Geometry
Problem booklet

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Contents

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1 Week 4: Analytic conditions of parallelism. Projections and symmetries

1.1 Brief theoretical background. Analytic conditions of parallelism

The equation $AX + BY + CZ = 0$ is a necessary and sufficient condition for the vector $\overrightarrow{A_0M}\ (X, Y, Z)$ to be contained within the direction of the plane

$$\pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Thus the *equation of the director subspace* $\overrightarrow{\pi} = \{\overrightarrow{A_0M} \mid M \in \pi\}$ is $AX + BY + CZ = 0$.

Proposition 1.1. *The straight line*

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

is parallel to the plane $\pi : Ax + By + Cz + D = 0$ *iff*

$$Ap + Bq + Cr = 0 \tag{1.1}$$

Proposition 1.2. *Consider the planes*

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0,\ (\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0.$$

Then $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) \in \{1, 2\}$ *and the following statemenets are equivalent*

1. $\pi_1 \| \pi_2$.
2. $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) = 2$, *i.e.* $\overrightarrow{\pi}_1 = \overrightarrow{\pi}_2$.
3. *rang* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1$.
4. *The vectors* $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly dependent.*

Corollary 1.3. *Consider the planes*

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0,\ (\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0.$$

The following statements are equivalent

1. $\pi_1 \nparallel \pi_2$.
2. $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) = 1$.
3. *rang* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$.
4. *The vectors* $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly independent.*

By using the characterization of parallelism between a line and a plane (Proposition 1.1), we shall find a necessary and sufficient condition for a vector to be contained within the direction of a straight line which is given as the intersection of two planes.

Consider the planes $(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0$, $(\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0$ such that

$$\text{rang}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2,$$

alongside their intersectio9n straight line $\Delta = \pi_1 \cap \pi_2$ of equations

$$(\Delta)\ \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_1 = 0. \end{cases}$$

Thus, $\overrightarrow{\Delta} = \overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2$ and therefore, by means of some previous Proposition, it follows that the equations of $\overrightarrow{\Delta}$ are

$$(\overrightarrow{\Delta})\ \begin{cases} A_1X + B_1Y + C_1Z = 0 \\ A_2X + B_2Y + C_2Z = 0. \end{cases} \tag{1.2}$$

By solving the system (1.2) one can therefore deduce that $\overrightarrow{d}\ (p,q,r) \in \overrightarrow{\Delta} \Leftrightarrow\ \exists \lambda \in \mathbb{R}$ such that

$$(p,q,r) = \lambda\left(\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \right). \tag{1.3}$$

The relation is usually (1.3) written in the form

$$\frac{p}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{q}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}} = \frac{r}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \tag{1.4}$$

Let us mention that the chosen values for (p,q,r) are usually precisely

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \text{ şi } \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}.$$

1.2 Projections and symmetries. Pencils of planes

1.2.1 The intersection point of a straight line and a plane

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

The parametric equations of d are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases},\ t \in \mathbb{R}. \tag{1.5}$$

The value of $t \in \mathbb{R}$ for which this line (1.5) punctures the plane π can be determined by imposing the condition on the point of coordinates

$$(x_0 + pt, y_0 + qt, z_0 + rt)$$

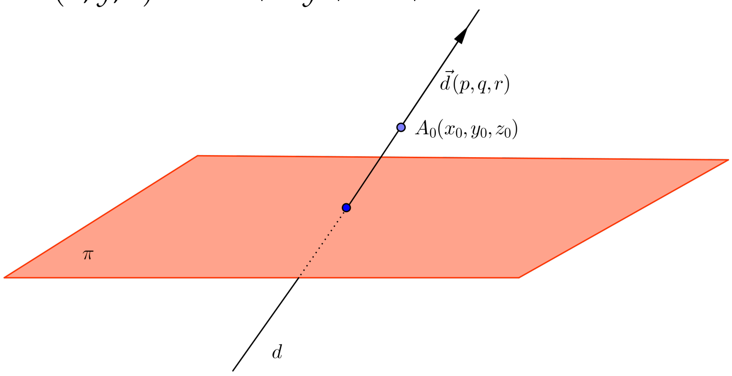
to verify the equation of the plane, namely

$$A(x_0 + pt) + B(y_0 + qt) + C(z_0 + Ct) + D = 0.$$

Thus

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr} = -\frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr},$$

where $F(x, y, z) = Ax + By + Cz + D$.



The coordinates of the intersection point $d \cap \pi$ are

$$\begin{cases} x_0 - p\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ y_0 - q\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ z_0 - r\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr}. \end{cases} \tag{1.6}$$

1.2.2 The projection on a plane parallel to a given line

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

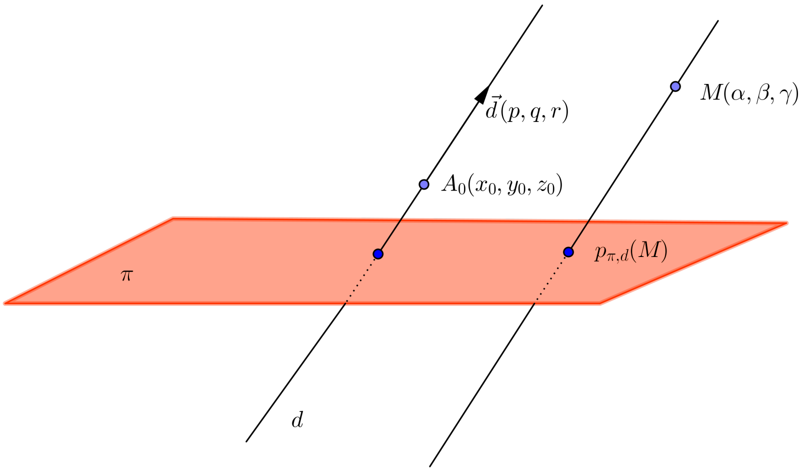
and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{\pi,d} : \mathcal{P} \longrightarrow \pi$ of $\mathcal{P}$ on π parallel to d, whose value $p_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the intersection point between π and the line through M which is parallel to d. Due to relations (1.6), the coordinates of $p_{\pi,d}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_M - p\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ y_M - q\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ z_M - r\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr}, \end{cases} \tag{1.7}$$

where $F(x, y, z) = Ax + By + Cz + D$.



Consequently, the position vector of $p_{\pi,d}(M)$ is

$$\overrightarrow{Op_{\pi,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{Ap + Bq + Cr} \vec{d}. \quad (1.8)$$

1.2.3 The symmetry with respect to a plane parallel to a line

We call the function $s_{\pi,d} : \mathcal{P} \longrightarrow \mathcal{P}$, whose value $s_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $p_{\pi,d}(M)$ *the symmetry of* $\mathcal{P}$ *with respect to* π *parallel to* d. The direction of d is equally called the *direction* of the symmetry and π is called the *axis* of the symmetry. For the position vector of $s_{\pi,d}(M)$ we have

$$\overrightarrow{Op_{\pi,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\pi,d}(M)}}{2}, \text{ i.e.} \quad (1.9)$$

$$\overrightarrow{Os_{\pi,d}(M)} = 2\,\overrightarrow{Op_{\pi,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{Ap + Bq + Cr} \vec{d}. \quad (1.10)$$

1.2.4 The projection on a straight line parallel to a given plane

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

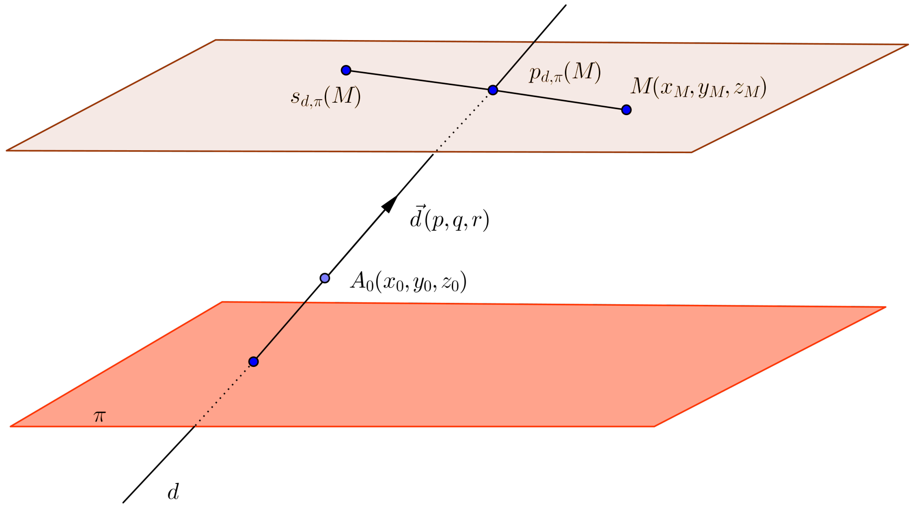
$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{d,\pi} : \mathcal{P} \longrightarrow d$ of $\mathcal{P}$ on d, whose value $p_{d,\pi}(M)$ at $M \in \mathcal{P}$ is the intersection point between d and the plane through M which is parallel to π. Due to relations (1.6), the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_0 + p\dfrac{G(x_M, y_M, z_M)}{Ap+Bq+Cr} = x_0 + p\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ y_0 + q\dfrac{G(x_M, y_M, z_M)}{Ap+Bq+Cr} = y_0 + q\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ z_0 + r\dfrac{G(x_M, y_M, z_M)}{Ap+Bq+Cr} = z_0 + r\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr}, \end{cases} \tag{1.11}$$

where $G(x,y,z) = A(x-x_0)+B(y-y_0)+C(z-z_0) = F(M)-F(A_0)$ and $F(x,y,z) = Ax+By+Cz+D$ The position vector of $p_{d,\pi}(M)$ is

$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} + \frac{F(M)-F(A_0)}{Ap+Bq+Cr}\vec{d}, \text{ where } A_0(x_0,y_0,z_0). \tag{1.12}$$



1.2.5 The symmetry with respect to a line parallel to a plane

We call the function $s_{d,\pi} : \mathcal{P} \longrightarrow \mathcal{P}$, whose value $s_{d,\pi}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $p_{d,\pi}(M)$, *the symmetry of $\mathcal{P}$ with respect to d parallel to π*. The direction of π is equally called the *direction* of the symmetry and d is called the *axis* of the symmetry. For the position vector of $s_{d,\pi}(M)$ we have

$$\overrightarrow{Op_{d,\pi}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{d,\pi}(M)}}{2}, \text{ i.e.} \tag{1.13}$$

$$\begin{aligned} \overrightarrow{Os_{d,\pi}(M)} &= 2\,\overrightarrow{Op_{d,\pi}(M)} - \overrightarrow{OM} \\ &= 2\,\overrightarrow{OA_0} - \overrightarrow{OM} + 2\frac{F(M)-F(A_0)}{Ap+Bq+Cr}\vec{d}\,. \end{aligned} \tag{1.14}$$

1.3 Pencils of planes

Definition 1.4. *The collection of all planes containing a given straight line*

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

is called the pencil of planes *through* Δ.

Proposition 1.5. *The plane π belongs to the pencil of planes through the straight line Δ if and only if there exists $\lambda, \mu \in \mathbb{R}$, $\lambda^2 + \mu^2 > 0$ such that the equation of the plane π is*

$$\lambda(A_1x + B_1y + C_1z + D_1) + \mu(A_2x + B_2y + C_2z + D_2) = 0. \tag{1.15}$$

Remark 1.6. *The family of planes*

$$A_1x + B_1y + C_1z + D_1 + \lambda(A_2x + B_2y + C_2z + D_2) = 0,$$

where λ covers the whole real line $\mathbb{R}$, is the so called reduced pencil of planes *through Δ and it consists in all planes through Δ except the plane of equation $A_2x + B_2y + C_2z + D_2 = 0$.*

Remark 1.7. *If*

$$(\Delta_1) \left\{ \begin{array}{l} A_1x + B_1y + C_1z + D_1 = 0 \\ Ax + By + Cz + D = 0 \end{array} \right. \quad (\Delta_2) \left\{ \begin{array}{l} A_2x + B_2y + C_2z + D_2 = 0 \\ Ax + By + Cz + D = 0 \end{array} \right.$$

are two concurrent lines at some point P, then the family of lines

$$(\Delta_{\lambda,\mu}) \left\{ \begin{array}{l} A_1x + B_1y + C_1z + D_1 + \lambda(Ax + By + Cz + D) = 0 \\ A_2x + B_2y + C_2z + D_2 + \mu(Ax + By + Cz + D) = 0 \end{array} \right.$$

consists in all lines through P except those lines which lie in the plane

$$\pi : Ax + By + Cz + D = 0.$$

1.4 Problems

1. Write the equation of the line which passes through tht point $M(1, 0, 7)$, is parallel to the plane (π) $3x - y + 2z - 15 = 0$ and intersects the line

$$(d)\ \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

2. Write the equations of the projection of the line

$$(d) \left\{ \begin{array}{rrrrrrrl} 2x & - & y & + & z & - & 1 & = 0 \\ x & + & y & - & z & + & 1 & = 0 \end{array} \right.$$

   on the plane $\pi : x + 2y - z = 0$ paralel to the direction $\overrightarrow{u}\ (1, 1, -2)$. Write the equations of the symmetry of the line d with respect to the plane π paralel to the direction $\overrightarrow{u}\ (1, 1, -2)$.

3. Write the equation of the plane determined by the line

$$(d) \left\{ \begin{array}{rrrrrrrl} x & - & 2y & + & 3z & & & = 0 \\ 2x & & & + & z & - & 3 & = 0 \end{array} \right.$$

   and the point $A(-1, 2, 6)$.

4. Find the equation of the plane containing the origin and the line
$$(d)\ \frac{x-2}{3}=\frac{y-4}{1}=\frac{z-1}{1}.$$

5. Show that two different parallel lines are either projected onto parallel lines or on two points by a projection $p_{\pi,d}$, where
$$(\pi)\ Ax+By+Cz+D=0,\quad (d)\ \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$$
and $\pi \nparallel d$.

6. Show that two parallel lines are mapped onto parallel lines by a symmetry $s_{\pi,d}$, where
$$(\pi)\ Ax+By+Cz+D=0,\quad (d)\ \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$$
and $\pi \nparallel d$.

7. Consider a plane $(\pi)\ Ax+By+Cz+D=0$ and a line
$$(d)\ \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$$
with respect a coordinate Cartesian reference system $R=(O,b)$, where $b=[\mathbf{u},\mathbf{v},\mathbf{w}]$ is a basis of $\mathcal{V}$. If $\pi \nparallel d$, show that:

   (a) $\overrightarrow{p_{\pi,d}(M)p_{\pi,d}(N)}=p(\overrightarrow{MN})$, $\forall M,N\in\mathcal{P}$, where $p:\mathcal{V}\longrightarrow\mathcal{V}$ is the linear transformation with the matrix representation
$$[p]_b=\frac{1}{Ap+Bq+Cr}\begin{pmatrix} Bq+Cr & -Bp & -Cp\\ -Aq & Ap+Cr & -Cq\\ -Ar & -Br & Ap+Bq\end{pmatrix}.$$

   (b) $\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)}=s(\overrightarrow{MN})$, $\forall M,N\in\mathcal{P}$, where $s:\mathcal{V}\longrightarrow\mathcal{V}$ is the linear transformation with the matrix representation
$$[s]_b=\frac{1}{Ap+Bq+Cr}\begin{pmatrix} -Ap+Bq+Cr & -2Bp & -2Cp\\ -2Aq & Ap-Bq+Cr & -2Cq\\ -2Ar & -2Br & Ap+Bq-Cr\end{pmatrix}.$$

8. Consider a plane $(\pi)\ Ax+By+Cz+D=0$ and a line
$$(d)\ \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}.$$
If $\pi \nparallel d$, show that:

   (a) $p_{\pi,d}\circ p_{\pi,d}=p_{\pi,d}$;

   (b) $s_{\pi,d}\circ s_{\pi,d}=id_{\mathcal{P}}$.

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