Seminar 5

1. Determine the following generated subspaces:
(i) $< 1, X, X^2 >$ in the real vector space $\mathbb{R}[X]$.
(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in the real vector space $M_2(\mathbb{R})$.

2. Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces with a minimal number of generators.

3. Consider the following vectors in the real vector space $\mathbb{R}^3$:

$$a = (-2, 1, 3), b = (3, -2, -1), c = (1, -1, 2), d = (-5, 3, 4), e = (-9, 5, 10).$$

Show that $\langle a, b\rangle = \langle c, d, e\rangle$.

4. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that S and T are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

5. Let S and T be the set of all even functions and of all odd functions in $\mathbb{R}^{\mathbb{R}}$ respectively. Prove that S and T are subspaces of the real vector space $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{R}} = S \oplus T$.

6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x + y, x - y),$$
$$g(x, y) = (2x - y, 4x - 2y),$$
$$h(x, y, z) = (x - y, y - z, z - x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

7. Which ones of the following functions are endomorphisms of the real vector space $\mathbb{R}^2$:
(i) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (ax + by, cx + dy)$, where $a, b, c, d \in \mathbb{R}$;
(ii) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (a + x, b + y)$, where $a, b \in \mathbb{R}$?

8. Let $a \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x \cos a - y \sin a, x \sin a + y \cos a)\,.$$

Prove that $f \in End_{\mathbb{R}}(\mathbb{R}^2)$.

9. Determine the kernel and the image of the endomorphisms from Exercise **6.**

10. Let V be a vector space over K and $f \in End_K(V)$. Show that the set

$$S = \{x \in V \mid f(x) = x\}$$

of fixed points of f is a subspace of V.