Lab 7

Quadrature formulas (1)

Repeated trapezium formula:

$$\int_a^b f(x)dx = \frac{b-a}{2n}[f(a)+f(b)+2\sum_{k=1}^{n-1} f(x_k)] + R_n(f);$$

with

$$x_k = a+kh, \quad k=0,1,...,n; \quad h=\frac{b-a}{n}.$$

Repeated Simpson's formula:

$$\int_a^b f(x)dx = \frac{b-a}{6n}[f(a)+f(b)+4\sum_{k=1}^{n} f(\frac{x_{k-1}+x_k}{2})+2\sum_{k=1}^{n-1} f(x_k)] + R_n(f),$$

with

$$x_k = a+kh, \quad k=0,1,...,n; \quad h=\frac{b-a}{n}.$$

Trapezium formula for double integral

Applying succesively trapezium formula with respect to y, and with respect to x, we have:

$$\begin{aligned}
\int_a^b \int_c^d f(x,y)dydx &\approx \int_a^b \left(\frac{d-c}{4}\right)\left[f(x,c)+2f\left(x,\frac{c+d}{2}\right)+f(x,d)\right]dx \\
&= \frac{b-a}{4}\cdot\frac{d-c}{4}\left[f(a,c)+2f\left(a,\frac{c+d}{2}\right)+f(a,d)\right] \\
&+ \frac{b-a}{4}\cdot 2\cdot\frac{d-c}{4}\left[f\left(\frac{a+b}{2},c\right)\right. \\
&\left.+2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\left(\frac{a+b}{2},d\right)\right] \\
&+ \frac{b-a}{4}\cdot\frac{d-c}{4}\left[f(b,c)+2f\left(b,\frac{c+d}{2}\right)+f(b,d)\right].
\end{aligned}$$

We get

$$\int_a^b \int_c^d f(x,y)dydx \approx \frac{(b-a)(d-c)}{16}\Big[f(a,c)+f(a,d)+f(b,c)+f(b,d) \\ + 2f\left(\frac{a+b}{2},c\right)+2f\left(\frac{a+b}{2},d\right)+2f\left(a,\frac{c+d}{2}\right) \\ + 2f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\Big] \tag{1}$$

Simpson's formula for double integral

Consider the integral $I=\int_a^b\int_c^d f(x,y)dydx$. Let $m,n\in\mathbb{N}$ and the equidistant points $x_0,...,x_{2m}$ in $[a,b]$, with step $h=\frac{b-a}{2m}$, respectively $y_0,...,y_{2n}$ in $[c,d]$, with step $k=\frac{d-c}{2n}$.

We apply the repeated Simpson's formula to the integral $\int_c^d f(x,y)dy$ and then to the integral $\int_a^b\int_c^d f(x,y)dydx$.

Algorithm:

INPUT: a,b,c,d,m,n

OUTPUT: the approximant J of the integral I

```
h=(b-a)/(2*n);
j1=0; j2=0; j3=0
for i=0,1,...,2*n
        Let x=a+i*h;
         hx=(d-c)/(2*m);
         k1=f(x,c)+f(x,d);
         k2=0;
         k3=0;
         for j=1,2,...,2*m-1
                 y=c+j*hx;
                 z=f(x,y);
                 if j is even do k2=k2+z;
                     else k3=k3+z;
                end{if}
         end{for}
        l=(k1+2*k2+4*k3)*hx/3;
         if (i==0)| (i==2*n) do j1=j1+l;
              else if i is even do j2=j2+l;
                     else j3=j3+l;
                     end{if}
             end{if}
end
J=(j1+2*j2+4*j3)*h/3
```

Problems

1. a) Approximate the integral

$$I = \int_0^1 f(x)dx, \qquad \text{for } f(x) = \frac{2}{1+x^2},$$

using trapezium formula.

b) Plot the graph of the function f and the graph of the trapezium with vertices $(0,0), (0, f(0)), (1, f(1))$ and $(1,0)$.

c) Approximate the integral I using Simpson's formula.

2. Approximate the following double integral

$$\int_{1.4}^{2} \int_{1}^{1.5} \ln(x+2y)dydx$$

using trapezium formula for double integral, given in (1). (Exact value is: 0.4295545)

3. Evaluate the integral that arises in electrical field theory:

$$H(x,r) = \frac{60r}{r^2 - x^2} \int_0^{2\pi} \left[1 - \left(\frac{x}{r}\right)^2 \sin\phi\right]^{1/2} d\phi,$$

for $r = 110$, $x = 75$, using the repeated trapezium formula. (*Result:* 6.3131)

4. Evaluate the integral

$$\int_0^{\pi} \frac{dx}{4 + \sin 20x}$$

using the repeated Simpson's formula for $n = 10$ and 30. (*Result:* 0.78;0.81)

5. The volume of a solid is given by $\int_{0.1}^{0.5} \int_{0.01}^{0.25} e^{\frac{y}{x}} dydx$. Approximate this volume applying Simpson's algorithm for double integrals for $m = n = 10$. (*Result*: 0.178571)