CHAPTER 2

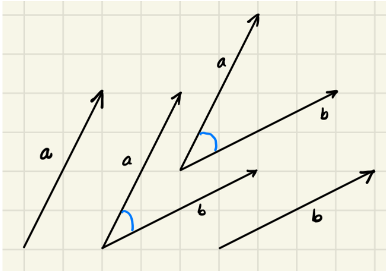
Scalar product and vector product

Contents

We denote by $\mathbb{E}^n$ the n-dimensional Euclidean space and by $\mathbb{V}^n$ the vector space associated to $\mathbb{E}^n$. For us $n = 1,2,3$.

2.1 Scalar product in $\mathbb{E}^n$

Definition. Let $\mathbf{a},\mathbf{b} \in \mathbb{V}^n$ be two non-zero distinct vectors. Given a point $O \in \mathbb{E}^n$, there are unique points $A,B \in \mathbb{E}^n$ such that $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$. The *angle* between $\mathbf{a}$ and $\mathbf{b}$ is the angle $\measuredangle AOB$ and we denote it by $\measuredangle(\mathbf{a},\mathbf{b})$.



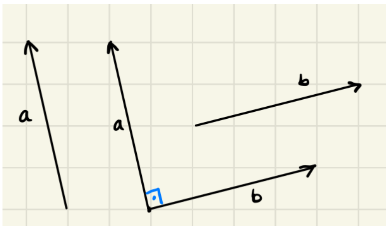
- The angle between two vectors is well defined: it does not depend on the choice of representatives of the vectors, it is a property of the two equipollence classes **a** and **b**.

Definition. The *scalar product* (or, *dot product*) of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{V}^n$ is the real number

$$\mathbf{a} \cdot \mathbf{b} = \begin{cases} 0, & \text{if one of the two vector is zero} \\ \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos \measuredangle(\mathbf{a}, \mathbf{b}) & \text{if both vectors are non-zero.} \end{cases}$$

- The scalar product is a map $\mathbb{V}^n \times \mathbb{V}^n \to \mathbb{R}$.
- Notice that $\mathbf{a}^2$ denotes $\mathbf{a} \cdot \mathbf{a}$ which by definition equals $\|\mathbf{a}\|^2$.
- For any vector $\mathbf{a} \in \mathbb{V}^n$ the *unit vector corresponding to* **a** is $\frac{1}{\|\mathbf{a}\|}\mathbf{a} = \frac{1}{\sqrt{\mathbf{a} \cdot \mathbf{a}}}\mathbf{a}$.
- When we divide a vector by its norm we say that we *normalize* the vector.
- A vector **a** is called a *unit vector* if $\|\mathbf{a}\| = 1$.
- By normalizing a vector we obtain a unit vector with the same direction as the initial vector.

Definition. Let $\mathbf{a}, \mathbf{b} \in \mathbb{V}^n$ be two vectors. We say that **b** is *orthogonal to* **a** if $\mathbf{b} \cdot \mathbf{a} = 0$. When this is the case, we say that the two vectors are orthogonal to each other and we write $\mathbf{b} \perp \mathbf{a}$.



Definition. For $\mathbf{v} \in \mathbb{V}^n$, let $\mathbf{v}^\perp$ be the set of all vectors which are orthogonal to **v**. One can show that if **v** is non-zero then $\mathbf{v}^\perp$ is an $(n-1)$-dimensional vector subspace of $\mathbb{V}^n$ and that $\mathbb{V}^n = \mathbf{v}^\perp \oplus \langle \mathbf{v} \rangle$. In particular, $\mathbb{V}^n$ has a basis $\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ with $\mathbf{v}_1, \dots, \mathbf{v}_{n-1} \in \mathbf{v}^\perp$. Thus, any vector $\mathbf{w} \in \mathbb{V}^n$ has a unique decomposition

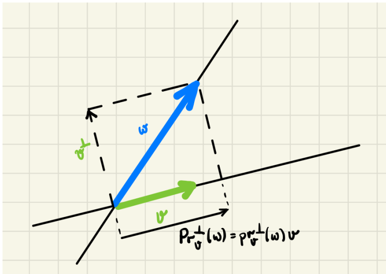
$$\mathbf{w} = w_0\mathbf{v} + w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + \cdots + w_n\mathbf{v}_n.$$

The map $\mathrm{Pr}^\perp_{\mathbf{v}} : \mathbb{V}^n \to \langle \mathbf{v} \rangle$ defined by $\mathrm{Pr}^\perp_{\mathbf{v}}(\mathbf{w}) = w_0\mathbf{v}$ is the *orthogonal projection on* $\langle \mathbf{v} \rangle$. The map $\mathrm{pr}^\perp_{\mathbf{v}} : \mathbb{V}^n \to \mathbb{R}$ defined by $\mathrm{pr}^\perp_{\mathbf{v}}(\mathbf{w}) = w_0$ is the *orthogonal projection on the direction of* **v**.

- The two maps defined above are very similar. They are linked by the following relation

$$\mathrm{Pr}_{\mathbf{v}}^{\perp}(\mathbf{w}) = \mathrm{pr}_{\mathbf{v}}^{\perp}(\mathbf{w})\mathbf{v}.$$

  So, the first one maps vectors to vectors, the second one maps vectors to scalars.



Proposition 2.1. Let $\mathbf{a}, \mathbf{b} \in \mathbb{V}^n$ be two non-zero vectors and let $\mathbf{a}_1 = \frac{1}{\|\mathbf{a}\|}$ and $\mathbf{b}_1 = \frac{1}{\|\mathbf{b}\|}\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \mathrm{pr}_{\mathbf{a}_1}^{\perp}\, \mathbf{b} = \|\mathbf{b}\| \cdot \mathrm{pr}_{\mathbf{b}_1}^{\perp}\, \mathbf{a}.$$

Proposition 2.2. For a non-zero vector $\mathbf{v} \in \mathbb{V}^n$, the maps $\mathrm{Pr}_{\mathbf{v}}^{\perp} : \mathbb{V}^n \to \langle \mathbf{v} \rangle$ and $\mathrm{pr}_{\mathbf{v}}^{\perp} : \mathbb{V}^n \to \mathbb{R}$ are linear.

Proposition 2.3. For any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}^n$ and any $\lambda \in \mathbb{R}$ we have

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$,
2. $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$,
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$,
4. $\mathbf{a} \cdot \mathbf{a} \geq 0$,
5. $\mathbf{a} \cdot \mathbf{a} = 0 \Leftrightarrow \mathbf{a} = 0$.

Definition. A basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of $\mathbb{V}^n$ is called *orthogonal* if $\mathbf{e}_i \perp \mathbf{e}_j$ for $i \neq j$. A basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is called *orthonormal* if it is orthogonal and all $\mathbf{e}_i$ are unit vectors.

A coordinate system $(O, \mathbf{e}_1, \ldots, \mathbf{e}_n)$ is called *orthogonal* or *orthonormal* if the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is orthogonal or respectively orthonormal.

2.1.1 Scalar product in an orthonormal coordinate system

Proposition 2.4. Consider two vectors $\mathbf{a}(a_1, a_2, \ldots, a_n), \mathbf{b}(b_1, b_2, \ldots, b_n) \in \mathbb{V}^n$ with components relative to an *orthonormal basis*. Their scalar product is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \tag{2.1}$$

- For an orthonormal basis $(\mathbf{i},\mathbf{j},\mathbf{k})$ of $\mathbb{V}^3$ and two vectors $\mathbf{a}(a_1,a_3,a_3),\mathbf{b}(b_1,b_2,b_3)\in\mathbb{V}^3$, we have

$$\|\mathbf{a}\| = \sqrt{a_1^2+a_2^2+a_3^2} \tag{2.2}$$

$$\cos\measuredangle(\mathbf{a},\mathbf{b}) = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}} \tag{2.3}$$

$$\mathbf{a}\perp\mathbf{b} \Leftrightarrow a_1b_1+a_2b_2+a_3b_3=0 \tag{2.4}$$

- In particular, for two points $A(A_1,A_2,A_3)$ and $B(B_1,B_2,B_3)$ we have

$$|AB| = \sqrt{(B_1-A_1)^2+(B_2-A_2)^2+(B_3-A_3)^2} = \|\overrightarrow{AB}\|.$$

- For dimension 2, we have similar formulas with respect to an orthonormal basis of $\mathbb{V}^2$.

- In your Algebra 1 course you encountered the notion of a scalar product. You defined it as in (2.1). When the scalar product is defined like that, the basis with respect to which it is defined is orthonormal.

2.1.2 Gram-Schmidt process

It is possible to express different formulas as above for the norm of a vector or the cosine of an angle in a coordinate system which is *not* orthonormal. However, the simplest expressions for these formulas are obtained for orthonormal bases, which is why everybody uses them.

If we have a basis $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n$ of $\mathbb{V}^n$, how do we obtain an orthonormal basis? We can do this in two steps:

1. Construct an orthogonal basis $\mathbf{e}'_1,\mathbf{e}'_2,\ldots,\mathbf{e}'_n$ as follows

$$\begin{aligned}
\mathbf{e}'_1 &= \mathbf{e}_1\\
\mathbf{e}'_2 &= \mathbf{e}_2 - \mathrm{Pr}^{\perp}_{\mathbf{e}'_1}(\mathbf{e}_2)\\
\mathbf{e}'_3 &= \mathbf{e}_3 - \mathrm{Pr}^{\perp}_{\mathbf{e}'_1}(\mathbf{e}_3) - \mathrm{Pr}^{\perp}_{\mathbf{e}'_2}(\mathbf{e}_3)\\
&\vdots
\end{aligned}$$

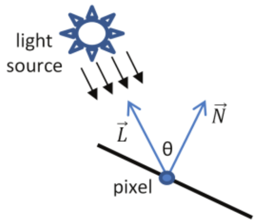
2. Normalize the vectors to obtain the basis

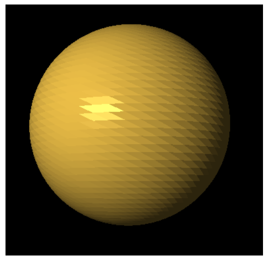
$$\left\{\frac{1}{\|\mathbf{e}'_1\|}\mathbf{e}'_1,\ldots,\frac{1}{\|\mathbf{e}'_n\|}\mathbf{e}'_n\right\}.$$

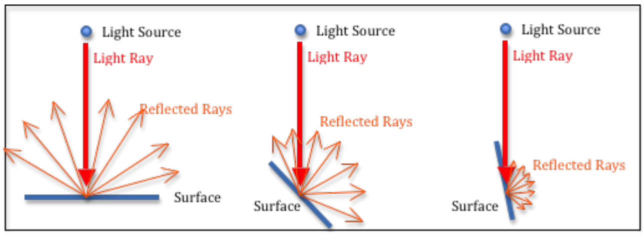
This process of obtaining an orthonormal basis from a given basis is the *Gram-Schmidt process.*

2.1.3 Applications

- [Lambert's cosine law] This is a law in optics, published by Johan Heinrich Lambert in 1760. It is used in computer graphics to model diffuse light. The law says that the luminous intensity observed from an ideal diffusely reflecting surface is directly proportional to the cosine of the angle θ between the direction of the incident light and the surface normal.





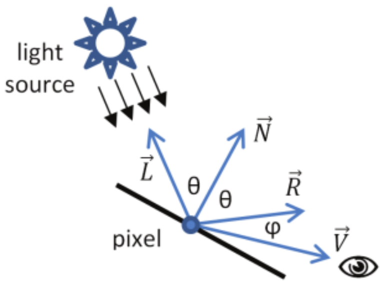


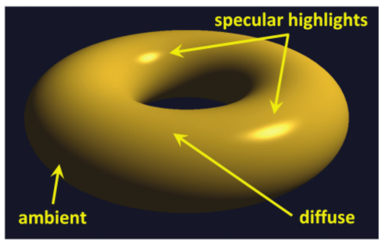
We will discuss normal vectors later during the course.

- [Specular light] The orthogonal reflection of a vector **b** in the vector **a** is

$$\mathrm{Ref}_{\mathbf{a}}^{\perp}(\mathbf{b}) = -\mathbf{b} + 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$$

In computer graphics specular light is modeled per pixel/primitiv as follows. One calculates the vector $\overrightarrow{L}$ to the light source and the vector $\overrightarrow{V}$ to the camera. The normal $\overrightarrow{N}$ to the surface is given. Using the above formula one calculates the vector $\overrightarrow{R}$ which gives the direction in which the light is reflected. The intensity of the pixel is then proportional to $\cos(\varphi)^n$ where n is some constant which depends on the surface.





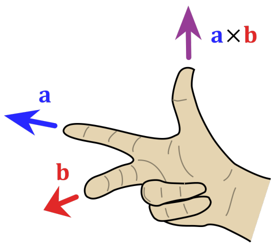
We will discuss normal vectors later during the course.

2.2 Vector product in $\mathbb{E}^3$

Definition. Let $\mathbf{u}, \mathbf{v}$ be a basis of $\mathbb{V}^2$. If $O \in \mathbb{E}^2$ is any point, then there are unique points $X, Y \in \mathbb{E}^2$ such that $\mathbf{u} = \overrightarrow{OX}$ and $\mathbf{v} = \overrightarrow{OY}$. Rotate the plane such that $\mathbf{u}$ points downwards. If Y is in the right half-plane determined by the line OX, then we say that the basis $\mathbf{u}, \mathbf{v}$ is *right oriented*. If Y lies in the left half-plane we say that the basis $\mathbf{u}, \mathbf{v}$ is *left oriented*. A coordinate system $(O, \mathbf{u}, \mathbf{v})$ is left or right oriented if the basis $\mathbf{u}, \mathbf{v}$ is left respectively right oriented.

Let $\mathbf{u} = \overrightarrow{OX}, \mathbf{v} = \overrightarrow{OY}, \mathbf{w} = \overrightarrow{OZ}$ be a basis of $\mathbb{V}^3$. We say that the basis is *right oriented* if $\mathbf{u}$, $\mathbf{v}$ is a right oriented basis when observed from the point Z. We say that the basis is *left oriented* if $\mathbf{u}$, $\mathbf{v}$ is a left oriented basis when observed from the point Z. A coordinate system $(O, \mathbf{u}, \mathbf{v}, \mathbf{w})$ is left or right oriented if the basis $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is left respectively right oriented.

- There are many equivalent ways of deciding if a basis is left or right oriented. The Swiss liked the three-finger rule so much, they put it on their 200-franc banknotes:



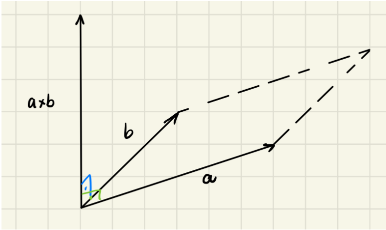
- A numerical way of checking this is by calculating a determinant (see Box product section).

Definition. Let $\mathbf{a}, \mathbf{b} \in \mathbb{V}^3$ be two vectors. The *vector product* (or *cross product*) of $\mathbf{a}$ and $\mathbf{b}$ is the vector denoted by $\mathbf{a} \times \mathbf{b}$ and defined by the following properties:

1. if $\mathbf{a}$ and $\mathbf{b}$ are collinear then $\mathbf{a} \times \mathbf{b} = 0$.
2. if $\mathbf{a}$ and $\mathbf{b}$ are not collinear, then
   (a) $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \sin \measuredangle(\mathbf{a}, \mathbf{b})$,
   (b) $\mathbf{a} \times \mathbf{b} \perp \mathbf{a}$ and $\mathbf{a} \times \mathbf{b} \perp \mathbf{b}$,
   (c) $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ is a right oriented basis of $\mathbb{V}^3$.

Notice that

- The vector product is a map $\mathbb{V}^3 \times \mathbb{V}^3 \to \mathbb{V}^3$, $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \times \mathbf{b}$.
- The formula in (a) is valid also when $\mathbf{a} \parallel \mathbf{b}$.
- The angle $\measuredangle(\mathbf{a}, \mathbf{b})$ lies between 0 and π, so $\sin \measuredangle(\mathbf{a}, \mathbf{b}) \geq 0$.



- The norm $\|\mathbf{a} \times \mathbf{b}\|$ equals the area of the parallelogram spanned by the two vectors.
- We make the convention that the zero vector $0 \in \mathbb{V}^3$ is collinear with any other vector (as in linear algebra where a set containing the zero vector is linearly dependent).
- Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = 0$.

Proposition 2.5. For any $\mathbf{a}, \mathbf{b} \in \mathbb{V}^3$ and any $\lambda \in \mathbb{R}$ we have

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
2. $(\lambda \mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$.
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$.

2.2.1 Vector product in a right oriented orthonormal coordinate system

Let $Oxyz = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ be a right oriented orthonormal reference frame (coordinate system). The values of the vector product on the basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of $\mathbb{V}^3$ are

$\times$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	0	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	0	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	0

Given two vectors $\mathbf{a}(a_1, a_2, a_3)$ and $\mathbf{b}(b_1, b_2, b_3)$ in $\mathbb{V}^3$ we can calculate $\mathbf{a} \times \mathbf{b}$ in terms of the components of $\mathbf{a}$ and $\mathbf{b}$ relative to $Oxyz$:

$$\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = (a_2b_3 - a_3b_2)\mathbf{i} + (-a_1b_3 + a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

This can be arranged as a determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}.$$

Therefore, the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ has area

$$\mathrm{Area}_{\mathrm{Parall}}(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} \times \mathbf{b}\| = \sqrt{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2}$$

and the triangle ABC where $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$ has area

$$\text{Area}_{\triangle}(\mathbf{a},\mathbf{b}) = \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\| = \frac{1}{2}\sqrt{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2}.$$

If the three points lie in the Oxy plane, i.e. $A(x_A, y_A, 0)$, $B(x_B, y_B, 0)$, $C(x_C, y_C, 0)$ then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_B - x_A & y_B - y_A & 0 \\ x_C - x_A & y_C - y_A & 0 \end{vmatrix} = \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}\mathbf{k} = \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}\mathbf{k}.$$

Thus

$$\|\mathbf{a}\times\mathbf{b}\| = \pm\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} \quad \text{and} \quad \text{Area}_{\triangle}(\mathbf{a},\mathbf{b}) = \pm\frac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}$$

where the sign '$\pm$' is chosen such that the value is positive. The value of the determinant could be negative. Without taking absolute value, we call the above determinant the *oriented area* of the triangle (or parallelogram).

This also gives a criterion for the collinearity of the points $A, B, C \in Oxy$: they are collinear if the above determinant is zero, i.e. if the area of the triangle ABC is zero.

2.2.2 Grassmann's vector product formula and Iacobi identity

The vector product is not associative. If $(\mathbf{i},\mathbf{j},\mathbf{k})$ is a right oriented orthonormal basis then

$$(\mathbf{i}\times\mathbf{j})\times\mathbf{j} = \mathbf{k}\times\mathbf{j} = -\mathbf{i} \quad \text{and} \quad \mathbf{i}\times(\mathbf{j}\times\mathbf{j}) = 0.$$

Definition. In order to fix ideas, we call $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ the *double vector product* of the vectors $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{V}^3$.

Theorem 2.6 (Grassmann's vector product formula)**.** For $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{V}^3$ we have

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}.$$

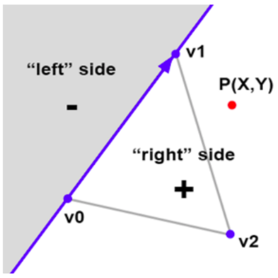
- The above theorem has computational value: instead of calculating two determinants we may calculate two scalar products.
- The property that makes up for the lack of associativity is the Jacobi identity.

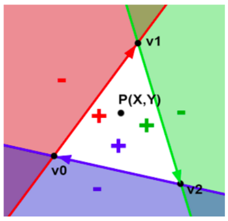
Proposition 2.7 (Jacobi identity)**.** For $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{V}^3$ we have

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{b}\times\mathbf{c})\times\mathbf{a} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b} = 0.$$

2.2.3 Applications

- [Edge function] In computer graphics one often has to decide if a point lies inside a triangle/primitive or not.





In such cases one uses a so-called *edge function*. The oriented area of a triangle gives a way of constructing such an edge function.

Suppose you have a triangle with vertices $v_0(x_0,y_0)$, $v_1(x_1,y_1)$ und $v_2(x_2,y_2)$. A point $P(x_P,y_P)$ lies inside the triangle if and only if

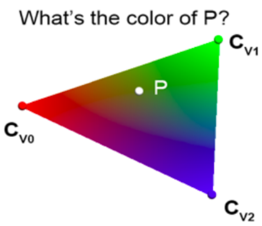
$$\begin{cases} A_{01}(x_P,y_P)A_{01}(x_2,y_2)>0\\ A_{12}(x_P,y_P)A_{12}(x_0,y_0)>0\\ A_{20}(x_P,y_P)A_{20}(x_1,y_1)>0\end{cases}$$

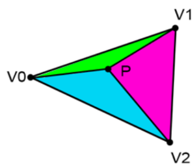
where $A_{01}(x_P,y_P)$= 2 times the oriented area of the triangle v_0v_1P, i.e.

$$A_{ij}(x,y)=\begin{vmatrix} x & y & 1\\ x_i & y_i & 1\\ x_j & y_j & 1\end{vmatrix}=\begin{vmatrix} x-x_i & y-y_i & 0\\ x_i & y_i & 1\\ x_j-x_i & y_j-y_i & 0\end{vmatrix}=\begin{vmatrix} x-x_i & y-y_i\\ x_j-x_i & y_j-y_i\end{vmatrix}.$$

Can you find a more efficient edge function?

- [Barycentric coordinates] In computer graphics one uses the vertices of a triangle/primitive to deduce attributes of the points inside the triangle/primitive. By attributes we mean color, normal vectors, texture coordinates, etc.





Definition. Consider a triangle ABC and a point P in the plane of the triangle. There exist unique scalars $\lambda_0,\lambda_1,\lambda_2$ such that $\lambda_0+\lambda_1+\lambda_2=1$ and

$$P=\lambda_0A+\lambda_1B+\lambda_2C.$$

You read the above equality as follows: for any point O we have $\overrightarrow{OP}=\lambda_0\overrightarrow{OA}+\lambda_1\overrightarrow{OB}+\lambda_2\overrightarrow{OC}$. These scalars are called barycentric coordinates of the point P.

If $F = 2\times$area of the triangle $V_0V_1V_2$ (Fig.) then we have

$$P = \lambda_0 V_0 + \lambda_1 V_1 + \lambda_2 V_2$$

where

$$\lambda_0 = \frac{A_{12}(P)}{F}, \quad \lambda_1 = \frac{A_{20}(P)}{F}, \quad \lambda_2 = \frac{A_{01}(P)}{F} \quad \text{with } A_{ij} \text{ as before.}$$

Now, if you have attributes for the vertices of the triangle encoded as numerical values A_0, A_1, A_2, then one calculates the corresponding attribute for the point P like this:

$$A_P = \lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2.$$

2.3 Box product in $\mathbb{E}^3$

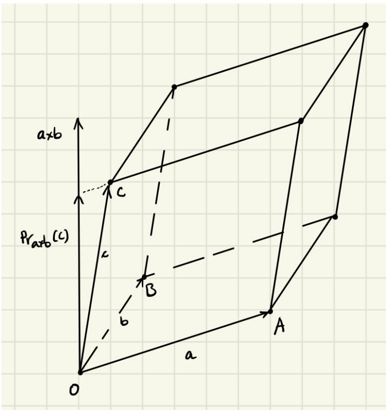
Definition. The *box product* of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}^3$ is $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ and we denote it by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

- Notice that the box product is a map

$$[\_, \_, \_] : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R} \quad (\mathbf{a}, \mathbf{b}, \mathbf{c}) \mapsto [\mathbf{a}, \mathbf{b}, \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

- The geometric interpretation of this product is the following.

Theorem 2.8. Let $\mathbf{a} = \overrightarrow{OA}, \mathbf{b} = \overrightarrow{OB}, \mathbf{c} = \overrightarrow{OC} \in \mathbb{V}^3$ be non-collinear vectors and let $\mathcal{P}$ be the parallelepiped spanned by the three vectors (i.e. $[OA]$, $[OB]$ and $[OC]$ are sides of the parallelepiped $\mathcal{P}$). Then $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is the volume of $\mathcal{P}$ if $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is right oriented; it is minus the volume of $\mathcal{P}$ if $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is left oriented.



- It follows that the volume of the tetrahedron $OABC$ ist

$$\text{Vol}_{OABC} = \left| \frac{1}{6} [\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}] \right|.$$

- The basis $(\mathbf{a},\mathbf{b},\mathbf{c})$ is right oriented if and only if $[\mathbf{a},\mathbf{b},\mathbf{c}] > 0$. It is left oriented if and only if $[\mathbf{a},\mathbf{b},\mathbf{c}] < 0$.

- In particular, an orthonormal basis $(\mathbf{a},\mathbf{b},\mathbf{c})$ is right oriented if and only if $[\mathbf{a},\mathbf{b},\mathbf{c}] = 1$. It is left oriented if and only if $[\mathbf{a},\mathbf{b},\mathbf{c}] = -1$.

- Three vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ are coplanar if and only if $[\mathbf{a},\mathbf{b},\mathbf{c}] = 0$.

2.3.1 Box product in an orthonormal coordinate system

Let $(O,\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal reference frame (coordinates system) and consider three vectors

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \quad \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

We have

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = (\mathbf{a}\times\mathbf{b})\mathbf{c} = (a_2b_3 - a_3b_2)c_1 + (-a_1b_3 + a_3b_1)c_2 + (a_1b_2 - a_2b_1)c_3.$$

or equivalently

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

From the properties of the determinants we have

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{c},\mathbf{a},\mathbf{b}] = -[\mathbf{b},\mathbf{a},\mathbf{c}] = -[\mathbf{a},\mathbf{c},\mathbf{b}] = -[\mathbf{c},\mathbf{b},\mathbf{a}].$$

The coplanarity condition for $\mathbf{a},\mathbf{b},\mathbf{c}$ is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

The orientation of the basis is determined as follows

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{cases} > 0 & \text{then } (\mathbf{a},\mathbf{b},\mathbf{c}) \text{ is right oriented} \\ = 0 & \text{then } (\mathbf{a},\mathbf{b},\mathbf{c}) \text{ is not a basis} \\ < 0 & \text{then } (\mathbf{a},\mathbf{b},\mathbf{c}) \text{ is left oriented} \end{cases}.$$

2.3.2 Applications

Look, this section is just saying that determinants of 3×3 matrices = volume in dimension 3. So this gives a geometric meaning to 3×3 determinants. Any applications of such determinants and any need to calculate volumes in dimension 3 is an application of the box product.

2.4 Further identities

Next to the Grassman identity and the Jacobi identity we have the following remarkable identities:

Theorem 2.9 (Lagrange identity)**.** For $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \mathbb{V}^3$ we have

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) = \begin{vmatrix} (\mathbf{a}\cdot\mathbf{c}) & (\mathbf{a}\cdot\mathbf{d}) \\ (\mathbf{b}\cdot\mathbf{c}) & (\mathbf{b}\cdot\mathbf{d}) \end{vmatrix}.$$

Theorem 2.10 (Triple vector product)**.** For $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \mathbb{V}^3$ we have

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{b}\cdot[\mathbf{a},\mathbf{c},\mathbf{d}] - \mathbf{a}\cdot[\mathbf{b},\mathbf{c},\mathbf{d}] = \mathbf{c}\cdot[\mathbf{a},\mathbf{b},\mathbf{d}] - \mathbf{d}\cdot[\mathbf{a},\mathbf{b},\mathbf{c}].$$

- You can prove such identities by writing both the left- and the right-hand side in coordinates.