Geometry
Problem booklet

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1 Week 5: Products of vectors

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background. Products of vectors

1.1.1 The dot product

Definition 1.1. The real number

$$\vec{a} \cdot \vec{b} = \begin{cases} 0 \text{ if } \vec{a} = 0 \text{ or } \vec{b} = 0 \\ \|\vec{a}\| \cdot \|\vec{b}\| cos(\widehat{\vec{a}, \vec{b}}) \text{ if } \vec{a} \neq 0 \text{ and } \vec{b} \neq 0 \end{cases} \tag{1.1}$$

is called the *dot product* of the vectors $\vec{a}$, $\vec{b}$.

Remark 1.2. 1. $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0.$

2. $\vec{a} \cdot \vec{a} = \|\vec{a}\| \cdot \|\vec{a}\| \cos 0 = \|\vec{a}\|^2.$

Proposition 1.3. *The dot product has the following properties:*

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a},\ \forall\ \vec{a}, \vec{b} \in \mathcal{V}.$

2. $\vec{a} \cdot (\lambda \vec{b}) = \lambda(\vec{a} \cdot \vec{b}),\ \forall \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}.$

3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c},\ \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}.$

4. $\vec{a} \cdot \vec{a} \geq 0,\ \forall\ \vec{a} \in \mathcal{V}.$

5. $\vec{a} \cdot \vec{a} = 0 \Leftrightarrow \vec{a} = \vec{0}.$

Definition 1.4. A basis of the vector space $\mathcal{V}$ is said to be *orthonormal*, if $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$, $\vec{i} \perp \vec{j}$, $\vec{j} \perp \vec{k}$, $\vec{k} \perp \vec{i}$ ($\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$, $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$). A cartesian reference system $R = (O, \vec{i}, \vec{j}, \vec{k})$ is said to be *orthonormal* if the basis $[\vec{i}, \vec{j}, \vec{k}]$ is orthonormal.

Proposition 1.5. *Let $[\vec{i}, \vec{j}, \vec{k}]$ be an orthonormal basis and $\vec{a}, \vec{b} \in \mathcal{V}$. If $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, then*

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.2}$$

Remark 1.6 1.6. *Let $[\vec{i}, \vec{j}, \vec{k}]$ be an orthonormal basis and $\vec{a}, \vec{b} \in \mathcal{V}$. If $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, then*

1. $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2$ *and we conclude that* $\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$

2.

$$\begin{aligned}\cos(\widehat{\vec{a},\vec{b}}) &= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}\\ &= \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}.\end{aligned} \tag{1.3}$$

In particular

$$\begin{aligned}\cos(\widehat{\vec{a},\vec{i}}) &= \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}};\\ \cos(\widehat{\vec{a},\vec{j}}) &= \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}};\\ \cos(\widehat{\vec{a},\vec{k}}) &= \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}.\end{aligned}$$

3. $\vec{a}\perp\vec{b} \Leftrightarrow a_1b_1+a_2b_2+a_3b_3=0$

1.1.2 Applications of the dot product

• **The distance between two points.** Consider two points $A(x_A,y_A,z_A),\ B(x_B,y_B,z_B)\in\mathcal{P}$. The norm of the vector $\overrightarrow{AB}\ (x_B-x_A,y_B-y_A,z_B-z_A)$ is

$$\|\overrightarrow{AB}\| = \sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}.$$

• **The normal vector of a plane.** Consider the plane $\pi\ :\ Ax+By+Cz+D=0$ and the point $P(x_0,y_0,z_0)\in\pi$. The equation of π becomes

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0. \tag{1.4}$$

If $M(x,y,z)\in\pi$, the coordinates of $\overrightarrow{PM}$ are $(x-x_0,y-y_0,z-z_0)$ and the equation (1.4) tells us that $\vec{n}\cdot\overrightarrow{PM}=0$, for every $M\in\pi$, that is $\vec{n}\perp\overrightarrow{PM}=0$, for every $M\in\pi$, which is equivalent to $\vec{n}\perp\vec{\pi}$, where $\vec{n}\ (A,B,C)$. This is the reason to call $\vec{n}\ (A,B,C)$ *the normal vector* of the plane π.

• **The distance from a point to a plane.** Consider the plane $\pi\ :\ Ax+By+Cz+D=0$, a point $P(x_P,y_P,z_P)\in\mathcal{P}$ and M the orthogonal projection of P on π. The real number δ given by $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$ is called the *oriented distance* from P to the plane π, where $\overrightarrow{n}_0=\frac{1}{\|\vec{n}\|}\overrightarrow{n}$ is the versor of the normal vector $\overrightarrow{n}\ (A,B,C)$. Since $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$, it follows that $\delta(P,M)=\|\overrightarrow{MP}\|=|\delta|$, where $\delta(P,M)$ stands for the distance from P to π. We shall show that

$$\delta=\frac{Ax_P+By_P+Cz_P+D}{\sqrt{A^2+B^2+C^2}}.$$

Indeed, since $\overrightarrow{MP} = \delta \cdot \overrightarrow{n}_0$, we get successively:

$$\begin{aligned}
\delta &= \overrightarrow{n}_0 \cdot \overrightarrow{MP} = \left(\frac{1}{\|\overrightarrow{n}\|}\overrightarrow{n}\right) \cdot \overrightarrow{MP} = \frac{\overrightarrow{n} \cdot \overrightarrow{MP}}{\|\overrightarrow{n}\|} \\
&= \frac{A(x_P - x_M) + B(y_P - y_M) + C(z_P - z_M)}{\sqrt{A^2 + B^2 + C^2}} \\
&= \frac{Ax_P + By_P + Cz_P - (Ax_M + By_M + Cz_M)}{\sqrt{A^2 + B^2 + C^2}} \\
&= \frac{Ax_P + By_P + Cz_P + D}{\sqrt{A^2 + B^2 + C^2}}.
\end{aligned}$$

Consequently

$$\delta(P, M) = \|\overrightarrow{MP}\| = |\delta| = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

1.1.3 The vector product

Definition 1.7. The *vector product* or the *cross product* of the vectors $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ is a vector, denoted by $\overrightarrow{a} \times \overrightarrow{b}$, which is defined to be zero if $\overrightarrow{a}, \overrightarrow{b}$ are linearly dependent (collinear), and if $\overrightarrow{a}, \overrightarrow{b}$ are linearly independent (noncollinear), then it is defined by the following data:

1. $\overrightarrow{a} \times \overrightarrow{b}$ is a vector orthogonal on the two-dimensional subspace $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$ of $\mathcal{V}$;

2. if $\overrightarrow{a} = \overrightarrow{OA}$, $\overrightarrow{b} = \overrightarrow{OB}$, then the sense of $\overrightarrow{a} \times \overrightarrow{b}$ is the one in which a right-handed screw, placed along the line passing through O orthogonal to the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, advances when it is being rotated simultaneously with the vector $\overrightarrow{a}$ from $\overrightarrow{a}$ towards $\overrightarrow{b}$ within the vector subspace $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$ and the support half line of $\overrightarrow{a}$ sweeps the interior of the angle $\widehat{AOB}$ (Screw rule).

3. the *norm* (*magnitude* or *length*) of $\overrightarrow{a} \times \overrightarrow{b}$ is defined by

$$\|\overrightarrow{a} \times \overrightarrow{b}\| = \|\overrightarrow{a}\| \cdot \|\overrightarrow{b}\| \sin(\widehat{\overrightarrow{a}, \overrightarrow{b}}).$$

Remarks 1.8. *1.* The *norm* (*magnitude* or *length*) of the vector $\overrightarrow{a} \times \overrightarrow{b}$ is actually the area of the parallelogram constructed on the vectors $\overrightarrow{a}, \overrightarrow{b}$.

2. The vectors $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ are linearly dependent (collinear) if and only if $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$.

Proposition 1.9. *The vector product has the following properties:*

1. $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}, \forall \overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$;

2. $(\lambda \overrightarrow{a}) \times \overrightarrow{b} = \overrightarrow{a} \times (\lambda \overrightarrow{b}) = \lambda(\overrightarrow{a} \times \overrightarrow{b}), \forall \lambda \in \mathbb{R}, \overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$;

3. $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{c}, \forall \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in \mathcal{V}$.

1.2 Problems

1. Write the equation of the line which passes through tht point $M(1,0,7)$, is parallel to the plane (π) $3x - y + 2z - 15 = 0$ and intersects the line
$$(d)\ \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

2. Consider the points $A(\alpha,0,0)$, $B(0,\beta,0)$ and $C(0,0,\gamma)$ such that
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{a} \text{ where } a \text{ is a constatnt.}$$
Show that the plane (ABC) passes through a fixed point.

3. Write the equation of the plane which passes through $M_0(1,-2,3)$ and cuts the positive coordinate axes through congruent segments.

4. Write the equation of the plane which passes through $A(1,2,1)$ and is parallel to the straight lines
$$(d_1) \begin{cases} x + 2y - z + 1 = 0 \\ x - y + z - 1 = 0 \end{cases} \quad (d_2) \begin{cases} 2x - y + z = 0 \\ x - y + z = 0. \end{cases}$$

5. Write the equation of the plane determined by the line
$$(d) \begin{cases} x - 2y + 3z = 0 \\ 2x + z - 3 = 0 \end{cases}$$
and the point $A(-1,2,6)$.

6. Consider the rtiangle ABC and the midpoint A' of the side $[BC]$. Show that
$$4\, \overrightarrow{AA'}^2 - \overrightarrow{BC}^2 = 4\, \overrightarrow{AB} \cdot \overrightarrow{AC}.$$

7. Consider the rectangle $ABCD$ and the arbitrary point M witin the space. Show that
   (a) $\overrightarrow{MA} \cdot \overrightarrow{MC} = \overrightarrow{MB} \cdot \overrightarrow{MD}$.
   (b) $\overrightarrow{MA}^2 + \overrightarrow{MC}^2 = \overrightarrow{MB}^2 + \overrightarrow{MD}^2$.

8. Consider the noncoplanar vectors $\overrightarrow{OA}\,(1,-1,-2)$, $\overrightarrow{OB}\,(1,0,-1)$, $\overrightarrow{OC}\,(2,2,-1)$ related to an orthonormal basis $\vec{i}, \vec{j}, \vec{k}$. Let H be the foot of the perpendicular through O on the plane ABC. Determine the components of the vectors $\overrightarrow{OH}$.

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