Geometry
Problem booklet

Assoc. Prof. Cornel Pintea

E-mail: `cpintea math.ubbcluj.ro`

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Module leader: Assoc. Prof. Cornel Pintea

Department of Mathematics,
"Babeş-Bolyai" University
400084 M. Kogălniceanu 1,
Cluj-Napoca, Romania

1 Week 8: Conics

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background. Conics

1.2 Conics

1.2.1 The Ellipse

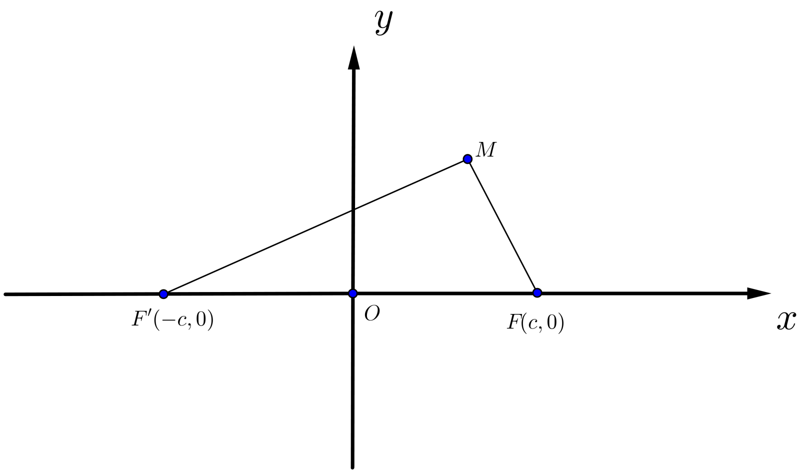
Definition 1.1 1.1. *An* ellipse *is the locus of points in a plane, the sum of whose distances from two fixed points, say F and F', called* foci *is constant.*

The distance between the two fixed points is called the *focal distance*

Let F and F' be the two foci of an ellipse and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition of the ellipse is $2a$. If M is an arbitrary point of the ellipse, it must verify the condition

$$|MF| + |MF'| = 2a.$$

One may chose a Cartesian system of coordinates centered at the midpoint of the segment $[F'F]$, so that $F(c, 0)$ and $F'(-c, 0)$.



Remark 1.2. *In $\Delta MFF'$ the following inequality $|MF| + |MF'| > |FF'|$ holds. Hence $2a > 2c$. Thus, the constants a and c must verify $a > c$.*

Thus, for the generic point $M(x, y)$ of the ellipse we have succesively:

$$\begin{aligned}
|MF| + |MF'| = 2a \quad \Leftrightarrow \quad & \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a \\
& \sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2} \\
& x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \\
& a\sqrt{(x+c)^2 + y^2} = cx + a^2 \\
& a^2(x^2 + 2xc + c^2) + a^2y^2 = c^2x^2 + 2a^2cx + a^2 \\
& (a^2 - c^2)x^2 + a^2y^2 - a^2(a^2 - c^2) = 0.
\end{aligned}$$

Denote $a^2 - c^2$ by b^2, as $(a > c)$. Thus $b^2x^2 + a^2y^2 - a^2b^2 = 0$, i.e.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{1.1}$$

Remark 1.3. *The equation (1.1) is equivalent to*

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}; \qquad x = \pm\frac{a}{b}\sqrt{b^2 - y^2},$$

which means that the ellipse is symmetric with respect to both the x and the y axes. In fact, the line FF', determined by the foci of the ellipse, and the perpendicular line on the midpoint of the segment $[FF']$ are axes of symmetry for the ellipse. Their intersection point, which is the midpoint of $[FF']$, is the center of symmetry of the ellipse, or, simply, its center.

Remark 1.4. *In order to sketch the graph of the ellipse, observe that it is enough to represent the function*

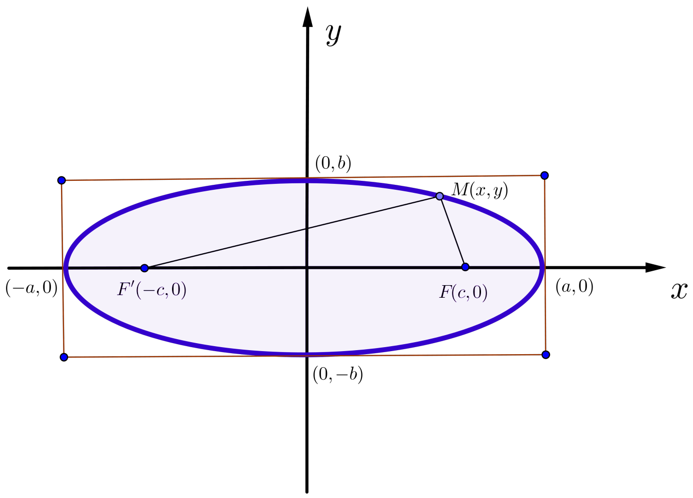
$$f : [-a, a] \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{a^2 - x^2},$$

and to complete the ellipse by symmetry with respect to the x-axis.

One has

$$f'(x) = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}, \qquad f''(x) = -\frac{ab}{(a^2 - x^2)\sqrt{a^2 - x^2}}.$$

x	$-a$		0		a
$f'(x)$	$\vert$	$+\ +\ +$	0	$-\ -\ -$	$\vert$
$f(x)$	0	$\nearrow$	b	$\searrow$	0
$f''(x)$	$\vert$	$-\ -\ -$	$-$	$-\ -\ -$	$\vert$



1.2.2 The Hyperbola

Definiţia 1.5. *The* hyperbola *is defined as the geometric locus of the points in the plane, whose absolute value of the difference of their distances to two fixed points, say F and F' is constant.*

The two fixed points are called the *foci* of the hyperbola, and the distance $|FF'| = 2c$ between the foci is the *focal distance*.

Suppose that the constant in the definition is $2a$. If $M(x,y)$ is an arbitrary point of the hyperbola, then

$$||MF| - |MF'|| = 2a.$$

Choose a Cartesian system of coordinates, having the origine at the midpoint of the segment $[FF']$ and such that $F(c,0)$, $F'(-c,0)$.

Remark 1.6. *In the triangle $\Delta MFF'$, $||MF| - |MF'|| < |FF'|$, so that $a < c$.*

Let us determine the equation of a hyperbola. By using the definition we get $|MF| - |MF'| = \pm 2a$, namely

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a,$$

or, equivalently

$$\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2}.$$

We therefore have successively

$$\begin{aligned}
&x^2 - 2cx + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2\\
&cx + a^2 = \pm a\sqrt{(x+c)^2 + y^2}\\
&c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2\\
&(c^2 - a^2)x^2 - a^2y^2 - a^2(c^2 - a^2) = 0.
\end{aligned}$$

By using the notation $c^2 - a^2 = b^2$ ($c > a$) we obtain the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0. \tag{1.2}$$

The equation (1.2) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}; \qquad x = \pm\frac{a}{b}\sqrt{y^2 + b^2}.$$

Therefore, the coordinate axes are axes of symmetry of the hyperbola and the origin is a center of symmetry equally called the *center of the hyperbola*.

Remark 1.7. *To sketch the graph of the hyperbola, is it enough to represent the function*

$$f : (-\infty, -a] \cup [a, \infty) \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{x^2 - a^2},$$

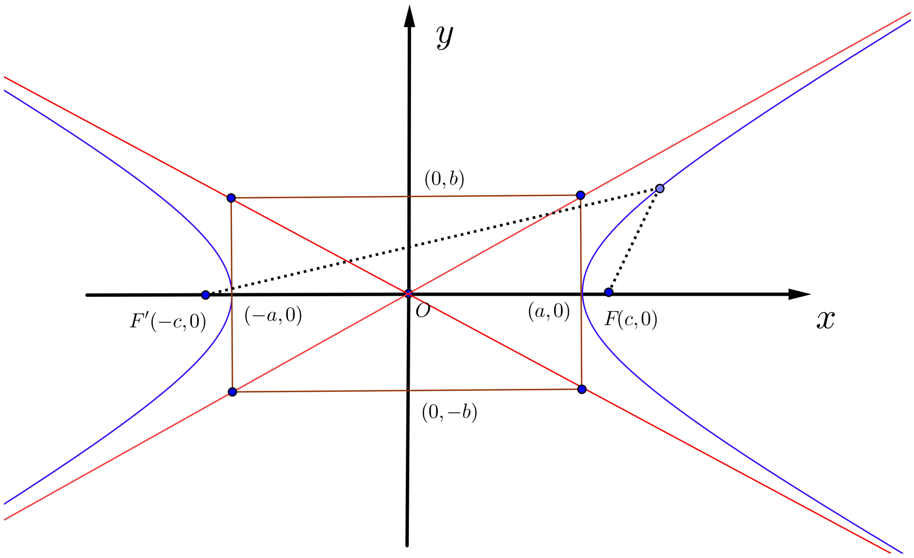
by taking into account that the hyperbola is symmetric with respect to the x-axis.

Since $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \dfrac{b}{a}$ and $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = -\dfrac{b}{a}$, it follows that $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$ are asymptotes of f.

One has, also

$$f'(x) = \frac{b}{a}\frac{x}{\sqrt{x^2 - a^2}}, \qquad f''(x) = -\frac{ab}{(x^2 - a^2)\sqrt{x^2 - a^2}}.$$

x	$-\infty$		$-a$		a		∞
$f'(x)$	$-$	$- - -$	$\vert$	$/ / /$	$\vert$	$+ + +$	$+$
$f(x)$	∞	$\searrow$	$0\vert$	$/ / /$	$\vert 0$	$\nearrow$	∞
$f''(x)$	$-$	$- - -$	$\vert$	$/ / /$	$\vert$	$- - -$	$-$



1.2.3 The Parabola

Definiţia 1.8. *The* parabola *is a plane curve defined to be the geometric locus of the points in the plane, whose distance to a fixed line d is equal to its distance to a fixed point F.*

The line d is the *director line* and the point F is the *focus*. The distance between the focus and the director line is denoted by p and represents the *parameter* of the parabola.

Consider a Cartesian system of coordinates xOy, in which $F\left(\frac{p}{2},0\right)$ and $d: x = -\frac{p}{2}$. If $M(x,y)$ is an arbitrary point of the parabola, then it verifies

$$|MN| = |MF|,$$

where N is the orthogonal projection of M on Oy.

Thus, the coordinates of a point of the parabola verify

$$\sqrt{\left(x+\frac{p}{2}\right)^2+0} = \sqrt{\left(x-\frac{p}{2}\right)^2+y^2}$$

$$\left(x+\frac{p}{2}\right)^2 = \left(x-\frac{p}{2}\right)^2 = y^2$$

$$x^2+px+\frac{p^2}{4} = x^2-px+\frac{p^2}{4}+y^2,$$

and the equation of the parabola is

$$y^2 = 2px. \tag{1.3}$$

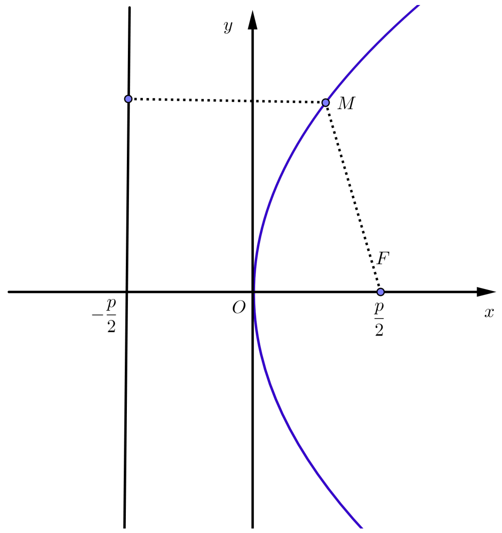
Remark 1.9. *The equation (1.3) is equivalent to $y = \pm\sqrt{2px}$, so that the parabola is symmetric with respect to the x-axis.*

Representing the graph of the function $f : [0, \infty) \to [0, \infty)$ and using the symmetry of the curve with respect to he x-axis, one obtains the graph of the parabola.

One has

$$f'(x) = \frac{p}{\sqrt{2px_0}}; \qquad f''(x) = -\frac{p}{2x\sqrt{2x}}.$$

x	0		∞
$f'(x)$	$\vert$	$+ + +$	$+$
$f(x)$	0	$\nearrow$	∞
$f''(x)$	$-$	$- - -$	$-$



Theorem 1.10. *(The preimage theorem) If $I \subseteq \mathbb{R}$ is an open set, $f : U \longrightarrow R$ is a C^1-smooth function and $a \in Imf$ is a regular value*[1] *of f, then te inverse image of a through f,*

$$f^{-1}(a) = \{(x, y) \in U | f(x, y) = a\}$$

is a planar regular curve called the regular curve of implicit cartezian equation $f(x, y) = a$.

Proposition 1.11. *The equation of the tangent line $T_{(x_0,y_0)}(C)$ of the planar egular curve C of implicit cartezian equation $f(x, y) = a$ at the point $p = (x_0, y_0) \in C$, is*

$$T_{(x_0,y_0)}(C) : f'_x(p)(x - x_0) + f'_y(p)(y - y_0) = 0,$$

and the equation of the normal line $N_{(x_0,y_0)}(C)$ of C at p is

$$N_{(x_0,y_0)}(C) : \frac{x - x_0}{f'_x(p)} = \frac{y - y_0}{f'_y(p)}.$$

[1]The value $a \in \mathrm{Im}(f)$ of the function f is said to be *regular* if $(\nabla f)(x, y) \neq 0$, $\forall (x, y) \in f^{-1}(a)$

1.3 Problems

1. Find the angle between:

   (a) the straight lines

$$(d_1)\begin{cases} x+2y+z-1=0\\ x-2y+z+1=0\end{cases}\qquad (d_2)\begin{cases} x-y-z-1=0\\ x-y+2z+1=0.\end{cases}$$

   (b) the planes

$$\pi_1:\ x+3y+2z+1=0 \text{ and } \pi_2: 3x+2y-z=6.$$

   (c) the plane xOy and the straight line M_1M_2, where $M_1(1,2,3)$ and $M_2(-2,1,4)$.

2. Two triangles ABC şi $A'B'C'$ are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices A', B', C' on the sides BC, CA, AB are concurrent. Show that, in this case, the perpendicular lines from the vertices A, B, C on the sides $B'C'$, $C'A'$, $A'B'$ are concurrent too.

3. Compute the distance:

   (a) from the point $P(1,2,-1)$ to the line

$$(d_1)\begin{cases} 3x-2y=5\\ x-2y-1=0\end{cases}$$

   (b) from the point $A(3,1,-1)$ to the plane

$$\pi: 22x+4y-20z-45=0.$$

   (c) between the lines

$$(d_1)\begin{cases} 3x-2y=5\\ x-2y-1=0\end{cases}\qquad (d_2)\begin{cases} 4x-3y+4=0\\ x-z+2=0.\end{cases}$$

   (d) Find the point on the z-axis which is equidistant with respect to the planes

$$\pi_1:\ 12x+9y-20z-19=0 \text{ and } \pi_2:\ 16x+12y+15z-9=0.$$

4. Consider two planes

$$(\pi_1)\ A_1x+B_1y+C_1z+D_1=0$$
$$(\pi_2)\ A_2x+B_2y+C_2z+D_2=0$$

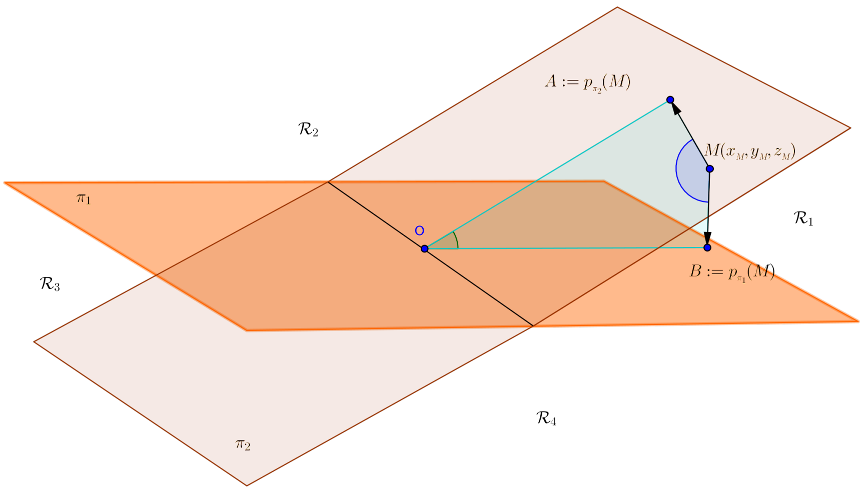
   which are not parallel and not perpendicular as well. The two planes π_1,π_2 devide the space into four regions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$, two of which, say $\mathcal{R}_1$ and $\mathcal{R}_3$, correspond to the acute dihedral angle of the two planes. Show that $M(x,y,z)\in\mathcal{R}_1\cup\mathcal{R}_3$, if and only if

$$F_1(x,y,z)\cdot F_2(x,y,z)(A_1A_2+B_1B_2+C_1C_2)<0,$$

   where $F_1(x,y,z)=A_1x+B_1y+C_1z+D_1$ and $F_2(x,y,z)=A_2x+B_2y+C_2z+D_2$.

   Hint. The non-paralellism relation between the two planes is equivalent with the condition

$$\operatorname{rank}\begin{pmatrix} A_1 & B_1 & C_1\\ A_2 & B_2 & C_2\end{pmatrix}=2.$$



The point M belongs to the union $\mathcal{R}_1 \cup \mathcal{R}_3$ if and only if the angle of the vectors $\overrightarrow{Mp_{\pi_1}(M)}$ and $\overrightarrow{Mp_{\pi_2}(M)}$ is at leat $90°$, as the quadrilateral $OAMB$ is inscriptible. More formally

$$M(x,y,z) \in \mathcal{R}_1 \cup \mathcal{R}_3 \Leftrightarrow m(\widehat{\overrightarrow{Mp_{\pi_1}(M)}, \overrightarrow{Mp_{\pi_2}(M)}}) > 90°$$
$$\Leftrightarrow \overrightarrow{Mp_{\pi_1}(M)} \cdot \overrightarrow{Mp_{\pi_2}(M)} < 0,$$

where $p_{\pi_1}(M), p_{\pi_2}(M)$ are the orthogonal projections of M on the planes π_1 and π_2 respectively.

5. Consider the planes $(\pi_1)\ 2x + y - 3z - 5 = 0$, $(\pi_2)\ x + 3y + 2z + 1 = 0$. Find the equations of the bisector planes of the dihedral angles formed by the planes π_1 and π_2 and select the one contained into the acute regions of the dihedral angles formed by the two planes.

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