Seminar 1

1. Which ones of the usual symbols of addition, subtraction, multiplication and division define an operation (composition law) on the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$?

2. Let $A = \{a_1, a_2, a_3\}$. Determine the number of:
(i) operations on A;
(ii) commutative operations on A;
(iii) operations on A with identity element.
Generalization for a set A with n elements ($n \in \mathbb{N}^*$).

3. Decide which ones of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are groups together with the usual addition or multiplication.

4. Let "$*$" be the operation defined on $\mathbb{R}$ by $x * y = x + y + xy$. Prove that:
(i) $(\mathbb{R}, *)$ is a commutative monoid.
(ii) The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

5. Let "$*$" be the operation defined on $\mathbb{N}$ by $x * y = \text{g.c.d.}(x, y)$.
(i) Prove that $(\mathbb{N}, *)$ is a commutative monoid.
(ii) Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ $(n \in \mathbb{N}^*)$ is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid.
(iii) Fill in the table of the operation "$*$" on D_6.

6. Determine the finite stable subsets of $(\mathbb{Z}, \cdot)$.

7. Let $(G, \cdot)$ be a group. Show that:
(i) G is abelian $\Longleftrightarrow \forall x, y \in G$, $(xy)^2 = x^2y^2$.
(ii) If $x^2 = 1$ for every $x \in G$, then G is abelian.

8. Let "$\cdot$" be an operation on a set A and let $X, Y \subseteq A$. Define an operation "$*$" on the power set $\mathcal{P}(A)$ by

$$X * Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

Prove that:
(i) If $(A, \cdot)$ is a monoid, then $(\mathcal{P}(A), *)$ is a monoid.
(ii) If $(A, \cdot)$ is a group, then in general $(\mathcal{P}(A), *)$ is not a group.