Geometry
Problem booklet

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Week 11

1 Generated surfaces (Brief theoretical background)

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Conical Surfaces

Definition 1.1. *The surface generated by a variable line, called* generatrix, *which passes through a fixed point V and intersects a given curve $\mathcal{C}$, is called* conical surface. *The point V is called the* vertex *of the surface and the curve $\mathcal{C}$* director curve.

Theorem 1.2. *The conical surface, of vertex $V(x_0, y_0, z_0)$ and director curve*

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

(V and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0. \tag{1.1}$$

Proof. The equations of an arbitrary line through $V(x_0, y_0, z_0)$ are

$$d_{\lambda\mu} : \begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \end{cases}.$$

A generatrix has to intersect the curve $\mathcal{C}$, hence the system of equations

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

must be compatible. This happens for some values of the parameters λ and μ, which verify a *compatibility condition*

$$\varphi(\lambda, \mu),$$

obtained by eliminating x, y and z in the the previous system of equations. In these conditions, the equation of the conical surface rises from the system

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ \varphi(\lambda, \mu) = 0 \end{cases},$$

i.e.

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0.$$

□

Remark 1.3. *If φ is a polynomial function, then the equation (1.1) can be written in the form*

$$\phi(x - x_0, y - y_0, z - z_0) = 0,$$

where ϕ is homogeneous with respect to $x - x_0$, $y - y_0$ and $z - z_0$. If φ is polynomial and V is the origin of the system of coordinates, then the equation of the conical surface is $\phi(x, y, z) = 0$, with ϕ a homogeneous polynomial. Conversely, an algebraic homogeneous equation in x, y and z represents a conical surface with the vertex at the origin.

Example 1.4. *Let us determine the equation of the conical surface, having the vertex $V(1, 1, 1)$ and the director curve*

$$\mathcal{C} : \begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \end{cases}.$$

The family of lines passing through V has the equations

$$d_{\lambda\mu} : \begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases}.$$

The system of equations

$$\begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \\ x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases}$$

must be compatible. A solution is

$$\begin{cases} x = 1 - \lambda \\ y = 1 - \mu \\ z = 0 \end{cases},$$

and, replaced in the first equation of the system, gives the compatibility condition

$$[(1 - \lambda)^2 + (1 - \mu)^2]^2 - (1 - \lambda)(1 - \mu) = 0.$$

The equation of the conical surface is obtained by eliminating the parameters λ and μ in

$$\begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \\ ((1 - \lambda)^2 + (1 - \mu)^2)^2 - (1 - \lambda)(1 - \mu) = 0 \end{cases}.$$

Expressing $\lambda = \dfrac{x - 1}{z - 1}$ and $\mu = \dfrac{y - 1}{z - 1}$ and replacing in the compatibility condition, one obtains

$$\left[\left(\frac{z - x}{z - 1}\right)^2 + \left(\frac{z - y}{z - 1}\right)^2\right]^2 - \left(\frac{z - x}{z - 1}\right)\left(\frac{z - y}{z - 1}\right) = 0,$$

or

$$[(z - x)^2 + (z - y)^2]^2 - (z - x)(z - y)(z - 1)^2 = 0.$$

1.2 Conoidal Surfaces

Definition 1.5. *The surface generated by a variable line, which intersects a given line d and a given curve $\mathcal{C}$, and remains parallel to a given plane π, is called* conoidal surface. *The curve $\mathcal{C}$ is the* director curve *and the plane π is the* director plane *of the conoidal surface.*

Theorem 1.6. *The conoidal surface whose generatrix intersects the line*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$$

and the curve

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases}$$

and has the director plane $\pi = 0$, (π is not parallel to d and that $\mathcal{C}$ is not contained into π), is characterized by an equation of the form

$$\varphi\left(\pi, \frac{\pi_1}{\pi_2}\right) = 0. \tag{1.2}$$

Proof. An arbitrary generatrix of the conoidal surface is contained into a plane parallel to π and, on the other hand, comes from the bundle of planes containing d. Then, the equations of a generatrix are

$$d_{\lambda\mu} : \begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \end{cases}.$$

Again, the generatrix must intersect the director curve, hence the system of equations

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases}$$

has to be compatible. This leads to a compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

and the equation of the conoidal surface is obtained from

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ \varphi(\lambda,\mu) = 0 \end{cases}.$$

By expressing λ and μ, one obtains (1.2). □

Example 1.7. *Let us find the equation of the conoidal surface, whose generatrices are parallel to xOy and intersect Oz and the curve*

$$\begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}.$$

The equations of xOy and Oz are, respectively,

$$xOy : z = 0, \quad \text{and} \quad Oz : \begin{cases} x = 0 \\ z = 0 \end{cases},$$

so that the equations of the generatrix are

$$d_{\lambda,\mu} : \begin{cases} x = \lambda y \\ z = \mu \end{cases}.$$

From the compatibility of the system of equations

$$\begin{cases} x = \lambda y \\ z = \mu \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases},$$

one obtains the compatibility condition

$$2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0,$$

and, replacing $\lambda = \dfrac{y}{x}$ and $\mu = z$, the equation of the conoidal surface is

$$2x^2z - 2y^2z - 2x^2 + y^2 = 0.$$

1.3 Revolution Surfaces

Definition 1.8. *The surface generated after the rotation of a given curve $\mathcal{C}$ around a given line d is said to be a* revolution surface*.*

Theorem 1.9. *The equation of the revolution surface generated by the curve*

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

in its rotation around the line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r},$$

is of the form

$$\varphi((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2, px + qy + rz) = 0. \tag{1.3}$$

Proof. An arbitrary point on the curve $\mathcal{C}$ will describe, in its rotation around d, a circle situated into a plane orthogonal on d and having the center on the line d. This circle can be seen as the intersection between a sphere, having the center on d and of variable radius, and a plane, orthogonal on d, so that its equations are

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}.$$

The circle has to intersect the curve $\mathcal{C}$, therefore the system

$$\begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \\ (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}$$

must be compatible. One obtains the compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

which, after replacing the parameters, gives the equation of the surface (1.3). □

2 Problems

1. Show that a ray of light through a focus of an ellipse reflects to a ray that passes through the other focus (optical property of the ellipse).

   Solution. Let $F_1(-c,0)$, $F_2(c,0)$ be the foci of the ellipse $\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Recall that the gradient $\mathrm{grad}(f)(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0))$ is a normal vector of the ellipse $\mathcal{E}$ to its point $M_0(x_0,y_0)$, where

   $$f(x,y) = \delta(F_1,M) + \delta(F_2,M) = \sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2}$$

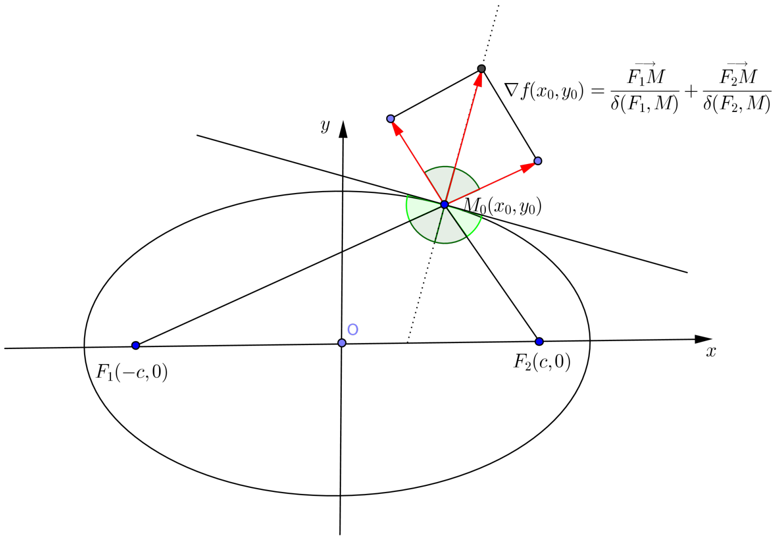
   and $M(x,y)$, as the ellipse is a level set of f. Note that

   $$f_x(x_0,y_0) = \frac{x_0+c}{\delta(F_1,M_0)} + \frac{x_0-c}{\delta(F_2,M_0)} \text{ and } f_y(x_0,y_0) = \frac{y}{\delta(F_1,M_0)} + \frac{y}{\delta(F_2,M_0)},$$

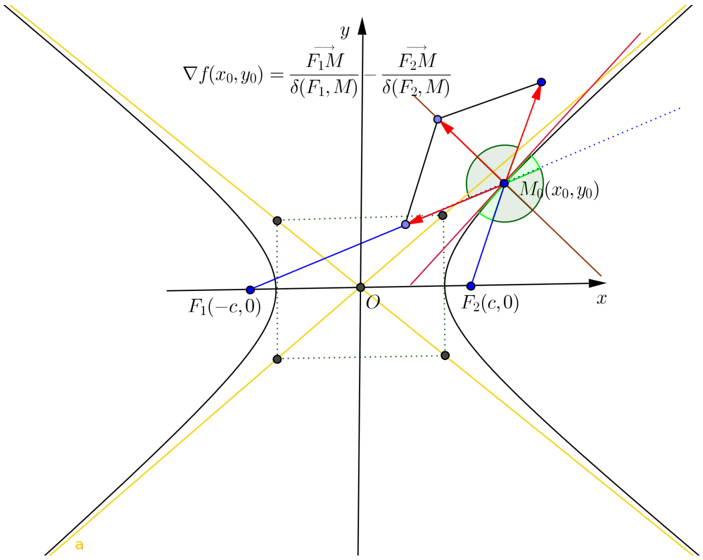
   and shows that

   $$\begin{aligned} \mathrm{grad}(f) &= (f_x(x_0,y_0), f_y(x_0,y_0)) = \left(\frac{x_0+c}{\delta(F_1,M_0)} + \frac{x_0-c}{\delta(F_2,M_0)}, \frac{y_0}{\delta(F_1,M_0)} + \frac{y_0}{\delta(F_2,M_0)} \right) \\ &= \frac{(x_0+c,y)}{\delta(F_1,M_0)} + \frac{(x_0-c,y)}{\delta(F_2,M_0)} = \frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)} + \frac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}. \end{aligned}$$

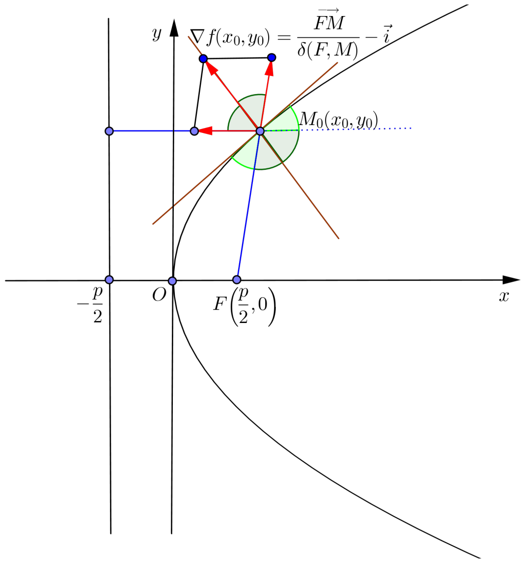
   The versors $\frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)}$ and $\frac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}$ point towards the exterior of the ellipse $\mathcal{E}$ and their sum make obviously equal angles with the directions of the vectors $\overrightarrow{F_1M_0}$ and $\overrightarrow{F_2M_0}$ and (the sum) is also orthogonal to the tangent $T_{M_0}(\mathcal{E})$ of the ellipse at $M_0(x_0,y_0)$. This shows that the angle between the ray F_1M and the tangent $T_{M_0}(\mathcal{E})$ equals the angle between the ray F_2M and the tangent $T_{M_0}(\mathcal{E})$.



2. Show that a ray of light through a focus of a hyperbola reflects to a ray that passes through the other focus (optical property of the hyperbola).



3. Show that a ray of light through a focus of a parabola reflects to a ray parallel to the axis of the parabola (optical property of the parabola).



4. Find the rectilinear generatrices of the hyperboloid of one sheet

$$(\mathcal{H}_1)\ \frac{x^2}{36} + \frac{y^2}{9} - \frac{z^2}{4} = 1$$

which are parallel to the plane $(\pi)\ x + y + z = 0$.

5. Find the locus of points on the hyperbolic paraboloid $(\mathcal{P}_h)\ y^2 - z^2 = 2x$ through which the rectilinear generatrices are perpendicular.

6. Find the locus of points in the space equidistant to two given straight lines.

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