Course 4

2.3 Generated subspace

For a vector space V over K, we denote by $S(V)$ the set of all subspaces of V. Sometimes, this set is denoted by $S_K(V)$ if we like to emphasize the field K.

Theorem 2.3.1 *Let V be a vector space over K and let $(S_i)_{i\in I}$ be a family of subspaces of V. Then $\bigcap_{i\in I} S_i \in S(V)$.*

Proof. For each $i \in I$, we have $S_i \in S(V)$, hence $0 \in S_i$. Then $0 \in \bigcap_{i\in I} S_i \neq \emptyset$. Now let $k_1, k_2 \in K$ and $x, y \in \bigcap_{i\in I} S_i$. Then $x, y \in S_i$, $\forall i \in I$. But $S_i \in S(V)$, $\forall i \in I$. It follows that $k_1x + k_2y \in S_i$, $\forall i \in I$, hence $k_1x + k_2y \in \bigcap_{i\in I} S_i$. Therefore, $\bigcap_{i\in I} S_i \in S(V)$. $\square$

Remark 2.3.2 In general, the union of two subspaces of a vector space is not a subspace. For instance, $S = \{(x, 0) \mid x \in \mathbb{R}\}$ and $T = \{(0, y) \mid y \in \mathbb{R}\}$ are subspaces of the canonical real vector space $\mathbb{R}^2$, but $S \cup T$ is not a subspace of $\mathbb{R}^2$. Indeed, for instance, we have $(1, 0), (0, 1) \in S \cup T$, but $(1, 0)+(0, 1) = (1, 1) \notin S \cup T$.

Now we are interested in how to "complete" a given subset of a vector space to a subspace in a minimal way. This is the motivation for the following definition.

Definition 2.3.3 Let V be a vector space and let $X \subseteq V$. Then we denote

$$\langle X\rangle = \bigcap\{S \leq V \mid X \subseteq S\}$$

and we call it the *subspace generated by* X or the *subspace spanned by* X.
Here X is called the *generating set* of $\langle X\rangle$.
If $X = \{v_1, \dots, v_n\}$, we denote $\langle v_1, \dots, v_n\rangle = \langle\{v_1, \dots, v_n\}\rangle$.

Remark 2.3.4 (1) $\langle X\rangle$ is the "smallest" (with respect to inclusion) subspace of V containing X.

(2) $\langle\emptyset\rangle = \{0\}$.

(3) If $S \leq V$, then $\langle S\rangle = S$.

Definition 2.3.5 A vector space V over K is called *finitely generated* if $\exists v_1, \dots, v_n \in V$ $(n \in \mathbb{N})$ such that $V = \langle v_1, \dots, v_n\rangle$. Then the set $\{v_1, \dots, v_n\}$ is called a *system of generators for* V.

Definition 2.3.6 Let V be a vector space over K and $v_1, \dots, v_n \in V$ $(n \in \mathbb{N})$. A finite sum of the form

$$k_1v_1 + \cdots + k_nv_n\,,$$

where $k_i \in K$ $(i = 1, \dots, n)$, is called a (finite) *linear combination* of the vectors $v_1, \dots, v_n$.

Let us now determine how the elements of a generated subspace look like.

Theorem 2.3.7 *Let V be a vector space over K and let $\emptyset \neq X \subseteq V$. Then*

$$\langle X\rangle = \{k_1v_1 + \cdots + k_nv_n \mid k_i \in K\,,\ v_i \in X\,, i = 1, \dots, n\,,\ n \in \mathbb{N}^*\}\,,$$

that is, the set of all finite linear combinations of vectors of X.

Proof. We prove the result in 3 steps, by showing that

$$L = \{k_1v_1 + \cdots + k_nv_n \mid k_i \in K\,,\ v_i \in X\,, i = 1, \dots, n\,,\ n \in \mathbb{N}^*\}$$

is the smallest subspace of V containing X.

(i) Let $v \in X$. Then $v = 1 \cdot v \in L$, hence $L \neq \emptyset$. Now let $k, k' \in K$ and $v, v' \in L$. Then $v = \sum_{i=1}^n k_i v_i$ and $v' = \sum_{j=1}^m k'_j v'_j$ for some $k_1, \dots, k_n, k'_1, \dots, k'_m \in K$ and $v_1, \dots, v_n, v'_1, \dots, v'_m \in X$. Hence

$$kv + k'v' = k \sum_{i=1}^n k_i v_i + k' \sum_{j=1}^m k'_j v'_j = \sum_{i=1}^n (kk_i) v_i + \sum_{j=1}^m (k'k'_j) v'_j \in L\,,$$

because it is a finite linear combination of vectors of X. Hence we have $L \leq V$.

(ii) Choose $n = 1$ and $k_1 = 1$ in order to see that $X \subseteq L$.

(iii) Let $S \leq V$ be such that $X \subseteq S$. Let $k_1, \dots, k_n \in K$ and $v_1, \dots, v_n \in X$. Since $X \subseteq S$ and $S \leq V$, it follows that

$$k_1 v_1 + \dots + k_n v_n \in S.$$

Hence $L \subseteq S$.

Thus, we have $\langle X \rangle = L$ by the remark from the beginning of the proof. □

Corollary 2.3.8 *Let V be a vector space over K and let $x_1, \dots, x_n \in V$. Then*

$$\langle x_1, \dots, x_n \rangle = \{k_1 x_1 + \dots + k_n x_n \mid k_i \in K\,,\ x_i \in X\,, i = 1, \dots, n\}\,.$$

Example 2.3.9 (a) Consider the canonical real vector space $\mathbb{R}^3$. Then

$$\begin{aligned}\langle (1,0,0), (0,1,0), (0,0,1) \rangle &= \{k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) \mid k_1, k_2, k_3 \in \mathbb{R}\} \\ &= \{(k_1,0,0) + (0,k_2,0) + (0,0,k_3) \mid k_1, k_2, k_3 \in \mathbb{R}\} \\ &= \{(k_1,k_2,k_3) \mid k_1, k_2, k_3 \in \mathbb{R}\} = \mathbb{R}^3\,.\end{aligned}$$

Hence $\mathbb{R}^3$ is generated by the three vectors $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, and thus it is finitely generated.

(b) Consider the canonical vector space $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$. Then

$$\begin{aligned}\langle (\widehat{1},\widehat{0},\widehat{0}), (\widehat{0},\widehat{1},\widehat{0}) \rangle &= \{k_1(\widehat{1},\widehat{0},\widehat{0}) + k_2(\widehat{0},\widehat{1},\widehat{0}) \mid k_1, k_2 \in \mathbb{Z}_2\} \\ &= \{(k_1,\widehat{0},\widehat{0}) + (\widehat{0},k_2,\widehat{0}) \mid k_1, k_2 \in \mathbb{Z}_2\} = \{(k_1,k_2,\widehat{0}) \mid k_1, k_2 \in \mathbb{Z}_2\} \neq \mathbb{Z}_2^3\,.\end{aligned}$$

Hence $\mathbb{Z}_2^3$ is not generated by the two vectors $(\widehat{1},\widehat{0},\widehat{0})$ and $(\widehat{0},\widehat{1},\widehat{0})$. But it is generated by $(\widehat{1},\widehat{0},\widehat{0})$, $(\widehat{0},\widehat{1},\widehat{0})$ and $(\widehat{0},\widehat{0},\widehat{1})$, hence it is finitely generated.

(c) Consider the subspace

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x - y - z = 0\}$$

of the canonical real vector space $\mathbb{R}^3$. Let us write it as a generated subspace. Expressing $x = y + z$, we have:

$$\begin{aligned}S &= \{(y+z,y,z) \mid y,z \in \mathbb{R}\} = \{(y,y,0) + (z,0,z) \mid y,z \in \mathbb{R}\} \\ &= \{y(1,1,0) + z(1,0,1) \mid y,z \in \mathbb{R}\} = \langle (1,1,0), (1,0,1) \rangle.\end{aligned}$$

Alternatively, one may express y or z by using the other two components and get other writings of S as a generated subspace, namely $S = \langle (1,1,0), (0,-1,1) \rangle = \langle (1,0,1), (0,1,-1) \rangle$. We see that S is finitely generated.

In what follows we shall be interested in "decomposing" a vector space into subspaces. This allows one to study the component subspaces and then deduce properties of the whole vector space.

Let us first define the sum and the direct sum of two subspaces of a vector space.

Definition 2.3.10 Let V be a vector space over K and let $S, T \leq V$.
We define the *sum* of the subspaces S and T as the set

$$S + T = \{s + t \mid s \in S\,,\ t \in T\}\,.$$

If $S \cap T = \{0\}$, then $S + T$ is denoted by $S \oplus T$ and is called the *direct sum* of the subspaces S and T.

Theorem 2.3.11 *Let V be a vector space over K and let $S, T \leq V$. Then*

$$S + T = \langle S \cup T \rangle\,.$$

Proof. We prove the equality by double inclusion.

First, let $v = s + t \in S + T$, for some $s \in S$ and $t \in T$. Then

$$v = 1 \cdot s + 1 \cdot t$$

is a linear combination of the vectors $s, t \in S \cup T$, hence $v \in \langle S \cup T \rangle$. Thus, $S + T \subseteq \langle S \cup T \rangle$.

Now let $v \in \langle S \cup T \rangle$. Then

$$v = \sum_{i=1}^{n} k_i v_i = \sum_{i \in I} k_i v_i + \sum_{j \in J} k_j v_j\,,$$

where $I = \{i \in \{1, \dots, n\} \mid v_i \in S\}$ and $J = \{j \in \{1, \dots, n\} \mid v_j \in T \setminus S\}$. But the first sum is a linear combination of vectors of S, hence it belongs to S, while the second sum is a linear combination of vectors of T, hence it belongs to T. Thus, $v \in S + T$ and consequently $\langle S \cup T \rangle \subseteq S + T$.

Therefore, $S + T = \langle S \cup T \rangle$. $\square$

Corollary 2.3.12 *Let V be a vector space over K and let $S, T \leq V$. Then $S + T \leq V$.*

Proof. By Theorem 2.3.11. $\square$

Theorem 2.3.13 *Let V be a vector space over K and let $S, T \leq V$. Then*

$$V = S \oplus T \Longleftrightarrow \forall v \in V\,, \exists! s \in S\,, t \in T\ : v = s + t\,.$$

Proof. $\boxed{\Longrightarrow}$ Assume that $V = S \oplus T$. Let $v \in V$. Then $\exists s \in S\,, t \in T$ such that $v = s + t$. Now suppose that $\exists s' \in S\,, t' \in T$ such that $v = s' + t'$. Then $s + t = s' + t'$, whence

$$s - s' = t' - t \in S \cap T = \{0\}\,.$$

Hence $s = s'$ and $t = t'$, that show the uniqueness.

$\boxed{\Longleftarrow}$ Assume that $\forall v \in V$, $\exists! s \in S\,, t \in T$ such that $v = s + t$. Then $V \subseteq S + T$. Clearly, we have $S + T \subseteq V$ and consequently $V = S + T$. Now suppose that $0 \neq v \in S \cap T$. Then

$$v = v + 0 = 0 + v.$$

But this is a contradiction, since we have the uniqueness of writing of v as a sum of an element of S and an element of T. Therefore, $S \cap T = \{0\}$ and thus, $V = S \oplus T$. $\square$

Example 2.3.14 Consider the canonical real vector space $\mathbb{R}^2$. Then $\mathbb{R}^2 = S \oplus T$, where $S = \{(x, 0) \mid x \in \mathbb{R}\}$ and $T = \{(0, y) \mid y \in \mathbb{R}\}$.

2.4 Linear maps

Definition 2.4.1 Let V and V' be vector spaces over the same field K. A function $f : V \to V'$ is called:
(1) *(K-)linear map* (or *(vector space) homomorphism* or *linear transformation*) if

$$\begin{aligned} f(v_1 + v_2) &= f(v_1) + f(v_2)\,, \quad \forall v_1, v_2 \in V, \\ f(kv) &= kf(v)\,, \quad \forall k \in K, \forall v \in V. \end{aligned}$$

(2) *isomorphism* if it is a bijective K-linear map.
(3) *endomorphism* if it is a K-linear map and $V = V'$.
(4) *automorphism* if it is a bijective K-linear map and $V = V'$.

Remark 2.4.2 If $f : V \to V'$ is a K-linear map, then the first condition from its definition tells us that f is a group homomorphism between the groups $(V, +)$ and $(V', +)$. Then we have $f(0) = 0'$ and $f(-v) = -f(v)$, $\forall v \in V$.

We denote by $V \simeq V'$ the fact that two vector spaces V and V' are isomorphic. We also denote

$$\begin{aligned} \mathrm{Hom}_K(V, V') &= \{f : V \to V' \mid f \text{ is } K\text{-linear}\}\,, \\ \mathrm{End}_K(V) &= \{f : V \to V \mid f \text{ is } K\text{-linear}\}\,, \\ \mathrm{Aut}_K(V) &= \{f : V \to V \mid f \text{ is bijective } K\text{-linear}\}\,. \end{aligned}$$

Let us now give a characterization theorem for linear maps.

Theorem 2.4.3 *Let V and V' be vector spaces over K and $f : V \to V'$. Then*

$$f \text{ is a } K\text{-linear map} \iff f(k_1v_1 + k_2v_2) = k_1f(v_1) + k_2f(v_2)\,, \forall k_1, k_2 \in K, \forall v_1, v_2 \in V.$$

Proof. $\boxed{\Longrightarrow}$ Let $k_1, k_2 \in K$ and $v_1, v_2 \in V$. Then

$$f(k_1v_1 + k_2v_2) = f(k_1v_1) + f(k_2v_2) = k_1f(v_1) + k_2f(v_2)\,.$$

$\boxed{\Longleftarrow}$ Choose $k_1 = k_2 = 1$ and then $k_2 = 0$ to get the two conditions of a K-linear map. □

Example 2.4.4 (a) Let V and V' be vector spaces over K and let $f : V \to V'$ be defined by $f(v) = 0'$, $\forall v \in V$. Then f is a K-linear map, called the *trivial linear map*.

(b) Let V be a vector space over K. Then the identity map $1_V : V \to V$ is an automorphism of V.

(c) Let V be a vector space and $S \le V$. Define $i : S \to V$ by $i(v) = v$, $\forall v \in S$. Then i is a K-linear map, called the *inclusion linear map*.

(d) Let V be a vector space over K and $a \in K$. Define $t_a : V \to V$ by $t_a(v) = av$, $\forall v \in V$. Then t_a is an endomorphism of V.

(e) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x + y$. Then f is an $\mathbb{R}$-linear map, because we have

$$\begin{aligned} f(k_1(x_1, y_1) + k_2(x_2, y_2)) &= f(k_1x_1 + k_2x_2, k_1y_1 + k_2y_2) \\ &= (k_1x_1 + k_2x_2) + (k_1y_1 + k_2y_2) \\ &= k_1(x_1 + y_1) + k_2(x_2 + y_2) \\ &= k_1f(x_1, y_1) + k_2f(x_2, y_2) \end{aligned}$$

for every $k_1, k_2 \in K$ and for every $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

On the other hand, $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy$ is not an $\mathbb{R}$-linear map, because, for instance, we have

$$f((1, 0) + (0, 1)) = f(1, 1) = 1 \neq 0 = f(1, 0) + f(0, 1).$$

(f) Let $\theta \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta),$$

which is the counterclockwise rotation of angle θ about the origin in the plane. Then f is an $\mathbb{R}$-linear map. In particular, for $\theta = \frac{\pi}{2}$, we have $f(x, y) = (-y, x)$.

(g) For an interval $I = [a, b] \subseteq \mathbb{R}$ we considered the real vector space

$$\mathbb{R}^I = \{f \mid f : I \to \mathbb{R}\}$$

and its subspaces

$$C(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ continuous on } I\},$$
$$D(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ derivable on } I\}.$$

Then

$$F : D(I, \mathbb{R}) \to \mathbb{R}^I, \quad F(f) = f',$$
$$G : C(I, \mathbb{R}) \to \mathbb{R}, \quad G(f) = \int_a^b f(t)\mathrm{d}t,$$

are $\mathbb{R}$-linear maps.

Theorem 2.4.5 *(i) Let $f : V \to V'$ be an isomorphism of vector spaces over K. Then $f^{-1} : V' \to V$ is again an isomorphism of vector spaces over K.*
(ii) Let $f : V \to V'$ and $g : V' \to V''$ be K-linear maps. Then $g \circ f : V \to V''$ is a K-linear map.

Proof. (i) Since f is an isomorphism of vector spaces over K, f is bijective, hence so is f^{-1}.

Now let $k_1, k_2 \in K$ and $v_1', v_2' \in V'$. We have to prove that

$$f^{-1}(k_1 v_1' + k_2 v_2') = k_1 f^{-1}(v_1') + k_2 f^{-1}(v_2').$$

Let us denote $v_1 = f^{-1}(v_1')$ and $v_2 = f^{-1}(v_2')$. Then $f(v_1) = v_1'$ and $f(v_2) = v_2'$, hence

$$k_1 v_1' + k_2 v_2' = k_1 f(v_1) + k_2 f(v_2) = f(k_1 v_1 + k_2 v_2).$$

Thus we have

$$f^{-1}(k_1 v_1' + k_2 v_2') = k_1 v_1 + k_2 v_2 = k_1 f^{-1}(v_1') + k_2 f^{-1}(v_2').$$

Hence f^{-1} is an isomorphism of vector spaces over K.

(ii) Let $k_1, k_2 \in K$ and $v_1, v_2 \in V$. We have:

$$\begin{aligned}(g \circ f)(k_1 v_1 + k_2 v_2) &= g(f(k_1 v_1 + k_2 v_2)) \\ &= g(k_1 f(v_1) + k_2 f(v_2)) \\ &= k_1 g(f(v_1)) + k_2 g(f(v_2)) \\ &= k_1 (g \circ f)(v_1) + k_2 (g \circ f)(v_2).\end{aligned}$$

Hence $g \circ f$ is a K-linear map. □

Definition 2.4.6 Let $f : V \to V'$ be a K-linear map. Then the set

$$\operatorname{Ker} f = \{v \in V \mid f(v) = 0'\}$$

is called the *kernel* (or the *null space*) of the K-linear map f and the set

$$\operatorname{Im} f = \{f(v) \mid v \in V\}$$

is called the *image* (or the *range space*) of the K-linear map f.

Theorem 2.4.7 *Let $f : V \to V'$ be a K-linear map. Then*

$$\mathrm{Ker} f \leq V \text{ and } \mathrm{Im} f \leq V' .$$

Proof. First, note that $f(0) = 0'$, hence $0 \in \mathrm{Ker}\, f \neq \emptyset$. Let $k_1, k_2 \in K$ and $v_1, v_2 \in \mathrm{Ker}\, f$. We prove that $k_1v_1 + k_2v_2 \in \mathrm{Ker}\, f$. Indeed, we have:

$$f(k_1v_1 + k_2v_2) = k_1f(v_1) + k_2f(v_2) = k_1 \cdot 0' + k_2 \cdot 0' = 0',$$

and thus $k_1v_1 + k_2v_2 \in \mathrm{Ker}\, f$. Hence $\mathrm{Ker}\, f \leq V$.

Now note that $0' = f(0) \in \mathrm{Im}\, f \neq \emptyset$. Let $k_1, k_2 \in K$ and $v_1', v_2' \in \mathrm{Im}\, f$. We prove that $k_1v_1' + k_2v_2' \in \mathrm{Im}\, f$. We have $v_1' = f(v_1)$ and $v_2' = f(v_2)$ for some $v_1, v_2 \in V$. Then:

$$k_1v_1' + k_2v_2' = k_1f(v_1) + k_2f(v_2) = f(k_1v_1 + k_2v_2) \in \mathrm{Im}\, f.$$

Hence $\mathrm{Im}\, f \leq V'$. □

Theorem 2.4.8 *Let $f : V \to V'$ be a K-linear map. Then*

$$\mathrm{Ker}\, f = \{0\} \Longleftrightarrow f \text{ is injective}.$$

Proof. $\boxed{\Longrightarrow}$ Assume that $\mathrm{Ker}\, f = \{0\}$. Let $v_1, v_2 \in V$ be such that $f(v_1) = f(v_2)$. It follows that $f(v_1 - v_2) = 0$, hence $v_1 - v_2 \in \mathrm{Ker}\, f = \{0\}$, and thus $v_1 = v_2$. Therefore, f is injective.

$\boxed{\Longleftarrow}$ Assume that f is injective. Clearly, we have $\{0\} \subseteq \mathrm{Ker}\, f$. Now let $v \in \mathrm{Ker}\, f$. Then $f(v) = 0' = f(0)$. By the injectivity of f, we deduce that $v = 0$. Thus $\mathrm{Ker}\, f \subseteq \{0\}$, and consequently, $\mathrm{Ker}\, f = \{0\}$. □

Theorem 2.4.9 *Let $f : V \to V'$ be a K-linear map and let $X \subseteq V$. Then*

$$f(\langle X\rangle) = \langle f(X)\rangle .$$

Proof. If $X = \emptyset$, then we have:

$$f(\langle\emptyset\rangle) = f(\{0\}) = \{f(0)\} = \{0'\} = \langle\emptyset\rangle = \langle f(\emptyset)\rangle.$$

Now assume that $X \neq \emptyset$. By Theorem 2.3.7 we have

$$\langle X\rangle = \{k_1v_1 + \dots + k_nv_n \mid k_i \in K\,,\ v_i \in X\,, i = 1, \dots, n\,,\ n \in \mathbb{N}^*\}\,.$$

Since f is a K-linear map, it follows by Theorem 2.4.3 that

$$\begin{aligned} f(\langle X\rangle) &= \{f(k_1v_1 + \dots + k_nv_n) \mid k_i \in K\,,\ v_i \in X\,, i = 1, \dots, n\,,\ n \in \mathbb{N}^*\} \\ &= \{k_1f(v_1) + \dots + k_nf(v_n) \mid k_i \in K\,,\ v_i \in X\,, i = 1, \dots, n\,,\ n \in \mathbb{N}^*\} \\ &= \langle f(X)\rangle\,, \end{aligned}$$

which proves the result. □

Theorem 2.4.10 *Let V and V' be vector spaces over K. Consider on $\mathrm{Hom}_K(V, V')$ the operations: $\forall f, g \in \mathrm{Hom}_K(V, V')$ and $\forall k \in K$, $f + g, k \cdot f \in \mathrm{Hom}_K(V, V')$, where*

$$\begin{aligned} (f + g)(v) &= f(v) + g(v)\,, \\ (kf)(v) &= kf(v) \end{aligned}$$

$\forall v \in V$. Then $\mathrm{Hom}_K(V, V')$ is a vector space over K.

Proof. Let $k \in K$ and $f, g \in \mathrm{Hom}_K(V, V')$. Let us prove first that the operations are well-defined, that is, $f + g, kf \in \mathrm{Hom}_K(V, V')$. Let $k_1, k_2 \in K$ and $v_1, v_2 \in V$. Then:

$$\begin{aligned}(f+g)(k_1xv_1 + k_2v_2) &= f(k_1v_1 + k_2v_2) + g(k_1v_1 + k_2v_2)\\ &= k_1f(v_1) + k_2f(v_2) + k_1g(v_1) + k_2g(v_2)\\ &= k_1(f(v_1) + g(v_1)) + k_2(f(v_2) + g(v_2))\\ &= k_1(f+g)(v_1) + k_2(f+g)(v_2)\,.\end{aligned}$$

We also have:

$$\begin{aligned}(kf)(k_1v_1 + k_2v_2) &= kf(k_1v_1 + k_2v_2)\\ &= k(k_1f(v_1)) + k(k_2f(v_2))\\ &= (kk_1)f(v_1) + (kk_2)f(v_2)\\ &= k_1(kf(v_1)) + k_2(kf(v_2))\,.\end{aligned}$$

Therefore, $f + g, kf \in \mathrm{Hom}_K(V, V')$.

It is easy to check that $(\mathrm{Hom}_K(V, V'), +)$ is an abelian group, where the identity element is the trivial linear map

$$\theta : V \to V', \quad \theta(v) = 0', \quad \forall v \in V$$

and every element $f \in \mathrm{Hom}_K(V, V')$ has a symmetric

$$-f \in \mathrm{Hom}_K(V, V'), \quad (-f)(v) = -f(v), \quad \forall v \in V.$$

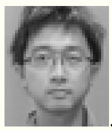
Checking the axioms of the vector space for $\mathrm{Hom}_K(V, V')$ reduces, by the definitions of operations, to the axioms for the vector space V'. □

Corollary 2.4.11 *Let V be a vector space over K. Then $\mathrm{End}_K(V)$ is a vector space over K.*

Proof. Take $V = V'$ in Theorem 2.4.10. □

Extra: Image crossfade

Following [Klein], we describe a way to achieve an image crossfade effect.
A black-and-white image of (say) $n = 1024 \times 768$ pixels can be viewed as a vector in the real canonical vector space $\mathbb{R}^n$, where each component of the vector is the intensity of the corresponding pixel.
Let us consider two vectors representing images:

$v_1 =$ , $v_2 =$ .

Now consider the following intermediate images:

The vectors corresponding to the above images are the following linear combinations of the vectors v_1 and v_2:

$$v_1, \quad \frac{8}{9}v_1 + \frac{1}{9}v_2, \quad \frac{7}{9}v_1 + \frac{2}{9}v_2, \quad \frac{6}{9}v_1 + \frac{3}{9}v_2, \quad \frac{5}{9}v_1 + \frac{4}{9}v_2,$$
$$\frac{4}{9}v_1 + \frac{5}{9}v_2, \quad \frac{3}{9}v_1 + \frac{6}{9}v_2, \quad \frac{2}{9}v_1 + \frac{7}{9}v_2, \quad \frac{1}{9}v_1 + \frac{8}{9}v_2, \quad v_2.$$

One may use these images as frames in a video in order to get a crossfade effect.