3 Week 3: Cartesian equations of lines and planes

3.1 Cartesian and affine reference systems

If $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is an ordered basis of $\mathcal{V}$ and $\overrightarrow{x} \in \mathcal{V}$, recall that the column vector of the coordinates of $\overrightarrow{x}$ with respect to b is denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

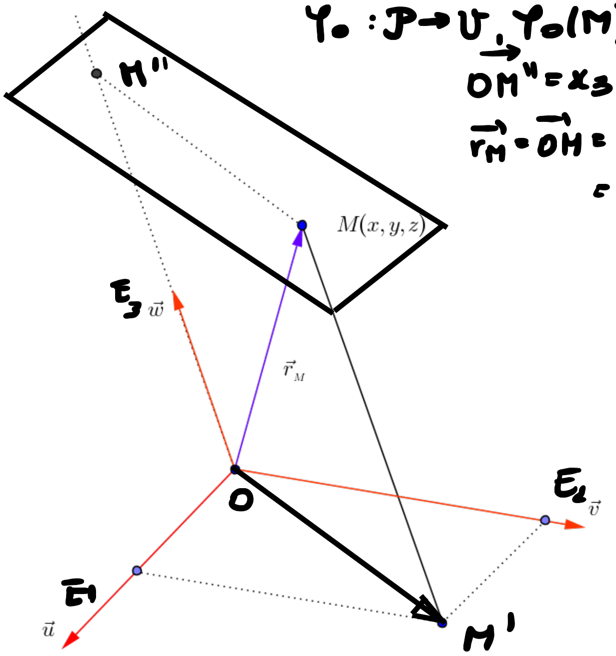
whenever $\overrightarrow{x} = x_1 \overrightarrow{u} + x_2 \overrightarrow{v} + x_3 \overrightarrow{w}$. To emphasize the coordinates of $\overrightarrow{x}$ with respect to b, we shall use the notation $\overrightarrow{x}(x_1, x_2, x_3)$.

$\varphi_0 : \mathcal{P} \to \mathcal{V},\ \varphi_0(M) = \overrightarrow{OM} = \overrightarrow{r_M}$

$\overrightarrow{OM''} = x_3 \overrightarrow{w}$

$\overrightarrow{r_M} = \overrightarrow{OM} = \overrightarrow{OM'} + \overrightarrow{OM''}$

$= x_1 \overrightarrow{u} + x_2 \overrightarrow{v} + x_3 \overrightarrow{w}$



Definition 3.1. A *cartesian reference system* $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ of the space $\mathcal{P}$, consists in a point $O \in \mathcal{P}$ called the *origin* of the reference system and an ordered basis $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ of the vector space $\mathcal{V}$.

Denote by E_1, E_2, E_3 the points for which $\overrightarrow{u} = \overrightarrow{OE_1},\ \overrightarrow{v} = \overrightarrow{OE_2},\ \overrightarrow{w} = \overrightarrow{OE_3}$.

Definition 3.2. The system of points (O, E_1, E_2, E_3) is called *the affine reference system associated to the cartesian reference system* $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$.

The straight lines $OE_i,\ i \in \{1, 2, 3\}$, oriented from O to E_i are called *the coordinate axes*. The coordinates x, y, z of the position vector $\overrightarrow{r}_M = \overrightarrow{OM}$ with respect to the basis $[\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y, z)$. Also, for the column matrix of coordinates of the vector $\overrightarrow{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\overrightarrow{r}_M = x\overrightarrow{u} + y\overrightarrow{v} + z\overrightarrow{w}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Remark 3.1. If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are two points, then $R=(O,\vec{u},\vec{v},\vec{w})$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= x_B\,\vec{u} + y_B\,\vec{v} + z_B\,\vec{w} - (x_A\,\vec{u} + y_A\,\vec{v} + z_A\,\vec{w})\\ &= (x_B - x_A)\,\vec{u} + (y_B - y_A)\,\vec{v} + (z_B - z_A)\,\vec{w},\end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

Remark 3.2. If $R = (O, b)$ is a cartesian reference system, where $b = [\vec{u}, \vec{v}, \vec{w}]$ is an ordered basis of $\mathcal{V}$, recall that $\varphi_O : \mathcal{P} \longrightarrow \mathcal{V}$, $\varphi_O(M) = \overrightarrow{OM}$ is bijective and $\psi_b : \mathbb{R}^3 \longrightarrow \mathcal{V}$, $\psi_b(x, y, z) = x\,\vec{u} + y\,\vec{v} + z\,\vec{w}$ is a linear isomorphism. The bijection φ_O defines a unique vector structure over $\mathcal{P}$ such that φ_O becomes an isomorphism. This vector structure depends on the choice of $O \in \mathcal{P}$. Therefore a point $M \in \mathcal{P}$ could be identified either with its position vector $\vec{r}_M = \varphi_O(M)$, or, with the triplet $(\psi_b^{-1} \circ \varphi_O)(M) \in \mathbb{R}^3$ of its coordinates with respect to the reference system R. If $f : X \longrightarrow \mathbb{R}^3$ is a given application, then $\varphi_O^{-1} \circ \psi_b \circ f : X \longrightarrow \mathcal{P}$ will be denoted by M_f. A similar discussion can be done for a cartesian reference system $R' = (O', b')$ of a plane π, where $b' = [\vec{u}', \vec{v}']$ is an ordered basis of $\vec{\pi}$.

Example 3.1 (Homework). Consider the tetrahedron $ABCD$, where $A(1, -1, 1)$, $B(-1, 1, -1)$, $C(2, 1, -1)$ and $D(1, 1, 2)$. Find the coordinates of:

1. the centroids G_A, G_B, G_C, G_D of the triangles BCD, ACD, ABD and ABC[1] respectively.
2. the midpoints M, N, P, Q, R and S of its edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[CD]$ and $[DB]$ respectively.

SOLUTION.

[1]The centroids of its faces

3.2 The Cylindrical Coordinate System

In order to have a valid coordinate system in the 3-dimensional case, each point of the space must be associated with a unique triple of real numbers (the coordinates of the point) and each triple of real numbers must determine a unique point, as in the case of the Cartesian system of coordinates.

Let $P(x,y,z)$ be a point in a Cartesian system of coordinates $Oxyz$ and P' be the orthogonal projection of P on the plane xOy. One can associate to the point P the triple (r,θ,z), where (r,θ) are the polar coordinates of P' (see Figure 1). The polar coordinates of P' can be obtained by specifying the distance ρ from O to P' and the angle θ (measured in radians), whose "initial" side is the polar axis, i.e. the x-axis, and whose "terminal" side is the ray OP. The *polar coordinates* of the point P are (ρ,θ) (See also section (3.6.2) of the Appendix). The triple (r,θ,z) gives the *cylindrical coordinates* of the point P. There is the bijection

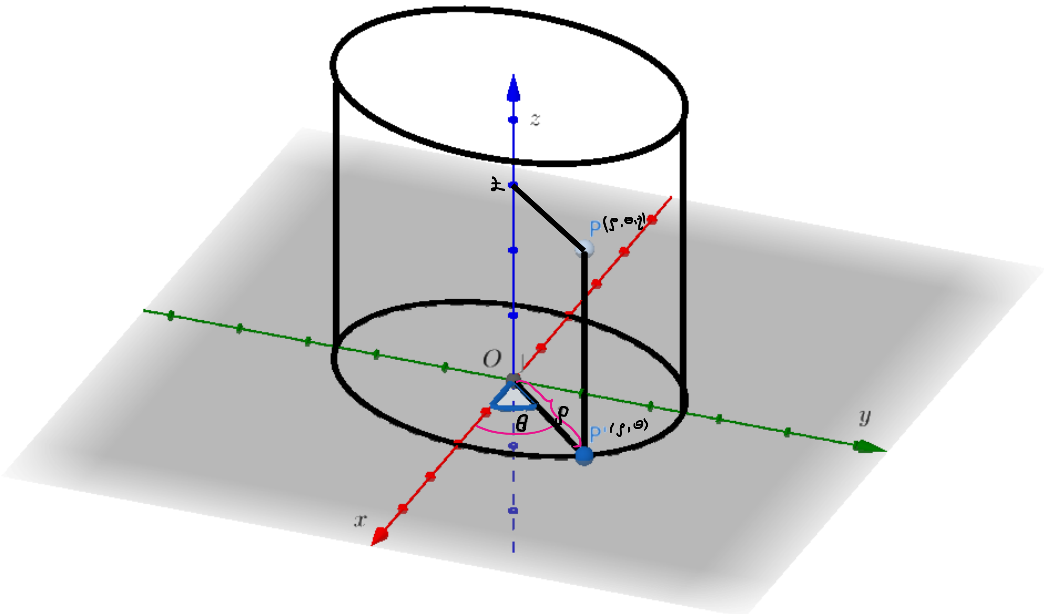


Figure 1: cylindrical coordinates

$$h_1 : \mathcal{P} \setminus \{O\} \to \mathbb{R}_+ \times [0,2\pi) \times \mathbb{R},\ P \to (r,\theta,z)$$

and one obtains a new coordinate system, named the *cylindrical coordinate system* in $\mathcal{P}$. For the conversion formulas between the cylindrical coordinates and the Cartesian coordinates we refere the reader to [1, p. 19]. Note however that once we have the cylindrical coordinates (r,θ,z) of a point P, then its Cartesian coordinates are

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}.$$

3.3 The Spherical Coordinate System

Another way to associate to each point P in $\mathcal{P}$ a triple of real numbers is illustrated in Figure 2. If $P(x,y,z)$ is a point in a rectangular system of coordinates $Oxyz$ and P' its orthogonal projection on Oxy, let ρ be the length of the segment $[OP]$, θ be the oriented angle determined by $[Ox$ and $[OP'$ and φ be the oriented angle between $[Oz$ and $[OP$. The triple

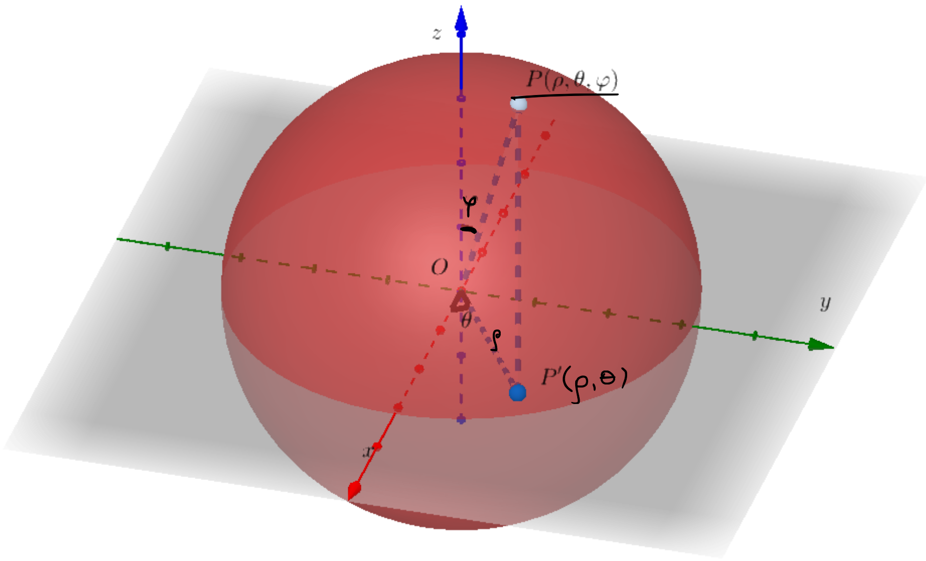


Figure 2: spherical coordinates

(ρ, θ, φ) gives the *spherical coordinates* of the point P. This way, one obtains the bijection

$$h_2 : \mathcal{P} \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi], P \to (\rho, \theta, \varphi),$$

which defines a new coordinate system in $\mathcal{P}$, called the *spherical coordinate system* . For the conversion formulas between the spherical coordinate system and the Cartesian coordinate system we refere the reader to [1, p. 20]. Note however that once we have the cylindrical coordinates (ρ, θ, φ) of a point P, then its Cartesian coordinates are

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}.$$

3.4 The cartesian equations of the straight lines

Let Δ be the straight line passing through the point $A_0(x_0, y_0, z_0)$ which is parallel to the vector $\overrightarrow{d}\,(p, q, r)$. Its vector equation is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda\,\overrightarrow{d},\ \lambda \in \mathbb{R}. \tag{3.1}$$

Denoting by x, y, z the coordinates of the generic point M of the straight line Δ, its vector equation (3.1) is equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases},\ \lambda \in \mathbb{R} \tag{3.2}$$

Indeed, the vector equation of Δ can be written, in terms of the coordinates of the vectors $\vec{r}_M$, $\vec{r}_{A_0}$ and $\vec{d}$, as follows:

$$\begin{aligned} & x\,\vec{u} + y\,\vec{v} + z\,\vec{w} = x_0\,\vec{u} + y_0\,\vec{v} + z_0\,\vec{w} + \lambda(p\,\vec{u} + q\,\vec{v} + r\,\vec{w}) \\ \Longleftrightarrow\ & x\,\vec{u} + y\,\vec{v} + z\,\vec{w} = (x_0 + p\lambda)\,\vec{u} + (y_0 + q\lambda)\,\vec{v} + (z_0 + r\lambda)\,\vec{w},\ \lambda \in \mathbb{R} \end{aligned}$$

which is obviously equivalent to (3.2). The relations (3.2) are called the *parametric equations* of the straight line Δ and they are equivalent to the following relations

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r} \tag{3.3}$$

If $r = 0$, for instance, the canonical equations of the straight line Δ are

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} \ \wedge\ z = z_0.$$

If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are different points of the line Δ, then

$$\overrightarrow{AB}\ (x_B - x_A, y_B - y_A, z_B - z_A)$$

is a director vector of Δ, its canonical equations having, in this case, the form

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}. \tag{3.4}$$

Example 3.2. Consider the tetrahedrom $ABCD$, where $A(1, -1, 1)$, $B(-1, 1, -1)$, $C(2, 1, -1)$ and $D(1, 1, 2)$, as well as the centroids G_A, G_B, G_C, G_D of the triangles BCD, ACD, ABD and ABC[2] respectively. Show that the medians AG_A, BG_B, CG_C and DG_D are concurrent and find the coordinates of their intersection point.

SOLUTION. One can easily see that the coordinates of the centroids G_A, G_B, G_C, G_D are $(2/3, 1, 0)$, $(4/3, 1/3, 2/3)$, $(1/3, 1/3, 2/3)$ and $(2/3, 1/3, -1/3)$ respectively. The equations of the medians AG_A and BG_B are

$$\begin{aligned} (AG_A)\ & \frac{x-1}{2/3-1} = \frac{y+1}{1-(-1)} = \frac{z-1}{0-1} \Longleftrightarrow \frac{x-1}{-1/3} = \frac{y+1}{2} = \frac{z-1}{-1} \\ (BG_B)\ & \frac{x+1}{4/3+1} = \frac{y-1}{1/3-1} = \frac{z+1}{2/3+1} \Longleftrightarrow \frac{x+1}{7/3} = \frac{y-1}{-2/3} = \frac{z+1}{5/3}. \end{aligned}$$

Thus, the director space of the median AG_A is $\left\langle \left(-\frac{1}{3}, 2, -1\right) \right\rangle = \langle(-1, 6, -3)\rangle$ and the director space of the median BG_B is $\left\langle \left(\frac{7}{3}, -\frac{2}{3}, \frac{5}{3}\right) \right\rangle = \langle(7, -2, 5)\rangle$. Consequently, the parametric equations of the medians AG_A and BG_B are

$$(AG_A) \begin{cases} x = 1 - t \\ y = -1 + 6t \\ z = 1 - 3t \end{cases},\ t \in \mathbb{R} \text{ and } (BG_B) \begin{cases} x = -1 + 7s \\ y = 1 - 2s \\ z = -1 + 5s \end{cases},\ s \in \mathbb{R}.$$

Thus, the two medians AG_A and BG_B are concurrent if and only if there exist $s, t \in \mathbb{R}$ such that

$$\begin{cases} 1 - t = -1 + 7s \\ -1 + 6t = 1 - 2s \\ 1 - 3t = -1 + 5s \end{cases} \Longleftrightarrow \begin{cases} 7s + t = 2 \\ 2s + 6t = 2 \\ 5s + 3t = 2 \end{cases} \Longleftrightarrow \begin{cases} 7s + t = 2 \\ s + 3t = 1 \\ 5s + 3t = 2. \end{cases}$$

[2] The centroids of its faces

This system is compatible and has the unique solution $s = t = \frac{1}{4}$, which shows that the two medians AG_A and BG_B are concurrent and

$$AG_A \cap BG_B = \left\{ G\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right) \right\}.$$

One can similarly show that $BG_B \cap CG_C = CG_C \cap AG_A = \left\{ G\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right) \right\}$.

Example 3.3 (Homework)**.** Consider the tetrahedrom $ABCD$, where $A(1,-1,1)$, $B(-1,1,-1)$, $C(2,1,-1)$ and $D(1,1,2)$, as well as the midpoints M, N, P, Q, R and S of its edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[CD]$ and $[DB]$ respectively. Show that the lines MR, PQ and NS are concurrent and find the coordinates of their intersection point.

SOLUTION.

3.5 The cartesian equations of the planes

Let $A_0(x_0, y_0, z_0) \in \mathcal{P}$ and $\overrightarrow{d}_1 (p_1, q_1, r_1), \overrightarrow{d}_2 (p_2, q_2, r_2) \in \mathcal{V}$ be linearly independent vectors, that is

$$\operatorname{rank} \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix} = 2.$$

The vector equation of the plane π passing through A_0 which is parallel to the vectors $\overrightarrow{d}_1$ (p_1, q_1, r_1), $\overrightarrow{d}_2$ (p_2, q_2, r_2) is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda_1 \overrightarrow{d}_1 + \lambda_2 \overrightarrow{d}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.5}$$

If we denote by x, y, z the coordinates of the generic point M of the plane π, then the vector equation (3.5) is the equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda_1 p_1 + \lambda_2 p_2 \\ y = y_0 + \lambda_1 q_1 + \lambda_2 q_2 \\ z = z_0 + \lambda_1 r_1 + \lambda_2 r_2 \end{cases},\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.6}$$

Indeed, the vector equation of π can be written, in terms of the coordinates of the vectors $\vec{r}_M$, $\vec{r}_{A_0}$, $\vec{d}_1$ and $\vec{d}_2$, as follows:

$$x\vec{u}+y\vec{v}+z\vec{w}=x_0\vec{u}+y_0\vec{v}+z_0\vec{w}+\lambda_1(p_1\vec{u}+q_1\vec{v}+r_1\vec{w})+\lambda_2(p_2\vec{u}+q_2\vec{v}+r_2\vec{w})$$
$$\Longleftrightarrow x\vec{u}+y\vec{v}+z\vec{w}=(x_0+\lambda_1p_1+\lambda_2p_2)\vec{u}+(y_0+\lambda_1q_1+\lambda_2q_2)\vec{v}+(z_0+\lambda_1r_1+\lambda_2r_2)\vec{w},$$
$\lambda_1,\lambda_2\in\mathbb{R}$,

which is obviously equivalent to (3.6). The relations (3.6) characterize the points of the plane π and are called the *parametric equations* of the plane π. More precisely, the compatibility of the linear system (3.6) with the unknowns λ_1,λ_2 is a necessary and sufficient condition for the point $M(x,y,z)$ to be contained within the plane π. On the other hand the compatibility of the linear system (3.6) is equivalent to

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}=0, \tag{3.7}$$

which expresses the equality between the rank of the coefficient matrix of the system and the rank of the extended matrix of the system. The equation (3.7) is a characterization of the points of the plane π in terms of the Cartesian coordinates of the generic point M and is called the *cartesian equation* of the plane π. On can put the equation (3.7) in the form

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \text{ or} \tag{3.8}$$

$$Ax+By+Cz+D=0, \tag{3.9}$$

where the coefficients A,B,C satisfy the relation $A^2+B^2+C^2>0$. It is also easy to show that every equation of the form (3.9) represents the equation of a plane. Indeed, if $A\neq 0$, then the equation (3.9) is equivalent to

$$\begin{vmatrix} x+\frac{D}{A} & y & z \\ B & -A & 0 \\ C & 0 & -A \end{vmatrix}=0.$$

We observe that one can put the equation (3.8) in the form

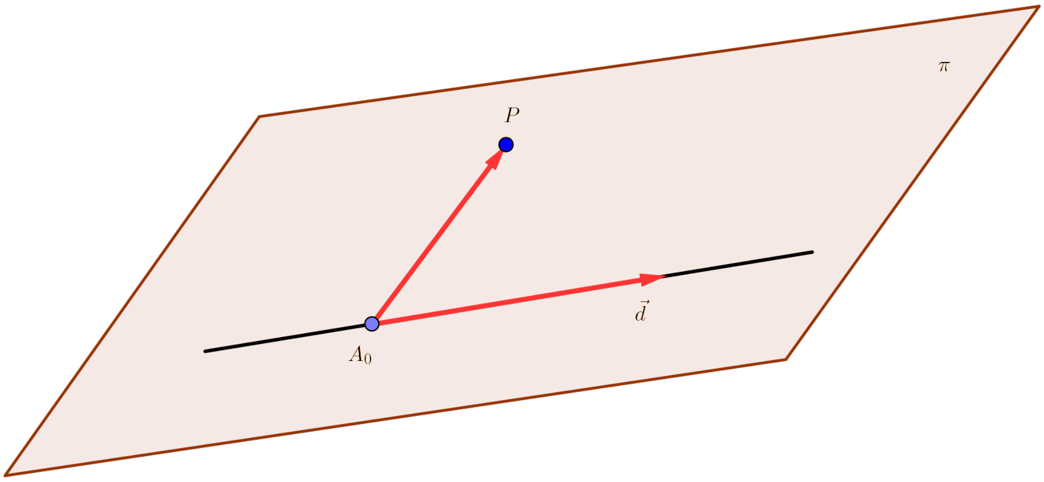
$$AX+BY+CZ=0 \tag{3.10}$$

where $X=x-x_0$, $Y=y-y_0$, $Z=z-z_0$ are the coordinates of the vector $\overrightarrow{A_0M}$.

Example 3.4. Write the equation of the plane determined by the point $P(-1,1,2)$ and the line (Δ) $\frac{x-1}{3}=\frac{y}{2}=\frac{z+1}{-1}$.

SOLUTION. Note that $P\notin\Delta$, as $\frac{-1-1}{3}\neq\frac{1}{2}\neq-3=\frac{2+1}{-1}$, i.e. the point P and the line Δ determine, indeed, a plane, say π. One can regard π as the plane through the point $A_0(1,0,-1)$ which is parallel to the vectors $\overrightarrow{A_0P}\ (-1-1,1-0,2-(-1))=\overrightarrow{A_0P}\ (-2,1,3)$ and $\vec{d}\ (3,2,-1)$. Thus, the equation of π is

$$\begin{vmatrix} x-1 & y & z+1 \\ -2 & 1 & 3 \\ 3 & 2 & -1 \end{vmatrix}=0 \Longleftrightarrow x-y+z=0.$$



Example 3.5 (Homework)**.** Generalize Example 3.4: Write the equation of the plane determined by the line $(\Delta)\ \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$ and the point $M(x_M, y_M, z_M) \notin \Delta$.

SOLUTION.

Remark 3.3. If $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ are noncollinear points, then the plane (ABC) determined by the three points can be viewed as the plane passing through the point A which is parallel to the vectors $\vec{d}_1 = \overrightarrow{AB}$, $\vec{d}_2 = \overrightarrow{AC}$. The coordinates of the vectors $\vec{d}_1$ şi $\vec{d}_2$ are

$$(x_B - x_A, y_B - y_A, z_B - z_A) \text{ and } (x_C - x_A, y_C - y_A, z_C - z_A) \text{ respectively.}$$

Thus, the equation of the plane (ABC) is

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0, \tag{3.11}$$

or, echivalently

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0. \tag{3.12}$$

Thus, four points $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ and $D(x_D, y_D, z_D)$ are coplanar if and ony if

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \tag{3.13}$$

Example 3.6 (Homework)**.** Write the equation of the plane determined by the points $M_1(3,-2,1)$, $M_2(5,4,1)$ and $M_3(-1,-2,3)$.

SOLUTION.

Remark 3.4. If $A(a,0,0),\ B(0,b,0),\ C(0,0,c)$ are three points ($abc \neq 0$), then for the equation of the plane (ABC) we have successively:

$$\begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = 0 \iff \begin{vmatrix} x & y & z-c & 1 \\ a & 0 & -c & 1 \\ 0 & b & -c & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0 \iff \begin{vmatrix} x & y & z-c \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = 0$$

$$\iff ab(z-c) + bcx + acy = 0 \iff bcx + acy + abz = abc$$

$$\iff \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{3.14}$$

The equation (3.14) of the plane (ABC) is said to be in *intercept form* and the x,y,z-intercepts of the plane (ABC) are a,b,c respectively.

Example 3.7 (Homework)**.** Write the equation of the plane (π) $3x - 4y + 6z - 24 = 0$ in intercept form.

SOLUTION.

3.6 Appendix: The Cartesian equations of lines in the two dimensional setting

3.6.1 Cartesian and affine reference systems

If $b = [\overrightarrow{e}, \overrightarrow{f}]$ is an ordered basis of the director subspace $\overrightarrow{\pi}$ of the plane π and $\overrightarrow{x} \in \overrightarrow{\pi}$, recall that the column vector of $\overrightarrow{x}$ with respect to b is being denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

whenever $\overrightarrow{x} = x_1 \overrightarrow{e} + x_2 \overrightarrow{f}$.

Definition 3.3. A *cartesian reference system* of the plane π, is a system $R = (O, \overrightarrow{e}, \overrightarrow{f})$, where O is a point from π called the *origin* of the reference system and $b = [\overrightarrow{e}, \overrightarrow{f}]$ is a basis of the vector space $\overrightarrow{\pi}$.

Denote by E, F the points for which $\overrightarrow{e} = \overrightarrow{OE}$, $\overrightarrow{f} = \overrightarrow{OF}$.

Definition 3.4. The system of points (O, E, F) is called *the affine reference system associated to the cartesian reference system* $R = (O, \overrightarrow{e}, \overrightarrow{f})$.

The straight lines OE, OF, oriented from O to E and from O to F respectively, are called *the coordinate axes*. The coordinates x, y of the position vector $\overrightarrow{r}_M = \overrightarrow{OM}$ with respect to the basis $[\overrightarrow{e}, \overrightarrow{f}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y)$. Also, for the column matrix of coordinates of the vector $\overrightarrow{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\overrightarrow{r}_M = x \overrightarrow{e} + y \overrightarrow{f}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Remark 3.5. If $A(x_A, y_A)$, $B(x_B, y_B)$ are two points, then

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = x_B \overrightarrow{e} + y_B \overrightarrow{f} - (x_A \overrightarrow{e} + y_A \overrightarrow{f}) \\ &= (x_B - x_A) \overrightarrow{e} + (y_B - y_A) \overrightarrow{f}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

3.6.2 The Polar Coordinate System [1, p. 17]

As an alternative to a Cartesian coordinate system (RS) one considers in the plane π a fixed point O, called *pole* and a half-line directed to the right of O, called *polar axis* (see Figure 3). By specifying the distance ρ from O to a point P and an angle θ (measured in radians), whose

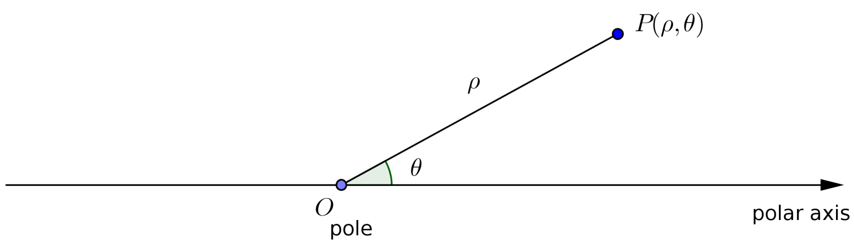


Figure 3: The pole and the polar axis related to a polar coordinate system

”initial” side is the polar axis and whose ”terminal” side is the ray OP, the *polar coordinates* of the point P are (ρ, θ). One obtains a bijection

$$\pi \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi), \quad P \to (\rho, \theta)$$

which associates to any point P in $\pi \setminus \{O\}$ the pair (ρ, θ) (suppose that $O(0,0)$). The positive real number ρ is called the *polar ray* of P and θ is called the *polar angle* of P.

Consider the Cartesian coordinate system in π, whose origin O is the pole and whose positive half-axis Ox is the polar axis (see Figure 4). The following transformation formulas give the connection between the coordinates of an arbitrary point in the two systems of coordinates.

Let P be a given point of polar coordinates (ρ, θ) . Its Cartesian coordinates are

$$\begin{cases} x_P = \rho \cos\theta \\ y_P = \rho \sin\theta \end{cases}. \tag{3.15}$$

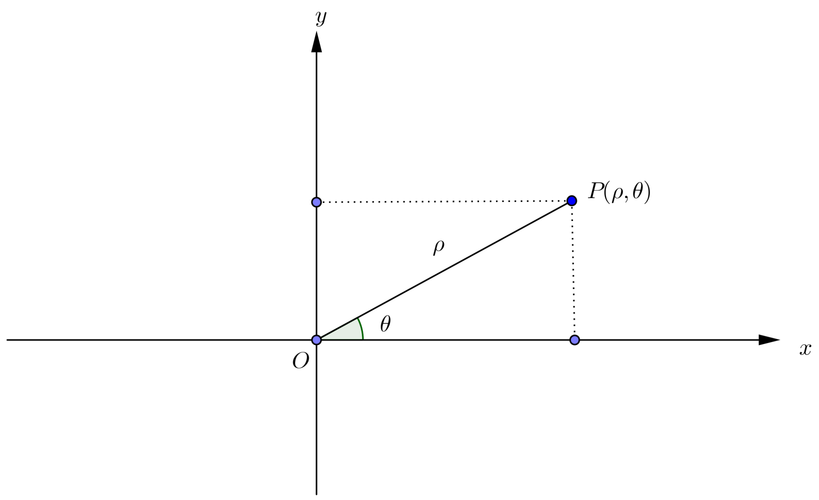


Figure 4: Polar coordinates

Let P be a point of Cartesian coordinates (x, y). It is clear that the polar ray of P is given by the formula

$$\rho = \sqrt{x^2 + y^2}. \tag{3.16}$$

In order to obtain the polar angle of P, it must be considered the quadrant where P is situated. One obtains the following formulas:

Case 1. If $x \neq 0$, then using $\tan\theta = \dfrac{y}{x}$, one has

$$\theta = \arctan\frac{y}{x} + k\pi, \quad \text{where} \quad k = \begin{cases} 0 \text{ if } P \in I \cup]Ox \\ 1 \text{ if } P \in II \cup III \cup]Ox' \\ 2 \text{ if } P \in IV; \end{cases};$$

Case 2. If $x = 0$ and $y \neq 0$, then

$$\theta = \begin{cases} \dfrac{\pi}{2} \text{ when } P \in]Oy \\ \dfrac{3\pi}{2} \text{ when } P \in]Oy'; \end{cases}$$

Case 3. If $x = 0$ and $y = 0$, then $\theta = 0$.

3.6.3 Parametric and Cartesian equations of Lines

Let Δ be a line passing through the point $A_0(x_0, y_0) \in \pi$ which is parallel to the vector $\overrightarrow{d}\ (p, q)$. Its vector equation is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + t\,\overrightarrow{d},\ t \in \mathbb{R}. \tag{3.17}$$

If (x, y) are the coordinates of a generic point $M \in \Delta$, then its vector equation (3.17) is equivalent to the following system

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \end{cases},\ t \in \mathbb{R}. \tag{3.18}$$

The relations are called the *parametric equations* of the line Δ and they are equivalent to the following equation

$$\frac{x - x_0}{p} = \frac{y - y_0}{q}, \tag{3.19}$$

called the *canonical equation*of Δ. If $q = 0$, then the equation (3.19) becomes $y = y_0$.

If $A(x_A, y_A)$ are two different points of the plane π, then $\overrightarrow{AB}\ (x_B - x_A, y_B - y_A)$ is a director vector of the line AB and the canonical equation of the line AB is

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}. \tag{3.20}$$

The equation (3.20) is equivalent to

$$\begin{vmatrix} x - x_A & y - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x - x_A & y - y_A & 1 \\ x_B - x_A & y_B - y_A & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} = 0.$$

Thus, three poins $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{3.21}$$

3.6.4 General Equations of Lines

We can put the equation (3.19) in the form

$$ax + by + c = 0, \qquad \text{with} \quad a^2 + a^2 > 0, \tag{3.22}$$

which means that any line from π is characterized by a first degree equation. Conversely, such of an equation represents a line, since the formula (3.22) is equivalent to

$$\frac{x + \frac{c}{a}}{-\frac{b}{a}} = \frac{y}{1}$$

and this is the *symmetric equation* of the line passing through $P_0\left(-\frac{c}{a}, 0\right)$ and parallel to $\overline{v}\left(-\frac{b}{a}, 1\right)$. The equation (3.22) is called *general equation* of the line.

Remark 3.6. The lines

$$(d)\ ax+by+c=0 \text{ and } (\Delta)\ \frac{x-x_0}{p}=\frac{x-x_0}{q}$$

are parallel if and only if $ap+bq=0$. Indeed, we have:

$$\begin{aligned} d\|\Delta \ &\Longleftrightarrow \overrightarrow{d}=\overrightarrow{\Delta} \Longleftrightarrow \langle \overrightarrow{u}\ (p,q)\rangle = \langle \overrightarrow{v}\ \left(-\tfrac{b}{a},1\right)\rangle \Longleftrightarrow \exists t\in\mathbb{R} \text{ s.t. } \overrightarrow{u}\ (p,q)=t\ \overrightarrow{v}\ \left(-\tfrac{b}{a},1\right) \\ &\Longleftrightarrow \exists t\in\mathbb{R} \text{ s.t. } =-t\tfrac{b}{a} \text{ and } q=t \Longleftrightarrow ap+bq=0. \end{aligned}$$

3.6.5 Reduced Equations of Lines

Consider a line given by its general equation $Ax+By+C=0$, where at least one of the coefficients A and B is nonzero. One may suppose that $B\neq 0$, so that the equation can be divided by B. One obtains

$$y=mx+n \tag{3.23}$$

which is said to be the *reduced equation* of the line.

Remark: If $B=0$, (3.22) becomes $Ax+C=0$, or $x=-\dfrac{C}{A}$, a line parallel to Oy. (In the same way, if $A=0$, one obtains the equation of a line parallel to Ox).

Let d be a line of equation $y=mx+n$ in a Cartesian system of coordinates and suppose that the line is not parallel to Oy. Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be two different points on d and φ be the angle determined by d and Ox (see Figure 5); $\varphi\in[0,\pi]\setminus\{\pi/2\}$. The points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ belong to d, hence

$$\begin{cases} y_1=mx_1+n \\ y_2=mx_2+n, \end{cases}$$

and $x_2\neq x_1$, since d is not parallel to Oy. Then,

$$m=\frac{y_2-y_1}{x_2-x_1}=\tan\varphi. \tag{3.24}$$

The number $m=\tan\varphi$ is called the *angular coefficient* of the line d. It is immediate that the equation of the line passing through the point $P_0(x_0,y_0)$ and of the given angular coefficient m is

$$y-y_0=m(x-x_0). \tag{3.25}$$

3.6.6 Intersection of Two Lines

Let $d_1: a_1x+b_1y+c_1=0$ and $d_2: a_2x+b_2y+c_2=0$ be two lines in $\mathcal{E}_2$. The solution of the system of equation

$$\begin{cases} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{cases}$$

will give the set of the intersection points of d_1 and d_2.

1) If $\dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2}$, the system has a unique solution (x_0,y_0) and the lines have a unique intersection point $P_0(x_0,y_0)$. They are *secant*.

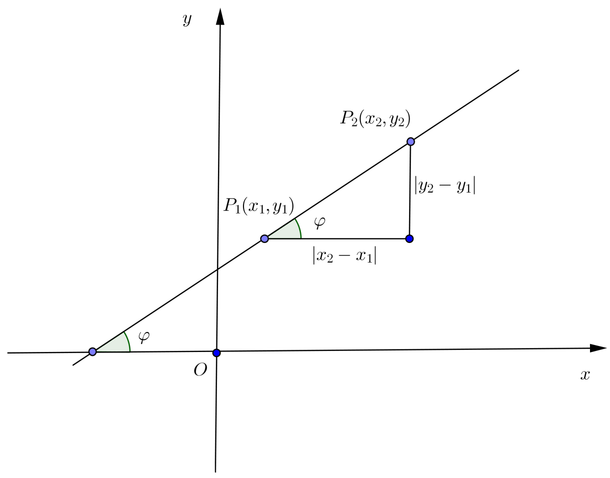


Figure 5:

2) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$, the system is not compatible, and the lines have no points in common. They are *parallel*.

3) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, the system has an infinity of solutions, and the lines coincide. They are *identical*.

If $d_i : a_i x + b_i y + c_i = 0$, $i = \overline{1,3}$ are three lines in $\mathcal{E}_2$, then they are concurrent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{3.26}$$

3.6.7 Bundles of Lines ([1])

The set of all the lines passing through a given point P_0 is said to be a *bundle* of lines. The point P_0 is called the *vertex* of the bundle.

If the point P_0 is of coordinates $P_0(x_0, y_0)$, then the equation of the bundle of vertex P_0 is

$$r(x - x_0) + s(y - y_0) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{3.27}$$

Remark: Tthe *reduced bundle* of line through P_0 is,

$$y - y_0 = m(x - x_0), \qquad m \in \mathbb{R}, \tag{3.28}$$

and covers the bundle of lines through P_0, except the line $x = x_0$. Similarly, the family of lines

$$x - x_0 = k(y - y_0), \qquad k \in \mathbb{R}, \tag{3.29}$$

covers the bundle of lines through P_0, exceptthe line $y = y_0$.

If the point P_0 is given as the intersection of two lines, then its coordinates are the solution of the system

$$\begin{cases} d_1 : a_1 x + b_1 y + c_1 = 0 \\ d_2 : a_2 x + b_2 y + c_2 = 0 \end{cases},$$

assumed to be compatible. The equation of the bundle of lines through P_0 is

$$r(a_1x + b_1y + c_1) + s(a_2x + b_2y + c_2) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \tag{3.30}$$

Remark: As before, if $r \neq 0$ (or $s \neq 0$), one obtains the reduced equation of the bundle, containing all the lines through P_0, except d_1 (respectively d_2).

3.6.8 The Angle of Two Lines ([1])

Let d_1 and d_2 be two concurrent lines, given by their reduced equations:

$$d_1 : y = m_1x + n_1 \quad \text{and} \quad d_2 : y = m_2x + n_2.$$

The angular coefficients of d_1 and d_2 are $m_1 = \tan\varphi_1$ and $m_2 = \tan\varphi_2$ (see Figure 6). One may suppose that $\varphi_1 \neq \frac{\pi}{2}$, $\varphi_2 \neq \frac{\pi}{2}$, $\varphi_2 \geq \varphi_1$, such that $\varphi = \varphi_2 - \varphi_1 \in [0, \pi] \setminus \{\frac{\pi}{2}\}$.

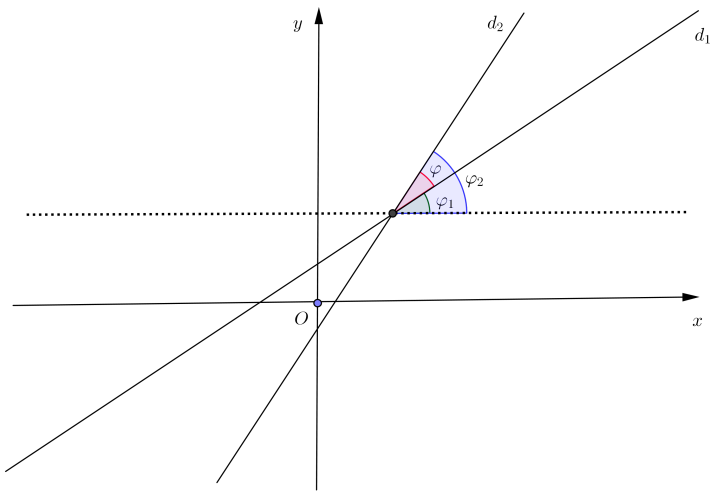


Figure 6:

The angle determined by d_1 and d_2 is given by

$$\tan\varphi = \tan(\varphi_2 - \varphi_1) = \frac{\tan\varphi_2 - \tan\varphi_1}{1 + \tan\varphi_1 \tan\varphi_2},$$

hence

$$\tan\varphi = \frac{m_2 - m_2}{1 + m_1m_2}. \tag{3.31}$$

1) The lines d_1 and d_2 are parallel if and only if $\tan\varphi = 0$, therefore

$$d_1 \parallel d_2 \iff m_1 = m_2. \tag{3.32}$$

2) The lines d_1 and d_2 are orthogonal if and only if they determine an angle of $\frac{\pi}{2}$, hence

$$d_1 \perp d_2 \iff m_1m_2 + 1 = 0. \tag{3.33}$$

3.7 Problems

1. Write the equation of the plane which passes through $M_0(1, -2, 3)$ and is parallel to the vectors $\overrightarrow{v}_1\ (1, -1, 0)$ and $\overrightarrow{v}_2\ (-3, 2, 4)$.

   HINT.
$$\begin{vmatrix} x-0 & y+2 & z-3 \\ 1 & -1 & 0 \\ -3 & 2 & 4 \end{vmatrix} = 0.$$

   vspace*2cm

2. Write the equation of the line which passes through $A(1, -2, 6)$ and is parallel to

   (a) The x-axis;

   (b) The line $(d_1)\ \dfrac{x-1}{2} = \dfrac{y+5}{-3} = \dfrac{z-1}{4}$.

   (c) The vector $\overrightarrow{v}\ (1, 0, 2)$.

   SOLUTION.

3. Write the equation of the plane which contains the line

$$(d_1)\ \frac{x-3}{2} = \frac{y+4}{1} = \frac{z-2}{-3}$$

and is parallel to the line

$$(d_2)\ \frac{x+5}{2} = \frac{y-2}{2} = \frac{z-1}{2}.$$

HINT.

$$\begin{vmatrix} x-3 & y+4 & z-2 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{vmatrix} = 0.$$

4. Consider the points $A(\alpha, 0, 0)$, $B(0, \beta, 0)$ and $C(0, 0, \gamma)$ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{a} \text{ where } a \text{ is a constatnt.}$$

Show that the plane (A, B, C) passes through a fixed point.

SOLUTION. The equation of the plane (ABC) can be written in intercept form, namely

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1.$$

The given relation shows that the point $P(a, a, a) \in (ABC)$ whenever α, β, γ verifies the given relation.

5. Write the equation of the line which passes through the point $M(1, 0, 7)$, is parallel to the plane (π) $3x - y + 2z - 15 = 0$ and intersects the line

$$(d)\ \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

Solution.

6. Write the equation of the plane which passes through $M_0(1,-2,3)$ and cuts the positive coordinate axes through equal intercepts.

   SOLUTION. The general equation of such a plane is $x+y+z=a$. In this particular case $a=1+(-2)+3=2$ and the equation of the required plane is $x+y+z=2$.

7. Write the equation of the plane which passes through $A(1,2,1)$ and is parallel to the straight lines

$$(d_1)\begin{cases} x+2y-z+1=0 \\ x-y+z-1=0 \end{cases} \quad (d_2)\begin{cases} 2x-y+z=1 \\ x-y+z=0. \end{cases}$$

   SOLUTION. We need to find some director parameters of the lines (d_1) and (d_2). In this respect we may solve the two systems. The general solution of the first system is

$$\begin{cases} x=-\frac{1}{3}t+\frac{1}{3} \\ y=\frac{2}{3}t-\frac{2}{3} \\ z=t \end{cases}, \ t\in\mathbb{R}$$

   and the general solution of the second system is

$$\begin{cases} x=1 \\ y=t+1 \\ z=t \end{cases}, \ t\in\mathbb{R}$$

   and these are the parametric equations of the lines (d_1) and (d_2). Thus, the direction of the line (d_1) is the 1-dimensional subspace

$$\left\langle\left(-\frac{1}{3},\frac{2}{3},1\right)\right\rangle=\langle(-1,2,3)\rangle,$$

   and the direction of the line (d_2) is the 1-dimensional subspace $\langle(0,1,1)\rangle$.

   Consequently, some director parameters of the line (d_1) are $p_1=-1, q_1=2, r_1=3$ and some director parameters of the line (d_2) are $p_2=0, q_2=r_2=1$. Finaly, the equation of the required plane is

$$\begin{vmatrix} x-1 & y-2 & z-1 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}=0.$$

   The computation of the determinant is left to the reader.

   A few questions in the two dimensional setting ([1])

8. The sides $[BC]$, $[CA]$, $[AB]$ of the triangle ΔABC are divided by the points M, N respectively P into the same ratio k. Prove that the triangles ΔABC and ΔMNP have the same center of gravity.

   SOLUTION.

9. Sketch the graph of $x^2 - 4xy + 3y^2 = 0$.

   SOLUTION.

10. Find the equation of the line passing through the intersection point of the lines

$$d_1 : 2x - 5y - 1 = 0, \quad d_2 : x + 4y - 7 = 0$$

and through a point M which divides the segment $[AB]$, $A(4, -3)$, $B(-1, 2)$, into the ratio $k = \frac{2}{3}$.

SOLUTION.

11. Let A be a mobile point on the Ox axis and B a mobile point on Oy, so that $\dfrac{1}{OA} + \dfrac{1}{OB} = k$ (constant). Prove that the lines AB passes through a fixed point.

    SOLUTION.

12. Find the equation of the line passing through the intersection point of

$$d_1 : 3x - 2y + 5 = 0, \quad d_2 : 4x + 3y - 1 = 0$$

and crossing the positive half axis of Oy at the point A with $OA = 3$.

SOLUTION.

13. Find the parametric equations of the line through P_1 and P_2, when
    (a) $P_1(3,-2)$, $P_2(5,1)$;
    (b) $P_1(4,1)$, $P_2(4,3)$.

    SOLUTION.

14. Find the parametric equations of the line through $P(-5, 2)$ and parallel to $\overline{v}(2, 3)$.
    SOLUTION.

15. Show that the equations

$$x = 3 - t, y = 1 + 2t \qquad \text{and} \qquad x = -1 + 3t, y = 9 - 6t$$

represent the same line.

SOLUTION.

16. Find the vector equation of the line P_1P_2, where
    (a) $P_1(2,-1)$, $P_2(-5,3)$;
    (b) $P_1(0,3)$, $P_2(4,3)$.

    SOLUTION.

17. Given the line $d : 2x + 3y + 4 = 0$, find the equation of a line d_1 through the point $M_0(2, 1)$, in the following situations:

    (a) d_1 is parallel with d;

    (b) d_1 is orthogonal on d;

    (c) the angle determined by d and d_1 is $\varphi = \frac{\pi}{4}$.

    SOLUTION.

18. The vertices of the triangle ΔABC are the intersection points of the lines

$$d_1 : 4x + 3y - 5 = 0, \quad d_2 : x - 3y + 10 = 0, \quad d_3 : x - 2 = 0.$$

(a) Find the coordinates of A, B, C.

(b) Find the equations of the median lines of the triangle.

(c) Find the equations of the heights of the triangle.

SOLUTION.

