Geometry
Problem booklet

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Week 14

1 Transformations

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Transformations of the plane in homogeneous coordinates

In this section we shall identify an affine transformation of $\mathbb{RP}^2$ with its homogeneous transformation matrix

1.1.1 Translations and scalings

• The homogeneous transformation matrix of the translation $T(h,k)$ is

$$T(h,k) = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of the scaling $S(s_x, s_y)$ is

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.1.2 Reflections

• The homogeneous transformation matrix of reflection r_x about the x-axis is

$$r_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of reflection r_y about the y-axis is

$$r_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of reflection r_l about the line $l : ax + by + c = 0$ is

$$r_l = \begin{bmatrix} \dfrac{b^2 - a^2}{a^2 + b^2} & -\dfrac{2ab}{a^2 + b^2} & -\dfrac{2ac}{a^2 + b^2} \\ -\dfrac{2ab}{a^2 + b^2} & \dfrac{a^2 - b^2}{a^2 + b^2} & -\dfrac{2bc}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since in homogeneous coordinates multiplication by a factor does not affect the result, the above matrix can be multiplied by a factor $a^2 + b^2 + c^2$ to give the homogeneous matrix of a general reflection

$$\begin{bmatrix} b^2 - a^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 & -2bc \\ 0 & 0 & a^2 + b^2 \end{bmatrix}.$$

Example 1.1. *Consider a line (d) $ax + by + c$ whose slope is* $\text{tg}\theta = -\frac{a}{b}$. *By using the observation that the reflection r_d in the line d is the following concatenation (product)*

$$T(0, -c/b) \circ R_\theta \circ r_x \circ R_{-\theta} \circ T(0, c/b),$$

one can show that the homogeneous transformation matrix of r_d is

$$\begin{bmatrix} \frac{b^2 - a^2}{a^2 + b^2} & -\frac{2ab}{a^2 + b^2} & -\frac{2ac}{a^2 + b^2} \\ -\frac{2ab}{a^2 + b^2} & \frac{a^2 - b^2}{a^2 + b^2} & -\frac{2bc}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. The homogeneous matrix of the concatenation

$$T(0, -c/b) \circ R_\theta \circ r_x \circ R_{-\theta} \circ T(0, c/b)$$

is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c/b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c/b \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta & 2\frac{c}{b}\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & \frac{c}{b}\left(\sin^2\theta - \cos^2\theta - 1\right) \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.1}$$

Since $\text{tg}\theta = -\frac{a}{b}$, it follows that $\frac{a^2}{b^2} = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{1 - \sin^2\theta} = \frac{1 - \cos^2\theta}{\cos^2\theta}$, namely

$$\sin^2\theta = \frac{a^2}{a^2 + b^2} \text{ and } \cos^2\theta = \frac{b^2}{a^2 + b^2}.$$

Thus

$$\sin\theta = \pm\frac{a}{\sqrt{a^2 + b^2}} \text{ and } \cos\theta = \mp\frac{b}{\sqrt{a^2 + b^2}}, \text{ as } \frac{\sin\theta}{\cos\theta} = \text{tg}\theta = -\frac{a}{b}.$$

Therefore $\sin\theta\cos\theta = -\frac{ab}{a^2 + b^2}$ and the matrix (1.1) becomes

$$\begin{bmatrix} \frac{b^2 - a^2}{a^2 + b^2} & -\frac{2ab}{a^2 + b^2} & -\frac{c}{b}\frac{2ab}{a^2 + b^2} \\ -\frac{2ab}{a^2 + b^2} & \frac{a^2 - b^2}{a^2 + b^2} & \frac{c}{b}\left(\frac{a^2 - b^2}{a^2 + b^2} - 1\right) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{b^2 - a^2}{a^2 + b^2} & -\frac{2ab}{a^2 + b^2} & -\frac{2ac}{a^2 + b^2} \\ -\frac{2ab}{a^2 + b^2} & \frac{a^2 - b^2}{a^2 + b^2} & -\frac{2bc}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

1.1.3 Rotations

The homogeneous transformation matrix of the rotation R_θ about the origin through an angle θ is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 1.2. *The homogeneous transformation matrix of the product (concatenation) homogeneous transformation $T(h,k) \circ R_\theta$ is the product*

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & h \\ \sin\theta & \cos\theta & k \\ 0 & 0 & 1 \end{bmatrix}.$$

In order to find the homogeneous transformation matrix of the inverse transformation

$$(T(h,k) \circ R_\theta)^{-1} = R_\theta^{-1} \circ T(h,k)^{-1} = R_{-\theta} \circ T(-h,-k)$$

of the product (concatenation) homogeneous transformation $T(h,k) \circ R_\theta$ we can either multiply the homogeneous transformation matrices of the inverse transformations $R_\theta^{-1} = R_\theta^{-1}$ and $T(h,k)^{-1} = T(-h,-k)$ or use the next proposition. The product of the homogeneous transformation matrices of the inverse transformations $R_\theta^{-1} = R_\theta^{-1}$ and $T(h,k)^{-1} = T(-h,-k)$ is

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} \cos\theta & \sin\theta & -h\cos\theta - k\sin\theta \\ -\sin\theta & \cos\theta & h\sin\theta - k\cos\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 1.3. *A homogeneous transformation L is invertible if and only if its homogeneous transformation matrix, say T, is invertible and T^{-1} is the transformation matrix of L^{-1}.*

Proof. Suppose that L has an inverse L^{-1} with transformation matrix T_1. The product transformation $L \circ L^{-1} = id$ has the transformation matrix $TT_1 = I_3$. Similarly, $L^{-1} \circ L = I_3$ has the transformation matrix $T_1 T = I_3$. Thus $T_1 = T^{-1}$. Conversely, assume that T has an inverse T^{-1}, and let L_1 be the homogeneous transformation defined by T^{-1}. Since $TT^{-1} = I_3$ and $T^{-1}T = I_3$, it follows that $L \circ L_1 = I$ and $L_1 \circ L = I$. Hence L_1 is the inverse transformation of L.

Example 1.4. *The homogeneous transformation matrix of inverse*

$$(T(h,k) \circ R_\theta)^{-1} = R_\theta^{-1} \circ T(h,k)^{-1} = R_{-\theta} \circ T(-h,-k)$$

of the product (concatenation) homogeneous transformation $T(h,k) \circ R_\theta$ is the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & h \\ \sin\theta & \cos\theta & k \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & -h\cos\theta - k\sin\theta \\ -\sin\theta & \cos\theta & h\sin\theta - k\cos\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

1.2 Transformations of the space

Definition 1.5. *An affine transformation of the plane is a mapping*

$$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ T(x,y,z) = (ax+by+cz+d, ex+fy+gz+h, kx+ly+mz+n), \tag{1.2}$$

for some constant real numbers a,b,c,d,e,f,g,h,k,l,m,n.

By using the matrix language, the action of the map L can be written in the form

$$L(x,y,z) = [x\ y\ z] \begin{bmatrix} a & e & k \\ b & f & l \\ c & g & m \end{bmatrix} + [d\ h\ n].$$

The affine transformation L can be also identified with the map $L^c : \mathbb{R}^{3\times1} \longrightarrow \mathbb{R}^{3\times1}$ given by

$$\begin{aligned} L^c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) &= \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix} \\ &= [L] \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}, \text{ where } [L] = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}. \end{aligned}$$

Definition 1.6. *An affine transformation (1.2) is said to be* singular *if*

$$\begin{vmatrix} a & b & c \\ e & f & g \\ k & l & m \end{vmatrix} = 0.$$

and non-singular otherwise.

1.2.1 Translations

The *translation of* $\mathbb{R}^3$ *of vector* $(h,k,l) \in \mathbb{R}^3$ is the affine transformation

$$T(h,k,l) : \mathbb{R}^3 \to \mathbb{R}^3,\ T(h,k,l)(x_1,x_2,x_3) = (x_1+h, x_2+k, x_3+l).$$

Its associated transformation is

$$T(h,k,l)^c : \mathbb{R}^{3\times1} \to \mathbb{R}^{3\times1},\ T(h,k,l)^c\Big(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\Big) = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} h \\ k \\ l \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[T(h,k,l)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 + h \\ w_2 = x_2 + k \\ w_3 = x_3 + l \end{cases}.$$

1.2.2 Scaling about the origin

The *scaling about the origin* by non-zero scaling factors $(s_x, s_y, s_z) \in \mathbb{R}^3$ is the affine transformation

$$S(s_x, s_y, s_z) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ \left[S(s_x, s_y, s_z)\right](x, y, z) = (s_x \cdot x, s_y \cdot y, s_z \cdot z).$$

Thus

$$\left[S(s_x, s_y, s_z)^c\right]\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \\ s_z \cdot z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

i.e.

$$[S(s_x, s_y, s_z)] = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}.$$

Note that the scaling about the origin by non-zero scaling factors $(s_x, s_y, s_z) \in \mathbb{R}^3$ is non-singular (invertible) and $(S(s_x, s_y, s_z))^{-1} = S(s_x^{-1}, s_y^{-1}, s_z^{-1})$.

1.2.3 Reflections about planes

1. The *reflection of* $\mathbb{R}^3$ *through the xy-plane is* $r_{xy} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{xy}(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. Its associated transformation is

$$r_{xy}^c : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r_{xy}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[r_{xy}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 \\ w_2 = x_2 \\ w_3 = -x_3 \end{cases}.$$

2. The *reflection of* $\mathbb{R}^3$ *through the xz-plane is* $r_{xz} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{xz}(x_1, x_2, x_3) = (x_1, -x_2, x_3)$. Its associated transformation is

$$r_{xz} : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r_{xz}^c\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[r_{xz}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = x_1 \\ w_2 = -x_2 \\ w_3 = x_3 \end{cases}.$$

3. The *reflection of* $\mathbb{R}^3$ *through the yz-plane is* $r_{yz} : \mathbb{R}^3 \to \mathbb{R}^3$, $r_{yz}(x_1, x_2, x_3) = (-x_1, x_2, x_3)$. Its associated transformation is

$$r_{yz}^c : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1},\ r_{yz}^c\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which shows that its standard matrix and equations are:

$$[r_{yz}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{cases} w_1 = -x_1 \\ w_2 = x_2 \\ w_3 = x_3 \end{cases}.$$

4. The *reflection of* $\mathbb{R}^3$ *through an arbitrary plane* $\pi : ax_1 + bx_2 + cx_3 + d = 0$ *is* $r_\pi : \mathbb{R}^3 \to \mathbb{R}^3$, given by

$$r_\pi(x,y,z) = \left(\frac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2}, \right.$$
$$\frac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2},$$
$$\left. \frac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2} \right).$$

Its associated transformation $r_\pi : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1}$ is given by

$$\begin{aligned} r_\pi^c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) &= \begin{bmatrix} \frac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2} \\ \frac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2} \\ \frac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2} \end{bmatrix} \\ &= \frac{1}{a^2+b^2+c^2} \left(\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 2d \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right). \end{aligned}$$

which shows that its standard matrix and equations are:

$$[r_\pi] = \frac{1}{a^2+b^2+c^2} \begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

and

$$\begin{cases} w_1 &= \frac{(-a^2+b^2+c^2)x - 2aby - 2acz - 2ad}{a^2+b^2+c^2} \\ w_2 &= \frac{-2abx + (a^2-b^2+c^2)y - 2bcz - 2bd}{a^2+b^2+c^2} \\ w_3 &= \frac{-2acx - 2bcy + (a^2+b^2-c^2)z - 2cd}{a^2+b^2+c^2} \end{cases}$$

1.2.4 Rotations

The *rotation operator of* $\mathbb{R}^3$ *through a fixed angle* θ *about an oriented axis*, rotates about the axis of rotation each point of $\mathbb{R}^3$ in such a way that its associated vector sweeps out some portion of the cone determine by the vector itself and by a vector which gives the direction and the orientation of the considered oriented axis. The angle of the rotation is measured at the base of the cone and it is measured clockwise or counterclockwise in relation with a viewpoint along the axis looking toward the origin. As in $\mathbb{R}^2$, the positives angles generates counterclockwise rotations and negative angles generates clockwise roattions. The counterclockwise sense of rotaion can be determined by the right-hand rule: If the thumb of the right hand points the direction of the direction of the oriented axis, then the cupped fingers points in a counterclockwise direction. The rotation operators in $\mathbb{R}^3$ are linear.

For example

1. The counterclockwise rotation about the positive x-axis through an angle θ has the equations
$$\begin{array}{lcl} w_1 &=& x \\ w_2 &=& y\cos\theta - z\sin\theta \\ w_3 &=& y\sin\theta + z\cos\theta \end{array},$$
its standard matrix is
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

2. The counterclockwise rotation about the positive y-axis through an angle θ has the equations
$$\begin{array}{lcl} w_1 &=& x\cos\theta + z\sin\theta \\ w_2 &=& y \\ w_3 &=& -x\sin\theta + z\cos\theta \end{array},$$
its standard matrix is
$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

3. The counterclockwise rotation about the positive z-axis through an angle θ has the equations
$$\begin{array}{lcl} w_1 &=& x\cos\theta - y\sin\theta \\ w_2 &=& x\sin\theta + y\cos\theta \\ w_3 &=& z \end{array},$$
its standard matrix is
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.3 Homogeneous coordinates

The affine transformation

$$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ T(x,y,z) = (ax+by+cz+d, ex+fy+gz+h, kx+ly+mz+n),$$

can be written by using the matrix language and by equations:

1. (a) indentifying the vectors $(x,y,z) \in \mathbb{R}^3$ with the line matrices $[x\ y\ z] \in \mathbb{R}^{1\times 3}$ and implicitely $\mathbb{R}^3$ with $\mathbb{R}^{1\times 3}$. With this identification, the action of L is given by

$$L[x\ y\ z] = [x\ y\ z]\begin{bmatrix} a & e & k \\ b & f & l \\ c & g & m \end{bmatrix} + [d\ h\ n].$$

(b) indentifying the vectors $(x, y, z) \in \mathbb{R}^3$ with the column matrices $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^{3\times 1}$ and implicitely $\mathbb{R}^3$ with $\mathbb{R}^{3\times 1}$. We de note by $L^c : \mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}^{3\times 1}$ the associated map via this identification, and its action is given by

$$\begin{aligned} L^c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) &= \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix} \\ &= [L] \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}, \text{ where } [L] = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix}. \end{aligned}$$

2. $\begin{cases} x' &= ax+by+cz+d \\ y' &= ex+fy+gz+h \\ z' &= kx+ly+mz+n \end{cases} \Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}$

Observe that the representation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & l & m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ h \\ n \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

In this section we identify the points $(x, y, z) \in \mathbb{R}^3$ with the points $(x, y, z, 1) \in \mathbb{R}^4$ and even with the punctured lines of $\mathbb{R}^4$, (rx, ry, rz, r), $r \in \mathbb{R}^*$. Due to technical reasons we shall actually identify the points $(x, y, z) \in \mathbb{R}^3$ with the punctured lines of $\mathbb{R}^4$ represented in the form

$$\begin{bmatrix} rx \\ ry \\ rz \\ r \end{bmatrix}, \ r \in \mathbb{R}^*,$$

and the latter ones we shall call *homogeneous coordinates* of the point $(x, y, z) \in \mathbb{R}^3$. The set of homogeneous coordinates (x, y, z, w) will be denoted by $\mathbb{RP}^3$ and call it the real *projective space*. The homogeneous coordinates $(x, y, z, w) \in \mathbb{RP}^3$, $w \neq 0$ and $\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, 1\right)$ represent the same element of $\mathbb{RP}^3$.

Observation 1.7. *The projective space $\mathbb{RP}^3$ is actually the quotient set $(\mathbb{R}^4 \setminus \{0\})/\sim$, where $'\sim'$ is the following equivalence relation on $\mathbb{R}^4 \setminus \{0\}$:*

$$(x, y, z, w) \sim (\alpha, \beta, \gamma, \delta) \Leftrightarrow \exists r \in \mathbb{R}^* \text{ a.î. } (x, y, z, w) = r(\alpha, \beta, \gamma, \delta).$$

Observe that the equivalence classes of the equivalence relation $\sim'$ are the punctured lines of $\mathbb{R}^3$ through the origin without the origin itself, i.e. the elements of the real projective plane $\mathbb{RP}^3$.

Definition 1.8. *A* projective transformation *of the projective space* $\mathbb{RP}^3$ *is a transformation*

$$L : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3,\ L\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} ax+by+cz+dw \\ ex+fy+gz+hw \\ kx+ly+mz+nw \\ px+qy+rz+sw \end{bmatrix}, \tag{1.3}$$

where $a, b, c, d, e, f, g, h, k, l, m, n, p, q, r, s \in \mathbb{R}$. *Note that*

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{bmatrix}$$

is called the homogeneous transformation matrix *of* L.

Observe that a projective transformation (1.3) is well defined since

$$L\begin{bmatrix} tx \\ ty \\ tz \\ tw \end{bmatrix} = \begin{bmatrix} atx+bty+ctz+dtw \\ etx+fty+gtz+htw \\ ktx+lty+mtz+ntw \\ ptx+qty+rtz+tsw \end{bmatrix} = \begin{bmatrix} t(ax+by+cz+dw) \\ t(ex+fy+gz+hw) \\ t(kx+ly+mz+nw) \\ t(px+qy+rz+sw) \end{bmatrix}.$$

If $p = q = r = 0$ and $s \neq 0$, then the projective transformation (1.3) is said to be *affine*. The restriction of the affine transformation (1.3), which corresponds to the situation $p = q = r = 0$ and $s = 1$, to the subspace $w = 1$, has the form

$$L\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by+cz+d \\ ex+fy+gz+h \\ kx+ly+mz+n \\ 1 \end{bmatrix}, \tag{1.4}$$

i.e.

$$\begin{cases} x' &= ax+by+cz+d \\ y' &= ex+fy+gz+h \\ z' &= kx+ly+mz+n. \end{cases} \tag{1.5}$$

Observation 1.9. *If* $L_1, L_2 : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3$ *are two projective applications, then their product (concatenation) transformation* $L_1 \circ L_2$ *is also a projective transformation and its homogeneous transformation matrix is the product of the homogeneous transformation matrices of* L_1 *and* L_2 .

Indeed, if

$$L_1\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ e_1 & f_1 & g_1 & h_1 \\ k_1 & l_1 & m_1 & n_1 \\ p_1 & q_1 & r_1 & s_1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

and

$$L_2\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ e_2 & f_2 & g_2 & h_2 \\ k_2 & l_2 & m_2 & n_2 \\ p_2 & q_2 & r_2 & s_2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

then

$$(L_1 \circ L_2)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \left(\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ e_1 & f_1 & g_1 & h_1 \\ k_1 & l_1 & m_1 & n_1 \\ p_1 & q_1 & r_1 & s_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ e_2 & f_2 & g_2 & h_2 \\ k_2 & l_2 & m_2 & n_2 \\ p_2 & q_2 & r_2 & s_2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Observation 1.10. *If $L_1, L_2 : \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3$ are two affine applications, then their product $L_1 \circ L_2$ is also an affine transformation.*

1.4 Transformations of the space in homogeneous coordinates

1.4.1 Translations

The homogeneous transformation matrix of the translation

$$T(h,k,l) : \mathbb{R}^3 \to \mathbb{R}^3,\ T(h,k,l)(x_1,x_2,x_3) = (x_1+h, x_2+k, x_3+l)$$

is

$$\begin{bmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1.4.2 Scaling about the origin

The homogeneous transformation matrix of the scaling

$$S(s_x,s_y,s_z) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ \left[S(s_x,s_y,s_z)\right](x,y,z) = (s_x \cdot x, s_y \cdot y, s_z \cdot z)$$

is

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1.4.3 Reflections about planes

1. The homogeneous transformation matrix of the reflection

$$r_{xy} : \mathbb{R}^3 \to \mathbb{R}^3,\ r_{xy}(x_1,x_2,x_3) = (x_1,x_2,-x_3)$$

of $\mathbb{R}^3$ through the xy-plane is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. The homogeneous transformation matrix of the reflection

$$r_{yz} : \mathbb{R}^3 \to \mathbb{R}^3,\ r_{yz}(x_1,x_2,x_3) = (-x_1,x_2,x_3)$$

of $\mathbb{R}^3$ through the yz-plane is

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. The homogeneous transformation matrix of the reflection

$$r_{xz} : \mathbb{R}^3 \to \mathbb{R}^3,\ r_{yz}(x_1, x_2, x_3) = (x_1, -x_2, x_3)$$

of $\mathbb{R}^3$ through the xz-plane is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. The homogeneous transformation matrix of the reflection $r_\pi : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{aligned} r_\pi(x, y, z) = \Bigg(& \frac{(-a^2 + b^2 + c^2)x - 2aby - 2acz - 2ad}{a^2 + b^2 + c^2}, \\ & \frac{-2abx + (a^2 - b^2 + c^2)y - 2bcz - 2bd}{a^2 + b^2 + c^2}, \\ & \frac{-2acx - 2bcy + (a^2 + b^2 - c^2)z - 2cd}{a^2 + b^2 + c^2} \Bigg). \end{aligned}$$

through an arbitrary plane $\pi : ax_1 + bx_2 + cx_3 + d = 0$ is

$$\begin{bmatrix} \frac{-a^2 + b^2 + c^2}{a^2 + b^2 + c^2} & \frac{-2ab}{a^2 + b^2 + c^2} & \frac{-2ac}{a^2 + b^2 + c^2} & \frac{-2ad}{a^2 + b^2 + c^2} \\ \frac{-2ab}{a^2 + b^2 + c^2} & \frac{a^2 - b^2 + c^2}{a^2 + b^2 + c^2} & \frac{-2bc}{a^2 + b^2 + c^2} & \frac{-2bd}{a^2 + b^2 + c^2} \\ \frac{-2ac}{a^2 + b^2 + c^2} & \frac{-2bc}{a^2 + b^2 + c^2} & \frac{a^2 + b^2 - c^2}{a^2 + b^2 + c^2} & \frac{-2cd}{a^2 + b^2 + c^2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since in homogeneous coordinates multiplication by a factor does not affect the result, the above matrix can be multiplied by a factor $a^2 + b^2 + c^2$ to give the homogeneous matrix of a general reflection

$$\begin{bmatrix} -a^2 + b^2 + c^2 & -2ab & -2ac & -2ad \\ -2ab & a^2 - b^2 + c^2 & -2bc & -2bd \\ -2ac & -2bc & a^2 + b^2 - c^2 & -2cd \\ 0 & 0 & 0 & a^2 + b^2 + c^2 \end{bmatrix}.$$

1.4.4 Rotations

1. The homogeneous transformation matrix of the counterclockwise rotation about the positive x-axis through an angle θ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. The homogeneous transformation matrix of the counterclockwise rotation about the positive y-axis through an angle θ is

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. The homogeneous transformation matrix of the counterclockwise rotation about the positive z-axis through an angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1.5 Problems

1. Find the homogeneous transformation matrix of the product (concatenation)

$$T(1,1,-2) \circ Rot_y(\pi/6),$$

where $Rot_y(\pi/6)$ stands for the rotation about the positive y-axis through an angle θ.

2. Find the homogeneous transformation matrix of the rotation through an angle θ, of the space, about an arbitrary line.

3. Find the homogeneous transformation matrix of the rotation through an angle θ about the line PQ, where $P(2,1,5)$ and $Q(4,7,2)$.

