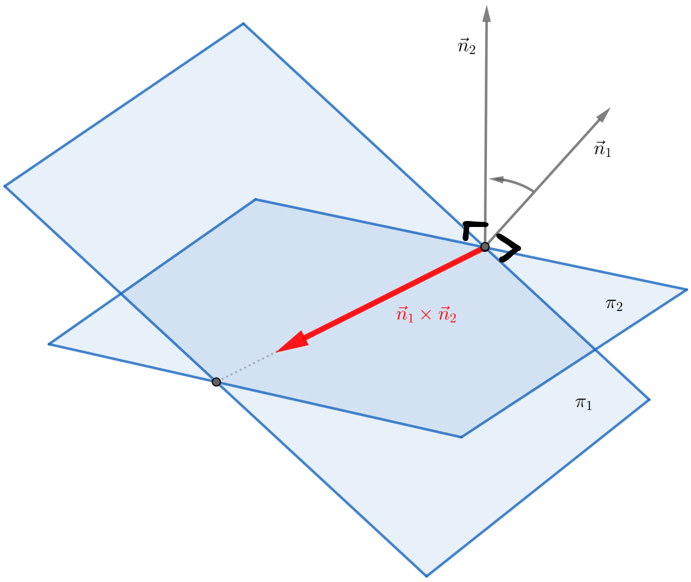
the right hand side determinant being understood in the sense of its cofactor expansion along the first line.

Remark 6.2. If $R = (O, \vec{i}, \vec{j}, \vec{k})$ is the direct Cartesian orthonormal reference system behind the equations of the line

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_1 = 0, \end{cases}$$

then we can recover the director parameters (4.10) of Δ, in this particular case of orthonormal Cartesian reference systems, by observing that $\vec{n}_1 \times \vec{n}_2$ is a director vector of Δ, where

$$\vec{n}_1 = A_1 \vec{i} + B_1 \vec{j} + C_1 \vec{k}$$
$$\vec{n}_2 = A_2 \vec{i} + B_2 \vec{j} + C_2 \vec{k}.$$



Recall that

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \vec{i} + \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \vec{j} + \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \vec{k}.$$

Note however that the director parameters were obtained before for arbitrary Cartesian reference systems (See (4.10)).

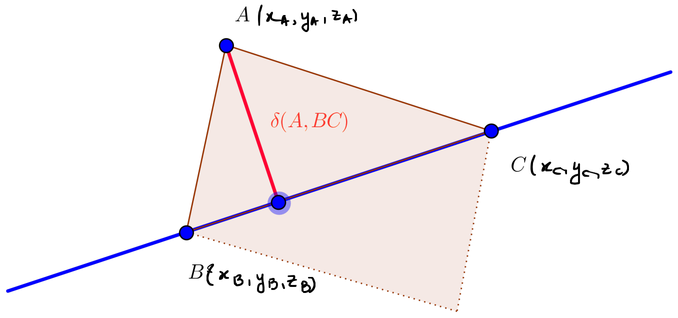
6.3 Applications of the vector product

- **The area of the triangle ABC.** $S_{ABC} = \frac{1}{2} || \overrightarrow{AB} || \cdot || \overrightarrow{AC} || \sin \widehat{BAC} = \frac{1}{2} || \overrightarrow{AB} \times \overrightarrow{AC} ||$. On the other hand

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_B - x_A & y_B - x_A & z_B - z_A \\ x_C - x_A & y_C - x_A & z_C - z_A \end{vmatrix},$$

as the coordinates of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are $(x_B - x_A, y_B - x_A, z_B - z_A)$ and $(x_C - x_A, y_C - x_A, z_C - z_A)$ respectively. Thus,

$$4S^2_{ABC} = \begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2.$$



- **The distance from one point to a straight line.**

(a) *The distance $\delta(A, BC)$ from the point $A(x_A, y_A, z_A)$ to the straight line BC*, where $B(x_B, y_B, z_B)$ and $C(x_C, y_C, z_C)$. Since

$$S_{ABC} = \frac{\| \overrightarrow{BC} \| \cdot \delta(A, BC)}{2}$$

it follos that

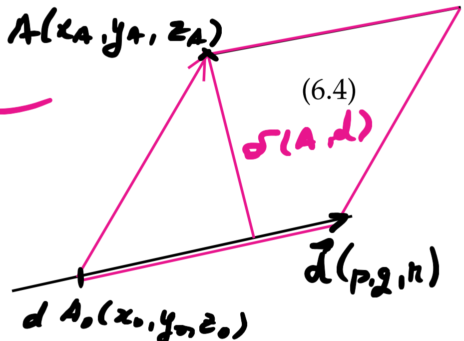
$$\delta^2(A, BC) = \frac{4S^2_{ABC}}{\| \overrightarrow{BC} \|^2}.$$

Thus, we obtain

$$\delta^2(A, BC) = \frac{\begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2}{(x_C - x_B)^2 + (y_C - y_B)^2 + (z_C - z_B)^2}.$$

(b) The distance from $\delta(A, d)$ from one point $A(A_A, y_A, z_A)$ to the straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$



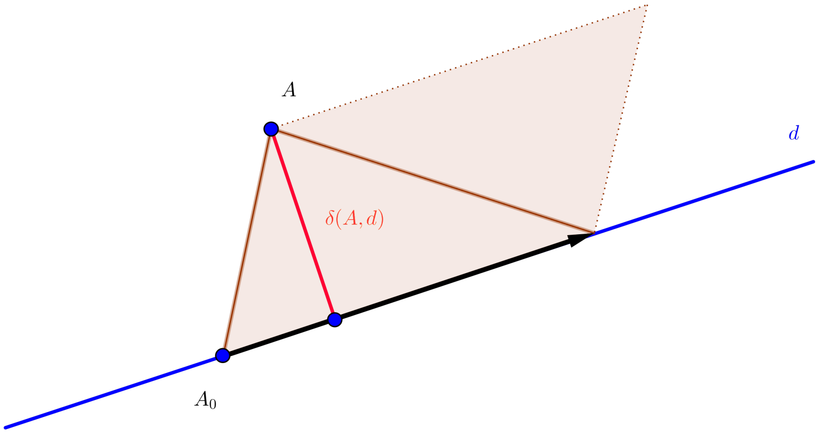
$$\delta(A, d) = \frac{\| \vec{d} \times \overrightarrow{A_0 A} \|}{\|\vec{d}\|}, \tag{6.4}$$

where $A_0(x_0, y_0, z_0) \in \delta$.
Since

$$\begin{aligned} \vec{d} \times \overrightarrow{A_0 A} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ x_A - x_0 & y_A - y_0 & z_A - z_0 \end{vmatrix} \\ &= \begin{vmatrix} q & r \\ y_A - y_0 & z_A - z_0 \end{vmatrix} \vec{i} + \begin{vmatrix} r & p \\ z_A - z_0 & x_A - x_0 \end{vmatrix} \vec{j} + \begin{vmatrix} p & q \\ x_A - x_0 & y_A - y_0 \end{vmatrix} \vec{k} \end{aligned}$$

it follows that

$$\delta(A, d) = \frac{\sqrt{\begin{vmatrix} q & r \\ y_A - y_0 & z_A - z_0 \end{vmatrix}^2 + \begin{vmatrix} r & p \\ z_A - z_0 & x_A - x_0 \end{vmatrix}^2 + \begin{vmatrix} p & q \\ x_A - x_0 & y_A - y_0 \end{vmatrix}^2}}{\sqrt{p^2 + q^2 + r^2}}.$$



6.4 The double vector (cross) product

The *double vector (cross) product* of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the vector $\vec{a} \times (\vec{b} \times \vec{c}) \underset{\text{usually}}{\neq} \underbrace{(\vec{a} \times \vec{b}) \times \vec{c}}_{\in \langle \vec{a}, \vec{b} \rangle}$

Proposition 6.3.

$$\vec{a} \times (\vec{b} \times \vec{c}) = \underbrace{(\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c}}_{\in \langle \vec{b}, \vec{c} \rangle} = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \end{vmatrix}, \quad \forall\, \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}. \tag{6.5}$$

Proof. (Sketch) If the vectors $\vec{b}$ and $\vec{c}$ are linearly dependent, then both sides are obviously zero. Otherwise one can choose an orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$, related to the vectors $\vec{a}, \vec{b}$ and $\vec{c}$, such that

$\vec{i} = \dfrac{\vec{b}}{\|\vec{b}\|}$

$$\vec{b} = b_1 \vec{i}, \vec{c} = c_1 \vec{i} + c_2 \vec{j}, \vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}.$$

For example one can choose $\vec{i}$ to be $\vec{b} / \| \vec{b} \|$ and $\vec{j}$ a unit vector in the subspace $\langle \vec{b}, \vec{c} \rangle$ which is perpendicular on $\vec{b}$. Finally, one can choose $\vec{k} = \vec{i} \times \vec{j}$. By computing the two sides of the equality 6.5, in terms of coordinates and the vectors $\vec{i}, \vec{j}, \vec{k}$, one gets the same result. □

Corollary 6.4. 1. $(\vec{a} \times \vec{b}) \times \vec{c} = \underbrace{(\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a}}_{\in \langle \vec{a}, \vec{b} \rangle} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c} \cdot \vec{b} & \vec{c} \cdot \vec{a} \end{vmatrix}, \forall\, \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$;

2. $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}, \forall\, \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$ *(Jacobi's identity).*

Proof. While the first identity follows immediately via 6.5, for the Jacobi's identity we get successively:

$$\begin{aligned} &\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) \\ &= (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c} + (\vec{b} \cdot \vec{a})\, \vec{c} - (\vec{b} \cdot \vec{c})\, \vec{a} + (\vec{c} \cdot \vec{b})\, \vec{a} - (\vec{c} \cdot \vec{a})\, \vec{b} = \vec{0}. \end{aligned}$$

□

6.5 Problems

1. **(2p)** Show that $\| \vec{a} \times \vec{b} \| \leq \| \vec{a} \| \cdot \| \vec{b} \|, \forall\, \vec{a}, \vec{b}, \in \mathcal{V}$.

   Solution. if $\vec{a}, \vec{b}$ are linearly dependent $\Rightarrow$ $\left.\begin{array}{l} \|\vec{a} \times \vec{b}\| = \|0\| = 0 \\ \|\vec{a}\| \geq 0 \\ \|\vec{b}\| \geq 0 \end{array}\right\} \Rightarrow \|\vec{a} \times \vec{b}\| \leq \|\vec{a}\| \cdot \|\vec{b}\|, \forall\, \vec{a}, \vec{b} \in \mathcal{V}$

if $\vec{a}, \vec{b}$ are linearly independent

$\Rightarrow \|\vec{a}\times\vec{b}\| = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\sin(\widehat{\vec{a},\vec{b}})$

but $\sin x \in [-1,1], \forall x \in \mathbb{R} \Rightarrow \sin(\widehat{\vec{a},\vec{b}}) \leq 1$ $\Big\}\Rightarrow$

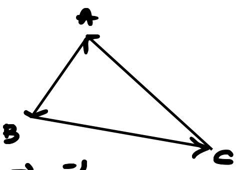
$\Rightarrow \|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\cdot\sin(\widehat{\vec{a},\vec{b}}) \leq \|\vec{a}\|\cdot\|\vec{b}\|, \forall \vec{a},\vec{b}\in V$

2. **(3p)** Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be pairwise noncollinear vectors. Show that the necesssary and sufficient condition for the existence of a triangle ABC with the properties $\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$, $\overrightarrow{AB}=\vec{c}$ is

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}.$$

From the equalities of the norms deduce the low of sines.

Solution.



$\vec{a}, \vec{b}$ - noncollinear $\Leftrightarrow \vec{a}\times\vec{b}\neq\vec{0}$

$\vec{b}, \vec{c}$ - noncollinear $\Leftrightarrow \vec{b}\times\vec{c}\neq\vec{0}$

$\vec{c}, \vec{a}$ - noncollinear $\Leftrightarrow \vec{c}\times\vec{a}\neq\vec{0}$

ABC - triangle $\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0} \Leftrightarrow \vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\vec{a}+\vec{b}=-\vec{c}\,/\times\vec{c} \Rightarrow \vec{a}\times\vec{c}+\vec{b}\times\vec{c}=-\underbrace{\vec{c}\times\vec{c}}_{\vec{0}} \Leftrightarrow \vec{a}\times\vec{c}=\vec{c}\times\vec{b}$ $\Big\}\Rightarrow$

$\vec{b}+\vec{c}=-\vec{a}\,/\times\vec{a} \Rightarrow \vec{b}\times\vec{a}+\vec{c}\times\vec{a}=-\underbrace{\vec{a}\times\vec{a}}_{\vec{0}} \Leftrightarrow \vec{c}\times\vec{a}=\vec{a}\times\vec{b}$

$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

Convercly, we assume that $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

$\vec{a}\times\vec{b}=\vec{b}\times\vec{c} \Leftrightarrow \vec{a}\times\vec{b}-\vec{b}\times\vec{c}=\vec{0} \Leftrightarrow \vec{a}\times\vec{b}+\vec{c}\times\vec{b}=\vec{0} \Leftrightarrow (\vec{a}+\vec{c})\times\vec{b}=\vec{0} \Leftrightarrow$

$\Leftrightarrow \vec{a}+\vec{c}$ and $\vec{b}$ are L.D $\Rightarrow \vec{a}+\vec{c}=\beta\vec{b}$ (1)

$\vec{b}\times\vec{c}=\vec{c}\times\vec{a} \Leftrightarrow \vec{b}\times\vec{c}+\vec{a}\times\vec{c}=\vec{0} \Leftrightarrow (\vec{b}+\vec{a})\times\vec{c}=\vec{0} \Rightarrow \vec{b}+\vec{a}$ and $\vec{c}$ are LD $\Rightarrow$

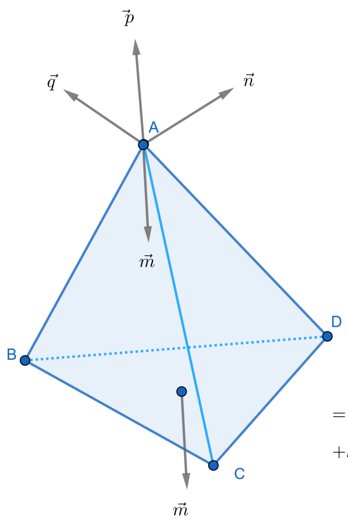
$\Leftrightarrow \vec{a}+\vec{b}=\gamma\vec{c}$ (2), $\vec{b}+\vec{c}=\alpha\vec{a}$ (similarly)

$\vec{b}+\vec{a}=\gamma\vec{c}\,/\times\vec{a} \Rightarrow \vec{b}\times\vec{a}+\vec{a}\times\vec{a}=\gamma\vec{c}\times\vec{a} \Leftrightarrow \vec{b}\times\vec{a}=\gamma\vec{a}\times\vec{b} \Rightarrow \gamma=-1$

$\vec{a}+\vec{b}=\gamma\vec{c} \Rightarrow \vec{a}+\vec{b}=-\vec{c} \Rightarrow \vec{a}+\vec{b}+\vec{c}=\vec{0} \Leftrightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0} \Rightarrow ABC$ - triangle

3. **(3p)** Show that the sum of some outer-pointing vectors perpendicular on the faces of a tetrahedron which are proportional to the areas of the faces is the zero vector.

Solution.



The proportionality of $\vec{m},\vec{n},\vec{p},\vec{q}$ with the areas of the corresponding faces of the tetrahedron show that

$\vec{m}=k\overrightarrow{BD}\times\overrightarrow{BC},\ \vec{n}=k\overrightarrow{AC}\times\overrightarrow{AD}$

$\vec{p}=k\overrightarrow{AD}\times\overrightarrow{AB},\ \vec{q}=k\overrightarrow{AB}\times\overrightarrow{AC}$

Thus, $\vec{m}+\vec{n}+\vec{p}+\vec{q}$

$=k\overrightarrow{BD}\times\overrightarrow{BC}+k\overrightarrow{AC}\times\overrightarrow{AD}+$

$+k\overrightarrow{AD}\times\overrightarrow{AB}+k\overrightarrow{AB}\times\overrightarrow{AC}$

$=k(\overrightarrow{AD}-\overrightarrow{AB})\times(\overrightarrow{AC}-\overrightarrow{AB})+k\overrightarrow{AC}\times\overrightarrow{AD}$

$+k\overrightarrow{AD}\times\overrightarrow{AB}+k\overrightarrow{AB}\times\overrightarrow{AC}=$

$=k\overrightarrow{AD}\times\overrightarrow{AC}-k\overrightarrow{AD}\times\overrightarrow{AB}-k\overrightarrow{AB}\times\overrightarrow{AC}+k\overrightarrow{AB}\times\overrightarrow{AB}=$

$+k\overrightarrow{AC}\times\overrightarrow{AD}+k\overrightarrow{AD}\times\overrightarrow{AB}+k\overrightarrow{AB}\times\overrightarrow{AC}=\vec{0}.$

4. **(2p)** Find the distance from the point $P(1,2,-1)$ to the straight line (d) $x=y=z$.

Solution. $\vec{v}_\ell(1,1,1), A_0(1,1,1) \in \ell, \vec{PA_0}(0,-1,2)$

$$\vec{PA_0} \times \vec{v}_\ell = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -\vec{i} + 2\vec{j} + \vec{k} - 2\vec{i} = -3\vec{i} + 2\vec{j} + \vec{k} \Rightarrow \|\vec{PA_0} \times \vec{v}_\ell\| = \sqrt{9+4+1} = \sqrt{14}$$

$$\|\vec{v}_\ell\| = \sqrt{3}$$

$$\Rightarrow dist(P, \ell) = \frac{\|\vec{PA_0} \times \vec{v}_\ell\|}{\|\vec{v}_\ell\|} = \frac{\sqrt{14}}{\sqrt{3}}$$

5. **(3p)** Find the area of the triangle ABC and the lengths of its hights, where $A(-1, 1, 2)$, $B(2, -1, 1)$ and $C(2, -3, -2)$.

$$A_{ABC} = \frac{1}{2} \|\vec{AB} \times \vec{AC}\|, \quad \vec{AB}(3,-2,-1), \vec{AC}(3,-4,-4), \vec{BC}(0,-2,-3)$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -1 \\ 3 & -4 & -4 \end{vmatrix} = 4\vec{i} + 9\vec{j} - 6\vec{k}$$

$$\|\vec{AB} \times \vec{AC}\| = \sqrt{16 + 81 + 36} = \sqrt{133} \Rightarrow A_{ABC} = \frac{\sqrt{133}}{2}$$

$$h_A = \frac{2A_{ABC}}{\|\vec{BC}\|} = \frac{2\frac{\sqrt{133}}{2}}{\sqrt{13}} = \frac{\sqrt{133}}{\sqrt{13}}$$

$$h_B = \frac{2A_{ABC}}{\|\vec{AC}\|} = \frac{\sqrt{133}}{\sqrt{41}}$$

$$h_C = \frac{2A_{ABC}}{\|\vec{AB}\|} = \frac{\sqrt{133}}{\sqrt{14}}$$

6. **(3p)** Let d_1, d_2, d_3, d_4 be pairwise skew straight lines. Assuming that $d_{12} \perp d_{34}$ and $d_{13} \perp d_{24}$, show that $d_{14} \perp d_{23}$, where d_{ik} is the common perpendicular of the lines d_i and d_k.

Solution. A director vector of the common perpendicular d_{ij} is $\vec{d}_i \times \vec{d}_j$, where $\vec{d}_r$ stands for a director vector of d_r. Therefore we have successively:

$$\begin{aligned} d_{12} \perp d_{34} \quad & \Leftrightarrow \vec{d}_1 \times \vec{d}_2 \perp \vec{d}_3 \times \vec{d}_4 \Leftrightarrow (\vec{d}_1 \times \vec{d}_2) \cdot (\vec{d}_3 \times \vec{d}_4) = 0 \\ & \Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_3 & \vec{d}_1 \cdot \vec{d}_4 \\ \vec{d}_2 \cdot \vec{d}_3 & \vec{d}_2 \cdot \vec{d}_4 \end{vmatrix} = 0 \Leftrightarrow (\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_2 \cdot \vec{d}_3). \end{aligned}$$

Similalry

$$\begin{aligned} d_{13} \perp d_{24} \quad & \Leftrightarrow \vec{d}_1 \times \vec{d}_3 \perp \vec{d}_2 \times \vec{d}_4 \Leftrightarrow (\vec{d}_1 \times \vec{d}_3) \cdot (\vec{d}_2 \times \vec{d}_4) = 0 \\ & \Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_2 & \vec{d}_1 \cdot \vec{d}_4 \\ \vec{d}_3 \cdot \vec{d}_2 & \vec{d}_3 \cdot \vec{d}_4 \end{vmatrix} = 0 \Leftrightarrow (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_3 \cdot \vec{d}_2). \end{aligned}$$

Therefore we have

$$(\overrightarrow{d}_1 \cdot \overrightarrow{d}_3)(\overrightarrow{d}_2 \cdot \overrightarrow{d}_4) = (\overrightarrow{d}_1 \cdot \overrightarrow{d}_4)(\overrightarrow{d}_2 \cdot \overrightarrow{d}_3) = (\overrightarrow{d}_1 \cdot \overrightarrow{d}_2)(\overrightarrow{d}_3 \cdot \overrightarrow{d}_4),$$

which shows that

$$(\overrightarrow{d}_1 \cdot \overrightarrow{d}_3)(\overrightarrow{d}_2 \cdot \overrightarrow{d}_4) - (\overrightarrow{d}_1 \cdot \overrightarrow{d}_2)(\overrightarrow{d}_3 \cdot \overrightarrow{d}_4) = 0 \Leftrightarrow \begin{vmatrix} \overrightarrow{d}_1 \cdot \overrightarrow{d}_2 & \overrightarrow{d}_1 \cdot \overrightarrow{d}_3 \\ \overrightarrow{d}_4 \cdot \overrightarrow{d}_2 & \overrightarrow{d}_4 \cdot \overrightarrow{d}_3 \end{vmatrix} = 0 \Leftrightarrow d_{14} \perp d_{23}.$$

Seminar W6 - Nicu - 31/03/2021

Cross product (vector product)

$\vec{v}, \vec{w}$ vectors

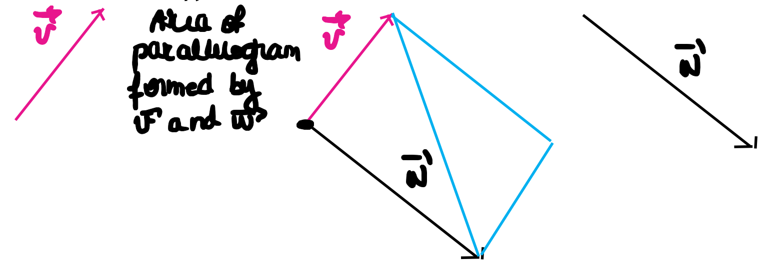
- if $\vec{v}, \vec{w}$ lin dep $\Rightarrow \vec{v} \times \vec{w} = \vec{0}$
- if $\vec{v}, \vec{w}$ lin. indep $\Rightarrow \vec{v} \times \vec{w} \in \mathcal{V}$

$\rightarrow$ direction: perp to $\vec{v}$ and $\vec{w}$, it is actually perp to $\langle \vec{v}, \vec{w} \rangle$

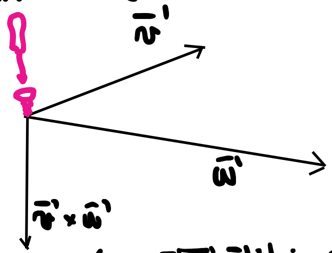
$\rightarrow$ norm: $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin(\widehat{\vec{v}, \vec{w}})$

$\quad\quad\quad\quad \shortparallel$

Area of parallelogram formed by $\vec{v}$ and $\vec{w}$

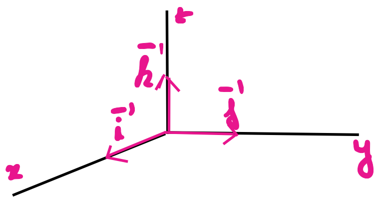


$\rightarrow$ orientation: the screw rule



If the ref. system $(O, [\vec{i}, \vec{j}, \vec{k}])$ is orthonormal and direct (for us all the time)

$\hookrightarrow \vec{i} \times \vec{j} = \vec{k}$



, then the cross product is computed as follows:

$\vec{v}(a_1, b_1, c_1), \vec{w}(a_2, b_2, c_2)$

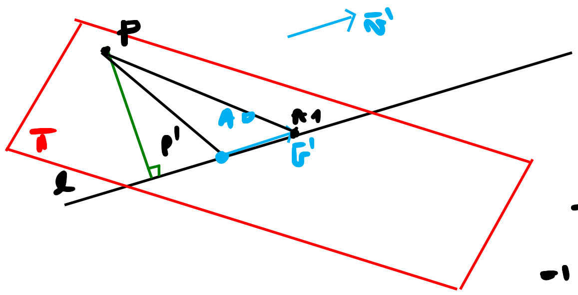
$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1c_2 - b_2c_1)\vec{i} - (a_1c_2 - a_2c_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k} =$$

$$= (b_1c_2 - b_2c_1, a_1c_2 - a_2c_1, a_1b_2 - a_2b_1)$$

$\rightarrow$ the cross product is anti-commutative, bilinear

$\forall \alpha, \beta \in \mathbb{R}, \forall v_1, v_2, w \in \mathcal{V}: (\alpha\vec{v}_1 + \beta\vec{v}_2) \times \vec{w} = \alpha\vec{v}_1 \times \vec{w} + \beta \cdot \vec{v}_2 \times \vec{w}$

The distance from a point to a line in 3D



π plane so that $\pi \perp \ell, \pi \ni P$

$\{P'\} = \ell \cap \pi \Rightarrow \ell \perp PP' \Rightarrow PP'$ is the perpendicular from P to the line ℓ

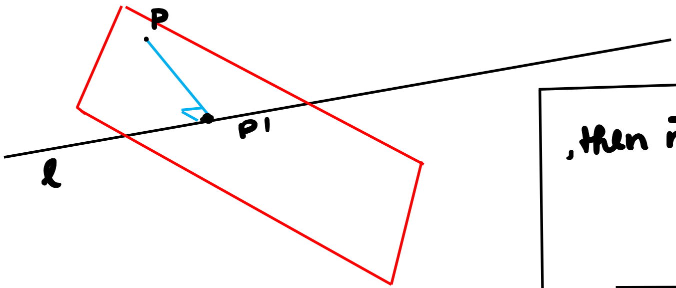
Take $A_0 \in \ell$, $\vec{v} \parallel \ell \Rightarrow \exists A_1 \in \ell$ so that: $\overrightarrow{A_0A_1} = \vec{v} \Rightarrow$

$\Rightarrow PP'$ height in $\Delta PA_0A_1 \Rightarrow$

$$\Rightarrow PP' = \frac{2\mathcal{A}_{PA_0A_1}}{A_0A_1} = \frac{\|\overrightarrow{PA_0} \times \overrightarrow{A_0A_1}\|}{\|\overrightarrow{A_0A_1}\|} = \frac{\|\overrightarrow{PA_0} \times \vec{v}\|}{\|\vec{v}\|}$$

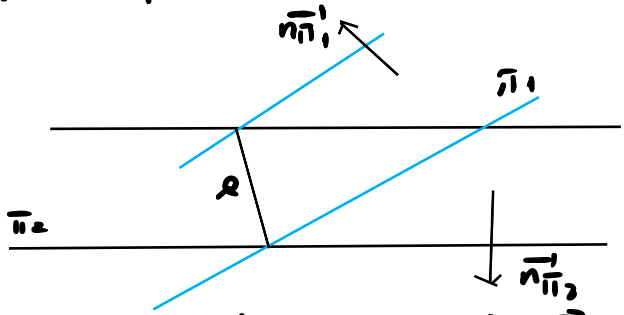
Extra: Consider the line:

$$l:\begin{cases}\pi_1: x+2y-8z+5=0\\ \pi_2: 2x+y+z+1=0\end{cases}$$, and the point $P(1,2,3)$. Find the eq of the perp from the point P onto the line l.



If $l:\begin{cases}\pi_1: A_1x+B_1y+C_1z+D_1=0\\ \pi_2: A_2x+B_2y+C_2z+D_2=0\end{cases}$

, then $\vec{n_{\pi_1}}\times\vec{n_{\pi_2}}$ is a director vector of the line l.



$$\left.\begin{array}{l}\vec{n_{\pi_1}}\perp\pi_1 \Rightarrow \vec{n_{\pi_1}}\perp l\\ \vec{n_{\pi_2}}\perp\pi_2\Rightarrow\vec{n_{\pi_2}}\perp l\end{array}\right\}\Rightarrow l \parallel \vec{n_{\pi_1}}\times\vec{n_{\pi_2}}$$

$\vec{n_{\pi_1}}(1,2,-8)$, $\vec{n_{\pi_2}}(2,1,1) \Rightarrow$

$\Rightarrow \vec{v_l} = \vec{n_{\pi_1}}\times\vec{n_{\pi_2}} =$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 2 & -8\\ 2 & 1 & 1\end{vmatrix} = 2\vec{i}+\vec{k}-16\vec{j}-4\vec{k}+8\vec{i}-\vec{j} =$$

$= 10\vec{i}-3\vec{k}-17\vec{j}$

$\Rightarrow \vec{v_l}(10,-17,-3)$

We now write the eq of the plane π that is perp to l and contains P.

$\vec{v_l}\perp\pi \Rightarrow \vec{v_l}\parallel\vec{n_\pi} \Rightarrow \pi: 10x-17y-3z+D=0$

$P\in\pi \Rightarrow 10\cdot 1-17\cdot 2-3\cdot 3+D=0 \Rightarrow D=34+9-10 \Rightarrow D=33 \Rightarrow$

$\Rightarrow \pi: 10x-17y-3z+33=0$

$$P'=\pi\cap l: \begin{cases}10x-17y-3z+33=0\\ x+2y-8z+5=0\\ 2x+y+z+1=0\end{cases} \Rightarrow \begin{pmatrix}10 & -17 & -3 & -33\\ 1 & 2 & -8 & -5\\ 2 & 1 & 1 & -1\end{pmatrix} \overset{L_1\leftrightarrow L_2}{\sim}$$

$$\sim\begin{pmatrix}1 & 2 & -8 & -5\\ 10 & -17 & -3 & -33\\ 2 & 1 & 1 & -1\end{pmatrix} \underset{L_3\leftarrow L_3-2L_1}{\overset{L_2\leftarrow L_2-10L_1}{\sim}} \begin{pmatrix}1 & 2 & -8 & -5\\ 0 & -37 & 77 & 17\\ 0 & -3 & 17 & 9\end{pmatrix} \overset{L_3\leftarrow L_3-\frac{3}{37}L_2}{\sim}$$

$$\sim\begin{pmatrix}1 & 2 & -8 & -5\\ \cdot & - & - & -\end{pmatrix} \rightarrow \begin{array}{l}x_{P'}= -\cdot\\ y_{P'}=\cdot\ -\\ z_{P'}= --\end{array}$$

$$\Rightarrow \frac{x-x_P}{x_{P'}-x_P}=\frac{y-y_P}{y_{P'}-y_P}=\frac{z-z_P}{z_{P'}-z_P}$$