Seminar 4

1. Let K be a field. Show that $K[X]$ is a K-vector space, where the addition is the usual addition of polynomials and the scalar multiplication is defined as follows: $\forall\ k \in K$, $\forall f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$,

$$k \cdot f = (ka_0) + (ka_1)X + \cdots + (ka_n)X^n\,.$$

2. Let K be a field and $m, n \in \mathbb{N}$, $m, n \geq 2$. Show that $M_{m,n}(K)$ is a K-vector space, with the usual addition and scalar multiplication of matrices.

3. Let K be a field, $A \neq \emptyset$ and denote $K^A = \{f \mid f : A \to K\}$. Show that K^A is a K-vector space, where the addition and the scalar multiplication are defined as follows: $\forall f, g \in K^A$, $\forall k \in K$, $f + g \in K^A$, $kf \in K^A$,

$$(f + g)(x) = f(x) + g(x)\,, \quad (k \cdot f)(x) = k \cdot f(x)\,, \forall x \in A\,.$$

4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define the operations: $x \perp y = xy$ and $k \top x = x^k$, $\forall k \in \mathbb{R}$ and $\forall x, y \in V$. Prove that V is a vector space over $\mathbb{R}$.

5. Let K be a field and let $V = K \times K$. Decide whether V is a K-vector space with respect to the following addition and scalar multiplication:

(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + 2y_2)$ and $k \cdot (x_1, y_1) = (kx_1, ky_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

(ii) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k \cdot (x_1, y_1) = (kx_1, y_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

6. Let p be a prime number and let V be a vector space over the field $\mathbb{Z}_p$.

(i) Prove that $\underbrace{x + \cdots + x}_{p \text{ times}} = 0$, $\forall x \in V$.

(ii) Is there a scalar multiplication endowing $(\mathbb{Z}, +)$ with a structure of a vector space over $\mathbb{Z}_p$?

7. Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:

(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;

(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;

(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;

(iv) $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;

(v) $E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;

(vi) $F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$?

8. Which ones of the following sets are subspaces:

(i) $[-1, 1]$ of the real vector space $\mathbb{R}$;

(ii) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ of the real vector space $\mathbb{R}^2$;

(iii) $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Q} \right\}$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;

(iv) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

9. Which ones of the following sets are subspaces of the K-vector space $K[X]$:

(i) $K_n[X] = \{f \in K[X] \mid \text{degree}(f) \leq n\}$ $(n \in \mathbb{N})$;

(ii) $K'_n[X] = \{f \in K[X] \mid \text{degree}(f) = n\}$ $(n \in \mathbb{N})$.

10. Show that the set of all solutions of a homogeneous system of two equations and two unknowns with real coefficients is a subspace of the real vector space $\mathbb{R}^2$.