Geometry
Problem booklet

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Contents

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1 Week 3: Cartezian equations of lines and planes

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background

1.1.1 The cartesian equations of the straight lines

Let Δ be a straight line passing through the point $A_0(x_0, y_0, z_0)$ which is parallel to the vector $\overrightarrow{d}\,(p, q, r)$. Its vector equation is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda \overrightarrow{d}\,. \tag{1.1}$$

Denoting by x, y, z the coordinates of the generic point M of the straight line Δ, its vector equation (1.1) is equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases}, \ \lambda \in \mathbb{R} \tag{1.2}$$

The relations (1.2) are being called the *parametric equations* of the straight line Δ and they are equivalent to the following relations

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r} \tag{1.3}$$

If $r = 0$, for instance, the canonical equations of the straight line Δ are

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} \ \wedge \ z = z_0.$$

If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are different pointsof the straight line Δ, then $\overrightarrow{AB}\,(x_B - x_A, y_B - y_A, z_B - z_A)$ is a director vector of Δ, its canonical equations having, in this case, the form

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}. \tag{1.4}$$

1.1.2 Te cartesian equations of the planes

Let $A_0(x_0, y_0, z_0) \in \mathcal{P}$ and $\overrightarrow{d}_1\,(p_1, q_1, r_1)$, $\overrightarrow{d}_2\,(p_2, q_2, r_2) \in \mathcal{V}$ be linearly independent vectors, that is

$$\text{rang} \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix} = 2.$$

The vector equation of the plane π passing through A_0 which is parallel to the vectors $\overrightarrow{d}_1\,(p_1, q_1, r_1)$, $\overrightarrow{d}_2\,(p_2, q_2, r_2)$ is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda_1 \overrightarrow{d}_1 + \lambda_2 \overrightarrow{d}_2, \ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{1.5}$$

If we denote by x, y, z the coordinates of the generic point M of the plane π, then the vector equation (1.5) is the equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda_1 p_1 + \lambda_2 p_2 \\ y = y_0 + \lambda_1 q_1 + \lambda_2 q_2 \\ z = z_0 + \lambda_1 r_1 + \lambda_2 r_2 \end{cases}, \ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{1.6}$$

The relations (1.6) reprezent a characterization of the points of the plane π called the *parametric equations* of the plane π. More precisely, the compatibility of the linear system (1.6) with the unknowns λ_1, λ_2 is a necessary and sufficient condition for the point $M(x, y, z)$ to be contained within the plane π. On the other hand the compatibility of the linear system (1.6) is equivalent to the relations

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ p_1 & q_1 & z_1 \\ p_2 & q_2 & z_2 \end{vmatrix} = 0. \tag{1.7}$$

and express the fact that the rank of the matrix of the system is equal to the rank of the augumented matrix of the system. The condition (1.7) is a characterization of the points of the plane π expressed in terms of the cartesian coordinates of the generic point M and is called the *cartesian equation* of the plane π.

If $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ are noncollinear points, then the plane (ABC) determined by the three points can be viewed as the plane passing through the point A which is parallel to the vectors $\vec{d}_1 = \overrightarrow{AB}$, $\vec{d}_2 = \overrightarrow{AC}$. The coordinates of the vectors $\vec{d}_1$ şi $\vec{d}_2$ are

$$(x_B - x_A, y_B - y_A, z_B - z_A) \text{ and } (x_C - x_A, y_C - y_A, z_C - z_A) \text{ respectively.}$$

Thus, the equation of the plane (ABC) is

$$\begin{vmatrix} x-x_A & y-y_A & z-z_A \\ x_B-x_A & y_B-y_A & z_B-z_A \\ x_C-x_A & y_C-y_A & z_C-z_A \end{vmatrix} = 0, \tag{1.8}$$

or, echivalently

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0. \tag{1.9}$$

On can put the equation (1.7) in the form

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0 \text{ or} \tag{1.10}$$

$$Ax + By + Cz + D = 0, \tag{1.11}$$

where the coefficients A, B, C satisfy the relation $A^2 + B^2 + C^2 > 0$. It is also easy to show that every equation of the form (1.11) represents the equation of a plane. Indeed, if $A \neq 0$, then the equation (1.11) is equivalent to

$$\begin{vmatrix} x+\frac{D}{A} & y & z \\ B & -A & 0 \\ C & 0 & -A \end{vmatrix} = 0.$$

We observe that one can put the equation (1.10) in the form

$$AX + BY + CZ = 0 \tag{1.12}$$

where $X = x - x_0$, $Y = y - y_0$, $Z = z - z_0$ are the coordinates of the vector $\overrightarrow{A_0M}$.

1.1.3 Analytic conditions of parallelism

The equation $AX + BY + CZ = 0$ is a necessary and sufficient condition for the vector $\overrightarrow{A_0M}\ (X, Y, Z)$ to be contained within the direction of the plane

$$\pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Thus the *equation of the director subspace* $\vec{\pi} = \{\overrightarrow{A_0M} \mid M \in \pi\}$ is $AX + BY + CZ = 0$.

Proposition 1.1. *The straight line*

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

is parallel to the plane $\pi : Ax + By + Cz + D = 0$ *iff*

$$Ap + Bq + Cr = 0 \tag{1.13}$$

Proposition 1.2. *Consider the planes*

$$\begin{array}{l} (\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0 \\ (\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0. \end{array}$$

Then $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) \in \{1, 2\}$ *and the following statemenets are equivalent*

1. $\pi_1 \| \pi_2$.
2. $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 2$, *i.e.* $\vec{\pi}_1 = \vec{\pi}_2$.
3. *rang* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1$.
4. *The vectors* $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly dependent.*

Corollary 1.3. *Consider the planes*

$$\begin{array}{l} (\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0 \\ (\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0. \end{array}$$

The following statements are equivalent

1. $\pi_1 \nparallel \pi_2$.
2. $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 1$.
3. *rang* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$.
4. *The vectors* $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly independent.*

By using Proposition the characterization of parallelism between a line and a plane, we shall find a necessary and sufficient condition for a vector to be contained within the direction of a straight line which is given as the intersection of two planes.

Consider the planes

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0$$
$$(\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0$$

such that

$$\operatorname{rang}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2,$$

alongside their intersectio9n straight line $\Delta = \pi_1 \cap \pi_2$ of equations

$$(\Delta)\ \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_1 = 0. \end{cases}$$

Thus, $\overrightarrow{\Delta} = \overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2$ and therefore, by means of some previous Proposition, it follows that the equations of $\overrightarrow{\Delta}$ are

$$(\overrightarrow{\Delta})\ \begin{cases} A_1X + B_1Y + C_1Z = 0 \\ A_2X + B_2Y + C_2Z = 0. \end{cases} \tag{1.14}$$

By solving the system (1.14) one can therefore deduce that $\overrightarrow{d}\ (p, q, r) \in \overrightarrow{\Delta} \Leftrightarrow\ \exists \lambda \in \mathbb{R}$ such that

$$(p, q, r) = \lambda \left(\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \right). \tag{1.15}$$

The relation is usually (1.15) written in the form

$$\frac{p}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{q}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}} = \frac{r}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \tag{1.16}$$

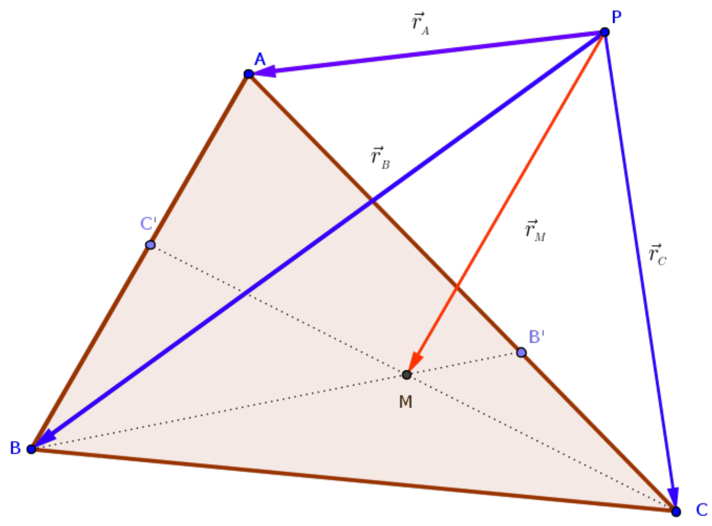
Let us mention that the chosen values for (p, q, r) are usually precisely

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \text{ şi } \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}.$$

1.2 Problems

1. ([4, Problema 16, p. 5]) Consider the points C' and B' on the sides AB and AC of the triangle ABC such that $\overrightarrow{AC'} = \lambda\ \overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\ \overrightarrow{CB'}$. The lines BB' and CC' meet at M. If $P \in \mathcal{P}$ is a given point and $\overrightarrow{r}_A = \overrightarrow{PA}$, $\overrightarrow{r}_B = \overrightarrow{PB}$, $\overrightarrow{r}_C = \overrightarrow{PC}$ are the position vectors, with respect to P, of the vertices A, B, C respectively, show that

$$\overrightarrow{r}_M = \frac{\overrightarrow{r}_A - \lambda\ \overrightarrow{r}_B - \mu\ \overrightarrow{r}_C}{1 - \lambda - \mu}. \tag{1.17}$$



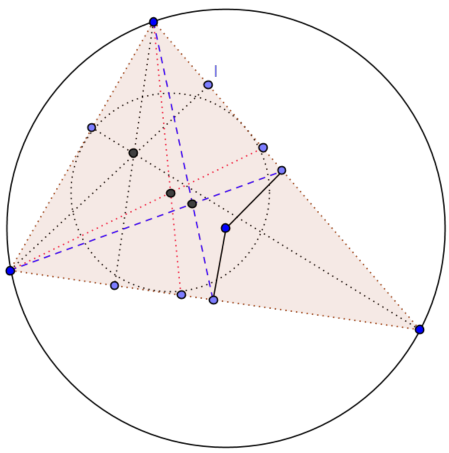
2. ([4, Problema 17, p. 5]) Consider the triangle ABC, its centroid G, its orthocenter H, its incenter I and its circumcenter O. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$, $\vec{r}_C = \overrightarrow{PC}$ are the position vectors with respect to P of the vertices A, B, C respectively, show that:

(a) $\vec{r}_G := \overrightarrow{PG} = \dfrac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}$.

(b) $\vec{r}_I := \overrightarrow{PI} = \dfrac{a\,\vec{r}_A + b\,\vec{r}_B + c\,\vec{r}_C}{a+b+c}$.

(c) $\vec{r}_H := \overrightarrow{PH} = \dfrac{(\tan A)\,\vec{r}_A + (\tan B)\,\vec{r}_B + (\tan C)\,\vec{r}_C}{\tan A + \tan B + \tan C}$.

(d) $\vec{r}_O := \overrightarrow{PO} = \dfrac{(\sin 2A)\,\vec{r}_A + (\sin 2B)\,\vec{r}_B + (\sin 2C)\,\vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}$.



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