Geometry
Problem booklet

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1 Transformations

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Transformations of the plane

Definition 1.1. *An affine transformation of the plane is a mapping*

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ L(x,y) = (ax+by+c, dx+ey+f), \tag{1.1}$$

for some constant real numbers a,b,c,d,e,f.

By using the matrix language, the action of the map L can be written in the form

$$L(x,y) = [x\ y]\begin{bmatrix} a & d \\ b & e \end{bmatrix} + [c\ f].$$

The affine transformation L can be also identified with the map $L^c : \mathbb{R}^{2\times 1} \longrightarrow \mathbb{R}^{2\times 1}$ given by

$$\begin{aligned} L^c\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} ax+by+c \\ dx+ey+f \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix} \\ &= [L]\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}, \text{ where } [L] = \begin{bmatrix} a & b \\ d & e \end{bmatrix}. \end{aligned}$$

Lemma 1.2. *If $(aB-bA)^2 + (dB-eA)^2 > 0$, then the affine transformation (1.1) maps the line $(d)\ Ax+By+C=0$ to the line*

$$(eA-dB)x + (aB-bA)y + (bf-ce)A - (af-cd)B + (ae-bd)C = 0.$$

If $aB-bA = dB-eA = 0$, then $ae-bd=0$ and L is the constant map $\left(\frac{cB-bC}{B}, \frac{fB-eC}{B}\right)$.

Definition 1.3. *An affine transformation (1.1) is said to be* singular *if*

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0 \text{ i.e. } ae-bd=0.$$

and non-singular otherwise.

1.1.1 Translations

Note that the affine transformation L is nonsingular if and only if it is invertible. In such a case the inverse L^{-1} is a non-singular affine transformation and $[L^{-1}] = [L]^{-1}$.

Definition 1.4. *The* translation *of vector $(h,k) \in \mathbb{R}^2$ is the affine transformation*

$$T(h,k) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ [T(h,k)]\,(x,y) = (x+h, y+k).$$

Thus

$$[T(h,k)^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+h \\ y+k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix},$$

i.e.

$$[T(h,k)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the translation $T(h,k)$ is non-singular (invertible) and $(T(h,k))^{-1} = T(-h,-k)$.

1.1.2 Scaling about the origine

Definition 1.5. *The* scaling about the origine *by non-zero scaling factors* $(s_x, s_y) \in \mathbb{R}^2$ *is the affine transformation*

$$S(s_x, s_y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ \left[S(s_x, s_y)\right](x, y) = (s_x \cdot x, s_y \cdot y).$$

Thus

$$\left[S(s_x, s_y)^c\right]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[S(s_x, s_y)] = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}.$$

Note that the scaling about the origin by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ is non-singular (invertible) and $(S(s_x, s_y))^{-1} = S(s_x^{-1}, s_y^{-1})$.

1.1.3 Reflections

Definition 1.6. *The* reflections about the x-axis *and the y-axis respectively are the affine transformation*

$$r_x, r_y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ r_x(x, y) = (x, -y),\ r_y = (-x, y).$$

Thus

$$[r_x^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[r_x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.\ \textit{Similarly}\ [r_y] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $r_x = S(-1, 1)$ and $r_y = S(1, -1)$. Thus the two reflections are non-singular (invertible) and $r_x^{-1} = r_x$, $r_y^{-1} = r_y$.

Definition 1.7. *The* reflection $r_l : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the line l *maps a given point M to the point M' defined by the property that l is the perpendicular bisector of the segment MM'. One can show that the action of the reflection about the line $l : ax + by + c = 0$ is*

$$r_l(x, y) = \left(\frac{b^2 - a^2}{a^2 + b^2}x - \frac{2ab}{a^2 + b^2}y - \frac{2ac}{a^2 + b^2}, -\frac{2ab}{a^2 + b^2}x + \frac{a^2 - b^2}{a^2 + b^2}y - \frac{2bc}{a^2 + b^2}\right).$$

Thus

$$\begin{aligned} [r_l^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} \dfrac{b^2 - a^2}{a^2 + b^2}x - \dfrac{2ab}{a^2 + b^2}y - \dfrac{2ac}{a^2 + b^2} \\ -\dfrac{2ab}{a^2 + b^2}x + \dfrac{a^2 - b^2}{a^2 + b^2}y - \dfrac{2bc}{a^2 + b^2} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{b^2 - a^2}{a^2 + b^2} & -\dfrac{2ab}{a^2 + b^2} \\ -\dfrac{2ab}{a^2 + b^2} & \dfrac{a^2 - b^2}{a^2 + b^2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \dfrac{2ac}{a^2 + b^2} \\ \dfrac{2bc}{a^2 + b^2} \end{bmatrix}, \end{aligned}$$

i.e.

$$[r_l] = \frac{1}{a^2 + b^2}\begin{bmatrix} b^2 - a^2 & -2ab \\ -2ab & a^2 - b^2 \end{bmatrix}.$$

Note that the reflection r_l is non-singular (invertible) and $r_l^{-1} = r_l$.

1.1.4 Rotations

Definition 1.8. *The* rotation $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *about the origin through an angle* θ *maps a point* $M(x,y)$ *into a point* $M'(x',y')$ *with the properties that the segments* $[OM]$ *and* $[OM']$ *are congruent and the* $m(\widehat{MOM'}) = \theta$*. If* $\theta > 0$ *the rotation is supposed to be* anticlockwise *and for* $\theta < 0$ *the rotation is* clockwise*. If* $(x,y) = (r\cos\varphi, r\sin\varphi)$*, then the coordinates of the rotated point are* $(r\cos(\theta+\varphi), r\sin(\theta+\varphi)) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$*, i.e.*

$$R_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Thus

$$\begin{aligned}[R_\theta^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},\end{aligned}$$

i.e.

$$[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Note that the rotation R_θ *is non-singular (invertible) and* $R_\theta^{-1} = R_{-\theta}$*.*

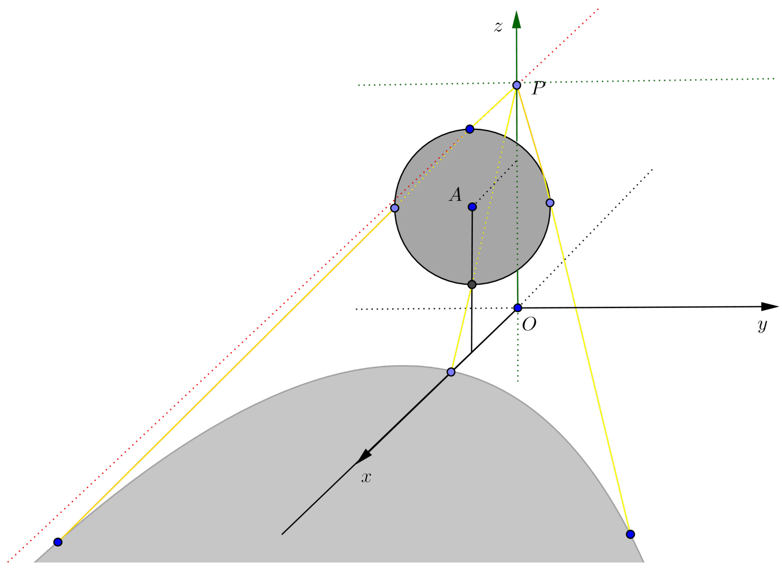
1.2 Problems

1. Find the equation of the cylindrical surface whose director curve is the planar curve

$$(C)\ \begin{cases} y^2 + z^2 = x \\ x = 2z \end{cases}$$

   and the generatrix is perpendicular to the plane of the director curve.

2. A disk of radius 1 is centered at the point $A(1,0,2)$ and is parallel to the plane yOz. A source of light is placed at the point $P(0,0,3)$. Characterize analitically the shadow of the disk rushed over the plane xOy.



3. Consider a circle and a line parallel with the plane of the circle. Find the equation of the conoidal surface generated by a variable line which intersects the line (d) and the circle (C) and remains orthogonal to (d). (The Willis conoid)

4. Find the equation of the revolution surface generated by the rotation of a variable line through a fixed line.

5. The *torus* is the revolution surface obtained by the rotation of a circle C about a fixed line (d) within the plane of the circle such that $d \cap C = \emptyset$. Find the equation of the torus.

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