# Geometry
# Problem booklet

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## Contents



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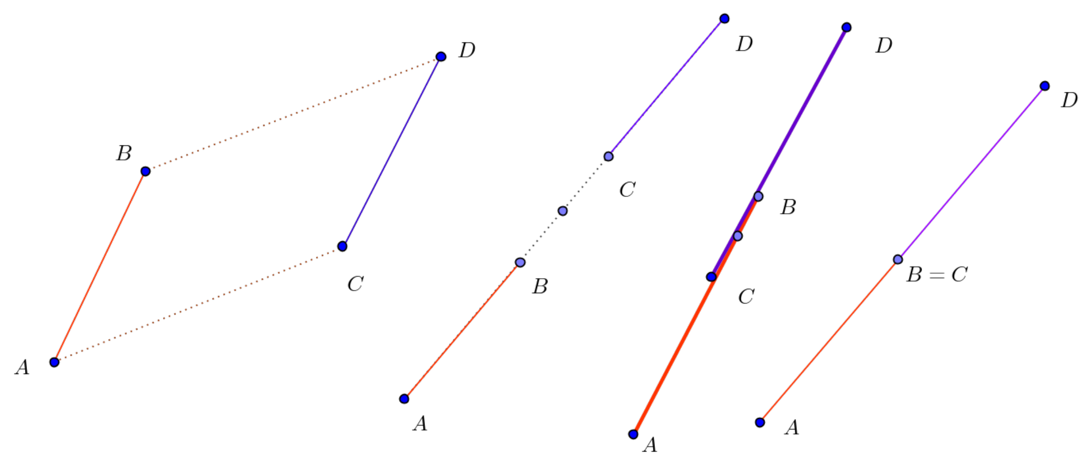
# 1 Week 1: Vector algebra

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

## 1.1 Brief theoretical background. Free vectors

**Vectors** Let $\mathcal{P}$ be the three dimensional physical space in which we can talk about points, lines, planes and various relations among them. If $(A,B) \in \mathcal{P} \times \mathcal{P}$ is an ordered pair, then $A$ is called the *original point* or the *origin* and $B$ is called the *terminal point* or the *extremity* of $(A,B)$.

**Definition 1.1.1.** *The ordered pairs $(A,B)$, $(C,D)$ are said to be equipollent, written $(A,B) \sim (C,D)$, if the segments $[AD]$ and $[BC]$ have the same midpoint.*




*Pairs of equipollent points* $(A,B) \sim (C,D)$

**Remark 1.1.2.** If the points $A$, $B$, $C$, $D \in \mathcal{P}$ are not collinear, then $(A,B) \sim (C,D)$ if and only if $ABDC$ is a parallelogram. In fact the length of the segments $[AB]$ and $[CD]$ is the same whenever $(A,B) \sim (C,D)$.

**Proposition 1.1.3.** *If $(A,B)$ is an ordered pair and $O \in \mathcal{P}$ is a given point, then there exists a unique point $X$ such that $(A,B) \sim (O,X)$.*

**Proposition 1.1.4.** *The equipollence relation is an equivalence relation on $\mathcal{P} \times \mathcal{P}$.*

**Definition 1.1.5.** *The equivalence classes with respect to the equipollence relation are called* (free) vectors.

Denote by $\overrightarrow{AB}$ the equivalence class of the ordered pair $(A,B)$, that is $\overrightarrow{AB} = \{(X,Y) \in \mathcal{P} \times \mathcal{P} \mid (X,Y) \sim (A,B)\}$ and let $\mathcal{V} = \mathcal{P} \times \mathcal{P}/_{\sim} = \{\overrightarrow{AB} \mid (A,B) \in \mathcal{P} \times \mathcal{P}\}$ be the set of (free) vectors. The *length* or the *magnitude* of the vector $\overrightarrow{AB}$, denoted by $\| \overrightarrow{AB} \|$ or by $| \overrightarrow{AB} |$, is the length of the segment $[AB]$.

**Remark 1.1.6.** *If two ordered pairs* $(A, B)$ *and* $(C, D)$ *are equippllent, i.e. the vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$ *are equal, then they have the same length, the same direction and the same sense. In fact a vector is determined by these three items.*

**Proposition 1.1.7.** 1. $\overrightarrow{AB}=\overrightarrow{CD} \Leftrightarrow \overrightarrow{AC}=\overrightarrow{BD}$.

2. $\forall A, B, O \in \mathcal{P}$, $\exists ! X \in \mathcal{P}$ *such that* $\overrightarrow{AB}=\overrightarrow{OX}$.

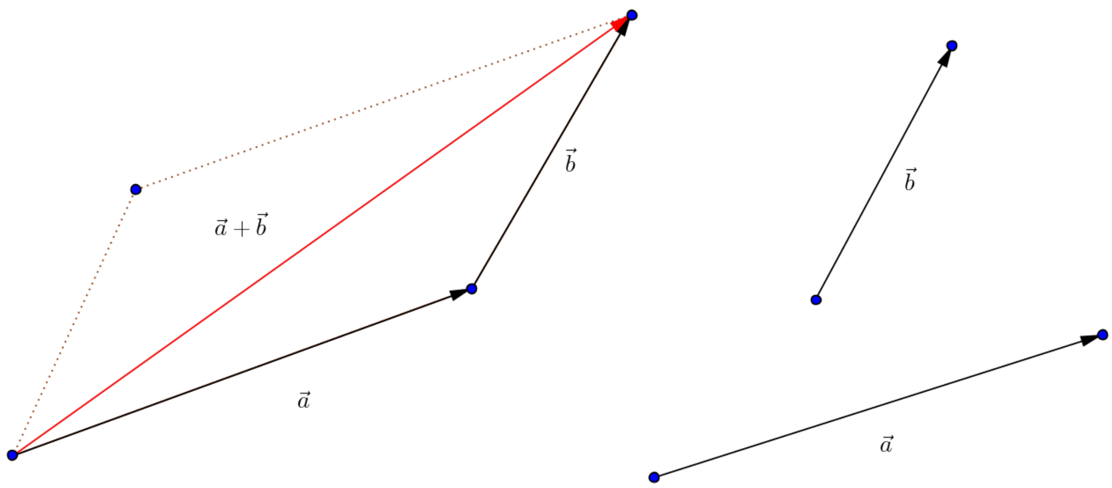
3. $\overrightarrow{AB}=\overrightarrow{A'B'}$, $\overrightarrow{BC}=\overrightarrow{B'C'} \Rightarrow \overrightarrow{AC}=\overrightarrow{A'C'}$.

**Definition 1.1.8.** If $O$, $M \in \mathcal{P}$, the the vector $\overrightarrow{OM}$ is denoted by $\overrightarrow{r}_M$ and is called the *position vector of* $M$ *with respect to* $O$.

**Corollary 1.1.9.** *The map* $\varphi_O : \mathcal{P} \to \mathcal{V}$, $\varphi_O(M) = \overrightarrow{r}_M$ *is one-to-one and onto, i.e bijective.*

### 1.1.1 Operations with vectors

• **The addition of vectors** Let $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ and $O \in \mathcal{P}$ be such that $\overrightarrow{a}=\overrightarrow{OA}$, $\overrightarrow{b}=\overrightarrow{AB}$. The vector $\overrightarrow{OB}$ is called the *sum* of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ and is written $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{a}+\overrightarrow{b}$.




Let $O'$ be another point and $A'$, $B' \in \mathcal{P}$ be such that $\overrightarrow{O'A'}=\overrightarrow{a}$, $\overrightarrow{A'B'}=\overrightarrow{b}$. Since $\overrightarrow{OA}=\overrightarrow{O'A'}$ and $\overrightarrow{AB}=\overrightarrow{A'B'}$ it follows, according to Proposition 1.1.4 (3), that $\overrightarrow{OB}=\overrightarrow{O'B'}$. Therefore the vector $\overrightarrow{a}+\overrightarrow{b}$ is independent on the choice of the point $O$.

**Proposition 1.1.10.** *The set* $\mathcal{V}$ *endowed to the binary operation* $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, $(\overrightarrow{a}, \overrightarrow{b}) \mapsto \overrightarrow{a}+\overrightarrow{b}$, *is an abelian group whose zero element is the vector* $\overrightarrow{AA}=\overrightarrow{BB}=\overrightarrow{0}$ *and the opposite of* $\overrightarrow{AB}$, *denoted by* $-\overrightarrow{AB}$, *is the vector* $\overrightarrow{BA}$.

In particular the addition operation is associative and the vector

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

is usually denoted by $\vec{a} + \vec{b} + \vec{c}$. Moreover the expression

$$((\cdots(\vec{a}_1 + \vec{a}_2) + \vec{a}_3 + \cdots + \vec{a}_n)\cdots), \tag{1.1}$$

is independent of the distribution of paranthesis and it is usually denoted by

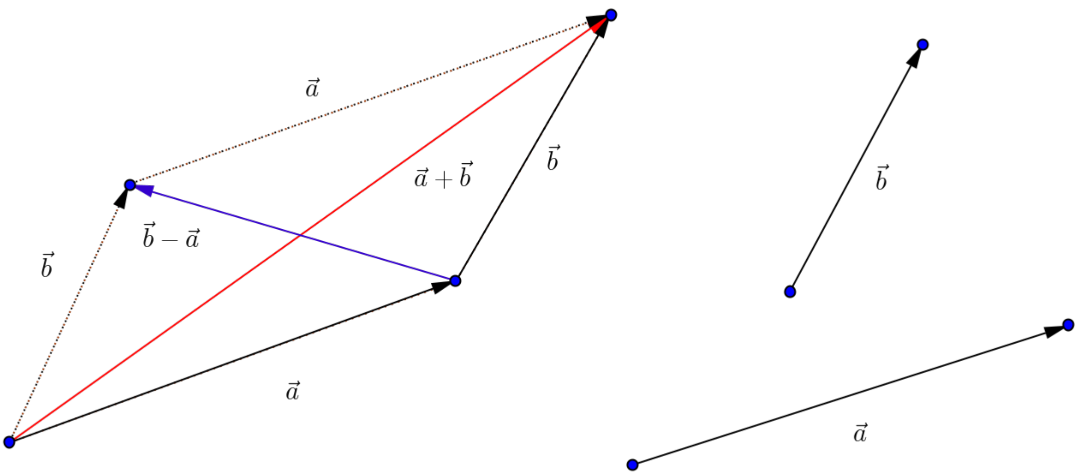
$$\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n .$$

**Example 1.1.11.** *If $A_1, A_2, A_3, \ldots, A_n \in \mathcal{P}$ are some given points, then*

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{A_1A_n} .$$

*This shows that $\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1} = \vec{0}$, namely the sum of vectors constructed on the edges of a closed broken line is zero.*

**Corolarul 1.1.12.** *If $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$ are given vectors, there exists a unique vector $\vec{x} \in \mathcal{V}$ such that $\vec{a} + \vec{x} = \vec{b}$. In fact $\vec{x} = \vec{b} + (-\vec{a}) = \overrightarrow{AB}$ and is denoted by $\vec{b} - \vec{a}$.*




## • The multiplication of vectors with scalars

Let $\alpha \in \mathbb{R}$ be a scalar and $\vec{a} = \overrightarrow{OA} \in \mathcal{V}$ be a vector. We define the vector $\alpha \cdot \vec{a}$ as follows: $\alpha \cdot \vec{a} = \vec{0}$ if $\alpha = 0$ or $\vec{a} = \vec{0}$; if $\vec{a} \neq \vec{0}$ and $\alpha > 0$, there exists a unique point on the half line $]OA$ such that $||OB|| = \alpha \cdot ||OA||$ and define $\alpha \cdot \vec{a} = \overrightarrow{OB}$; if $\alpha < 0$ we define $\alpha \cdot \vec{a} = -(|\alpha| \cdot \vec{a})$. The external binary operation

$$\mathbb{R} \times \mathcal{V} \to \mathcal{V}, \ (\alpha, \vec{a}) \mapsto \alpha \cdot \vec{a}$$

is called the *multiplication of vectors with scalars*.

**Proposition 1.1.13.** *The following properties hold:*

$(v1)$ $(\alpha+\beta)\cdot \overrightarrow{a} = \alpha\cdot \overrightarrow{a} + \beta\cdot \overrightarrow{a}$, $\forall \alpha,\beta \in \mathbb{R}$, $\overrightarrow{a} \in \mathcal{V}$.

$(v2)$ $\alpha\cdot(\overrightarrow{a} + \overrightarrow{b}) = \alpha\cdot \overrightarrow{a} + \alpha\cdot \overrightarrow{b}$, $\forall \alpha \in \mathbb{R}$, $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$.

$(v3)$ $\alpha\cdot(\beta\cdot \overrightarrow{a}) = (\alpha\beta)\cdot \overrightarrow{a}$, $\forall \alpha,\beta \in \mathbb{R}$.

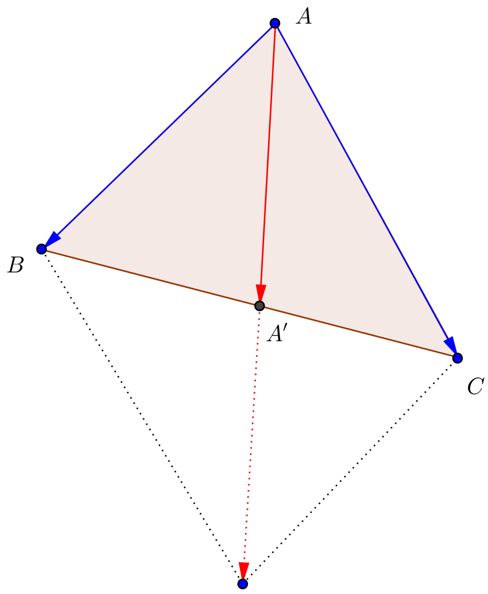
$(v4)$ $1\cdot \overrightarrow{a} = \overrightarrow{a}$, $\forall\ \overrightarrow{a} \in \mathcal{V}$.

### 1.1.2 The vector structure on the set of vectors

**Theorem 1.1.14.** *The set of (free) vectors endowed with the addition binary operation of vectors and the external binary operation of multiplication of vectors with scalars is a real vector space.*

**Example 1.1.15.** *If $A'$ is the midpoint of the egde $[BC]$ of the triangle $ABC$, then*

$$\overrightarrow{AA'} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}).$$




## 1.2 Problems

1. ([4, Problema 3, p. 1]) Let $OABCDE$ be a regular hexagon in which $\overrightarrow{OA} = \overrightarrow{a}$ and $\overrightarrow{OE} = \overrightarrow{b}$. Express the vectors $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$. Show that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OE} = 3\,\overrightarrow{OC}$.

2. ([4, Problema 11, p. 3]) Consider two parallelograms, $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ in $\mathcal{P}$, and $M_1$, $M_2$, $M_3$, $M_4$ the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$, $[A_4B_4]$ respectively. Show that:

   - $2\,\overrightarrow{M_1M_2} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2}$ and $2\,\overrightarrow{M_3M_4} = \overrightarrow{A_3A_4} + \overrightarrow{B_3B_4}$.

- $M_1, M_2, M_3, M_4$ are the vertices of a parallelogram.

3. ([4, Problema 12, p. 3]) Let $M$, $N$ be the midpoints of two opposite edges of a given quadrilateral $ABCD$ and $P$ be the midpoint of $[MN]$. Show that

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0$$

4. ([4, Problema 13, p. 3]) If $G$ is the centroid of a tringle $ABC$ and $O$ is a given point, show that

$$\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}.$$

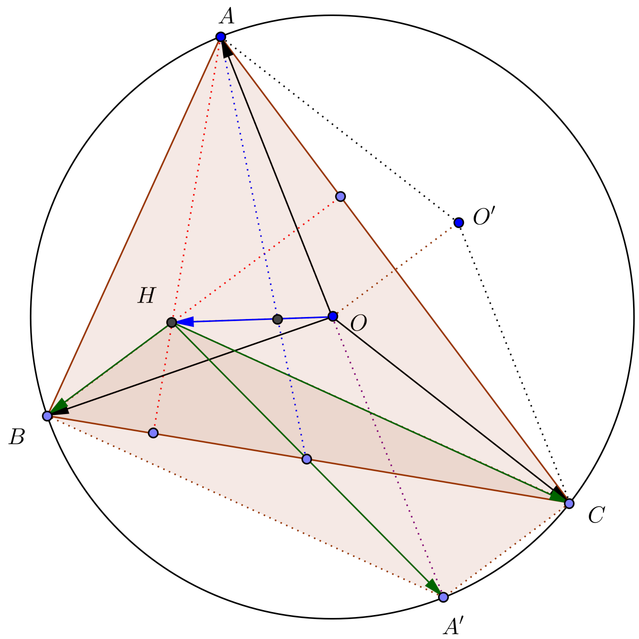
5. ([4, Problema 14, p. 4]) Consider the triangle $ABC$ alongside its orthocenter $H$, its circumcenter $O$ and the diametrically opposed point $A'$ of $A$ on the latter circle. Show that:

   (a) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$.

   (b) $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$.

   (c) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\,\overrightarrow{HO}$.

*Solution.* (5a) Let $M$ be the point with the property $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OM}$, namely $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OM} - \overrightarrow{OB} = \overrightarrow{BM}$. But $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OO'} \perp \overrightarrow{AC}$, i.e. $\overrightarrow{BM} \perp \overrightarrow{AC}$. One can similarly show that $\overrightarrow{CM} \perp \overrightarrow{AB}$ and $\overrightarrow{AM} \perp \overrightarrow{BC}$. Consequently $M = H$.




(5b) $A'BHC$ is a parallelogram as the two pairs of opposite edges are parallel. Indeed one of the pairs is orhtogonal to $AC$ and the other one is orthogonal to $AB$. Consequently $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$.

(5c) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA} + \overrightarrow{HA'} = 2\,\overrightarrow{HO}$. For an alternative solution we may observe:

$$\begin{aligned}\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} &= \overrightarrow{HO} + \overrightarrow{OA} + \overrightarrow{HO} + \overrightarrow{OB} + \overrightarrow{HO} + \overrightarrow{OC} \\ &= 3\,\overrightarrow{HO} + \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 3\,\overrightarrow{HO} + \overrightarrow{OH} = 2\,\overrightarrow{HO}.\end{aligned}$$

6. ([4, Problema 15, p. 4]) Consider the triangle $ABC$ alongside its centroid $G$, its orthocenter $H$ and its circumcenter $O$. Show that $O, G, H$ are collinear and $3\overrightarrow{HG} = 2\overrightarrow{HO}$.

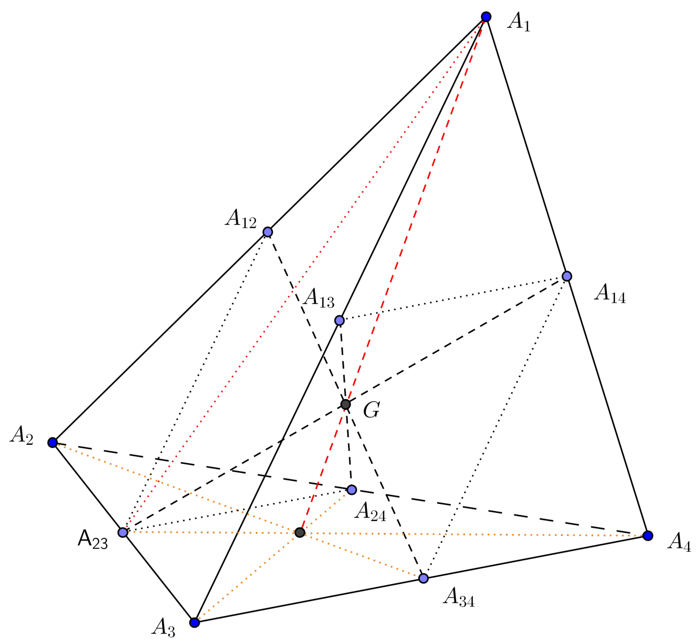
7. ([4, Problema 12, p. 7]) Consider two perpendicular chords $AB$ and $CD$ of a given circle and $\{M\} = AB \cap CD$. Show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2\overrightarrow{OM}.$$

8. ([4, Problema 21(a), p. 9]) In a triangle $ABC$ consider the points $M, L$ on the side $AB$ and $N, T$ on the side $AC$ such that $3\overrightarrow{AL} = 2\overrightarrow{AM} = \overrightarrow{AB}$ and $3\overrightarrow{AT} = 2\overrightarrow{AN} = \overrightarrow{AC}$. Show that $\overrightarrow{AB} + \overrightarrow{AC} = 5\overrightarrow{AS}$, where $\{S\} = MT \cap LN$.

9. ([4, Problema 11, p. 94]) Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$, not necesarily in the same plane, alongside their centroids $G_1, G_2$. Show that $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3\overrightarrow{G_1G_2}$.

10. ([4, Problema 27, p. 13]) Consider a tetrahedron $A_1A_2A_3A_4$ and the midpoints $A_{ij}$ of the edges $A_iA_j$, $i \neq j$. Show that:
    (a) The lines $A_{12}A_{34}$, $A_{13}A_{24}$ and $A_{14}A_{23}$ are concurrent in a point $G$.
    (b) The medians of the tetrahedron (the lines passing through the vertices and the centroids of the opposite faces) are also concurrent at $G$.
    (c) Determine the ratio in which the point $G$ divides each median.
    (d) Show that $\overrightarrow{GA_1} + \overrightarrow{GA_2} + \overrightarrow{GA_3} + \overrightarrow{GA_4} = \overrightarrow{0}$.
    (e) If $M$ is an arbitrary point, show that $\overrightarrow{MA_1} + \overrightarrow{MA_2} + \overrightarrow{MA_3} + \overrightarrow{MA_4} = 4\overrightarrow{MG}$.

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