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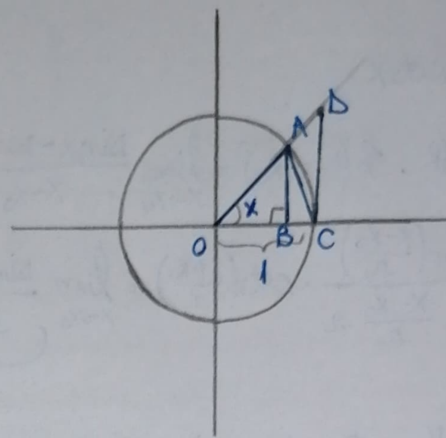
913

Homework

5) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$?



We will first prove that $\lim\limits_{\substack{x\to 0\\ x>0}} \frac{\sin x}{x} = 1$

$\Delta ABO: \sin x = \frac{AB}{OA} = \frac{AB}{1} = AB \Rightarrow AB = \sin x$

$\Delta DCO: tg\, x = \frac{DC}{OC} = \frac{DC}{1} \Rightarrow DC = tg\, x$

$AC = x$

$A_{\Delta AOC} = \frac{b \cdot h}{2} = \frac{OC \cdot AB}{2} = \frac{1 \cdot \sin x}{2} = \frac{\sin x}{2}$

$A_{\widehat{AOC}} = \frac{r^2 \cdot x}{2} = \frac{x}{2}$

$A_{\Delta DOC} = \frac{DC \cdot CO}{2} = \frac{tg\, x \cdot 1}{2} = \frac{tg\, x}{2}$

From the drawing we observe that : $A_{\Delta AOC} < A_{\widehat{AOC}} < A_{\Delta DOC}$

$\Leftrightarrow \frac{\sin x}{2} < \frac{x}{2} < \frac{tg\, x}{2} \,/\cdot 2>0 \Leftrightarrow \sin x < x < tg\, x$ $\Big\}\Rightarrow 1 < \frac{x}{\sin x} < \frac{tg\, x}{\sin x} \Leftrightarrow$

Since $x>0$ and $x\to 0 \Rightarrow x \in \left(0; \frac{\pi}{2}\right) \Rightarrow \sin x > 0$, $\cos x > 0$ $\Big\} \Leftrightarrow 1 < \frac{x}{\sin x} < \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x} \Leftrightarrow$

$\Leftrightarrow \underbrace{1}_{>0} < \underbrace{\frac{x}{\sin x}}_{>0} < \underbrace{\frac{1}{\cos x}}_{>0} \Leftrightarrow \cos x < \frac{\sin x}{x} < 1$

$\lim\limits_{\substack{x\to 0\\ x>0}} \cos x = \cos 0 = 1$

$\lim\limits_{\substack{x\to 0\\ x>0}} 1 = 1$

$\Bigg\} \xrightarrow[\text{THMR}]{\text{SQUEEZE}} \exists \lim\limits_{\substack{x\to 0\\ x>0}} \frac{\sin x}{x} = 1$

Now we need to show that $\lim\limits_{\substack{x\to 0\\ x<0}} \frac{\sin x}{x} = 1$

Let $t = -x \Rightarrow x = -t$
$x \to 0 \Rightarrow t \to 0$
$x<0 \Rightarrow t>0$
$\Bigg\} \Rightarrow \lim\limits_{\substack{x\to 0\\ x<0}} \frac{\sin x}{x} = \lim\limits_{\substack{t\to 0\\ t>0}} \frac{\sin(-t)}{-t} = \lim\limits_{\substack{t\to 0\\ t>0}} \frac{-\sin t}{-t} = \lim\limits_{\substack{t\to 0\\ t>0}} \frac{\sin t}{t} = 1 \Rightarrow \lim\limits_{\substack{x\to 0\\ x<0}} \frac{\sin x}{x} = 1 \Bigg\} \Rightarrow$

$\Rightarrow \exists \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

$(\sin x)' \overset{?}{=} \cos x$

Let $x_0 \in \mathbb{R}$. $(\sin x_0)' = \lim\limits_{x\to x_0} \frac{\sin x - \sin x_0}{x - x_0} = \lim\limits_{x\to x_0} \frac{2\cdot \sin\left(\frac{x-x_0}{2}\right)\cos\left(\frac{x+x_0}{2}\right)}{x-x_0} =$

$= 2\lim\limits_{x\to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}\cdot 2}\cdot \cos\left(\frac{x+x_0}{2}\right) = \lim\limits_{x\to x_0} \underbrace{\frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}}}_{\to 1}\cdot \cos\left(\frac{x+x_0}{2}\right) = \lim\limits_{x\to x_0} \cos\frac{x+x_0}{2} = \cos\frac{x_0+x_0}{2} = \cos x_0$

Therefore $\forall x_0 \in \mathbb{R} \Rightarrow (\sin x_0)' = \cos x_0 \Rightarrow (\sin x)' = \cos x$

$(\cos x)' \overset{?}{=} -\sin x$.

Let $x_0 \in \mathbb{R} \Rightarrow (\cos x_0)' = \lim\limits_{x\to x_0} \frac{\cos x - \cos x_0}{x - x_0} = \lim\limits_{x\to x_0} \frac{-2\sin\left(\frac{x-x_0}{2}\right)\sin\left(\frac{x+x_0}{2}\right)}{x-x_0} =$

$= -\lim\limits_{x\to x_0} \underbrace{\frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}}}_{\to 1}\cdot \sin\left(\frac{x+x_0}{2}\right) = -\lim\limits_{x\to x_0} \sin\left(\frac{x+x_0}{2}\right) = -\sin\frac{x_0+x_0}{2} = -\sin x_0$

Therefore $\forall x_0 \in \mathbb{R} \Rightarrow (\cos x_0)' = -\sin x_0 \Rightarrow (\cos x)' = -\sin x$