Geometry
Problem booklet

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1 Transformations

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Transformations of the plane

1.1.1 Reflections

Definition 1.1. *The* reflections about the x-axis *and the y-axis respectively are the affine transformation*

$$r_x, r_y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ r_x(x,y) = (x,-y),\ r_y = (-x,y).$$

Thus

$$[r_x^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[r_x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Similarly

$$[r_y] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $r_x = S(-1,1)$ and $r_y = S(1,-1)$. Thus the two reflections are non-singular (invertible) and $r_x^{-1} = r_x$, $r_y^{-1} = r_y$.

Definition 1.2. *The* reflection $r_l : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the line l *maps a given point M to the point M' defined by the property that l is the perpendicular bisector of the segment MM'. One can show that the action of the reflection about the line $l : ax+by+c=0$ is*

$$r_l(x,y) = \left(\frac{b^2-a^2}{a^2+b^2}x - \frac{2ab}{a^2+b^2}y - \frac{2ac}{a^2+b^2}, -\frac{2ab}{a^2+b^2}x + \frac{a^2-b^2}{a^2+b^2}y - \frac{2bc}{a^2+b^2}\right).$$

$$\begin{aligned} \text{Thus } [r_l^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2}x - \frac{2ab}{a^2+b^2}y - \frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2}x + \frac{a^2-b^2}{a^2+b^2}y - \frac{2bc}{a^2+b^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{2ac}{a^2+b^2} \\ \frac{2bc}{a^2+b^2} \end{bmatrix}, \end{aligned}$$

i.e. $[r_l] = \frac{1}{a^2+b^2}\begin{bmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}$. *Note that the reflection r_l is non-singular (invertible) and $r_l^{-1} = r_l$.*

1.1.2 Shears

Definition 1.3. *Given a fixed direction in the plane specified by a unit vector $v = (v_1, v_2)$, consider the lines d with direction v and the oriented distance d from the origin. The* shear *about the origin of factor r in the direction v is defined to be the transformation which maps a point $M(x,y)$ on d to the point $M' = M + rdv$. The equation of the line through M of direction v is $v_2X - v_1Y + (v_1y - v_2x) = 0$. The oriented distacnce from the origine to this line is $v_1y - v_2x$. Thus the action of the shear $Sh(v,r) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the origin of factor r in the direction v is*

$$\begin{aligned} Sh(v,r)(x,y) &= (x,y) + rd(v_1,v_2) \\ &= (x,y) + (r(v_1y - v_2x)v_1, r(v_1y - v_2x)v_2) \\ &= (x,y) + \left(-rv_1v_2x + rv_1^2y, -rv_2^2x + rv_1v_2y\right) \\ &= \left((1 - rv_1v_2)x + rv_1^2y, -rv_2^2x + (1 + rv_1v_2)y\right) \end{aligned}$$

Thus

$$\begin{aligned} [Sh(v,r)^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} (1 - rv_1v_2)x + rv_1^2y \\ -rv_2^2x + (1 + rv_1v_2)y \end{bmatrix} \\ &= \begin{bmatrix} 1 - rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1 + rv_1v_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \end{aligned}$$

i.e. $[Sh(v,r)] = \begin{bmatrix} 1 - rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1 + rv_1v_2 \end{bmatrix}$.

1.2 Homogeneous coordinates

The affine transformation

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ L(x,y) = (ax + by + c, dx + ey + f)$$

can be written by using the matrix language and by equations:

1. (a) indentifying the vectors $(x,y) \in \mathbb{R}^2$ with the line matrices $[x\ y] \in \mathbb{R}^{1\times 2}$ and implicitely $\mathbb{R}^2$ with $\mathbb{R}^{1\times 2}$:

$$L[x\ y] = [x\ y]\begin{bmatrix} a & d \\ b & e \end{bmatrix} + [c\ f].$$

   (b) indentifying the vectors $(x,y) \in \mathbb{R}^2$ with the column matrices $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1}$ and implicitely $\mathbb{R}^2$ cu $\mathbb{R}^{2\times 1}$:

$$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}.$$

2. $\left\{\begin{array}{l} x' = ax + by + c \\ y' = dx + ey + f. \end{array}\right. \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$

   Observe that the representation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

In this lesson we identify the points $(x, y) \in \mathbb{R}^2$ with the points $(x, y, 1) \in \mathbb{R}^3$ and even with the punctured lines of $\mathbb{R}^3$, (rx, ry, r), $r \in \mathbb{R}^*$. Due to technical reasons we shall actually identify the points $(x, y) \in \mathbb{R}^2$ with the punctured lines of $\mathbb{R}^3$ represented in the form

$$\begin{bmatrix} rx \\ ry \\ r \end{bmatrix}, \ r \in \mathbb{R}^*,$$

and the latter ones we shall call *homogeneous coordinates* of the point $(x, y) \in \mathbb{R}^2$. The set of homogeneous coordinates (x, y, w) will be denoted by $\mathbb{RP}^2$ and call it the real *projective plane*. The homogeneous coordinates $(x, y, w) \in \mathbb{RP}^2$, $w \neq 0$ şi $\left(\frac{x}{w}, \frac{y}{w}, 1\right)$represent the same element of $\mathbb{RP}^2$.

Observation 1.4. *The projective plane $\mathbb{RP}^2$ is actually the quotient set $(\mathbb{R}^3 \setminus \{0\})/\sim$, where $'\sim'$ is the following equivalence relation on $\mathbb{R}^3 \setminus \{0\}$:*

$$(x, y, w) \sim (\alpha, \beta, \gamma) \Leftrightarrow \exists r \in \mathbb{R}^* \text{ a.î. } (x, y, w) = r(\alpha, \beta, \gamma).$$

Observe that the equivalence classes of the equivalence relation $\sim'$ are the punctured lines of $\mathbb{R}^3$ through the origin without the origin itself, i.e. the elements of the real projective plane $\mathbb{RP}^2$.

Definition 1.5. *A* projective transformation *of the projective plane $\mathbb{RP}^2$ is a transformation*

$$L : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2, \ L \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} ax + by + cw \\ dx + ey + fw \\ gx + hy + kw \end{bmatrix}, \tag{1.1}$$

where $a, b, c, d, e, f, g, h, k \in \mathbb{R}$. Note that

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

is called the homogeneous transformation matrix *of L.*

Observe that a projective transformation (1.1) is well defined since

$$L \begin{bmatrix} rx \\ ry \\ rw \end{bmatrix} = \begin{bmatrix} arx + bry + crw \\ drx + ery + frw \\ grx + hry + krw \end{bmatrix} = \begin{bmatrix} r(ax + by + cw) \\ r(dx + ey + fw) \\ r(gx + hy + kw) \end{bmatrix}.$$

If $g = h = 0$ and $k \neq 0$, then the projective transformation (1.1) is said to be *affine*. The restriction of the affine transformation (1.1), which corresponds to the situation $g = h = 0$ and $k = 1$, to the subspace $w = 1$, has the form

$$L \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + cw \\ dx + ey + fw \\ 1 \end{bmatrix}, \tag{1.2}$$

i.e.

$$\begin{cases} x' = ax + by + c \\ y' = dx + ey + f. \end{cases} \tag{1.3}$$

Observation 1.6. *If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two projective applications, then their product (concatenation) transformation $L_1 \circ L_2$ is also a projective transformation and its homogeneous transformation matrix is the product of the homogeneous transformation matrices of L_1 and L_2 .*

Indeed, if

$$L_1 \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

and

$$L_2 \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

then

$$(L_1 \circ L_2) \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \left(\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Observation 1.7. *If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two affine applications, then their product $L_1 \circ L_2$ is also an affine transformation.*

1.3 Problems

1. Consider a quadrilateral with vertices $A(1,1)$, $B(3,1)$, $C(2,2)$, and $D(1.5,3)$. Find the image quadrilaterals through the translation $T(1,2)$, the scaling $S(2,2.5)$, the reflections about the x and y-axes, the clockwise and anticlockwise rotations through the angle $\pi/2$ and the shear $Sh\left(\left(2/\sqrt{5}, 1/\sqrt{5}\right), 1.5\right)$.

2. Find the concatenation (product) of an anticlockwise rotation about the origin through an angle of $\frac{3\pi}{2}$ followed by a scaling by a factor of 3 units in the x-direction and 2 units in the y-direction. (Hint: $S(3,2)R_{3\pi/2}$)

3. Find the homogeneous matrix of the product (concatenation) $S(3,2) \circ R_{\frac{3\pi}{2}}$.

4. Find the equations of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ.

   Solution The homogeneous transformation matrix of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ is

$$\begin{aligned} R_\theta(x_0, y_0) &= T(x_0, y_0) R_\theta T(-x_0, -y_0) \\ &= \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta & -x_0\cos\theta + y_0\sin\theta + x_0 \\ \sin\theta & \cos\theta & -x_0\sin\theta - y_0\cos\theta + y_0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Thus, the equations of the required rotation are:

$$\begin{cases} x' &= x\cos\theta - y\sin\theta - x_0\cos\theta + y_0\sin\theta + x_0 \\ y' &= x\sin\theta + y\cos\theta - x_0\sin\theta - y_0\cos\theta + y_0. \end{cases}.$$

5. Show that the concatenation (product) of two rotations, the first through an angle θ about a point $P(x_0, y_0)$ and the second about a point $Q(x_1, y_1)$ (distinct from P) through an angle $-\theta$ is a translation.

