Geometry
Problem booklet

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1 Week 7: Products of vectors

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background. Products of vectors

1.1.1 The triple scalar product

The *triple scalar product* $(\vec{a}, \vec{b}, \vec{c})$ of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the real number $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

Proposition 1.1. *If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$ şi $\vec{c} = c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}$ then*

$$(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{1.1}$$

Corollary 1.2.
1. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent (collinear) if and only if $(\vec{a}, \vec{b}, \vec{c}) = 0$*
2. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly independent (noncollinear) if and only if $(\vec{a}, \vec{b}, \vec{c}) \neq 0$*
3. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ form a basis of the space $\mathcal{V}$ if and only if $(\vec{a}, \vec{b}, \vec{c}) \neq 0$.*
4. *The correspondence $F : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$, $F(\vec{a}, \vec{b}, \vec{c}) = (\vec{a}, \vec{b}, \vec{c})$ is a skew-symmetric, i..e*

$$\begin{array}{l} (\alpha \vec{a} + \alpha' \vec{a}\,', \vec{b}, \vec{c}) = \alpha(\vec{a}, \vec{b}, \vec{c}) + \alpha'(\vec{a}\,', \vec{b}, \vec{c}) \\ (\vec{a}, \beta \vec{b} + \beta' \vec{b}\,', \vec{c}) = \beta(\vec{a}, \vec{b}, \vec{c}) + \beta'(\vec{a}, \vec{b}\,', \vec{c}) \\ (\vec{a}, \vec{b}, \gamma \vec{c} + \gamma' \vec{c}\,') = \gamma(\vec{a}, \vec{b}, \vec{c}) + \gamma'(\vec{a}, \vec{b}, \vec{c}\,') \end{array} \tag{1.2}$$

$\forall \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{R}$, $\forall \vec{a}, \vec{b}, \vec{c}, \vec{a}\,', \vec{b}\,', \vec{c}\,' \in \mathcal{V}$ *şi*

$$(\vec{a}_1, \vec{a}_2, \vec{a}_3) = sgn(\sigma)(\vec{a}_{\sigma(1)}, \vec{a}_{\sigma(2)}, \vec{a}_{\sigma(3)}), \ \forall \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V} \text{ şi } \forall \sigma \in S_3 \tag{1.3}$$

Remark 1.3. One can rewrite the relations (1.3) as follows:

$$\begin{aligned} (\vec{a}_1, \vec{a}_2, \vec{a}_3) &= (\vec{a}_2, \vec{a}_3, \vec{a}_1) = (\vec{a}_3, \vec{a}_1, \vec{a}_2) \\ &= -(\vec{a}_2, \vec{a}_1, \vec{a}_3) = (\vec{a}_1, \vec{a}_3, \vec{a}_2) = -(\vec{a}_3, \vec{a}_2, \vec{a}_1), \end{aligned}$$

$\forall \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V}$

Corollary 1.4.
1. $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) \ \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.
2. *For every $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathcal{V}$ the* Laplace formula

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

holds.

Proof. 1. $(\vec{a} \times \vec{b})\cdot \vec{c} = (\vec{a}, \vec{b}, \vec{c}) = (\vec{b}, \vec{c}, \vec{a}) = (\vec{b} \times \vec{c})\ \vec{a} = \vec{a} \cdot (\vec{b} \times \vec{c})$.

2. Indeed, we have successively:

$$\begin{aligned}(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= (\vec{a}, \vec{b}, \vec{c} \times \vec{d}) = (\vec{c} \times \vec{d}, \vec{a}, \vec{b}) = [(\vec{c} \times \vec{d}) \times \vec{a}] \cdot \vec{b} \\ &= (\vec{a} \cdot \vec{c})\ \vec{d} - (\vec{a} \cdot \vec{d})\ \vec{c}] \cdot \vec{b} = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}] \\ &= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}.\end{aligned}$$

□

Definition 1.1. The basis $[\vec{a}, \vec{b}, \vec{c}]$ of the space $\mathcal{V}$ is said to be *directe* if $(\vec{a}, \vec{b}, \vec{c}) > 0$. If, on the contrary, $(\vec{a}, \vec{b}, \vec{c}) < 0$, we say that the basis $[\vec{a}, \vec{b}, \vec{c}]$ is *inverse*

Definition 1.2. The *oriented volume* of the parallelipiped constructed on the noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ is $\varepsilon \cdot V$, where V is the volume of this parallelepiped and $\varepsilon = +1$ or -1 insomuch as the basis $[\vec{a}, \vec{b}, \vec{c}]$ is directe or inverse respectively.

Propoziţia 1.3. *The triple scalar product $(\vec{a}, \vec{b}, \vec{c})$ of the noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ equals the oriented volume of the parallelepiped constructed on these vectors.*

1.1.2 Applications of the triple scalar product

• The distance between two straight lines.

If $d_1,\ d_2$ are two straight lines, then the distance between them, denoted by $\delta(d_1, d_2)$, is being defined as

$$\min\{||\ \overrightarrow{M_1M_2}\ ||\ |\ M_1 \in d_1,\ M_2 \in d_2\}.$$

1. If $d_1 \cap d_2 \neq \emptyset$, then $\delta(d_1, d_2) = 0$.

2. If $d_1 || d_2$, then $\delta(d_1, d_2) = ||\ \overrightarrow{MN}\ ||$ where $\{M\} = d \cap d_1$, $\{N\} = d \cap d_2$ and d is a straight line perpendicular to the lines d_1 and d_2. Obviously $||\ \overrightarrow{MN}\ ||$ is independent on the choice of the line d.

3. We now assume that the straight lines d_1, d_2 are noncoplanar (skew lines). In this case there exits a unique straight line d such that $d \perp d_1, d_2$ and $d \cap d_1 = \{M_1\}$, $d \cap d_2 = \{M_2\}$. The straight line d is called the *common perpendicular* of the lines d_1, d_2 and obviously $\delta(d_1, d_2) = ||\ \overrightarrow{M_1M_2}\ ||$.

Assume that the straight lines d_1, d_2 are given by their points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$ and their vectors şi au vectorii directori $\vec{d}_1\ (p_1, q_1, r_1)$ $\vec{d}_2\ (p_2, q_2, r_2)$, that is, thei equations are

$$d_1 : \frac{x - x_1}{p_1} = \frac{y - y_1}{q_1} = \frac{z - z_1}{r_1}$$

$$d_2 : \frac{x - x_2}{p_2} = \frac{y - y_2}{q_2} = \frac{z - z_2}{r_2}.$$

The common perpendicular of the lines d_1, d_2 is the intersection line between the plane containing the line d_1 which is parallel to the vector $\overrightarrow{d}_1 \times \overrightarrow{d}_2$, and the plane containing the line d_2 which is parallel to $\overrightarrow{d}_1 \times \overrightarrow{d}_2$. Since

$$\overrightarrow{d}_1 \times \overrightarrow{d}_2 = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} \overrightarrow{i} + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \overrightarrow{k}$$

it follows that the equations of the common perpendicular are

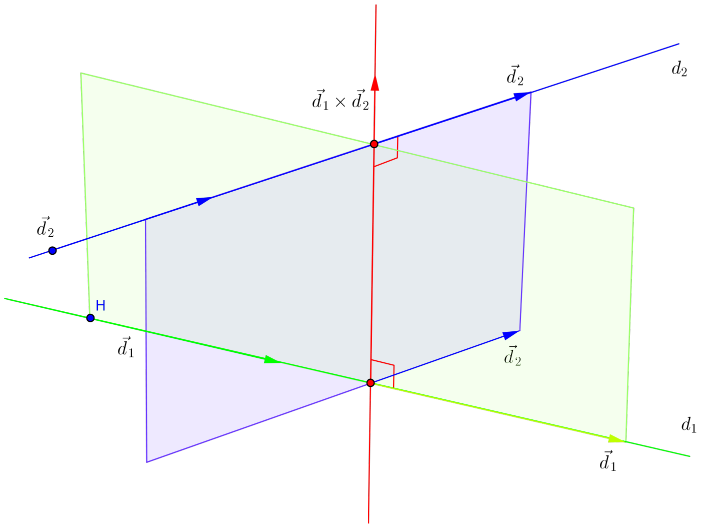


Figure 1: Prependiculara comună a dreptelor d_1 şi d_2

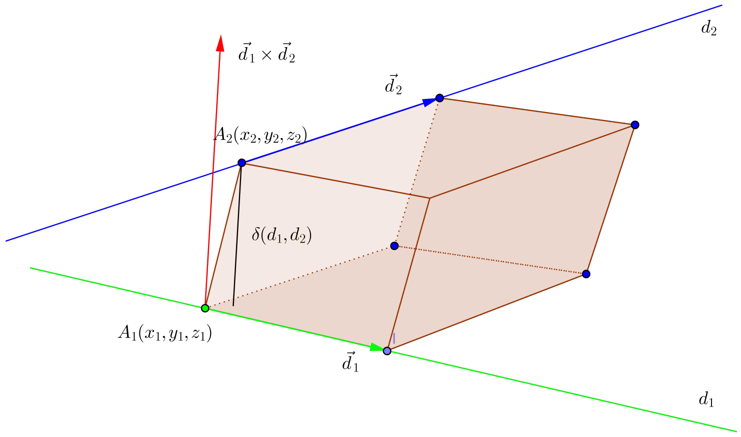
$$\begin{cases} \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ p_1 & q_1 & r_1 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0 \\ \\ \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ p_2 & q_2 & r_2 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0. \end{cases} \tag{1.4}$$

The distance between the straight lines d_1, d_2 can be also regarded as the height of the parallelogram constructed on the vectors $\overrightarrow{d}_1$, $\overrightarrow{d}_2$, $\overrightarrow{d}_1 \times \overrightarrow{d}_2$. Thus

$$\delta(d_1, d_2) = \frac{|(\overrightarrow{A_1A_2}, \overrightarrow{d}_1, \overrightarrow{d}_2)|}{|| \overrightarrow{d}_1 \times \overrightarrow{d}_2 ||}. \tag{1.5}$$

Therefore we obtain

$$\delta(d_1, d_2) = \frac{\left| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} \right|}{\sqrt{\begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}^2 + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}^2 + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}^2}} \tag{1.6}$$



• **The coplanarity condition of two straight lines.**

Using the notations of the previous section, observe that the straight lines d_1, d_2 are coplanar if and only if the vectors $\overrightarrow{A_1A_2}, d_1, d_2$ are linearly dependent (coplanar), or equivalently $(\overrightarrow{A_1A_2}, \overrightarrow{d}_1, \overrightarrow{d}_2) = 0$. Consequently the stright lines d_1, d_2 are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0 \tag{1.7}$$

1.2 Problems

1. If two pairs of opposite edges of the tetrahedron $ABCD$ are perpendicular ($AB \perp CD$, $AD \perp BC$), show that
   (a) The third pair of opposite edges are perpendicular too ($AC \perp BD$).
   (b) $AB^2 + CD^2 = AC^2 + BD^2 = BC^2 + AD^2$.
   (c) The heights of the tetrahedron are concurrent.
   (Such a tetrahedron is said to be orthocentric)

2. Two triangles ABC şi $A'B'C'$ are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices A', B', C' on the sides BC, CA, AB are concurrent. Show that, in this case, the perpendicular lines from the vertices A, B, C on the sides $B'C'$, $C'A'$, $A'B'$ are concurrent too.

3. Let $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ be noncollinear vectors. Show that the necesssary and sufficient condition for the existence of a triangle ABC with the properties $\overrightarrow{BC}=\overrightarrow{a}$, $\overrightarrow{CA}=\overrightarrow{b}$, $\overrightarrow{AB}=\overrightarrow{c}$ is

$$\overrightarrow{a} \times \overrightarrow{b}=\overrightarrow{b} \times \overrightarrow{c}=\overrightarrow{c} \times \overrightarrow{a}.$$

   From the equalities of the norms deduce the low of sines.

4. Show that the sum of some outer-pointing vectors perpendicular on the faces of a tetrahedron which are proportional to the areas of the faces is the zero vector.

5. Find the distance from the point $P(1, 2, -1)$ to the line $(d)\ x = y = z$.

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