Seminar 11

1. In the real vector space $\mathbb{R}^3$ consider the bases $B = (v_1, v_2, v_3) = ((1,0,1),(0,1,1),(1,1,1))$ and $B' = (v'_1, v'_2, v'_3) = ((1,1,0),(-1,0,0),(0,0,1))$. Determine the matrices of change of basis $T_{BB'}$ and $T_{B'B}$, and compute the coordinates of the vector $u = (2,0,-1)$ in both bases.

2. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v'_1, v'_2) = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

3. In the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ consider the bases $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)\,(a \in \mathbb{R})$ and $B' = (1, X-b, (X-b)^2)\,(b \in \mathbb{R})$. Determine the matrices of change of bases T_{EB}, T_{BE} and $T_{BB'}$.

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^2)$ be defined by $f(x,y) = (3x+3y, 2x+4y)$.
(i) Determine the eigenvalues and the eigenvectors of f.
(ii) Write a basis B of $\mathbb{R}^2$ consisting of eigenvectors of f and $[f]_B$.

Compute the eigenvalues and the eigenvectors of the (endomorphisms having) matrices:

$$\textbf{5.}\ \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}. \quad \textbf{6.}\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$\textbf{7.}\ \begin{pmatrix} x & 0 & y \\ 0 & x & 0 \\ y & 0 & x \end{pmatrix} (x, y \in \mathbb{R}^*). \quad \textbf{8.}\ \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} (x \in \mathbb{R}).$$

9. Let $A \in M_2(\mathbb{R})$ and let λ_1, λ_2 be the eigenvalues of A in $\mathbb{C}$. Prove that:
(i) $\lambda_1 + \lambda_2 = Tr\,(A)$ and $\lambda_1 \cdot \lambda_2 = det\,(A)$, where $Tr\,(A)$ denotes the trace of A, that is, the sum of the elements of the principal diagonal. Generalization.
(ii) A has all the eigenvalues in $\mathbb{R} \iff (Tr\,(A))^2 - 4 \cdot det\,(A) \geq 0$.
(iii) Show that A is a root of its characteristic polynomial.

10. Let $A \in M_2(\mathbb{R})$ be such that $det(A + iI_2) = 0$. Show that $det(A + 2I_2) = 5$.