Geometry
Problem booklet

Assoc. Prof. Cornel Pintea

E-mail: `cpintea math.ubbcluj.ro`

Contents

Module leader: Assoc. Prof. Cornel Pintea

Department of Mathematics,
"Babeş-Bolyai" University
400084 M. Kogălniceanu 1,
Cluj-Napoca, Romania

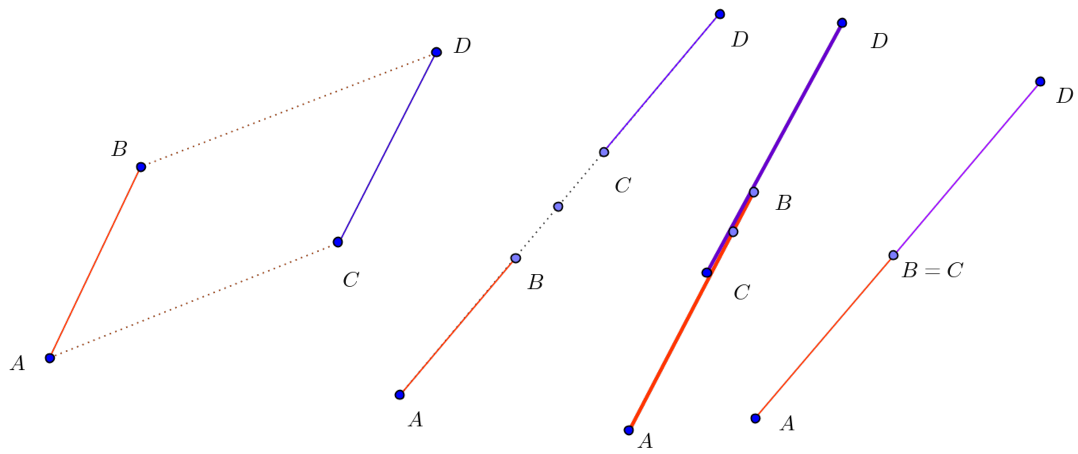
1 Week 1: Vector algebra

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background. Free vectors

Vectors Let $\mathcal{P}$ be the three dimensional physical space in which we can talk about points, lines, planes and various relations among them. If $(A, B) \in \mathcal{P} \times \mathcal{P}$ is an ordered pair, then A is called the *original point* or the *origin* and B is called the *terminal point* or the *extremity* of (A, B).

Definition 1.1. *The ordered pairs (A, B), (C, D) are said to be equipollent, written $(A, B) \sim (C, D)$, if the segments $[AD]$ and $[BC]$ have the same midpoint.*



Pairs of equipollent points $(A,B) \sim (C,D)$

Remark 1.2. If the points A, B, C, $D \in \mathcal{P}$ are not collinear, then $(A, B) \sim (C, D)$ if and only if $ABDC$ is a parallelogram. In fact the length of the segments $[AB]$ and $[CD]$ is the same whenever $(A, B) \sim (C, D)$.

Proposition 1.3. *If (A, B) is an ordered pair and $O \in \mathcal{P}$ is a given point, then there exists a unique point X such that $(A,B) \sim (O, X)$.*

Proposition 1.4. *The equipollence relation is an equivalence relation on $\mathcal{P} \times \mathcal{P}$.*

Definition 1.5. *The equivalence classes with respect to the equipollence relation are called* (free) vectors.

Denote by $\overrightarrow{AB}$ the equivalence class of the ordered pair (A, B), that is $\overrightarrow{AB} = \{(X, Y) \in \mathcal{P} \times \mathcal{P} \mid (X, Y) \sim (A, B)\}$ and let $\mathcal{V} = \mathcal{P} \times \mathcal{P}\big/_{\sim} = \{\overrightarrow{AB} \mid (A, B) \in \mathcal{P} \times \mathcal{P}\}$ be the set of (free) vectors. The *length* or the *magnitude* of the vector $\overrightarrow{AB}$, denoted by $\| \overrightarrow{AB} \|$ or by $| \overrightarrow{AB} |$, is the length of the segment $[AB]$.

Remark 1.6. *If two ordered pairs (A, B) and (C, D) are equipplent, i.e. the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal, then they have the same length, the same direction and the same sense. In fact a vector is determined by these three items.*

Proposition 1.7. 1. $\overrightarrow{AB}=\overrightarrow{CD}\Leftrightarrow\overrightarrow{AC}=\overrightarrow{BD}$.

2. $\forall A, B, O \in \mathcal{P}$, $\exists ! X \in \mathcal{P}$ *such that* $\overrightarrow{AB}=\overrightarrow{OX}$.

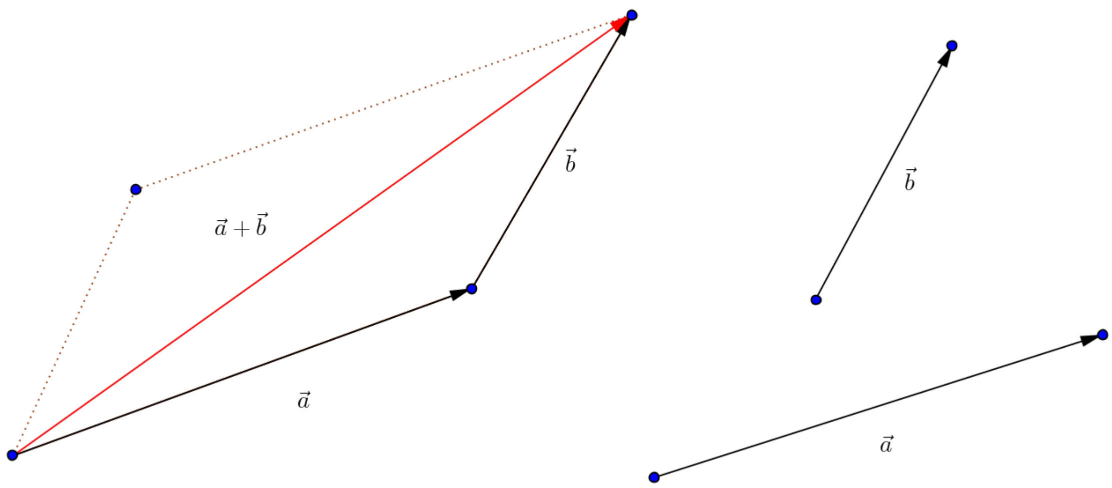
3. $\overrightarrow{AB}=\overrightarrow{A'B'}$, $\overrightarrow{BC}=\overrightarrow{B'C'}\Rightarrow\overrightarrow{AC}=\overrightarrow{A'C'}$.

Definition 1.8. If O, $M \in \mathcal{P}$, the the vector $\overrightarrow{OM}$ is denoted by $\vec{r}_M$ and is called the *position vector of M with respect to O.*

Corollary 1.9. *The map $\varphi_O : \mathcal{P} \to \mathcal{V}$, $\varphi_O(M) = \vec{r}_M$ is one-to-one and onto, i.e bijective.*

1.1.1 Operations with vectors

• **The addition of vectors** Let $\vec{a}, \vec{b} \in \mathcal{V}$ and $O \in \mathcal{P}$ be such that $\vec{a}=\overrightarrow{OA}$, $\vec{b}=\overrightarrow{AB}$. The vector $\overrightarrow{OB}$ is called the *sum* of the vectors $\vec{a}$ and $\vec{b}$ and is written $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\vec{a}+\vec{b}$.



Let O' be another point and A', $B' \in \mathcal{P}$ be such that $\overrightarrow{O'A'}=\vec{a}$, $\overrightarrow{A'B'}=\vec{b}$. Since $\overrightarrow{OA}=\overrightarrow{O'A'}$ and $\overrightarrow{AB}=\overrightarrow{A'B'}$ it follows, according to Proposition 1.4 (3), that $\overrightarrow{OB}=\overrightarrow{O'B'}$. Therefore the vector $\vec{a}+\vec{b}$ is independent on the choice of the point O.

Proposition 1.10. *The set $\mathcal{V}$ endowed to the binary operation $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, $(\vec{a}, \vec{b}) \mapsto \vec{a}+\vec{b}$, is an abelian group whose zero element is the vector $\overrightarrow{AA}=\overrightarrow{BB}=\vec{0}$ and the opposite of $\overrightarrow{AB}$, denoted by $-\overrightarrow{AB}$, is the vector $\overrightarrow{BA}$.*

In particular the addition operation is associative and the vector

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$

is usually denoted by $\vec{a}+\vec{b}+\vec{c}$. Moreover the expression

$$((\cdots(\vec{a}_1+\vec{a}_2)+\vec{a}_3+\cdots+\vec{a}_n)\cdots), \tag{1.1}$$

is independent of the distribution of paranthesis and it is usually denoted by

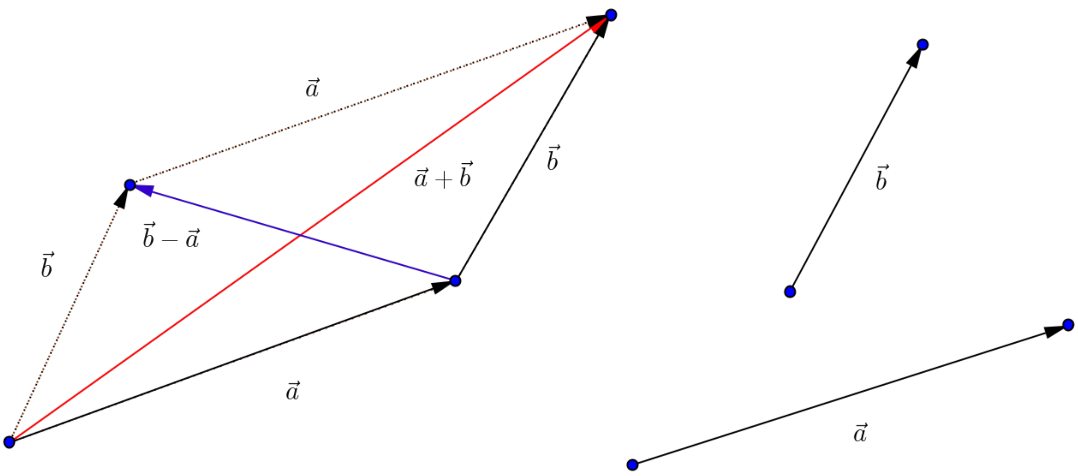
$$\vec{a}_1+\vec{a}_2+\cdots+\vec{a}_n\ .$$

Example 1.11. *If $A_1, A_2, A_3, \ldots, A_n \in \mathcal{P}$ are some given points, then*

$$\overrightarrow{A_1A_2}+\overrightarrow{A_2A_3}+\cdots+\overrightarrow{A_{n-1}A_n}=\overrightarrow{A_1A_n}\ .$$

This shows that $\overrightarrow{A_1A_2}+\overrightarrow{A_2A_3}+\cdots+\overrightarrow{A_{n-1}A_n}+\overrightarrow{A_nA_1}=\vec{0}$, namely the sum of vectors constructed on the edges of a closed broken line is zero.

Corolarul 1.12. *If $\vec{a}=\overrightarrow{OA}$, $\vec{b}=\overrightarrow{OB}$ are given vectors, there exists a unique vector $\vec{x}\in\mathcal{V}$ such that $\vec{a}+\vec{x}=\vec{b}$. In fact $\vec{x}=\vec{b}+(-\vec{a})=\overrightarrow{AB}$ and is denoted by $\vec{b}-\vec{a}$.*



• **The multiplication of vectors with scalars**

Let $\alpha\in\mathbb{R}$ be a scalar and $\vec{a}=\overrightarrow{OA}\in\mathcal{V}$ be a vector. We define the vector $\alpha\cdot\vec{a}$ as follows: $\alpha\cdot\vec{a}=\vec{0}$ if $\alpha=0$ or $\vec{a}=\vec{0}$; if $\vec{a}\neq\vec{0}$ and $\alpha>0$, there exists a unique point on the half line $]OA$ such that $||OB||=\alpha\cdot||OA||$ and define $\alpha\cdot\vec{a}=\overrightarrow{OB}$; if $\alpha<0$ we define $\alpha\cdot\vec{a}=-(|\alpha|\cdot\vec{a})$. The external binary operation

$$\mathbb{R}\times\mathcal{V}\to\mathcal{V},\ (\alpha,\vec{a})\mapsto\alpha\cdot\vec{a}$$

is called the *multiplication of vectors with scalars*.

Proposition 1.13. *The following properties hold:*

$(v1)$ $(\alpha+\beta)\cdot \vec{a} = \alpha\cdot \vec{a} + \beta\cdot \vec{a}$, $\forall\, \alpha,\beta \in \mathbb{R}$, $\vec{a} \in \mathcal{V}$.

$(v2)$ $\alpha\cdot(\vec{a} + \vec{b}) = \alpha\cdot \vec{a} + \alpha\cdot \vec{b}$, $\forall\, \alpha \in \mathbb{R}$, $\vec{a}, \vec{b} \in \mathcal{V}$.

$(v3)$ $\alpha\cdot(\beta\cdot \vec{a}) = (\alpha\beta)\cdot \vec{a}$, $\forall\, \alpha,\beta \in \mathbb{R}$.

$(v4)$ $1\cdot \vec{a} = \vec{a}$, $\forall\, \vec{a} \in \mathcal{V}$.

1.1.2 The vector structure on the set of vectors

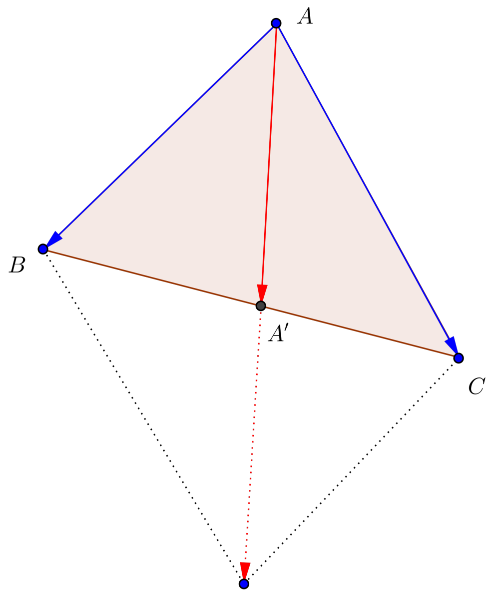
Theorem 1.14. *The set of (free) vectors endowed with the addition binary operation of vectors and the external binary operation of multiplication of vectors with scalars is a real vector space.*

1.2 Examples

Examples 1.15. 1. *([4, Problema 3, p. 1]) Let $OABCDE$ be a regular hexagon in which $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OE} = \vec{b}$. Express the vectors $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of the vectors $\vec{a}$ and $\vec{b}$. Show that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OE} = 3\,\overrightarrow{OC}$.*

2. *If A' is the midpoint of the egde $[BC]$ of the triangle ABC, then*

$$\overrightarrow{AA'} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}).$$



3. *([4, Problema 12, p. 3]) Let M, N be the midpoints of two opposite edges of a given quadrilateral $ABCD$ and P be the midpoint of $[MN]$. Show that*

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0$$

4. *([4, Problema 12, p. 7]) Consider two perpendicular chords AB and CD of a given circle and $\{M\} = AB \cap CD$. Show that*

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2\,\overrightarrow{OM}.$$

5. *([4, Problema 13, p. 3]) If G is the centroid of a tringle ABC and O is a given point, show that*

$$\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}.$$

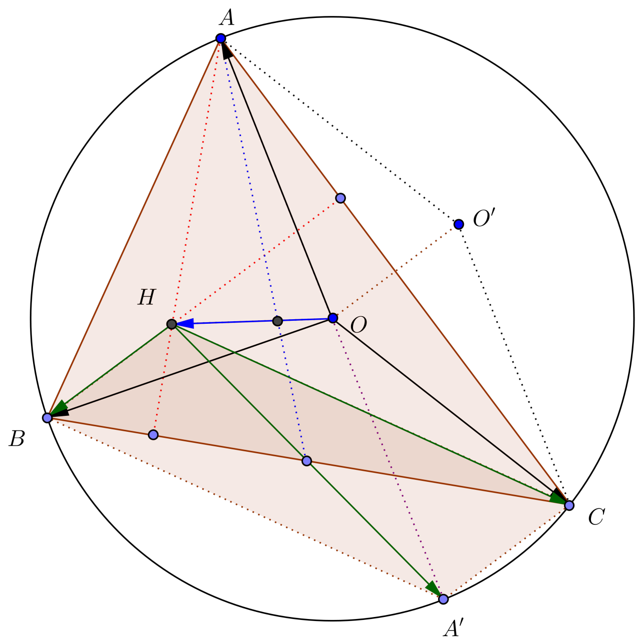
6. *([4, Problema 14, p. 4]) Consider the triangle ABC alongside its orthocenter H, its circumcenter O and the diametrically opposed point A' of A on the latter circle. Show that:*

   (a) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$.

   (b) $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$.

   (c) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\,\overrightarrow{HO}$.

Solution. (6a) *Let M be the point with the property $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OM}$, namely $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OM} - \overrightarrow{OB} = \overrightarrow{BM}$. But $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OO'} \perp \overrightarrow{AC}$, i.e. $\overrightarrow{BM} \perp \overrightarrow{AC}$. One can similarly show that $\overrightarrow{CM} \perp \overrightarrow{AB}$ and $\overrightarrow{AM} \perp \overrightarrow{BC}$. Consequently $M = H$.*



(6b) *$A'BHC$ is a parallelogram as the two pairs of opposite edges are parallel. Indeed one of the pairs is orhtogonal to AC and the other one is orthogonal to AB. Consequently $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$.*

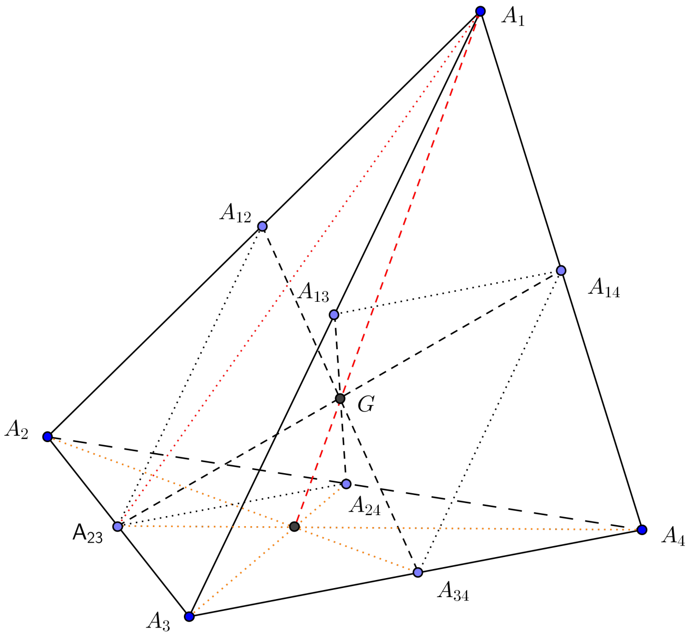
(6c) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA} + \overrightarrow{HA'} = 2\,\overrightarrow{HO}$. *For an alternative solution we may observe:*

$$\begin{aligned}\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} &= \overrightarrow{HO} + \overrightarrow{OA} + \overrightarrow{HO} + \overrightarrow{OB} + \overrightarrow{HO} + \overrightarrow{OC} \\ &= 3\,\overrightarrow{HO} + \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 3\,\overrightarrow{HO} + \overrightarrow{OH} = 2\,\overrightarrow{HO}.\end{aligned}$$

7. *([4, Problema 15, p. 4]) Consider the triangle ABC alongside its centroid G, its orthocenter H and its circumcenter O. Show that O, G, H are collinear and $3\overrightarrow{HG} = 2\overrightarrow{HO}$.*

8. *([4, Problema 11, p. 3]) Consider two parallelograms, $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ in $\mathcal{P}$, and M_1, M_2, M_3, M_4 the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$, $[A_4B_4]$ respectively. Show that:*
   - $2\overrightarrow{M_1M_2} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2}$ *and* $2\overrightarrow{M_3M_4} = \overrightarrow{A_3A_4} + \overrightarrow{B_3B_4}$.
   - M_1, M_2, M_3, M_4 *are the vertices of a parallelogram.*

9. *([4, Problema 27, p. 13]) Consider a tetrahedron $A_1A_2A_3A_4$ and the midpoints A_{ij} of the edges A_iA_j, $i \neq j$. Show that:*
   (a) *The lines $A_{12}A_{34}$, $A_{13}A_{24}$ and $A_{14}A_{23}$ are concurrent in a point G.*
   (b) *The medians of the tetrahedron (the lines passing through the vertices and the centroids of the opposite faces) are also concurrent at G.*
   (c) *Determine the ratio in which the point G divides each median.*
   (d) *Show that $\overrightarrow{GA_1} + \overrightarrow{GA_2} + \overrightarrow{GA_3} + \overrightarrow{GA_4} = \overrightarrow{0}$.*
   (e) *If M is an arbitrary point, show that $\overrightarrow{MA_1} + \overrightarrow{MA_2} + \overrightarrow{MA_3} + \overrightarrow{MA_4} = 4\overrightarrow{MG}$.*



10. *In a triangle ABC consider the points M, L on the side AB and N, T on the side AC such that $3\overrightarrow{AL} = 2\overrightarrow{AM} = \overrightarrow{AB}$ and $3\overrightarrow{AT} = 2\overrightarrow{AN} = \overrightarrow{AC}$. Show that $\overrightarrow{AB} + \overrightarrow{AC} = 5\overrightarrow{AS}$, where $\{S\} = MT \cap LN$.*

11. *Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$, not necesarily in the same plane, alongside their centroids G_1, G_2. Show that $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3\overrightarrow{G_1G_2}$.*

2 Week 2: Straight lines and planes

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

2.1 Brief theoretical background

2.1.1 Linear dependence and linear independence of vectors

Definition 2.1. 1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *collinear* if the points O, A, B are collinear. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *noncollinear*.

2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}, \overrightarrow{OC}$ are said to be *coplanar* if the points O, A, B, C are coplanar. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are *noncoplanar*.

Remark 2.2. 1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are linearly (in)dependent if and only if they are (non)collinear.

2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are linearly (in)dependent if and only if they are (non)coplanar.

Proposition 2.3. *The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ form a basis of $\mathcal{V}$ if and only if they are noncoplanar.*

Corollary 2.4. *The dimension of the vector space of free vectors $\mathcal{V}$ is three.*

2.1.2 Cartesian and affine reference systems

A basis of the direction $\overrightarrow{\pi}$ of the plane π, or for the vector space $\mathcal{V}$ is an ordered basis $[\overrightarrow{e}, \overrightarrow{f}]$ of π, or an ordered basis $[\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ a of $\mathcal{V}$.

If $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is a basis of $\mathcal{V}$ and $\overrightarrow{x} \in \mathcal{V}$, recall that the column vector of $\overrightarrow{x}$ with respect to b is being denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

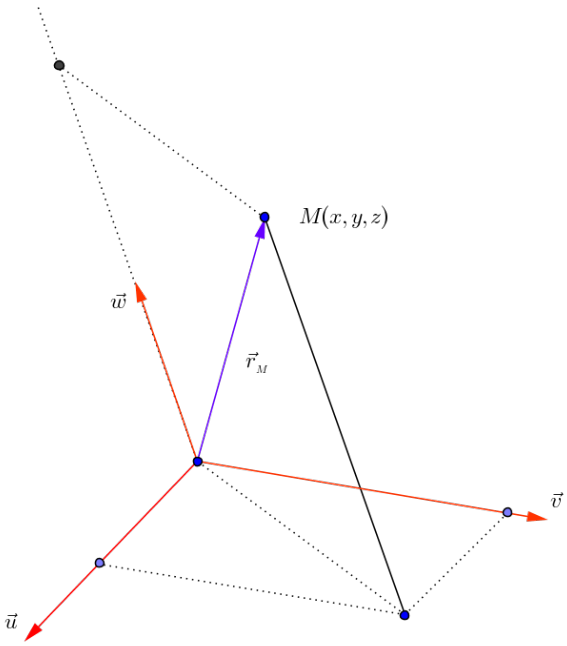
whenever $\overrightarrow{x} = x_1 \overrightarrow{u} + x_2 \overrightarrow{v} + x_3 \overrightarrow{w}$.

Definition 2.5. *A* cartesian reference system *of the space $\mathcal{P}$, is a system $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ where O is a point from $\mathcal{P}$ called the* origin *of the reference system and $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is a basis of the vector space $\mathcal{V}$.*

Denote by E_1, E_2, E_3 the points for which $\overrightarrow{u} = \overrightarrow{OE_1}$, $\overrightarrow{v} = \overrightarrow{OE_2}$, $\overrightarrow{w} = \overrightarrow{OE_3}$.

Definition 2.6. *The system of points (O, E_1, E_2, E_3) is called* the affine reference system associated to the cartesian reference system *$R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$.*

The straight lines OE_i, $i \in \{1, 2, 3\}$, oriented from O to E_i are called *the coordinate axes*. The coordinates x, y, z of the position vector $\overrightarrow{r}_M = \overrightarrow{OM}$ with respect to the basis $[\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y, z)$.



Also, for the column matrix of coordinates of the vector $\vec{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\vec{r}_M = x\vec{u} + y\vec{v} + z\vec{w}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Remark 2.7. *If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are two points, then*

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= x_B\vec{u} + y_B\vec{v} + z_B\vec{w} - (x_A\vec{u} + y_A\vec{v} + z_A\vec{w}) \\ &= (x_B - x_A)\vec{u} + (y_B - y_A)\vec{v} + (z_B - z_A)\vec{w}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

Proposition 2.8. *Let Δ be a straight line and let $A \in \Delta$ be a given point. The set*

$$\vec{\Delta} = \{\overrightarrow{AM} \mid M \in \Delta\}$$

is an one dimensional subspace of $\mathcal{V}$. It is independent on the choice of $A \in \Delta$ and is called the director subspace *of Δ or the* direction *of Δ.*

Remark 2.9. *The straight lines Δ, Δ' are parallel if and only if $\vec{\Delta} = \vec{\Delta}'$*

Definition 2.10. We call *director vector* of the straigh line Δ every nonzero vector $\{\vec{d}\} \in \vec{\Delta}$.

If $\vec{d} \in \mathcal{V}$ is a nonzero vector and $A \in \mathcal{P}$ is a given point, then there exits a unique straight line which passes through A and has the direction $\langle\vec{d}\rangle$. This stright line is

$$\Delta = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle\vec{d}\rangle\}.$$

Δ is called the straight line which passes through O and is parallel to the vector $\vec{d}$.

Proposition 2.11. *Let π be a plane and let $A \in \pi$ be a given point. The set $\overrightarrow{\pi}= \{\overrightarrow{AM} \in \mathcal{V} \mid M \in \pi\}$ is a two dimensional subspace of $\mathcal{V}$. It is independent on the position of A inside π and is called the* director subspace, the director plane *or the* direction *of the plane π.*

Remark 2.12. • The planes π, π' are parallel if and only if $\overrightarrow{\pi}=\overrightarrow{\pi}'$.

• If $\overrightarrow{d}_1$, $\overrightarrow{d}_2$ are two linearly independent vectors and $A \in \mathcal{P}$ is a fixed point, then there exists a unique plane through A whose direction is $\langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle$. This plane is $\pi = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle\}$.

We say that π is the plane which passes through the point A and is parallel to the vectors $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$.

2.1.3 The vector ecuation of the straight lines and planes

Let Δ be a straight line and let $A \in \Delta$ be a given point.

$$\overrightarrow{r}_M = \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{r}_A + \overrightarrow{AM}.$$

Thus

$$\begin{aligned} \{\overrightarrow{r}_M \mid M \in \Delta\} &= \{\overrightarrow{r}_A + \overrightarrow{AM} \mid M \in \Delta\} \\ &= \overrightarrow{r}_A + \{\overrightarrow{AM} \mid M \in \Delta\} \\ &= \overrightarrow{r}_A + \overrightarrow{\Delta}. \end{aligned}$$

Similarly, for a plane π and $B \in \pi$ a given point, then

$$\{\overrightarrow{r}_M \mid M \in \pi\} = \overrightarrow{r}_B + \overrightarrow{\pi}.$$

Generally speaking, a subset X of a vector space is called *affine variety* if either $X = \emptyset$ or there exists $a \in V$ and a vector subspace U of V, such that $X = a + U$.

$$\dim(X) = \begin{cases} -1 & \text{dacă } X = \emptyset \\ \dim(U) & \text{dacă } X = a + U, \end{cases}$$

Proposition 2.13. *The bijection φ_O transforms the straight lines and the planes of the space $\mathcal{P}$ into the one and two dimnensional affine varieties of the vector space $\mathcal{V}$.*

Let Δ be a straight line, let π be a plane, $\{\overrightarrow{d}\}$ be a basis of $\overrightarrow{\Delta}$ and let $[\overrightarrow{d}_1, \overrightarrow{d}_2]$ be a casis of $\overrightarrow{\pi}$. Then for $A \in \Delta$, we obtain the equivalence $M \in \Delta$ if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\overrightarrow{r}_M = \overrightarrow{r}_A + \lambda \overrightarrow{d}. \tag{2.1}$$

The relation (2.1) is called *the vector equation* of the straight line Δ. Similalrly, for $B \in \pi$, we obtain the equivalence $M \in \pi$ if and only if there exists $\lambda_2, \lambda_2 \in \mathbb{R}$ such that

$$\overrightarrow{r}_M = \overrightarrow{r}_A + \lambda_1 \overrightarrow{d}_1 + \lambda_2 \overrightarrow{d}_2. \tag{2.2}$$

The relation (2.2) is called the *vector equation* of the plane π.

Proposition 2.14. *If A, B are different points of a straight line Δ, then its vector equation can be put in the form*

$$\overrightarrow{r}_M = (1-\lambda)\overrightarrow{r}_A + \lambda \overrightarrow{r}_B,\ \lambda \in \mathbb{R}. \tag{2.3}$$

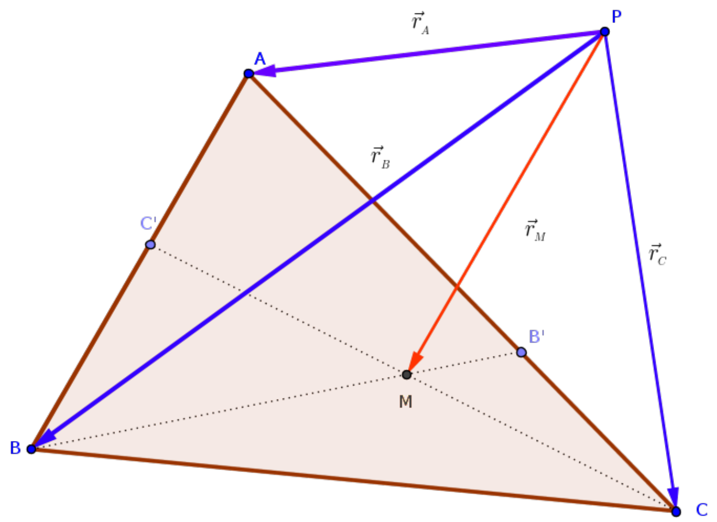
Proposition 2.15. *If A, B, C are three noncolinear points within the plane π, then the vector equation of the plane π can be put in the form*

$$\vec{r}_M = (1-\lambda_1-\lambda_2)\,\vec{r}_A + \lambda_1\,\vec{r}_B + \lambda_2\,\vec{r}_C,\ \lambda_1,\ \lambda_2 \in \mathbb{R}. \tag{2.4}$$

2.2 Problems

1. ([4, Problema 16, p. 5]) Consider the points C' and B' on the sides AB and AC of the triangle ABC such that $\overrightarrow{AC'} = \lambda\,\overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\,\overrightarrow{CB'}$. The lines BB' and CC' meet at M. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$, $\vec{r}_C = \overrightarrow{PC}$ are the position vectors, with respect to P, of the vertices A, B, C respectively, show that

$$\vec{r}_M = \frac{\vec{r}_A - \lambda\,\vec{r}_B - \mu\,\vec{r}_C}{1-\lambda-\mu}. \tag{2.5}$$



Solution. The equations of the lines BB' and CC' are:

$$BB' : \vec{r}_X = (1-t)\,\vec{r}_B + t\,\vec{r}_{B'},\ CC' : \vec{r}_Y = (1-s)\,\vec{r}_C + s\,\vec{r}_{C'}\,.$$

In order to express $\vec{r}_{B'}$ in terms of $\vec{r}_A$ and $\vec{r}_C$ we observe that:

$$\overrightarrow{AB'} = \mu\,\overrightarrow{CB'} \Leftrightarrow \overrightarrow{PB'} - \overrightarrow{PA} = \mu\left(\overrightarrow{PB'} - \overrightarrow{PC}\right) \Leftrightarrow \vec{r}_{B'} = \frac{\vec{r}_A - \mu\,\vec{r}_C}{1-\mu}.$$

One can similarly show that $\vec{r}_{C'} = \dfrac{\vec{r}_A - \lambda\,\vec{r}_B}{1-\lambda}$. Thus, the vector equations of the lines BB' and CC' become:

$$BB' : \vec{r}_X = \frac{t}{1-\mu}\,\vec{r}_A + (1-t)\,\vec{r}_B - \frac{t\mu}{1-\mu}\,\vec{r}_C$$
$$CC' : \vec{r}_Y = \frac{s}{1-\lambda}\,\vec{r}_A - \frac{s\lambda}{1-\lambda}\,\vec{r}_B + (1-s)\,\vec{r}_C\,.$$

Since $BB' \cap CC' = \{M\}$, it follows that

$$\overrightarrow{r}_M = \frac{s_0}{1-\lambda}\overrightarrow{r}_A - \frac{s_0\lambda}{1-\lambda}\overrightarrow{r}_B + (1-s_0)\overrightarrow{r}_C = \frac{t_0}{1-\mu}\overrightarrow{r}_A + (1-t_0)\overrightarrow{r}_B - \frac{t_0\mu}{1-\mu}\overrightarrow{r}_C,$$

for some $s_0, t_0 \in \mathbb{R}$.

Taking into account that the system

$$\begin{cases} \dfrac{t}{1-\mu} &= \dfrac{s}{1-\lambda} \\ 1-t &= \dfrac{s\lambda}{\lambda-1} \\ \dfrac{t\mu}{\mu-1} &= 1-s \end{cases}$$

has the unique solution $s_0 = \dfrac{1-\lambda}{1-\lambda-\mu}$, $t_0 = \dfrac{1-\mu}{1-\lambda-\mu}$, it follows that

$$\begin{aligned} \overrightarrow{r}_M &= \frac{s_0}{1-\lambda}\overrightarrow{r}_A - \frac{s_0\lambda}{1-\lambda}\overrightarrow{r}_B + (1-s_0)\overrightarrow{r}_C \\ &= \frac{t_0}{1-\mu}\overrightarrow{r}_A + (1-t_0)\overrightarrow{r}_B - \frac{t_0\mu}{1-\mu}\overrightarrow{r}_C \\ &= \frac{\overrightarrow{r}_A - \lambda\overrightarrow{r}_B - \mu\overrightarrow{r}_C}{1-\lambda-\mu}. \end{aligned}$$

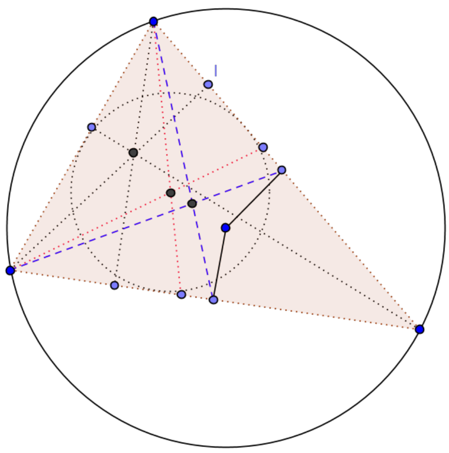
2. ([4, Problema 17, p. 5]) Consider the triangle ABC, its centroid G, its orthocenter H, its incenter I and its circumcenter O. If $P \in \mathcal{P}$ is a given point and $\overrightarrow{r}_A = \overrightarrow{PA}$, $\overrightarrow{r}_B = \overrightarrow{PB}$, $\overrightarrow{r}_C = \overrightarrow{PC}$ are the position vectors with respect to P of the vertices A, B, C respectively, show that:

(a) $\overrightarrow{r}_G := \overrightarrow{PG} = \dfrac{\overrightarrow{r}_A + \overrightarrow{r}_B + \overrightarrow{r}_C}{3}$.

(b) $\overrightarrow{r}_I := \overrightarrow{PI} = \dfrac{a\overrightarrow{r}_A + b\overrightarrow{r}_B + c\overrightarrow{r}_C}{a+b+c}$.

(c) $\overrightarrow{r}_H := \overrightarrow{PH} = \dfrac{(\tan A)\overrightarrow{r}_A + (\tan B)\overrightarrow{r}_B + (\tan C)\overrightarrow{r}_C}{\tan A + \tan B + \tan C}$.

(d) $\overrightarrow{r}_O := \overrightarrow{PO} = \dfrac{(\sin 2A)\overrightarrow{r}_A + (\sin 2B)\overrightarrow{r}_B + (\sin 2C)\overrightarrow{r}_C}{\sin 2A + \sin 2B + \sin 2C}$.



Solution. (**??**) Taking into accout the property of the centroid to be the intersection point of the medians BB' and CC', $C' \in [AC]$, $B' \in [AB]$, it follows that

$$\overrightarrow{AC'} = -\overrightarrow{BC'},\ \overrightarrow{AB'} = -\overrightarrow{CB'},$$

i.e. we may obtain $\overrightarrow{r}_G$ simply by taking $\lambda = -1 = \mu$ within the formula (2.5). By doing so we obtain

$$\overrightarrow{r}_G = \frac{\overrightarrow{r}_A + \overrightarrow{r}_B + \overrightarrow{r}_C}{3}.$$

(**??**) Recall that the incenter I is the intersection point of the angle bisectors BB' and CC'. In order to express $\overrightarrow{r}_I$ in terms $\overrightarrow{r}_A$, $\overrightarrow{r}_B$ and $\overrightarrow{r}_C$, we only need to find λ and μ with the properties $\overrightarrow{AC'} = \lambda\ \overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\ \overrightarrow{CB'}$. Since $C' \in]AC[$, $B' \in]AB[$ it follows that $\lambda, \mu < 0$. On the other hand the equalities $\overrightarrow{AC'} = \lambda\ \overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\ \overrightarrow{CB'}$ imply that

$$\| \overrightarrow{AC'} \| = |\lambda| \cdot \| \overrightarrow{BC'} \| = -\lambda \cdot \| \overrightarrow{BC'} \| \text{ and } \| \overrightarrow{AB'} \| = |\mu| \cdot \| \overrightarrow{CB'} \| = -\mu \cdot \| \overrightarrow{CB'} \|,$$

i.e.

$$\lambda = -\frac{\| \overrightarrow{AC'} \|}{\| \overrightarrow{BC'} \|} = -\frac{b}{a} \text{ and } \mu = -\frac{\| \overrightarrow{AB'} \|}{\| \overrightarrow{CB'} \|} = -\frac{c}{a}.$$

If we replace these values within the formula (2.5) we obtain

$$\overrightarrow{r}_I = \frac{a\ \overrightarrow{r}_A + b\ \overrightarrow{r}_B + c\ \overrightarrow{r}_C}{a+b+c}.$$

3. Consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\overrightarrow{r}_M = m\frac{1-n}{1-mn}\ \overrightarrow{u} + n\frac{1-m}{1-mn}\ \overrightarrow{v} \tag{2.6}$$

and

$$\overrightarrow{r}_N = m\frac{n-1}{n-m}\ \overrightarrow{u} + n\frac{m-1}{m-n}\ \overrightarrow{v}, \tag{2.7}$$

where $\{M\} = AB' \cap A'B$, $\{N\} = AA' \cap BB'$, $\overrightarrow{u} = \overrightarrow{OA}$, $\overrightarrow{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\ \overrightarrow{OA}$ and $\overrightarrow{OB'} = n\ \overrightarrow{OA'}$. In other words

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\ \overrightarrow{OA} \quad + \quad n\frac{1-m}{1-mn}\ \overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\ \overrightarrow{OA} \quad + \quad n\frac{m-1}{m-n}\ \overrightarrow{OA'}\ .$$

Solution. (2.6) The vector equations of the lines AB' and $A'B$ are:

$$AB' : \overrightarrow{r}_X = (1-\lambda)\ \overrightarrow{r}_A + \lambda\ \overrightarrow{r}_{B'},\ A'B : \overrightarrow{r}_Y = (1-\mu)\ \overrightarrow{r}_{A'} + \lambda\ \overrightarrow{r}_B,$$

or, equivalently $AB' : \overrightarrow{r}_X = (1-\lambda)\ \overrightarrow{u} + \lambda n\ \overrightarrow{v}$, $A'B : \overrightarrow{r}_Y = (1-\mu)\ \overrightarrow{v} + \lambda m\ \overrightarrow{u}$. Since $\{M\} = AB' \cap A'B$, it follows that $\overrightarrow{r}_M$ admits both a representation in the form $(1-\lambda)\ \overrightarrow{u} + \lambda n\ \overrightarrow{v}$ and a representation in the form $(1-\mu)\ \overrightarrow{v} + \mu m\ \overrightarrow{u}$, i.e.

$$\overrightarrow{r}_M = (1-\lambda)\ \overrightarrow{u} + \lambda n\ \overrightarrow{v} = (1-\mu)\ \overrightarrow{v} + \mu m\ \overrightarrow{u},\ \lambda, \mu \in \mathbb{R}.$$

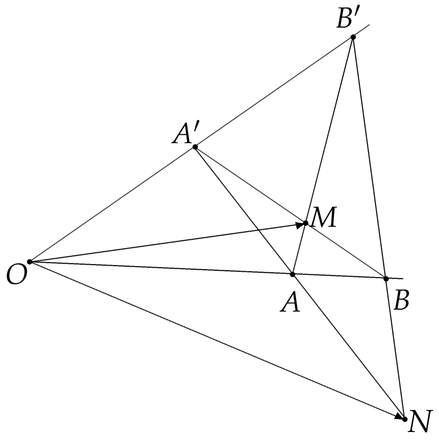


Figure 1:

The linear independence of the vectors $\vec{u}=\overrightarrow{OA}$, $\vec{v}=\overrightarrow{OA'}$ leads us to the compatible linear system

$$\begin{array}{l}1-\lambda=\mu m\\ \lambda n=1-\mu,\end{array}$$

whose solution is

$$\lambda=\frac{1-m}{1-mn}, \mu=\frac{1-n}{1-mn},$$

i.e.

$$1-\mu=n\frac{1-m}{1-mn}.$$

Thus,

$$\vec{r}_M = m\frac{1-n}{1-mn}\,\vec{u} + n\frac{1-m}{1-mn}\,\vec{v}\ .$$

(2.7) The vector equation of the straight lines AA' and BB' are:

$$AA' : \vec{r}_X = (1-\lambda)\ \vec{r}_A + \lambda\ \vec{r}_{A'},\ BB' : \vec{r}_Y = (1-\mu)\ \vec{r}_B + \lambda\ \vec{r}_{B'},$$

$AA' : \vec{r}_X = (1-\lambda)\ \vec{u} + \lambda\ \vec{v}, BB' : \vec{r}_Y = (1-\mu)m\ \vec{u} + \lambda n\ \vec{v}$. Since $\{N\} = AA' \cap BB'$, we deduce that $\vec{r}_N$ admits both a representation of the form $(1-\lambda)\ \vec{u} + \lambda\ \vec{v}$, and a representation of the form $(1-\mu)m\ \vec{u} + \mu n\ \vec{v}$, that is

$$\vec{r}_M = (1-\lambda)\ \vec{u} + \lambda\ \vec{v} = (1-\mu)m\ \vec{u} + \mu n\ \vec{v},\ \lambda, \mu \in \mathbb{R}.$$

The linear independence of the vectors $\vec{u}=\overrightarrow{OA}$, $\vec{v}=\overrightarrow{OA'}$ leads us to the compatible linear system

$$\begin{array}{l}1-\lambda=m(1-\mu)\\ \lambda=\mu n,\end{array}$$

whose solution is

$$\lambda=n\frac{1-m}{n-m},\ \mu=\frac{1-m}{n-m}$$

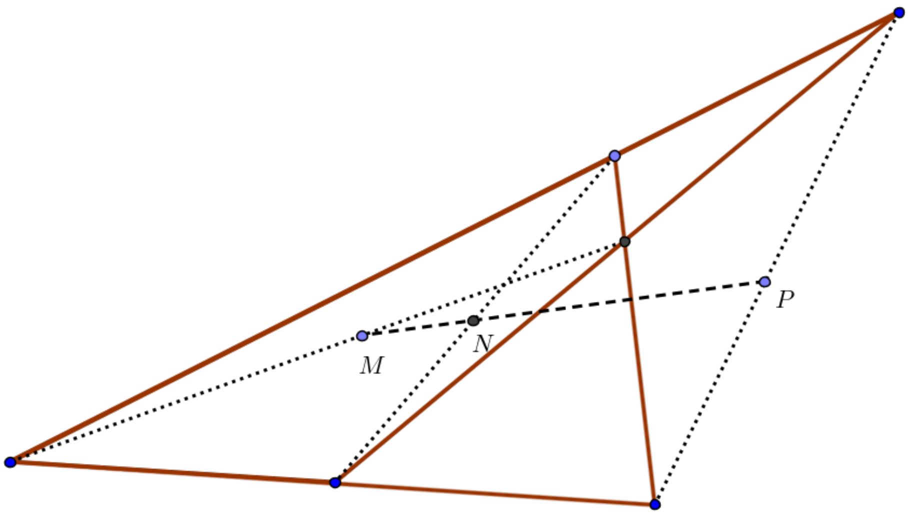
and thus

$$1-\mu=n\frac{n-1}{n-m}.$$

Therefore,

$$\vec{r}_M = m\frac{n-1}{n-m}\,\vec{u} + n\frac{m-1}{m-n}\,\vec{v}\,.$$

4. Show that the midpoints of the diagonals of a complet quadrilateral are collinear (Newton's theorem).



Solution. Consider the convex quadrilater $OABC$ with pairwise unparallel opposite sides. Let us also consider $\{D\} = OC \cap AB$ and $\{E\} = OA \cap BC$. The figure $OABCDE$ is called *complete quadrilateral*, and its diagonals are OB, AC and DE. Denote by M, N and P the midpoints of the diagonals $[OB]$, $[AC]$ and $[DE]$ and observe that

$$\vec{r}_N = \frac{1}{2}(\vec{a} + \vec{c}) \text{ şi } \vec{r}_P = \frac{1}{2}(m\,\vec{a} + n\,\vec{c}),$$

where $\vec{a} = \overrightarrow{OA}$, $\vec{c} = \overrightarrow{OC}$, $\overrightarrow{OE} = m\,\vec{a}$ and $\overrightarrow{OD} = n\,\vec{c}$. Using the relation (2.6), we conclude that

$$\vec{r}_B = m\frac{1-n}{1-mn}\,\vec{a} + n\frac{1-m}{1-mn}\,\vec{c},$$

i.e.

$$\vec{r}_M = \frac{1}{2}\left(m\frac{1-n}{1-mn}\,\vec{a} + n\frac{1-m}{1-mn}\,\vec{c}\right).$$

Therefore

$$\begin{aligned}\overrightarrow{MP} &= \vec{r}_P - \vec{r}_M = \frac{mn}{2(mn-1)}\left((m-1)\,\vec{a} + (n-1)\,\vec{c}\right)\\ \overrightarrow{NP} &= \vec{r}_P - \vec{r}_N = \frac{1}{2}\left((m-1)\,\vec{a} + (n-1)\,\vec{c}\right),\end{aligned}$$

which implies the equality $\overrightarrow{MP} = \frac{mn}{mn-1}\,\overrightarrow{NP}$ and shows the colinearity of M, N şi P.

3 Week 3: Cartezian equations of lines and planes

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

3.1 Brief theoretical background

3.1.1 The cartesian equations of the straight lines

Let Δ be a straight line passing through the point $A_0(x_0, y_0, z_0)$ which is parallel to the vector $\overrightarrow{d}\ (p, q, r)$. Its vector equation is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda \overrightarrow{d}\ . \tag{3.1}$$

Denoting by x, y, z the coordinates of the generic point M of the straight line Δ, its vector equation (3.1) is equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases},\ \lambda \in \mathbb{R} \tag{3.2}$$

The relations (3.2) are being called the *parametric equations* of the straight line Δ and they are equivalent to the following relations

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r} \tag{3.3}$$

If $r = 0$, for instance, the canonical equations of the straight line Δ are

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} \ \wedge\ z = z_0.$$

If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are different pointsof the straight line Δ, then $\overrightarrow{AB}\ (x_B - x_A, y_B - y_A, z_B - z_A)$ is a director vector of Δ, its canonical equations having, in this case, the form

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}. \tag{3.4}$$

3.1.2 Te cartesian equations of the planes

Let $A_0(x_0, y_0, z_0) \in \mathcal{P}$ and $\overrightarrow{d}_1\ (p_1, q_1, r_1), \overrightarrow{d}_2\ (p_2, q_2, r_2) \in \mathcal{V}$ be linearly independent vectors, that is

$$\text{rang} \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix} = 2.$$

The vector equation of the plane π passing through A_0 which is parallel to the vectors $\overrightarrow{d}_1\ (p_1, q_1, r_1), \overrightarrow{d}_2\ (p_2, q_2, r_2)$ is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda_1 \overrightarrow{d}_1 + \lambda_2 \overrightarrow{d}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.5}$$

If we denote by x, y, z the coordinates of the generic point M of the plane π, then the vector equation (3.5) is the equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda_1 p_1 + \lambda_2 p_2 \\ y = y_0 + \lambda_1 q_1 + \lambda_2 q_2 \\ z = z_0 + \lambda_1 r_1 + \lambda_2 r_2 \end{cases},\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.6}$$

The relations (3.6) reprezent a characterization of the points of the points of the plane π called the *parametric equations* of the plane π. More precisely, the compatibility of the linear system (3.6) with the unknowns λ_1, λ_2 is a necessary and sufficient condition for the point $M(x, y, z)$ to be contained within the plane π. On the other hand the compatibility of the linear system (3.6) is equivalent to the relations

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ p_1 & q_1 & z_1 \\ p_2 & q_2 & z_2 \end{vmatrix} = 0. \tag{3.7}$$

and express the fact that the rank of the matrix of the system is equal to the rank of the extended matrix of the system. The condition (3.7) is a characterization of the points of the plane π expressed in terms of the cartesian coordinates of the generic point M and is called the *cartesian equation* of the plane π.

If $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ are noncollinear points, then the plane (ABC) determined by the three points can be viewed as the plane passing through the point A which is parallel to the vectors $\vec{d}_1 = \overrightarrow{AB}$, $\vec{d}_2 = \overrightarrow{AC}$. The coordinates of the vectors $\vec{d}_1$ şi $\vec{d}_2$ are

$$(x_B - x_A, y_B - y_A, z_B - z_A) \text{ and } (x_C - x_A, y_C - y_A, z_C - z_A) \text{ respectively.}$$

Thus, the equation of the plane (ABC) is

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0, \tag{3.8}$$

or, echivalently

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0. \tag{3.9}$$

On can put the equation (3.7) in the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \text{ or} \tag{3.10}$$

$$Ax + By + Cz + D = 0, \tag{3.11}$$

where the coefficients A, B, C satisfy the relation $A^2 + B^2 + C^2 > 0$. It is also easy to show that every equation of the form (3.11) represents the equation of a plane. Indeed, if $A \neq 0$, then the equation (3.11) is equivalent to

$$\begin{vmatrix} x + \frac{D}{A} & y & z \\ B & -A & 0 \\ C & 0 & -A \end{vmatrix} = 0.$$

We observe that one can put the equation (3.10) in the form

$$AX + BY + CZ = 0 \tag{3.12}$$

where $X = x - x_0$, $Y = y - y_0$, $Z = z - z_0$ are the coordinates of the vector $\overrightarrow{A_0M}$.

3.1.3 Analytic conditions of parallelism

The equation $AX + BY + CZ = 0$ is a necessary and sufficient condition for the vector $\overrightarrow{A_0M}\ (X, Y, Z)$ to be contained within the direction of the plane

$$\pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Thus the *equation of the director subspace* $\overrightarrow{\pi} = \{\overrightarrow{A_0M} \mid M \in \pi\}$ is $AX + BY + CZ = 0$.

Proposition 3.1. *The straight line*

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

is parallel to the plane $\pi : Ax + By + Cz + D = 0$ *iff*

$$Ap + Bq + Cr = 0 \tag{3.13}$$

Proposition 3.2. *Consider the planes*

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0,\ (\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0.$$

Then $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) \in \{1, 2\}$ *and the following statemenets are equivalent*

1. $\pi_1 \| \pi_2$.
2. $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) = 2$, *i.e.* $\overrightarrow{\pi}_1 = \overrightarrow{\pi}_2$.
3. *rang* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1$.
4. *The vectors* $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly dependent.*

Corollary 3.3. *Consider the planes*

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0,\ (\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0.$$

The following statements are equivalent

1. $\pi_1 \nparallel \pi_2$.
2. $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) = 1$.
3. *rang* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$.
4. *The vectors* $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly independent.*

By using Proposition the characterization of parallelism between a line and a plane, we shall find a necessary and sufficient condition for a vector to be contained within the direction of a straight line which is given as the intersection of two planes.

Consider the planes $(\pi_1)\ A_1x+B_1y+C_1z+D_1=0$, $(\pi_2)\ A_2x+B_2y+C_2z+D_2=0$ such that

$$\text{rang}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}=2,$$

alongside their intersectio9n straight line $\Delta=\pi_1\cap\pi_2$ of equations

$$(\Delta)\ \begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_1=0. \end{cases}$$

Thus, $\overrightarrow{\Delta}=\overrightarrow{\pi}_1\cap\overrightarrow{\pi}_2$ and therefore, by means of some previous Proposition, it follows that the equations of $\overrightarrow{\Delta}$ are

$$(\overrightarrow{\Delta})\ \begin{cases} A_1X+B_1Y+C_1Z=0 \\ A_2X+B_2Y+C_2Z=0. \end{cases} \tag{3.14}$$

By solving the system (3.14) one can therefore deduce that $\overrightarrow{d}\ (p,q,r)\in\overrightarrow{\Delta}\Leftrightarrow\ \exists\lambda\in\mathbb{R}$ such that

$$(p,q,r)=\lambda\left(\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix},\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix},\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}\right). \tag{3.15}$$

The relation is usually (3.15) written in the form

$$\frac{p}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}}=\frac{q}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}}=\frac{r}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \tag{3.16}$$

Let us mention that the chosen values for (p,q,r) are usually precisely

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix},\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}\ \text{şi}\ \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}.$$

3.2 Problems

1. Write the equation of the line which passes through $A(1,-2,6)$ and is parallel to
   (a) The x-axis;
   (b) The line $(d_1)\ \dfrac{x-1}{2}=\dfrac{y+5}{-3}=\dfrac{z-1}{4}$.
   (c) The vector $\overrightarrow{v}\ (1,0,2)$.
2. Write the equation of the plane which contains the line
   $$(d_1)\ \frac{x-3}{2}=\frac{y+4}{1}=\frac{z-2}{-3}$$
   and is parallel to the line
   $$(d_2)\ \frac{x+5}{2}=\frac{y-2}{2}=\frac{z-1}{2}.$$

3. Consider the points $A(\alpha, 0, 0)$, $B(0, \beta, 0)$ and $C(0, 0, \gamma)$ such that
$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{1}{a} \text{ where } a \text{ is a constatnt.}$$
Show that the plane (A, B, C) passes through a fixed point.

4. Write the equation of the line which passes through tht point $M(1, 0, 7)$, is parallel to the plane (π) $3x - y + 2z - 15 = 0$ and intersects the line
$$(d)\ \frac{x-1}{4}=\frac{y-3}{2}=\frac{z}{1}.$$

5. Write the equation of the plane which passes through $M_0(1, -2, 3)$ and is parallel to the vectors $\vec{v}_1\ (1, -1, 0)$ and $\vec{v}_2\ (-3, 2, 4)$.

6. Write the equation of the plane which passes through $M_0(1, -2, 3)$ and cuts the positive coordinate axes through congruent segments.

7. Write the equation of the plane which passes through $A(1, 2, 1)$ and is parallel to the straight lines
$$(d_1) \begin{cases} x + 2y - z + 1 = 0 \\ x - y + z - 1 = 0 \end{cases} \quad (d_2) \begin{cases} 2x - y + z = 0 \\ x - y + z = 0. \end{cases}$$

4 Week 4: Projections and symmetries. Pencils of planes

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

4.1 Brief theoretical background. Projections and symmetries

4.1.1 The intersection point of a straight line and a plane

Consider a straight line
$$d : \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$$
and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.
$$Ap + Bq + Cr \neq 0.$$

The parametric equations of d are
$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases},\ t \in \mathbb{R}. \tag{4.1}$$

The value of $t \in \mathbb{R}$ for which this line (4.1) punctures the plane π can be determined by imposing the condition on the point of coordinates
$$(x_0 + pt, y_0 + qt, z_0 + rt)$$

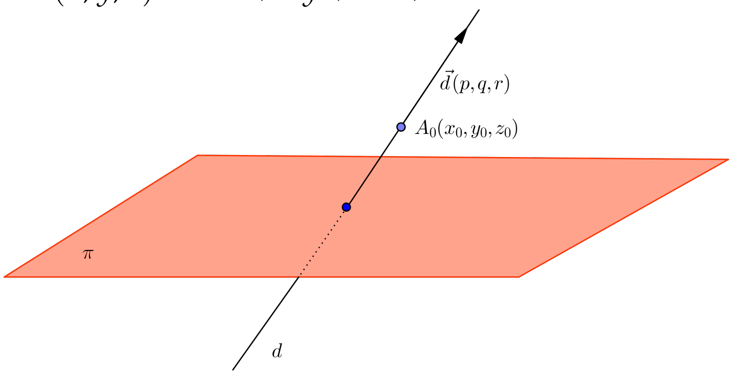
to verify the equation of the plane, namely

$$A(x_0 + pt) + B(y_0 + qt) + C(z_0 + Ct) + D = 0.$$

Thus

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr} = -\frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr},$$

where $F(x, y, z) = Ax + By + Cz + D$.



The coordinates of the intersection point $d \cap \pi$ are

$$\begin{cases} x_0 - p\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ y_0 - q\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ z_0 - r\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr}. \end{cases} \tag{4.2}$$

4.1.2 The projection on a plane parallel to a given line

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

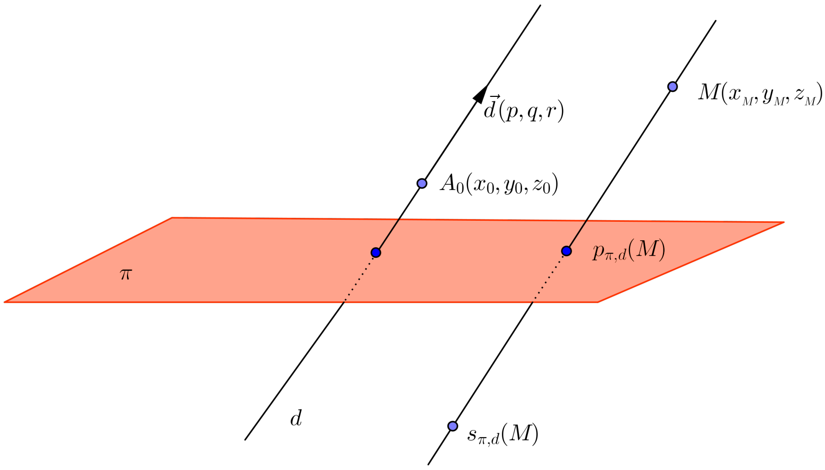
and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{\pi,d} : \mathcal{P} \longrightarrow \pi$ of $\mathcal{P}$ on π parallel to d, whose value $p_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the intersection point between π and the line through M which is parallel to d. Due to relations (4.2), the coordinates of $p_{\pi,d}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_M - p\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ y_M - q\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ z_M - r\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr}, \end{cases} \tag{4.3}$$

where $F(x, y, z) = Ax + By + Cz + D$.



Consequently, the position vector of $p_{\pi,d}(M)$ is

$$\overrightarrow{Op_{\pi,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{Ap+Bq+Cr}\,\vec{d}\,. \tag{4.4}$$

4.1.3 The symmetry with respect to a plane parallel to a line

We call the function $s_{\pi,d} : \mathcal{P} \longrightarrow \mathcal{P}$, whose value $s_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $p_{\pi,d}(M)$ *the symmetry of* $\mathcal{P}$ *with respect to* π *parallel to* d. The direction of d is equally called the *direction* of the symmetry and π is called the *axis* of the symmetry. For the position vector of $s_{\pi,d}(M)$ we have

$$\overrightarrow{Op_{\pi,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\pi,d}(M)}}{2}, \text{ i.e.} \tag{4.5}$$

$$\overrightarrow{Os_{\pi,d}(M)} = 2\,\overrightarrow{Op_{\pi,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{Ap+Bq+Cr}\,\vec{d}\,. \tag{4.6}$$

4.1.4 The projection on a straight line parallel to a given plane

Consider a straight line

$$d : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{d,\pi} : \mathcal{P} \longrightarrow d$ of $\mathcal{P}$ on d, whose value $p_{d,\pi}(M)$ at $M \in \mathcal{P}$ is the intersection point between d and the plane through M which is parallel to π. Due to relations (4.2), the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_0 - p\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr} \\ y_0 - q\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr} \\ z_0 - r\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr}, \end{cases} \tag{4.7}$$

where $G_M(x,y,z) = A(x-x_M) + B(y-y_M) + C(z-z_M)$. Consequently, the position vector of $p_{d,\pi}(M)$ is

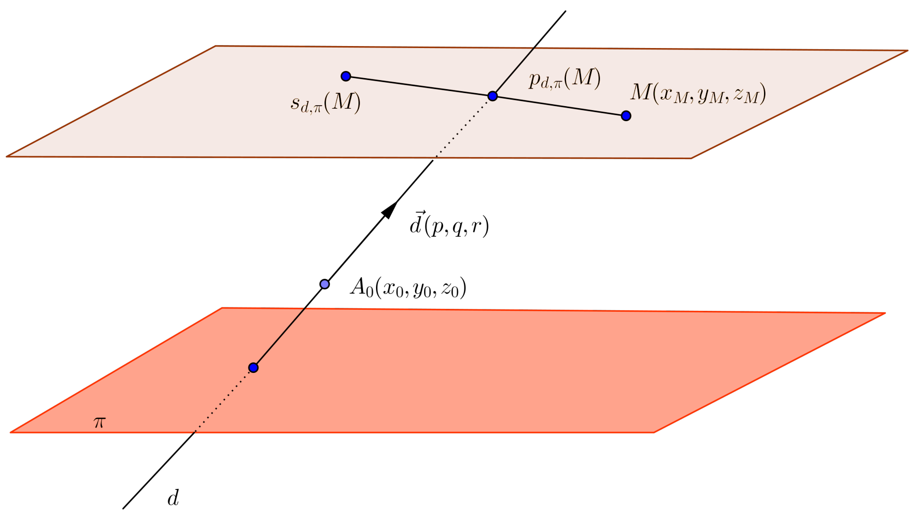
$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} - \frac{G_M(A_0)}{Ap+Bq+Cr}\vec{d}, \text{ where } A_0(x_0,y_0,z_0). \tag{4.8}$$

Note that $G_M(A_0) = A(x_0-x_M) + B(y_0-y_M) + C(z_0-z_M) = F(A_0) - F(M)$, where $F(x,y,z) = Ax+By+Cz+D$. Consequently the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_0 + p\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ y_0 + q\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ z_0 + r\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr}, \end{cases} \tag{4.9}$$

and the position vector of $p_{d,\pi}(M)$ is

$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} + \frac{F(M)-F(A_0)}{Ap+Bq+Cr}\vec{d}, \text{ where } A_0(x_0,y_0,z_0). \tag{4.10}$$



4.1.5 The symmetry with respect to a line parallel to a plane

We call the function $s_{d,\pi} : \mathcal{P} \longrightarrow \mathcal{P}$, whose value $s_{d,\pi}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $p_{d,\pi}(M)$, *the symmetry of $\mathcal{P}$ with respect to d parallel to π*. The direction of π is equally called the *direction* of the symmetry and d is called the *axis* of the symmetry. For the position vector of $s_{d,\pi}(M)$ we have

$$\overrightarrow{Op_{d,\pi}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{d,\pi}(M)}}{2}, \text{ i.e.} \tag{4.11}$$

$$\begin{aligned}\overrightarrow{Os_{d,\pi}(M)} &= 2\,\overrightarrow{Op_{d,\pi}(M)} - \overrightarrow{OM}\\ &= 2\,\overrightarrow{OA_0} - \overrightarrow{OM} + 2\frac{F(M)-F(A_0)}{Ap+Bq+Cr}\overrightarrow{d}\,.\end{aligned}\tag{4.12}$$

4.2 Pencils of planes

Definition 4.1. *The collection of all planes containing a given straight line*

$$(\Delta)\ \begin{cases} A_1x+B_1y+C_1z+D_1=0\\ A_2x+B_2y+C_2z+D_2=0\end{cases}$$

is called the pencil of planes *through* Δ.

Proposition 4.2. *The plane* π *belongs to the pencil of planes through the straight line* Δ *if and only if there exists* $\lambda,\mu\in\mathbb{R}$ *such that the equation of the plane* π *is*

$$\lambda(A_1x+B_1y+C_1z+D_1)+\mu(A_2x+B_2y+C_2z+D_2)=0.\tag{4.13}$$

Remark 4.3. *The family of planes*

$$A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0,$$

where λ *covers the whole real line* $\mathbb{R}$, *is the so called* reduced pencil of planes *through* Δ *and it consists in all planes through* Δ *except the plane of equation* $A_2x+B_2y+C_2z+D_2=0$.

4.3 Problems

1. Write the equations of the projection of the line

$$(d)\ \begin{cases} 2x - y + z - 1 = 0\\ x + y - z + 1 = 0\end{cases}$$

on the plane $\pi : x+2y-z=0$ paralel to the direction $\overrightarrow{u}\,(1,1,-2)$. Write the equations of the symmetry of the line d with respect to the plane π paralel to the direction $\overrightarrow{u}\,(1,1,-2)$.

2. Write the equation of the plane determined by the line

$$(d)\ \begin{cases} x - 2y + 3z = 0\\ 2x + z - 3 = 0\end{cases}$$

and the point $A(-1,2,6)$.

5 Week 5: Products of vectors

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

5.1 Brief theoretical background. Products of vectors

5.1.1 The dot product

Definition 5.1. The real number

$$\overrightarrow{a} \cdot \overrightarrow{b} = \begin{cases} 0 \text{ if } \overrightarrow{a} = 0 \text{ or } \overrightarrow{b} = 0 \\ || \overrightarrow{a} || \cdot || \overrightarrow{b} || cos(\widehat{\overrightarrow{a}, \overrightarrow{b}}) \text{ if } \overrightarrow{a} \neq 0 \text{ and } \overrightarrow{b} \neq 0 \end{cases} \tag{5.1}$$

is called the *dot product* of the vectors $\overrightarrow{a}, \overrightarrow{b}$.

Remark 5.2. *1.* $\overrightarrow{a} \perp \overrightarrow{b} \Leftrightarrow \overrightarrow{a} \cdot \overrightarrow{b} = 0.$

2. $\overrightarrow{a} \cdot \overrightarrow{a} = || \overrightarrow{a} || \cdot || \overrightarrow{a} || \cos 0 = || \overrightarrow{a} ||^2.$

Proposition 5.3. *The dot product has the following properties:*

1. $\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{b} \cdot \overrightarrow{a}, \ \forall \ \overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}.$
2. $\overrightarrow{a} \cdot (\lambda \overrightarrow{b}) = \lambda(\overrightarrow{a} \cdot \overrightarrow{b}), \ \forall \lambda \in \mathbb{R}, \overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}.$
3. $\overrightarrow{a} \cdot (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{a} \cdot \overrightarrow{c}, \ \forall \ \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in \mathcal{V}.$
4. $\overrightarrow{a} \cdot \overrightarrow{a} \geq 0, \ \forall \ \overrightarrow{a} \in \mathcal{V}.$
5. $\overrightarrow{a} \cdot \overrightarrow{a} = 0 \Leftrightarrow \overrightarrow{a} = \overrightarrow{0}.$

Definition 5.4. A basis of the vector space $\mathcal{V}$ is said to be *orthonormal*, if $|| \overrightarrow{i} || = || \overrightarrow{j} || = || \overrightarrow{k} || = 1$, $\overrightarrow{i} \perp \overrightarrow{j}, \overrightarrow{j} \perp \overrightarrow{k}, \overrightarrow{k} \perp \overrightarrow{i}$ $(\overrightarrow{i} \cdot \overrightarrow{i} = \overrightarrow{j} \cdot \overrightarrow{j} = \overrightarrow{k} \cdot \overrightarrow{k} = 1, \ \overrightarrow{i} \cdot \overrightarrow{j} = \overrightarrow{j} \cdot \overrightarrow{k} = \overrightarrow{k} \cdot \overrightarrow{i} = 0)$. A cartesian reference system $R = (O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ is said to be *orthonormal* if the basis $[\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}]$ is orthonormal.

Proposition 5.5. *Let* $[\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}]$ *be an orthonormal basis and* $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$. *If* $\overrightarrow{a} = a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{k}$, $\overrightarrow{b} = b_1 \overrightarrow{i} + b_2 \overrightarrow{j} + b_3 \overrightarrow{k}$, *then*

$$\overrightarrow{a} \cdot \overrightarrow{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{5.2}$$

Remark 5.6 5.6. *Let* $[\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}]$ *be an orthonormal basis and* $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$. *If* $\overrightarrow{a} = a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{k}$, $\overrightarrow{b} = b_1 \overrightarrow{i} + b_2 \overrightarrow{j} + b_3 \overrightarrow{k}$, *then*

1. $\overrightarrow{a} \cdot \overrightarrow{a} = a_1^2 + a_2^2 + a_3^2$ *and we conclude that* $|| \overrightarrow{a} || = \sqrt{\overrightarrow{a} \cdot \overrightarrow{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$
2. $$\begin{aligned} \cos(\widehat{\overrightarrow{a}, \overrightarrow{b}}) &= \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|| \overrightarrow{a} || \cdot || \overrightarrow{b} ||} \\ &= \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}. \end{aligned} \tag{5.3}$$

In particular

$$\cos(\widehat{\overrightarrow{a},\overrightarrow{i}}) = \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}};$$
$$\cos(\widehat{\overrightarrow{a},\overrightarrow{j}}) = \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}};$$
$$\cos(\widehat{\overrightarrow{a},\overrightarrow{k}}) = \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}.$$

3. $\overrightarrow{a} \perp \overrightarrow{b} \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

5.1.2 Applications of the dot product

• **The distance between two points.** Consider two points $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B) \in \mathcal{P}$. The norm of the vector $\overrightarrow{AB}\,(x_B - x_A, y_B - y_A, z_B - z_A)$ is

$$\| \overrightarrow{AB} \| = \sqrt{(x_B-x_A)^2 + (y_B-y_A)^2 + (z_B-z_A)^2}.$$

• **The normal vector of a plane.** Consider the plane $\pi : Ax + By + Cz + D = 0$ and the point $P(x_0, y_0, z_0) \in \pi$. The equation of π becomes

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0. \tag{5.4}$$

If $M(x,y,z) \in \pi$, the coordinates of $\overrightarrow{PM}$ are $(x-x_0, y-y_0, z-z_0)$ and the equation (5.4) tells us that $\overrightarrow{n} \cdot \overrightarrow{PM} = 0$, for every $M \in \pi$, that is $\overrightarrow{n} \perp \overrightarrow{PM} = 0$, for every $M \in \pi$, which is equivalent to $\overrightarrow{n} \perp \overrightarrow{\pi}$, where $\overrightarrow{n}\,(A,B,C)$. This is the reason to call $\overrightarrow{n}\,(A,B,C)$ *the normal vector* of the plane π.

• **The distance from a point to a plane.** Consider the plane $\pi : Ax + By + Cz + D = 0$, a point $P(x_P, y_P, z_P) \in \mathcal{P}$ and M the orthogonal projection of P on π. The real number δ given by $\overrightarrow{MP} = \delta \cdot \overrightarrow{n}_0$ is called the *oriented distance* from P to the plane π, where $\overrightarrow{n}_0 = \frac{1}{\|\overrightarrow{n}\|}\overrightarrow{n}$ is the versor of the normal vector $\overrightarrow{n}\,(A,B,C)$. Since $\overrightarrow{MP} = \delta \cdot \overrightarrow{n}_0$, it follows that $\delta(P,M) = \|\overrightarrow{MP}\| = |\delta|$, where $\delta(P,M)$ stands for the distance from P to π. We shall show that

$$\delta = \frac{Ax_P + By_P + Cz_P + D}{\sqrt{A^2+B^2+C^2}}.$$

Indeed, since $\overrightarrow{MP} = \delta \cdot \overrightarrow{n}_0$, we get successively:

$$\begin{aligned}
\delta &= \overrightarrow{n}_0 \cdot \overrightarrow{MP} = \left(\frac{1}{\|\overrightarrow{n}\|}\overrightarrow{n}\right) \cdot \overrightarrow{MP} = \frac{\overrightarrow{n} \cdot \overrightarrow{MP}}{\|\overrightarrow{n}\|} \\
&= \frac{A(x_P - x_M) + B(y_P - y_M) + C(z_P - z_M)}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{Ax_P + By_P + Cz_P - (Ax_M + By_M + Cz_M)}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{Ax_P + By_P + Cz_P + D}{\sqrt{A^2+B^2+C^2}}.
\end{aligned}$$

Consequently

$$\delta(P, M) = || \overrightarrow{MP} || = |\delta| = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

5.1.3 The vector product

Definition 5.7. The *vector product* or the *cross product* of the vectors $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ is a vector, denoted by $\overrightarrow{a} \times \overrightarrow{b}$, which is defined to be zero if $\overrightarrow{a}, \overrightarrow{b}$ are linearly dependent (collinear), and if $\overrightarrow{a}, \overrightarrow{b}$ are linearly independent (noncollinear), then it is defined by the following data:

1. $\overrightarrow{a} \times \overrightarrow{b}$ is a vector orthogonal on the two-dimensional subspace $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$ of $\mathcal{V}$;
2. if $\overrightarrow{a} = \overrightarrow{OA}$, $\overrightarrow{b} = \overrightarrow{OB}$, then the sense of $\overrightarrow{a} \times \overrightarrow{b}$ is the one in which a right-handed screw, placed along the line passing through O orthogonal to the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, advances when it is being rotated simultaneously with the vector $\overrightarrow{a}$ from $\overrightarrow{a}$ towards $\overrightarrow{b}$ within the vector subspace $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$ and the support half line of $\overrightarrow{a}$ sweeps the interior of the angle $\widehat{AOB}$ (Screw rule).
3. the *norm* (*magnitude* or *length*) of $\overrightarrow{a} \times \overrightarrow{b}$ is defined by

$$|| \overrightarrow{a} \times \overrightarrow{b} || = || \overrightarrow{a} || \cdot || \overrightarrow{b} || \sin(\widehat{\overrightarrow{a}, \overrightarrow{b}}).$$

Remarks 5.8. *1.* The *norm* (*magnitude* or *length*) of the vector $\overrightarrow{a} \times \overrightarrow{b}$ is actually the area of the parallelogram constructed on the vectors $\overrightarrow{a}, \overrightarrow{b}$.

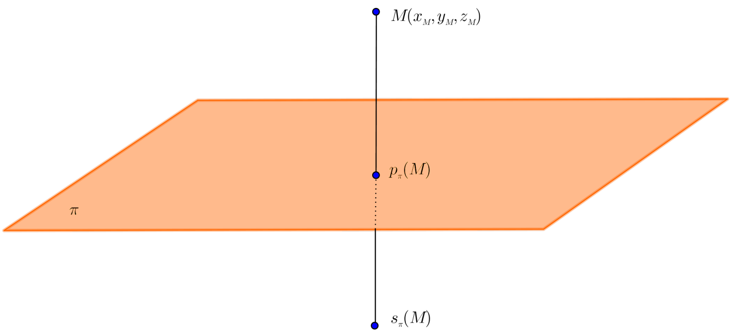
2. The vectors $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ are linearly dependent (collinear) if and only if $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$.

Proposition 5.9. *The vector product has the following properties:*

1. $\overrightarrow{a} \times \overrightarrow{b} = - \overrightarrow{b} \times \overrightarrow{a}, \forall\ \overrightarrow{a}, \overrightarrow{b} \in \mathcal{V};$
2. $(\lambda \overrightarrow{a}) \times \overrightarrow{b} = \overrightarrow{a} \times (\lambda \overrightarrow{b}) = \lambda(\overrightarrow{a} \times \overrightarrow{b}), \forall \lambda \in \mathbb{R}, \overrightarrow{a}, \overrightarrow{b} \in \mathcal{V};$
3. $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{c}, \forall\ \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in \mathcal{V}.$

5.1.4 Appendix: Orthogonal projections and orthogonal symmetries

• **The orthogonal projection on a plane π.** For a given plane $\pi : Ax + By + Cz + D = 0$ and a given point $M(x_M, y_M, z_M)$, we shall determine the coordinates of its orthogonal projection on the plane π, as well as the coordinates of its (orthogonal) symmetric with respect to π. The equation of the plane and the coordinates of M are considered with respect to some cartezian coordinate system $R = (O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$. In this respect we consider the orthogonal line on π which passes through M.



Its parametric equations are

$$\begin{cases} x = x_M + At \\ y = y_M + Bt \\ z = z_M + Ct \end{cases}, \ t \in \mathbb{R}. \tag{5.5}$$

The orthogonal projection $p_\pi(M)$ of M on the plane π is at its intersection point with the orthogonal line (5.5) and the value of $t \in \mathbb{R}$ for which this orthogonal line (5.5) puncture the plane π can be determined by imposing the condition on the point of coordinates $(x_M + At, y_M + Bt, z_M + Ct)$ to verify the equation of the plane, namely $A(x_M + At) + B(y_M + Bt) + C(z_M + Ct) + D = 0$. Thus

$$t = -\frac{Ax_M + By_M + Cz_M + D}{A^2 + B^2 + C^2} = -\frac{F(x_M, y_M, z_M)}{\| \overrightarrow{n}_\pi \|^2},$$

where $F(x, y, z) = Ax + By + Cz + D$ şi $\overrightarrow{n}_\pi = A \overrightarrow{i} + B \overrightarrow{j} + C \overrightarrow{k}$ is the normal vector of the plasne π.

• **The orthogonal projection on the plane π.**

The coordinates of the orthogonal projection $p_\pi(M)$ of M on th eplane π are

$$\begin{cases} x_M - A\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ y_M - B\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ z_M - C\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2}. \end{cases}$$

Therefore, the position vector of the orthogonal projection $p_\pi(M)$ is

$$\overrightarrow{Op_\pi(M)} = \overrightarrow{OM} - \frac{F(M)}{\| \overrightarrow{n}_\pi \|^2} \overrightarrow{n}_\pi \ . \tag{5.6}$$

• **The orthogonal symmetry with respect to the plane π.** In order to find the position vector of the orthogonally symmetric point $s_\pi(M)$ of M w.r.t. π, we use the relation

$$\overrightarrow{Op_\pi(M)} = \frac{1}{2}\left(\overrightarrow{OM} + \overrightarrow{Os_\pi(M)}\right),$$

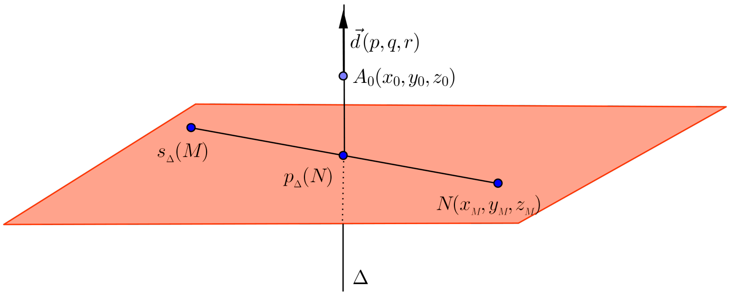
namely

$$\overrightarrow{Os_\pi(M)} = 2\,\overrightarrow{Op_\pi(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{\| \overrightarrow{n}_\pi \|^2} \overrightarrow{n}_\pi \ .$$

• **The orthogonal projection on a line Δ.** For a given line

$$\Delta : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

and a point $N(x_N, y_N, z_N)$, we shall find the coordinates of its orthogonal projection on the line Δ, as well as the coordinates of the orthogonally symmetric point M with respect to Δ. The equations of the line and the coordinates of the point N are considered with respect to an orthonormal coordinate system $R = (O, \vec{i}, \vec{j}, \vec{k})$. In this respect we consider the plane $p(x-x_N) + q(y-y_N) + r(z-z_N) = 0$ orthogonal on the line Δ which passes through the point N.



The parametric equations of the line Δ are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \ t \in \mathbb{R}. \tag{5.7}$$

The orthogonal projection of N on the line Δ is at its intersection point and the plane $p(x-x_N) + q(y-y_N) + r(z-z_N) = 0$, and the value of $t \in \mathbb{R}$ for which the line Δ puncture the orthogonal plane $p(x-x_N) + q(y-y_N) + r(z-z_N) = 0$ can be found by imposing the condition on the point of coordinate $(x_0 + pt, y_0 + qt, z_0 + rt)$ to verify the equation of the plane, namely $p(x_0 + pt - x_N) + q(y_0 + qt - y_N) + r(z_0 + rt - z_N) = 0$. Thus

$$t = -\frac{p(x_0 - x_N) + q(y_0 - y_N) + r(z_0 - z_N)}{p^2 + q^2 + r^2} = -\frac{G(x_0, y_0, z_0)}{\| \vec{d}_\Delta \|^2},$$

where $G(x,y,z) = p(x-x_N) + q(y-y_N) + r(z-z_N)$ and $\vec{d}_\pi = p\ \vec{i}\ + q\ \vec{j}\ + r\ \vec{k}$ is the director vectoir of the line Δ. Ths coordinates of the orthogonal projection $p_\Delta(N)$ of N on the line Δ are therefore

$$\begin{cases} x_0 - p\dfrac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ y_0 - q\dfrac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ z_0 - r\dfrac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2}. \end{cases}$$

Thus, the position vector of the orthogonal projection $p_\Delta(N)$ is

$$\overrightarrow{Op_\Delta(N)} = \overrightarrow{OA_0} - \frac{G(A_0)}{\| \vec{d}_\Delta \|^2} \vec{d}_\Delta, \tag{5.8}$$

where $A_0(x_0, y_0, z_0) \in \Delta$.

• **The orthogonal symmetry with respect to a line Δ.** In order to find the position vector of the orthogonally symmetric point $s_\Delta(N)$ of N with respect to the line Δ we use the relation

$$\overrightarrow{Op_\Delta(N)} = \frac{1}{2}\left(\overrightarrow{ON} + \overrightarrow{Os_\Delta(N)}\right)$$

i.e.

$$\overrightarrow{Os_{\Delta}(N)} = 2\,\overrightarrow{Op_{\Delta}(N)} - \overrightarrow{ON} = 2\,\overrightarrow{OA_0} - 2\frac{G(A_0)}{\|\overrightarrow{d}_{\Delta}\|^2}\,\overrightarrow{d}_{\Delta} - \overrightarrow{ON}.$$

5.2 Problems

1. Consider the triangle ABC and the midpoint A' of the side $[BC]$. Show that

$$4\,\overrightarrow{AA'}^{\,2} - \overrightarrow{BC}^{\,2} = 4\,\overrightarrow{AB} \cdot \overrightarrow{AC}.$$

2. Consider the rectangle $ABCD$ and the arbitrary point M witin the space. Show that

   (a) $\overrightarrow{MA} \cdot \overrightarrow{MC} = \overrightarrow{MB} \cdot \overrightarrow{MD}$.

   (b) $\overrightarrow{MA}^{\,2} + \overrightarrow{MC}^{\,2} = \overrightarrow{MB}^{\,2} + \overrightarrow{MD}^{\,2}$.

3. Compute the distance from the point $A(3,1,-1)$ to the plane

$$\pi : 22x + 4y - 20z - 45 = 0.$$

4. Find the angle between:

   (a) the straight lines

$$(d_1)\begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases} \qquad (d_2)\begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0. \end{cases}$$

   (b) the planes

$$\pi_1 :\ x + 3y + 2z + 1 = 0 \text{ and } \pi_2 : 3x + 2y - z = 6.$$

   (c) the plane xOy and the straight line M_1M_2, where $M_1(1,2,3)$ and $M_2(-2,1,4)$.

5. Consider the noncoplanar vectors $\overrightarrow{OA}\ (1,-1,-2)$, $\overrightarrow{OB}\ (1,0,-1)$, $\overrightarrow{OC}\ (2,2,-1)$ related to an orthonormal basis $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$. Let H be the foot of the perpendicular through O on the plane ABC. Determine the components of the vectors $\overrightarrow{OH}$.

6. Find the point on the z-axis which is equidistant with respect to the planes

$$\pi_1 :\ 12x + 9y - 20z - 19 = 0 \text{ and } \pi_2 :\ 16x + 12y + 15z - 9 = 0.$$

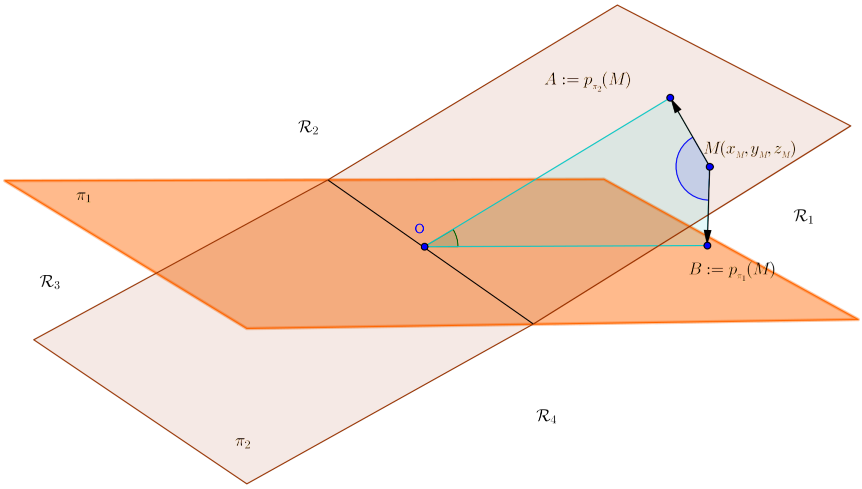
7. Consider two planes

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0$$
$$(\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0$$

   which are not parallel and not perpendicular as well. The two planes π_1, π_2 devide the space into four regions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$, two of which, say $\mathcal{R}_1$ and $\mathcal{R}_3$, correspond to the acute dihedral angle of the two planes. Show that $M(x,y,z) \in \mathcal{R}_1 \cup \mathcal{R}_3$, if and only if

$$F_1(x,y,z) \cdot F_2(x,y,z)(A_1A_2 + B_1B_2 + C_1C_2) < 0,$$

where $F_1(x,y,z) = A_1x + B_1y + C_1z + D_1$ and $F_2(x,y,z) = A_2x + B_2y + C_2z + D_2$.



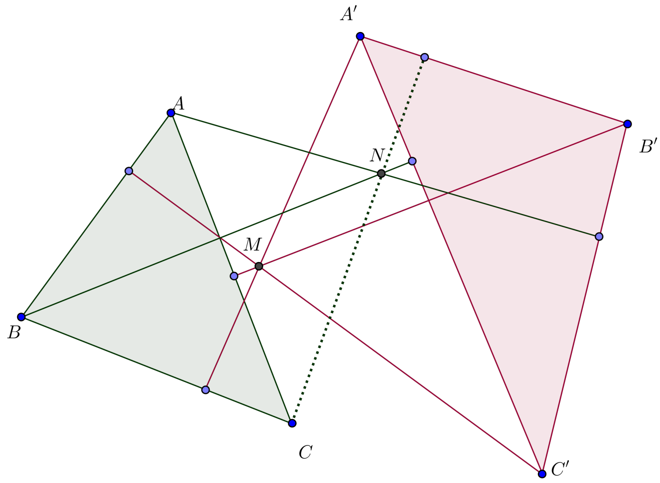
8. Consider the planes $(\pi_1)\ 2x + y - 3z - 5 = 0$, $(\pi_2)\ x + 3y + 2z + 1 = 0$. Find the equations of the bisector planes of the dihedral angles formed by the planes π_1 and π_2 and select the one contained into the acute regions of the dihedral angles formed by the two planes.

9. Let a, b be two real numbers such that $a^2 \neq b^2$. Consider the planes:
$$(\alpha_1) ax + by - (a+b)z = 0$$
$$(\alpha_2) ax - by - (a-b)z = 0$$
and the quadric $(\mathcal{C}) : a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 - 2a^2xz + 2b^2yz - a^2b^2 = 0$. If $a^2 < b^2$, show that the quadric $\mathcal{C}$ is contained in the acute regions of the dihedral angles formed by the two planes. If, on the contrary, $a^2 > b^2$, show that the quadric $\mathcal{C}$ is contained in the obtuse regions of the dihedral angles formed by the two planes.

10. If two pairs of opposite edges of the tetrahedron $ABCD$ are perpendicular ($AB \perp CD$, $AD \perp BC$), show that
    (a) The third pair of opposite edges are perpendicular too ($AC \perp BD$).
    (b) $AB^2 + CD^2 = AC^2 + BD^2 = BC^2 + AD^2$.
    (c) The heights of the tetrahedron are concurrent.
    (Such a tetrahedron is said to be orthocentric)

11. Two triangles ABC şi $A'B'C'$ are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices A', B', C' on the sides BC, CA, AB are concurrent. Show that, in this case, the perpendicular lines from the vertices A, B, C on the sides $B'C'$, $C'A'$, $A'B'$ are concurrent too.

    Solution Due to the given hypothesis, we have
$$\overrightarrow{MA'} \cdot \overrightarrow{BC} = \overrightarrow{MB'} \cdot \overrightarrow{CA} = \overrightarrow{MC'} \cdot \overrightarrow{AB} = 0 \tag{5.9}$$
    We now consider the perpendicular lines from the vertices A and B on the edges $B'C'$ and $C'A'$ and denote by N their intersection point.



Thus

$$\overrightarrow{NA} \cdot \overrightarrow{B'C'} = \overrightarrow{NB} \cdot \overrightarrow{C'A'} = 0.$$

By using the relations (5.9) we obtain

$$\begin{aligned}
& \overrightarrow{MA'} \cdot \overrightarrow{BC} + \overrightarrow{MB'} \cdot \overrightarrow{CA} + \overrightarrow{MC'} \cdot \overrightarrow{AB} = 0 \\
\Leftrightarrow\ & \overrightarrow{MA'} \cdot (\overrightarrow{NC} - \overrightarrow{NB}) + \overrightarrow{MB'} \cdot (\overrightarrow{NA} - \overrightarrow{NC}) + \overrightarrow{MC'} \cdot (\overrightarrow{NB} - \overrightarrow{NA}) = 0 \\
\Leftrightarrow\ & (\overrightarrow{MB'} - \overrightarrow{MC'}) \cdot \overrightarrow{NA} + (\overrightarrow{MC'} - \overrightarrow{MA'}) \cdot \overrightarrow{NB} + (\overrightarrow{MA'} - \overrightarrow{MB'}) \cdot \overrightarrow{NC} = 0 \\
\Leftrightarrow\ & \overrightarrow{C'B'} \cdot \overrightarrow{NA} + \overrightarrow{A'C'} \cdot \overrightarrow{NB} + \overrightarrow{B'A'} \cdot \overrightarrow{NC} = 0 \\
\Leftrightarrow\ & \overrightarrow{B'A'} \cdot \overrightarrow{NC} = 0 \Leftrightarrow NC \perp A'B'.
\end{aligned}$$

6 Week 6: Products of vectors

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

6.1 Brief theoretical background. Products of vectors

6.1.1 The vector product

If $[\vec{i}, \vec{j}, \vec{k}]$ is an orthonormal basis, observe that $\vec{i} \times \vec{j} \in \{-\vec{k}, \vec{k}\}$. We say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *direct* if $\vec{i} \times \vec{j} = \vec{k}$. If, on the contrary, $\vec{i} \times \vec{j} = -\vec{k}$, we say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *inverse*. Therefore, if $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis, then $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$ and obviously $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

Proposition 6.1. *If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, then*

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2) \vec{i} + (a_3 b_1 - a_1 b_3) \vec{j} + (a_1 b_2 - a_2 b_1) \vec{k}, \tag{6.1}$$

or, equivalently,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \tag{6.2}$$

One can rewrite formula (6.1) in the form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{6.3}$$

the right hand side determinant being understood in the sense of its cofactor expansion along the first line.

6.1.2 Applications of the vector product

• **The area of the triangle ABC.** $S_{ABC} = \frac{1}{2} || \overrightarrow{AB} || \cdot || \overrightarrow{AC} || \sin \widehat{BAC} = \frac{1}{2} || \overrightarrow{AB} \times \overrightarrow{AC} ||$. Since the coordinates of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are $(x_B - x_A, y_B - x_A, z_B - z_A)$ and $(x_C - x_A, y_C - x_A, z_C - z_A)$ respectively, we deduce that

$$S_{ABC} = \frac{1}{2} || \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_B - x_A & y_B - x_A & z_B - z_A \\ x_C - x_A & y_C - x_A & z_C - z_A \end{vmatrix} ||,$$

or, equivalently

$$4S^2_{ABC} = \begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2.$$

• **The distance from one point to a straight line.**

a) *The distance $\delta(A, BC)$ from the point $A(x_A, y_A, z_A)$ to the straight line BC*, where $B(x_B, y_B, z_B)$ şi $C(x_C, y_C, z_C)$. Since

$$S_{ABC} = \frac{|| \overrightarrow{BC} || \cdot \delta(A, BC)}{2}$$

rezultă că

$$\delta^2(A, BC) = \frac{4S^2_{ABC}}{|| \overrightarrow{BC} ||^2}.$$

Thus, we obtain

$$\delta^2(A, BC) = \frac{\begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2}{(x_C - x_B)^2 + (y_C - y_B)^2 + (z_C - z_B)^2}.$$

(b) The distance from $\delta(A, d)$ from one point $A(A_A, y_A, z_A)$ to the straight line $d : \frac{x - x_0}{p} + \frac{y - y_0}{p} + \frac{z - z_0}{p}$.

$$\delta(A, d) = \frac{|| \vec{d} \times \overrightarrow{A_0 A} ||}{\vec{d}}, \quad \text{where } A_0(x_0, y_0, z_0) \in \delta.$$

Since

$$\begin{aligned} \vec{d} \times \overrightarrow{A_0 A} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ x_A - x_0 & y_A - y_0 & z_A - z_0 \end{vmatrix} \\ &= \begin{vmatrix} q & r \\ y_A - y_0 & z_A - z_0 \end{vmatrix} \vec{i} + \begin{vmatrix} r & p \\ z_A - z_0 & x_A - x_0 \end{vmatrix} \vec{j} + \begin{vmatrix} p & q \\ x_A - x_0 & y_A - y_0 \end{vmatrix} \vec{k} \end{aligned}$$

it follows that

$$\delta(A,d)=\frac{\sqrt{\begin{vmatrix} q & r \\ y_A-y_0 & z_A-z_0\end{vmatrix}^2+\begin{vmatrix} r & p \\ z_A-z_0 & x_A-x_0\end{vmatrix}^2+\begin{vmatrix} p & q \\ x_A-x_0 & y_A-y_0\end{vmatrix}^2}}{\sqrt{p^2+q^2+r^2}}.$$

6.1.3 The double vector (cross) product

The *double vector (cross) product* of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the vector $\vec{a}\times(\vec{b}\times\vec{c})$

Proposition 6.2 6.2. $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{a}\cdot\vec{b})\,\vec{c}=\begin{vmatrix}\vec{b} & \vec{c}\\ \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\end{vmatrix}$, $\forall\ \vec{a},\vec{b},\vec{c}\in\mathcal{V}$.

Corollary 6.3. *1.* $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{b}\cdot\vec{c})\,\vec{a}=\begin{vmatrix}\vec{b} & \vec{a}\\ \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{a}\end{vmatrix}, \forall\ \vec{a},\vec{b},\vec{c}\in\mathcal{V}$;

2. $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$, $\forall\ \vec{a},\vec{b},\vec{c}\in\mathcal{V}$ *(Jacobi's identity).*

Proof. (1)

$$\begin{aligned}(\vec{a}\times\vec{b})\times\vec{c} &= -\vec{c}\cdot(\vec{a}\times\vec{b})=-[(\vec{c}\cdot\vec{b})\,\vec{a}-(\vec{c}\cdot\vec{a})\,\vec{b}]\\ &=(\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{b}\cdot\vec{c})\,\vec{a}=\begin{vmatrix}\vec{b} & \vec{a}\\ \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{a}\end{vmatrix}.\end{aligned}$$

(2)

$$\begin{aligned}&\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})\\ =&(\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{a}\cdot\vec{b})\,\vec{c}+(\vec{b}\cdot\vec{a})\,\vec{c}-(\vec{b}\cdot\vec{c})\,\vec{a}+(\vec{c}\cdot\vec{b})\,\vec{a}-(\vec{c}\cdot\vec{a})\,\vec{b}=\vec{0}.\end{aligned}$$

□

6.1.4 The triple scalar product

The *triple scalar product* $(\vec{a},\vec{b},\vec{c})$ of the vectors $\vec{a},\vec{b},\vec{c}$ is the real number $(\vec{a}\times\vec{b})\cdot\vec{c}$.

Proposition 1. *If* $[\vec{i},\vec{j},\vec{k}]$ *is a direct orthonormal basis and*

$$\begin{aligned}\vec{a}&=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}\\ \vec{b}&=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}\\ \vec{c}&=c_1\vec{i}+c_2\vec{j}+c_3\vec{k}\end{aligned}$$

then

$$(\vec{a},\vec{b},\vec{c})=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \tag{6.4}$$

Proof.

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k}.$$

Thus

$$\begin{aligned}(\vec{a},\vec{b},\vec{c}) &= (\vec{a}\times\vec{b})\cdot\vec{c}=\vec{c}\cdot(\vec{a}\times\vec{b})\\ &= c_1\begin{vmatrix} a_2 & a_3\\ b_2 & b_3\end{vmatrix}-c_2\begin{vmatrix} a_1 & a_3\\ b_1 & b_3\end{vmatrix}+c_3\begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.\end{aligned}$$

□

Remark 6.4. *1. The distance from the point $M(x_M, y_M, z_M)$ to the plane $\pi : Ax + By + Cz + D = 0$ can be equally computed by means of* (5.6). *Indeed,*

$$\begin{aligned}\delta(M,\pi) &= \| \overrightarrow{Mp_\pi(M)} \| = \| \overrightarrow{Op_\pi(M)} - \overrightarrow{OM} \|\\ &= \left| -\tfrac{F(M)}{\|\vec{n}_\pi\|^2}\right| \cdot \| \vec{n}_\pi \| = \frac{|F(M)|}{\| \vec{n}_\pi \|}.\end{aligned}$$

2. *The distance from the point $N(x_N, y_N, z_N)$ to the straight line $\Delta : \dfrac{x-x_0}{p} = \dfrac{y-y_0}{q} = \dfrac{z-z_0}{r}$ can be computed by means of* (5.8). *Indeed,*

$$\begin{aligned}\delta(M,\Delta) &= \| \overrightarrow{Np_\Delta(N)} \| = \| \overrightarrow{NO} + \overrightarrow{Op_\Delta(N)} \| \qquad (6.5)\\ &= \left\| \overrightarrow{NA_0} - \frac{G(A_0)}{\| \vec{d}_\Delta \|^2}\vec{d}_\Delta \right\| = \left\| \overrightarrow{NA_0} - \frac{\vec{d}_\Delta\cdot\overrightarrow{NA_0}}{\| \vec{d}_\Delta \|^2}\vec{d}_\Delta \right\|.\end{aligned}$$

Proposition 6.5. *Taking into account the formula* (6.5) *for the distance $\delta(M,\Delta)$ from the point $N(x_N, y_N, z_N)$ to the straight line $\Delta : \dfrac{x-x_0}{p} = \dfrac{y-y_0}{q} = \dfrac{z-z_0}{r}$ as well as Proposition 6.2 we deduce that*

$$\begin{aligned}\delta(M,\Delta) &= \left\| \overrightarrow{NA_0} - \frac{\vec{d}_\Delta\cdot\overrightarrow{NA_0}}{\| \vec{d}_\Delta \|^2}\vec{d}_\Delta \right\| = \frac{\| \left(\vec{d}_\Delta\cdot\vec{d}_\Delta\right)\overrightarrow{NA_0} - \left(\vec{d}_\Delta\cdot\overrightarrow{NA_0}\right)\vec{d}_\Delta \|}{\| \vec{d}_\Delta \|^2}\\ &= \frac{\| \vec{d}_\Delta \times(\overrightarrow{NA_0}\times\vec{d}_\Delta)\|}{\| \vec{d}_\Delta \|^2} = \frac{\| \overrightarrow{NA_0}\times\vec{d}_\Delta \|}{\| \vec{d}_\Delta \|}.\end{aligned}$$

6.2 Problems

1. Show that $\| \vec{a}\times\vec{b} \| \leq \| \vec{a} \| \cdot \| \vec{b} \|, \forall\, \vec{a},\vec{b},\in \mathcal{V}$.

   Solution. $\| \vec{a}\times\vec{b} \| = \| \vec{a} \| \cdot \| \vec{b} \| \sin(\widehat{\vec{a},\vec{b}}) \leq \| \vec{a} \| \cdot \| \vec{b} \|$.

2. Let $\vec{a},\vec{b},\vec{c}$ be noncollinear vectors. Show that the necesssary and sufficient condition for the existence of a triangle ABC with the properties $\overrightarrow{BC}=\vec{a}, \overrightarrow{CA}=\vec{b}, \overrightarrow{AB}=\vec{c}$ is

   $$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}.$$

   From the equalities of the norms deduce the low of sines.

3. Show that the sum of some outer-pointing vectors perpendicular on the faces of a tetrahedron which are proportional to the areas of the faces is the zero vector.

4. Find the orthogonal projection

   (a) of the point $A(1,2,1)$ on the plane $\pi: \ x+y+3z+5=0$.

   (b) of the point $B(5,0,-2)$ on the straight line $(d)\ \dfrac{x-2}{3}=\dfrac{y-1}{2}=\dfrac{z-3}{4}$.

   (c) Let d_1, d_2, d_3, d_4 be pairwise skew straight lines. Assuming that $d_{12} \perp d_{34}$ and $d_{13} \perp d_{24}$, show that $d_{14} \perp d_{23}$, where d_{ik} is the common perpendicular of the lines d_i and d_k.

References

[1] Andrica, D., Ţopan, L., Analytic geometry, Cluj University Press, 2004.

[2] Galbură Gh., Radó, F., Geometrie, Editura didactică şi pedagogică-Bucureşti, 1979.

[3] Pintea, C. Geometrie. Elemente de geometrie analitică. Elemente de geometrie diferenţială a curbelor şi suprafeţelor, Presa Universitară Clujeană, 2001.

[4] Radó, F., Orban, B., Groze, V., Vasiu, A., Culegere de Probleme de Geometrie, Lit. Univ. ”Babeş-Bolyai”, Cluj-Napoca, 1979.

