Seminar 8

1. Prove that for any $x, y \in \mathbb{R}^n$ the following hold:
   (a) $\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big)$ (the parallelogram identity).
   (b) $\langle x, y\rangle = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big)$.
2. ★ For $x, y \in \mathbb{R}^n$ prove that the following statements are equivalent:
   (a) $\langle x, y\rangle = 0$.
   (b) $\|x+y\| = \|x-y\|$.
   (c) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.
3. For $x, y \in \mathbb{R}^n$ prove the Cauchy-Schwarz inequality $\langle x, y\rangle \le \|x\|\|y\|$.
4. Find the orthogonal projection of a vector v onto a vector a in $\mathbb{R}^2$.
5. ★ Using the dot product, show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix with angle θ in $\mathbb{R}^2$. What is the inverse of this matrix?
6. Consider the p-norm $\|x\|_p := \big(|x_1|^p + \ldots + |x_n|^p\big)^{\frac{1}{p}}$, $p \ge 1$ and $\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$. Represent the unit ball in $\mathbb{R}^2$ for the p-norm with $p \in \{1, 2, \infty\}$.
   ★ [Python] Represent the unit ball in $\mathbb{R}^2$ for the p-norm with $p \in \{1.25, 1.5, 3, 8\}$.
7. Find the interior, the closure and the boundary for each of the following sets:
   (a) $[0,1) \times (1,2]$.
   (b) $\{(x,y) \in \mathbb{R}^2 \mid |x| < |y|\}$.
   (c) $\{(x,y) \in \mathbb{R}^2 \mid |x| + |y| < 1\}$.
   (d) $\{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \le 1,\ x \le 1\}$.
8. Study the convergence of the sequence (x^k) in $\mathbb{R}^2$ for:
   (a) $x^k = \big(\frac{1}{k}, \frac{2^k}{k!}\big)$.
   (b) $x^k = \big((-1)^k, -\big(\frac{1}{2}\big)^k\big)$.
   (c) $x^k = \big(e^{-k}\cos k, k\big)$.

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.