

Geometry

Problem booklet

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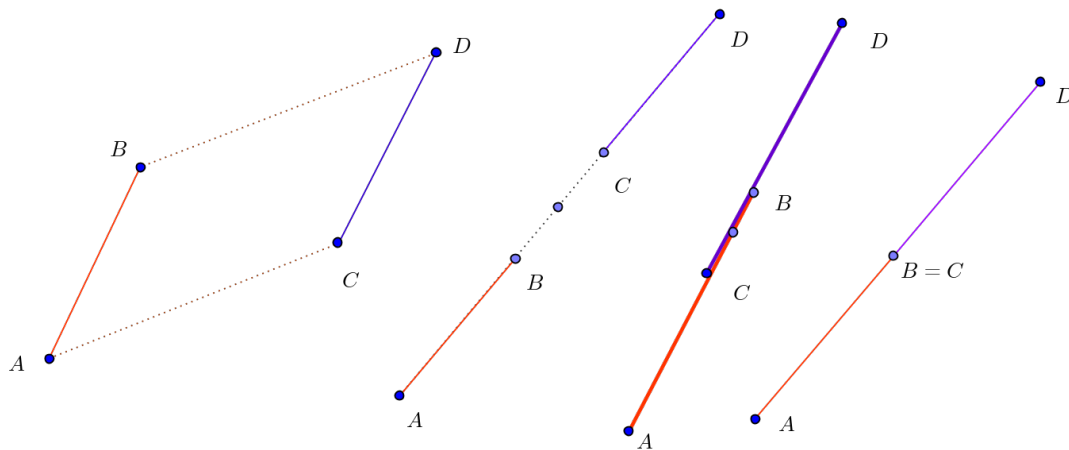
1 Week 1: Vector algebra

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background. Free vectors

Vectors Let \mathcal{P} be the three dimensional physical space in which we can talk about points, lines, planes and various relations among them. If $(A, B) \in \mathcal{P} \times \mathcal{P}$ is an ordered pair, then A is called the *original point* or the *origin* and B is called the *terminal point* or the *extremity* of (A, B) .

Definition 1.1.1. The ordered pairs (A, B) , (C, D) are said to be equipollent, written $(A, B) \sim (C, D)$, if the segments $[AD]$ and $[BC]$ have the same midpoint.



Pairs of equipollent points $(A, B) \sim (C, D)$

Remark 1.1.2. If the points $A, B, C, D \in \mathcal{P}$ are not collinear, then $(A, B) \sim (C, D)$ if and only if $ABDC$ is a parallelogram. In fact the length of the segments $[AB]$ and $[CD]$ is the same whenever $(A, B) \sim (C, D)$.

Proposition 1.1.3. If (A, B) is an ordered pair and $O \in \mathcal{P}$ is a given point, then there exists a unique point X such that $(A, B) \sim (O, X)$.

Proposition 1.1.4. The equipollence relation is an equivalence relation on $\mathcal{P} \times \mathcal{P}$.

Definition 1.1.5. The equivalence classes with respect to the equipollence relation are called (free) vectors.

Denote by \overrightarrow{AB} the equivalence class of the ordered pair (A, B) , that is $\overrightarrow{AB} = \{(X, Y) \in \mathcal{P} \times \mathcal{P} \mid (X, Y) \sim (A, B)\}$ and let $\mathcal{V} = \mathcal{P} \times \mathcal{P} / \sim = \{\overrightarrow{AB} \mid (A, B) \in \mathcal{P} \times \mathcal{P}\}$ be the set of (free) vectors. The *length* or the *magnitude* of the vector \overrightarrow{AB} , denoted by $\|\overrightarrow{AB}\|$ or by $|\overrightarrow{AB}|$, is the length of the segment $[AB]$.

Remark 1.1.6. If two ordered pairs (A, B) and (C, D) are equipollent, i.e. the vectors \overrightarrow{AB} and \overrightarrow{CD} are equal, then they have the same length, the same direction and the same sense. In fact a vector is determined by these three items.

Proposition 1.1.7. 1. $\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \overrightarrow{AC} = \overrightarrow{BD}$.

2. $\forall A, B, O \in \mathcal{P}, \exists ! X \in \mathcal{P}$ such that $\overrightarrow{AB} = \overrightarrow{OX}$.

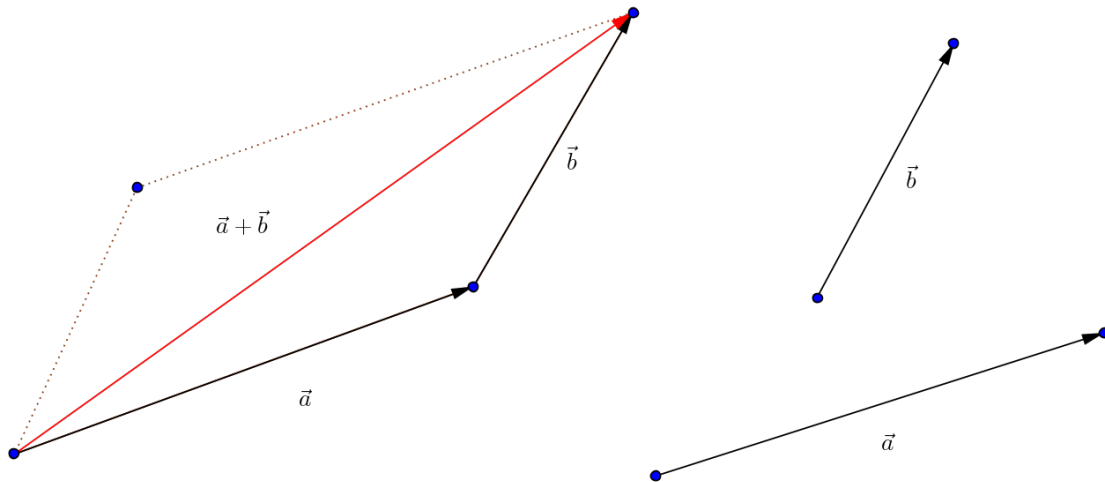
3. $\overrightarrow{AB} = \overrightarrow{A'B'}, \overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$.

Definition 1.1.8. If $O, M \in \mathcal{P}$, the vector \overrightarrow{OM} is denoted by \vec{r}_M and is called the *position vector* of M with respect to O .

Corollary 1.1.9. The map $\varphi_O : \mathcal{P} \rightarrow \mathcal{V}, \varphi_O(M) = \vec{r}_M$ is one-to-one and onto, i.e. bijective.

1.1.1 Operations with vectors

• **The addition of vectors** Let $\vec{a}, \vec{b} \in \mathcal{V}$ and $O \in \mathcal{P}$ be such that $\vec{a} = \overrightarrow{OA}, \vec{b} = \overrightarrow{AB}$. The vector \overrightarrow{OB} is called the *sum* of the vectors \vec{a} and \vec{b} and is written $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$.



Let O' be another point and $A', B' \in \mathcal{P}$ be such that $\overrightarrow{O'A'} = \vec{a}, \overrightarrow{A'B'} = \vec{b}$. Since $\overrightarrow{OA} = \overrightarrow{O'A'}$ and $\overrightarrow{AB} = \overrightarrow{A'B'}$ it follows, according to Proposition 1.1.4 (3), that $\overrightarrow{OB} = \overrightarrow{O'B'}$. Therefore the vector $\vec{a} + \vec{b}$ is independent on the choice of the point O .

Proposition 1.1.10. The set \mathcal{V} endowed to the binary operation $\mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}, (\vec{a}, \vec{b}) \mapsto \vec{a} + \vec{b}$, is an abelian group whose zero element is the vector $\overrightarrow{AA} = \overrightarrow{BB} = \vec{0}$ and the opposite of \overrightarrow{AB} , denoted by $-\overrightarrow{AB}$, is the vector \overrightarrow{BA} .

In particular the addition operation is associative and the vector

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

is usually denoted by $\vec{a} + \vec{b} + \vec{c}$. Moreover the expression

$$((\cdots (\vec{a}_1 + \vec{a}_2) + \vec{a}_3 + \cdots + \vec{a}_n) \cdots), \quad (1.1)$$

is independent of the distribution of paranthesis and it is usually denoted by

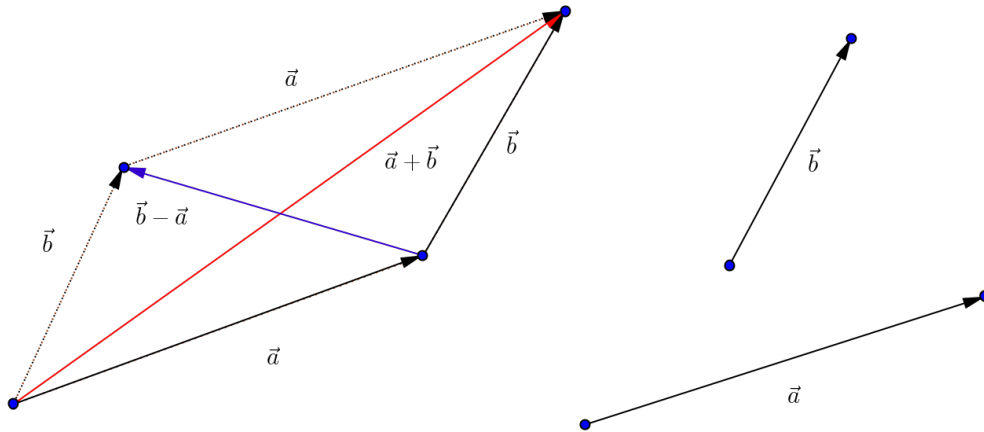
$$\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n.$$

Example 1.1.11. If $A_1, A_2, A_3, \dots, A_n \in \mathcal{P}$ are some given points, then

$$\vec{A_1A_2} + \vec{A_2A_3} + \cdots + \vec{A_{n-1}A_n} = \vec{A_1A_n}.$$

This shows that $\vec{A_1A_2} + \vec{A_2A_3} + \cdots + \vec{A_{n-1}A_n} + \vec{A_nA_1} = \vec{0}$, namely the sum of vectors constructed on the edges of a closed broken line is zero.

Corolarul 1.1.12. If $\vec{a} = \vec{OA}$, $\vec{b} = \vec{OB}$ are given vectors, there exists a unique vector $\vec{x} \in \mathcal{V}$ such that $\vec{a} + \vec{x} = \vec{b}$. In fact $\vec{x} = \vec{b} + (-\vec{a}) = \vec{AB}$ and is denoted by $\vec{b} - \vec{a}$.



• The multiplication of vectors with scalars

Let $\alpha \in \mathbb{R}$ be a scalar and $\vec{a} = \vec{OA} \in \mathcal{V}$ be a vector. We define the vector $\alpha \cdot \vec{a}$ as follows:

- $\alpha \cdot \vec{a} = \vec{0}$ if $\alpha = 0$ or $\vec{a} = \vec{0}$;
- if $\vec{a} \neq \vec{0}$ and $\alpha > 0$, there exists a unique point on the half line $]OA$ such that $\|OB\| = \alpha \cdot \|OA\|$ and define $\alpha \cdot \vec{a} = \vec{OB}$;
- if $\alpha < 0$ we define $\alpha \cdot \vec{a} = -(|\alpha| \cdot \vec{a})$.

The external binary operation

$$\mathbb{R} \times \mathcal{V} \rightarrow \mathcal{V}, (\alpha, \vec{a}) \mapsto \alpha \cdot \vec{a}$$

is called the *multiplication of vectors with scalars*.

Proposition 1.1.13. *The following properties hold:*

$$(v1) (\alpha + \beta) \cdot \vec{a} = \alpha \cdot \vec{a} + \beta \cdot \vec{a}, \forall \alpha, \beta \in \mathbb{R}, \vec{a} \in \mathcal{V}.$$

$$(v2) \alpha \cdot (\vec{a} + \vec{b}) = \alpha \cdot \vec{a} + \alpha \cdot \vec{b}, \forall \alpha \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}.$$

$$(v3) \alpha \cdot (\beta \cdot \vec{a}) = (\alpha\beta) \cdot \vec{a}, \forall \alpha, \beta \in \mathbb{R}.$$

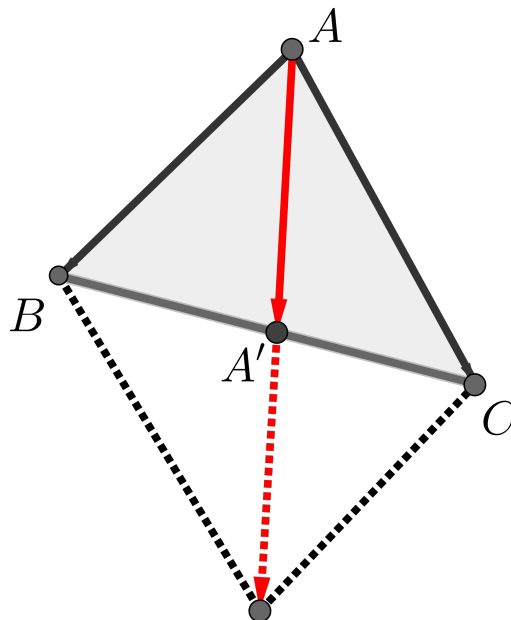
$$(v4) 1 \cdot \vec{a} = \vec{a}, \forall \vec{a} \in \mathcal{V}.$$

1.1.2 The vector structure on the set of vectors

Theorem 1.1.14. *The set of (free) vectors endowed with the addition binary operation of vectors and the external binary operation of multiplication of vectors with scalars is a real vector space.*

Example 1.1.15. *If A' is the midpoint of the edge $[BC]$ of the triangle ABC , then*

$$\vec{AA'} = \frac{1}{2} (\vec{AB} + \vec{AC}).$$



1.2 Problems

1. ([4, Problema 3, p. 1]) Let $OABCDE$ be a regular hexagon in which $\vec{OA} = \vec{a}$ and $\vec{OE} = \vec{b}$. Express the vectors $\vec{OB}, \vec{OC}, \vec{OD}$ in terms of the vectors \vec{a} and \vec{b} . Show that $\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} + \vec{OE} = 3 \vec{OC}$.

2. Consider a tetrahedron $ABCD$. Find the the following sums of vectors:

(a) $\vec{AB} + \vec{BC} + \vec{CD}$.

(b) $\vec{AD} + \vec{CB} + \vec{DC}$.

(c) $\vec{AB} + \vec{BC} + \vec{DA} + \vec{CD}$.

3. Consider a pyramid with the vertex at S and the basis a parallelogram $ABCD$ whose diagonals are concurrent at O . Show the equality $\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4 \vec{SO}$.

4. Let E and F be the midpoints of the diagonals of a quadrilateral $ABCD$. Show that

$$\vec{EF} = \frac{1}{2} (\vec{AB} + \vec{CD}) = \frac{1}{2} (\vec{AD} + \vec{CB}).$$

5. In a triangle ABC we consider the height AD from the vertex A ($D \in BC$). Find the decomposition of the vector AD in terms of the vectors $\vec{c} = \vec{AB}$ and $\vec{b} = \vec{AC}$.

6. ([4, Problema 12, p. 3]) Let M, N be the midpoints of two opposite edges of a given quadrilateral $ABCD$ and P be the midpoint of $[MN]$. Show that

$$\vec{PA} + \vec{PB} + \vec{PC} + \vec{PD} = 0$$

7. ([4, Problema 12, p. 7]) Consider two perpendicular chords AB and CD of a given circle and $\{M\} = AB \cap CD$. Show that

$$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 2 \vec{OM}.$$

8. ([4, Problema 13, p. 3]) If G is the centroid of a triangle ABC and O is a given point, show that

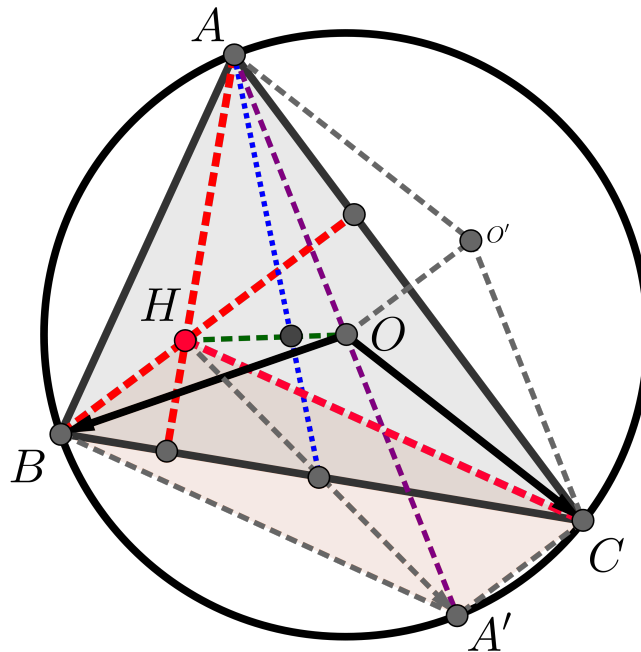
$$\vec{OG} = \frac{\vec{OA} + \vec{OB} + \vec{OC}}{3}.$$

9. ([4, Problema 14, p. 4]) Consider the triangle ABC alongside its orthocenter H , its circumcenter O and the diametrically opposed point A' of A on the latter circle. Show that:

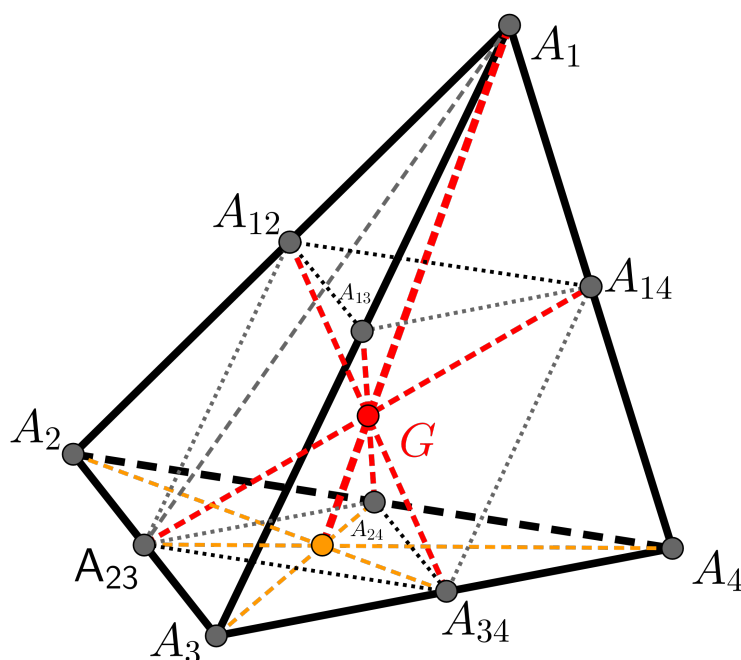
(a) $\vec{OA} + \vec{OB} + \vec{OC} = \vec{OH}$.

(b) $\vec{HB} + \vec{HC} = \vec{HA'}$.

(c) $\vec{HA} + \vec{HB} + \vec{HC} = 2 \vec{HO}$.



10. ([4, Problema 15, p. 4]) Consider the triangle ABC alongside its centroid G , its orthocenter H and its circumcenter O . Show that O, G, H are collinear and $3 \overrightarrow{HG} = 2 \overrightarrow{HO}$.
11. ([4, Problema 11, p. 3]) Consider two parallelograms, $A_1A_2A_3A_4, B_1B_2B_3B_4$ in \mathcal{P} , and M_1, M_2, M_3, M_4 the midpoints of the segments $[A_1B_1], [A_2B_2], [A_3B_3], [A_4B_4]$ respectively. Show that:
- $2 \overrightarrow{M_1M_2} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2}$ and $2 \overrightarrow{M_3M_4} = \overrightarrow{A_3A_4} + \overrightarrow{B_3B_4}$.
 - M_1, M_2, M_3, M_4 are the vertices of a parallelogram.
12. ([4, Problema 27, p. 13]) Consider a tetrahedron $A_1A_2A_3A_4$ and the midpoints A_{ij} of the edges $A_iA_j, i \neq j$. Show that:
- The lines $A_{12}A_{34}, A_{13}A_{24}$ and $A_{14}A_{23}$ are concurrent in a point G .
 - The medians of the tetrahedron (the lines passing through the vertices and the centroids of the opposite faces) are also concurrent at G .
 - Determine the ratio in which the point G divides each median.
 - Show that $\overrightarrow{GA_1} + \overrightarrow{GA_2} + \overrightarrow{GA_3} + \overrightarrow{GA_4} = \vec{0}$.
 - If M is an arbitrary point, show that $\overrightarrow{MA_1} + \overrightarrow{MA_2} + \overrightarrow{MA_3} + \overrightarrow{MA_4} = 4 \overrightarrow{MG}$.



13. In a triangle ABC consider the points M, L on the side AB and N, T on the side AC such that $3 \overrightarrow{AL} = 2 \overrightarrow{AM} = \overrightarrow{AB}$ and $3 \overrightarrow{AT} = 2 \overrightarrow{AN} = \overrightarrow{AC}$. Show that $\overrightarrow{AB} + \overrightarrow{AC} = 5 \overrightarrow{AS}$, where $\{S\} = MT \cap LN$.
14. Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$, not necessarily in the same plane, alongside their centroids G_1, G_2 . Show that $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3 \overrightarrow{G_1G_2}$.

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