CHAPTER 4

Lines and planes in dimension 3

Contents

Unless otherwise stated, any coordinate system that we fix will be a right oriented orthonormal coordinate system (reference frame). In this document $Oxyz = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ is a coordinate system of $\mathbb{E}^3$. Recall that $\mathbb{V}^3$ is the vector space of vectors which can be represented with points in $\mathbb{E}^3$. Recall also that we have a bijective map $\phi_O : \mathbb{E}^3 \to \mathbb{V}^3$ given by $\phi_O(P) = \overrightarrow{OP}$.

4.1 Planes in $\mathbb{E}^3$

4.1.1 Describing surfaces in $\mathbb{E}^3$

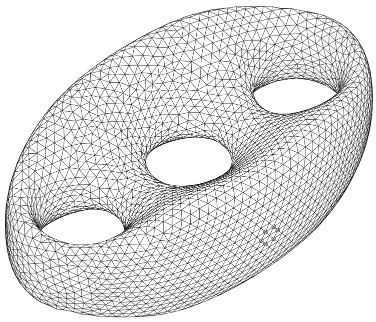
There are essentially three ways in which we can describe a surface in $\mathbb{E}^3$:

1. Via parametrizations, where, for two intervals $I_1, I_2 \subseteq \mathbb{R}$, you use a map $\varphi : I_1 \times I_2 \to \mathbb{E}^3$ to specify for each pair $(s,t) \in I_1 \times I_2$ a point in $\mathbb{E}^3$. For example, the sphere of radius 1 centered at the origin is given by $\phi : [0, 2\pi) \times [0, \pi] \to \mathbb{E}^3$ with $\phi(s,t) = (\cos(s)\sin(t), \sin(s)\sin(t), \cos(t))$.

2. Via global equations, where you describe the surface as the set of all points whose coordinates satisfy a given equation. For example, the sphere of radius 1 centered at the origin is the set $\{P(x,y,z) \in \mathbb{E}^3 : x^2 + y^2 + z^2 = 1\}$.

3. Via geometric properties, where you describe the curve as the set of all points satisfying a geometric property. For example, the sphere of radius 1 centered at the point O is the set $\{P \in \mathbb{E}^3 : P \text{ is at distance 1 from } O\}$.

Computationally you will most often use an approximation of the surface that you are interested in. The simplest form of approximation is via triangles.



4.1.2 Geometric description

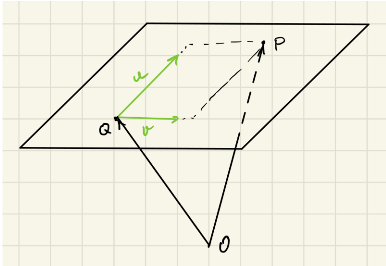
Recall from Lecture 1 that we can describe planes as sets of points S in $\mathbb{E}^3$ such that the set of vectors which can be represented by points in S form a 2-dimensional vector subspace of $\mathbb{V}^3$, i.e. for any $A \in S$

$$\phi_A(S) = \{\overrightarrow{AB} : B \in S\} \quad \text{is a 2-dimensional vector subspace of } \mathbb{V}^3.$$

4.1.3 Parametric equations - local description

Since $\phi_A(S)$ is 2-dimensional, any basis will contain two vectors. Let $\mathbf{v}$, $\mathbf{w}$ be a basis of the vector space $\phi_A(S)$. Thus, for any two points $A, B \in S$, the vector $\overrightarrow{AB}$ is a linear combination of the basis vectors, i.e. there exist unique scalars $s, t \in \mathbb{R}$ such that

$$\overrightarrow{AB} = s\mathbf{v} + t\mathbf{w}.$$



Now, if you fix A and let B vary in the plane S then s and t vary in $\mathbb{R}$. Since ϕ_A is a bijection, the plane S can be described as

$$S = \left\{ B \in \mathbb{E}^3 : \overrightarrow{AB} = s\mathbf{v} + t\mathbf{w} \text{ for some } s, t \in \mathbb{R} \right\}.$$

In this description the point $A \in S$ is arbitrary but fixed. We sometimes refer to it as the *base point*. Now, the coordinates of a point B are the components of the vector $\overrightarrow{OB}$ and, rearranging the above equation we have

$$\overrightarrow{OB} = \overrightarrow{OA} + s\mathbf{v} + t\mathbf{w}. \tag{4.1}$$

So, we can describe the plane S as the set of points B in $\mathbb{E}^3$ which satisfy Equation (4.1) for some $s, t \in \mathbb{R}$. This is called the *vector equation* of the plane S.

- The vector equation depends on the choice of the base point A.
- The vector equation depends on the choice of the vector $\mathbf{v}$ and $\mathbf{w}$. In analogy with the case of the line in $\mathbb{E}^2$ we may call such vectors direction vectors for the plane S.
- Hence a plane does not have a unique vector equation.
- The vector equation does not depend on the coordinate system. In the above description O can be any point in $\mathbb{E}^3$.

If we write Equation (4.1) in coordinates (relative to the coordinate system $Oxyz$) then we obtain

$$\begin{cases} x = x_A + sv_x + tw_x \\ y = y_A + sv_y + tw_y \\ z = z_A + sv_z + tw_z \end{cases} \quad \text{or, in matrix form} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} + s \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + t \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} \tag{4.2}$$

where $A = A(x_A, y_A, z_A)$, $\mathbf{v} = \mathbf{v}(v_x, v_y, v_z)$, $\mathbf{w} = \mathbf{v}(w_x, w_y, w_z)$ and for different values s and t we obtain different points (x, y, z) in the plane S. The three equations in the system (4.2) are called *parametric equations* for the plane S.

4.1.4 Cartesian equations - global description

As in the case of the line in $\mathbb{E}^2$, it is possible to eliminate the parameters s, t in (4.2) to obtain

$$\left(\frac{v_x}{w_x} - \frac{v_z}{w_z}\right)\left(\frac{x - x_A}{w_x} - \frac{y - y_A}{w_y}\right) = \left(\frac{v_x}{w_x} - \frac{v_y}{w_y}\right)\left(\frac{x - x_A}{w_x} - \frac{z - z_A}{w_z}\right). \tag{4.3}$$

We will not give this equation a name, because it is a bit much to keep in mind, and one has to make sense of what happens when the denominators are zero. The whole point here, is that *it is a linear equation* in x, y and z and that it can be obtained by eliminating the parameters in (4.2).

There is an easier way of describing S with a linear equation. For this you can interpret (4.2) as saying that the vector $\overrightarrow{AB}$ is linearly dependent on the vectors $\mathbf{v}$ and $\mathbf{w}$. With this in mind, the point $B(x,y,z)$ lies in the plane S if and only if

$$\begin{vmatrix} x-x_A & y-y_A & z-z_A \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0 \quad \Leftrightarrow \quad [\overrightarrow{AB},\mathbf{v},\mathbf{w}] = 0 \tag{4.4}$$

Recall that this was the coplanarity condition which we deduced when we discussed the box product. It says that the volume of the parallelepiped spanned by the vectors $\overrightarrow{AB},\mathbf{v},\mathbf{w}$ is zero. We may refer to this expression as the *zero-volume equation* of the plane S.

Proposition 4.1. Every plane in $\mathbb{E}^3$ can be described with a linear equation in three variables

$$ax+by+cz+d=0 \tag{4.5}$$

relative to a fixed coordinate system and any linear equation in two variables describes a plane relative to a fixed coordinate system if the constants a,b,c are not all zero.

- Equation (4.5) is called the *Cartesian equation* of the plane it describes.
- Notice that there are infinitely many Cartesian equations describing the same plane, since you can multiply one equation by a non-zero constant.
- In a fixed coordinate system equations of a plane are the same up to multiplication by a non-zero scalar.

If a plane is given with the Cartesian equation (4.5), you can rearrange it in the form

$$\frac{x}{A}+\frac{y}{B}+\frac{z}{C}=1 \quad \text{where} \quad A=-\frac{d}{a}, B=-\frac{d}{b}. \quad \text{and} \quad C=-\frac{d}{c}.$$

In this form we have the *equation of the plane where we can read off the intersection points with the coordinate axes* since the line intersects Ox in $(A,0,0)$, it intersects Oy in $(0,B,0)$ and it intersects Oz in $(0,0,C)$.

4.1.5 Normal vectors

Normal vectors play an important role both in abstract geometry and in applications. Here we consider normal vectors of planes in $\mathbb{E}^3$.

The idea is again simple: we discussed in Lecture 2 that if we fix a vector $\mathbf{n}\in\mathbb{V}^3$ then $\mathbf{n}^\perp$ is a vector subspace of $\mathbb{V}^3$ and $\mathbb{V}^3=\langle\mathbf{n}\rangle\oplus\mathbf{n}^\perp$. The vector subspace $\langle\mathbf{n}\rangle$ is 1-dimensional (it is spanned by one vector) and $\dim(\mathbb{V}^3) = 3$. Therefore $\mathbf{n}^\perp$ must have dimension 2. So, returning to the description of a plane as a set of points of the form

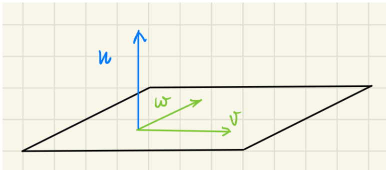
$$S = \left\{B \in \mathbb{E}^3 : \overrightarrow{AB} = s\mathbf{v} + t\mathbf{w} \text{ for some } s, t \in \mathbb{R}\right\},$$

we may choose $\mathbf{n}$ perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. In other words, we may choose $\mathbf{n}$ such that $\mathbf{v}, \mathbf{w}$ are a basis of $\mathbf{n}^\perp$. Then, we have

$$S = \left\{B \in \mathbb{E}^3 : \mathbf{n} \perp \overrightarrow{AB}\right\} = \left\{B \in \mathbb{E}^3 : \mathbf{n} \cdot \overrightarrow{AB} = 0\right\},$$

where A is some point in S. In other words, the plane S can be described as the set of point $B \in \mathbb{E}^3$ such that

$$\mathbf{n} \cdot \overrightarrow{AB} = 0 \quad \text{or, equivalently} \quad \mathbf{n} \cdot (\overrightarrow{OB} - \overrightarrow{OA}) = 0.$$



The vector $\mathbf{n}$ is perpendicular to the plane S and any non-zero vector with this property is called a *normal vector* for the plane S. If we look at this equation in coordinates, then $\overrightarrow{OA} = (x_A, y_A, z_A) = A \in S$ and with $\mathbf{n} = \mathbf{n}(n_x, n_y, n_z)$ we have

$$n_x(x - x_A) + n_y(y - y_A) + n_z(z - z_A) = 0 \tag{4.6}$$

where (x, y, z) is an arbitrary point in the plane S. This is a linear equation for S and since all other equations of S (in the coordinate system Oxy) are obtained from (4.6) by multiplying with a non-zero constant, we see that the coefficients of x, y and z in a Cartesian equation of S can be interpreted as the components of a normal vector.

If a plane is given via parametric equations, with $\mathbf{v}$ and $\mathbf{w}$ as direction vectors. How can we obtain a normal vector? You can transform the parametric equations in a Cartesian equation and read off the coefficients of x, y, z or you recall that $\mathbf{v} \times \mathbf{w}$ is a vector which is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, thus, it is a normal vector for the plane.

4.1.6 Relative positions of two planes

- [Intersections] Assume that we have two planes

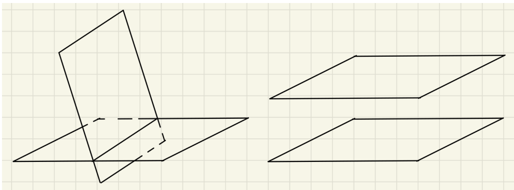
  $$\pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \quad \text{and} \quad \pi_2 : a_2x + b_2y + c_2z + d_2 = 0.$$

  In order to determine if they intersect or not one has to discuss the system:

  $$\begin{cases} \pi_1 : a_1x + b_1y + c_1z + d_1 & = & 0 \\ \pi_2 : a_2x + b_2y + c_2z + d_2 & = & 0 \end{cases} \tag{4.7}$$

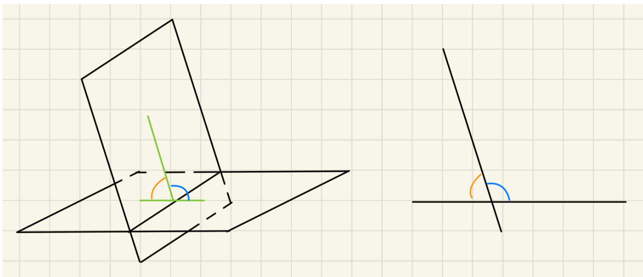
  Recall Lecture 12 of your Algebra course last semester. Let A be the matrix of the system and $\tilde{A}$ the extended matrix of the system.

– two planes either intersect in a line, the coordinates of the points on the line will be solutions to (4.7), this happens if the rank of A and the rank of $\tilde{A}$ equals 2; or
– they don't intersect and (4.7) doesn't have solutions, in which case the planes are parallel, this happens if the rank of A is strictly less than the rank of $\tilde{A}$; or
– the solution to system (4.7) depends on two parameters in which case $\pi_1 = \pi_2$, this happens if the rank of A and the rank of $\tilde{A}$ are equal to 1.



Notice that (a_1, b_1, c_1) is a normal vector for π_1 and (a_2, b_2, c_2) is a normal vector for π_2. These are the rows of the matrix A, so the last two cases above occur if these two vectors are proportional, since the rank of A equals 1.

- [Angles]



For planes we have the notion of *dihedral angle*. Two planes π_1 and π_2 define two dihedral angles which are supplementary.

These two angles can also be described with normal vectors: if $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal vectors for π_1 and π_2 respectively, then the two angles between π_1 and π_2 are congruent to $\measuredangle(\mathbf{n}_1, \mathbf{n}_2)$ and $\measuredangle(-\mathbf{n}_1, \mathbf{n}_2)$. So, if these vectors are known we may calculate

$$\cos\measuredangle(\mathbf{n}_1, \mathbf{n}_2) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_2\|}. \tag{4.8}$$

Clearly, if the two planes are parallel, then the two angles are 0° and 180°.

- [Distances] We first consider the distance from a point to a plane

Proposition 4.2. Suppose you have a plane $\pi : ax + by + cz + d = 0$ and a point $P(x_P, y_P, z_P)$ in $\mathbb{E}^3$. The distance from P to π is

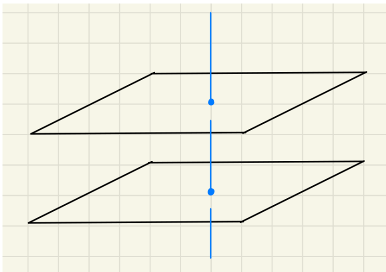
$$d(P, \pi) = \frac{|ax_P + by_P + cz_P + d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{4.9}$$

Now, considering the distances between two planes π_1 and π_2 we have the following cases:

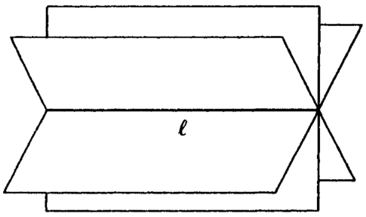
- if π_1 and π_2 intersect then the distance between them is zero $d(\pi_1,\pi_2) = 0$.
- if π_1 and π_2 do not intersect, i.e. if they are parallel, then

$$d(\pi_1,\pi_2) = d(P_1,\pi_2) = d(\pi_1,P_2)$$

  for any point $P_1 \in \pi_1$ and any point $P_2 \in \pi_2$.



4.1.7 Bundles of planes



Definition. Let $\ell \subseteq \mathbb{E}^3$ be a line. The set Π_ℓ of all planes in $\mathbb{E}^3$ containing ℓ is called a *bundle of planes* and ℓ is called the *axis* of the bundle Π_ℓ.

Proposition 4.3. If $\pi_1 : a_1x + b_1y + c_1z + d_1 = 0$ and $\pi_2 : a_2x + b_2y + c_2z + d_2 = 0$ are two distinct planes in the bundle Π_ℓ, then Π_ℓ consists of planes having equations of the form

$$\pi_{\lambda,\mu} : \lambda(a_1x + b_1y + c_1z + d_1) + \mu(a_2x + b_2y + c_2z + d_2) = 0.$$

where $\lambda, \mu \in \mathbb{R}$ not both zero.

- Bundles of planes are useful in practice when a line ℓ is given as the intersection of two planes and one wants to find the equation of a plane containing ℓ and satisfying some other conditions. For example, the condition that it contains some point P which does not belong to ℓ or that it is parallel to a given line.

- There is redundancy in the two parameters λ, μ, meaning that there are not two independent parameters here. If $\lambda \neq 0$ then one can divide the equation of $\pi_{\lambda,\mu}$ by λ to obtain

$$\pi_{1,t} : (a_1x + b_1y + c_1z + d_1) + t(a_2x + b_2y + c_2z + d_2) = 0.$$

where $\frac{\mu}{\lambda} = t \in \mathbb{R}$. So $\pi_{1,\frac{\mu}{\lambda}}$ and $\pi_{\lambda,\mu}$ are in fact the same planes.

- A *reduced bundle* is the set of all planes Π_ℓ with axis ℓ from which *we remove one plane*, i.e. it is $\Pi_\ell \setminus \{\pi_2\}$ for some $\pi_2 \in \Pi_\ell$. With the above notation and discussion, it is the set

$$\{\pi_{1,t} : (a_1x + b_1y + c_1z + d_1) + t(a_2x + b_2y + c_2z + d_2) = 0 : t \in \mathbb{R}\}.$$

The fact that we use one parameter instead of two, to describe almost all planes containing ℓ, greatly simplifies calculations.

Definition. Let $\mathbb{W} \subseteq \mathbb{V}^3$ be a vector subspace of dimension 2. The set $\Pi_{\mathbb{W}}$ of all planes in $\mathbb{A}$ having associated vector subspace $\mathbb{W}$ is called an *improper bundle of planes*, and $\mathbb{W}$ is called the vector subspace associated to the bundle $\Pi_{\mathbb{W}}$.

- The connection between bundles of planes and improper bundles of planes is best understood through projective geometry, where we can think about the improper bundle of planes as the set of all planes intersecting in a line at infinity.

4.2 Lines in $\mathbb{E}^3$

4.2.1 Describing curves in $\mathbb{E}^3$

There are essentially three ways in which we can describe a curve in $\mathbb{E}^3$:

1. Via parametrizations, where, for an interval $I \subseteq \mathbb{R}$, you use a map $\varphi : I \to \mathbb{E}^3$ to specify for each $t \in I$ a point in $\mathbb{E}^3$. For example, *a* circle of radius 1 centered at the origin is given by $\phi : [0, 2\pi) \to \mathbb{E}^3$ with $\phi(s,t) = (\cos(t), \sin(t), 0)$.

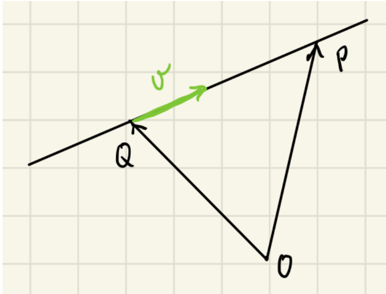
2. Via global equations, where you describe the curve as the set of all points whose coordinates satisfy *two* given equations. For example, the above circle can be described as the set $\{P(x,y,z) \in \mathbb{E}^3 : x^2 + y^2 = 1 \text{ and } z = 0\}$.

3. Via geometric properties, where you describe the curve as the set of all points satisfying some geometric properties. For example, the above circle can be described as the set of points $\{P \in \mathbb{E}^3 :\ P \text{ is at distance 1 from } O \text{ and } P \text{ lies in the plane } Oxy\}$.

4.2.2 Geometric description

Recall from Lecture 1 that we can describe lines in $\mathbb{E}^3$ as being sets of points S such that the set of vectors which can be represented by points in S form a 1-dimensional vector subspace of $\mathbb{V}^3$, i.e. for any $A \in S$

$$\phi_A(S) = \{\overrightarrow{AB} : B \in S\} \quad \text{is a 1-dimensional vector subspace of } \mathbb{V}^3.$$



4.2.3 Parametric equations - local description

If S is a line then for any two distinct points A, B in S the vector $\overrightarrow{AB}$ is called a *direction vector* of S. Since $\phi_A(S)$ is 1-dimensional, all direction vectors are linearly dependent and $\mathbf{v}$ is a direction vector for S if and only if it is linearly dependent on $\overrightarrow{AB}$. So, for any direction vector $\mathbf{v}$ of S there is a scalar $t \in \mathbb{R}$ such that

$$\overrightarrow{AB} = t\mathbf{v}.$$

Now, if you fix A and let B vary on the line then t varies in $\mathbb{R}$. Since ϕ_A is a bijection, the line S can be described as

$$S = \left\{ B \in \mathbb{E}^3 : \overrightarrow{AB} = t\mathbf{v} \text{ for some } t \in \mathbb{R} \right\}.$$

In this description the point $A \in S$ is arbitrary but fixed. We sometimes refer to it as the *base point*. Now, the coordinates of a point B are the components of the vector $\overrightarrow{OB}$ and, rearranging the above equation we have

$$\overrightarrow{OB} = \overrightarrow{OA} + t\mathbf{v}. \tag{4.10}$$

So, we can describe the line S as being the set of points B in $\mathbb{E}^3$ which satisfy Equation (4.10) for some $t \in \mathbb{R}$. This is called the *vector equation* of the line S.

- The vector equation depends on the choice of the base point A.
- The vector equation depends on the choice of the direction vector $\mathbf{v}$.
- Hence, a line does not have a unique vector equation.
- The vector equation does not depend on the coordinate system. In the above description O can be any point in $\mathbb{E}^3$.

If we write Equation (4.10) in coordinates (relative to the coordinate system $Oxyz$) then we obtain

$$\begin{cases} x = x_A + tv_x \\ y = y_A + tv_y \\ z = z_A + tv_z \end{cases} \quad \text{or, in matrix form} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{4.11}$$

where $A = A(x_A, y_A, z_A)$, $\mathbf{v} = \mathbf{v}(v_x, v_y, v_z)$ and for different values t we obtain different points (x, y, z) on the line. The three equations in the system (4.11) are called *parametric equations* for the line S.

4.2.4 Cartesian equations - global description

It is possible to eliminate the parameter t in (4.11) in order to obtain

$$\frac{x-x_A}{v_x}=\frac{y-y_A}{v_y}=\frac{z-z_A}{v_z}. \tag{4.12}$$

We refer to the Equations (4.12) as the *symmetric equations* of the line S. It could happen that v_x, v_y or v_z are zero. In that case, translate back to the parametric equations to understand what happens.

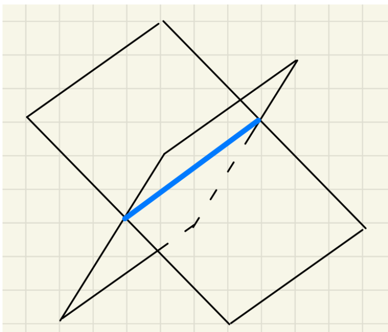
- For lines in $\mathbb{E}^3$ there is no notion of 'slope'.
- We have just described a line with *two* linear equations (relative to the coordinate system $Oxyz$).

Proposition 4.4. Every line in $\mathbb{E}^3$ can be described with two linear equations in three variables

$$\begin{cases} a_1x+b_1y+c_1z+d_1 &= 0\\ a_2x+b_2y+c_2z+d_2 &= 0\end{cases} \tag{4.13}$$

relative to a fixed coordinate system and any compatible system of two linear equations of rank 2 in three variable describes a line relative to a fixed coordinate system.

- The Equations (4.5) are called *Cartesian equations* of the line it describes.
- The Equations (4.5) describe a line as an intersection of two planes.



4.2.5 Relative positions of two lines

- [Intersections] Assume we have two lines

$$\ell_1:\begin{cases} a_1x+b_1y+c_1z+d_1 &= 0\\ a_2x+b_2y+c_2z+d_2 &= 0\end{cases} \quad\text{and}\quad \ell_2:\begin{cases} a_3x+b_3y+c_3z+d_3 &= 0\\ a_4x+b_4y+c_4z+d_4 &= 0\end{cases}.$$

One way to determine if they intersect is to discuss the system:

$$\begin{cases} a_1x+b_1y+c_1z+d_1 &= 0\\ a_2x+b_2y+c_2z+d_2 &= 0\\ a_3x+b_3y+c_3z+d_3 &= 0\\ a_4x+b_4y+c_4z+d_4 &= 0\end{cases}. \tag{4.14}$$

As you did in Lecture 12 of your Algebra course last semester. However, it is easier to discuss the relative positions of lines in $\mathbb{E}^3$ via their parametric equations:

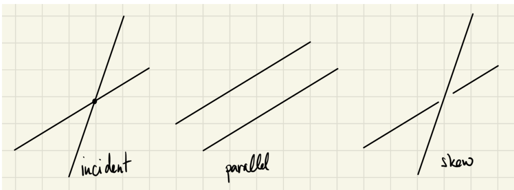
$$\ell_1 : \begin{cases} x = x_1 + tv_x \\ y = y_1 + tv_y \\ z = z_1 + tv_z \end{cases} \quad \text{und} \quad \ell_2 : \begin{cases} x = x_2 + tu_x \\ y = y_2 + tu_y \\ z = z_2 + tu_z \end{cases}$$

We have the following cases

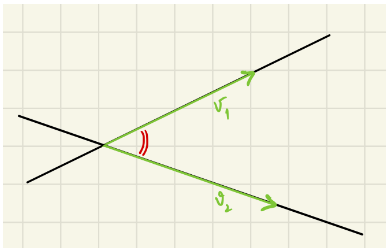
- If the direction vectors $\mathbf{v}$ and $\mathbf{u}$ are proportional then the two lines are parallel.
- If they are parallel and have a point in common then the two lines are equal.
- If they are not parallel then they are coplanar (they lie in the same plane) if

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0.$$

  in which case they intersect in exactly one point.
- If they are not parallel and they don't intersect then we say that the two lines ℓ_1 and ℓ_2 are *skew* relative to each other.



- [Angles]



Two lines define two angles: if $\mathbf{v}_1$ is a direction vector for ℓ_1 and if $\mathbf{v}_2$ is a direction vector for ℓ_2 then the two angles described by ℓ_1 and ℓ_2 are $\measuredangle(\mathbf{v}_1, \mathbf{v}_2)$ and $\measuredangle(-\mathbf{v}_1, \mathbf{v}_2)$. They are supplementary angles so if you can measure one of them you know the other one. Recall from Lecture 2 that

$$\cos \measuredangle(\mathbf{v}_1, \mathbf{v}_2) = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\|}. \tag{4.15}$$

Clearly, if the two lines are parallel, then the two angles are $0°$ and $180°$.

- [Distances] We first consider the distance from a point to a lines

  Proposition 4.5. Suppose you have a line

  $$\ell_1 : \begin{cases} x = x_A + tv_x \\ y = y_A + tv_y \\ z = z_A + tv_z \end{cases} \quad \text{and a point} \quad P(x_P, y_P, z_P) \in \mathbb{E}^3.$$

  The distance from P to ℓ is

  $$d(P,\ell) = \frac{\|\overrightarrow{PA} \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$

  Now, considering the distances between two lines ℓ_1 and ℓ_2. We have the following cases:

  – if ℓ_1 and ℓ_2 intersect then the distance between them is zero $d(\ell_1,\ell_2) = 0$.

  – if ℓ_1 and ℓ_2 do not intersect and they are parallel, then

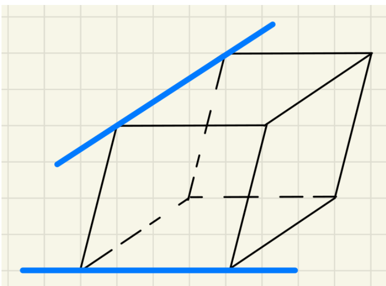
  $$d(\ell_1,\ell_2) = d(P_1,\ell_2) = d(\ell_1,P_2)$$

  for any point $P_1 \in \ell_1$ and any point $P_2 \in \ell_2$.

  – if ℓ_1 and ℓ_2 are skew relative to each other, there is a unique plane π_1 containing ℓ_1 which is parallel to ℓ_2 and there is a unique plane π_2 which contains ℓ_2 and is parallel to ℓ_1, thus

  $$d(\ell_1,\ell_2) = d(\pi_1,P_2) = d(P_1,\pi_2)$$

  for any point $P_1 \in \ell_1$ and any point $P_2 \in \ell_2$.

  More concretely, if the two lines are

  $$\ell_1 : \begin{cases} x = x_1 + tv_x \\ y = y_1 + tv_y \\ z = z_1 + tv_z \end{cases} \quad \text{and} \quad \ell_2 : \begin{cases} x = x_2 + tu_x \\ y = y_2 + tu_y \\ z = z_2 + tu_z \end{cases}$$

  then

  $$\pi_1 : \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0$$

Hence, by (4.9),

$$d(\ell_1,\ell_2) = d(\pi_1, P_2) = \left| \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix}}{\sqrt{a_x^2+a_y^2+a_z^2}} \right|.$$

where $\mathbf{a}$ is the normal vector of the plane π_1 corresponding to the above equation, i.e.

$$a_x = \begin{vmatrix} v_y & v_z \\ u_y & u_z \end{vmatrix}, \quad a_y = \begin{vmatrix} v_z & v_x \\ u_z & u_x \end{vmatrix} \quad \text{and} \quad a_z = \begin{vmatrix} v_x & v_y \\ u_x & u_y \end{vmatrix}.$$

- [Common perpendicular line of two skew lines] With the notation in the previous paragraph consider the line

$$\ell : \begin{cases} \pi_1' : \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ v_x & v_y & v_z \\ a_x & a_y & a_z \end{vmatrix} = 0, \\ \pi_2' : \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ u_x & u_y & u_z \\ a_x & a_y & a_z \end{vmatrix} = 0. \end{cases}$$

viewed as the intersection of the two planes π_1' and π_2'. It is a calculation to check that

  - $\ell \perp \ell_1$ and $\ell \perp \ell_2$, and that
  - $\ell \cap \ell_1 \neq \emptyset$ and $\ell \cap \ell_2 \neq \emptyset$.

4.3 Relative positions of a line and a plane

- [Intersection] Consider the plane

$$\pi : ax + by + cz + d = 0$$

and the line

$$\ell : \begin{cases} x = x_0 + tv_x \\ y = y_0 + tv_y \\ z = z_0 + tv_z. \end{cases}$$

In order to see if they intersect, we check which points in ℓ satisfy the equation of π:

$$a(x_0+tv_x)+b(y_0+tv_y)+c(z_0+tv_z)+d=0 \quad \Leftrightarrow \quad (av_x+bv_y+cv_z)t+ax_0+by_0+cz_0+d=0. \tag{4.16}$$

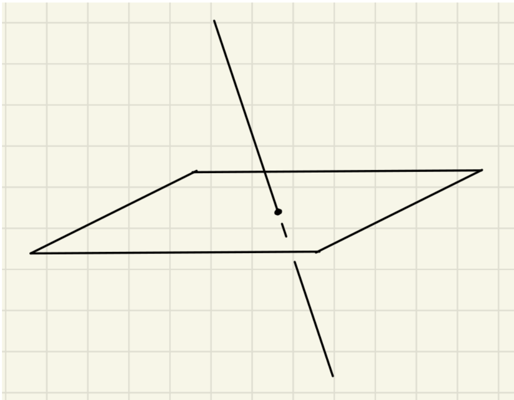
Notice that $\mathbf{n}(a,b,c)$ is a normal vector for π. The possibilities are

  - $\mathbf{n}\cdot\mathbf{v} = 0$ in which case the line is parallel to the plane.
  - If $\ell \parallel \pi$ and Equation (4.16) is not satisfied, then the line lies outside the plane π.

  - If $\ell \parallel \pi$ and Equation (4.16) is satisfied, then any $t \in \mathbb{R}$ is a solution. In this case the line lies in the plane π.
  - If $\ell \nparallel \pi$ then Equation (4.16) has the unique solution

$$t_0 = -\frac{ax_0 + by_0 + cz_0 + d}{av_x + bv_y + cv_z}.$$

    Hence, the point corresponding to the parammeter t_0 is the intersection point $\ell \cap \pi$.



- [Angle] Let $\mathbf{n}$ be a normal vector for the plane π and $\mathbf{v}$ be a direction vector for the line ℓ. If $\mathbf{n} \cdot \mathbf{v} < 0$ then the angle $\measuredangle(\mathbf{n}, \mathbf{v})$ is obtuse, so, replace $\mathbf{n}$ by $-\mathbf{n}$ or $\mathbf{v}$ by $-\mathbf{v}$ so that the angle $\measuredangle(\mathbf{n}, \mathbf{v})$ is acute and $\mathbf{n} \cdot \mathbf{v} > 0$.

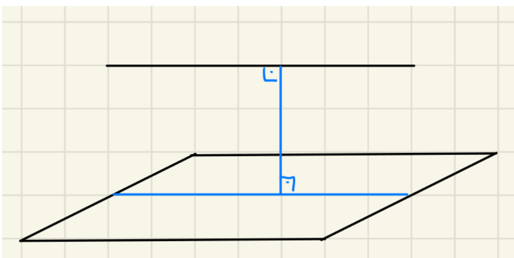
  The line ℓ and the plane π determine two angles: $90° \pm \measuredangle(\mathbf{n}, \mathbf{v})$ which can be calculated as usual with the scalar product if $\mathbf{n}$ and $\mathbf{v}$ are known.

- [Distance] For the distances between a plane π and a line ℓ we have the following cases:

  - if π and ℓ intersect, then the distance between them is zero $d(\ell, \pi) = 0$.
  - if π and ℓ do not intersect, i.e. if they are parallel, then

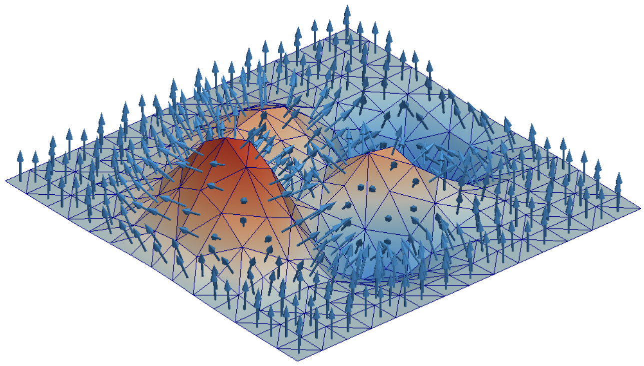
$$d(\ell, \pi) = d(P, \pi)$$

    for any point $P \in \ell$.



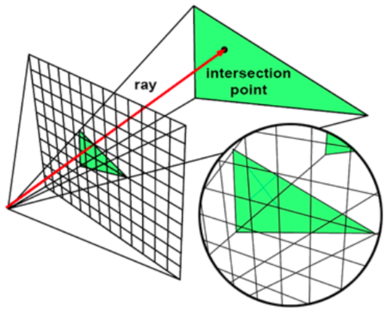
4.4 Applications

- [Triangulation] In computer graphics it is common to describe a surface by a triangulation, i.e. as a union of triangles:



- [Light] When modeling light on a surface one uses the normal vectors to the surface as described in Lecture 2. If the surface is given by a triangulation, the luminosity of each triangle can be calculated with the normal vector corresponding to that triangle, more precisely, with the normal vector to the plane determined by the corresponding triangle. Normal vectors are generally pre-computed, and are part of the 'model'.

- [Ray tracing]



Ray tracing algorithms consider rays passing through the camera and each pixel of the screen. These rays are intersected with the objects in the scene in order to decide what color to choose for the pixels. For this, one intersects the rays with the planes of the triangles. After finding the intersection points P one has to decide if P is inside or outside the triangle.