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Homework - Seminar 4

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1) d) $\sum_{n\geq 2} \frac{1}{(\ln n)^{\ln n}}$

~~Cauchy-Condensation Test:~~ $\cancel{\sum_{n\geq 1} \frac{2^n}{(\ln(2^n))^{\ln(2^n)}} = \sum_{n\geq 1} \frac{2^n}{(n\ln 2)^{(n\ln 2)}}}$

$(\ln n)^{\ln n} = \left(e^{\ln(\ln n)}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln(\ln(n))} = n^{\ln\ln n}$

$\ln\ln n > 2 \Leftrightarrow \ln n > e^2 \Leftrightarrow n > e^{e^2}$ $\Rightarrow \ln\ln n > 2, \forall n$ ~~> 27~~ sufficiently large $(n \approx 1600)$

$\cancel{e^e < 27 = 3^3}$

$\Rightarrow n^{\ln\ln n} > n^2 \Rightarrow \frac{1}{n^{\ln\ln n}} < \frac{1}{n^2} \Rightarrow$

$\Rightarrow$ Therefore, $\sum_{n\geq 2} \frac{1}{n^{\ln\ln n}}$ converges.

2) c) $\sum_{n\geq 1} \frac{\sin n}{n^2}$

$\frac{\sin n}{n^2} \leq \frac{1}{n^2} \Rightarrow \sum_{n\geq 1} \frac{\sin n}{n^2}$ converges

$\sum_{n\geq 1} \left|\frac{\sin n}{n^2}\right| = \sum_{n\geq 1} \frac{|\sin n|}{n^2}$

$\frac{|\sin n|}{n^2} \leq \frac{1}{n^2} \Rightarrow \sum_{n\geq 1} \frac{|\sin n|}{n^2}$ converges

$\Rightarrow \sum_{n\geq 1} \frac{\sin n}{n^2}$ - absolutely convergent

d) $\sum_{n\geq 1} \sin\left(\pi\sqrt{n^2+1}\right)$

$\sin\left(\pi\sqrt{n^2+1}\right) = \sin\left(\pi\sqrt{n^2+1} - n\pi + n\pi\right) = \sin\left[\pi\left(\sqrt{n^2+1} - n\right) + n\pi\right] \underset{\sqrt{n^2+1}+n}{=}$

$= \sin\left[\pi\left(\sqrt{n^2+1} - n\right)\right] \cdot \underbrace{\cos n\pi}_{(-1)^n} + \underbrace{\sin n\pi}_{0} \cdot \cos\left[\pi\left(\sqrt{n^2+1} - n\right)\right] = (-1)^n \cdot \sin\left[\pi\left(\sqrt{n^2+1} - n\right)\right] =$

$= (-1)^n \sin\pi \cdot \frac{1}{\sqrt{n^2+1}+n} = \cancel{\sin (-1)^n \frac{1}{\sin}}\ (-1)^n \cdot \sin\frac{\pi}{\sqrt{n^2+1}+n}$

$\sqrt{n^2+1} + n$ - increasing $\Rightarrow \frac{\pi}{\sqrt{n^2+1}+n}$ - decreasing

$\frac{\pi}{\sqrt{n^2+1}+n} \in \left(0; \frac{\pi}{2}\right), \forall n \geq 1$

$\Rightarrow \sin\left(\frac{\pi}{\sqrt{n^2+1}+n}\right)$ decreasing

$\lim_{n\to\infty} \sin\frac{\pi}{\sqrt{n^2+1}+n} = 0$

$\Rightarrow \sum_{n\geq 1} \sin\left(\pi\sqrt{n^2+1}\right)$ - converges

$$\sum_{n\geq 1} \left|\sin\left(\pi\sqrt{n^2+1}\right)\right|$$

$$\left|\sin\left(\pi\sqrt{n^2+1}\right)\right| = \left|(-1)^n \cdot \sin\left(\frac{\pi}{\sqrt{n^2+1}+n}\right)\right| = \sin\frac{\pi}{\sqrt{n^2+1}+n}$$

$$\lim_{n\to\infty} \frac{\sin\frac{\pi}{\sqrt{n^2+1}+n}}{\frac{\pi}{\sqrt{n^2+1}+n}} = 1 \Rightarrow \sum_{n\geq 1}\left|\sin\left(\pi\sqrt{n^2+1}\right)\right| \text{ "like" } \sum_{n\geq 1}\frac{\pi}{\sqrt{n^2+1}+n}$$

$$\sum_{n\geq 1}\frac{\pi}{\sqrt{n^2+1}+n} = \pi\sum_{n\geq 1}\frac{1}{\sqrt{n^2+1}+n}$$

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n^2+1}+n}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}+n} \Rightarrow \frac{1}{2} > 0 \Rightarrow \sum_{n\geq 1}\frac{1}{\sqrt{n^2+1}+n} \text{ "like" } \sum_{n\geq 1}\frac{1}{n} = \infty \Rightarrow \sum_{n\geq 1}\left|\sin\left(\pi\sqrt{n^2+1}\right)\right| = \infty$$

3) c) $\sum_{n\geq 1} a^{\ln n}$, $a>0$

Ratio Test: $\frac{x_{n+1}}{x_n} = \frac{a^{\ln(n+1)}}{a^{\ln n}} = a^{\ln(n+1)-\ln n} = a^{\ln\frac{n+1}{n}}$

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty} a^{\ln\frac{n+1}{n}} = a^{\ln 1} = a^0 = 1$$

R.D. Test: $\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\underbrace{a^{\ln\frac{n}{n+1}}}_{\to 1}-1\right) = [\infty\cdot 0] = \lim_{n\to\infty} n\cdot \underbrace{\frac{a^{\ln\frac{n}{n+1}}-1}{\ln\frac{n}{n+1}}}_{\to \ln a}\cdot\ln\frac{n}{n+1} =$

$= \lim_{n\to\infty} \ln a \cdot n\cdot\ln\frac{n}{n+1} = \ln a\cdot\lim_{n\to\infty} n\ln\left(1+\frac{-1}{n+1}\right) \overset{[\infty\cdot 0]}{=} \ln a\cdot\lim_{n\to\infty} n\cdot\underbrace{\frac{\ln\left(1+\frac{-1}{n+1}\right)}{\frac{-1}{n+1}}}_{\to 1}\cdot\frac{-1}{n+1} =$

$= \ln a\cdot\lim_{n\to\infty} n\cdot\frac{-1}{n+1} = \ln a\cdot(-1) = -1\cdot\ln a = -\ln a$

$-\ln a > 1 \Leftrightarrow \ln a < -1 \Leftrightarrow \ln a < \ln e^{-1} \Leftrightarrow a < e^{-1} \Leftrightarrow a < \frac{1}{e}$

For $a\in\left(0;\frac{1}{e}\right) \Rightarrow \sum_{n\geq 1} a^{\ln n}$ converges

For $a\in\left(\frac{1}{e};+\infty\right) \Rightarrow \sum_{n\geq 1} a^{\ln n}$ diverges

If $a=\frac{1}{e} \Rightarrow \sum_{n\geq 1}\left(\frac{1}{e}\right)^{\ln n} = \sum_{n\geq 1}\left(e^{-1}\right)^{\ln n} = \sum_{n\geq 1}\left(e^{\ln n}\right)^{-1} = \sum_{n\geq 1} n^{-1} = \sum_{n\geq 1}\frac{1}{n}$ diverges

$$\sum_{n\geq 1} a^{\ln n} \begin{cases} \text{converges, for } a\in\left(0;\frac{1}{e}\right) \\ \text{diverges, for } a\in\left[\frac{1}{e};+\infty\right) \end{cases}$$

3) b) $\sum_{n\geq 1} \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\cdot\frac{1}{n^2}$

Ratio Test : $\frac{x_{n+1}}{x_n} = \frac{\cancel{1\cdot 3\cdot\ldots\cdot(2n-1)}(2n+1)}{\cancel{2\cdot 4\cdot\ldots\cdot 2n}\cdot(2n+2)}\cdot\frac{1}{(n+1)^2}\cdot\frac{\cancel{2\cdot 4\cdot\ldots\cdot 2n}}{\cancel{1\cdot 3\cdot\ldots\cdot(2n-1)}}\cdot n^2 = \frac{2n+1}{2n+2}\cdot\frac{n^2}{(n+1)^2}\to 1$

R.D Test : $\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{2n+2}{2n+1}\cdot\frac{(n+1)^2}{n^2}-1\right) = \lim_{n\to\infty} \cancel{n}\cdot\frac{(2n+2)(n+1)^2-(2n+1)\cdot n^2}{(2n+1)\cdot n^{\cancel{2}}} =$

$= \lim_{n\to\infty}\frac{(2n+2)(n^2+2n+1)-2n^3-n^2}{(2n+1)\cdot n} = \lim_{n\to\infty}\frac{\cancel{2n^3}+4n^2+2n+2n^2+4n+2-\cancel{2n^3}-n^2}{(2n+1)\cdot n} =$

$= \lim_{n\to\infty}\frac{5n^2+6n+2}{2n^2+n} = \frac{5}{2} > 1 \Rightarrow \sum_{n\geq 1}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\cdot\frac{1}{n^2}$ converges

d) $\sum_{n\geq 1}\frac{a^n\cdot n!}{n^n}, a>0$

Ratio Test : $\frac{x_{n+1}}{x_n} = \frac{a^{n+1}\cdot\cancel{(n+1)!}}{(n+1)^{n+1}}\cdot\frac{n^n}{a^n\cdot\cancel{n!}} = \frac{a^{n+1}\cdot\cancel{(n+1)}}{(n+1)^{\cancel{n+1}}}\cdot\frac{n^n}{a^n} = a\left(\frac{n}{n+1}\right)^n = \cancel{a\left(1+\frac{-1}{n+1}\right)}$

$= a\cdot\left[\left(\frac{n+1}{n}\right)^n\right]^{-1} \Rightarrow a\cdot e^{-1} = \frac{a}{e}$

For $a<e \Rightarrow \sum_{n\geq 1}\frac{a^n\cdot n!}{n^n}$ converges

For $a>e \Rightarrow \sum_{n\geq 1}\frac{a^n\cdot n!}{n^n}$ diverges

For $a=e \Rightarrow \sum_{n\geq 1}\frac{e^n\cdot n!}{n^n}$ ?

R.D Test : $\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\underbrace{\frac{1}{e}\cdot\left(\frac{n+1}{n}\right)^n-1}_{\to 0}\right)$