Seminar 1

1. Find the lower bound and the upper bound, then sup, inf, max, min for each of the following:

   (a) $(-1,1)\cup(2,\infty)$.
   (b) $(-3,2)\cup\{3\}$.
   (c) $(-5,5)\cap\mathbb{Z}$.
   (d) $\emptyset$.

2. Find the sup, inf, max, min for each of the following sets:

   (a) $\{x\in\mathbb{Q}\mid x^2<2\}$.
   (b) $\{x^2-4x+3\mid x\in\mathbb{R}\}$.
   (c) $\{\frac{n}{n+1}\mid n\in\mathbb{N}\}$.
   (d) $\{2^{-k}+3^{-m}\mid k,m\in\mathbb{N}\}$.

3. ★ Let $A=(0,1)\cap\mathbb{Q}$. Show that $\inf A=0$, $\sup A=1$, $\mathrm{int}A=\emptyset$ and $\mathrm{cl}A=[0,1]$.

4. Suppose that S is nonempty and bounded above. Show that the set $-S:=\{-x\mid x\in S\}$ is bounded below and $\inf(-S)=-\sup(S)$.

5. ★ Let $a,b\in\mathbb{R}$ with $a>0$. If S is nonempty and bounded above, prove that

$$\sup_{x\in S}(ax+b)=a\sup_{x\in S}+b.$$

6. Let $f:D\to\mathbb{R}$ and $g:D\to\mathbb{R}$ be two functions defined on a nonempty set D. Prove that

$$\inf_{x\in D}\big(f(x)+g(x)\big)\geq\inf_{x\in D}f(x)+\inf_{x\in D}g(x)\quad\text{and}\quad\sup_{x\in D}\big(f(x)+g(x)\big)\leq\sup_{x\in D}f(x)+\sup_{x\in D}g(x).$$

   Give examples where the above inequalities are strict.

7. Which of the following sets are neighborhoods of 0?

$$[-1,1]\cup\{2\};\quad(-1,1)\cap\mathbb{Q};\quad\bigcap_{n=1}^{\infty}[-\frac{1}{n},\frac{1}{n}].$$

8. $*$ Let α be an irrational number and consider the set $S_\alpha:=\big\{\{n\alpha\}\mid n\in\mathbb{N}\big\}$, where $\{\cdot\}$ denotes the fractional part. Show that the set S_α is dense in $[0,1]$, meaning that its closure is $[0,1]$. Then show that the set $\big\{\{n\alpha\}+m\mid n,m\in\mathbb{Z}\big\}$ is dense in $\mathbb{R}$.

Homework questions are marked with ★. Bonus questions are marked with $*$.
Solutions should be handed in at the beginning of next week's lecture.