

## ANSWER SHEET 3

**Assignment 1.** (i) If  $X \sim \text{Pois}(\lambda)$  then

$$\begin{aligned} f(x; \lambda) &= \frac{e^{-\lambda} \lambda^x}{x!} \\ &= \exp \left( \ln \left( \frac{e^{-\lambda} \lambda^x}{x!} \right) \right) \\ &= \exp(-\lambda + x \ln(\lambda) - \ln(x!)). \end{aligned}$$

So we can set  $\phi = \ln(\lambda)$ ,  $T(x) = x$ ,  $\gamma(\phi) = e^\phi$  and  $S(x) = -\ln(x!)$  for a natural parametrisation. Observe that the support of  $f$  is  $\mathcal{X} = \{0\} \cup \mathbb{N}$ , thus doesn't depend on  $\phi$ .

For the usual parametrisation we take  $\vartheta = \lambda$  and consequently  $\eta(\vartheta) = \log(\vartheta)$  and  $d(\vartheta) = \vartheta$ .

(ii) If  $X \sim \text{Geom}(p)$  then

$$\begin{aligned} f(x; p) &= (1-p)^x p \\ &= \exp(x \ln(1-p) + \ln(p)). \end{aligned}$$

Set  $\phi = \ln(1-p)$ ,  $T(x) = x$ ,  $\gamma(\phi) = -\ln(1-e^\phi)$  and  $S(x) = 0$  to obtain the natural parametrisation. Observe that the support of  $f$ , given by  $\mathcal{X} = \{0\} \cup \mathbb{N}$ , does not depend on  $\phi$ .

For the usual parametrisation, call  $\vartheta = p$  and define  $\eta(\vartheta) = \log(1-p)$  and  $d(\vartheta) = \gamma(\eta(\vartheta)) = -\log(1 - \exp(\log(1-p))) = -\log(p)$ .

(iii) If  $X \sim \text{Exp}(\lambda)$  then for  $x \geq 0$ ,

$$\begin{aligned} f(x; \lambda) &= \lambda e^{-\lambda x} \\ &= \exp(\ln(\lambda) - \lambda x). \end{aligned}$$

Set  $\phi = \lambda$ ,  $T(x) = -x$ ,  $\gamma(\phi) = -\ln(\phi)$  and  $S(x) = 0$  and observe that the support of  $f$  is given by  $\mathcal{X} = [0, \infty)$  and doesn't depend on  $\phi$ .

(iv) If  $X \sim \text{Gamma}(r, \lambda)$ , then for  $x \geq 0$ ,

$$\begin{aligned} f(x; r, \lambda) &= \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \\ &= \exp \left( \ln \left( \frac{\lambda^r}{\Gamma(r)} \right) + (r-1) \ln(x) - \lambda x \right) \\ &= \exp(r \ln(\lambda) - \ln(\Gamma(r)) + r \ln(x) - \ln(x) - \lambda x) \end{aligned}$$

Observe that here, as in the Normal example seen in class,  $k = 2$ , while in all the previous cases  $k$  was equal to 1.

Set  $\phi = (\phi_1, \phi_2) = (\lambda, r)$ ,  $T_1(x) = -x$ ,  $T_2(x) = \ln(x)$ ,  $\gamma(\phi) = -\phi_2 \ln(\phi_1) + \ln(\Gamma(\phi_2))$  and  $S(x) = -\ln(x)$ . Finally observe that the support of  $f$  is  $\mathcal{X} = [0, \infty)$  and it doesn't depend on  $\phi$ .

(Note : we could have set instead  $\phi = (\phi_1, \phi_2) = (\lambda, r-1)$ ,  $T_1(x) = -x$ ,  $T_2(x) = \ln(x)$ ,  $\gamma(\phi) = -(\phi_2 + 1) \ln(\phi_1) + \ln(\Gamma(\phi_2 + 1))$  et  $S(x) = 0$ ).

**Assignment 2.** (i) Due to independence, the joint probability of  $Y$  is

$$f(y; \lambda) = \prod_{i=1}^n f(y_i; \lambda) = (e^{-n\lambda} \lambda^{\sum_{i=1}^n y_i}) \left( \frac{1}{y_1! \dots y_n!} \right).$$

In particular we split  $f$  into two functions, one of which doesn't depend on  $\lambda$  and the other that is a function of the statistics  $\sum_{i=1}^n y_i$ . Hence by the factorisation theorem the statistics  $T(y)$  is sufficient for the Poisson distribution.

(ii) The joint probability mass function of  $Y$  is

$$\prod_{i=1}^n p(1-p)^{y_i} = p^n (1-p)^{\sum_{i=1}^n y_i}.$$

Therefore the factorisation theorem with

$$g(T(y); p) = p^n (1-p)^{\sum_{i=1}^n y_i} \\ h(y) = 1,$$

tells us that  $T(y) = \sum_{i=1}^n y_i$  is sufficient for the geometric distribution.

(iii) The joint probability density function of  $Y$  is written as

$$f(y; \vartheta) = \prod_{i=1}^n \frac{1}{\vartheta} \exp\left(\frac{-y_i}{\vartheta}\right) = \frac{1}{\vartheta^n} \exp\left(-\frac{1}{\vartheta} \sum_{i=1}^n y_i\right).$$

By the factorisation theorem a sufficient statistics is then  $T(y) = \sum_{i=1}^n y_i$ .

(iv) If  $r$  is unknown and  $\lambda$  known the joint density of  $Y$  writes as

$$f(y; r) = \frac{\lambda^{nr}}{\Gamma(r)^n} \left( \prod_{i=1}^n y_i^{r-1} \right) \exp(-\lambda \sum_{i=1}^n y_i).$$

Write

$$\prod_{i=1}^n y_i^{r-1} = \exp\left((r-1) \sum_{i=1}^n \log(y_i)\right).$$

Hence by the factorisation theorem  $T(y) = \sum_{i=1}^n \log(y_i)$  is a sufficient statistics.

**Assignment 3.** (a) Note that

$$\begin{aligned} \mathbb{P}[X_{(1)} > y] &= \mathbb{P}[X_1 > y, X_2 > y, \dots, X_n > y] \\ &= \prod_{i=1}^n \mathbb{P}[X_i > y] = (\mathbb{P}[X_1 > y])^n = [1 - F(y)]^n. \end{aligned}$$

Thus,  $\mathbb{P}[X_{(1)} \leq y] = 1 - [1 - F(y)]^n$ . Hence,  $f_{X_{(1)}}(y) = n[1 - F(y)]^{n-1} f(y)$ .

(b) Note that

$$\begin{aligned} \mathbb{P}[X_{(n)} \leq z] &= \mathbb{P}[X_1 \leq z, X_2 \leq z, \dots, X_n \leq z] \\ &= \prod_{i=1}^n \mathbb{P}[X_i \leq z] = (\mathbb{P}[X_1 \leq z])^n = [F(z)]^n. \end{aligned}$$

Thus,  $f_{X_{(n)}}(y) = n[F(z)]^{n-1}f(z)$ .

(c) Note that

$$\begin{aligned}\mathbb{P}[X_{(1)} > y, X_{(n)} \leq z] &= \mathbb{P}[y < X_1 \leq z, y < X_2 \leq z, \dots, y < X_n \leq z] \\ &= \prod_{i=1}^n \mathbb{P}[y < X_i \leq z] = (\mathbb{P}[y < X_1 \leq z])^n = [F(z) - F(y)]^n \quad \text{if } y < z.\end{aligned}$$

Also,  $\mathbb{P}[X_{(1)} > y, X_{(n)} \leq z] = 0$  if  $y \geq z$ . Thus,

$$\mathbb{P}[X_{(1)} \leq y, X_{(n)} \leq z] = [F(z)]^n - [F(z) - F(y)]^n \quad \text{if } y < z,$$

and equals  $[F(z)]^n$  otherwise.

(d) Using (c), we get that

$$\begin{aligned}f_{(X_{(1)}, X_{(n)})}(y, z) &= \frac{\partial^2}{\partial y \partial z} \mathbb{P}[X_{(1)} \leq y, X_{(n)} \leq z] \\ &= \begin{cases} n(n-1)f(y)f(z)[F(z) - F(y)]^{n-2}, & y < z \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

No,  $X_{(1)}$  and  $X_{(n)}$  are not independent.

(e) For the  $\text{Unif}(0, \theta)$  distribution, we have  $F(x) = (x/\theta)\mathbb{I}(0 \leq x \leq \theta) + \mathbb{I}(x > \theta)$ , where  $\mathbb{I}(\cdot)$  is the indicator function. So,

$$\begin{aligned}F_{X_{(1)}}(y) &= \begin{cases} 0, & y < 0 \\ 1 - [1 - (y/\theta)]^n, & 0 \leq y \leq \theta \\ 1, & y \geq \theta. \end{cases} \\ f_{X_{(1)}}(y) &= (n/\theta)[1 - (y/\theta)]^{n-1}\mathbb{I}(0 \leq y \leq \theta). \\ F_{X_{(n)}}(z) &= \begin{cases} 0, & z < 0 \\ (z/\theta)^n, & 0 \leq z \leq \theta \\ 1, & z \geq \theta. \end{cases} \\ f_{X_{(n)}}(z) &= (n/\theta)(z/\theta)^{n-1}\mathbb{I}(0 \leq z \leq \theta). \\ f_{(X_{(1)}, X_{(n)})}(y, z) &= \{n(n-1)/\theta^n\}(z-y)^{n-2}\mathbb{I}(0 \leq y < z \leq \theta).\end{aligned}$$

(f) As  $n \rightarrow \infty$ , we have  $F_{X_{(n)}}(z) \rightarrow 0$  if  $z < \theta$  and  $F_{X_{(n)}}(z) \rightarrow 1$  if  $z \geq \theta$ . Thus, the c.d.f. of  $X_{(n)}$  converges to the c.d.f. of a discrete distribution which puts probability one at a single point  $\theta$ .

**Assignment 4.** (a) Note that

$$\begin{aligned}\mathbb{P}[V^2 \leq w] &= \mathbb{P}[-\sqrt{w} \leq V \leq \sqrt{w}] \\ &= \Phi\left(\frac{\sqrt{w} - \mu}{\sigma}\right) - \Phi\left(\frac{-\sqrt{w} - \mu}{\sigma}\right) \\ \Rightarrow f_{V^2}(w) &= \phi\left(\frac{\sqrt{w} - \mu}{\sigma}\right) \frac{1}{2\sqrt{w}\sigma} + \phi\left(\frac{-\sqrt{w} - \mu}{\sigma}\right) \frac{1}{2\sqrt{w}\sigma}.\end{aligned}$$

(b) When  $\mu = 0$  and  $\sigma = 1$ , we have

$$f_{V^2}(w) = \frac{1}{\sqrt{2\pi w}} \exp(-w/2) = \frac{1}{2^{\frac{1}{2}} \Gamma(\frac{1}{2})} w^{\frac{1}{2}-1} \exp(-w/2),$$

which is the density function of a  $\chi^2$  distribution with one degree of freedom.

(c) Note that

$$\mathbb{P}[U \leq u, V^2 \leq v] = \mathbb{P}[U \leq u, -\sqrt{v} \leq V \leq \sqrt{v}] = \mathbb{P}[U \leq u] \mathbb{P}[-\sqrt{v} \leq V \leq \sqrt{v}] = \mathbb{P}[U \leq u] \mathbb{P}[V^2 \leq v],$$

where the second inequality follows from the independence of  $U$  and  $V$ .

(d) Here  $\bar{X} = (X_1 + X_2)/2$ . So,

$$\begin{aligned} S^2 &= \frac{1}{2-1} \sum_{i=1}^2 [X_i - (X_1 + X_2)/2]^2 \\ &= [(X_1 - X_2)/2]^2 + [(X_2 - X_1)/2]^2 = (X_1 - X_2)^2/2. \end{aligned}$$

(e) Define  $Y_1 = (X_1 - \mu)/\sigma$  and  $Y_2 = (X_2 - \gamma)/\eta$ . So,  $Y_1, Y_2$  are i.i.d.  $N(0, 1)$  variables. Set  $U = (\sqrt{2}/\eta)(\bar{X} - \gamma)$ , which also equals  $(Y_1 + Y_2)/\sqrt{2}$ . Set  $V = (Y_1 - Y_2)/\sqrt{2}$ , which also equals  $(\sqrt{2}/\eta)(X_1 - X_2)$ .

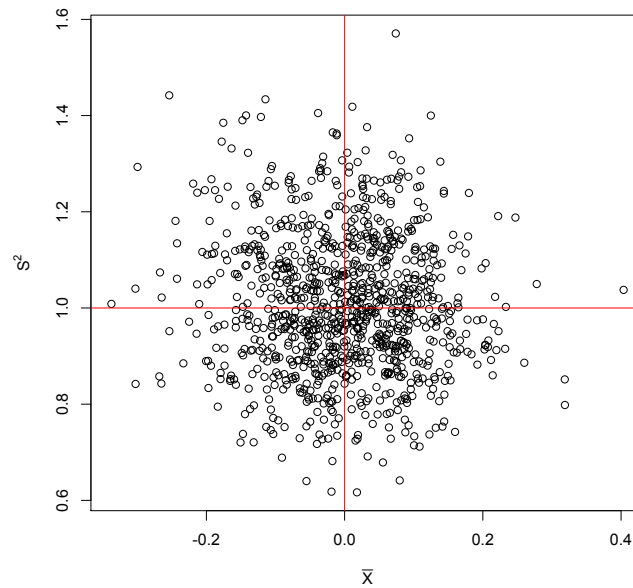
Using Exercise (1) in Week 3, it follows that  $Y_1 + Y_2$  and  $Y_1 - Y_2$  are independent. Thus,  $U = (Y_1 + Y_2)/\sqrt{2}$  and  $V = (Y_1 - Y_2)/\sqrt{2}$  are also independent. They are also normally distributed. Now using part (c), it follows that  $U = (\sqrt{2}/\eta)(\bar{X} - \gamma)$  and  $V^2 = (2/\eta^2)(X_1 - X_2)^2 = 4S^2/\eta^2$  are independent. So,  $\bar{X}$  and  $S^2$  are independent (since these are functions of  $U$  and  $V^2$ ).

(f) If  $\gamma = 0$ , observe that

$$T = \frac{X_1 + X_2}{|X_1 - X_2|} = \frac{2\bar{X}}{\sqrt{2}S} = \frac{\bar{X} - \gamma}{S/\sqrt{2}}.$$

Hence,  $T$  has a Student's  $t$  distribution with two degrees of freedom.

(g)



Since the scatter-plot shows that  $\bar{X}$  and  $S^2$  are distributed almost evenly in all the four quadrants when the center is shifted to the true value  $(0, 1)^\top$  (which is close to the empirical values), we may guess that the covariance/correlation between the two should be close to zero. This is also because the correlation is a measure of the strength of linear relationship between the two variables, and the scatter-plot indicates the lack thereof.

**Assignment 5.** (a) Since  $0 \leq p_k \leq 1$ ,  $\log p_k \leq 0$  and  $-p_k \log p_k \geq 0$ .

The entropy is infinite if  $p_k$  behaves like  $[k \log^2(k)]^{-1}$  (meant in the limit, the discrete random variable must have infinite support).

(b) Since  $g$  is injective for any  $y \in \mathcal{Y} = g(\mathcal{X})$  there is a unique  $x = g^{-1}(y) \in \mathcal{X}$  such that  $y = g(x)$ . Then

$$-H(g(X)) = \sum_{y \in \mathcal{Y}} f_Y(y) \log f_Y(y) = \sum_{x \in \mathcal{X}} f_Y(g(x)) \log f_Y(g(x)) = \sum_{x \in \mathcal{X}} f_X(x) \log f_X(x) = -H(X).$$

(c)  $X^2$  takes the values 0 and 1 with probabilities  $p_2$  and  $p_1 + p_3$ . Since  $p_1, p_3 > 0$  the superadditivity gives

$$-H(X^2) = h(p_1 + p_3) + h(p_2) > h(p_1) + h(p_3) + h(p_2) = -H(X).$$

For the general case one applies the same idea by “stacking” for each  $y$  those  $x \in \mathcal{X}$  for which  $g(x) = y$ .

(d) Here we have

$$H(X) = -\int_0^\theta \frac{1}{\theta} \log \frac{1}{\theta} dx = \log \theta.$$

(e) No. Take  $\theta < 1$  above.

(f) No. Take  $X \sim \text{Unif}[0, 1]$  and  $g(x) = \theta x$ . Then  $H(g(X)) > H(X)$  if  $\theta > 1$ . Note that  $g$  is injective!

*Remark : if  $X$  has density that behaves like  $[x \log^2 x]^{-1}$  for  $x > 2$ , then  $H(X) = \infty$ . If we have the same behaviour but only on  $(0, 1/2)$  then  $H(X) = -\infty$ . If we have the same behaviour on  $(0, 1/2) \cup (2, \infty)$  then  $H(X)$  is undefined. Thus the continuous entropy can take any value in  $[-\infty, \infty]$  or be undefined!*

**Assignment 6.** The complete R code for this assignment is available on the course website at <http://smat.epfl.ch/courses/datasci/corrections/3.R>

(e) We see that the values of `small` increase towards the value of `small.norm`.

(f) Similarly, the densities of the  $t$  distribution converge to that of the Gaussian distribution.