Geometry
Problem booklet

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Contents

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1 Week 2: Straight lines and planes

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background

1.1.1 Linear dependence and linear independence of vectors

Definition 1.1.
1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *collinear* if the points O, A, B are collinear. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *noncollinear*.
2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}, \overrightarrow{OC}$ are said to be *coplanar* if the points O, A, B, C are coplanar. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are *noncoplanar*.

Remark 1.2.
1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are linearly (in)dependent if and only if they are (non)collinear.
2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are linearly (in)dependent if and only if they are (non)coplanar.

Proposition 1.3. *The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ form a basis of $\mathcal{V}$ if and only if they are noncoplanar.*

Corollary 1.4. *The dimension of the vector space of free vectors $\mathcal{V}$ is three.*

1.1.2 Cartesian and affine reference systems

A basis of the direction $\overrightarrow{\pi}$ of the plane π, or for the vector space $\mathcal{V}$ is an ordered basis $[\overrightarrow{e}, \overrightarrow{f}]$ of π, or an ordered basis $[\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ a of $\mathcal{V}$.

If $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is a basis of $\mathcal{V}$ and $\overrightarrow{x} \in \mathcal{V}$, recall that the column vector of $\overrightarrow{x}$ with respect to b is being denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

whenever $\overrightarrow{x} = x_1 \overrightarrow{u} + x_2 \overrightarrow{v} + x_3 \overrightarrow{w}$.

Definition 1.5. *A* cartesian reference system *of the space $\mathcal{P}$, is a system $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ where O is a point from $\mathcal{P}$ called the* origin *of the reference system and $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is a basis of the vector space $\mathcal{V}$.*

Denote by E_1, E_2, E_3 the points for which $\overrightarrow{u} = \overrightarrow{OE_1}$, $\overrightarrow{v} = \overrightarrow{OE_2}$, $\overrightarrow{w} = \overrightarrow{OE_3}$.

Definition 1.6. *The system of points (O, E_1, E_2, E_3) is called* the affine reference system associated to the cartesian reference system $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$.

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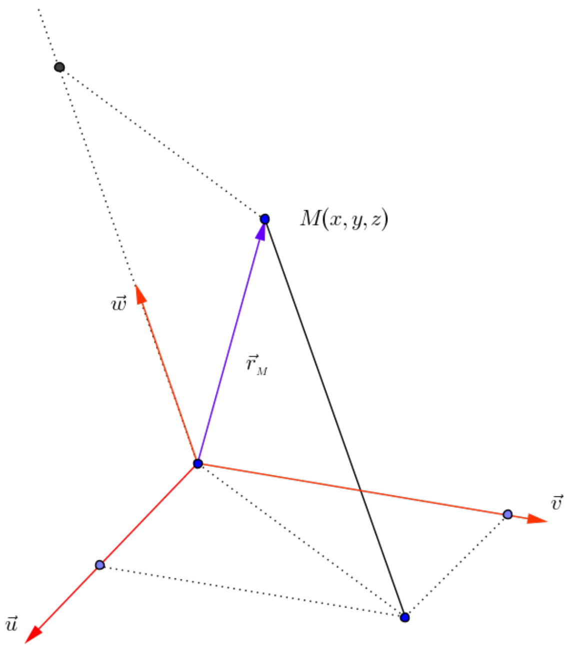
The straight lines OE_i, $i \in \{1,2,3\}$, oriented from O to E_i are called *the coordinate axes*. The coordinates x, y, z of the position vector $\overrightarrow{r}_M = \overrightarrow{OM}$ with respect to the basis $[\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x,y,z)$.



Also, for the column matrix of coordinates of the vector $\overrightarrow{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\overrightarrow{r}_M = x\,\overrightarrow{u} + y\,\overrightarrow{v} + z\,\overrightarrow{w}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Remark 1.7. *If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are two points, then*

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= x_B\,\overrightarrow{u} + y_B\,\overrightarrow{v} + z_B\,\overrightarrow{w} - (x_A\,\overrightarrow{u} + y_A\,\overrightarrow{v} + z_A\,\overrightarrow{w}) \\ &= (x_B - x_A)\,\overrightarrow{u} + (y_B - y_A)\,\overrightarrow{v} + (z_B - z_A)\,\overrightarrow{w}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

Proposition 1.8. *Let Δ be a straight line and let $A \in \Delta$ be a given point. The set*

$$\overrightarrow{\Delta} = \{\overrightarrow{AM} \mid M \in \Delta\}$$

is an one dimensional subspace of $\mathcal{V}$. It is independent on the choice of $A \in \Delta$ and is called the director subspace *of Δ or the* direction *of Δ.*

Remark 1.9. *The straight lines Δ, Δ' are parallel if and only if $\overrightarrow{\Delta} = \overrightarrow{\Delta}'$*

Definition 1.10. We call *director vector* of the straigh line Δ every nonzero vector $\{\overrightarrow{d}\} \in \overrightarrow{\Delta}$.

If $\overrightarrow{d} \in \mathcal{V}$ is a nonzero vector and $A \in \mathcal{P}$ is a given point, then there exits a unique straight line which passes through A and has the direction $\langle \overrightarrow{d} \rangle$. This stright line is

$$\Delta = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d} \rangle\}.$$

Δ is called the straight line which passes through O and is parallel to the vector $\overrightarrow{d}$.

Proposition 1.11. *Let π be a plane and let $A \in \pi$ be a given point. The set $\overrightarrow{\pi} = \{\overrightarrow{AM} \in \mathcal{V} \mid M \in \pi\}$ is a two dimensional subspace of $\mathcal{V}$. It is independent on the position of A inside π and is called the* director subspace, the director plane *or the* direction *of the plane π.*

Remark 1.12. • The planes π, π' are parallel if and only if $\overrightarrow{\pi} = \overrightarrow{\pi}'$.

• If $\overrightarrow{d}_1, \overrightarrow{d}_2$ are two linearly independent vectors and $A \in \mathcal{P}$ is a fixed point, then there exists a unique plane through A whose direction is $\langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle$. This plane is $\pi = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle\}$.

We say that π is the plane which passes through the point A and is parallel to the vectors $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$.

1.1.3 The vector ecuation of the straight lines and planes

Let Δ be a straight line and let $A \in \Delta$ be a given point.

$$\overrightarrow{r}_M = \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{r}_A + \overrightarrow{AM}.$$

Thus

$$\begin{aligned} \{\overrightarrow{r}_M \mid M \in \Delta\} &= \{\overrightarrow{r}_A + \overrightarrow{AM} \mid M \in \Delta\} \\ &= \overrightarrow{r}_A + \{\overrightarrow{AM} \mid M \in \Delta\} \\ &= \overrightarrow{r}_A + \overrightarrow{\Delta}. \end{aligned}$$

Similarly, for a plane π and $B \in \pi$ a given point, then

$$\{\overrightarrow{r}_M \mid M \in \pi\} = \overrightarrow{r}_B + \overrightarrow{\pi}.$$

Generally speaking, a subset X of a vector space is called *affine variety* if either $X = \emptyset$ or there exists $a \in V$ and a vector subspace U of V, such that $X = a + U$.

$$\dim(X) = \begin{cases} -1 & \text{dacă } X = \emptyset \\ \dim(U) & \text{dacă } X = a + U, \end{cases}$$

Proposition 1.13. *The bijection φ_O transforms the straight lines and the planes of the space $\mathcal{P}$ into the one and two dimnensional affine varieties of the vector space $\mathcal{V}$.*

Let Δ be a straight line, let π be a plane, $\{\overrightarrow{d}\}$ be a basis of $\overrightarrow{\Delta}$ and let $[\overrightarrow{d}_1, \overrightarrow{d}_2]$ be a casis of $\overrightarrow{\pi}$. Then for $A \in \Delta$, we obtain the equivalence $M \in \Delta$ if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\overrightarrow{r}_M = \overrightarrow{r}_A + \lambda \overrightarrow{d}. \tag{1.1}$$

The relation (1.1) is called *the vector equation* of the straight line Δ. Similalrly, for $B \in \pi$, we obtain the equivalence $M \in \pi$ if and only if there exists $\lambda_2, \lambda_2 \in \mathbb{R}$ such that

$$\overrightarrow{r}_M = \overrightarrow{r}_A + \lambda_1 \overrightarrow{d}_1 + \lambda_2 \overrightarrow{d}_2. \tag{1.2}$$

The relation (1.2) is called the *vector equation* of the plane π.

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Proposition 1.14. *If A, B are different points of a straight line Δ, then its vector equation can be put in the form*

$$\vec{r}_M = (1-\lambda)\,\vec{r}_A + \lambda\,\vec{r}_B,\ \lambda \in \mathbb{R}. \tag{1.3}$$

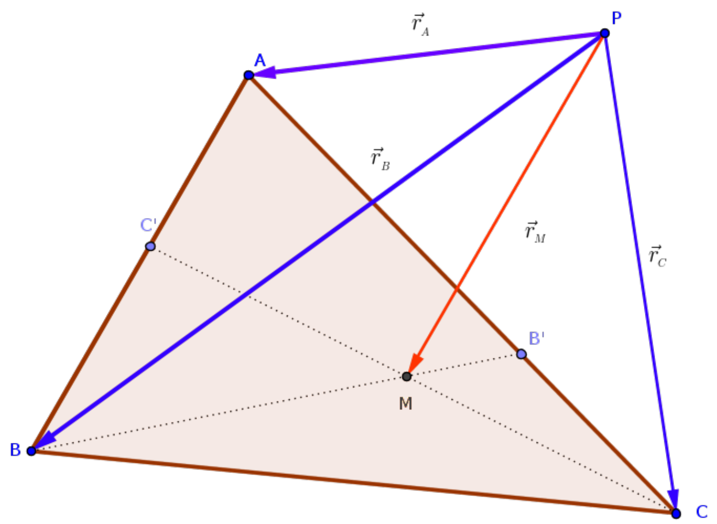
Proposition 1.15. *If A, B, C are three noncolinear points within the plane π, then the vector equation of the plane π can be put in the form*

$$\vec{r}_M = (1-\lambda_1-\lambda_2)\,\vec{r}_A + \lambda_1\,\vec{r}_B + \lambda_2\,\vec{r}_C,\ \lambda_1,\ \lambda_2 \in \mathbb{R}. \tag{1.4}$$

1.2 Problems

1. ([4, Problema 16, p. 5]) Consider the points C' and B' on the sides AB and AC of the triangle ABC such that $\overrightarrow{AC'} = \lambda\,\overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\,\overrightarrow{CB'}$. The lines BB' and CC' meet at M. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$, $\vec{r}_C = \overrightarrow{PC}$ are the position vectors, with respect to P, of the vertices A, B, C respectively, show that

$$\vec{r}_M = \frac{\vec{r}_A - \lambda\,\vec{r}_B - \mu\,\vec{r}_C}{1-\lambda-\mu}. \tag{1.5}$$



2. ([4, Problema 17, p. 5]) Consider the triangle ABC, its centroid G, its orthocenter H, its incenter I and its circumcenter O. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$, $\vec{r}_C = \overrightarrow{PC}$ are the position vectors with respect to P of the vertices A, B, C respectively, show that:

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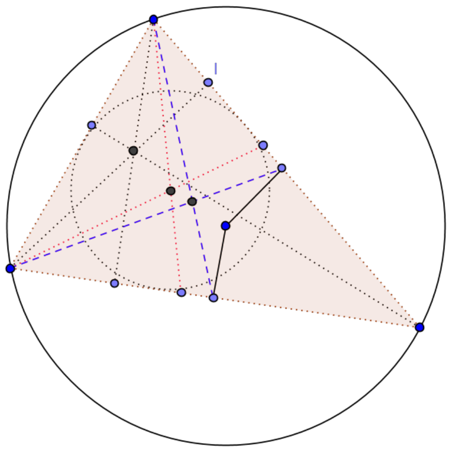
(a) $\vec{r}_G := \overrightarrow{PG} = \dfrac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}$.

(b) $\vec{r}_I := \overrightarrow{PI} = \dfrac{a\,\vec{r}_A + b\,\vec{r}_B + c\,\vec{r}_C}{a+b+c}$.

(c) $\vec{r}_H := \overrightarrow{PH} = \dfrac{(\tan A)\,\vec{r}_A + (\tan B)\,\vec{r}_B + (\tan C)\,\vec{r}_C}{\tan A + \tan B + \tan C}$.

(d) $\vec{r}_O := \overrightarrow{PO} = \dfrac{(\sin 2A)\,\vec{r}_A + (\sin 2B)\,\vec{r}_B + (\sin 2C)\,\vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}$.



3. Consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\vec{r}_M = m\frac{1-n}{1-mn}\,\vec{u} + n\frac{1-m}{1-mn}\,\vec{v} \tag{1.6}$$

and

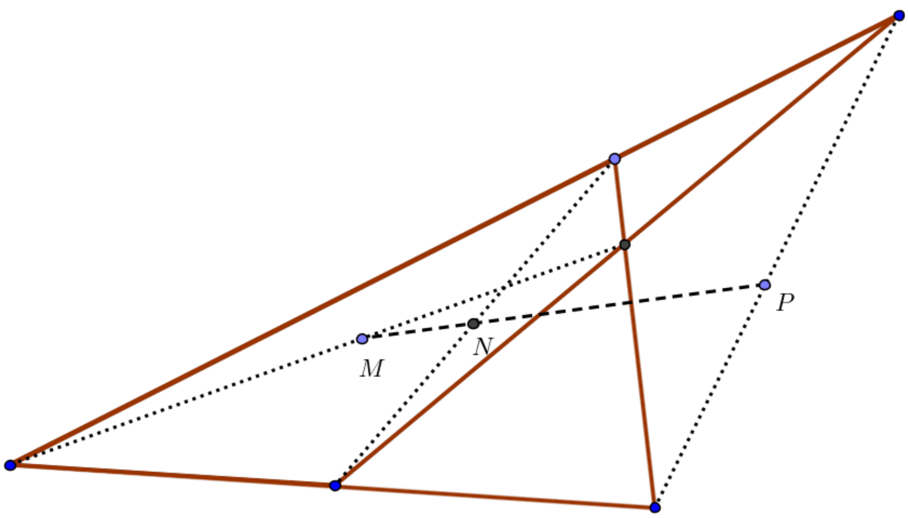
$$\vec{r}_N = m\frac{n-1}{n-m}\,\vec{u} + n\frac{m-1}{m-n}\,\vec{v}, \tag{1.7}$$

where $\{M\} = AB' \cap A'B$, $\{N\} = AA' \cap BB'$, $\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\,\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\,\overrightarrow{OA'}$. In other words

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\,\overrightarrow{OA} + n\frac{1-m}{1-mn}\,\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\,\overrightarrow{OA} + n\frac{m-1}{m-n}\,\overrightarrow{OA'}.$$

4. Show that the midpoints of the diagonals of a complet quadrilateral are collinear (Newton's theorem).



5. Let d, d' be concurrent straight lines and $A, B, C \in d$, $A', B', C' \in d'$. If $AB' \nparallel A'B$, $AC' \nparallel A'C$, $BC' \nparallel B'C$, show that the points $\{M\} := AB' \cap A'B$, $\{N\} := AC' \cap A'C$, $\{P\} := BC' \cap B'C$ are collinear (Pappus' theorem).

6. Let d, d' be two straight lines and $A, B, C \in d$, $A', B', C' \in d'$ three points on each line such that $AB' \| BA'$, $AC' \| CA'$. Show that $BC' \| CB'$ (the affine Pappus' theorem).

7. Let us consider two triangles ABC and $A'B'C'$ such that the lines AA', BB', CC' are concurrent at a point O and $AB \nparallel A'B'$, $BC \nparallel B'C'$ and $CA \nparallel C'A'$. Show that the points $\{M\} = AB \cap A'B'$, $\{N\} = BC \cap B'C'$ and $\{P\} = CA \cap C'A'$ are collinear (Desargues).

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