Geometry
Problem booklet

Assoc. Prof. Cornel Pintea

E-mail: `cpintea math.ubbcluj.ro`

Contents

Module leader: Assoc. Prof. Cornel Pintea

Department of Mathematics,
"Babeş-Bolyai" University
400084 M. Kogălniceanu 1,
Cluj-Napoca, Romania

1 Week 4: Projections and symmetries. Pencils of planes

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

2 Brief theoretical background

2.1 Projections and symmetries

2.1.1 The intersection point of a straight line and a plane

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

The parametric equations of d are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \; t \in \mathbb{R}. \tag{2.1}$$

The value of $t \in \mathbb{R}$ for which this line (2.1) punctures the plane π can be determined by imposing the condition on the point of coordinates

$$(x_0 + pt, y_0 + qt, z_0 + rt)$$

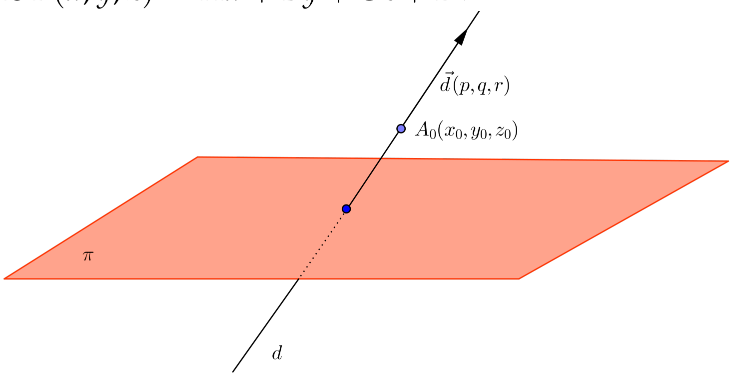
to verify the equation of the plane, namely

$$A(x_0 + pt) + B(y_0 + qt) + C(z_0 + Ct) + D = 0.$$

Thus

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr} = -\frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr},$$

where $F(x, y, z) = Ax + By + Cz + D$.



The coordinates of the intersection point $d \cap \pi$ are

$$\begin{cases} x_0 - p\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ y_0 - q\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ z_0 - r\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr}. \end{cases} \tag{2.2}$$

2.1.2 The projection on a plane parallel to a given line

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

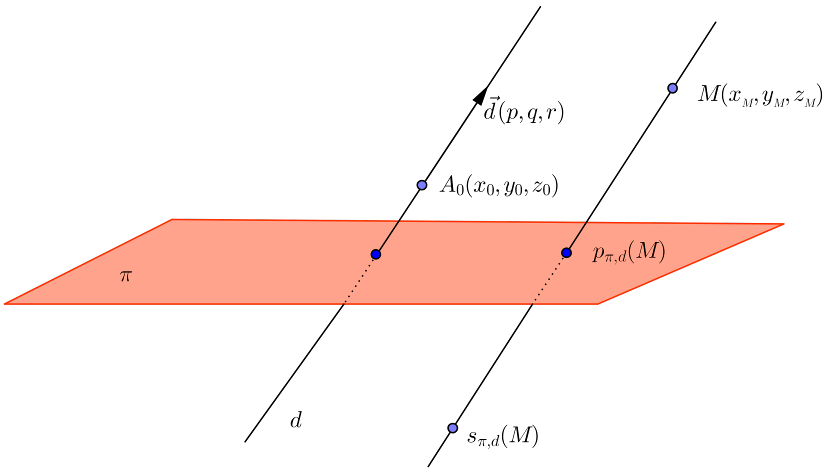
and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{\pi,d} : \mathcal{P} \longrightarrow \pi$ of $\mathcal{P}$ on π parallel to d, whose value $p_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the intersection point between π and the line through M which is parallel to d. Due to relations (2.2), the coordinates of $p_{\pi,d}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_M - p\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ y_M - q\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ z_M - r\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr}, \end{cases} \tag{2.3}$$

where $F(x, y, z) = Ax + By + Cz + D$.



Consequently, the position vector of $p_{\pi,d}(M)$ is

$$\overrightarrow{Op_{\pi,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{Ap + Bq + Cr}\vec{d}\ . \tag{2.4}$$

2.1.3 The symmetry with respect to a plane parallel to a line

We call the function $s_{\pi,d} : \mathcal{P} \longrightarrow \mathcal{P}$, whose value $s_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $p_{\pi,d}(M)$ *the symmetry of $\mathcal{P}$ with respect to π parallel to d*. The direction of d is equally called the *direction* of the symmetry and π is called the *axis* of the symmetry. For the position vector of $s_{\pi,d}(M)$ we have

$$\overrightarrow{Op_{\pi,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\pi,d}(M)}}{2}, \text{ i.e.} \tag{2.5}$$

$$\overrightarrow{Os_{\pi,d}(M)} = 2\,\overrightarrow{Op_{\pi,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{Ap+Bq+Cr}\,\overrightarrow{d}\,. \tag{2.6}$$

2.1.4 The projection on a straight line parallel to a given plane

Consider a straight line

$$d : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{d,\pi} : \mathcal{P} \longrightarrow d$ of $\mathcal{P}$ on d, whose value $p_{d,\pi}(M)$ at $M \in \mathcal{P}$ is the intersection point between d and the plane through M which is parallel to π. Due to relations (2.2), the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_0 - p\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr} \\ y_0 - q\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr} \\ z_0 - r\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr}, \end{cases} \tag{2.7}$$

where $G_M(x,y,z) = A(x-x_M) + B(y-y_M) + C(z-z_M)$. Consequently, the position vector of $p_{d,\pi}(M)$ is

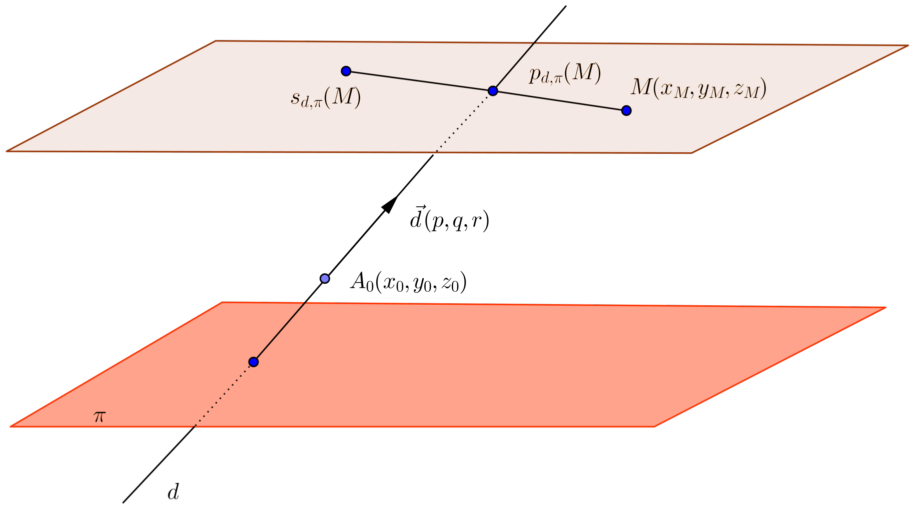
$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} - \frac{G_M(A_0)}{Ap+Bq+Cr}\,\overrightarrow{d}, \text{ where } A_0(x_0,y_0,z_0). \tag{2.8}$$

Note that $G_M(A_0) = A(x_0-x_M) + B(y_0-y_M) + C(z_0-z_M) = F(A_0) - F(M)$, where $F(x,y,z) = Ax + By + Cz + D$. Consequently the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_0 + p\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ y_0 + q\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ z_0 + r\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr}, \end{cases} \tag{2.9}$$

and the position vector of $p_{d,\pi}(M)$ is

$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} + \frac{F(M)-F(A_0)}{Ap+Bq+Cr}\,\overrightarrow{d}, \text{ where } A_0(x_0,y_0,z_0). \tag{2.10}$$



2.1.5 The symmetry with respect to a line parallel to a plane

We call the function $s_{d,\pi} : \mathcal{P} \longrightarrow \mathcal{P}$, whose value $s_{d,\pi}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $p_{d,\pi}(M)$, *the symmetry of $\mathcal{P}$ with respect to d parallel to π*. The direction of π is equally called the *direction* of the symmetry and d is called the *axis* of the symmetry. For the position vector of $s_{d,\pi}(M)$ we have

$$\overrightarrow{Op_{d,\pi}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{d,\pi}(M)}}{2}, \text{ i.e.} \tag{2.11}$$

$$\begin{aligned} \overrightarrow{Os_{d,\pi}(M)} &= 2\,\overrightarrow{Op_{d,\pi}(M)} - \overrightarrow{OM} \\ &= 2\,\overrightarrow{OA_0} - \overrightarrow{OM} + 2\frac{F(M) - F(A_0)}{Ap + Bq + Cr}\,\vec{d}\ . \end{aligned} \tag{2.12}$$

2.2 Pencils of planes

Definition 2.1. *The collection of all planes containing a given straight line*

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

is called the pencil of planes *through* Δ.

Proposition 2.2. *The plane π belongs to the pencil of planes through the straight line Δ if and only if there exists $\lambda, \mu \in \mathbb{R}$ such that the equation of the plane π is*

$$\lambda(A_1x + B_1y + C_1z + D_1) + \mu(A_2x + B_2y + C_2z + D_2) = 0. \tag{2.13}$$

Remark 2.3. *The family of planes*

$$A_1x + B_1y + C_1z + D_1 + \lambda(A_2x + B_2y + C_2z + D_2) = 0,$$

where λ covers the whole real line $\mathbb{R}$, is the so called reduced pencil of planes *through Δ and it consists in all planes through Δ except the plane of equation $A_2x + B_2y + C_2z + D_2 = 0$.*

2.3 Problems

1. Consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\vec{r}_M = m\frac{1-n}{1-mn}\vec{u} + n\frac{1-m}{1-mn}\vec{v} \tag{2.14}$$

and

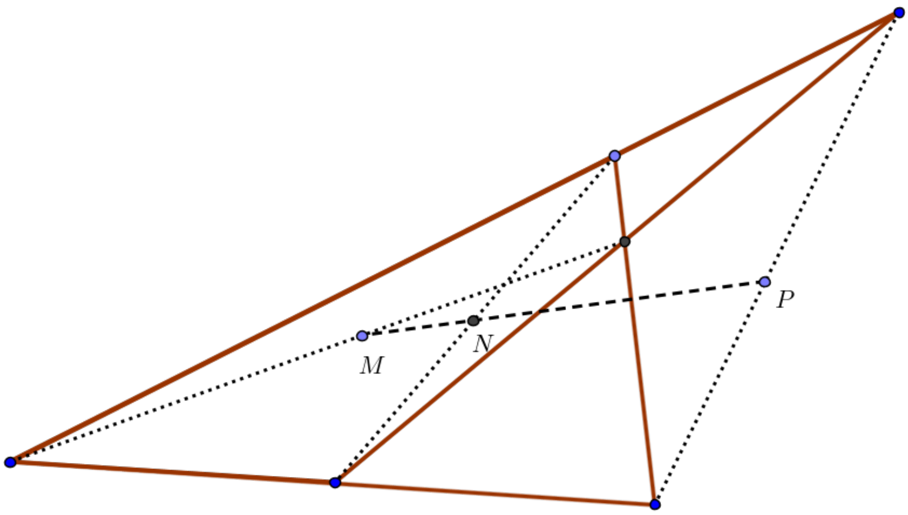
$$\vec{r}_N = m\frac{n-1}{n-m}\vec{u} + n\frac{m-1}{m-n}\vec{v}, \tag{2.15}$$

where $\{M\} = AB' \cap A'B$, $\{N\} = AA' \cap BB'$, $\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\,\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\,\overrightarrow{OA'}$. In other words

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\overrightarrow{OA} + n\frac{m-1}{m-n}\overrightarrow{OA'}.$$

2. Show that the midpoints of the diagonals of a complet quadrilateral are collinear (Newton's theorem).



3. Let d, d' be concurrent straight lines and $A, B, C \in d$, $A', B', C' \in d'$. If $AB' \nparallel A'B$, $AC' \nparallel A'C$, $BC' \nparallel B'C$, show that the points $\{M\} := AB' \cap A'B$, $\{N\} := AC' \cap A'C$, $\{P\} := BC' \cap B'C$ are collinear (Pappus' theorem).

4. Let d, d' be two straight lines and $A, B, C \in d$, $A', B', C' \in d'$ three points on each line such that $AB' \| BA'$, $AC' \| CA'$. Show that $BC' \| CB'$ (the affine Pappus' theorem).

5. Let us consider two triangles ABC and $A'B'C'$ such that the lines AA', BB', CC' are concurrent at a point O and $AB \nparallel A'B'$, $BC \nparallel B'C'$ and $CA \nparallel C'A'$. Show that the points $\{M\} = AB \cap A'B'$, $\{N\} = BC \cap B'C'$ and $\{P\} = CA \cap C'A'$ are collinear (Desargues).

6. Write the equation of the line which passes through $A(1, -2, 6)$ and is parallel to

   (a) The x-axis;

   (b) The line (d_1) $\dfrac{x-1}{2} = \dfrac{y+5}{-3} = \dfrac{z-1}{4}$.

(c) The vector $\overrightarrow{v}\ (1,0,2)$.

7. Write the equation of the plane which contains the line

$$(d_1)\ \frac{x-3}{2}=\frac{y+4}{1}=\frac{z-2}{-3}$$

and is parallel to the line

$$(d_2)\ \frac{x+5}{2}=\frac{y-2}{2}=\frac{z-1}{2}.$$

8. Consider the points $A(\alpha,0,0)$, $B(0,\beta,0)$ and $C(0,0,\gamma)$ such that

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{1}{a}\ \text{ where } a \text{ is a constatnt.}$$

Show that the plane (A,B,C) passes through a fixed point.

9. Write the equation of the line which passes through tht point $M(1,0,7)$, is parallel to the plane $(\pi)\ 3x-y+2z-15=0$ and intersects the line

$$(d)\ \frac{x-1}{4}=\frac{y-3}{2}=\frac{z}{1}.$$

10. Write the equation of the plane which passes through $M_0(1,-2,3)$ and is parallel to the vectors $\overrightarrow{v}_1\ (1,-1,0)$ and $\overrightarrow{v}_2\ (-3,2,4)$.

11. Write the equation of the plane which passes through $M_0(1,-2,3)$ and cuts the positive coordinate axes through congruent segments.

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