

Course 5

2.5 Linear independence

Definition 2.5.1 Let V be a vector space over K . We say that the vectors $v_1, \dots, v_n \in V$ are (or the set of vectors $\{v_1, \dots, v_n\}$ is):

(1) *linearly independent* in V if for every $k_1, \dots, k_n \in K$,

$$k_1 v_1 + \dots + k_n v_n = 0 \implies k_1 = \dots = k_n = 0.$$

(2) *linearly dependent* in V if they are not linearly independent, that is, $\exists k_1, \dots, k_n \in K$ not all zero such that

$$k_1 v_1 + \dots + k_n v_n = 0.$$

Remark 2.5.2 (1) A set consisting of a single vector v is linearly dependent $\iff v = 0$.

(2) As an immediate consequence of the definition, we notice that if V is a vector space over K and $X, Y \subseteq V$ such that $X \subseteq Y$, then:

(i) If Y is linearly independent, then X is linearly independent.

(ii) If X is linearly dependent, then Y is linearly dependent. Thus, every set of vectors containing the zero vector is linearly dependent.

(3) More generally, an infinite set of vectors of V is called *linearly independent* if any finite subset is linearly independent, and *linearly dependent* if there exists a finite subset which is linearly dependent.

Theorem 2.5.3 Let V be a vector space over K . Then the vectors $v_1, \dots, v_n \in V$ are linearly dependent if and only if one of the vectors is a linear combination of the others, that is, $\exists j \in \{1, \dots, n\}$ such that

$$v_j = \sum_{\substack{i=1 \\ i \neq j}}^n \alpha_i v_i$$

for some $\alpha_i \in K$, where $i \in \{1, \dots, n\}$ and $i \neq j$.

Proof. \implies Assume that $v_1, \dots, v_n \in V$ are linearly dependent. Then $\exists k_1, \dots, k_n \in K$ not all zero, say $k_j \neq 0$, such that $k_1 v_1 + \dots + k_n v_n = 0$. But this implies

$$-k_j v_j = \sum_{\substack{i=1 \\ i \neq j}}^n k_i v_i$$

and further,

$$v_j = \sum_{\substack{i=1 \\ i \neq j}}^n (-k_j^{-1} k_i) v_i.$$

Now choose $\alpha_i = -k_j^{-1} k_i$ for each $i \neq j$ to get the conclusion.

\impliedby Assume that $\exists j \in \{1, \dots, n\}$ such that

$$v_j = \sum_{\substack{i=1 \\ i \neq j}}^n \alpha_i v_i$$

for some $\alpha_i \in K$, where $i \in \{1, \dots, n\}$ and $i \neq j$. Then

$$(-1)v_j + \sum_{\substack{i=1 \\ i \neq j}}^n \alpha_i v_i = 0.$$

Since there exists such a linear combination equal to zero and the scalars are not all zero, the vectors v_1, \dots, v_n are linearly dependent. \square

Example 2.5.4 (a) Let V_2 be the real vector space of all vectors (in the classical sense) in the plane with a fixed origin O . Recall that the addition is the usual addition of two vectors by the parallelogram rule and the external operation is the usual scalar multiplication of vectors by real scalars. Then:

- (i) one vector v is linearly dependent in $V_2 \iff v = 0$;
- (ii) two vectors are linearly dependent in $V_2 \iff$ they are collinear;
- (iii) three vectors (or more) are always linearly dependent in V_2 .

Now let V_3 be the real vector space of all vectors (in the classical sense) in the space with a fixed origin O . Then:

- (i) one vector v is linearly dependent in $V_3 \iff v = 0$;
- (ii) two vectors are linearly dependent in $V_3 \iff$ they are collinear;
- (iii) three vectors are linearly dependent in $V_3 \iff$ they are coplanar;
- (iv) four vectors (or more) are always linearly dependent in V_3 .

(b) If K is a field and $n \in \mathbb{N}^*$, then the vectors $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, \dots , $e_n = (0, 0, 0, \dots, 1) \in K^n$ are linearly independent in the canonical vector space K^n over K . In order to show that, let $k_1, \dots, k_n \in K$ be such that

$$k_1 e_1 + k_2 e_2 + \dots + k_n e_n = 0 \in K^n.$$

Then we have

$$k_1(1, 0, 0, \dots, 0) + k_2(0, 1, 0, \dots, 0) + \dots + k_n(0, 0, 0, \dots, 1) = (0, \dots, 0),$$

and furthermore

$$(k_1, \dots, k_n) = (0, \dots, 0).$$

This implies that $k_1 = \dots = k_n = 0$, and thus the vectors e_1, \dots, e_n are linearly independent in K^n .

(c) Let K be a field and $n \in \mathbb{N}$. Then the vectors $1, X, X^2, \dots, X^n$ are linearly independent in the vector space $K_n[X] = \{f \in K[X] \mid \text{degree}(f) \leq n\}$ over K . More generally, the vectors

$$1, X, X^2, \dots, X^n, \dots \quad (n \in \mathbb{N})$$

form an infinite linearly independent set in the vector space $K[X]$ over K .

Now let us give a very useful practical result on linear dependence.

Theorem 2.5.5 Let $n \in \mathbb{N}$, $n \geq 2$.

- (i) Two vectors in the canonical vector space K^n are linearly dependent \iff their components are respectively proportional.
- (ii) n vectors in the canonical vector space K^n are linearly dependent \iff the determinant consisting of their components is zero.

Proof. (i) Let $v = (x_1, \dots, x_n)$, $v' = (x'_1, \dots, x'_n) \in K^n$. By Theorem 2.5.3, the vectors v and v' are linearly dependent if and only if one of them is a linear combination of the other, say $v' = kv$ for some $k \in K$. That is, $x'_i = kx_i$ for each $i \in \{1, \dots, n\}$.

(ii) Let $v_1 = (x_{11}, x_{21}, \dots, x_{n1})$, \dots , $v_n = (x_{1n}, x_{2n}, \dots, x_{nn}) \in K^n$. The vectors v_1, \dots, v_n are linearly dependent if and only if $\exists k_1, \dots, k_n \in K$ not all zero such that

$$k_1 v_1 + \dots + k_n v_n = 0.$$

But this is equivalent to

$$k_1(x_{11}, x_{21}, \dots, x_{n1}) + \dots + k_n(x_{1n}, x_{2n}, \dots, x_{nn}) = (0, \dots, 0),$$

and further to

[illegible]

We are interested in the existence of a non-zero solution for this homogeneous linear system. We will see later on that such a solution does exist if and only if the determinant of the system is zero. \square

2.6 Bases

We are going to define a key notion related to a vector space, namely that of a *basis*, which will perfectly determine a vector space. For the sake of simplicity and because of our limited needs, til the end of the chapter, *by a vector space we will understand a finitely generated vector space*.

Definition 2.6.1 Let V be a vector space over K . A list of vectors $B = (v_1, \dots, v_n) \in V^n$ is called a *basis* of V if:

- (i) B is linearly independent in V ;
- (ii) B is a system of generators for V , that is, $\langle B \rangle = V$.

Theorem 2.6.2 *Every vector space has a basis.*

Proof. Let V be a vector space over K . If $V = \{0\}$, then it has the basis \emptyset .

Now let $V = \langle B \rangle \neq \{0\}$, where $B = (v_1, \dots, v_n)$. If B is linearly independent, then B is a basis and we are done. Suppose that the list B is linearly dependent. Then by Theorem 2.5.3, $\exists j_1 \in \{1, \dots, n\}$ such that

$$v_{j_1} = \sum_{\substack{i=1 \\ i \neq j_1}}^n k_i v_i$$

for some $k_i \in K$. It follows that $V = \langle B \setminus \{v_{j_1}\} \rangle$, because every vector of V can be written as a linear combination of the vectors of $B \setminus \{v_{j_1}\}$. If $B \setminus \{v_{j_1}\}$ is linearly independent, it is a basis and we are done. Otherwise, $\exists j_2 \in \{1, \dots, n\} \setminus \{j_1\}$ such that

$$v_{j_2} = \sum_{\substack{i=1 \\ i \neq j_1, j_2}}^n k'_i v_i$$

for some $k'_i \in K$. It follows that $V = \langle B \setminus \{v_{j_1}, v_{j_2}\} \rangle$, because every vector of V can be written as a linear combination of the vectors of $B \setminus \{v_{j_1}, v_{j_2}\}$. If $B \setminus \{v_{j_1}, v_{j_2}\}$ is linearly independent, then it is a basis and we are done. Otherwise, we continue the procedure. If all the previous intermediate subsets are linearly dependent, we get to the step

$$V = \langle B \setminus \{v_{j_1}, \dots, v_{j_{n-1}}\} \rangle = \langle v_{j_n} \rangle.$$

If v_{j_n} were linearly dependent, then $v_{j_n} = 0$, hence $V = \langle v_{j_n} \rangle = \{0\}$, contradiction. Hence v_{j_n} is linearly independent and thus forms a single element basis of V . \square

Remark 2.6.3 We are going to see that a vector space may have more than one basis.

Let us give now a characterization theorem for a basis of a vector space.

Theorem 2.6.4 *Let V be a vector space over K . A list $B = (v_1, \dots, v_n)$ of vectors in V is a basis of V if and only if every vector $v \in V$ can be uniquely written as a linear combination of the vectors v_1, \dots, v_n , that is,*

$$v = k_1 v_1 + \cdots + k_n v_n$$

for some unique $k_1, \dots, k_n \in K$.

Proof. \Rightarrow Assume that B is a basis of V . Hence B is linearly independent and $\langle B \rangle = V$. The second condition assures us that every vector $v \in V$ can be written as a linear combination of the vectors of B . Suppose now that $v = k_1 v_1 + \cdots + k_n v_n$ and $v = k'_1 v_1 + \cdots + k'_n v_n$ for some $k_1, \dots, k_n, k'_1, \dots, k'_n \in K$. It follows that

$$(k_1 - k'_1)v_1 + \cdots + (k_n - k'_n)v_n = 0.$$

By the linear independence of B , we must have $k_i = k'_i$ for each $i \in \{1, \dots, n\}$. Thus, we have proved the uniqueness of writing.

\Leftarrow Assume that every vector $v \in V$ can be uniquely written as a linear combination of the vectors of B . Then clearly, $V = \langle B \rangle$. For $k_1, \dots, k_n \in K$, we have by the uniqueness of writing

$$\begin{aligned} k_1 v_1 + \cdots + k_n v_n = 0 &\Rightarrow k_1 v_1 + \cdots + k_n v_n = 0 \cdot v_1 + \cdots + 0 \cdot v_n \Rightarrow \\ &\Rightarrow k_1 = \cdots = k_n = 0, \end{aligned}$$

hence B is linearly independent. Consequently, B is a basis of V . \square

Definition 2.6.5 Let V be a vector space over K , $B = (v_1, \dots, v_n)$ a basis of V and $v \in V$. Then the scalars $k_1, \dots, k_n \in K$ appearing in the unique writing of v as a linear combination

$$v = k_1 v_1 + \cdots + k_n v_n$$

of the vectors of B are called the *coordinates of v in the basis B* .

Example 2.6.6 (a) If K is a field and $n \in \mathbb{N}^*$, then the list $E = (e_1, \dots, e_n)$ of vectors of K^n , where

$$\begin{cases} e_1 = (1, 0, 0, \dots, 0) \\ e_2 = (0, 1, 0, \dots, 0) \\ \dots\dots\dots \\ e_n = (0, 0, 0, \dots, 1) \end{cases}$$

is a basis of the canonical vector space K^n over K , called the *canonical basis* (or *standard basis*). Indeed, each vector $v = (x_1, \dots, x_n) \in K^n$ has a unique writing $v = x_1 e_1 + \cdots + x_n e_n$ as a linear combination of the vectors of E , hence E is a basis of V by Theorem 2.6.4.

Notice that the coordinates of a vector in the canonical basis are just the components of that vector, fact that is not true in general.

In particular, the canonical vector space \mathbb{Z}_2^n over \mathbb{Z}_2 has the above canonical basis $E = (e_1, \dots, e_n)$, where 0 and 1 are just the elements $\widehat{0}$ and $\widehat{1}$ of \mathbb{Z}_2 .

Also, if $n = 1$, the set $\{1\}$ is a basis of the canonical vector space K over K . For instance, $\{1\}$ is a basis of the vector space \mathbb{C} over \mathbb{C} .

(b) Consider the canonical real vector space \mathbb{R}^2 . We already know a basis of \mathbb{R}^2 , namely the canonical basis $((1, 0), (0, 1))$. But it is easy to show that the list $((1, 1), (0, 1))$ is also a basis of \mathbb{R}^2 . Therefore, a vector space may have more than one basis.

Also, note that $\{e_1\}$ is linearly independent, but not a system of generators, while the list $(e_1, e_2, e_1 + e_2)$ is a system of generators, but not linearly independent. Hence none of the two lists is a basis of the canonical real vector space \mathbb{R}^2 .

(c) Let V_3 be the real vector space of all vectors (in the classical sense) in the space with a fixed origin O . Then a basis of V_3 consists of the three pairwise orthogonal *unit vectors* $\vec{i}, \vec{j}, \vec{k}$.

(d) Let K be a field and $n \in \mathbb{N}$. Then the list

$$E = (1, X, X^2, \dots, X^n)$$

is a basis of the vector space $K_n[X] = \{f \in K[X] \mid \deg(f) \leq n\}$ over K , because every vector (polynomial) $f \in K_n[X]$ can be uniquely written as a linear combination $a_0 \cdot 1 + a_1 \cdot X + \cdots + a_n \cdot X^n$ ($a_0, \dots, a_n \in K$) of the vectors of E (see Theorem 2.6.4).

In this case, the coordinates of a vector $f \in K_n[X]$ in the basis B are just its coefficients as a polynomial.

(e) Consider the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid \deg(f) \leq 2\}$. We have seen that the list $E = (1, X, X^2)$ is a basis of $\mathbb{R}_2[X]$. Let us show that the list

$$B = (1, X - 1, (X - 1)^2)$$

is also a basis of $\mathbb{R}_2[X]$. Let $g = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]$. We look for unique $k_1, k_2, k_3 \in \mathbb{R}$ such that

$$g = k_1 \cdot 1 + k_2 \cdot (X - 1) + k_3 \cdot (X - 1)^2.$$

The equality is equivalent to the linear system of equations

$$\begin{cases} k_1 - k_2 + k_3 &= a_0 \\ k_2 - 2k_3 &= a_1 \\ k_3 &= a_2 \end{cases}$$

which has the unique solution $k_1 = a_0 + a_1 + a_2$, $k_2 = a_1 + 2a_2$, $k_3 = a_2$. Hence B is a basis of $\mathbb{R}_2[X]$, and the coordinates of a vector $g = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]$ in the basis B are $a_0 + a_1 + a_2$, $a_1 + 2a_2$, a_2 .

(f) Let K be a field. The list

$$E = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

is a basis of the vector space $M_2(K)$ over K .

More generally, let $m, n \in \mathbb{N}$, $m, n \geq 2$ and consider the matrices $E_{ij} = (a_{kl})$, where

$$a_{kl} = \begin{cases} 1 & \text{if } k = i \text{ and } l = j \\ 0 & \text{otherwise} \end{cases}.$$

Then the list consisting of all matrices E_{ij} is a basis of the vector space $M_{m,n}(K)$ over K .

In this case, the coordinates of a vector $A \in M_{m,n}(K)$ in the above basis are just the entries of that matrix.

(g) Consider the real vector space $M_2(\mathbb{R})$. We have seen that

$$E = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

is a basis of $M_2(\mathbb{R})$. Let us show that the list

$$B = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

is also a basis of $M_2(\mathbb{R})$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$. We look for unique $k_1, k_2, k_3, k_4 \in \mathbb{R}$ such that

$$A = k_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_3 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

The equality is equivalent to the linear system of equations

$$\begin{cases} k_1 + k_2 + k_3 &= a \\ k_4 &= b \\ k_3 &= c \\ k_2 + k_4 &= d \end{cases}$$

which has the unique solution

$$\begin{cases} k_1 &= a - d + b - c \\ k_2 &= d - b \\ k_3 &= c \\ k_4 &= b \end{cases}.$$

Hence B is a basis of $M_2(\mathbb{R})$, and the coordinates of a vector $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$ in the basis B are $a - d + b - c, d - b, c, b$.

(h) Since $\forall z \in \mathbb{C}, \exists! x, y \in \mathbb{R}$ such that $z = x \cdot 1 + y \cdot i$, the list $B = (1, i)$ is a basis of the vector space \mathbb{C} over \mathbb{R} .

The coordinates of a vector $z \in \mathbb{C}$ in the basis B are just its real and its imaginary part.

Theorem 2.6.7 Let $f : V \rightarrow V'$ be a K -linear map and let $B = (v_1, \dots, v_n)$ be a basis of V . Then f is determined by its values on the vectors of the basis B .

Proof. Let $v \in V$. Since B is a basis of V , $\exists! k_1, \dots, k_n \in K$ such that $v = k_1 v_1 + \dots + k_n v_n$. Then

$$f(v) = f(k_1 v_1 + \dots + k_n v_n) = k_1 f(v_1) + \dots + k_n f(v_n),$$

that is, f is determined by $f(v_1), \dots, f(v_n)$. \square

Corollary 2.6.8 Let $f, g : V \rightarrow V'$ be K -linear maps and let $B = (v_1, \dots, v_n)$ be a basis of V . If $f(v_i) = g(v_i), \forall i \in \{1, \dots, n\}$, then $f = g$.

Proof. Let $v \in V$. Then $v = k_1 v_1 + \dots + k_n v_n$ for some $k_1, \dots, k_n \in K$, hence

$$f(v) = f(k_1 v_1 + \dots + k_n v_n) = k_1 f(v_1) + \dots + k_n f(v_n) = k_1 g(v_1) + \dots + k_n g(v_n) = g(v).$$

Therefore, $f = g$. \square

Theorem 2.6.9 Let $f : V \rightarrow V'$ be a K -linear map, and let $X = (v_1, \dots, v_n)$ be a list of vectors in V .

- (i) If f is injective and X is linearly independent in V , then $f(X)$ is linearly independent in V' .
- (ii) If f is surjective and X is a system of generators for V , then $f(X)$ is a system of generators for V' .
- (iii) If f is bijective and X is a basis of V , then $f(X)$ is a basis of V' .

Proof. We have $f(X) = (f(v_1), \dots, f(v_n))$.

(i) Let $k_1, \dots, k_n \in K$ be such that

$$k_1 f(v_1) + \dots + k_n f(v_n) = 0'.$$

Since f is a K -linear map, it follows that

$$f(k_1 v_1 + \dots + k_n v_n) = f(0),$$

whence by the injectivity of f we get

$$k_1 v_1 + \dots + k_n v_n = 0.$$

But since X is linearly independent in V , we have $k_1 = \dots = k_n = 0$. Hence $f(X)$ is linearly independent in V' .

(ii) Since X is a system of generators for V , we have $\langle X \rangle = V$. By the surjectivity of f we have:

$$\langle f(X) \rangle = f(\langle X \rangle) = f(V) = V',$$

that is, $f(X)$ is a system of generators for V' .

(iii) This follows by (i) and (ii). \square

EXTRA: LOSSY COMPRESSION

Following [Klein], we present a way of achieving lossy compression of images.

Definition 2.6.10 Let $k, n \in \mathbb{N}^*$ be such that $k < n$, and let u be a vector of the canonical vector space K^n over K . Then the *closest k -sparse* vector associated to u is defined as the vector obtained from u by replacing all but its k largest magnitude components by zero.

Example 2.6.11 Consider an image consisting of a single row of four pixels with intensities 200, 50, 200 and 75 respectively. We know that such an image can be viewed as a vector $u = (200, 50, 200, 75)$ in the real canonical vector space \mathbb{R}^4 . The closest 2-sparse vector associated to u is the vector $\tilde{u} = (200, 0, 200, 0)$.

Suppose that we need to store a grayscale image of (say) $n = 2000 \times 1000$ pixels more compactly. We can view it as a vector v in the real canonical vector space \mathbb{R}^n . If we just store its associated closest k -sparse vector, then the compressed image may be far from the original.

One may use the following *lossy compression algorithm*:

Step 1. Consider a suitable basis $B = (v_1, \dots, v_n)$ of the real canonical vector space \mathbb{R}^n .

Step 2. Determine the n -tuple u (which is desired to have as many zeros as possible) of the coordinates of v in the basis B .

Step 3. Replace u by the closest k -sparse n -tuple \tilde{u} for a suitable k , and store \tilde{u} .

Step 4. In order to recover an image from \tilde{u} , compute the corresponding linear combination of the vectors of B with scalars the components of \tilde{u} .

Consider the following image:



First, use the closest sparse vector which suppresses all but 10% of the components of v , and secondly, use the lossy compression algorithm which suppresses all but 10% of the components of u in order to get the following images respectively:

