Some Elements of Spectral Graph Theory

References:

First few chapters of

"Spectral graph theory", Fan Chung

Orientation-agnostic definitions

$$V = \{v_1, \ldots, v_N\} \qquad E = \{e_1, \ldots, e_M\}$$

Degrees and Degree Matrix:

$$d(v) = |\{u \in V \text{ s.t. } (u,v) \in E \text{ or } (v,u) \in E\}|$$

$$\mathbf{D}(G) = \operatorname{diag}(d_1, \ldots d_N)$$

Incidence Matrix:

$$\mathbf{S}(i,j) = \begin{cases} +1 & \text{if } e_j = (v_i, v_k) \text{ for some } k \\ -1 & \text{if } e_j = (v_k, v_i) \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

Orientation-agnostic definitions

$$V = \{v_1, \ldots, v_N\} \qquad E = \{e_1, \ldots, e_M\}$$

Adjacency matrix

$$\mathbf{A}(i,j) = \begin{cases} +1 & \text{if there is an edge } (v_i, v_j) \text{ or } (v_j, v_i) \in E \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{A}(i,j) = \begin{cases} +1 & \text{if there is an edge } \{v_i, v_j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Extensions to weighted graphs

$$V = \{v_1, \ldots, v_N\} \qquad E = \{e_1, \ldots, e_M\}$$

Weight Matrix:

A *symmetric* N-by-N matrix $\mathbf{W}$

$$\mathbf{W}(i,j) \geqslant 0 \qquad \mathbf{W}(i,i) = 0$$

$\mathbf{W}(i,j)$ is the weight ("strength") of the edge between i,j (if any)

Degrees:

$$d(v_i) = \sum_{j \sim i} \mathbf{W}(i,j)$$

Orientation-agnostic definitions

With these definitions we have:

$$\mathbf{S}\mathbf{S}^T = \mathbf{D} - \mathbf{A}$$

$\mathbf{L} = \mathbf{D}$ - $\mathbf{A}$ is called unnormalized Laplacian of G

$\mathbf{L}$ does not depend on the orientation (so OK for undirected)

For a weighted graph we have $\mathbf{L} = \mathbf{D}$ - $\mathbf{W}$ (attention to degrees)

$\mathbf{L}$ is a symmetric, positive semi-definite matrix

Graph Laplacian

Proposition: $\mathbf{L}$ is positive semi-definite

For any N-by-N weight matrix $\mathbf{W}$, if $\mathbf{L} = \mathbf{D}\text{-}\mathbf{W}$ where $\mathbf{D}$ is the degree matrix of $\mathbf{W}$, then

$$x^T \mathbf{L} x = \frac{1}{2} \sum_{i,j} \mathbf{W}(i,j)(x[i] - x[j])^2 \geqslant 0 \quad \forall x \in \mathbb{R}^N$$

Rem: to ease notations we will sometimes use $\quad w_{ij} = \mathbf{W}(i,j)$

Since $\mathbf{L}$ is real, symmetric and PSD:

- It has an eigendecomposition into real eigenvalues and eigenvectors λ_i, u_i
- The eigenvalues are non-negative

$$0 = \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_N$$

$$\mathbf{L1} = 0$$

What can be learned from eigenvectors and eigenvalues ?

Some examples

Path graph

DCT II transform

$$\begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & \\ & & & & & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}$$

$$\lambda_k = 2 - 2\cos\frac{\pi k}{N} = 4\sin^2\frac{\pi k}{2N},\ k = 0, \ldots, N-1$$

$$u_k[\ell] = \cos\left(\pi k(\ell + \frac{1}{2})/N\right),\ \ell = 0, \ldots, N-1$$

Some examples

Ring graph

$$\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ -1 & & & -1 & 2 \end{pmatrix}$$

DCT transform

$$\lambda_k = 2 - 2\cos\frac{\pi k}{N} = 4\sin^2\frac{\pi k}{2N},\ k = 0, \ldots, N-1$$

$$u_k^c[\ell] = \cos\left(2\pi k\ell/N\right),\ \ell = 0, \ldots, N-1$$

$$u_k^s[\ell] = \sin\left(2\pi k\ell/N\right),\ \ell = 0, \ldots, N-1$$

Proposition: eigendecomposition of $\mathbf{L}$ and structure of G

The number of connected components c of G is the dimension of the nullspace of $\mathbf{L}$. Furthermore the null space of $\mathbf{L}$ has a basis of indicator vectors of the connected components of G

Indicator of a subset H of V is

$$x \in \mathbf{R}^N \text{ s.t. } \begin{cases} x[i] = 1 & \text{if } \in H \\ x[i] = 0 & \text{otherwise} \end{cases}$$

Normalized Graph Laplacian

Note: we will sometimes need to consider the generalised problem

$$\mathbf{L}u = \lambda \mathbf{D} u$$

In this case it makes sense to introduce the normalised Laplacian

$$\mathbf{L}_{\text{norm}} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$$

Eigenvectors are closely related

$$\mathbf{L}_{\text{norm}} f = \lambda f \to u = D^{-1/2} f$$

Normalized Graph Laplacian

Eigenvalues of the normalised Laplacian

$$0 = \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_N \leqslant 2$$

Algebraic connectivity

IFF bipartite graph!

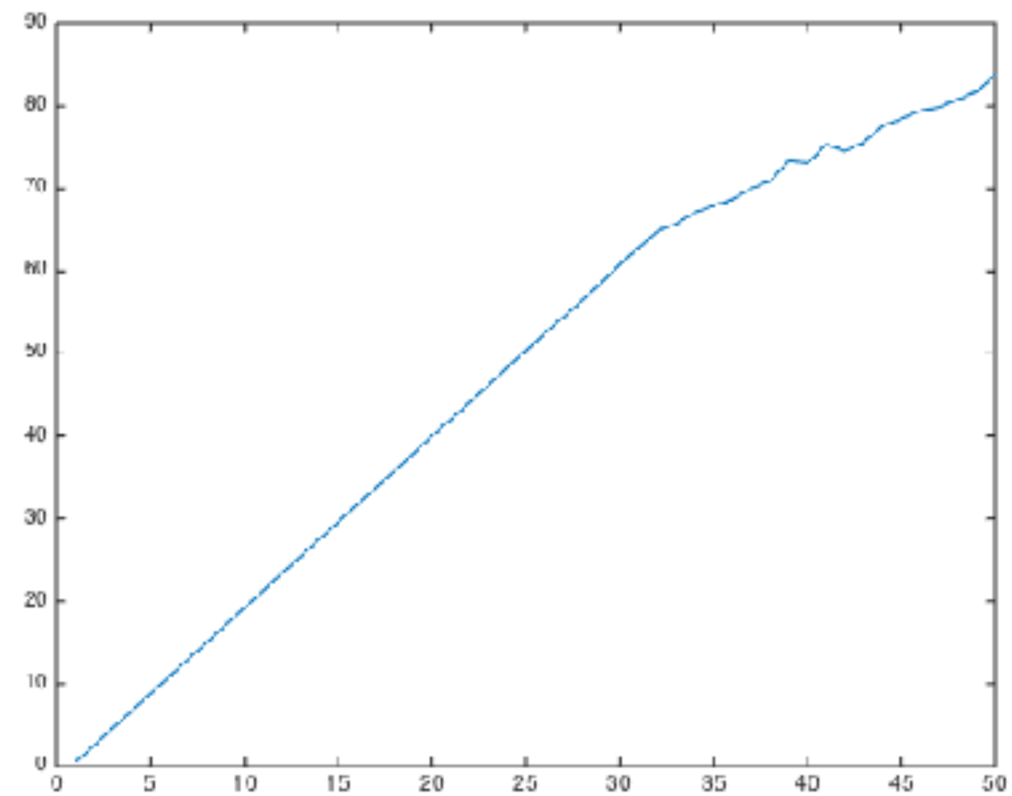
Algebraic Connectivity, Fiedler Vector

Multiplicity of eigenvalue 0 gives connectedness of graph

What if $\lambda_2 > 0$?

Experiment:

Gradually increase connections between two Erdos-Renyi subgraphs



$$\lambda_2 \geqslant \frac{1}{\mathrm{vol}(G)d(G)}$$ where $d(G)$ is the diameter of the graph

The Cheeger Constant

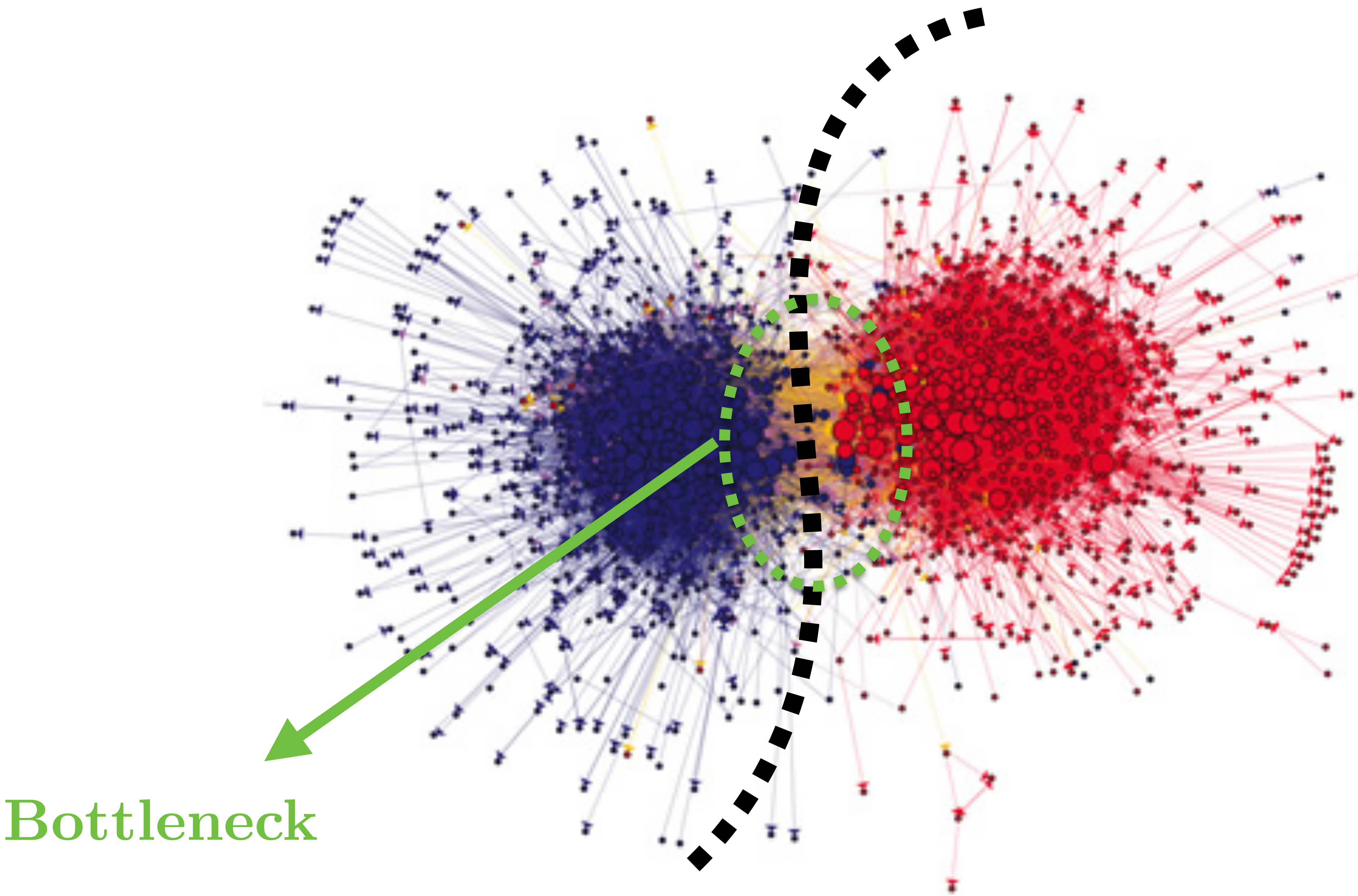
Cheeger constant measures presence of a “bottleneck”

$$A \subset V \qquad \partial A = \left\{(u,v) \in E \text{ s.t. } u \in A,\ v \in \overline{A}\right\}$$

$$\mathrm{vol}(A) = \sum_{u \in A} d(u)$$

$$h(G) = \min_{A \subset V} \left\{ \frac{|\partial A|}{\min\Big(\mathrm{vol}(A), \mathrm{vol}(\overline{A})\Big)} \text{ s.t. } 0 < |A| < \frac{1}{2}|V| \right\}$$

The Cheeger Constant



A Cheeger Inequality

The Cheeger constant and algebraic connectivity are related by Cheeger inequalities. A simple example:

Theorem: Cheeger Inequality[Polya, Szego]

For a general graph G,

$$2h(G) \geqslant \lambda_2 \geqslant \frac{h^2(G)}{2}$$

Remark: the eigenvector associated to the algebraic connectivity is called the Fiedler vector

Algebraic Connectivity, Fiedler Vector

Set of 1490 US political blogs, labelled “Dem” or “Rep”

Hyperlinks among blogs

Removed small degrees (<12), keep N = 622 vertices

Compute normalised Laplacian, Fiedler vector

Assign attributes +1, -1 by sign of Fiedler vector