Seminar 4 – dyn. sys.

Let $f \in C^1(\mathbb{R})$ and let the DE: $x' = f(x)$. (1)

- The flow $\equiv$ the unique solution of the IVP: $\begin{cases} x' = f(x) \\ x(0) = \eta \end{cases}$, $\eta \in \mathbb{R}$ fixed.

  solution denoted $\varphi(t, \eta)$

- The equilibrium point of eq (1) denoted η^*, $\varphi(t, \eta^*) = \eta^*$ $\xrightarrow{\text{Prop}}$ $\boxed{\eta^* = \text{sol of } f(\eta^*) = 0}$

- The orbits:

  $\gamma_\eta = \{\varphi(t, \eta),\ t \in (\alpha_\eta, \beta_\eta) := I_\eta\}$ – the orbit

  $\gamma_\eta^+ = \{\varphi(t, \eta),\ t \in (0, \beta_\eta)\}$ – the positive orbit

  $\gamma_\eta^- = \{\varphi(t, \eta),\ t \in (\alpha_\eta, 0)\}$ – the negative orbit

- The phase portrait of eq (1) is the representation of the real line ($\mathbb{R}$) of all the orbits, together with an arrow on each orbit that indicates the future.

  The arrows indicates:
  - to the right if $f > 0$
  - to the left if $f < 0$.

[→ Lecture 6–7: the algorithm to represent the phase portrait.]

ex 24: For each $k>0$, let the diff. eq.:

$$x' = -k(x-21)$$

(the model of Newton for cooling processes).

$x(t) =$ the temperature of a cup of tea at time t.

a) Find the flow.

b) The cup of tea has initial temp $49°C$; after 10 min the cup has temp $37°C$.

Find the initial temp of a cup of tea s.t. after 20 min the cup has $37°C$.

Solution:

The flow: let $\eta \in \mathbb{R}$ fixed.

↳ Find the solution of the IVP: $\begin{cases} x' = -k(x-21) \\ x(0) = \eta . \end{cases}$

- Solve the first order lin nonhomog eq:

$$x' = -k(x-21) \iff x' + kx = 21k$$

- st 1: solve the homog eq:

$$x' + kx = 0$$

$$x' = -kx \iff \frac{dx}{dt} = -kx \iff \frac{dx}{x} = -k\,dt$$

$$\int \frac{dx}{x} = -k \int dt \Rightarrow \ln|x| = -kt + \ln C$$

$$x_h = C \cdot e^{-kt}, \quad C \in \mathbb{R}$$

- $\underline{st2}$: $x_p = c(t)\cdot e^{-kt}$ - particular solution

$$x_p' = c'(t)\cdot e^{-kt} + c(t)\cdot(-k)e^{-kt}$$

$$\Rightarrow c'(t)e^{-kt} \cancel{- kc(t)e^{-kt}} \cancel{+ kc(t)\cdot e^{-kt}} = 21\cdot k$$

$$c'(t) = 21\cdot k\cdot e^{kt}$$

$$c(t) = 21k\int e^{kt}dt = 21k\cdot\frac{1}{k}\cdot e^{kt} = 21e^{kt}$$

$$\Rightarrow x_p = 21\cdot e^{kt}\cdot e^{-kt} = 21.$$

- $\underline{st3}$: $x = x_h + x_p$

$$\underline{x = c\cdot e^{-kt} + 21 \quad , c\in\mathbb{R}}$$

- Apply the condition: $x(0) = \eta$.

$$\Rightarrow x(0) = c\cdot e^0 + 21 = \eta \quad \Rightarrow \boxed{C = \eta - 21}$$

Then:

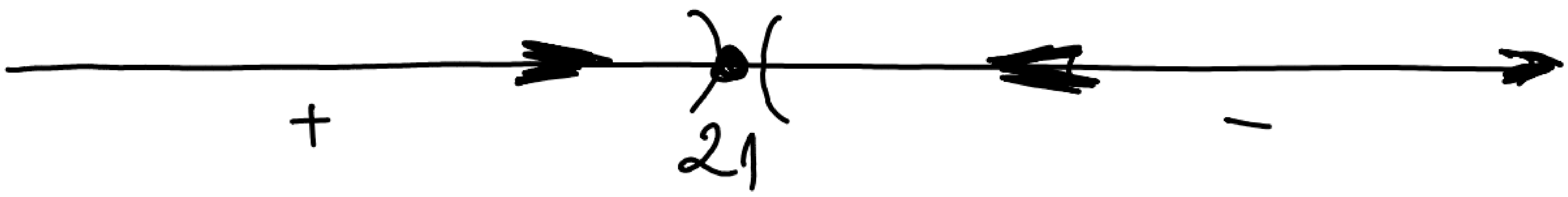
$$\underline{\varphi(t,\eta) = (\eta-21)\cdot e^{-kt} + 21 \quad , \forall \eta\in\mathbb{R}, \forall t\in\mathbb{R}}$$

(the expression of the flow)

The equilibrium point: $f(x^*) = 0$

$-k(x-21) = 0 \iff \underline{x^* = 21}$, $k > 0$

The phase portrait:



x	$-\infty$		21		$+\infty$
$f(x)$	$+\ +\ +$		0		$-\ -\ -$

The orbits are: $(-\infty, 21)$; $\{21\}$; $(21, \infty)$.

Notice that:

$$\lim_{t\to\infty} \varphi(t,\eta) = 21, \quad \forall \eta \in \mathbb{R},$$

follows that: $\eta^* = 21$ = global attractor equilibrium point

b) $49°C \xrightarrow{10'} 37°C$

$49°C$: initial state $(t=0)$

$\Updownarrow$

$x(0) = 49°$ (initial state)

$37°C$: state at time $t=10$.

$\Updownarrow$

$\varphi(10, 49°) = 37°$.

$k = ?$

Thus,

$$\varphi(10,49) \overset{(a)}{=} (49-21)\cdot e^{-k\cdot 10} + 21 = 37$$

$$\Rightarrow 28 \cdot e^{-k\cdot 10} = 16$$

$$\Rightarrow e^{-k\cdot 10} = \frac{16}{28} = \frac{4}{7} \quad \Rightarrow e^{-10k} = \frac{4}{7} \quad (*)$$

$$\Rightarrow \boxed{k = -\frac{1}{10} \ln\frac{4}{7}}$$

- Now, let's find other initial state s.t. $\varphi(20,\eta) = 37$

$? \xrightarrow{20'} 37°$, $\eta = ?$

Here: $\varphi(20,\eta) \overset{(a)}{=} (\eta-21)\, e^{-k\cdot 20} + 21 = 37$

$$\Rightarrow (\eta-21)\cdot e^{-k\cdot 20} = 16$$

-apply $(*)$ $\Rightarrow \eta-21 \cdot \left(\frac{4}{7}\right)^2 = 16$

$$\Rightarrow \eta = 49+21$$

$$\Rightarrow \boxed{\eta = 70°} \quad \checkmark$$

Theorem (the linearization method)

Let $f \in C^1(\mathbb{R})$ and $\eta^* =$ ech point of $x' = f(x)$

If : $\begin{cases} f'(\eta^*) < 0 & \Rightarrow \eta^* = \underline{\text{attractor}} \\ f'(\eta^*) > 0 & \Rightarrow \eta^* = \underline{\text{repeller}} \end{cases}$

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ex 2: Let $0 < c < 1$ and $x' = x(1-x) - cx$.

a) Find the equilibria, study stability (lin-meth)

b) Phase portrait

c) $x(t) > 0 \rightarrow$ the density of fish in a lake

$0 < c < 1 \rightarrow$ the rate of fishing.

Predict the fate of fish from lake from (a), (b).

Solution :

$x' = x(1-x) - cx \Leftrightarrow x' = x(1-x-c)$

• Equilibria : $x(1-x-c) = 0$. $\rightarrow x = 0$

$\rightarrow 1 - x - C = 0$.

$\Downarrow$

$\underline{\eta_1^* = 0}$ and $\eta_2^* = \underbrace{1 - C}_{\in(0,1)}$, $0 < c < 1$.

• Stability : $f(x) = x - x^2 - xc \Rightarrow f'(x) = 1 - 2x - c$

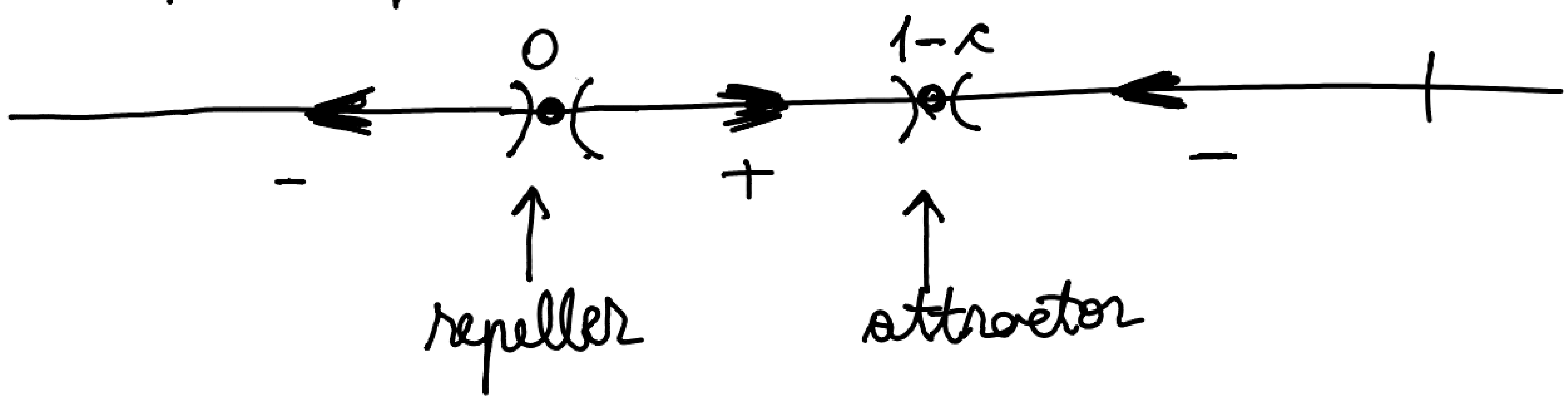
$f'(\eta_1^*) = 1 - 2\cdot 0 - c = 1 - c > 0$

$\Rightarrow \underline{\eta_1^* = 0 = \text{repeller}}$

$f'(\eta_2^*) = 1 - 2(1-c) - c = -(1-c) < 0$

$\Rightarrow \underline{\eta_2^* = 1 - c = \text{attractor}}$

b) The phase portrait:



The orbits are:

$(-\infty, 0)$; $\{0\}$; $(0, 1-c)$; $\{1-c\}$; $(1-c, +\infty)$.

c) $x(t)$ = fish density , $x(t) \geq 0$.

c = rate of fish , $0 < c < 1$

The optimal density : $N = 1-c \Rightarrow \boxed{c = 1-N}$?

Let's choose an initial value : $\underline{\eta \in (0, N)}$.

Let's suppose : $\eta = \frac{N}{2} \Rightarrow c = \frac{N}{2}$

$\Rightarrow$ Conclusion:

Knowing N, find out that the rate of fishing is
$c = 1 - N$.

Theoretically, we could choose for the initial state, η, any value between $(0, N)$, because (just) theoretically, in time, the density will increase and tend to N, when time $t \to \infty$.

Thus, we could give the advice : $\eta = \frac{N}{2} \in (0, N)$.

ex 6 : Represent the phase portrait; orbits, prop; equilibria, stability:

(b) $x' = x - x^3 + 1$

$\Rightarrow f(x) = x - x^3 + 1 \quad , f \in C^1(\mathbb{R})$

$f(x) = 0 \iff x - x^3 + 1 = 0.$

$x_1, x_2, x_3 = \text{roots}.$

$x_1 = \text{real}$, $x_{2,3} \in \mathbb{C}$ or $x_1, x_2, x_3 \in \mathbb{R}.$

$\Rightarrow$ There exist at least one real root.

$f'(x) = 1 - 3x^2 \quad , f'(x) = 0 \iff 1 - 3x^2 = 0. \Rightarrow x_{1,2} = \pm \frac{1}{\sqrt{3}}$

x	$-\infty$		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$		1		η^*		2		$+\infty$
f'		$-\ -\ -$	0	$+\ +$	0	$-$			|	$-$		$-\ -$	
f	$+\infty$	$\searrow$	$+$	$\nearrow$	$+$	$\searrow$			|			$\searrow$	$-\infty$
$\operatorname{sgn} f$		$+\ +$	$+$	$+$	$+$	$+$			0	$-$		$-\ -$	

$f(-\frac{1}{\sqrt{3}}) = -\frac{1}{\sqrt{3}} - \left(-\frac{1}{\sqrt{3}}\right)^3 + 1 = 1 - \frac{2}{3\sqrt{3}} > 0$

$f(\frac{1}{\sqrt{3}}) = \frac{1}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}}\right)^3 + 1 = 1 + \frac{2}{3\sqrt{3}} > 0$

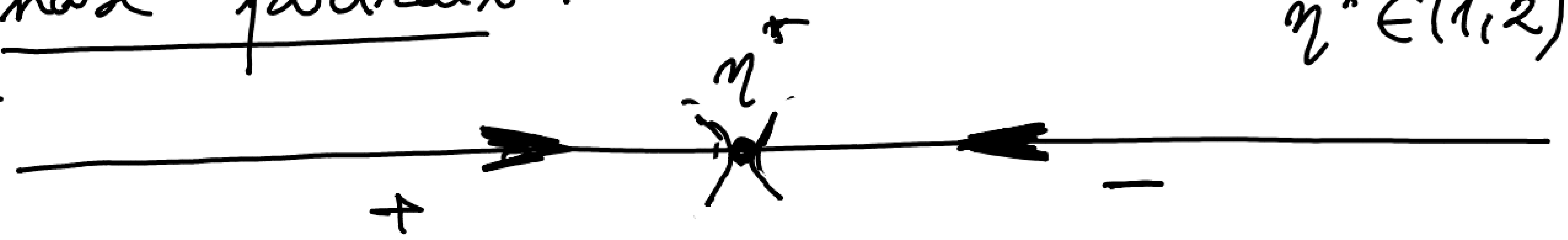
$f(1) = 1 > 0$

$f(2) = 2 - 2^3 + 1 < 0$

$\longrightarrow \underline{\eta^* \in (1, 2)}$

Phase portrait:

$\eta^* \in (1,2)$



$\Rightarrow A_{\eta^*} = \mathbb{R} \Rightarrow \eta^*$ is a global attractor

$\downarrow$
basin of attraction

$$\Rightarrow \lim_{t\to\infty} \varphi(t,\eta) = \eta^* \quad , \forall \eta \in \mathbb{R}$$

The orbits are: $(-\infty, \eta^*)$; $\{\eta^*\}$; $(\eta^*, +\infty)$

ex: Represent the phase portrait, study the stability of the equilibrium points for $\underline{\underline{x' = x - x^3}}$

Find: $\varphi(t;1)$ and $\varphi(t;2)$.

Deduce the property of $\varphi(t;2)$, $\varphi(t,-\frac{1}{2})$.

<u>Solution</u>

$x' = x - x^3 \Rightarrow f(x) = x - x^3$, $f \in C^1(\mathbb{R})$

<u>Eq points</u>: $f(x) = 0 \Leftrightarrow x(1-x^2) = 0$

$\Rightarrow \eta_1^* = 0$, $\eta_2^* = -1$, $\eta_3^* = 1$.

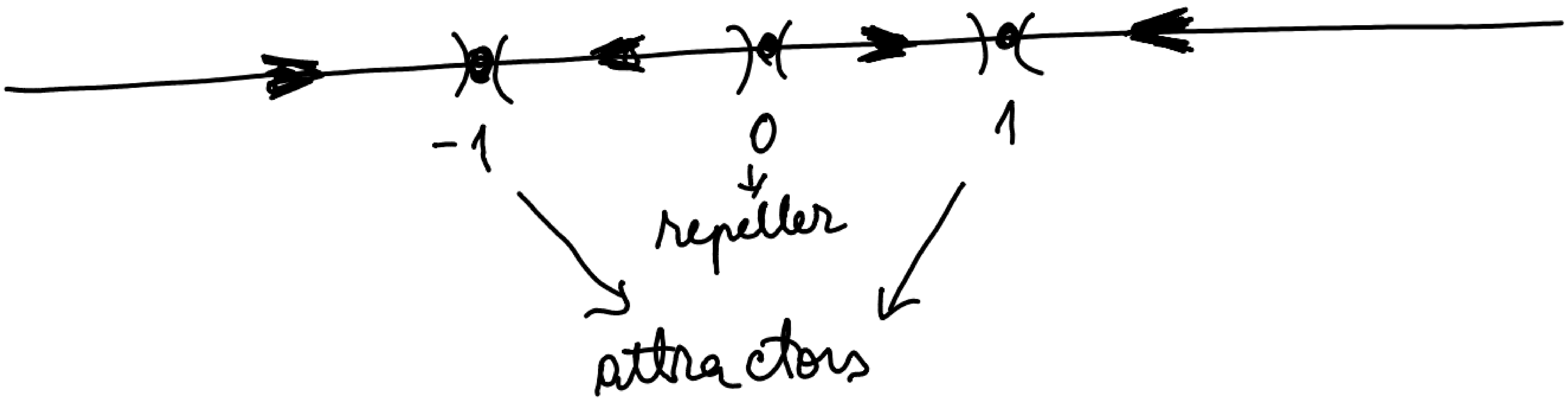
$f'(x) = 1 - 3x^2$

$f'(0) = 1 > 0 \Rightarrow \underline{\eta_1^* = 0 \text{ repeller}}$

$f'(-1) = 1 - 3 = -2 < 0 \Rightarrow \underline{\eta_2^* = -1 \text{ attractor}}$

$f'(1) = 1 - 3 = -2 < 0 \Rightarrow \underline{\eta_3^* = -1 \text{ attractor}}$.

Phase portrait:



The orbits:

$(-\infty, -1)$; $\{-1\}$; $(-1, 0)$; $\{0\}$; $(0, 1)$; $\{1\}$; $(1, \infty)$.

- 1 = equilibrium $\Rightarrow \varphi(t,1) = 1 \quad , \forall t \in \mathbb{R}$
- 0 = equilibrium $\Rightarrow \varphi(t,0) = 0 \quad , \forall t \in \mathbb{R}$
- $\varphi(t,2) \to$ study. ; $\eta = 2$ = initial state

$2 \in (1,+\infty) \Rightarrow \gamma_2 = (1,+\infty) \overset{(***)}{\Rightarrow}$ the image of $\varphi(\cdot,2)$ is $(1,+\infty)$.

$$\left.\begin{array}{l} I_2 = (\alpha_2,\infty) \\ \varphi(\cdot,2) = \text{strictly decreasing} \\ \lim\limits_{t\to\infty} \varphi(t,2) = 1 \\ \lim\limits_{t\to\alpha_2} \varphi(t,2) = +\infty \end{array}\right\} \Leftarrow$$

- $\varphi(t,-\frac{1}{2}) \to$ study ; $\eta = -\frac{1}{2} \in (-1,0)$ ↳ initial state

$$\overset{(***)}{\Downarrow}$$

$$\gamma_{-\frac{1}{2}} = (-1,0)$$

$$\left.\begin{array}{l} \varphi(\cdot,-\frac{1}{2}) = \text{strictly decreasing} \\ \lim\limits_{t\to\infty} (t, -\frac{1}{2}) = -1 \\ \lim\limits_{t\to-\infty} \varphi(t,-\frac{1}{2}) = 0 \end{array}\right\} \Leftrightarrow \text{the image of } \varphi(\cdot,-\tfrac{1}{2}) \text{ is } (-1,0).$$

! (***) Lect 7: Statement...