Geometry
Problem booklet

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Contents

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1 Week 9: Quadrics

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background. Quadrics

2 Quadrics

2.1 The ellipsoid

The *ellipsoid* is the quadric surface given by the equation

$$\mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0, \qquad a,b,c\in\mathbb{R}_+^*. \tag{2.1}$$

- The coordinate planes are all planes of symmetry of $\mathcal{E}$ since, for an arbitrary point $M(x,y,z)\in\mathcal{E}$, its symmetric points with respect to these planes, $M_1(-x,y,z)$, $M_2(x,-y,z)$ and $M_3(x,y,-z)$ belong to $\mathcal{E}$; therefore, the coordinate axes are axes of symmetry for $\mathcal{E}$ and the origin O is the center of symmetry of the ellipsoid (2.1);
- The traces in the coordinates planes are ellipses of equations

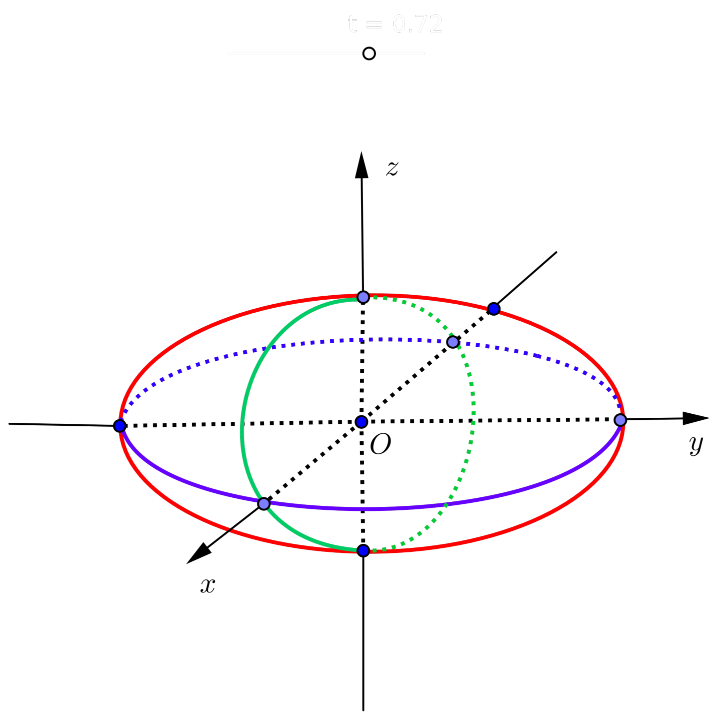
$$\begin{cases}\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1=0\\ x=0\end{cases},\begin{cases}\dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}-1=0\\ y=0\end{cases},\begin{cases}\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0\\ z=0.\end{cases}$$

- The sections with planes parallel to xOy are given by setting $z=\lambda$ in (2.1). Then, a section is of equations $\begin{cases}\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1-\dfrac{\lambda^2}{c^2}\\ z=\lambda\end{cases}$.
- If $|\lambda|<c$, the section is an ellipse

$$\begin{cases}\dfrac{x^2}{\left(a\sqrt{1-\dfrac{\lambda^2}{c^2}}\right)^2}+\dfrac{y^2}{\left(b\sqrt{1-\dfrac{\lambda^2}{c^2}}\right)^2}=1\\ z=\lambda\end{cases};$$

- If $|\lambda|=c$, the intersection is reduced to one (tangency) point $(0,0,\lambda)$;
- If $|\lambda|>c$, the plane $z=\lambda$ does not intersect the ellipsoid $\mathcal{E}$.

The sections with planes parallel to xOz or yOz are obtained in a similar way.



2.2 Hyperboloids of One Sheet

The surface of equation

$$\mathcal{H}_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{2.2}$$

is called *hyperboloid of one sheet*.

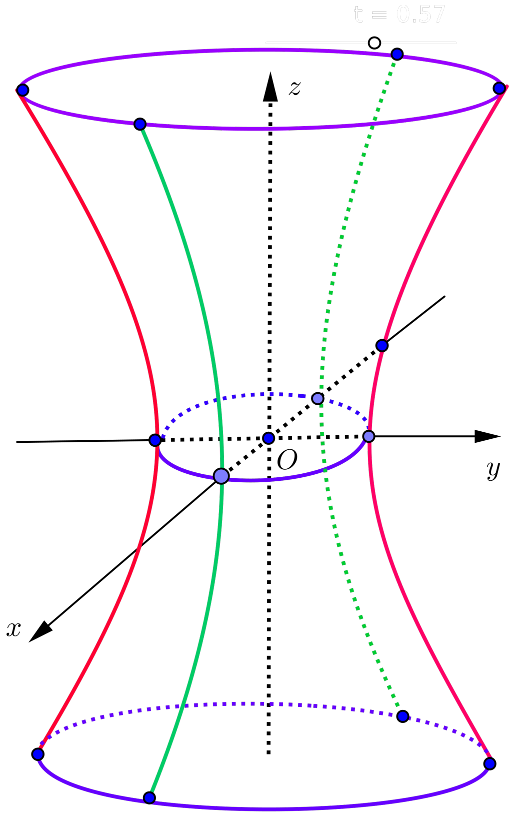
- The coordinate planes are planes of symmetry for $\mathcal{H}_1$; hence, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively, of equations

$$\left\{\begin{array}{c} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} - 1 = 0 \\ x = 0 \\ \text{a hyperbola} \end{array}\right. ; \left\{\begin{array}{c} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} - 1 = 0 \\ y = 0 \\ \text{a hyperbola} \end{array}\right. ; \left\{\begin{array}{c} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0 \\ z = 0 \\ \text{an ellipse} \end{array}\right. ;$$

- The intersections with planes parallel to the coordinate planes are

$$\left\{\begin{array}{c} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{array}\right. ; \left\{\begin{array}{c} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{array}\right. ;$$

$$\left\{\begin{array}{c} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{\lambda^2}{c^2} \\ z = \lambda \\ \text{ellipses} \end{array}\right. ;$$



Remark: The surface $\mathcal{H}_1$ contains two families of lines, as

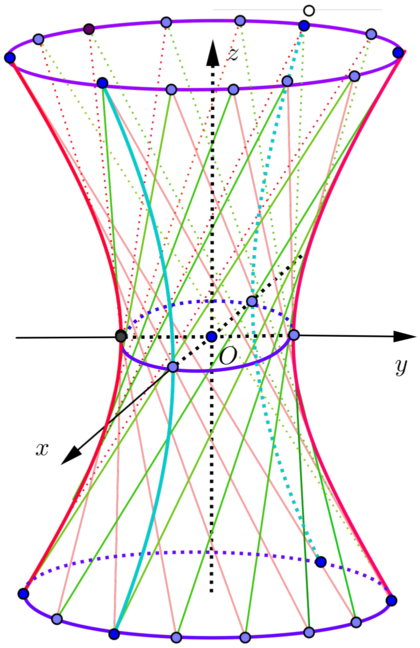
$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}\Leftrightarrow\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right).$$

The equations of the two families of lines are:

$$d_\lambda:\begin{cases}\lambda\left(\dfrac{x}{a}+\dfrac{z}{c}\right)=1+\dfrac{y}{b}\\ \dfrac{x}{a}-\dfrac{z}{c}=\lambda\left(1-\dfrac{y}{b}\right)\end{cases},\ \lambda\in\mathbb{R},$$

$$d'_\mu:\begin{cases}\mu\left(\dfrac{x}{a}+\dfrac{z}{c}\right)=1-\dfrac{y}{b}\\ \dfrac{x}{a}-\dfrac{z}{c}=\mu\left(1+\dfrac{y}{b}\right)\end{cases},\ \mu\in\mathbb{R}.$$

Through any point on $\mathcal{H}_1$ pass two lines, one line from each family.



2.3 Th hyperboloid of two sheets

The *hyperboloid of two sheets* is the surface of equation

$$\mathcal{H}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \tag{2.3}$$

- The coordinate planes are planes of symmetry for $\mathcal{H}_1$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively,

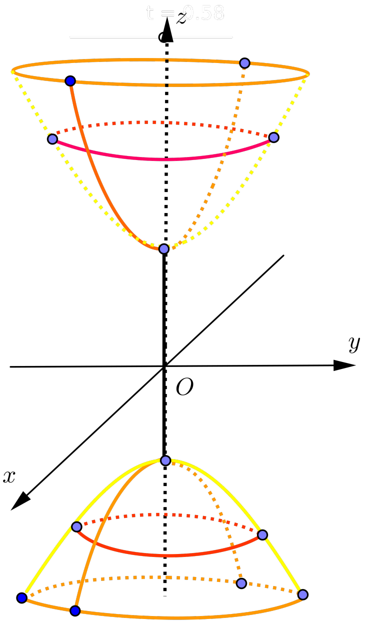
$$\left\{ \begin{array}{c} \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0 \\ x = 0 \\ \text{a hyperbola;} \end{array} \right. , \left\{ \begin{array}{c} \frac{x^2}{a^2} - \frac{z^2}{c^2} + 1 = 0 \\ y = 0 \\ \text{a hyperbola} \end{array} \right. , \left\{ \begin{array}{c} \frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = 0 \\ z = 0 \\ \text{the empty set} \end{array} \right. ;$$

- The intersections with planes parallel to the coordinate planes are

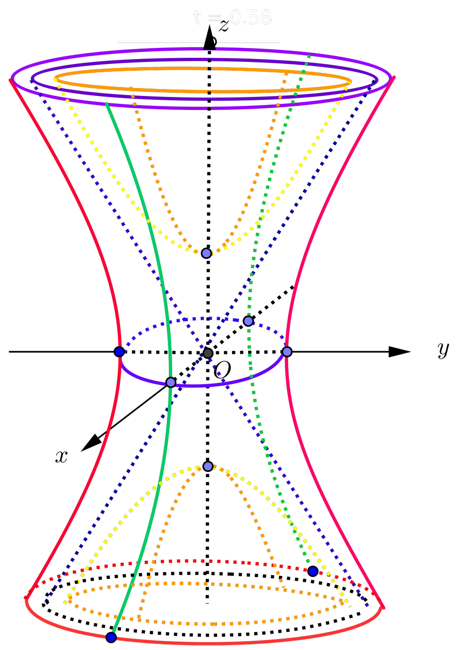
$$\left\{ \begin{array}{c} \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 - \frac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{array} \right. , \left\{ \begin{array}{c} \frac{x^2}{a^2} - \frac{z^2}{c^2} = -1 - \frac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{array} \right.$$

and $\left\{ \begin{array}{c} \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 + \frac{\lambda^2}{c^2} \\ z = \lambda \end{array} \right. .$

  - If $|\lambda| > c$, the section is an ellipse;
  - If $|\lambda| = c$, the intersection reduces to a point $(0, 0, \lambda)$;
  - If $|\lambda| < c$, one obtains the empty set.



The hyperboloid of two theets



The hyperboloids of one and two sheets and their common asymptotic cone

2.4 Problems

1. Show the following identities:

   (a) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a}, \vec{c}, \vec{d})\,\vec{b} - (\vec{b}, \vec{c}, \vec{d})\,\vec{a} = (\vec{a}, \vec{b}, \vec{d})\,\vec{c} - (\vec{a}, \vec{b}, \vec{c})\,\vec{d}$.

   (b) $(\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}) = (\vec{u}, \vec{v}, \vec{w})^2$.

2. The *reciprocal vectors* of the noncoplanar vectors $\vec{u}, \vec{v}, \vec{w}$ are defined by

$$\vec{u}' = \frac{\vec{v} \times \vec{w}}{(\vec{u}, \vec{v}, \vec{w})}, \quad \vec{v}' = \frac{\vec{w} \times \vec{u}}{(\vec{u}, \vec{v}, \vec{w})}, \quad \vec{w}' = \frac{\vec{u} \times \vec{v}}{(\vec{u}, \vec{v}, \vec{w})}.$$

Show that:

(a)

$$\begin{aligned}\vec{a} &= (\vec{a}\cdot\vec{u}')\,\vec{u} + (\vec{a}\cdot\vec{v}')\,\vec{v} + (\vec{a}\cdot\vec{w}')\,\vec{w}\\ &= \frac{(\vec{a},\vec{v},\vec{w})}{(\vec{u},\vec{v},\vec{w})}\,\vec{u} + \frac{(\vec{u},\vec{a},\vec{w})}{(\vec{u},\vec{v},\vec{w})}\,\vec{v} + \frac{(\vec{u},\vec{v},\vec{a})}{(\vec{u},\vec{v},\vec{w})}\,\vec{w}\,.\end{aligned}$$

(b) the reciprocal vectors of $\vec{u}', \vec{v}', \vec{w}'$ are the vectors $\vec{u}, \vec{v}, \vec{w}$.

3. Let d_1, d_2, d_3, d_4 be pairwise skew straight lines. Assuming that $d_{12} \perp d_{34}$ and $d_{13} \perp d_{24}$, show that $d_{14} \perp d_{23}$, where d_{ik} is the common perpendicular of the lines d_i and d_k.

4. Find the value of the parameter α for which the pencil of planes through the straight line AB has a common plane with the pencil of planes through the straight line CD, where $A(1, 2\alpha, \alpha)$, $B(3, 2, 1)$, $C(-\alpha, 0, \alpha)$ and $D(-1, 3, -3)$.

5. Find the value of the parameter λ for which the straight lines

$$(d_1)\ \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1},\ (d_2)\ \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{\lambda}$$

are coplanar. Find the coordinates of their intersection point in that case.

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