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913

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Homework - Seminar 8

2) I will prove that these statements are equivalent by proving that a) ⇒ b), b) ⇒ c) and c) ⇒ a).

I) a) ⇒ b)

From a) $\Rightarrow \langle x,y\rangle = 0 \Rightarrow \langle y,x\rangle = 0$

$\langle x,y\rangle + \langle y,x\rangle = 0 \ /+\langle x,x\rangle \Leftrightarrow \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle = \langle x,x\rangle \ /+\langle y,y\rangle \Leftrightarrow$

$\Leftrightarrow \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle = \langle x,x\rangle + \langle y,y\rangle \Leftrightarrow$

$\Leftrightarrow \langle x, x+y\rangle + \langle y, x+y\rangle = \langle x,x\rangle + \langle y,y\rangle \Leftrightarrow \langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,y\rangle \Leftrightarrow$

$\Leftrightarrow \|x+y\|^2 = \langle x,x\rangle + \langle y,y\rangle$ ①

~~$\langle x,y\rangle = \langle y,x\rangle = 0 \ /\cdot(-1) \Leftrightarrow -\langle x,y\rangle + \langle y,x\rangle = 0 \ /+\langle x,x\rangle \Leftrightarrow$~~

~~$\Leftrightarrow \langle x,x\rangle - \langle x,y\rangle + \langle y,x\rangle = \langle x,x\rangle \Leftrightarrow \langle x, x-y\rangle \ /+\langle y,y\rangle \Leftrightarrow$~~

~~$\Leftrightarrow \langle x,x\rangle - \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle = \langle x,x\rangle$~~

$-\langle x,y\rangle - \langle y,x\rangle = 0 \ /+\langle x,x\rangle \Leftrightarrow \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle = \langle x,x\rangle \ /+\langle y,y\rangle \Leftrightarrow$

$\Leftrightarrow \langle x,x\rangle - \langle x,y\rangle + \langle y,y\rangle - \langle y,x\rangle = \langle x,x\rangle + \langle y,y\rangle \Leftrightarrow$

$\Leftrightarrow \langle x, x-y\rangle - (\langle y,x\rangle - \langle y,y\rangle) = \langle x,x\rangle + \langle y,y\rangle \Leftrightarrow$

$\Leftrightarrow \langle x, x-y\rangle - \langle y, x-y\rangle = \langle x,x\rangle + \langle y,y\rangle \Leftrightarrow \langle x-y, x-y\rangle = \langle x,x\rangle + \langle y,y\rangle \Leftrightarrow$

$\Leftrightarrow \|x-y\|^2 = \langle x,x\rangle + \langle y,y\rangle$ ②

①, ② $\Rightarrow \|x+y\|^2 = \|x-y\|^2 \Leftrightarrow |\underbrace{\|x+y\|}_{\geq 0}| = |\underbrace{\|x-y\|}_{\geq 0}| \Leftrightarrow \|x+y\| = \|x-y\|$

Therefore, a) ⇒ b).

II) b) ⇒ c)

$\|x+y\| = \|x-y\| \Rightarrow \|x+y\|^2 = \|x-y\|^2 \Rightarrow \langle x+y, x+y\rangle = \langle x-y, x-y\rangle \Leftrightarrow$

$\Leftrightarrow \cancel{\langle x,x\rangle} + \langle x,y\rangle + \cancel{\langle y,y\rangle} + \langle y,x\rangle = \cancel{\langle x,x\rangle} - \langle x,y\rangle - \langle y,x\rangle + \cancel{\langle y,y\rangle}$

$\Leftrightarrow \langle x,y\rangle + \langle x,y\rangle + \langle x,y\rangle + \langle x,y\rangle = 0 \Rightarrow \langle x,y\rangle = 0$

$\|x+y\|^2 = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle = \langle x,x\rangle + \langle y,y\rangle = \|x\|^2 + \|y\|^2 \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$

Hence, b) $\Rightarrow$ c)

III) c) $\Rightarrow$ a)

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \Leftrightarrow \langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,y\rangle \Leftrightarrow$$

$$\Leftrightarrow \langle x, x+y\rangle + \langle y, x+y\rangle = \langle x,x\rangle + \langle y,y\rangle \Leftrightarrow$$

$$\Leftrightarrow \cancel{\langle x,x\rangle} + \langle x,y\rangle + \langle y,x\rangle + \cancel{\langle y,y\rangle} = \cancel{\langle x,x\rangle} + \cancel{\langle y,y\rangle}$$

$$\Leftrightarrow \langle x,y\rangle + \langle y,x\rangle = 0 \Leftrightarrow \langle x,y\rangle + \langle x,y\rangle = 0 \Leftrightarrow 2\langle x,y\rangle = 0 \Leftrightarrow \langle x,y\rangle = 0$$

Hence, c) $\Rightarrow$ a)

5) Let $v = \cancel{[x \quad y] \in \mathbb{R}^2} \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$

$$v' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix} \in \mathbb{R}^2$$

$v \cdot v' = \|v\| \cdot \|v'\| \cdot \cos\varphi$, where $\varphi = \widehat{v, v'}$

$$v \cdot v' = x \cdot (x\cos\theta + y\sin\theta) + y(-x\sin\theta + y\cos\theta) =$$

$$= x^2\cos\theta + \cancel{xy\sin\theta} - \cancel{xy\sin\theta} + y^2\cos\theta =$$

$$= (x^2+y^2)\cos\theta \quad ①$$

$$\|v\| = \sqrt{v \cdot v} = \sqrt{x^2+y^2}$$

$$\|v'\| = \sqrt{v' \cdot v'} = \sqrt{(x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2} =$$

$$= \sqrt{x^2\cos^2\theta + \cancel{2xy\sin\theta\cos\theta} + y^2\sin^2\theta + x^2\sin^2\theta - \cancel{2xy\sin\theta\cos\theta} + y^2\cos^2\theta} =$$

$$= \sqrt{x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta)}$$

$$= \sqrt{x^2+y^2}$$

$$v \cdot v' = \sqrt{x^2+y^2} \cdot \sqrt{x^2+y^2} \cdot \cos\varphi = (x^2+y^2) \cdot \cos\varphi \quad ②$$

$$①, ② \Rightarrow \left.\begin{matrix} \cos\theta = \cos\varphi \\ \theta, \varphi \in [0, \pi] \end{matrix}\right\} \Rightarrow \theta = \varphi$$

Hence, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix with angle θ in $\mathbb{R}^2$.

5) Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\det(A) = \cos\theta \cdot \cos\theta - \sin\theta \cdot (-\sin\theta) = \cos^2\theta + \sin^2\theta = 1 \neq 0 \Rightarrow \exists A^{-1}$ s.t. $A \cdot A^{-1} = A^{-1} \cdot A = I_2$

$A^{-1} = \frac{1}{\det A} \cdot A^* = A^*$

$A^* = \begin{bmatrix} \delta_{11} & \delta_{21} \\ \delta_{12} & \delta_{22} \end{bmatrix}$

$\delta_{11} = (-1)^{11} d_{11} = |\cos\theta| = \cos\theta$

$\delta_{12} = (-1)^{1+2} \cdot d_{12} = -|+\sin\theta| = -\sin\theta$

$\delta_{21} = (-1)^{2+1} \cdot d_{21} = -|-\sin\theta| = \sin\theta$

$\delta_{22} = (-1)^{2+2} \cdot d_{22} = |\cos\theta| = \cos\theta$

Hence, $A^* = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$