– Seminar 5 –

– gr 911 – 912 – 913 – 914 –

A. Type and stability of linear planar systems.

Let $X' = A \cdot X$, $A \in \mathcal{M}_2(\mathbb{R})$, $\det A \neq 0$.

Denote: $\lambda_1, \lambda_2 \in \mathbb{C}$ the eigenvalues of A.

- $\det(A - \lambda I_2) = 0 \longrightarrow$ eigenvalues.

Remark:

- $\det A \neq 0 \iff 0 \in \mathbb{R}^2$ the only eq. point.
- $\det A \neq 0 \iff \lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

Definition:

We say that the equilibrium $(0,0) \in \mathbb{R}^2$ is:

- NODE : when $\lambda_1, \lambda_2 \in \mathbb{R}$ and, either

  $\lambda_1 \leq \lambda_2 < 0$ or $0 < \lambda_1 \leq \lambda_2$
- SADDLE : when $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < 0 < \lambda_2$
- FOCUS : when $\lambda_{1,2} = \alpha \pm i\beta$, $\alpha \neq 0$, $\beta \neq 0$.
- CENTER : when $\lambda_{1,2} = \pm i\beta$, $\beta \neq 0$ /.

Theorem 1:

- If $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0 \Rightarrow$ eq = global attractor
- If $Re(\lambda_1) > 0$ and $Re(\lambda_2) > 0 \Rightarrow$ eq = global repeller

Theorem 2:

- Any CENTER is stable.
- Any SADDLE is unstable.

ex 1: Decide the type and the stability of the linear systems:

a) $\begin{cases} x' = -y \\ y' = 5x \end{cases}$ b) $\begin{cases} x' = -x \\ y' = 5y \end{cases}$ c) $\begin{cases} x' = -3x \\ y' = -2y \end{cases}$ d) $\begin{cases} x' = x-y \\ y' = x+y \end{cases}$

e) $\begin{cases} x' = 4x-5y \\ y' = x-2y. \end{cases}$

Also, decide whether the system has a global first integer. If there is a possibility to have, try to find it. Represent the phase portrait.

---

a) $\begin{cases} x' = -y \\ y' = 5x \end{cases}$ Here: $X' = A\cdot X$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}$

the matrix of the system.

- We find the eigenvalues of A:

$$\det(A - \lambda I_2) = 0 \iff \det\begin{pmatrix} -\lambda & -1 \\ 5 & -\lambda \end{pmatrix} = 0.$$

$$\iff \lambda^2 + 5 = 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{5}$$

The equilibrium is a CENTER $\rightarrow$ stable.

- We try to find the first integral:

$$\frac{dy}{dx} = \frac{5x}{-y} \xrightarrow[\text{variables}]{\text{separate}} -y\,dy = 5x\,dx$$

$$\xrightarrow{\text{integrate}} -\int y\,dy = 5\int x\,dx \Rightarrow -\frac{y^2}{2} = 5\frac{x^2}{2} + c,\ c\in\mathbb{R}$$

$$\Rightarrow \frac{-y^2 - 5x^2}{2} = c \Rightarrow \boxed{5x^2 + y^2 = c_1}\ c_1 \in \mathbb{R}$$

Define : $H(x,y) = 5x^2 + y^2 \quad , \forall (x,y) \in \mathbb{R}^2$

$H : \mathbb{R}^2 \to \mathbb{R}$

Note $H \in C^1(\mathbb{R})$ is not locally constant.

$\to$ It is a good candidate for a global first integr.

Let's check this :

$$\frac{\partial H}{\partial x} \cdot f_1 + \frac{\partial H}{\partial y} \cdot f_2 = 0 \quad \text{in } \mathbb{R}^2, \text{ where } \begin{cases} x' = -y = f_1 \\ y' = 5x = f_2 \end{cases}$$

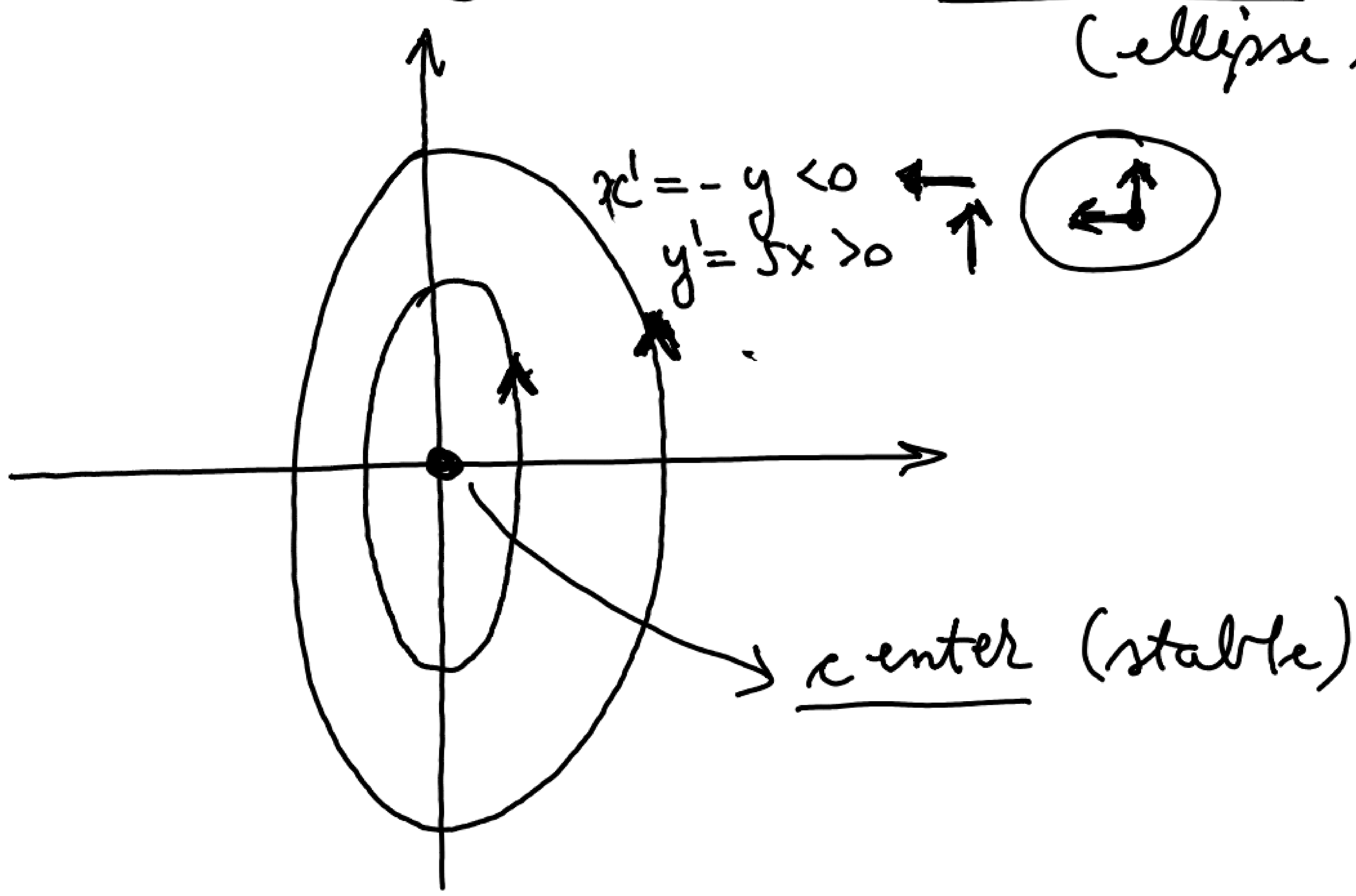
$\Leftrightarrow 10x \cdot (-y) + 2y \cdot (5x) = 0.$ TRUE

$\Rightarrow$ We can conclude that :

$H : \mathbb{R}^2 \to \mathbb{R}, \; H(x,y) = 5x^2 + y^2$ is a global first int. of the system.

- We want to draw the phase portrait.

$\to$ We know that the orbits lie on the level curves of the first integral. $\to$ The level curves of H are : $H(x,y) = c \Leftrightarrow 5x^2 + y^2 = c, \; c \in \mathbb{R}$ (ellipse)



b) $\begin{cases} x' = -x \\ y' = 5y \end{cases}$ $A = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$ = diagonal matrix.

$\Downarrow$

$\lambda_1 = -1$, $\lambda_2 = 5$.

Thus $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < 0 < \lambda_2$.

$\Rightarrow$ The equilibrium is SADDLE $\rightarrow$ unstable.

Theorem: Any linear system, either center or saddle has a global first integral.

-> Here, we try to find the global first integr.

$\begin{cases} x' = -x \\ y' = 5y \end{cases} \Leftrightarrow \begin{cases} \frac{dx}{dt} = -x \\ \frac{dy}{dt} = 5y \end{cases} \Rightarrow \frac{dy}{dx} = \frac{5y}{-x}$ $(=dt)$

$\Rightarrow \frac{dy}{y} = -5\frac{dx}{x} \Rightarrow \int \frac{dy}{y} = -5\int \frac{dx}{x}$

$\Rightarrow \ln|y| = -5\ln|x| + \ln c \Rightarrow \ln|y| + 5\ln|x| = \ln c$

$\Rightarrow y \cdot x^5 = c$, $c \in \mathbb{R}$

Take $H(x,y) = y \cdot x^5$, $\forall (x,y) \in \mathbb{R}^2$

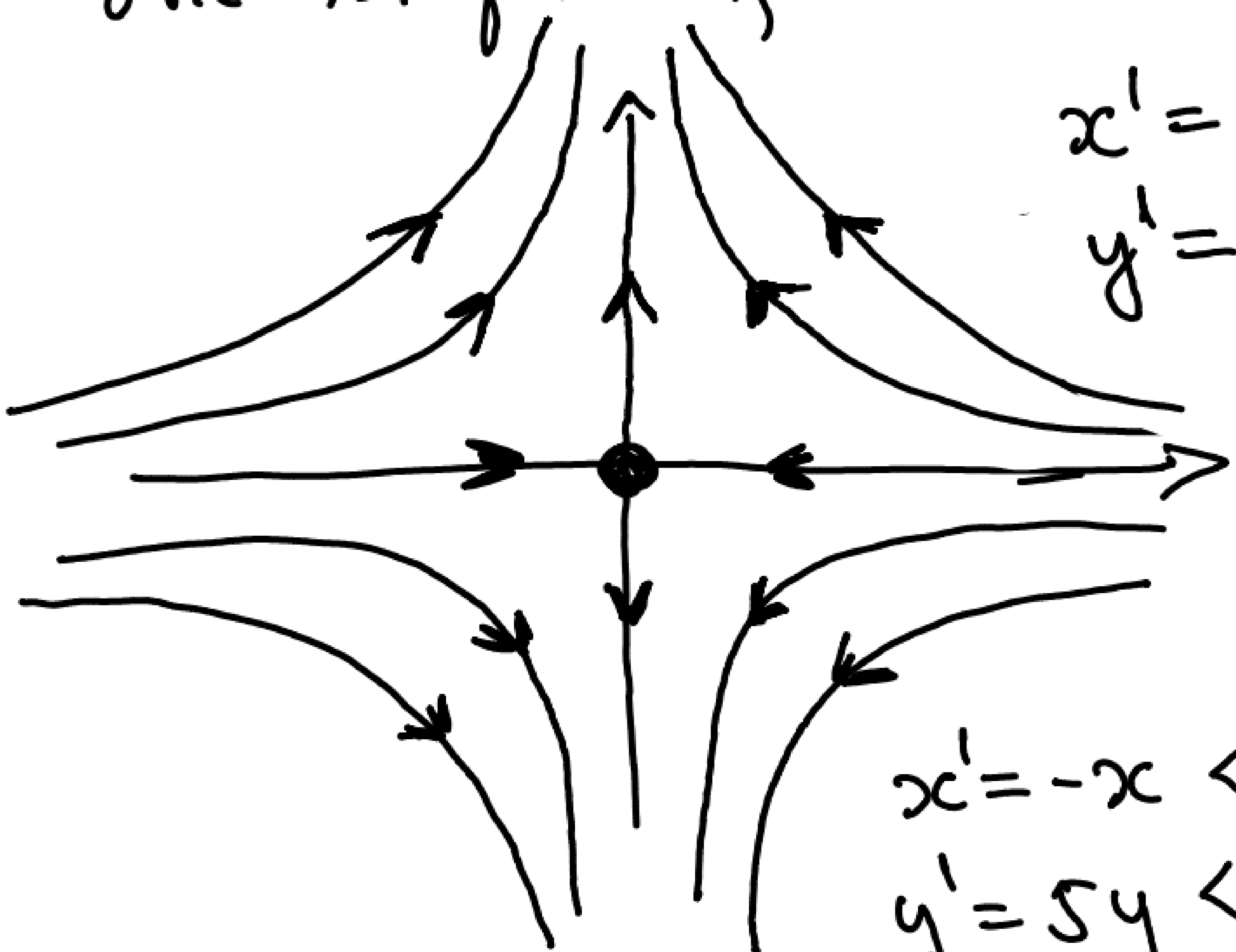
$H: \mathbb{R}^2 \to \mathbb{R}$

Check: $\frac{\partial H}{\partial x} \cdot f_1 + \frac{\partial H}{\partial y} \cdot f_2 = 0$ in $\mathbb{R}^2$

$5x^4 y \cdot (-x) + x^5 \cdot (5y) = 0$ ✓ TRUE

$\Rightarrow H(x,y) = x^5 \cdot y$ is a global first integral of the system, that is saddle, thus unstable.

• The shape of the orbits: $y = c \cdot \frac{1}{x^5}$, $c \in \mathbb{R}$



$x' = -x < 0$

$y' = 5y > 0$

$x' = -x < 0$

$y' = 5y < 0$

c) $\begin{cases} x' = -3x \\ y' = -2y \end{cases}$ $\quad A = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$ - diagonal matrix

$\Downarrow$

$\lambda_1 = -3, \ \lambda_2 = -2$

$\Rightarrow \lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1, \lambda_2 < 0 \Rightarrow$ <u>NODE</u> (global attractor)

If - global attractor $\Rightarrow$ No global first integral!

We try to find a first integral in a region U such that $O_2 \notin U$.

$$\Rightarrow \frac{dy}{dx} = \frac{-2y}{-3x} \Rightarrow 3\frac{dy}{y} = 2 \cdot \frac{dx}{x}$$

$$\Rightarrow \ln|y^3| = \ln|x^2| + \ln C \Rightarrow \ln\left|\frac{y^3}{x^2}\right| = \ln C, \ c \in \mathbb{R}$$

$$\Rightarrow \frac{y^3}{x^2} = c \quad , c \in \mathbb{R}$$

- We take $H(x,y) = \frac{y^3}{x^2} \quad , x \neq 0.$

$U_1 := \{ (x,y) \in \mathbb{R}^2 , x > 0 \}$

$U_2 := \{ (x,y) \in \mathbb{R}^2 , x < 0 \}$

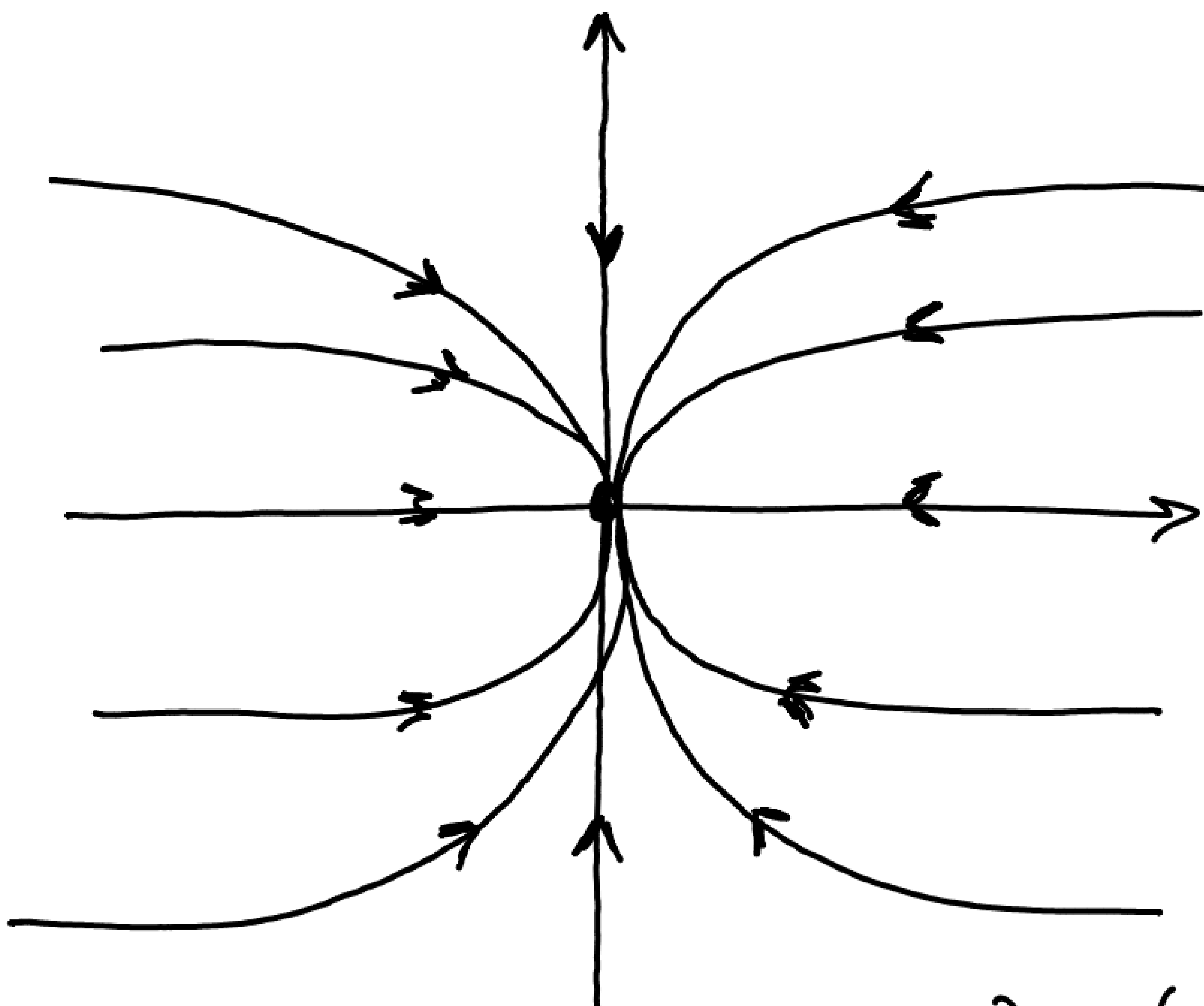
- Check that H is a first integral in U_1 and

$$\frac{\partial H}{\partial x} \cdot f_1 + \frac{\partial H}{\partial y} \cdot f_2 = 0. \quad \text{in } U_1 , U_2$$

$$\Leftrightarrow y^3 \cdot (-2x^{-3}) \cdot (-3x) + x^{-2} \cdot 3y^2 \cdot (-2y) = 0. \quad \underline{\text{TRUE}}.$$

- The shape of the orbits:

$$\frac{y^3}{x^2} = c \quad , c \in \mathbb{R} \quad \Rightarrow \quad y = c \cdot x^{\frac{2}{3}} , c \in \mathbb{R}$$



$x' = -3x < 0$

$y' = -2y < 0$

$x' = -3x < 0$

$y' = -2y > 0$

R: The flow: $\varphi(t, \eta_1, \eta_2) = (\eta_1 \cdot e^{-3t} , \eta_2 \cdot e^{-2t})$

$\Rightarrow \varphi(t, 0, \eta_2) = (0, \eta_2 \cdot e^{-2t})$

$\eta_2 > 0 \Rightarrow \gamma = (0, \infty)$

$\eta_2 < 0 \Rightarrow \gamma = (-\infty, 0)$

$\eta_2 = 0 \Rightarrow \gamma = (0,0)$

d) $\begin{cases} x' = x - y \\ y' = x + y \end{cases}$ $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\det(A - \lambda I_2) = 0 \iff \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$\iff (1-\lambda)^2 + 1 = 0 \iff \lambda^2 - 2\lambda + 2 = 0.$

$\Rightarrow \lambda_{1,2} = 1 \pm i \Rightarrow \text{Re}\, \lambda_{1,2} \neq 0.$, $\text{Re}\, \lambda_{1,2} > 0$

$\Downarrow$ $\Downarrow$

Focus, global repeller

The system does not have a global first integral.
We look if we can find a first integral in a region U such that $0_2 \notin U$.

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x+y}{x-y} \iff \dfrac{dy}{dx} = \dfrac{y+x}{y-x}$ = not separable

(we did not learn how to solve this, yet).

$\rightarrow$ Lecture 9 - the phase portrait with polar coordinates.

B. Type and stability of the equilibrium points of nonlinear planar systems using the linearization method.

$$X' = f(X) \qquad (1)$$

- Find the equilibrium : $f(\eta^*) = 0$, $\eta^* \in \mathbb{R}^2$
- The linearization method:

$\to$ the linearized system: $X' = Jf(\eta^*) \cdot X \qquad (2)$

The Jacobian matrix of f computed in (x, y)

$$Jf(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix}$$

Def: The equilibrium point η^* is hyperbolic if $Re(\lambda_1) \neq 0$ and $Re(\lambda_2) \neq 0$, where λ_1, λ_2 - are the eigenvalues of $Jf(\eta^*)$

Theorem: Let η^* = hyperbolic e.p..

- If the e.p. 0 of the linearized system is an attractor (repeller), follows that η^* is the same.
- If the e.p. 0 of the linearized system is a saddle, follows that η^* is unstable.

ex 2: Study the stability of the equilibrium points of the nonlinear system:

$$\begin{cases} x' = x(1-x) \\ y' = y(3-y) \end{cases}$$

- First we find the equilibrium points by finding the solutition of the system:

$$\begin{cases} x(1-x) = 0 \\ y(3-y) = 0 \end{cases} \Rightarrow \begin{matrix} x = 0 \ ; \ x = 1 \\ y = 0 \ , \ y = 3 \end{matrix}$$

$\Rightarrow$ The equilibrium points are:

$$\eta_1^* = (0,0) \ ; \ \eta_2^* = (0,3) \ ; \ \eta_3^* = (1,0) \ ; \ \eta_4^* = (1,3).$$

- The function: $f(x,y) = \begin{pmatrix} x - x^2 \\ 3y - y^2 \end{pmatrix}$

$\Rightarrow$ We find the Jacobian:

$$Jf(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1-2x & 0 \\ 0 & 3-2y \end{pmatrix}$$

- We study the stability of each eq. point.

$\rightarrow$ $\boxed{\eta_1^* = (0,0)}$ $\Rightarrow Jf(\eta_1^*) = Jf(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$\Rightarrow$ the eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 3$

$\Rightarrow$ the linearized system: $X' = Jf(0,0) \cdot X$

↳ NODE, global repeller

↳ η_1^* = repeller

$\rightarrow \boxed{\eta_2^* = (0,3)} \Rightarrow Jf(\eta_2^*) = Jf(0,3) = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$

$\Rightarrow \lambda_1 = 1 \quad , \lambda_2 = -3$

The linearized system : $X' = Jf(0,3) \cdot X$

$\hookrightarrow$ SADDLE , <u>unstable</u>

$\hookrightarrow \eta_2^* = (0,3) =$ <u>unstable</u>

$\rightarrow \boxed{\eta_3^* = (1,0)} \Rightarrow Jf(\eta_3^*) = Jf(1,0) = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$

$\Rightarrow \lambda_1 = -1 \quad , \lambda_2 = 3 > 0$

The linearized system: $X' = Jf(1,0) \cdot X$

$\hookrightarrow$ SADDLE , <u>unstable</u>

$\hookrightarrow \eta_3^* = (1,0) =$ <u>unstable</u>

$\rightarrow \boxed{\eta_4^* = (1,3)} \Rightarrow Jf(\eta_4^*) = Jf(1,3) = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$

$\Rightarrow \lambda_1 = -1 < 0 \quad , \lambda_2 = -3 < 0$

The linearized system : $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

$\hookrightarrow$ NODE , <u>stable</u> (global attractor)

$\Downarrow$

$\eta_4^* = (1,3) =$ <u>attractor</u>.