Seminar 5

1. Find the accumulation points for each of the following sets: $[0,1) \cup \{2\}$, $\mathbb{Z}$, $\mathbb{Q}$.
2. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous everywhere with $|f|$ continuous everywhere.
3. If $f : [a,b] \to [a,b]$ is continuous, then it has at least one fixed point x^* with $f(x^*) = x^*$.
4. Study the continuity and the differentiability for f and f', where $f : \mathbb{R} \to \mathbb{R}$,
$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$
5. ★ Prove (from scratch) that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$ and then that $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$.
6. Compute the following limits:
   (a) $\displaystyle\lim_{x\to\infty} \frac{\lfloor x \rfloor}{x}$.
   (b) $\displaystyle\lim_{x\to\infty} x\big(\ln(x+2) - \ln(x+1)\big)$.
   (c) $\displaystyle\lim_{x\to 0} \big(\cos x\big)^{\frac{1}{x^2}}$.
   (d) $\displaystyle\lim_{\substack{x\to 0\\ x>0}} x^x$.
   (e) $\displaystyle\lim_{\substack{x\to 0\\ x>0}} (\sin x)^x$.
   (f) $\displaystyle\lim_{x\to\infty} x\big((1+\tfrac{1}{x})^x - e\big)$.
7. Find the n^{th} derivative of the following functions:
   (a) $f : (-1,\infty) \to \mathbb{R},\ f(x) = \ln(1+x)$.
   (b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$.
   (c) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 \sin x$.
   (d) $f : \mathbb{R} \to \mathbb{R},\ f(x) = e^{2x} x^3$.
8. ★ [Python] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, $x_1 \in \mathbb{R}$ and consider the *gradient descent*
$$x_{n+1} = x_n - \eta f'(x_n),$$
to minimize f with the so-called learning rate $\eta > 0$. (a) Take f to be a convex function and show numerically (plots allowed) that for small η the iteration converges to the global minimum of f. (b) Show that by increasing η the algorithm can converge faster. (c) However, taking η too large might lead to divergence. (d) Take f to be nonconvex and show that the algorithm can get stuck in a local minimum.
9. $*$ Construct a function continuous everywhere but differentiable nowhere. *Hint: fractal.*

Homework questions are marked with ★. Bonus questions are marked with $*$.
Solutions should be handed in at the beginning of next week's lecture.