Seminar 2

1. Prove using the ε-definition that $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and $\lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$.

2. Find the limit points, $\liminf\limits_{n\to\infty}$ and $\limsup\limits_{n\to\infty}$ for each of the following sequences:

   (a) $\sin(n\pi)$.

   (b) $\frac{(-1)^n \cdot n}{n+1}$.

3. Study if the sequence (x_n) is bounded, monotone, and convergent, for each of the following:

   (a) $x_n = \sqrt{n+1} - \sqrt{n}$.

   (b) $x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$.

   (c) $x_n = \frac{\sin(n)}{n}$.

   (d) $x_n = \frac{2^n}{n!}$.

4. Find the limit for each of the following sequences:

   (a) $\sqrt{n}\big(\sqrt{n+1} - \sqrt{n}\big)$.

   (b) $\big(a_1^n + a_2^n + \ldots + a_k^n\big)^{\frac{1}{n}}$, with $a_i > 0$.

   (c) $\sqrt[n]{n}$.

   (d) $\sqrt[n]{1 + 2 + \ldots + n}$.

   (e) $\frac{2^n + (-1)^n}{3^n}$.

   (f) $\frac{(an+1)^2}{4n^2 - 2n + 1}$, $a \in \mathbb{R}$.

5. Consider the sequence (e_n) given by

$$e_n = \big(1 + \frac{1}{n}\big)^n.$$

   Prove that (e_n) is increasing and bounded – its limit is denoted by e (Euler's number).

6. Find the limit for each of the following sequences:

   (a) $\big(\frac{2n+1}{2n-1}\big)^n$.

   (b) $n\big(\ln(n+2) - \ln(n+1)\big)$.

   (c) $\big(\frac{\ln(n+1)}{\ln n}\big)^n$.

7. ★ Prove that the sequence (x_n) given by

$$x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n.$$

   is decreasing and bounded, hence convergent – its limit is denoted by γ (Euler's constant).

8. (Stolz-Cesàro lemma) Let $(a_n), (b_n)$ be two sequences such that
   (i) $a_n \to 0$ and $b_n \to 0$ with (b_n) decreasing; or (ii) $b_n \to \infty$ with (b_n) increasing.
   If

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell, \quad \text{then } \lim_{n\to\infty} \frac{a_n}{b_n} = \ell.$$

(a) Let (x_n) be a convergent sequence. What can you say about the sequence (a_n) of averages
$$a_n = \frac{x_1 + x_2 + \ldots + x_n}{n}?$$
Give an example where the averages converge, even though the sequence does not.

(b) Compare
$$1 + \frac{1}{2} + \ldots + \frac{1}{n}$$
with n and $\ln n$, respectively.

(c) Let (x_n) be a sequence such that
$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell.$$
Then
$$\lim_{n\to\infty} \sqrt[n]{x_n} = \ell.$$

9. Find the limit for each of the following sequences:

(a) $\frac{n}{\sqrt[n]{n!}}$.
(b) $\frac{n^n}{1+2^2+3^3+\ldots+n^n}$.
(c) $\frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}}$, $p \in \mathbb{N}$.

10. (Banach fixed point theorem) Let $f : [a,b] \to \mathbb{R}$ be a contraction, meaning that there exists $\alpha \in (0,1)$ such that
$$|f(x) - f(y)| \le \alpha\, |x - y|\,, \quad \forall x, y \in [a,b].$$
Let an arbitrary $x_1 \in [a,b]$ and consider the sequence (x_n) given by
$$x_{n+1} = f(x_n), \quad \forall n \in \mathbb{N}.$$
Prove that the sequence (x_n) is Cauchy, and that its limit x^* is a fixed point, i.e. $f(x^*) = x^*$.

11. ★ Let $\alpha \in (0,1)$ and $x_1, x_2 \in \mathbb{R}$. Study the convergence of the sequence (x_n) given by
$$x_{n+2} = \alpha x_{n+1} + (1-\alpha) x_n, \quad \forall n \in \mathbb{N}.$$

12. $*$ Give an example of a sequence having the set of limit points equal to $[0,1]$. Justify.

Homework questions are marked with ★. Bonus questions are marked with $*$.
Solutions should be handed in at the beginning of next week's lecture.