Geometry
Problem booklet

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Week 10

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1 Quadrics. Brief theoretical background

1.1 Th hyperboloid of two sheets

The *hyperboloid of two sheets* is the surface of equation

$$\mathcal{H}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \tag{1.1}$$

- The coordinate planes are planes of symmetry for $\mathcal{H}_1$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively,

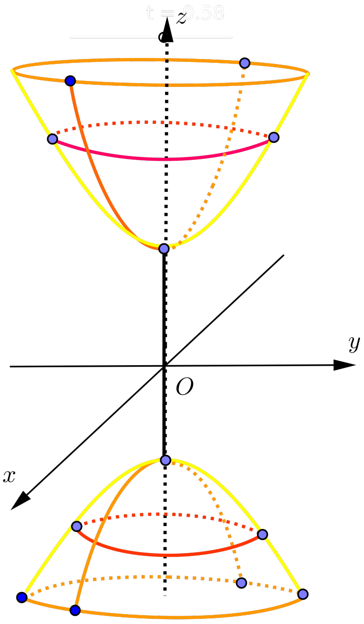
$$\left\{\begin{array}{c} \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0 \\ x = 0 \\ \text{a hyperbola;} \end{array}\right., \left\{\begin{array}{c} \frac{x^2}{a^2} - \frac{z^2}{c^2} + 1 = 0 \\ y = 0 \\ \text{a hyperbola} \end{array}\right., \left\{\begin{array}{c} \frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = 0 \\ z = 0 \\ \text{the empty set} \end{array}\right. ;$$

- The intersections with planes parallel to the coordinate planes are

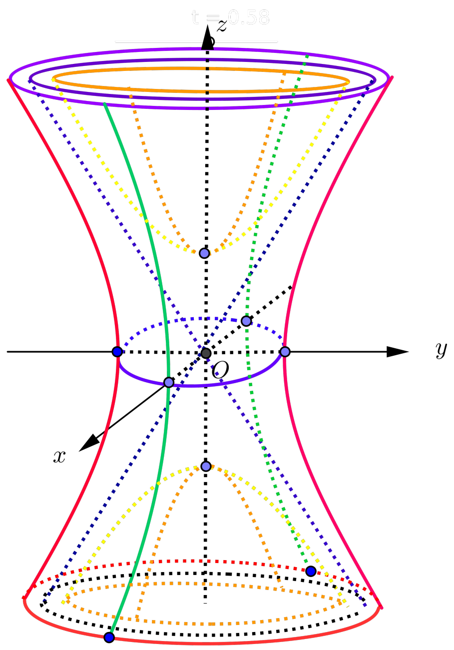
$$\left\{\begin{array}{c} \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 - \frac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{array}\right., \quad \left\{\begin{array}{c} \frac{x^2}{a^2} - \frac{z^2}{c^2} = -1 - \frac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{array}\right.$$

and $\left\{\begin{array}{c} \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 + \frac{\lambda^2}{c^2} \\ z = \lambda \end{array}\right.$.

  - If $|\lambda| > c$, the section is an ellipse;
  - If $|\lambda| = c$, the intersection reduces to a point $(0, 0, \lambda)$;
  - If $|\lambda| < c$, one obtains the empty set.



The hyperboloid of two sheets



The hyperboloids of one and two sheets and their common asymptotic cone

1.2 Elliptic Cones

The surface of equation

$$\mathcal{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{1.2}$$

is called *elliptic cone*.

- The coordinate planes are planes of symmetry for $\mathcal{C}$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{C}$;

- The intersections with the coordinates planes are

$$\begin{cases} \dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0 \\ x=0 \\ \text{two lines} \end{cases}, \begin{cases} \dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}-1=0 \\ y=0 \\ \text{two lines} \end{cases},$$

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=0 \\ z=0 \\ \text{the origin } O(0,0,0). \end{cases}$$

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=-\dfrac{\lambda^2}{a^2} \\ x=\lambda \\ \text{hyperbolas} \end{cases}; \quad \begin{cases} \dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}=-\dfrac{\lambda^2}{b^2} \\ y=\lambda \\ \text{hyperbolas.} \end{cases}$$

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{\lambda^2}{c^2} \\ z=\lambda \\ \text{ellipses} \end{cases}$$

1.3 Elliptic Paraboloids

The surface of equation

$$\mathcal{P}_e : \frac{x^2}{p}+\frac{y^2}{q}=2z, \qquad p,q\in\mathbb{R}_+^*, \tag{1.3}$$

is called *elliptic paraboloid*.

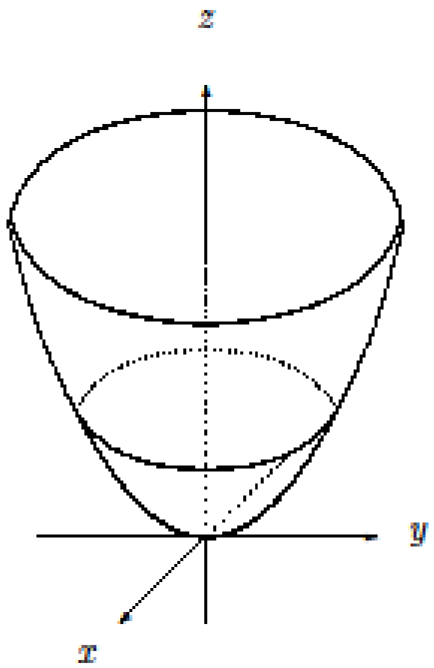
- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are

$$\begin{cases} \dfrac{y^2}{q}=2z \\ x=0 \\ \text{a parabola} \end{cases}, \begin{cases} \dfrac{x^2}{p}=2z \\ y=0 \\ \text{a parabola} \end{cases}, \begin{cases} \dfrac{x^2}{p}+\dfrac{y^2}{q}=0 \\ z=0 \\ \text{the origin } O(0,0,0). \end{cases}$$

- The intersection with the planes parallel to the coordinate planes are $\begin{cases} \dfrac{x^2}{p}+\dfrac{y^2}{q}=2\lambda \\ z=\lambda \end{cases}$,
  - If $\lambda > 0$, the section is an ellipse;
  - If $\lambda = 0$, the intersection reduces to the origin;
  - If $\lambda < 0$, one has the empty set;

  and

$$\begin{cases} \dfrac{y^2}{q}=2z-\dfrac{\lambda^2}{p} \\ x=\lambda \\ \text{parabolas} \end{cases}; \quad \begin{cases} \dfrac{x^2}{p}=2z-\dfrac{\lambda^2}{q} \\ y=\lambda \\ \text{parabolas} \end{cases};$$



1.4 Hyperbolic Paraboloids

The *hyperbolic paraboloid* is the surface given by the equation

$$\mathcal{P}_h : \frac{x^2}{p} - \frac{y^2}{q} = 2z, \qquad p, q > 0. \tag{1.4}$$

- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are, respectively,

$$\left\{ \begin{array}{c} -\dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{array} \right. ; \quad \left\{ \begin{array}{c} \dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{array} \right. ; \quad \left\{ \begin{array}{c} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{two lines.} \end{array} \right. ;$$

- The intersection with the planes parallel to the coordinate planes are

$$\left\{ \begin{array}{c} \dfrac{y^2}{q} = -2z + \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{array} \right. ; \quad \left\{ \begin{array}{c} \dfrac{x^2}{p} = 2z + \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas.} \end{array} \right.$$

$$\left\{ \begin{array}{c} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \\ \text{hyperbolas} \end{array} \right.$$

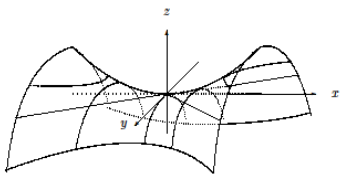
Remark: The hyperbolic paraboloid contains two families of lines. Since

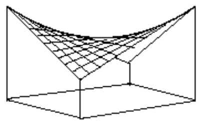
$$\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} \right) \left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} \right) = 2z,$$

then the two families are, respectively, of equations

$$d_\lambda : \begin{cases} \dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} = \lambda \\ \lambda\left(\dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}}\right) = 2z \end{cases}, \lambda \in \mathbb{R} \text{ and}$$

$$d'_\mu : \begin{cases} \dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} = \mu \\ \mu\left(\dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}}\right) = 2z \end{cases}, \mu \in \mathbb{R}.$$





1.5 Singular Quadrics

1.5.1 Elliptic Cylinder, Hyperbolic Cylinder, Parabolic Cylinder

- The *elliptic cylinder* is the surface of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0. \tag{1.5}$$

or

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0, \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

- The *hyperbolic cylinder* is the surface of equation

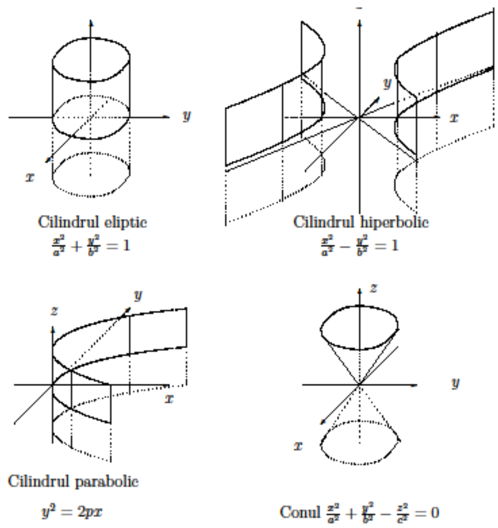
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0, \tag{1.6}$$

or

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0, \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0.$$

- The *parabolic cylinder* is the surface of equation

$$y^2 = 2px, \quad p > 0, \quad (\text{or an alternative equation}). \tag{1.7}$$



2 Generated Surfaces

Consider the 3-dimensional Euclidean space $\mathcal{E}_3$, together with a Cartesian system of coordinates $Oxyz$. Generally, the set

$$S = \{M(x,y,z) : F(x,y,z) = 0\},$$

where $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a real function and D is a domain, is called *surface* of implicit equation $F(x,y,z) = 0$. For example the quadric surfaces, defined in the previous chapter for F a polynomial of degree two, are such of surfaces. On the other hand, the set

$$S_1 = \{M(x,y,z) : x = x(u,v), y = y(u,v), z = z(u,v)\},$$

where $x, y, z : D_1 \subseteq \mathbb{R}^2 \to \mathbb{R}$, is a *parameterized surface*, of parametric equations

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}, \quad (u,v) \in D_1.$$

The intersection between two surfaces is a *curve* in 3-space (remember, for instance, that the intersection between a quadric surface and a plane is a conic section, hence the conics are plane curves). Then, the set

$$C = \{M(x,y,z) : F(x,y,z) = 0, G(x,y,z) = 0\},$$

where $F, G : D \subseteq \mathbb{R}^3 \to \mathbb{R}$, is the curve of *implicit* equations

$$\begin{cases} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{cases}.$$

As before, one can parameterize the curve. The set

$$C_1 = \{M(x,y,z) : x = x(t), y = y(t), z = z(t)\},$$

where $x, y, z : I \subseteq \mathbb{R} \to \mathbb{R}$ and I is open, is called *parameterized curve* of parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in I.$$

Let be given a family of curves, depending on one single parameter λ,

$$\mathcal{C}_\lambda : \begin{cases} F_1(x, y, z; \lambda) = 0 \\ F_2(x, y, z; \lambda) = 0 \end{cases}.$$

In general, the family $\mathcal{C}_\lambda$ does not cover the entire space. By eliminating the parameter λ between the two equations of the family, one obtains the equation of the surface *generated* by the family of curves.

Suppose now that the family of curves depends on two parameters λ, μ,

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} F_1(x, y, z; \lambda, \mu) = 0 \\ F_2(x, y, z; \lambda, \mu) = 0 \end{cases},$$

and that the parameters are related through $\varphi(\lambda, \mu) = 0$ If it can be obtained an equation which does not depend on the parameters (by eliminating the parameters between the three equations), then the set of all the points which verify it is called surface *generated* by the family (or the sub-family) of curves.

2.1 Cylindrical Surfaces

Definition 2.1. *The surface generated by a variable line, called* generatrix, *which remains parallel to a fixed line d and intersects a given curve $\mathcal{C}$, is called* cylindrical surface. *The curve $\mathcal{C}$ is called the* director curve *of the cylindrical surface.*

Theorem 2.2. *The cylindrical surface, with the generatrix parallel to the line*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases},$$

which has the director curve

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

(d and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi(\pi_1, \pi_2) = 0. \tag{2.1}$$

Proof. The equations of an arbitrary line, which is parallel to

$$d : \begin{cases} \pi_1(x, y, z) = 0 \\ \pi_2(x, y, z) = 0 \end{cases}, \text{ are } d_{\lambda,\mu} : \begin{cases} \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases}.$$

Not every line from the family $d_{\lambda,\mu}$ intersects the curve $\mathcal{C}$. This happens only when the system of equations

$$\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \\ \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases}$$

is compatible. By eliminating λ and μ between four equations of the system, one obtains a *necessary condition* $\varphi(\lambda, \mu) = 0$ for the parameters λ and μ in order to nonempty intersection between the line $d_{\lambda,\mu}$. The equation of the surface can be determined now from the system

$$\begin{cases} \pi_1(x,y,z) = \lambda \\ \pi_2(x,y,z) = \mu \\ \varphi(\lambda,\mu) = 0 \end{cases},$$

and it is immediate that $\varphi(\pi_1, \pi_2) = 0$. $\square$

Remark 2.3. *Any equation of the form (2.1), where π_1 and π_2 are linear function of x, y and z, represents a cylindrical surface, having the generatrices parallel to* $d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$.

Remark 2.4. *Let us find the equation of the cylindrical surface having the generatrices parallel to*

$$d : \begin{cases} x + y = 0 \\ z = 0 \end{cases}$$

and the director curve given by

$$\mathcal{C} : \begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \end{cases}.$$

The equations of the generatrices d are

$$d_{\lambda,\mu} : \begin{cases} x + y = \lambda \\ z = \mu \end{cases}.$$

They must intersect the curve $\mathcal{C}$, i.e. the system

$$\begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \\ x + y = \lambda \\ z = \mu \end{cases}$$

has to be compatible. A solution of the system can be obtained using the three last equations

$$\begin{cases} x = 1 \\ y = \lambda - 1 \\ z = \mu \end{cases}$$

and, replacing in the first one, one obtains the compatibility condition

$$2(\lambda - 1)^2 + \mu - 1 = 0.$$

Thus, the equation of the required cylindrical surface is

$$2(x + y - 1)^2 + x - 1 = 0.$$

3 Problems

1. Find the equations of the tangent lines to the ellipse $\mathcal{E}: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ having a given angular coefficient $m \in \mathbb{R}$.
2. Find the equations of the tangent lines to the ellipse $\mathcal{E}: x^2+4y^2-20=0$ which are orthogonal to the line $d: 2x-2y-13=0$.
3. Find the equations of the tangent lines to the ellipse $\mathcal{E}: \dfrac{x^2}{25}+\dfrac{y^2}{16}-1=0$, passing through $P_0(10,-8)$.
4. If $M(x,y)$ is a point of the tangent line $T_{M_0}(E)$ of the ellipse $\mathcal{E}: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at one of its points $M_0(x_0,y_0)\in\mathcal{E}$, show that $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\geq 1$.
5. Find the equations of the tangent lines to the hyperbola $\mathcal{H}: \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-1=0$ having a given angular coefficient $m \in \mathbb{R}$.
6. Find the equations of the tangent lines to the hyperbola $\mathcal{H}: \dfrac{x^2}{20}-\dfrac{y^2}{5}-1=0$ which are orthogonal to the line $d: 4x+3y-7=0$.
7. Find the equations of the tangent lines to the parabola $\mathcal{P}: y^2=2px$ having a given angular coefficient $m \in \mathbb{R}$.
8. Find the equation of the tangent line to the parabola $\mathcal{P}: y^2-8x=0$, parallel to $d: 2x+2y-3=0$.
9. Find the equation of the tangent line to the parabola $\mathcal{P}: y^2-36x=0$, passing through $P(2,9)$.
10. Find the locus of the orthogonal projections of the center $O(0,0)$ of the ellipse

$$E: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

on its tangents.

11. Find the locus of the orthogonal projections of the center $O(0,0)$ of the hyperbola

$$H: \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

on its tangents.

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