

Seminar Nr. 5,

Continuous Random Variables and Continuous Random Vectors

Theory Review

$X : S \rightarrow \mathbb{R}$ continuous random variable with pdf $f : \mathbb{R} \rightarrow \mathbb{R}$ and cdf $F : \mathbb{R} \rightarrow \mathbb{R}$. Properties:

1. $F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$
2. $f(x) \geq 0, \forall x \in \mathbb{R}, \int_{\mathbb{R}} f(x) = 1$
3. $P(X = x) = 0, \forall x \in \mathbb{R}, P(a < X < b) = \int_a^b f(t) dt$
4. $F(-\infty) = 0, F(\infty) = 1$

$(X, Y) : S \rightarrow \mathbb{R}^2$ continuous random vector with pdf $f = f_{(X,Y)} : \mathbb{R}^2 \rightarrow \mathbb{R}$ and

cdf $F = F_{(X,Y)} : \mathbb{R}^2 \rightarrow \mathbb{R}, F(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^x \int_{-\infty}^y f(u, v) dv du, \forall (x, y) \in \mathbb{R}^2$. Properties:

1. $P(a_1, X \leq b_1, a_2 < Y \leq b_2) = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$
2. $F(\infty, \infty) = 1, F(-\infty, y) = F(x, -\infty) = 0, \forall x, y \in \mathbb{R}$
3. $F_X(x) = F(x, \infty), F_Y(y) = F(\infty, y), \forall x, y \in \mathbb{R}$ (marginal cdf's)
4. $P((X, Y) \in D) = \int_D \int f(x, y) dy dx$
5. $f_X(x) = \int_{\mathbb{R}} f(x, y) dy, \forall x \in \mathbb{R}, f_Y(y) = \int_{\mathbb{R}} f(x, y) dx, \forall y \in \mathbb{R}$ (marginal densities)
6. X and Y are independent $\Leftrightarrow f_{(X,Y)}(x, y) = f_X(x)f_Y(y), \forall (x, y) \in \mathbb{R}^2$.

Function $Y = g(X)$: X r.v., $g : \mathbb{R} \rightarrow \mathbb{R}$ differentiable with $g' \neq 0$, strictly monotone

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}, y \in g(\mathbb{R})$$

Uniform distribution $\mathcal{U}(a, b), -\infty < a < b < \infty$: pdf $f(x) = \frac{1}{b-a}, x \in [a, b]$.

Normal distribution $N(\mu, \sigma), \mu \in \mathbb{R}, \sigma > 0$: pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}$.

Gamma distribution $\text{Gamma}(a, b), a, b > 0$: pdf $f(x) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-\frac{x}{b}}, x > 0$.

Exponential distribution $\text{Exp}(\lambda) = \text{Gamma}(1, 1/\lambda), \lambda > 0$: pdf $f(x) = \lambda e^{-\lambda x}, x > 0$.

- Exponential distribution models *time*: waiting time, interarrival time, failure time, time between rare events, etc.

The parameter λ represents the frequency of rare events, measured in time^{-1} .

- Gamma distribution models the *total* time of a multistage scheme.

- For $\alpha \in \mathbb{N}$, a $\text{Gamma}(\alpha, 1/\lambda)$ variable is the sum of α independent $\text{Exp}(\lambda)$ variables.

1. The lifetime, in years, of some electronic component is a random variable with density

$$f(x) = \begin{cases} \frac{k}{x^4}, & \text{for } x \geq 1 \\ 0, & \text{for } x < 1. \end{cases}$$

Find

a) the constant k ;

- b) the corresponding cdf F ;
- c) the probability for the lifetime of the component to exceed 2 years.

2. (The Uniform property) Let $X \in \mathcal{U}(a, b)$. For any $h > 0$ and $t, s \in [a, b - h]$,

$$P(s < X < s + h) = P(t < X < t + h).$$

The probability is only determined by the length of the interval, but not by its location.

Example: A certain flight can arrive at any time between 4:50 and 5:10 pm. Let X denote the arrival time of the flight.

- a) What distribution does X have?
- b) When is the flight more likely to arrive: between 4:50 and 4:55 or between 5 and 5:05; before 4:55 or after 5:05?

3. On the average, a computer experiences breakdowns every 5 months. The time until the first breakdown and the times between any two consecutive breakdowns are independent Exponential random variables. After the third breakdown, a computer requires a special maintenance.

- a) Find the probability that a special maintenance is required within the next 9 months;
- b) Given that a special maintenance was not required during the first 12 months, what is the probability that it will not be required within the next 4 months?

4. The joint density for (X, Y) is $f_{(X,Y)}(x, y) = \frac{1}{16}x^3y^3, x, y \in [0, 2]$.

- a) Find the marginal densities f_X, f_Y .
- b) Are X and Y independent?
- c) Find $P(X \leq 1)$.

5. Let X be a random variable with density $f_X(x) = \frac{1}{4}xe^{-\frac{x}{2}}, x \geq 0$ and let $Y = \frac{1}{2}X + 2$. Find f_Y .

6. Let $X \in N(0, 1)$. Find the probability density function of $Y = |X|$.

Bonus Problems:

7. An internet service provider has two connection lines for its customers. Eighty percent of customers are connected through Line I, the rest through Line II. Line I has a $\text{Gamma}(3, 1/2)$ connection time (in minutes), while Line II has a $\mathcal{U}(20, 50)$ connection time (in seconds). Find the probability that it takes a randomly selected customer more than 30 seconds to connect to the internet.

8. Let $X, Y \in N(0, 1)$ be independent random variables. Let D_r be the disk centered at the origin with radius r . Find r such that $P((X, Y) \in D_r) = 0.3$.