

Seminar 6

1. Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space \mathbb{R}^3 . Prove that:

- (i) v_1, v_2, v_3 are linearly dependent and determine a dependence relationship.
- (ii) v_1, v_2 are linearly independent.

2. Prove that the following vectors are linearly independent:

- (i) $v_1 = (1, 0, 2)$, $v_2 = (-1, 2, 1)$, $v_3 = (3, 1, 1)$ in \mathbb{R}^3 .
- (ii) $v_1 = (1, 2, 3, 4)$, $v_2 = (2, 3, 4, 1)$, $v_3 = (3, 4, 1, 2)$, $v_4 = (4, 1, 2, 3)$ in \mathbb{R}^4 .

3. Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in \mathbb{R}^3 . Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly independent.

4. Let $v_1 = (1, -2, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (0, a, 1, 2)$ be vectors in \mathbb{R}^4 . Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly dependent.

5. Let $v_1 = (1, 1, 0)$, $v_2 = (-1, 0, 2)$, $v_3 = (1, 1, 1)$ be vectors in \mathbb{R}^3 .

- (i) Show that the list (v_1, v_2, v_3) is a basis of the real vector space \mathbb{R}^3 .
- (ii) Express the vectors of the canonical basis (e_1, e_2, e_3) of \mathbb{R}^3 as a linear combination of the vectors v_1, v_2 and v_3 .
- (iii) Determine the coordinates of $u = (1, -1, 2)$ in each of the two bases.

6. Let $n \in \mathbb{N}^*$. Show that the vectors

$$v_1 = (1, \dots, 1, 1), v_2 = (1, \dots, 1, 2), v_3 = (1, \dots, 1, 2, 3), \dots, v_n = (1, 2, \dots, n-1, n)$$

form a basis of the real vector space \mathbb{R}^n and write the coordinates of a vector (x_1, \dots, x_n) in this basis.

7. Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the lists (E_1, E_2, E_3, E_4) and (A_1, A_2, A_3, A_4) are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in each of the two bases.

8. Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid \deg(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X - a, (X - a)^2) (a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial $f = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]$ in each basis.

9. Determine the number of bases of the vector space \mathbb{Z}_2^3 over \mathbb{Z}_2 .

10. Determine the number of elements of the general linear group $(GL_3(\mathbb{Z}_2), \cdot)$ of invertible 3×3 -matrices over \mathbb{Z}_2 .