Seminar 6

1. (a) Prove that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

   and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots.$$

   (b) Based on Taylor series, prove that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$.

   (c) Prove that $x - \frac{x^3}{6} < \sin x < x$, $\forall x > 0$ and $1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$, $\forall x \in \mathbb{R}$.

2. Prove, by computing higher order derivatives or by integrating the geometric series, that

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$$

   for $|x| < 1$, and

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots.$$

   What are the convergence sets for each of these power series?

3. (a) ★ Find the coefficients a_n in the Taylor series

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

   (b) ★ Prove that for $|x| < 1$,

$$\arctan x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

   and show that the convergence set of this power series is $[-1, 1]$.

4. Prove that the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{-1/x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

   is infinitely differentiable at 0, but f is not expandable in a Taylor series around 0.

5. (a) For $\alpha \in \mathbb{R}$ and $|x| < 1$, prove the generalized binomial expansion (binomial series)

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{k} x^k, \quad \binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}.$$

   (b) Find the first four terms in the binomial series of $\sqrt{1+x}$ and $1/\sqrt{1+x}$.

6. For each function $f : \mathbb{R} \to \mathbb{R}$ given below check that $f'(0) = 0$ and find the first $n \in \mathbb{N}$ such that $f^{(n)}(0) \neq 0$. Then, deduce whether 0 is a local extremum point of f or not; in the affirmative, specify if 0 is a global extremum point or just a local one.

   (a) $f(x) = e^x + e^{-x} - x^2$

   (b) ★ $f(x) = \cos(x^2)$

   (c) ★ $f(x) = 6\sin x - 6x + x^3$.

7. ★ Using Taylor series, prove that the forward difference $\big(f(x+h) - f(x)\big)/h$ approximates the derivative $f'(x)$ with an error of order h (first order approximation), i.e.

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h),$$

   and that the centered difference $\big(f(x+h) - f(x-h)\big)/2h$ approximates the derivative $f'(x)$ with an error of order h^2 (second order approximation), i.e.

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2).$$

   [Python] Choose a function f, a point x and compute $f'(x)$. By taking a range of small values h, show that the errors when approximating $f'(x)$ with the finite differences above are proportional to h and h^2, respectively.

8. $*$ Let C_n be the number of full binary trees with $n+1$ leaves (known as the Catalan number).

   (a) Find a recurrence relation for C_n.

   (b) Considering the generating function $f(x) = \sum_{n=0}^{\infty} C_n x^n$, prove that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Homework questions are marked with ★. Bonus questions are marked with $*$.
Solutions should be handed in at the beginning of next week's lecture.