- Seminar 7 -

Linear difference equations with constant coefficients.

1) First order scalar difference eq. with constant coefficients.

(1) $x_{k+1} - a x_k = f_k$

where - $a \in \mathbb{R}^*$ given

- f_k given (sequence)
  ↳ nonhomog part of eq (1)

- $(x_k)_{k \geq 0}$ - unknown of eq (1)
  $x : \mathbb{N} \to \mathbb{R}$

$f_k \neq 0 \Rightarrow$ eq (1) = nonhomog eq.

$f_k = 0 \Rightarrow$ eq (1) $\to$ eq (2) : $x_{k+1} - a \cdot x_k = 0$
homog eq.

solution of eq (2) :
$\boxed{x_k = c \cdot a^k.}$ $c \in \mathbb{R}$ arbitrary

<u>Theorem:</u>

Let x^h = the general sol of (2)

x^p = a particular sol of (1)

$\Rightarrow$ The general solution of (1):

$$x = x^h + x^p.$$

<u>Ex1:</u>

a) Find a particular solution of the form: $x_k^p = a\cdot(-5)^k$ of $x_{k+1} + 2x_k = (-5)^{k+1}$

b) Find the general solution of the eq from above, with $x_0 = 0$. (IVP)

<u>Solution:</u>

a) $x_k^p = a\cdot(-5)^k \Rightarrow x_{k+1}^p = a\cdot(-5)^{k+1}$

· replace in eq: $a(-5)^{k+1} + 2a(-5)^k = (-5)^{k+1}$

$$(-5)^k\left[a(-5) + 2a\right] = (-5)^k\cdot(-5) \quad \Big| : (-5)^k$$

$$-3a = -5 \Rightarrow \boxed{a = \frac{5}{3}}$$

$\Rightarrow x_k^p = \frac{5}{3}(-5)^k = -\frac{1}{3}\cdot(-5)\cdot(-5)^k =$

$$\boxed{x_k^p = -\frac{1}{3}(-5)^{k+1}}, \quad k \geq 0$$

b) The general solution:

$$x_k = x_k^h + x_k^p$$

x_k^h: we have to find

x_k^p: found at point (a)

$x_{k+1} + 2\cdot x_k = 0$ (the homog eq)

$x_{k+1} - \underbrace{(-2)}\, x_k = 0.$

$\hookrightarrow x_k^h = c\cdot(-2)^k, \quad c\in\mathbb{R}$

$\Rightarrow$ The general solution of the nonhomog eq:

$$x_k = c\cdot(-2)^k - \frac{1}{3}(-5)^{k+1}, \quad c\in\mathbb{R}$$

We put the condition: $x_0 = 0$.

$\Rightarrow k=0$: $x_0 = C(-2)^0 - \frac{1}{3}(-5)^1 = 0$.

$$C + \frac{5}{3} = 0 \Rightarrow C = -\frac{5}{3}$$

$\Rightarrow$ The solution of the IVP:

$$\boxed{x_k = -\frac{5}{3}\cdot(-2)^k - \frac{1}{3}\cdot(-5)^{k+1}, \quad k \geq 0}$$

2) Second order linear homogenous difference equations with constant coefficients.

(3) $x_{k+2} + a_1 \cdot x_{k+1} + a_2 x_k = 0.$

$a_1, a_2 \in \mathbb{R}.$

Theorem (The fundamental theorem for second order difference equations)

Let x^1, x^2 two linearly independent solutions of eq (3), then the gen sol:

$$x = c_1 \cdot x^1 + c_2 \cdot x^2 \quad , c_1, c_2 \in \mathbb{R}$$

Remark: Let $r \in \mathbb{R}^*$. If $x_k = r^k$ is a sol. of eq (3) $\Rightarrow r^2 + a_1 r + a_2 = 0$ (charact eq)

proof: $x_k = r^k \overset{(3)}{\Rightarrow} r^{k+2} + a_1 \cdot r^{k+1} + a_2 \cdot r^k = 0 \quad | : r^k$

$r^2 + a_1 r + a_2 = 0.$ qed

The characteristic equation method to find two linearly independent solutions of (3).

Step 1: write the charact eq:

$$r^2 + a_1 r + a_2 = 0$$

$\Rightarrow$ roots $r_1, r_2 \in \mathbb{C}$

Step 2: associate two sequences, as follows:

- If $r_1 \neq r_2$ real (simple root)

  $\rightarrow x_k^1 = r_1^k$, $x_k^2 = r_2^k$

- If $r_1 = r_2 = r \in \mathbb{R}$ (double root)

  $\rightarrow x_k^1 = r^k$, $x_k^2 = k \cdot r^k$

- If $r_{1,2} = \alpha \pm i\beta \in \mathbb{C}$, $\beta > 0$.

  $\rightarrow x_k^1 = \mathrm{Re}\left[(\alpha + i\beta)^k\right]$

  $x_k^2 = \mathrm{Im}\left[(\alpha + i\beta)^k\right]$

Theorem: Two sequences found at St2 are linearly independent sol. of (3).

Step 3: $x_k = c_1 \cdot x_k^1 + c_2 \cdot x_k^2$, $c_1, c_2 \in \mathbb{R}$

Ex 2: Find the solution of the IVP:

$$\begin{cases} x_{k+2} - x_{k+1} - x_k = 0. \\ x_0 = 0 \quad , x_1 = 1 \end{cases}$$

(Fibonacci sequence)

- The char eq: $r^2 - r - 1 = 0$.

$$\Delta = 5 \Rightarrow r_{1,2} = \frac{1 \pm \sqrt{5}}{2} \in \mathbb{R}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \Rightarrow x_k^1 = \left(\frac{1+\sqrt{5}}{2}\right)^k$$

$$r_2 = \frac{1-\sqrt{5}}{2} \rightarrow x_k^2 = \left(\frac{1-\sqrt{5}}{2}\right)^k$$

- The general solution: $x_k = c_1\left(\frac{1+\sqrt{5}}{2}\right)^k + c_2\left(\frac{1-\sqrt{5}}{2}\right)^k$, $c_1, c_2 \in \mathbb{R}$

$$\begin{cases} x_0 = 0 \\ x_1 = 1 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = -c_2 \\ c_2\left(-\frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}\right) = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = \frac{1}{\sqrt{5}} \\ c_2 = -\frac{1}{\sqrt{5}} \end{cases}$$

The solution of the IVP:

$$x_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$$

$k \geq 0$.

Ex 3: Find the general solution:

a) $x_{k+2} + x_k = 0$.

b) $x_{k+2} - x_k = 0$

c) $x_{k+2} + x_{k+1} + x_k = 0$.

b) $x_{k+2} - x_k = 0$.

$r^2 - 1 = 0 \Rightarrow r_1 = 1, \; r_2 = -1$

$$\Rightarrow \begin{cases} r_1 = 1 & \Rightarrow x_k^1 = 1^k = 1 \\ r_2 = -1 & \Rightarrow x_k^2 = (-1)^k \end{cases}$$

$\Rightarrow$ The general solution:

$$x_k = c_1 + c_2(-1)^k, \; c_1, c_2 \in \mathbb{R}, \; k \geq 0$$

a) $x_{k+2} + x_k = 0$

$r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i$

$\alpha = 0, \ \beta = 1 > 0 \rightarrow \ ? \ i^k \ ?$

$x_k^1 = \operatorname{Re} i^k \ ?$

$x_k^2 = \operatorname{Im} i^k \ ?$

$\rightarrow (0,1)$

$z = i = 0 + i \cdot 1 \Rightarrow x = 0, \ y = 1$

$= \rho(\cos\theta + i\sin\theta) = \cos\theta + i\sin\theta$

$(\rho = \sqrt{0^2 + 1^2} = 1)$ $\quad \Downarrow \quad \begin{cases} \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \\ \sin\theta = 1 \end{cases}$

$\theta = \sphericalangle(Ox, \overline{Oz}) = \frac{\pi}{2}$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$i^k = \cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}$$

$$z = \rho(\cos\theta + i\sin\theta)$$

$$z^n = \rho^n(\cos(n\theta) + i\sin(n\theta))$$

$$\mathrm{Re}(i^k) = \cos\frac{k\pi}{2} \longrightarrow x_k^1$$

$$\mathrm{Im}(i^k) = \sin\frac{k\pi}{2} \longrightarrow x_k^2$$

$$\Rightarrow x_k = c_1 \cdot \cos\frac{k\pi}{2} + c_2 \cdot \sin\frac{k\pi}{2}, \quad c_1, c_2 \in$$

c) $x_{k+2} + x_{k+1} + x_k = 0.$

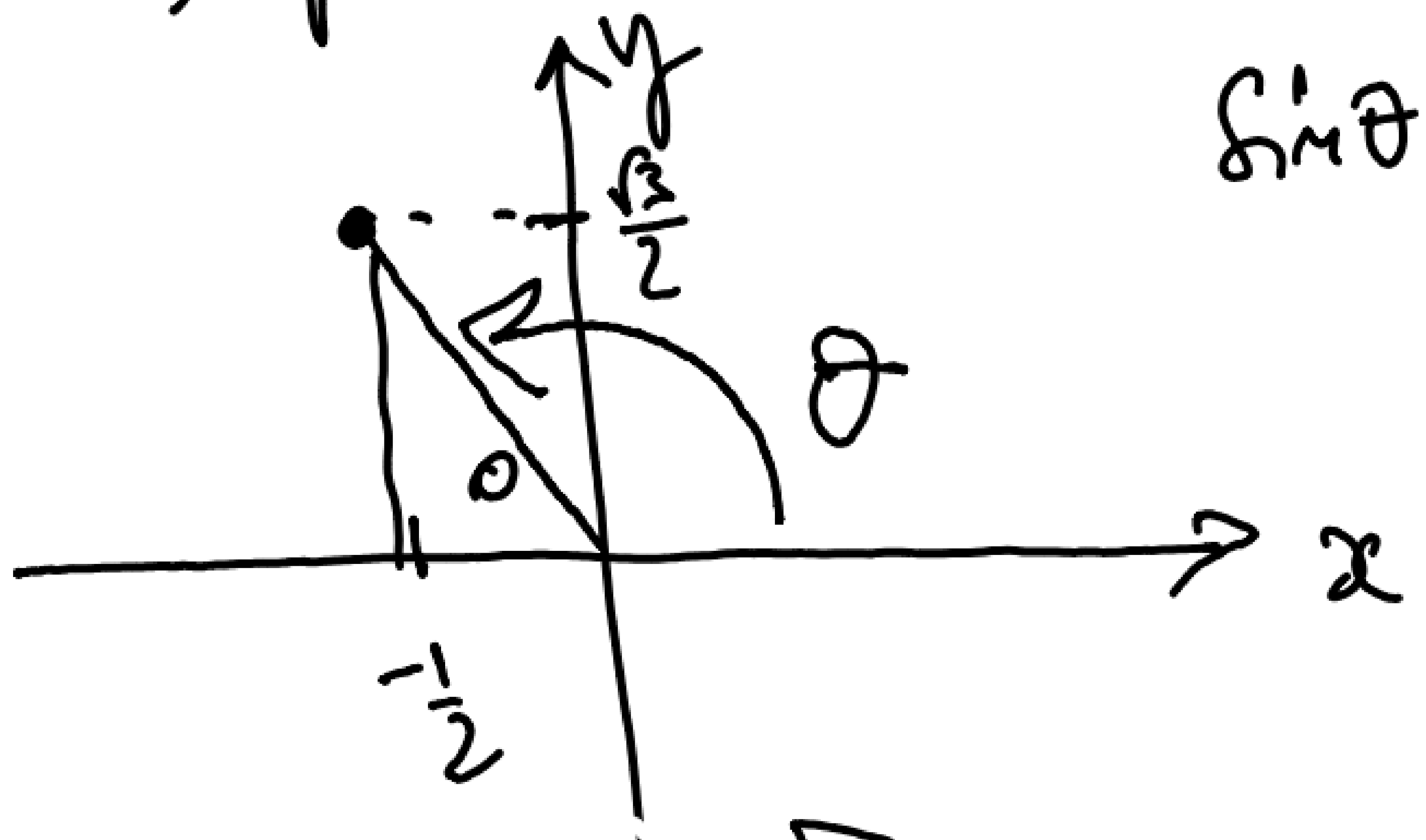
$r^2 + r + 1 = 0$

$\Delta = -3 < 0 \Rightarrow$

$r_{1,2} = \frac{-1 \pm \sqrt{3}\, i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\alpha = -\frac{1}{2}, \quad \beta = \frac{\sqrt{3}}{2} > 0 \qquad \cos\theta = -\frac{1}{2}$

$\sin\theta = \frac{\sqrt{3}}{2}$



$\theta = \frac{\pi}{2} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$

$$\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^k = \left(\underbrace{\cos\frac{2k\pi}{3}}_{x_k^1} + i\,\underbrace{\sin\frac{2k\pi}{3}}_{x_k^2}\right)$$

$\Rightarrow x_k^1 = \cos\frac{2k\pi}{3}$

$x_k^2 = \sin\frac{2k\pi}{3}$

The general solution:

$$x_k = c_1 \cdot \cos\frac{2k\pi}{3} + c_2 \cdot \sin\frac{2k\pi}{3}$$

$c_1, c_2 \in \mathbb{R}$, $k \geq 0$

3) Linear homogenous system with constant coefficients.

$$(4) \quad X_{k+1} = A \cdot X_k \quad , A \in M_n(\mathbb{R})$$

$$\Rightarrow \text{sol}: \boxed{X_k = A^k \cdot X_0}$$

$X_0 \in \mathbb{R}^n$ arbitrary

We need to find $A^k = ?$

Remark: One possibility is to find a simpler $B \in M_n(\mathbb{R})$ and an invertible $P \in M_n(\mathbb{R})$ such that: $A = P^{-1} \cdot B \cdot P$

Then: $A^k = P^{-1} B^k \cdot P$, $\forall k \geq 0$

For example if B = diagonal matrix

$$B = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

i.e. $B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{pmatrix}$

$$\Rightarrow B^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix}$$

<u>Remark:</u> The equation:

$$x_{k+2} + a_1 \cdot x_{k+1} + a_2 x_k = 0.$$

can be written in formula (4):

where $n = 2$.

$$X_k = \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} \Rightarrow$$

$x_{k+2} \cdots$

$$X_{k+1} = \begin{pmatrix} x_{k+1} \\ x_{k+2} \end{pmatrix} = \begin{pmatrix} x_{k+1} \\ -a_1 x_{k+1} - a_2 x_k \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}}_{} \cdot \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}$$

$$= A \cdot X_k$$

$$\Rightarrow X_{k+1} = A \cdot X_k$$

Ex 4 : Let A = diagonalizable matrix and eigenval of A satisfy : $|\lambda_i| < 1$ $i = \overline{1, n}$.

Prove that any solution of :

$$X_{k+1} = A \cdot X_k$$

satisfy : $\lim_{k \to \infty} X_k = 0 \in \mathbb{R}^n$

proof :

A = diagonalizable matrix :

$\Rightarrow \begin{cases} \exists \text{ diagonal matrix } B \\ \exists \text{ invertible matrix } P \end{cases}$

such that $A = P^{-1} B P$

$B = \text{diag}(\underbrace{\lambda_1, \lambda_2, \ldots, \lambda_n}_{\text{eigenvalues}})$

$\Rightarrow$

$$\Rightarrow A^k = P^{-1} \cdot B^k \cdot P$$

$$A^k = P^{-1} \cdot \operatorname{diag}(\lambda_1^k, \lambda_2^k, \ldots \lambda_n^k) P$$

The solution of the system:

$$X_k = A^k \cdot X_0 \quad , X_0 = \text{arbitrary}$$

$$\Rightarrow X_k = P^{-1} \operatorname{diag}(\lambda_1^k, \ldots, \lambda_n^k) P \cdot X_0$$

$$\lim_{k\to\infty} X_k = \lim_{k\to\infty} P^{-1} \cdot \underbrace{\operatorname{diag}(\lambda_1^k, \ldots, \lambda_n^k)}_{} P \cdot X_0$$

$$\begin{array}{ccc} \lim_{k\to\infty} X_k & & \operatorname{diag}(\lambda_1^k, \ldots, \lambda_n^k) \\ \Big\downarrow & & \Big\downarrow {\scriptstyle k\to\infty} \quad |\lambda_i|<1 \Rightarrow |\lambda_i|^k \to 0 \\ O_n & \Longleftarrow & O_n \end{array}$$