- Seminar 3 -

Linear nonhomogenous equations with constant coef.

1.5.1.

a) Our first aim is to find the general solution:

$x'' + 3x' + x = 1$

↳ second order LHHDE with. c.c

1 → the nonhomogenous part

Step 1: Solve the associated LHDE:

$x'' + 3x' + x = 0.$

- the characteristic equation: $r^2 + 3r + 1 = 0.$

$\Rightarrow r_{1,2} = \dfrac{-3 \pm \sqrt{5}}{2} \in \mathbb{R}$

$\Rightarrow \boxed{x_h = c_1 \cdot e^{r_1 t} + c_2 e^{r_2 t}}, \quad c_1, c_2 \in \mathbb{R}$

Step 2: Find the particular solution for LNDEcc.

Since the nonhomogenous part $f(t) = 1 =$ constant, we look for a constant solution $(x_p = k)$.

$x_p' = 0, \quad x_p'' = 0.$

$\Rightarrow 0 + 3 \cdot 0 + k = 1 \Rightarrow k = 1 \Rightarrow \boxed{x_p = 1}$

Step 3: The general solution:

$x(t) = x_h + x_p$

$= c_1 \cdot e^{t(-3+\sqrt{5})/2} + c_2 e^{t(-3-\sqrt{5})/2} + 1$

Notice $r_{1,2} < 0 \Rightarrow \lim\limits_{t \to \infty} x(t) = 0 + 0 + 1 = 1$

1.5.1.

b) First we look for the general solution of:

$x'' + 4x = 1$ (LNDE with c.c)

Step 1: we solve the LHDE : $x'' + 4x = 0$.

- the characteristic equation: $r^2 + 4 = 0$.

$\Rightarrow r_{1,2} = \pm 2i \Rightarrow \boxed{x_h = c_1 \cos(2t) + c_2 \cdot \sin(2t)}, c_1, c_2 \in \mathbb{R}$

Step 2: we look for a particular solution:

$f(t) = 1 \Rightarrow x_p = k \Rightarrow x_p' = 0, \; x_p'' = 0.$

Replace in LNDE : $0 + 4k = 1 \Rightarrow k = 1/4$

$\Rightarrow \boxed{x_p = 1/4}$

Step 3: The general solution is:

$$\boxed{x = c_1 \cdot \cos(2t) + c_2 \cdot \sin(2t) + \frac{1}{4}, \; c_1, c_2 \in \mathbb{R}}$$

- Let's put the conditions:

$x(0) = \frac{5}{4} \Rightarrow x(0) = c_1 \cos 0 + c_2 \sin 0 + \frac{1}{4} = \frac{5}{4}$

$\Rightarrow \boxed{c_1 = 1}$

$x'(0) = 0 \Rightarrow x'(0) = -2c_1 \cdot \sin 0 + 2c_2 \cdot \cos 0 = 0$

$\Rightarrow \boxed{c_2 = 0}$

$\Rightarrow$ The solution of the IVP:

$$x(t) = \cos 2t + \frac{1}{4}$$

Now we can see that:

$$x(\pi) = \cos(2\pi) + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4}$$

1.5.1.

c) Let's find the solution for : $x'-3\cdot x=t^3$

Step 1: Solve the LHDE associated : $x'-3x=0$.

- the characteristic equation : $r-3=0 \Rightarrow r=3$

$$\Rightarrow x_h = e^{3t}\cdot c_1 \quad , c_1 \in \mathbb{R}.$$

Step 2: We look for a particular solution : x_p.

Since $f(t) = t^3 \Rightarrow x_p =$ polinomial of degree 3.

$$\Rightarrow x_p = at^3 + bt^2 + ct + d.$$

$$x_p' = 3at^2 + 2bt + c$$

Replace x_p, x_p' in the LNDE:

$$\underbrace{3at^2+2bt+c}_{x_p'} - 3\,(\underbrace{at^3+bt^2+ct+d}_{x_p}) = t^3$$

$$-3at^3 + t^2(3a-3b) + t(2b-3c) + c-3d = t^3$$

$$\Rightarrow \begin{cases} -3a = 1 \\ 3a-3b=0 \\ 2b-3c=0 \\ c-3d=0 \end{cases}$$

$\Rightarrow \boxed{a=-\frac{1}{3}}$ $\Big\} \Rightarrow -\frac{3}{3} = 3b \Rightarrow \boxed{b=-\frac{1}{3}}$

$\Rightarrow -\frac{2}{3} = 3c \Rightarrow \boxed{c=-\frac{2}{9}}$

$\Rightarrow -\frac{2}{9} = 3d \Rightarrow \boxed{d=-\frac{2}{27}}$

$$\Rightarrow x_p = -\frac{1}{3}t^3 - t^2 - \frac{2}{3}t - \frac{2}{9}$$

Step 3: The general solution :

$$\boxed{x = c_1\cdot e^{3t} - \frac{1}{3}t^3 - \frac{1}{3}t^2 - \frac{2}{9}t - \frac{2}{27} \quad , c_1 \in \mathbb{R}}$$

1.5.2.

The equation : $x''-x=e^{\lambda t}$, $\lambda\in\mathbb{R}$ is LNDE c.c..

Step 1 : We solve the LHDE associated: $x''-x=0$.

$$r^2-1=0 \Leftrightarrow (r-1)(r+1)=0 \Rightarrow r_1=1,\ r_2=-1.$$

$$\Rightarrow x_h=c_1e^t+c_2e^{-t},\quad c_1,c_2\in\mathbb{R}.$$

Step 2 : We look for a particular solution of the form: $x_p=a\cdot e^{\lambda t}$. Let's determine a real.

$$x_p'=a\lambda e^{\lambda t}$$

$$x_p''=a\lambda^2e^{\lambda t}$$

$\xrightarrow[\text{eq}]{\text{replace}}$ $a\lambda^2e^{\lambda t}-ae^{\lambda t}=e^{\lambda t}\quad |:e^{\lambda t}$

$$a(\lambda^2-1)=1\Longrightarrow \boxed{a=\frac{1}{\lambda^2-1},\ \lambda\neq\pm1}$$

Here „a" is well-defined only when : $\lambda\in\mathbb{R}\setminus\{\pm1\}$

- When $\lambda\in\{\pm1\}$, we look for a particular solution of the form: $x_p=a\cdot t\cdot e^{\lambda t}$.

$$x_p'=ae^{\lambda t}+a\lambda te^{\lambda t}$$

$$x_p''=a\lambda e^{\lambda t}+a\lambda e^{\lambda t}+a\lambda^2te^{\lambda t}$$

$$=2a\lambda e^{\lambda t}+a\lambda^2te^{\lambda t}$$

$\xrightarrow[\text{eq}]{\text{replace}}$ $2a\lambda e^{\lambda t}+a\lambda^2te^{\lambda t}-ate^{\lambda t}=e^{\lambda t}\quad |:e^{\lambda t}$

$$2a\lambda+(a\lambda^2-a)t=1$$

$$2a\lambda+a\underbrace{(\lambda-1)(\lambda+1)}_{=0}\cdot t=1\quad (\lambda\in\{\pm1\})$$

$$2a\lambda=1\Rightarrow \boxed{a=\frac{1}{2\lambda},\ \lambda\in\{\pm1\}}$$

Conclusion: (Step 3)

The general solution of the given LNDEcc is:

- $x = c_1 e^{t} + c_2 e^{-t} + \frac{1}{\lambda^2-1} \cdot e^{\lambda t}$, when $\lambda \in \mathbb{R} \setminus \{\pm 1\}$
- $x = c_1 e^{t} + c_2 e^{-t} + \frac{t}{2\lambda} \cdot e^{\lambda t}$, when $\lambda \in \{\pm 1\}$

where $c_1, c_2 \in \mathbb{R}$.

1.5.3

Let $\omega > 0$ parameter;

(i) $\varphi(\cdot, \omega) =$ sol of IVP: $\begin{cases} x'' + x = \cos(\omega t) \\ x(0) = x'(0) = 0. \end{cases}$

Step 1:

The LHDE associated: $x'' + x = 0$.

- the characteristic equation: $r^2 + 1 = 0$.

$\Rightarrow r_{1,2} = \pm i \Rightarrow \boxed{x_h = c_1 \sin t + c_2 \cos t}$, $c_1, c_2 \in \mathbb{R}$

Step 2:

$\omega \neq 1$, $x_p = a\cos(\omega t) + b \cdot \sin(\omega t)$

where $x_p =$ particular solution for LNDE:

$x_p' = -a\omega \sin(\omega t) + b\omega \cdot \cos(\omega t)$

$x_p'' = -a \cdot \omega^2 \cdot \cos(\omega t) - b \cdot \omega^2 \cdot \sin(\omega t)$

$\xrightarrow[\text{eq}]{\text{replace}} -a\omega^2 \cdot \cos(\omega t) - b \cdot \omega^2 \cdot \sin(\omega t) + a \cdot \cos(\omega t) + b \cdot \sin(\omega t) = \cos(\omega t)$

$\Rightarrow \begin{cases} -a\omega^2 + a = 1 \\ -b\omega^2 + b = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{1-\omega^2} \\ b = 0 \end{cases}$, $\omega \neq 1$, $\omega > 0$

$$\Rightarrow x_p = \frac{1}{1-\omega^2}\cdot\cos(\omega t) \quad , \omega \neq 1, \omega > 0$$

Step 3:

The gen sol: $x = c_1 \cdot \sin t + c_2 \cdot \cos t + \frac{1}{1-\omega^2}\cdot\cos(\omega t)$

where $c_1, c_2 \in \mathbb{R}$, $\omega \neq 1$, $\omega > 0$

(ii) Here the LNDE is: $x'' + x = \cos t$.

Notice that we are in the case (i) with $\omega = 1$.

Step 1:

Solve $x'' + x = 0 \quad \overset{(i)}{\Longrightarrow} \quad x_h = c_1 \cdot \sin t + c_2 \cdot \cos t, \; c_1, c_2 \in \mathbb{R}$

Step 2:

Look for a particular solution:

$x_p(t) = t(a\cos t + b\sin t)$ for LNDE.

$x_p'(t) = a\cos t + b\sin t - ta\sin t + bt\cos t$

$x_p''(t) = -a\sin t + b\cos t - a\sin t + b\cos t - $

$- ta\cos t - bt\sin t$

$$\xrightarrow[\text{eq}]{\text{replace}} -2a\sin t + 2b\cos t \cancel{- ta\cos t - bt\sin t} +$$

$$+ \cancel{t(a\cos t + b\sin t)} = \cos t$$

$-2a = 0$

$2b = 1 \Rightarrow b = \frac{1}{2}$, $a = 0$.

$$\Rightarrow \boxed{x_p = \frac{1}{2}t\sin t}$$

Step 3: The general solution:

$x = c_1 \cdot \sin t + c_2 \cdot \cos t + \frac{1}{2}t\sin t$, $c_1, c_2 \in \mathbb{R}$

(iii) $\to$ From (i) $\Rightarrow$ $x = c_1 \sin t + c_2 \cos t + \dfrac{1}{1-\omega^2} \cdot \cos(\omega t)$

- $x(0) = 0 \Rightarrow c_1 \sin 0 + c_2 \cdot \cos 0 + \dfrac{1}{1-\omega^2} \cdot \cos 0 = 0$

$$\Rightarrow c_2 = \frac{1}{\omega^2 - 1}$$

$$\Rightarrow x(t) = c_1 \cdot \sin t + \frac{1}{\omega^2-1} \cos t - \frac{1}{\omega^2-1} \cdot \cos(\omega t)$$

$$x'(t) = c_1 \cos t - \frac{1}{\omega^2-1} \sin t + \frac{\omega}{\omega^2-1} \cdot \sin(\omega t)$$

- $x'(0) = c_1 \cdot \cos 0 - 0 + 0 = 0 \Rightarrow c_1 = 0.$

$$\Rightarrow \boxed{\varphi(\cdot,\omega) = \frac{1}{\omega^2-1} \cdot \cos t - \frac{1}{\omega^2-1} \cos(\omega t) \quad , \ \omega \neq 1. \ (\omega > 0)}$$

$\to$ From (ii) $\Rightarrow$ $x = c_1 \sin t + c_2 \cos t + \frac{1}{2} t \sin t.$

- $x(0) = c_1 \cdot \sin 0 + c_2 \cdot \cos 0 + \frac{1}{2} 0 = 0.$

$$\hookrightarrow c_2 = 0.$$

$$\Rightarrow x(t) = c_1 \cdot \sin t + \frac{1}{2} t \sin t$$

$$x'(t) = c_1 \cdot \cos t + \frac{1}{2} \sin t + \frac{1}{2} t \cos t$$

- $x'(0) = c_1 \cdot \cos 0 + \frac{1}{2} \sin 0 + \frac{1}{2} \cdot 0 = 0$

$$\hookrightarrow c_1 = 0$$

$$\Rightarrow \boxed{\varphi(\cdot,1) = \frac{1}{2} t \sin t \quad , \ \omega = 1.}$$

$$(iv)\quad \varphi(t,\omega) = \frac{1}{\omega^2-1}\cdot\cos t - \frac{1}{\omega^2-1}\cdot\cos(\omega t)$$

$$\varphi(t,1) = \frac{1}{2} t \sin t$$

$$\varphi(t,\omega) = \frac{1}{\omega^2-1}\left[\cos t - \cos(\omega t)\right]$$

$$= \frac{1}{\omega^2-1}\cdot(-2)\cdot\sin\frac{t+\omega t}{2}\cdot\sin\frac{t-\omega t}{2}.$$

$$= -\frac{2}{\omega^2-1}\cdot\sin\frac{t(\omega+1)}{2}\cdot\sin\frac{t(1-\omega)}{2}$$

$$\lim_{\omega\to1}\varphi(t,\omega) = \lim_{\omega\to1}\left[(-2)\cdot\underbrace{\frac{-\sin\frac{t(\omega-1)}{2}}{\omega-1}}_{\to t/2}\cdot\frac{\sin\frac{t(\omega+1)}{2}}{\omega+1}\right]$$

$$= +\cancel{2}\cdot\frac{t}{\cancel{2}}\cdot\lim_{\omega\to1}\frac{\sin\frac{t(\omega+1)}{2}}{\omega+1}$$

$$= t\cdot\frac{\sin\frac{2t}{2}}{2}$$

$$= \frac{1}{2}t\sin t = \boxed{\varphi(t,1)}$$

q.e.d.

1.3.2.

b) $x' + tx = e^{-t^2-t}$

Notice that the equation is a first order linear nonhomogenous differential equation, with variable coefficient: $a(t) = t$.

The nonhomogenous part is: $f(t) = e^{-t^2-t}$

Here $I := \mathbb{R}$

Let $A(t) = -\int_0^t a(s)\,ds = -\int_0^t s\,ds = -\frac{t^2}{2}$

Method 1 (Integrating factor method)

The integrating factor here is:

$$\mu(t) = e^{-A(t)} = e^{-(-\frac{t^2}{2})} = e^{t^2/2}$$

$$e^{\int_0^t a(s)ds}$$

Let's take the equation:

$$x' + tx = e^{-t^2-t} \quad \Big| \cdot e^{t^2/2}$$

$$x' \cdot e^{t^2/2} + t \cdot e^{t^2/2} \cdot x = e^{-t^2-t} \cdot e^{t^2/2}$$

$$\left(x \cdot e^{t^2/2} \right)' = e^{-\frac{t^2}{2} - t} \quad \Big| \int dt$$

$$x \cdot e^{t^2/2} = \int_0^t e^{-\frac{s^2}{2} - s}\,ds := I$$

$$x \cdot e^{t^2/2} = I + C$$

$$x = I \cdot e^{-t^2/2} + C \cdot e^{-t^2/2} \quad , C \in \mathbb{R}$$

Method 2 : (Separation of variables method & Lagrange method)

St1: We write the linear homog eq. ~~associated~~:

$$x' + tx = 0 \Rightarrow \frac{dx}{dt} = -t \cdot x \Rightarrow \frac{dx}{x} = -t\,dt$$

$$\Rightarrow \ln|x| = -\frac{t^2}{2} + \ln C \Rightarrow \underline{x_h = C \cdot e^{-t^2/2}}$$

(the general solution of the homog eq)

St2: We apply the Lagrange method to find a particular solution, denoted x_p, of the nonhomogenous equation.

We look for $C(t) \in C^1(\mathbb{R})$ such that:

$$x_p = C(t) \cdot e^{-t^2/2}$$

$$x_p' = C'(t) \cdot e^{-t^2/2} + C(t) \cdot e^{-t^2/2} \cdot \left(-\frac{2t}{2}\right)$$

$$x_p' = C'(t) \cdot e^{-t^2/2} - t \cdot C(t) \cdot e^{-t^2/2}$$

We replace x_p, x_p' in the LHDE:

$$C'(t) \cdot e^{-t^2/2} - t \cdot C(t) \cdot e^{-t^2/2} + t \cdot C(t) \cdot e^{-t^2/2} = e^{-t^2-t}$$

$$C'(t) = e^{-t^2-t} \cdot e^{t^2/2} \Rightarrow C'(t) = e^{-\frac{t^2}{2}-t}$$

$$\Rightarrow c(t) = \int_0^t e^{-\frac{s^2}{2} - s}\, ds = \dot{I}$$

$$\Rightarrow x_p = \dot{I} \cdot e^{-t^2/2}$$

<u>St3</u>: The general solution for LNDE:

$$x = x_h + x_p = C \cdot e^{-t^2/2} + \dot{I} \cdot e^{-t^2/2}, \quad C \in \mathbb{R}.$$

<u>Method 3</u>: (<u>direct aplications of Prop.</u>)

<u>Prop</u>: The general solution of the first order linear homogenous differential equation is:

$$x(t) = C \cdot e^{A(t)} + \int_{t_0}^{t} e^{-A(s) + A(t)} \cdot f(s)\, ds, \quad C \in \mathbb{R}$$

Here $A(t) = -\frac{t^2}{2}$

$f(t) = e^{-t^2 - t}$, $t_0 = 0$.

$$\Rightarrow x(t) = C \cdot e^{-t^2/2} + \int_0^t e^{-t^2/2 + s^2/2} \cdot e^{-s^2 - s}\, ds$$

$$x(t) = C \cdot e^{-t^2/2} + e^{-t^2/2} \cdot \underbrace{\int_0^t e^{-s^2/2 - s}\, ds}_{\dot{I}}, \quad C \in \mathbb{R}$$