

Geometry

Problem booklet

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1 Week 6: Products of vectors

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background. Products of vectors

1.1.1 The dot product

Definition 1.1. The real number

$$\vec{a} \cdot \vec{b} = \begin{cases} 0 & \text{if } \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \\ \|\vec{a}\| \cdot \|\vec{b}\| \cos(\widehat{\vec{a}, \vec{b}}) & \text{if } \vec{a} \neq \vec{0} \text{ and } \vec{b} \neq \vec{0} \end{cases} \quad (1.1)$$

is called the *dot product* of the vectors \vec{a}, \vec{b} .

Remark 1.2. 1. $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$.

$$2. \vec{a} \cdot \vec{a} = \|\vec{a}\| \cdot \|\vec{a}\| \cos 0 = \|\vec{a}\|^2.$$

Proposition 1.3. The dot product has the following properties:

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}, \forall \vec{a}, \vec{b} \in \mathcal{V}$.
2. $\vec{a} \cdot (\lambda \vec{b}) = \lambda(\vec{a} \cdot \vec{b}), \forall \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}$.
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}, \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.
4. $\vec{a} \cdot \vec{a} \geq 0, \forall \vec{a} \in \mathcal{V}$.
5. $\vec{a} \cdot \vec{a} = 0 \Leftrightarrow \vec{a} = \vec{0}$.

Definition 1.4. A basis of the vector space \mathcal{V} is said to be *orthonormal*, if $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1, \vec{i} \perp \vec{j}, \vec{j} \perp \vec{k}, \vec{k} \perp \vec{i}$ ($\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1, \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$). A cartesian reference system $R = (O, \vec{i}, \vec{j}, \vec{k})$ is said to be *orthonormal* if the basis $[\vec{i}, \vec{j}, \vec{k}]$ is orthonormal.

Proposition 1.5. Let $[\vec{i}, \vec{j}, \vec{k}]$ be an orthonormal basis and $\vec{a}, \vec{b} \in \mathcal{V}$. If $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (1.2)$$

Remark 1.6 1.6. Let $[\vec{i}, \vec{j}, \vec{k}]$ be an orthonormal basis and $\vec{a}, \vec{b} \in \mathcal{V}$. If $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, then

1. $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2$ and we conclude that $\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

2.

$$\begin{aligned}\cos(\widehat{\vec{a}, \vec{b}}) &= \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} \\ &= \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}.\end{aligned}\tag{1.3}$$

In particular

$$\begin{aligned}\cos(\widehat{\vec{a}, \vec{i}}) &= \frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}; \\ \cos(\widehat{\vec{a}, \vec{j}}) &= \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}; \\ \cos(\widehat{\vec{a}, \vec{k}}) &= \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}.\end{aligned}$$

$$3. \vec{a} \perp \vec{b} \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$$

1.1.2 Applications of the dot product

• **The distance between two points.** Consider two points $A(x_A, y_A, z_A), B(x_B, y_B, z_B) \in \mathcal{P}$. The norm of the vector \overrightarrow{AB} ($x_B - x_A, y_B - y_A, z_B - z_A$) is

$$\|\overrightarrow{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

• **The normal vector of a plane.** Consider the plane $\pi : Ax + By + Cz + D = 0$ and the point $P(x_0, y_0, z_0) \in \pi$. The equation of π becomes

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.\tag{1.4}$$

If $M(x, y, z) \in \pi$, the coordinates of \overrightarrow{PM} are $(x - x_0, y - y_0, z - z_0)$ and the equation (1.4) tells us that $\vec{n} \cdot \overrightarrow{PM} = 0$, for every $M \in \pi$, that is $\vec{n} \perp \overrightarrow{PM} = 0$, for every $M \in \pi$, which is equivalent to $\vec{n} \perp \pi$, where $\vec{n} (A, B, C)$. This is the reason to call $\vec{n} (A, B, C)$ the *normal vector* of the plane π .

• **The distance from a point to a plane.** Consider the plane $\pi : Ax + By + Cz + D = 0$, a point $P(x_P, y_P, z_P) \in \mathcal{P}$ and M the orthogonal projection of P on π . The real number δ given by $\overrightarrow{MP} = \delta \cdot \vec{n}_0$ is called the *oriented distance* from P to the plane π , where $\vec{n}_0 = \frac{1}{\|\vec{n}\|} \vec{n}$ is the versor of the normal vector $\vec{n} (A, B, C)$. Since $\overrightarrow{MP} = \delta \cdot \vec{n}_0$, it follows that $\delta(P, M) = \|\overrightarrow{MP}\| = |\delta|$, where $\delta(P, M)$ stands for the distance from P to π . We shall show that

$$\delta = \frac{Ax_P + By_P + Cz_P + D}{\sqrt{A^2 + B^2 + C^2}}.$$

Indeed, since $\vec{MP} = \delta \cdot \vec{n}_0$, we get successively:

$$\begin{aligned}\delta &= \vec{n}_0 \cdot \vec{MP} = \left(\frac{1}{\|\vec{n}\|} \vec{n} \right) \cdot \vec{MP} = \frac{\vec{n} \cdot \vec{MP}}{\|\vec{n}\|} \\ &= \frac{A(x_P - x_M) + B(y_P - y_M) + C(z_P - z_M)}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{Ax_P + By_P + Cz_P - (Ax_M + By_M + Cz_M)}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{Ax_P + By_P + Cz_P + D}{\sqrt{A^2 + B^2 + C^2}}.\end{aligned}$$

Consequently

$$\delta(P, M) = \|\vec{MP}\| = |\delta| = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

1.1.3 The vector product

Definition 1.7. The *vector product* or the *cross product* of the vectors $\vec{a}, \vec{b} \in \mathcal{V}$ is a vector, denoted by $\vec{a} \times \vec{b}$, which is defined to be zero if \vec{a}, \vec{b} are linearly dependent (collinear), and if \vec{a}, \vec{b} are linearly independent (noncollinear), then it is defined by the following data:

1. $\vec{a} \times \vec{b}$ is a vector orthogonal on the two-dimensional subspace $\langle \vec{a}, \vec{b} \rangle$ of \mathcal{V} ;
2. if $\vec{a} = \vec{OA}$, $\vec{b} = \vec{OB}$, then the sense of $\vec{a} \times \vec{b}$ is the one in which a right-handed screw, placed along the line passing through O orthogonal to the vectors \vec{a} and \vec{b} , advances when it is being rotated simultaneously with the vector \vec{a} from \vec{a} towards \vec{b} within the vector subspace $\langle \vec{a}, \vec{b} \rangle$ and the support half line of \vec{a} sweeps the interior of the angle \widehat{AOB} (Screw rule).
3. the *norm* (magnitude or length) of $\vec{a} \times \vec{b}$ is defined by

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \sin(\widehat{\vec{a}, \vec{b}}).$$

Remarks 1.8. 1. The *norm* (magnitude or length) of the vector $\vec{a} \times \vec{b}$ is actually the area of the parallelogram constructed on the vectors \vec{a}, \vec{b} .

2. The vectors $\vec{a}, \vec{b} \in \mathcal{V}$ are linearly dependent (collinear) if and only if $\vec{a} \times \vec{b} = \vec{0}$.

Proposition 1.9. The vector product has the following properties:

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}, \forall \vec{a}, \vec{b} \in \mathcal{V}$;
2. $(\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b}) = \lambda(\vec{a} \times \vec{b}), \forall \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}$;
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}, \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.

If $[\vec{i}, \vec{j}, \vec{k}]$ is an orthonormal basis, observe that $\vec{i} \times \vec{j} \in \{-\vec{k}, \vec{k}\}$. We say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *direct* if $\vec{i} \times \vec{j} = \vec{k}$. If, on the contrary, $\vec{i} \times \vec{j} = -\vec{k}$, we say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *inverse*. Therefore, if $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis, then $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$ and obviously $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

Proposition 1.10. If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}, \quad \vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k},$$

then

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2) \vec{i} + (a_3 b_1 - a_1 b_3) \vec{j} + (a_1 b_2 - a_2 b_1) \vec{k}, \quad (1.5)$$

or, equivalently,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \quad (1.6)$$

One can rewrite formula (1.5) in the form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (1.7)$$

the right hand side determinant being understood in the sense of its cofactor expansion along the first line.

1.2 Problems

1. If two pairs of opposite edges of the tetrahedron $ABCD$ are perpendicular ($AB \perp CD$, $AD \perp BC$), show that
 - (a) The third pair of opposite edges are perpendicular too ($AC \perp BD$).
 - (b) $AB^2 + CD^2 = AC^2 + BD^2 = BC^2 + AD^2$.
 - (c) The heights of the tetrahedron are concurrent.
(Such a tetrahedron is said to be orthocentric)
2. Two triangles ABC și $A'B'C'$ are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices A', B', C' on the sides BC, CA, AB are concurrent. Show that, in this case, the perpendicular lines from the vertices A, B, C on the sides $B'C', C'A', A'B'$ are concurrent too.

3. Show that $\|\vec{a} \times \vec{b}\| \leq \|\vec{a}\| \cdot \|\vec{b}\|$, $\forall \vec{a}, \vec{b} \in \mathcal{V}$.

$$\text{Solution. } \|\vec{a} \times \vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \sin(\widehat{\vec{a}, \vec{b}}) \leq \|\vec{a}\| \cdot \|\vec{b}\|.$$

4. Let $\vec{a}, \vec{b}, \vec{c}$ be noncollinear vectors. Show that the necessary and sufficient condition for the existence of a triangle ABC with the properties $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$, $\vec{AB} = \vec{c}$ is

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$$

From the equalities of the norms deduce the law of sines.

5. Show that the sum of some outer-pointing vectors perpendicular on the faces of a tetrahedron which are proportional to the areas of the faces is the zero vector.
6. Find the orthogonal projection
 - (a) of the point $A(1, 2, 1)$ on the plane $\pi : x + y + 3z + 5 = 0$.
 - (b) of the point $B(5, 0, -2)$ on the straight line $(d) \frac{x-2}{3} = \frac{y-1}{2} = \frac{z-3}{4}$.
7. Compute the distance from the point $A(3, 1, -1)$ to the plane $\pi : 22x + 4y - 20z - 45 = 0$.
8. Find the equations of the bisector planes of the dihedral angles of the planes

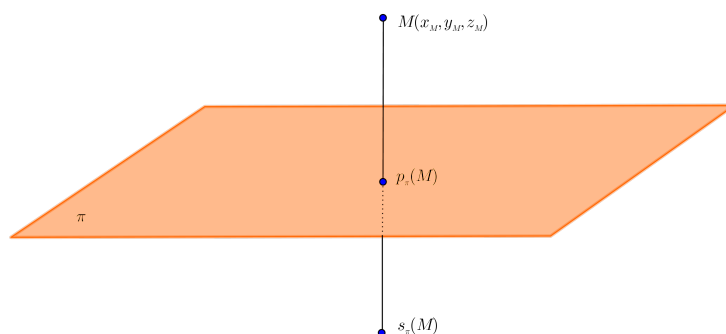
$$(\pi_1) 2x + y - 3z - 5 = 0, (\pi_2) x + 3y + 2z + 1 = 0.$$
9. Find the angle between:
 - (a) the straight lines

$$(d_1) \begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases} \quad (d_2) \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0 \end{cases}$$
 - (b) the planes

$$\pi_1 : x + 3y + 2z + 1 = 0 \text{ and } \pi_2 : 3x + 2y - z = 6.$$
 - (c) the plane xOy and the straight line M_1M_2 , where $M_1(1, 2, 3)$ and $M_2(-2, 1, 4)$.

1.2.1 Appendix: Orthogonal projections and orthogonal symmetries

• **The orthogonal projection on a plane π .** For a given plane $\pi : Ax + By + Cz + D = 0$ and a given point $M(x_M, y_M, z_M)$, we shall determine the coordinates of its orthogonal projection on the plane π , as well as the coordinates of its (orthogonal) symmetric with respect to π . The equation of the plane and the coordinates of M are considered with respect to some cartesian coordinate system $R = (O, \vec{i}, \vec{j}, \vec{k})$. In this respect we consider the orthogonal line on π which passes through M .



Its parametric equations are

$$\begin{cases} x = x_M + At \\ y = y_M + Bt \\ z = z_M + Ct \end{cases}, t \in \mathbb{R}. \quad (1.8)$$

The orthogonal projection $p_\pi(M)$ of M on the plane π is at its intersection point with the orthogonal line (1.8) and the value of $t \in \mathbb{R}$ for which this orthogonal line (1.8) puncture the plane π can be determined by imposing the condition on the point of coordinates $(x_M + At, y_M + Bt, z_M + Ct)$ to verify the equation of the plane, namely $A(x_M + At) + B(y_M + Bt) + C(z_M + Ct) + D = 0$. Thus

$$t = -\frac{Ax_M + By_M + Cz_M + D}{A^2 + B^2 + C^2} = -\frac{F(x_M, y_M, z_M)}{\|\vec{n}_\pi\|^2},$$

where $F(x, y, z) = Ax + By + Cz + D$ și $\vec{n}_\pi = A\vec{i} + B\vec{j} + C\vec{k}$ is the normal vector of the plane π .

• **The orthogonal projection on the plane π .**

The coordinates of the orthogonal projection $p_\pi(M)$ of M on the plane π are

$$\begin{cases} x_M - A \frac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ y_M - B \frac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ z_M - C \frac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2}. \end{cases}$$

Therefore, the position vector of the orthogonal projection $p_\pi(M)$ is

$$\overrightarrow{Op_\pi(M)} = \overrightarrow{OM} - \frac{F(M)}{\|\vec{n}_\pi\|^2} \vec{n}_\pi. \quad (1.9)$$

• **The orthogonal symmetry with respect to the plane π .** In order to find the position vector of the orthogonally symmetric point $s_\pi(M)$ of M w.r.t. π , we use the relation

$$\overrightarrow{Op_\pi(M)} = \frac{1}{2} \left(\overrightarrow{OM} + \overrightarrow{Os_\pi(M)} \right),$$

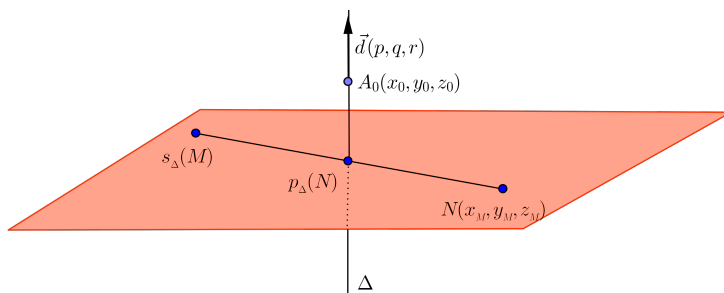
namely

$$\overrightarrow{Os_\pi(M)} = 2 \overrightarrow{Op_\pi(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2 \frac{F(M)}{\|\vec{n}_\pi\|^2} \vec{n}_\pi.$$

• **The orthogonal projection on a line Δ .** For a given line

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a point $N(x_N, y_N, z_N)$, we shall find the coordinates of its orthogonal projection on the line Δ , as well as the coordinates of the orthogonally symmetric point M with respect to Δ . The equations of the line and the coordinates of the point N are considered with respect to an orthonormal coordinate system $R = (O, \vec{i}, \vec{j}, \vec{k})$. In this respect we consider the plane $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$ orthogonal on the line Δ which passes through the point N .



The parametric equations

of the line Δ are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, t \in \mathbb{R}. \quad (1.10)$$

The orthogonal projection of N on the line Δ is at its intersection point and the plane $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$, and the value of $t \in \mathbb{R}$ for which the line Δ puncture the orthogonal plane $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$ can be found by imposing the condition on the point of coordinate $(x_0 + pt, y_0 + qt, z_0 + rt)$ to verify the equation of the plane, namely $p(x_0 + pt - x_N) + q(y_0 + qt - y_N) + r(z_0 + rt - z_N) = 0$. Thus

$$t = -\frac{p(x_0 - x_N) + q(y_0 - y_N) + r(z_0 - z_N)}{p^2 + q^2 + r^2} = -\frac{G(x_0, y_0, z_0)}{\|\vec{d}_\Delta\|^2},$$

where $G(x, y, z) = p(x - x_N) + q(y - y_N) + r(z - z_N)$ and $\vec{d}_\Delta = p\vec{i} + q\vec{j} + r\vec{k}$ is the director vector of the line Δ . The coordinates of the orthogonal projection $p_\Delta(N)$ of N on the line Δ are therefore

$$\begin{cases} x_0 - p \frac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ y_0 - q \frac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ z_0 - r \frac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2}. \end{cases}$$

Thus, the position vector of the orthogonal projection $p_\Delta(N)$ is

$$\overrightarrow{Op_\Delta(N)} = \overrightarrow{OA_0} - \frac{G(A_0)}{\|\vec{d}_\Delta\|^2} \vec{d}_\Delta, \quad (1.11)$$

where $A_0(x_0, y_0, z_0) \in \Delta$.

• **The orthogonal symmetry with respect to a line Δ .** In order to find the position vector of the orthogonally symmetric point $s_\Delta(N)$ of N with respect to the line Δ we use the relation

$$\overrightarrow{Op_\Delta(N)} = \frac{1}{2} \left(\overrightarrow{ON} + \overrightarrow{Os_\Delta(N)} \right)$$

i.e.

$$\overrightarrow{Os_\Delta(N)} = 2 \overrightarrow{Op_\Delta(N)} - \overrightarrow{ON} = 2 \overrightarrow{OA_0} - 2 \frac{G(A_0)}{\|\vec{d}_\Delta\|^2} \vec{d}_\Delta - \overrightarrow{ON}.$$

Remark 1.11. 1. The distance from the point $M(x_M, y_M, z_M)$ to the plane $\pi : Ax + By + Cz + D = 0$ can be equally computed by means of (1.9). Indeed,

$$\begin{aligned} \delta(M, \pi) &= \|\overrightarrow{Mp_\pi(M)}\| = \|\overrightarrow{Op_\pi(M)} - \overrightarrow{OM}\| \\ &= \left| -\frac{F(M)}{\|\vec{n}_\pi\|^2} \right| \cdot \|\vec{n}_\pi\| = \frac{|F(M)|}{\|\vec{n}_\pi\|}. \end{aligned}$$

2. The distance from the point $N(x_N, y_N, z_N)$ to the straight line $\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$

can be computed by means of (1.11). Indeed,

$$\begin{aligned}\delta(M, \Delta) &= \|\overrightarrow{Np_{\Delta}(N)}\| = \|\overrightarrow{NO} + \overrightarrow{Op_{\Delta}(N)}\| \\ &= \left\| \overrightarrow{NA_0} - \frac{G(A_0)}{\|\overrightarrow{d_{\Delta}}\|^2} \overrightarrow{d_{\Delta}} \right\| = \left\| \overrightarrow{NA_0} - \frac{\overrightarrow{d_{\Delta}} \cdot \overrightarrow{NA_0}}{\|\overrightarrow{d_{\Delta}}\|^2} \overrightarrow{d_{\Delta}} \right\|.\end{aligned}\quad (1.12)$$

Proposition 1.12. Taking into account the formula (1.12) for the distance $\delta(M, \Delta)$ from the point $N(x_N, y_N, z_N)$ to the straight line $\Delta : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$ as well as Proposition ?? we deduce that

$$\begin{aligned}\delta(M, \Delta) &= \left\| \overrightarrow{NA_0} - \frac{\overrightarrow{d_{\Delta}} \cdot \overrightarrow{NA_0}}{\|\overrightarrow{d_{\Delta}}\|^2} \overrightarrow{d_{\Delta}} \right\| = \frac{\left\| \left(\overrightarrow{d_{\Delta}} \cdot \overrightarrow{d_{\Delta}} \right) \overrightarrow{NA_0} - \left(\overrightarrow{d_{\Delta}} \cdot \overrightarrow{NA_0} \right) \overrightarrow{d_{\Delta}} \right\|}{\|\overrightarrow{d_{\Delta}}\|^2} \\ &= \frac{\|\overrightarrow{d_{\Delta}} \times (\overrightarrow{NA_0} \times \overrightarrow{d_{\Delta}})\|}{\|\overrightarrow{d_{\Delta}}\|^2} = \frac{\|\overrightarrow{NA_0} \times \overrightarrow{d_{\Delta}}\|}{\|\overrightarrow{d_{\Delta}}\|}.\end{aligned}$$

References

- [1] Andrica, D., Țopan, L., Analytic geometry, Cluj University Press, 2004.
- [2] Galbură Gh., Radó, F., Geometrie, Editura didactică și pedagogică-București, 1979.
- [3] Pinte, C. Geometrie. Elemente de geometrie analitică. Elemente de geometrie diferențială a curbelor și suprafețelor, Presa Universitară Clujeană, 2001.
- [4] Radó, F., Orban, B., Groze, V., Vasiu, A., Culegere de Probleme de Geometrie, Lit. Univ. "Babeş-Bolyai", Cluj-Napoca, 1979.