Geometry
Problem booklet

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1 Week 3: Cartesian equations of lines and planes

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background

1.1.1 Cartesian and affine reference systems

A basis of the direction $\overrightarrow{\pi}$ of the plane π, or for the vector space $\mathcal{V}$ is an ordered basis $[\overrightarrow{e}, \overrightarrow{f}]$ of π, or an ordered basis $[\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ a of $\mathcal{V}$.

If $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is a basis of $\mathcal{V}$ and $\overrightarrow{x} \in \mathcal{V}$, recall that the column vector of $\overrightarrow{x}$ with respect to b is being denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

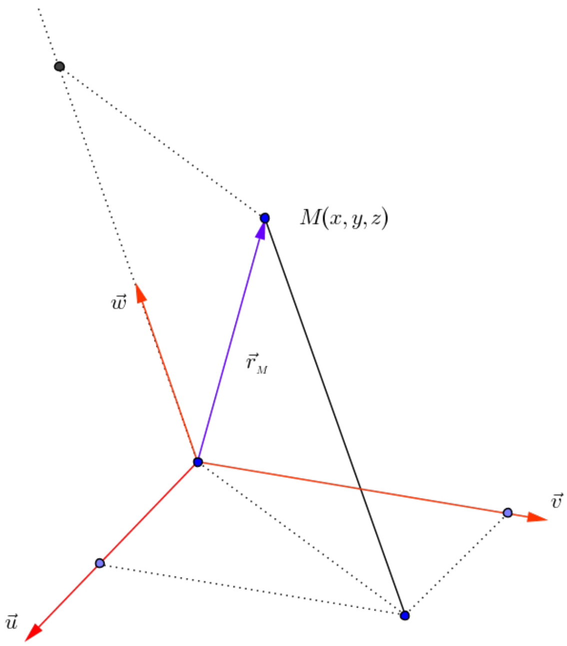
whenever $\overrightarrow{x} = x_1 \overrightarrow{u} + x_2 \overrightarrow{v} + x_3 \overrightarrow{w}$.

Definition 1.1. *A* cartesian reference system *of the space $\mathcal{P}$, is a system $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ where O is a point from $\mathcal{P}$ called the* origin *of the reference system and $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is a basis of the vector space $\mathcal{V}$.*

Denote by E_1, E_2, E_3 the points for which $\overrightarrow{u} = \overrightarrow{OE_1}$, $\overrightarrow{v} = \overrightarrow{OE_2}$, $\overrightarrow{w} = \overrightarrow{OE_3}$.

Definition 1.2. *The system of points (O, E_1, E_2, E_3) is called* the affine reference system associated to the cartesian reference system $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$.

The straight lines OE_i, $i \in \{1, 2, 3\}$, oriented from O to E_i are called *the coordinate axes*. The coordinates x, y, z of the position vector $\overrightarrow{r}_M = \overrightarrow{OM}$ with respect to the basis $[\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y, z)$.



Also, for the column matrix of coordinates of the vector $\vec{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\vec{r}_M = x\,\vec{u} + y\,\vec{v} + z\,\vec{w}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Remark 1.3. *If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are two points, then*

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= x_B\,\vec{u} + y_B\,\vec{v} + z_B\,\vec{w} - (x_A\,\vec{u} + y_A\,\vec{v} + z_A\,\vec{w}) \\ &= (x_B - x_A)\,\vec{u} + (y_B - y_A)\,\vec{v} + (z_B - z_A)\,\vec{w}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

1.1.2 The cartesian equations of the straight lines

Let Δ be a straight line passing through the point $A_0(x_0, y_0, z_0)$ which is parallel to the vector $\vec{d}\,(p, q, r)$. The parametric equations of Δ are:

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases}, \lambda \in \mathbb{R} \tag{1.1}$$

and they are equivalent with the following canonical equations of Δ

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r} \tag{1.2}$$

If $r = 0$, for instance, the canonical equations of the straight line Δ are

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} \wedge z = z_0.$$

If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are different pointsof the straight line Δ, then $\overrightarrow{AB}\,(x_B - x_A, y_B - y_A, z_B - z_A)$ is a director vector of Δ, its canonical equations having, in this case, the form

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}. \tag{1.3}$$

1.1.3 Te cartesian equations of the planes

Let $A_0(x_0, y_0, z_0) \in \mathcal{P}$ and $\vec{d}_1\,(p_1, q_1, r_1)$, $\vec{d}_2\,(p_2, q_2, r_2) \in \mathcal{V}$ be linearly independent vectors, that is

$$\text{rang} \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix} = 2.$$

The parametric equations of the plane π passing through A_0 which is parallel to the vectors $\vec{d}_1\,(p_1, q_1, r_1)$, $\vec{d}_2\,(p_2, q_2, r_2)$ are

$$\begin{cases} x = x_0 + \lambda_1 p_1 + \lambda_2 p_2 \\ y = y_0 + \lambda_1 q_1 + \lambda_2 q_2 \\ z = z_0 + \lambda_1 r_1 + \lambda_2 r_2 \end{cases}, \lambda_1, \lambda_2 \in \mathbb{R}. \tag{1.4}$$

The cartesian equation of the plane π is

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ p_1 & q_1 & z_1 \\ p_2 & q_2 & z_2 \end{vmatrix} = 0. \tag{1.5}$$

If $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ are noncollinear points, then the equation of the plane (ABC) is

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0. \tag{1.6}$$

Example 1.4. *If $A(a,0,0)$, $B(0,b,0)$ and $C(0,0,c)$, $(abc \neq 0)$ then for the equation of the plane (ABC) we have successively*

$$\begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = 0 \iff \begin{vmatrix} x & y & z-c & 1 \\ a & 0 & -c & 1 \\ 0 & b & -c & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0 \iff \begin{vmatrix} x & y & z-c \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = 0$$

$$\iff ab(z-c) + bcx + acy = 0 \iff bcx + acy + abz - abc = 0$$

$$\iff \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{1.7}$$

The equation (1.7) of the plane (ABC) is said to be in *intercept form* and the x, y, z-intercepts of the plane (ABC) are a, b, c respectively.

Thus, four points $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$, $C(x_C, y_C, z_C)$ and $D(x_D, y_D, z_D)$ are coplanar if and only if

$$\begin{vmatrix} x_D & y_D & z_D & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0 \text{ i.e. } \begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \tag{1.8}$$

One can put the equation (1.5) in the form

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0 \tag{1.9}$$

or

$$Ax + By + Cz + D = 0, \tag{1.10}$$

where the coefficients A, B, C satisfy the relation $A^2 + B^2 + C^2 > 0$. It is also easy to show that every equation of the form (1.10) represents the equation of a plane. Indeed, if $A \neq 0$, then the equation (1.10) is equivalent to

$$\begin{vmatrix} x + \frac{D}{A} & y & z \\ B & -A & 0 \\ C & 0 & -A \end{vmatrix} = 0.$$

We observe that one can put the equation (1.9) in the form

$$AX + BY + CZ = 0 \tag{1.11}$$

where $X = x - x_0$, $Y = y - y_0$, $Z = z - z_0$ are the coordinates of the vector $\overrightarrow{A_0M}$.

1.2 Problems

1. Write the equation of the line which passes through $A(1,-2,6)$ and is parallel to

   (a) The x-axis;

   (b) The line $(d_1)\ \dfrac{x-1}{2}=\dfrac{y+5}{-3}=\dfrac{z-1}{4}$.

   (c) The vector $\overrightarrow{v}\ (1,0,2)$.

2. Write the equation of the plane which contains the line

$$(d_1)\ \frac{x-3}{2}=\frac{y+4}{1}=\frac{z-2}{-3}$$

   and is parallel to the line

$$(d_2)\ \frac{x+5}{2}=\frac{y-2}{2}=\frac{z-1}{2}.$$

3. Consider the points $A(\alpha,0,0)$, $B(0,\beta,0)$ and $C(0,0,\gamma)$ such that

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{1}{a}\ \text{ where } a \text{ is a constatnt.}$$

   Show that the plane (ABC) passes through a fixed point.

4. Write the equation of the plane which passes through $M_0(1,-2,3)$ and is parallel to the vectors $\overrightarrow{v}_1\ (1,-1,0)$ and $\overrightarrow{v}_2\ (-3,2,4)$.

5. Write the equation of the plane which passes through $M_0(1,-2,3)$ and cuts the positive coordinate axes through congruent segments.

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