Lab 8

Quadrature formulas (2)

The rectangle quadrature formula is

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + R_1(f).$$

The repeated rectangle quadrature formula is

$$\int_a^b f(x)dx = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i) + R_n(f),$$

with $x_1 = a + \frac{b-a}{2n}$, $x_i = x_1 + (i-1)\frac{b-a}{n},\ i = 2, ..., n.$

Problems:

1. Use Romberg's algorithm for trapezium and Simpson's formulas to approximate the integral
$$\int_0^1 \frac{2}{1+x^2}dx,$$
with precision $\varepsilon = 10^{-5}$.

2. Plot the graph of $f : [1, 3] \to \mathbb{R},\ f(x) = \frac{100}{x^2}\sin\frac{10}{x}$. Use an adaptive quadrature algorithm for Simpson's formula to approximate the integral
$$\int_1^3 f(x)dx,$$
with precision $\varepsilon = 10^{-4}$. Compare the obtained result with the one obtained applying repeated Simpson formula for $n = 50$ and 100. (The exact value is -1.4260247818.)

3. a) Use the rectangle formula to evaluate the integral

$$\int_1^{1.5} e^{-x^2}\,dx.$$

b) Use the repeated rectangle formula, for $n = 150$ and 500, to evaluate the integral

$$\int_1^{1.5} e^{-x^2}\,dx.$$

(*Answer:* 0.1094)

Facultative problems

Quadrature formula of Gauss type for double integral

Consider the integral $I = \int_a^b \int_c^d f(x,y)dydx$.

We change the variable y from $[c,d]$, in variable t from $[-1,1]$. The linear transformation gives:

$$f(x,y) = f\left(\frac{(d-c)t+d+c}{2}\right) \quad dy = \frac{d-c}{2}dt.$$

$$\int_c^d f(x,y)dy = \int_{-1}^1 f\left(x, \frac{(d-c)t+d+c}{2}\right) dt.$$

We obtain

$$\int_a^b \int_c^d f(x,y)dx \approx \int_a^b \frac{d-c}{2} \sum_{j=1}^n c_{n,j} f\left(x, \frac{(d-c)r_{n,j}+d+c}{2}\right) dt,$$

with $c_{n,j}$ and $r_{n,j}$ given in tables. Then, it is changed the interval $[a,b]$ in the interval $[-1,1]$ and it is repeated the same procedure.

Algorithm:

INPUT: a,b,c,d,m,n

the coefficients $c_{i,j}$ and nodes r_{ij} for $i = \max\{m,n\}$ and $1 \leq j \leq i$

OUTPUT: the approximant J of the integral I

$h_1 = (b-a)/2;$

$h_2 = (b+a)/2;$

$J = 0.$

For $i = 1, 2, ..., m$ do

  $JX = 0$

  $x = h_1 r_{m,i} + h_2;$

  $k_1 = (d-c)/2;$

  $k_2 = (d+c)/2.$

  For $j = 1, 2, ..., n$ do

    $y = k_1 r_{n,j} + k_2;$

    $Q = f(x,y);$

    $JX = JX + c_{n,j} Q.$

  end{for}

  Let $J = J + c_{m,i} \cdot k_1 \cdot JX.$

end{for}

$J = h_1 J$

Romberg's algorithm for rectangle quadrature formula

Apply successively the rectangle formula on $[a,b]$, then on subintervals obtained by dividing in 3 equal parts, in 3^2 equal parts, and so on. We get

$$Q_{D_0}(f) = (b-a)f(x_1), \quad x_1 = \frac{a+b}{2} \tag{1}$$

$$Q_{D_1}(f) = \frac{1}{3}Q_{D_0}(f) + \frac{b-a}{3}\left[f(x_2) + f(x_3)\right], \quad x_2 = a + \frac{b-a}{6},\ x_3 = b - \frac{b-a}{6}.$$

Continuing in an analogous manner, we obtain the sequence

$$Q_{D_0}(f),\ Q_{D_1}(f), ..., Q_{D_k}(f), ... \tag{2}$$

which converges to the value I of the integral $\int_a^b f(x)dx$.

If we want to approximate the integral I with error less than ε, we compute successively the elements of (2) until the first index for which

$$\left|Q_{D_m}(f) - Q_{D_{m-1}}(f)\right| \le \varepsilon,$$

$Q_{D_m}(f)$ being the required value.

The algorithm for generating the elements of the sequence (2) is:

I. *Let* $h := b-a$, $h_1 := \frac{h}{2}$, $x_1 : a + h_1$ *and* $Q_{D_0}(f) := hf(x_1)$.

II. *For* $k := 1, 2, ...$ *do*

$h := \frac{h}{3}$, $h_1 := \frac{h_1}{3}$, $h_2 := 4h_1$, $h_3 := 2h_1$, $m := 3^{k-1}$, $x_1 := a + h_1$;

for $i = 1, ..., m-1$, *do* $x_{2i} := x_{2i-1} + h_2$, $x_{2i+1} := x_{2i} + h_3$;

$x_{2m} := x_{2m-1} + h_2$ and

$$Q_{D_k}(f) = \frac{1}{3}Q_{D_{k-1}}(f) + h\sum_{i=1}^{2m} f(x_i),$$

(for $k = 1$ $(m = 1)$ *the generation of* x_{2i}, x_{2i+1} *is missing).*

Problems

1. The volume of a solid is given by $\int_{0.1}^{0.5}\int_{0.01}^{0.25} e^{\frac{y}{x}}dydx$. Approximate the volume applying the algorithm for Gauss type quadratures for double integrals for $m = n = 5$. Compare the result with the one obtained applying Simpson's algorithm for double integrals for $m = n = 10$. (Result: 0.178571)

We know the following data:

Nodes $r_{5,i}$	*Coefficients* $c_{5,i}$
0.9062	0.2369
0.5385	0.4786
0	0.5689
-0.5385	0.4786
-0.9062	0.2369

2. Use the Romberg's iterations for rectangle formula to approximate the integral

$$\int_1^{1.5} e^{-x^2}dx,$$

with precision $\varepsilon = 10^{-4}$.