Geometry
Problem booklet

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Contents

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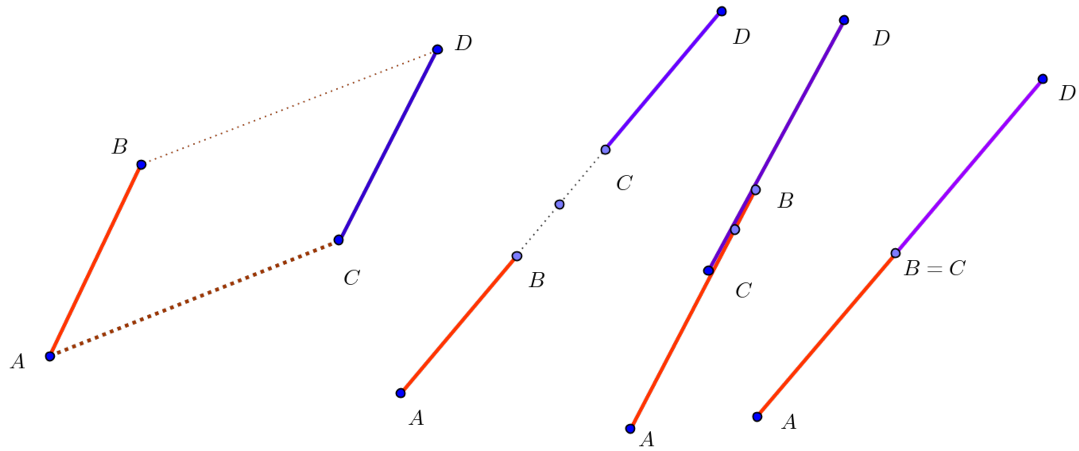
1 Week 1: Vector algebra

1.1 Free vectors

Vectors Let $\mathcal{P}$ be the three dimensional physical space in which we can talk about points, lines, planes and various relations among them. If $(A, B) \in \mathcal{P} \times \mathcal{P}$ is an ordered pair, then A is called the *original point* or the *origin* and B is called the *terminal point* or the *extremity* of (A, B).

Definition 1.1. The ordered pairs (A, B), (C, D) are said to be equipollent, written $(A, B) \sim (C, D)$, if the segments $[AD]$ and $[BC]$ have the same midpoint.



Pairs of equipollent points $(A, B) \sim (C, D)$

Remark 1.1. If the points A, B, C, $D \in \mathcal{P}$ are not collinear, then $(A, B) \sim (C, D)$ if and only if $ABDC$ is a parallelogram. In fact the length of the segments $[AB]$ and $[CD]$ is the same whenever $(A, B) \sim (C, D)$.

Proposition 1.1. *If (A, B) is an ordered pair and $O \in \mathcal{P}$ is a given point, then there exists a unique point X such that $(A,B) \sim (O, X)$.*

Proposition 1.2. *The equipollence relation is an equivalence relation on $\mathcal{P} \times \mathcal{P}$.*

Definition 1.2. The equivalence classes with respect to the equipollence relation are called *(free) vectors*.

Denote by $\overrightarrow{AB}$ the equivalence class of the ordered pair (A, B), that is $\overrightarrow{AB} = \{(X, Y) \in \mathcal{P} \times \mathcal{P} \mid (X, Y) \sim (A, B)\}$ and let $\mathcal{V} = \mathcal{P} \times \mathcal{P} \big/_{\sim} = \{\overrightarrow{AB} \mid (A, B) \in \mathcal{P} \times \mathcal{P}\}$ be the set of (free) vectors. The *length* or the *magnitude* of the vector $\overrightarrow{AB}$, denoted by $\| \overrightarrow{AB} \|$ or by $| \overrightarrow{AB} |$, is the length of the segment $[AB]$.

Remark 1.2. If two ordered pairs (A, B) and (C, D) are equippllent, i.e. the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal, then they have the same length, the same direction and the same sense. In fact a vector is determined by these three items.

Proposition 1.3. *1.* $\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \overrightarrow{AC} = \overrightarrow{BD}$.

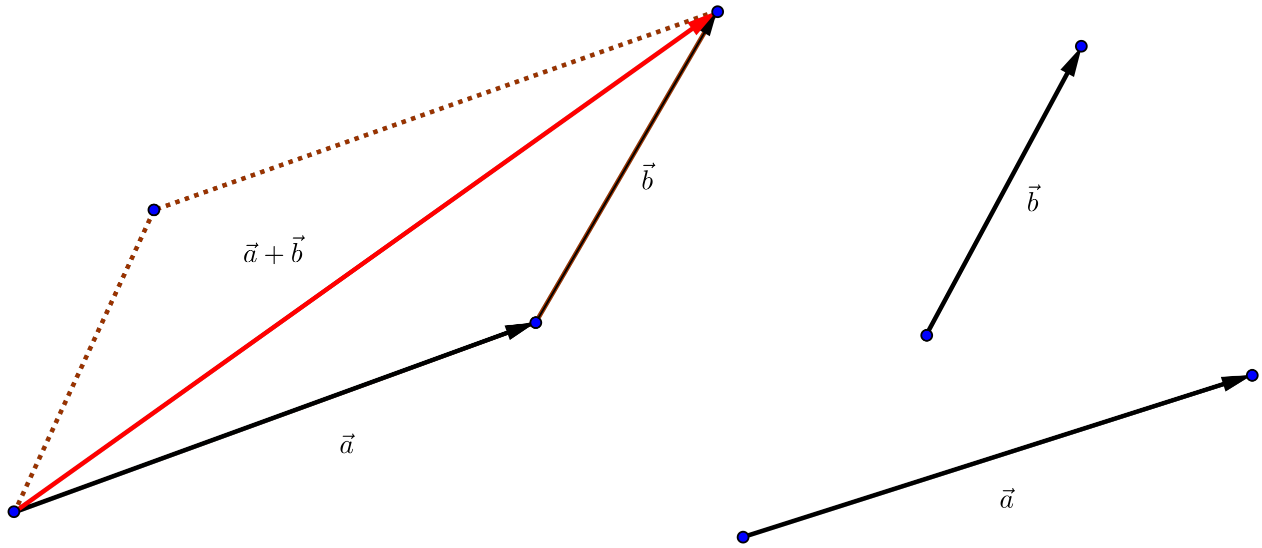
2. $\forall A, B, O \in \mathcal{P}$, $\exists! X \in \mathcal{P}$ *such that* $\overrightarrow{AB} = \overrightarrow{OX}$.
3. $\overrightarrow{AB} = \overrightarrow{A'B'}$, $\overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$.

Definition 1.3. If O, $M \in \mathcal{P}$, the the vector $\overrightarrow{OM}$ is denoted by $\vec{r}_M$ and is called the *position vector* of M with respect to O.

Corollary 1.4. *The map* $\varphi_O : \mathcal{P} \to \mathcal{V}$, $\varphi_O(M) = \vec{r}_M$ *is one-to-one and onto, i.e. bijective.*

1.1.1 Operations with vectors

• **The addition of vectors** Let $\vec{a}, \vec{b} \in \mathcal{V}$ and $O \in \mathcal{P}$ be such that $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{AB}$. The vector $\overrightarrow{OB}$ is called the *sum* of the vectors $\vec{a}$ and $\vec{b}$ and is written $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$.



Let O' be another point and A', $B' \in \mathcal{P}$ be such that $\overrightarrow{O'A'} = \vec{a}$, $\overrightarrow{A'B'} = \vec{b}$. Since $\overrightarrow{OA} = \overrightarrow{O'A'}$ and $\overrightarrow{AB} = \overrightarrow{A'B'}$ it follows, according to Proposition 1.3(3), that $\overrightarrow{OB} = \overrightarrow{O'B'}$. Therefore the vector $\vec{a} + \vec{b}$ is independent on the choice of the point O.

Proposition 1.5. *The set* $\mathcal{V}$ *endowed to the binary operation* $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, $(\vec{a}, \vec{b}) \mapsto \vec{a} + \vec{b}$, *is an abelian group whose zero element is the vector* $\overrightarrow{AA} = \overrightarrow{BB} = \vec{0}$ *and the opposite of* $\overrightarrow{AB}$, *denoted by* $-\overrightarrow{AB}$, *is the vector* $\overrightarrow{BA}$.

In particular the addition operation is associative and the vector

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

is usually denoted by $\vec{a} + \vec{b} + \vec{c}$. Moreover the expression

$$((\cdots(\vec{a}_1 + \vec{a}_2) + \vec{a}_3 + \cdots + \vec{a}_n)\cdots), \tag{1.1}$$

is independent of the distribution of paranthesis and it is usually denoted by

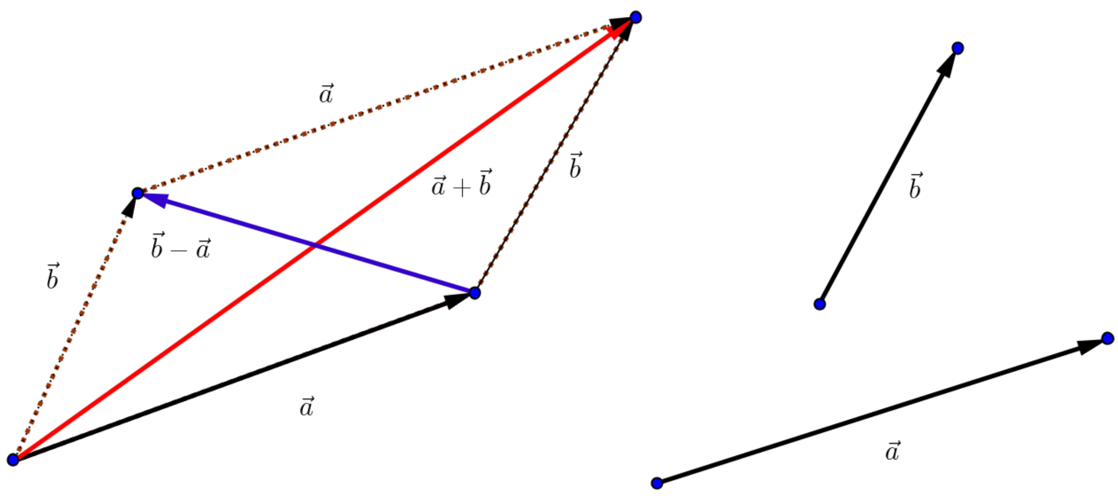
$$\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n .$$

Example 1.1. If $A_1, A_2, A_3, \ldots, A_n \in \mathcal{P}$ are some given points, then

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{A_1A_n} .$$

This shows that $\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1} = \overrightarrow{0}$, namely the sum of vectors constructed on the edges of a closed broken line is zero.

Corollary 1.6. *If $\overrightarrow{a} = \overrightarrow{OA}$, $\overrightarrow{b} = \overrightarrow{OB}$ are given vectors, there exists a unique vector $\overrightarrow{x} \in \mathcal{V}$ such that $\overrightarrow{a} + \overrightarrow{x} = \overrightarrow{b}$. In fact $\overrightarrow{x} = \overrightarrow{b} + (-\overrightarrow{a}) = \overrightarrow{AB}$ and is denoted by $\overrightarrow{b} - \overrightarrow{a}$.*



• **The multiplication of vectors with scalars**

Let $\alpha \in \mathbb{R}$ be a scalar and $\overrightarrow{a} = \overrightarrow{OA} \in \mathcal{V}$ be a vector. We define the vector $\alpha \cdot \overrightarrow{a}$ as follows: $\alpha \cdot \overrightarrow{a} = \overrightarrow{0}$ if $\alpha = 0$ or $\overrightarrow{a} = \overrightarrow{0}$; if $\overrightarrow{a} \neq \overrightarrow{0}$ and $\alpha > 0$, there exists a unique point on the half line $]OA$ such that $||OB|| = \alpha \cdot ||OA||$ and define $\alpha \cdot \overrightarrow{a} = \overrightarrow{OB}$; if $\alpha < 0$ we define $\alpha \cdot \overrightarrow{a} = -(|\alpha| \cdot \overrightarrow{a})$. The external binary operation

$$\mathbb{R} \times \mathcal{V} \to \mathcal{V}, \ (\alpha, \overrightarrow{a}) \mapsto \alpha \cdot \overrightarrow{a}$$

is called the *multiplication of vectors with scalars*.

Proposition 1.7. *The following properties hold:*

$(v1)$ $(\alpha+\beta)\cdot \vec{a} = \alpha\cdot \vec{a} + \beta\cdot \vec{a}$, $\forall\, \alpha, \beta \in \mathbb{R}$, $\vec{a} \in \mathcal{V}$.

$(v2)$ $\alpha\cdot(\vec{a} + \vec{b}) = \alpha\cdot \vec{a} + \alpha\cdot \vec{b}$, $\forall\, \alpha \in \mathbb{R}$, $\vec{a}, \vec{b} \in \mathcal{V}$.

$(v3)$ $\alpha\cdot(\beta\cdot \vec{a}) = (\alpha\beta)\cdot \vec{a}, \forall\, \alpha, \beta \in \mathbb{R}$.

$(v4)$ $1\cdot \vec{a} = \vec{a}$, $\forall\ \vec{a} \in \mathcal{V}$.

Application 1.1. Consider two parallelograms, $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ in $\mathcal{P}$, and M_1, M_2, M_3, M_4 the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$, $[A_4B_4]$ respectively. Then:

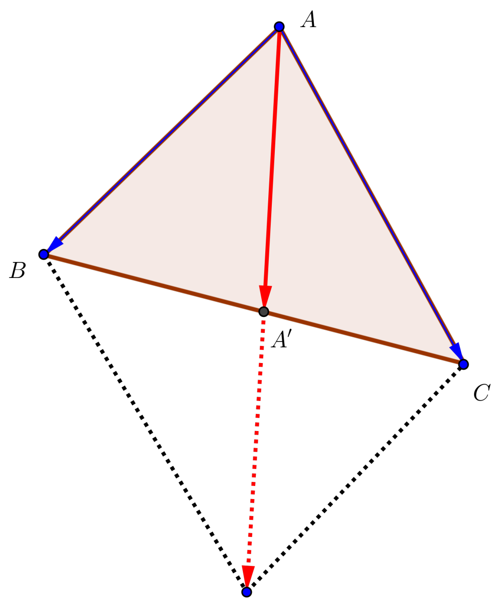
- $2\,\overrightarrow{M_1M_2} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2}$ and $2\,\overrightarrow{M_3M_4} = \overrightarrow{A_3A_4} + \overrightarrow{B_3B_4}$.
- M_1, M_2, M_3, M_4 are the vertices of a parallelogram.

1.1.2 The vector structure on the set of vectors

Theorem 1.8. *The set of (free) vectors endowed with the addition binary operation of vectors and the external binary operation of multiplication of vectors with scalars is a real vector space.*

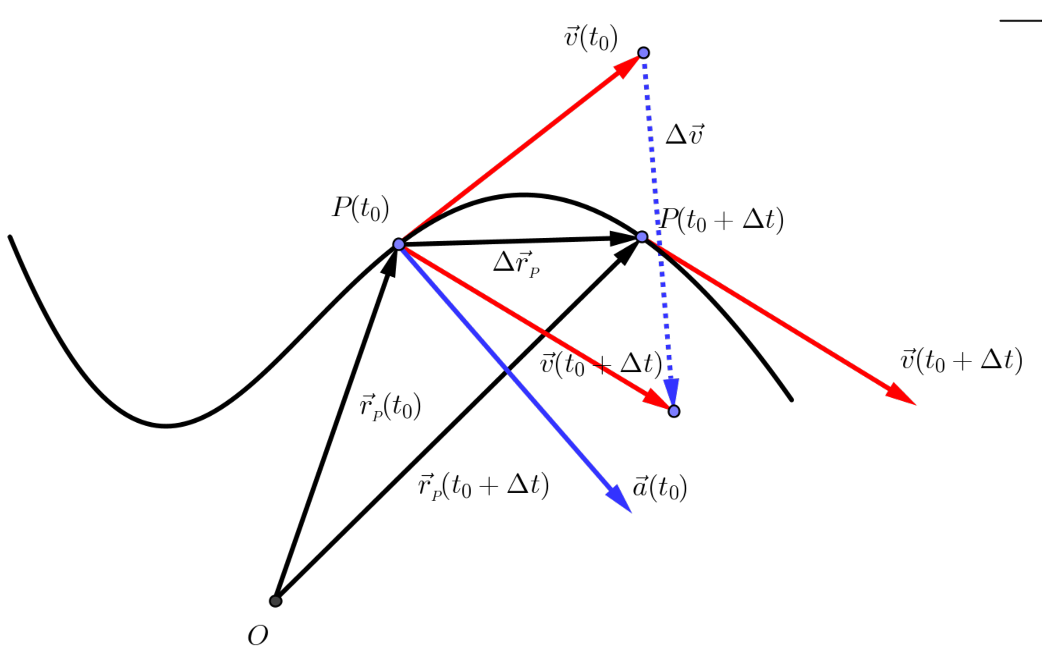
Example 1.2. If A' is the midpoint of the egde $[BC]$ of the triangle ABC, then

$$\overrightarrow{AA'} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}).$$



A few vector quantities:

1. The force, usually denoted by $\overrightarrow{F}$.
2. The velocity $\frac{d\overrightarrow{r}_P}{dt}$ of a moving particle P, is usually denoted by $\overrightarrow{v}_P$ or simply by $\overrightarrow{v}$.
3. The acceleration $\frac{d\overrightarrow{v}_P}{dt}$ of a moving particle P, is usually denoted by $\overrightarrow{a}_P$ or simply by $\overrightarrow{a}$.



• **Newton's law of gravitation**, statement that any particle of matter in the universe attracts any other with a force varying directly as the product of the masses and inversely as the square of the distance between them. In symbols, the magnitude of the attractive force F is equal to G (the gravitational constant, a number the size of which depends on the system of units used and which is a universal constant) multiplied by the product of the masses (m_1 and m_2) and divided by the square of the distance R: $F = G(m_1m_2)/R^2$. (Encyclopdia

Britannica)
• **Newton's second law** is a quantitative description of the changes that a force can produce on the motion of a body. It states that the time rate of change of the momentum of a body is equal in both magnitude and direction to the force imposed on it. The momentum of a body is equal to the product of its mass and its velocity. Momentum, like velocity, is a vector quantity, having both magnitude and direction. A force applied to a body can change the magnitude of the momentum, or its direction, or both. Newtons second law is one of the most important in all of physics. For a body whose mass m is constant, it can be written in the form $F = ma$, where F (force) and a (acceleration) are both vector quantities. If a body has a net force acting on it, it is accelerated in accordance with the equation. Conversely, if a body is not accelerated, there is no net force acting on it. (Encyclopdia Britannica)

1.2 Problems

1. Consider a tetrahedron $ABCD$. Find the the following sums of vectors:

   (a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$.

   (b) $\overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{DC}$.

   (c) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{CD}$.

   Solution.

2. ([4, Problem 3, p. 1]) Let $OABCDE$ be a regular hexagon in which $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OE}=\vec{b}$. Express the vectors $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of the vectors $\vec{a}$ and $\vec{b}$. Show that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OE} = 3\,\overrightarrow{OC}$.

   Solution.

3. Consider a pyramid with the vertex at S and the basis a parallelogram $ABCD$ whose diagonals are concurrent at O. Show the equality $\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\,\overrightarrow{SO}$.
   Solution.

4. Let E and F be the midpoints of the diagonals of a quadrilateral $ABCD$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right) = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{CB}\right).$$

Solution.

5. In a triangle ABC we consider the height AD from the vertex A ($D \in BC$). Find the decomposition of the vector AD in terms of the vectors $\vec{c} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AC}$.
   Solution.

6. ([4, Problem 12, p. 3]) Let M, N be the midpoints of two opposite edges of a given quadrilateral $ABCD$ and P be the midpoint of $[MN]$. Show that

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0$$

Solution.

7. ([4, Problem 12, p. 7]) Consider two perpendicular chords AB and CD of a given circle and $\{M\} = AB \cap CD$. Show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2\,\overrightarrow{OM}\,.$$

Solution.

8. ([4, Problem 13, p. 3]) If G is the centroid of a tringle ABC and O is a given point, show that

$$\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}.$$

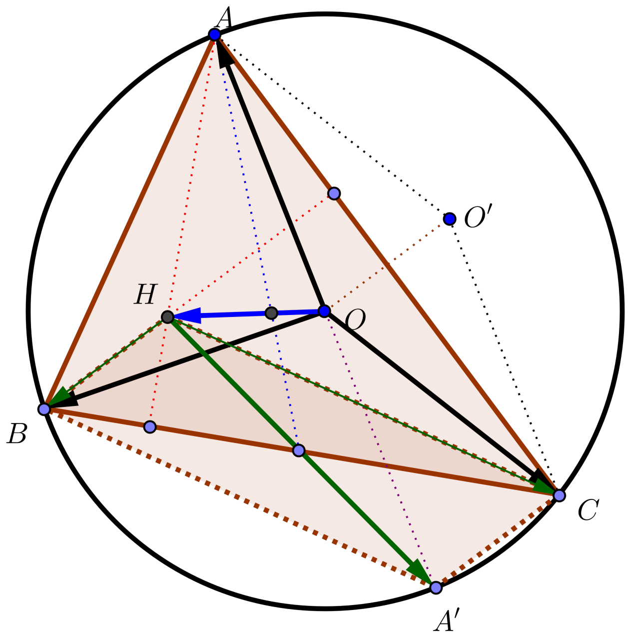
Solution.

9. ([4, Problem 14, p. 4]) Consider the triangle ABC alongside its orthocenter H, its circumcenter O and the diametrically opposed point A' of A on the latter circle. Show that:

   (a) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$.

   (b) $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$.

   (c) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\,\overrightarrow{HO}$.



Solution.

10. ([4, Problem 15, p. 4]) Consider the triangle ABC alongside its centroid G, its orthocenter H and its circumcenter O. Show that O, G, H are collinear and $3 \overrightarrow{HG} = 2 \overrightarrow{HO}$.

    Solution.

11. ([4, Problem 27, p. 13]) Consider a tetrahedron $A_1A_2A_3A_4$ and the midpoints A_{ij} of the edges A_iA_j, $i \neq j$. Show that:

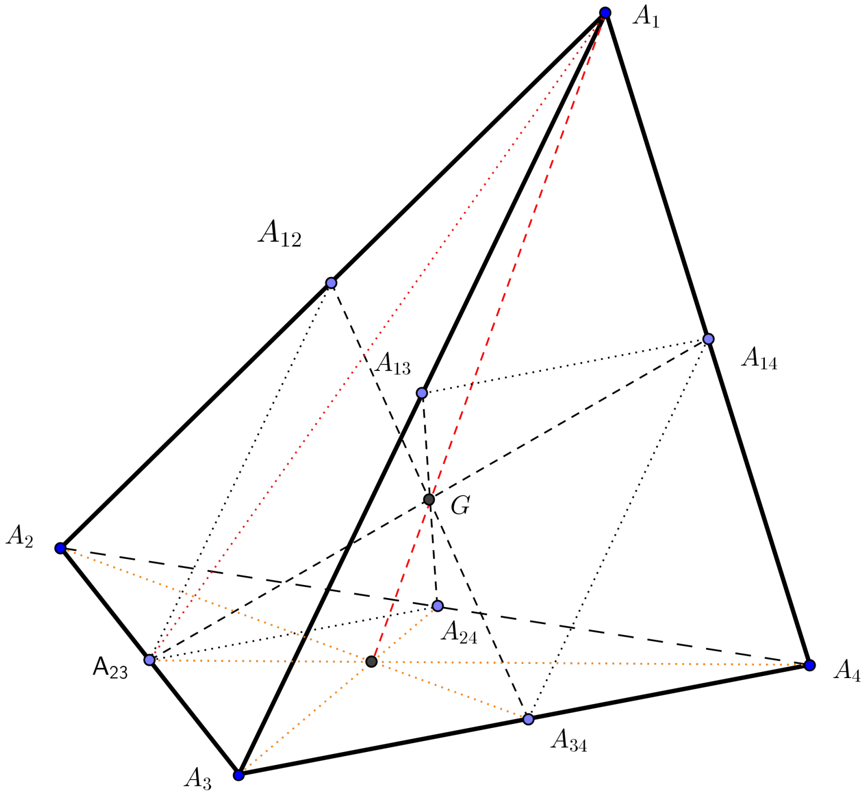
    (a) The lines $A_{12}A_{34}$, $A_{13}A_{24}$ and $A_{14}A_{23}$ are concurrent in a point G.

    (b) The medians of the tetrahedron (the lines passing through the vertices and the centroids of the opposite faces) are also concurrent at G.

    (c) Determine the ratio in which the point G divides each median.

    (d) Show that $\overrightarrow{GA_1} + \overrightarrow{GA_2} + \overrightarrow{GA_3} + \overrightarrow{GA_4} = \overrightarrow{0}$.

    (e) If M is an arbitrary point, show that $\overrightarrow{MA_1} + \overrightarrow{MA_2} + \overrightarrow{MA_3} + \overrightarrow{MA_4} = 4\,\overrightarrow{MG}$.



Solution.

12. In a triangle ABC consider the points M, L on the side AB and N, T on the side AC such that $3\overrightarrow{AL} = 2\overrightarrow{AM} = \overrightarrow{AB}$ and $3\overrightarrow{AT} = 2\overrightarrow{AN} = \overrightarrow{AC}$. Show that $\overrightarrow{AB} + \overrightarrow{AC} = 5\overrightarrow{AS}$, where $\{S\} = MT \cap LN$.

    Solution.

13. Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$, not necesarily in the same plane, alongside their centroids G_1, G_2. Show that $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3\,\overrightarrow{G_1G_2}$.

Solution.

2 Week 2: Straight lines and planes

2.1 Linear dependence and linear independence of vectors

Definition 2.1.
1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *collinear* if the points O, A, B are collinear. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *noncollinear*.
2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}, \overrightarrow{OC}$ are said to be *coplanar* if the points O, A, B, C are coplanar. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are *noncoplanar*.

Remark 2.1.
1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are linearly (in)dependent if and only if they are (non)collinear.
2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are linearly (in)dependent if and only if they are (non)coplanar.

Proposition 2.1. *The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ form a basis of $\mathcal{V}$ if and only if they are noncoplanar.*

Corollary 2.2. *The dimension of the vector space of free vectors $\mathcal{V}$ is three.*

Proposition 2.3. *Let Δ be a straight line and let $A \in \Delta$ be a given point. The set*

$$\overrightarrow{\Delta} = \{\overrightarrow{AM} \mid M \in \Delta\}$$

is an one dimensional subspace of $\mathcal{V}$. It is independent on the choice of $A \in \Delta$ and is called the director subspace *of Δ or the* direction *of Δ.*

Remark 2.2. The straight lines Δ, Δ' are parallel if and only if $\overrightarrow{\Delta} = \overrightarrow{\Delta}'$

Definition 2.2. *We call* director vector *of the straigh line Δ every nonzero vector $\overrightarrow{d} \in \overrightarrow{\Delta}$.*

If $\overrightarrow{d} \in \mathcal{V}$ is a nonzero vector and $A \in \mathcal{P}$ is a given point, then there exits a unique straight line which passes through A and has the direction $\langle \overrightarrow{d} \rangle$. This stright line is

$$\Delta = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d} \rangle\}.$$

Δ is called the straight line which passes through O and is parallel to the vector $\overrightarrow{d}$.

Proposition 2.4. *Let π be a plane and let $A \in \pi$ be a given point. The set $\overrightarrow{\pi} = \{\overrightarrow{AM} \in \mathcal{V} \mid M \in \pi\}$ is a two dimensional subspace of $\mathcal{V}$. It is independent on the position of A inside π and is called the* director subspace, the director plane *or the* direction *of the plane π.*

Remark 2.3. • *The planes π, π' are parallel if and only if $\overrightarrow{\pi} = \overrightarrow{\pi}'$.*

• If $\overrightarrow{d}_1, \overrightarrow{d}_2$ are two linearly independent vectors and $A \in \mathcal{P}$ is a fixed point, then there exists a unique plane through A whose direction is $\langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle$. This plane is

$$\pi = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle\}.$$

We say that π is the plane which passes through the point A and is parallel to the vectors $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$.

Remark 2.4. Let $\Delta \subset \mathcal{P}$ be a straight line and $\pi \subset \mathcal{P}$ be given plane.

1. If $A \in \Delta$ is a given point, then $\varphi_O(\Delta) = \overrightarrow{r}_A + \overrightarrow{\Delta}$.

2. If $B \in \Delta$ is a given point, then $\varphi_O(\pi) = \overrightarrow{r}_B + \overrightarrow{\pi}$.

Generally speaking, a subset X of a vector space is called *linear variety* if either $X = \emptyset$ or there exists $a \in V$ and a vector subspace U of V, such that $X = a + U$.

$$\dim(X) = \begin{cases} -1 & \text{dacă } X = \emptyset \\ \dim(U) & \text{dacă } X = a + U, \end{cases}$$

Proposition 2.5. *The bijection φ_O transforms the straight lines and the planes of the affine space $\mathcal{P}$ into the one and two dimnensional linear varieties of the vector space $\mathcal{V}$ respectively.*

2.2 The vector ecuations of the straight lines and planes

Proposition 2.6. *Let Δ be a straight line, let π be a plane, $\{\overrightarrow{d}\}$ be a basis of $\overrightarrow{\Delta}$ and let $[\overrightarrow{d}_1, \overrightarrow{d}_2]$ be an ordered basis of $\overrightarrow{\pi}$.*

1. *The points $M \in \Delta$ are characterized by the* vector equation of Δ

$$\overrightarrow{r}_M = \overrightarrow{r}_A + \lambda \overrightarrow{d}, \ \lambda \in \mathbb{R} \tag{2.1}$$

where $A \in \Delta$ is a given point.

2. *The points $M \in \pi$ are characterized by the* vector equation of π

$$\overrightarrow{r}_M = \overrightarrow{r}_A + \lambda_1 \overrightarrow{d}_1 + \lambda_2 \overrightarrow{d}_2, \ \lambda_1, \lambda_2 \in \mathbb{R}, \tag{2.2}$$

where $A \in \pi$ is a given point.

PROOF.

□

Corollary 2.7. *If $A, B \in \mathcal{P}$ are different points, then the vector equation of the line AB is*

$$\overrightarrow{r}_M = (1-\lambda)\overrightarrow{r}_A + \lambda \overrightarrow{r}_B,\ \lambda \in \mathbb{R}. \tag{2.3}$$

PROOF.

□

Corollary 2.8. *If A, B, $C \in \mathcal{P}$ are three noncolinear points, then the vector equation of the plane (ABC) is*

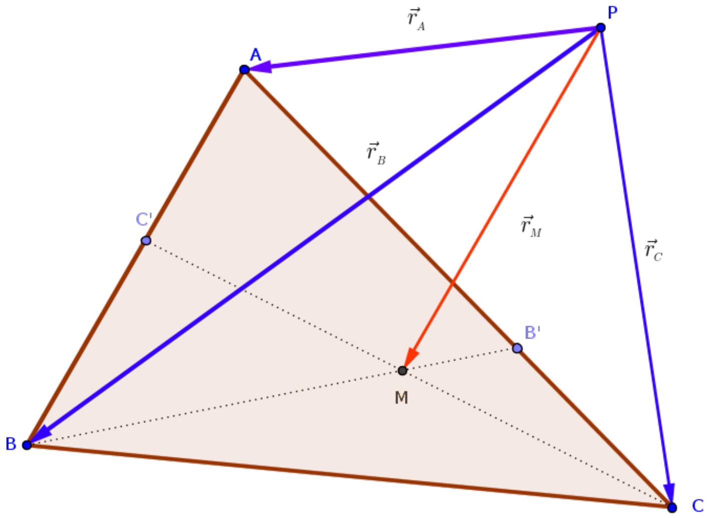
$$\overrightarrow{r}_M = (1-\lambda_1-\lambda_2)\overrightarrow{r}_A + \lambda_1 \overrightarrow{r}_B + \lambda_2 \overrightarrow{r}_C,\ \lambda_1,\ \lambda_2 \in \mathbb{R}. \tag{2.4}$$

PROOF.

□

Example 2.1. Consider the points C' and B' on the sides AB and AC of the triangle ABC such that $\overrightarrow{AC'} = \lambda\ \overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\ \overrightarrow{CB'}$. The lines BB' and CC' meet at M. If $P \in \mathcal{P}$ is a given point and $\overrightarrow{r}_A = \overrightarrow{PA}$, $\overrightarrow{r}_B = \overrightarrow{PB}$, $\overrightarrow{r}_C = \overrightarrow{PC}$ are the position vectors, with respect to P, of the vertices A, B, C respectively, show that

$$\overrightarrow{r}_M = \frac{\overrightarrow{r}_A - \lambda \overrightarrow{r}_B - \mu \overrightarrow{r}_C}{1-\lambda-\mu}. \tag{2.5}$$



SOLUTION.

□

2.3 Problems

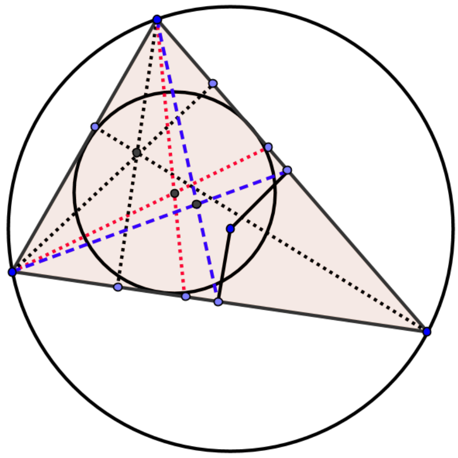
1. ([4, Problem 17, p. 5]) Consider the triangle ABC, its centroid G, its orthocenter H, its incenter I and its circumcenter O. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$, $\vec{r}_C = \overrightarrow{PC}$ are the position vectors with respect to P of the vertices A, B, C respectively, show that:

$$\vec{r}_G := \overrightarrow{PG} = \frac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}.$$

$$\vec{r}_I := \overrightarrow{PI} = \frac{a\,\vec{r}_A + b\,\vec{r}_B + c\,\vec{r}_C}{a+b+c}.$$

$$\vec{r}_H := \overrightarrow{PH} = \frac{(\tan A)\,\vec{r}_A + (\tan B)\,\vec{r}_B + (\tan C)\,\vec{r}_C}{\tan A + \tan B + \tan C}.$$

$$\vec{r}_O := \overrightarrow{PO} = \frac{(\sin 2A)\,\vec{r}_A + (\sin 2B)\,\vec{r}_B + (\sin 2C)\,\vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}.$$



Solution.

2. Consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

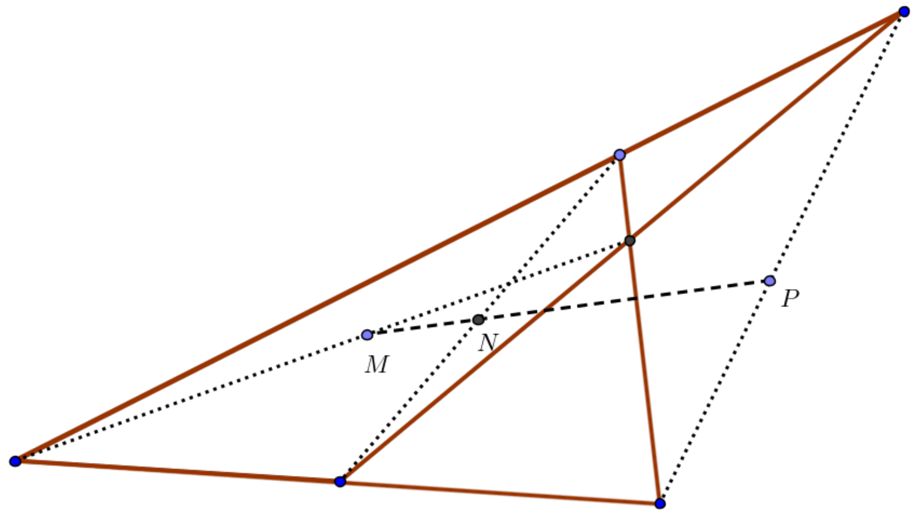
$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\,\overrightarrow{OA} \quad + \quad n\frac{1-m}{1-mn}\,\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\,\overrightarrow{OA} \quad + \quad n\frac{m-1}{m-n}\,\overrightarrow{OA'}\,.$$

where $\{M\} = AB' \cap A'B$, $\{N\} = AA' \cap BB'$, $\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\,\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\,\overrightarrow{OA'}$.

Solution.

3. Show that the midpoints of the diagonals of a complet quadrilateral are collinear (Newton’s theorem).



Solution.

4. Let d, d' be concurrent straight lines and $A, B, C \in d$, $A', B', C' \in d'$. If the following relations $AB' \nparallel A'B$, $AC' \nparallel A'C$, $BC' \nparallel B'C$ hold, show that the points $\{M\} := AB' \cap A'B$, $\{N\} := AC' \cap A'C$, $\{P\} := BC' \cap B'C$ are collinear (Pappus' theorem).

   SOLUTION.

5. Let d, d' be two straight lines and $A, B, C \in d$, $A', B', C' \in d'$ three points on each line such that $AB' \| BA'$, $AC' \| CA'$. Show that $BC' \| CB'$ (the affine Pappus' theorem).

   SOLUTION.

6. Let us consider two triangles ABC and $A'B'C'$ such that the lines AA', BB', CC' are concurrent at a point O and $AB \nparallel A'B'$, $BC \nparallel B'C'$ and $CA \nparallel C'A'$. Show that the points $\{M\} = AB \cap A'B'$, $\{N\} = BC \cap B'C'$ and $\{P\} = CA \cap C'A'$ are collinear (Desargues).

   SOLUTION.

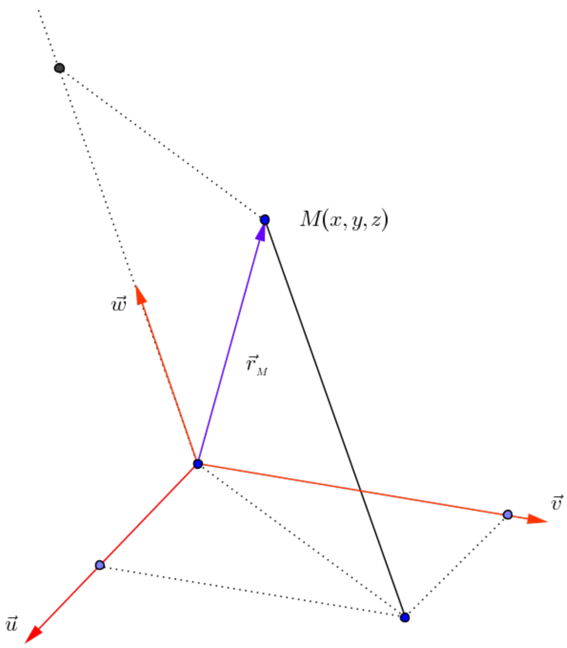
3 Week 3: Cartesian equations of lines and planes

3.1 Cartesian and affine reference systems

If $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is an ordered basis of $\mathcal{V}$ and $\overrightarrow{x} \in \mathcal{V}$, recall that the column vector of the coordinates of $\overrightarrow{x}$ with respect to b is denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

whenever $\overrightarrow{x} = x_1 \overrightarrow{u} + x_2 \overrightarrow{v} + x_3 \overrightarrow{w}$. To emphasize the coordinates of $\overrightarrow{x}$ with respect to b, we shall use the notation $\overrightarrow{x}(x_1, x_2, x_3)$.



Definition 3.1. A *cartesian reference system* $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ of the space $\mathcal{P}$, consists in a point $O \in \mathcal{P}$ called the *origin* of the reference system and an ordered basis $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ of the vector space $\mathcal{V}$.

Denote by E_1, E_2, E_3 the points for which $\overrightarrow{u} = \overrightarrow{OE_1}$, $\overrightarrow{v} = \overrightarrow{OE_2}$, $\overrightarrow{w} = \overrightarrow{OE_3}$.

Definition 3.2. The system of points (O, E_1, E_2, E_3) is called *the affine reference system associated to the cartesian reference system* $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$.

The straight lines OE_i, $i \in \{1, 2, 3\}$, oriented from O to E_i are called *the coordinate axes*. The coordinates x, y, z of the position vector $\overrightarrow{r}_M = \overrightarrow{OM}$ with respect to the basis $[\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y, z)$. Also, for the column matrix of coordinates of the vector $\overrightarrow{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\overrightarrow{r}_M = x \overrightarrow{u} + y \overrightarrow{v} + z \overrightarrow{w}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Remark 3.1. If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are two points, then

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= x_B \overrightarrow{u} + y_B \overrightarrow{v} + z_B \overrightarrow{w} - (x_A \overrightarrow{u} + y_A \overrightarrow{v} + z_A \overrightarrow{w}) \\ &= (x_B - x_A) \overrightarrow{u} + (y_B - y_A) \overrightarrow{v} + (z_B - z_A) \overrightarrow{w}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

Remark 3.2. If $R = (O, b)$ is a cartesian reference system, where $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is an ordered basis of $\mathcal{V}$, recall that $\varphi_O : \mathcal{P} \longrightarrow \mathcal{V}$, $\varphi_O(M) = \overrightarrow{OM}$ is bijective and $\psi_b : \mathbb{R}^3 \longrightarrow \mathcal{V}$, $\psi_b(x, y, z) = x \overrightarrow{u} + y \overrightarrow{v} + z \overrightarrow{w}$ is a linear isomorphism. The bijection φ_O defines a unique vector structure over $\mathcal{P}$ such that φ_O becomes an isomorphism. This vector structure depends on the choice of $O \in \mathcal{P}$. Therefore a point $M \in \mathcal{P}$ could be identified either with its position vector $\overrightarrow{r}_M = \varphi_O(M)$, or, with the triplet $(\psi_b^{-1} \circ \varphi_O)(M) \in \mathbb{R}^3$ of its coordinates with respect to the reference system R. If $f : X \longrightarrow \mathbb{R}^3$ is a given application, then $\varphi_O^{-1} \circ \psi_b \circ f : X \longrightarrow \mathcal{P}$ will be denoted by M_f. A similar discussion can be done for a cartesian reference system $R' = (O', b')$ of a plane π, where $b' = [\overrightarrow{u}', \overrightarrow{v}']$ is an ordered basis of $\overrightarrow{\pi}$.

Example 3.1 (Homework)**.** Consider the tetrahedron $ABCD$, where $A(1, -1, 1)$, $B(-1, 1, -1)$, $C(2, 1, -1)$ and $D(1, 1, 2)$. Find the coordinates of:

1. the centroids G_A, G_B, G_C, G_D of the triangles BCD, ACD, ABD and ABC[1] respectively.
2. the midpoints M, N, P, Q, R and S of its edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[CD]$ and $[DB]$ respectively.

SOLUTION.

[1] The centroids of its faces

3.2 The cylindrical coordinate System

In order to have a valid coordinate system in the 3-dimensional case, each point of the space must be associated with a unique triple of real numbers (the coordinates of the point) and each triple of real numbers must determine a unique point, as in the case of the Cartesian system of coordinates.

Let $P(x, y, z)$ be a point in a Cartesian system of coordinates $Oxyz$ and P' be the orthogonal projection of P on the plane xOy. One can associate to the point P the triple (r, θ, z), where (r, θ) are the polar coordinates of P' (see Figure 1). The polar coordinates of P' can be obtained by specifying the distance ρ from O to P' and the angle θ (measured in radians), whose ”initial” side is the polar axis, i.e. the x-axis, and whose ”terminal” side is the ray OP. The *polar coordinates* of the point P are (ρ, θ) (See also section (3.6.2) of the Appendix). The triple (r, θ, z) gives the *cylindrical coordinates* of the point P. There is the bijection

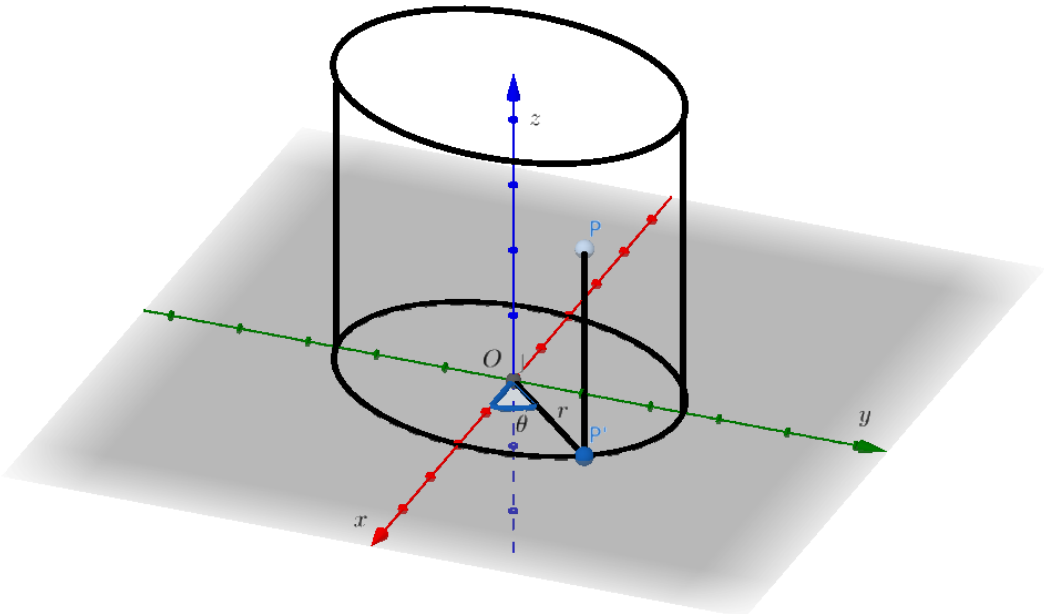


Figure 1: cylindrical coordinates

$$h_1 : \mathcal{P} \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R}, \ P \to (r, \theta, z)$$

and one obtains a new coordinate system, named the *cylindrical coordinate system* in $\mathcal{P}$. For the conversion formulas between the cylindrical coordinates and the Cartesian coordinates we refere the reader to [1, p. 19]. Note however that once we have the cylindrical coordinates (r, θ, z) of a point P, then its Cartesian coordinates are

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}.$$

3.3 The spherical coordinate system

Another way to associate to each point P in $\mathcal{P}$ a triple of real numbers is illustrated in Figure 2. If $P(x, y, z)$ is a point in a rectangular system of coordinates $Oxyz$ and P' its orthogonal projection on Oxy, let ρ be the length of the segment $[OP]$, θ be the oriented angle determined by $[Ox$ and $[OP'$ and φ be the oriented angle between $[Oz$ and $[OP$. The triple

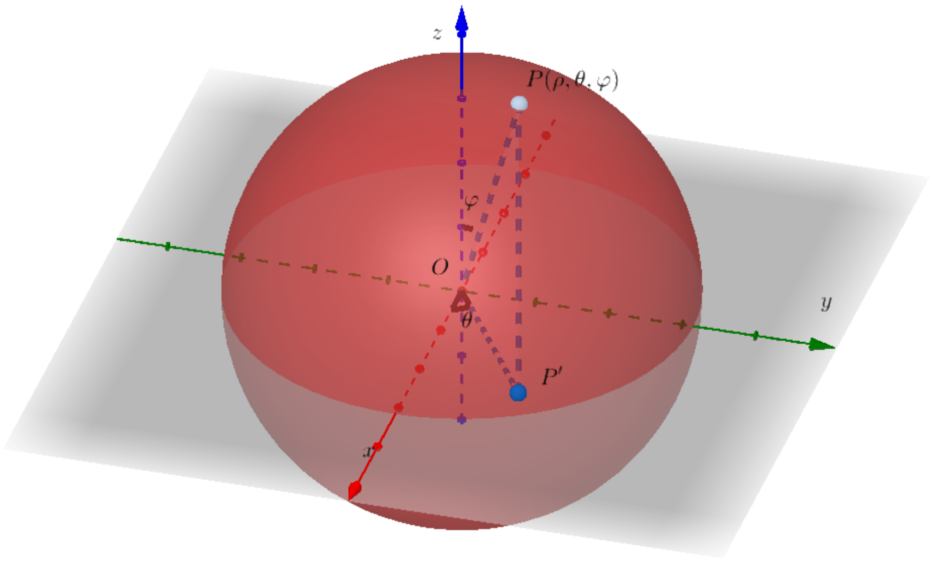


Figure 2: spherical coordinates

(ρ,θ,φ) gives the *spherical coordinates* of the point P. This way, one obtains the bijection

$$h_2 : \mathcal{P} \setminus \{O\} \to \mathbb{R}_+ \times [0,2\pi) \times [0,\pi], P \to (\rho,\theta,\varphi),$$

which defines a new coordinate system in $\mathcal{P}$, called the *spherical coordinate system* . For the conversion formulas between the spherical coordinate system and the Cartesian coordinate system we refere the reader to [1, p. 20]. Note however that once we have the cylindrical coordinates (ρ,θ,φ) of a point P, then its Cartesian coordinates are

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}.$$

3.4 The Cartesian equations of the straight lines

Let Δ be the straight line passing through the point $A_0(x_0,y_0,z_0)$ which is parallel to the vector $\overrightarrow{d}\,(p,q,r)$. Its vector equation is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda\,\overrightarrow{d},\ \lambda \in \mathbb{R}. \tag{3.1}$$

Denoting by x,y,z the coordinates of the generic point M of the straight line Δ, its vector equation (3.1) is equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases},\ \lambda \in \mathbb{R} \tag{3.2}$$

Indeed, the vector equation of Δ can be written, in terms of the coordinates of the vectors $\vec{r}_M$, $\vec{r}_{A_0}$ and $\vec{d}$, as follows:

$$\begin{aligned} & x\,\vec{u}+y\,\vec{v}+z\,\vec{w}=x_0\,\vec{u}+y_0\,\vec{v}+z_0\,\vec{w}+\lambda(p\,\vec{u}+q\,\vec{v}+r\,\vec{w}) \\ \Longleftrightarrow\ & x\,\vec{u}+y\,\vec{v}+z\,\vec{w}=(x_0+p\lambda)\,\vec{u}+(y_0+q\lambda)\,\vec{v}+(z_0+r\lambda)\,\vec{w},\ \lambda\in\mathbb{R} \end{aligned}$$

which is obviously equivalent to (3.2). The relations (3.2) are called the *parametric equations* of the straight line Δ and they are equivalent to the following relations

$$\frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r} \tag{3.3}$$

If $r=0$, for instance, the canonical equations of the straight line Δ are

$$\frac{x-x_0}{p}=\frac{y-y_0}{q}\ \wedge\ z=z_0.$$

If $A(x_A,y_A,z_A)$, $B(x_B,y_B,z_B)$ are different points of the line Δ, then

$$\overrightarrow{AB}\ (x_B-x_A,y_B-y_A,z_B-z_A)$$

is a director vector of Δ, its canonical equations having, in this case, the form

$$\frac{x-x_A}{x_B-x_A}=\frac{y-y_A}{y_B-y_A}=\frac{z-z_A}{z_B-z_A}. \tag{3.4}$$

Example 3.2. Consider the tetrahedrom $ABCD$, where $A(1,-1,1)$, $B(-1,1,-1)$, $C(2,1,-1)$ and $D(1,1,2)$, as well as the centroids G_A, G_B, G_C, G_D of the triangles BCD, ACD, ABD and ABC[2] respectively. Show that the medians AG_A, BG_B, CG_C and DG_D are concurrent and find the coordinates of their intersection point.

SOLUTION. One can easily see that the coordinates of the centroids G_A, G_B, G_C, G_D are $(2/3,1,0)$, $(4/3,1/3,2/3)$, $(1/3,1/3,2/3)$ and $(2/3,1/3,-1/3)$ respectively. The equations of the medians AG_A and BG_B are

$$\begin{aligned} &(AG_A)\ \frac{x-1}{2/3-1}=\frac{y+1}{1-(-1)}=\frac{z-1}{0-1}\Longleftrightarrow\frac{x-1}{-1/3}=\frac{y+1}{2}=\frac{z-1}{-1} \\ &(BG_B)\ \frac{x+1}{4/3+1}=\frac{y-1}{1/3-1}=\frac{z+1}{2/3+1}\Longleftrightarrow\frac{x+1}{7/3}=\frac{y-1}{-2/3}=\frac{z+1}{5/3}. \end{aligned}$$

Thus, the director space of the median AG_A is $\left\langle\left(-\frac{1}{3},2,-1\right)\right\rangle=\langle(-1,6,-3)\rangle$ and the director space of the median BG_B is $\left\langle\left(\frac{7}{3},-\frac{2}{3},\frac{5}{3}\right)\right\rangle=\langle(7,-2,5)\rangle$. Consequently, the parametric equations of the medians AG_A and BG_B are

$$(AG_A)\begin{cases} x=1-t \\ y=-1+6t \\ z=1-3t \end{cases},\ t\in\mathbb{R}\text{ and }(BG_B)\begin{cases} x=-1+7s \\ y=1-2s \\ z=-1+5s \end{cases},\ s\in\mathbb{R}.$$

Thus, the two medians AG_A and BG_B are concurrent if and only if there exist $s,t\in\mathbb{R}$ such that

$$\begin{cases} 1-t=-1+7s \\ -1+6t=1-2s \\ 1-3t=-1+5s \end{cases}\Longleftrightarrow\begin{cases} 7s+t=2 \\ 2s+6t=2 \\ 5s+3t=2 \end{cases}\Longleftrightarrow\begin{cases} 7s+t=2 \\ s+3t=1 \\ 5s+3t=2. \end{cases}$$

[2]The centroids of its faces

This system is compatible and has the unique solution $s = t = \frac{1}{4}$, which shows that the two medians AG_A and BG_B are concurrent and

$$AG_A \cap BG_B = \left\{ G\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right) \right\}.$$

One can similarly show that $BG_B \cap CG_C = CG_C \cap AG_A = \left\{ G\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right) \right\}$.

Example 3.3 (Homework)**.** Consider the tetrahedrom $ABCD$, where $A(1,-1,1)$, $B(-1,1,-1)$, $C(2,1,-1)$ and $D(1,1,2)$, as well as the midpoints M, N, P, Q, R and S of its edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[CD]$ and $[DB]$ respectively. Show that the lines MR, PQ and NS are concurrent and find the coordinates of their intersection point.

SOLUTION.

3.5 The Cartesian equations of the planes

Let $A_0(x_0, y_0, z_0) \in \mathcal{P}$ and $\overrightarrow{d}_1\,(p_1, q_1, r_1)$, $\overrightarrow{d}_2\,(p_2, q_2, r_2) \in \mathcal{V}$ be linearly independent vectors, that is

$$\operatorname{rank}\begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix} = 2.$$

The vector equation of the plane π passing through A_0 which is parallel to the vectors $\overrightarrow{d}_1$ (p_1, q_1, r_1), $\overrightarrow{d}_2\,(p_2, q_2, r_2)$ is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda_1 \overrightarrow{d}_1 + \lambda_2 \overrightarrow{d}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.5}$$

If we denote by x, y, z the coordinates of the generic point M of the plane π, then the vector equation (3.5) is the equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda_1 p_1 + \lambda_2 p_2 \\ y = y_0 + \lambda_1 q_1 + \lambda_2 q_2 \\ z = z_0 + \lambda_1 r_1 + \lambda_2 r_2 \end{cases},\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.6}$$

Indeed, the vector equation of π can be written, in terms of the coordinates of the vectors $\overrightarrow{r}_M$, $\overrightarrow{r}_{A_0}$, $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$, as follows:

$$x\overrightarrow{u}+y\overrightarrow{v}+z\overrightarrow{w}=x_0\overrightarrow{u}+y_0\overrightarrow{v}+z_0\overrightarrow{w}+\lambda_1(p_1\overrightarrow{u}+q_1\overrightarrow{v}+r_1\overrightarrow{w})+\lambda_2(p_2\overrightarrow{u}+q_2\overrightarrow{v}+r_2\overrightarrow{w})$$
$$\Longleftrightarrow x\overrightarrow{u}+y\overrightarrow{v}+z\overrightarrow{w}=(x_0+\lambda_1p_1+\lambda_2p_2)\overrightarrow{u}+(y_0+\lambda_1q_1+\lambda_2q_2)\overrightarrow{v}+(z_0+\lambda_1r_1+\lambda_2r_2)\overrightarrow{w},$$
$$\lambda_1,\lambda_2\in\mathbb{R},$$

which is obviously equivalent to (3.6). The relations (3.6) characterize the points of the plane π and are called the *parametric equations* of the plane π. More precisely, the compatibility of the linear system (3.6) with the unknowns λ_1, λ_2 is a necessary and sufficient condition for the point $M(x,y,z)$ to be contained within the plane π. On the other hand the compatibility of the linear system (3.6) is equivalent to

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}=0, \tag{3.7}$$

which expresses the equality between the rank of the coefficient matrix of the system and the rank of the extended matrix of the system. The equation (3.7) is a characterization of the points of the plane π in terms of the Cartesian coordinates of the generic point M and is called the *cartesian equation* of the plane π. On can put the equation (3.7) in the form

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \text{ or} \tag{3.8}$$

$$Ax+By+Cz+D=0, \tag{3.9}$$

where the coefficients A,B,C satisfy the relation $A^2+B^2+C^2>0$. It is also easy to show that every equation of the form (3.9) represents the equation of a plane. Indeed, if $A\neq 0$, then the equation (3.9) is equivalent to

$$\begin{vmatrix} x+\frac{D}{A} & y & z \\ B & -A & 0 \\ C & 0 & -A \end{vmatrix}=0.$$

We observe that one can put the equation (3.8) in the form

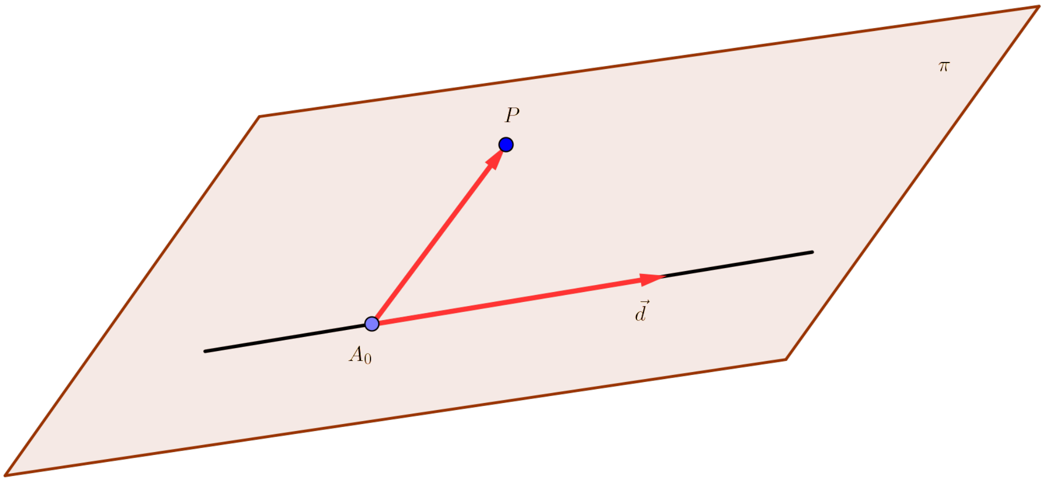
$$AX+BY+CZ=0 \tag{3.10}$$

where $X=x-x_0$, $Y=y-y_0$, $Z=z-z_0$ are the coordinates of the vector $\overrightarrow{A_0M}$.

Example 3.4. Write the equation of the plane determined by the point $P(-1,1,2)$ and the line (Δ) $\frac{x-1}{3}=\frac{y}{2}=\frac{z+1}{-1}$.

SOLUTION. Note that $P\notin\Delta$, as $\frac{-1-1}{3}\neq\frac{1}{2}\neq-3=\frac{2+1}{-1}$, i.e. the point P and the line Δ determine, indeed, a plane, say π. One can regard π as the plane through the point $A_0(1,0,-1)$ which is parallel to the vectors $\overrightarrow{A_0P}$ $(-1-1,1-0,2-(-1))=\overrightarrow{A_0P}$ $(-2,1,3)$ and $\overrightarrow{d}$ $(3,2,-1)$. Thus, the equation of π is

$$\begin{vmatrix} x-1 & y & z+1 \\ -2 & 1 & 3 \\ 3 & 2 & -1 \end{vmatrix}=0\Longleftrightarrow x-y+z=0.$$



Example 3.5 (Homework)**.** Generalize Example 3.4: Write the equation of the plane determined by the line $(\Delta)\ \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$ and the point $M(x_M, y_M, z_M) \notin \Delta$.

SOLUTION.

Remark 3.3. If $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ are noncollinear points, then the plane (ABC) determined by the three points can be viewed as the plane passing through the point A which is parallel to the vectors $\vec{d}_1 = \overrightarrow{AB}$, $\vec{d}_2 = \overrightarrow{AC}$. The coordinates of the vectors $\vec{d}_1$ şi $\vec{d}_2$ are

$$(x_B - x_A, y_B - y_A, z_B - z_A) \text{ and } (x_C - x_A, y_C - y_A, z_C - z_A) \text{ respectively.}$$

Thus, the equation of the plane (ABC) is

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0, \tag{3.11}$$

or, echivalently

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0. \tag{3.12}$$

Thus, four points $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ and $D(x_D, y_D, z_D)$ are coplanar if and ony if

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \tag{3.13}$$

Example 3.6 (Homework)**.** Write the equation of the plane determined by the points $M_1(3,-2,1)$, $M_2(5,4,1)$ and $M_3(-1,-2,3)$.

SOLUTION.

Remark 3.4. If $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ are three points ($abc \neq 0$), then for the equation of the plane (ABC) we have successively:

$$\begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & z-c & 1 \\ a & 0 & -c & 1 \\ 0 & b & -c & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & z-c \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = 0$$

$$\Longleftrightarrow ab(z-c) + bcx + acy = 0 \Longleftrightarrow bcx + acy + abz = abc$$

$$\Longleftrightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{3.14}$$

The equation (3.14) of the plane (ABC) is said to be in *intercept form* and the x, y, z-intercepts of the plane (ABC) are a, b, c respectively.

Example 3.7 (Homework)**.** Write the equation of the plane (π) $3x - 4y + 6z - 24 = 0$ in intercept form.

SOLUTION.

3.6 Appendix: The Cartesian equations of lines in the two dimensional setting

3.6.1 Cartesian and affine reference systems

If $b = [\overrightarrow{e}, \overrightarrow{f}]$ is an ordered basis of the director subspace $\overrightarrow{\pi}$ of the plane π and $\overrightarrow{x} \in \overrightarrow{\pi}$, recall that the column vector of $\overrightarrow{x}$ with respect to b is being denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

whenever $\overrightarrow{x}= x_1 \overrightarrow{e} + x_2 \overrightarrow{f}$.

Definition 3.3. A *cartesian reference system* of the plane π, is a system $R = (O, \overrightarrow{e}, \overrightarrow{f})$, where O is a point from π called the *origin* of the reference system and $b = [\overrightarrow{e}, \overrightarrow{f}]$ is a basis of the vector space $\overrightarrow{\pi}$.

Denote by E, F the points for which $\overrightarrow{e}=\overrightarrow{OE}$, $\overrightarrow{f}=\overrightarrow{OF}$.

Definition 3.4. The system of points (O, E, F) is called *the affine reference system associated to the cartesian reference system* $R = (O, \overrightarrow{e}, \overrightarrow{f})$.

The straight lines OE, OF, oriented from O to E and from O to F respectively, are called *the coordinate axes*. The coordinates x, y of the position vector $\overrightarrow{r}_M=\overrightarrow{OM}$ with respect to the basis $[\overrightarrow{e}, \overrightarrow{f}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y)$. Also, for the column matrix of coordinates of the vector $\overrightarrow{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\overrightarrow{r}_M= x \overrightarrow{e} + y \overrightarrow{f}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Remark 3.5. If $A(x_A, y_A)$, $B(x_B, y_B)$ are two points, then

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}= x_B \overrightarrow{e} + y_B \overrightarrow{f} - (x_A \overrightarrow{e} + y_A \overrightarrow{f}) \\ &= (x_B - x_A) \overrightarrow{e} + (y_B - y_A) \overrightarrow{f}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

3.6.2 The Polar Coordinate System [1, p. 17]

As an alternative to a Cartesian coordinate system (RS) one considers in the plane π a fixed point O, called *pole* and a half-line directed to the right of O, called *polar axis* (see Figure 3). By specifying the distance ρ from O to a point P and an angle θ (measured in radians), whose

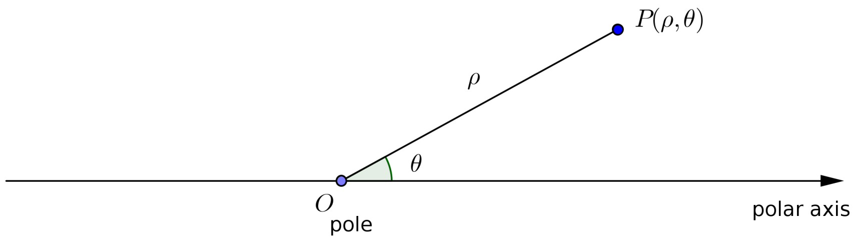


Figure 3: The pole and the polar axis related to a polar coordinate system

”initial” side is the polar axis and whose ”terminal” side is the ray OP, the *polar coordinates* of the point P are (ρ, θ). One obtains a bijection

$$\pi \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi), \quad P \to (\rho, \theta)$$

which associates to any point P in $\pi \setminus \{O\}$ the pair (ρ, θ) (suppose that $O(0,0)$). The positive real number ρ is called the *polar ray* of P and θ is called the *polar angle* of P.

Consider the Cartesian coordinate system in π, whose origin O is the pole and whose positive half-axis Ox is the polar axis (see Figure 4). The following transformation formulas give the connection between the coordinates of an arbitrary point in the two systems of coordinates.

Let P be a given point of polar coordinates (ρ, θ) . Its Cartesian coordinates are

$$\begin{cases} x_P = \rho\cos\theta \\ y_P = \rho\sin\theta \end{cases}. \tag{3.15}$$

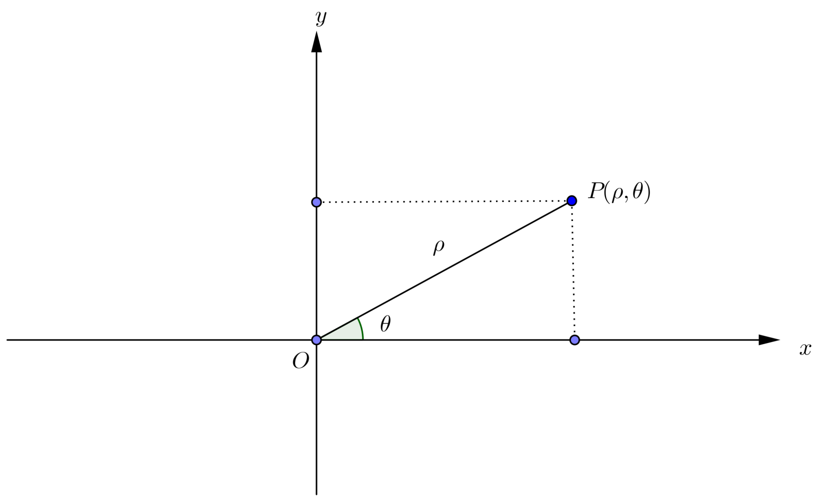


Figure 4: Polar coordinates

Let P be a point of Cartesian coordinates (x, y). It is clear that the polar ray of P is given by the formula

$$\rho = \sqrt{x^2 + y^2}. \tag{3.16}$$

In order to obtain the polar angle of P, it must be considered the quadrant where P is situated. One obtains the following formulas:

Case 1. If $x \neq 0$, then using $\tan\theta = \dfrac{y}{x}$, one has

$$\theta = \arctan\frac{y}{x} + k\pi, \quad \text{where} \quad k = \begin{cases} 0 \text{ if } P \in I \cup]Ox \\ 1 \text{ if } P \in II \cup III \cup]Ox' \\ 2 \text{ if } P \in IV; \end{cases};$$

Case 2. If $x = 0$ and $y \neq 0$, then

$$\theta = \begin{cases} \dfrac{\pi}{2} \text{ when } P \in]Oy \\ \dfrac{3\pi}{2} \text{ when } P \in]Oy'; \end{cases}$$

Case 3. If $x = 0$ and $y = 0$, then $\theta = 0$.

3.6.3 Parametric and Cartesian equations of Lines

Let Δ be a line passing through the point $A_0(x_0, y_0) \in \pi$ which is parallel to the vector $\overrightarrow{d}\ (p, q)$. Its vector equation is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + t\,\overrightarrow{d},\ t \in \mathbb{R}. \tag{3.17}$$

If (x, y) are the coordinates of a generic point $M \in \Delta$, then its vector equation (3.17) is equivalent to the following system

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \end{cases},\ t \in \mathbb{R}. \tag{3.18}$$

The relations are called the *parametric equations* of the line Δ and they are equivalent to the following equation

$$\frac{x - x_0}{p} = \frac{y - y_0}{q}, \tag{3.19}$$

called the *canonical equation*of Δ. If $q = 0$, then the equation (3.19) becomes $y = y_0$.

If $A(x_A, y_A)$ are two different points of the plane π, then $\overrightarrow{AB}\ (x_B - x_A, y_B - y_A)$ is a director vector of the line AB and the canonical equation of the line AB is

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}. \tag{3.20}$$

The equation (3.20) is equivalent to

$$\begin{vmatrix} x - x_A & y - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x - x_A & y - y_A & 1 \\ x_B - x_A & y_B - y_A & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} = 0.$$

Thus, three poins $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{3.21}$$

3.6.4 General Equations of Lines

We can put the equation (3.19) in the form

$$ax + by + c = 0, \qquad \text{with} \quad a^2 + a^2 > 0, \tag{3.22}$$

which means that any line from π is characterized by a first degree equation. Conversely, such of an equation represents a line, since the formula (3.22) is equivalent to

$$\frac{x + \frac{c}{a}}{-\frac{b}{a}} = \frac{y}{1}$$

and this is the *symmetric equation* of the line passing through $P_0\left(-\frac{c}{a}, 0\right)$ and parallel to $\overline{v}\left(-\frac{b}{a}, 1\right)$. The equation (3.22) is called *general equation* of the line.

Remark 3.6. The lines

$$(d)\ ax + by + c = 0 \text{ and } (\Delta)\ \frac{x - x_0}{p} = \frac{x - x_0}{q}$$

are parallel if and only if $ap + bq = 0$. Indeed, we have:

$$d \| \Delta \iff \overrightarrow{d} = \overrightarrow{\Delta} \iff \langle \overrightarrow{u}\ (p,q) \rangle = \langle \overrightarrow{v}\ \left(-\tfrac{b}{a}, 1\right) \rangle \iff \exists t \in \mathbb{R} \text{ s.t. } \overrightarrow{u}\ (p,q) = t\ \overrightarrow{v}\ \left(-\tfrac{b}{a}, 1\right)$$
$$\iff \exists t \in \mathbb{R} \text{ s.t. } = -t\tfrac{b}{a} \text{ and } q = t \iff ap + bq = 0.$$

3.6.5 Reduced Equations of Lines

Consider a line given by its general equation $Ax + By + C = 0$, where at least one of the coefficients A and B is nonzero. One may suppose that $B \neq 0$, so that the equation can be divided by B. One obtains

$$y = mx + n \tag{3.23}$$

which is said to be the *reduced equation* of the line.

Remark: If $B = 0$, (3.22) becomes $Ax + C = 0$, or $x = -\dfrac{C}{A}$, a line parallel to Oy. (In the same way, if $A = 0$, one obtains the equation of a line parallel to Ox).

Let d be a line of equation $y = mx + n$ in a Cartesian system of coordinates and suppose that the line is not parallel to Oy. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two different points on d and φ be the angle determined by d and Ox (see Figure 5); $\varphi \in [0, \pi] \setminus \{\pi/2\}$. The points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ belong to d, hence

$$\begin{cases} y_1 = mx_1 + n \\ y_2 = mx_2 + n, \end{cases}$$

and $x_2 \neq x_1$, since d is not parallel to Oy. Then,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan \varphi. \tag{3.24}$$

The number $m = \tan \varphi$ is called the *angular coefficient* of the line d. It is immediate that the equation of the line passing through the point $P_0(x_0, y_0)$ and of the given angular coefficient m is

$$y - y_0 = m(x - x_0). \tag{3.25}$$

3.6.6 Intersection of Two Lines

Let $d_1 : a_1x + b_1y + c_1 = 0$ and $d_2 : a_2x + b_2y + c_2 = 0$ be two lines in $\mathcal{E}_2$. The solution of the system of equation

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

will give the set of the intersection points of d_1 and d_2.

1) If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, the system has a unique solution (x_0, y_0) and the lines have a unique intersection point $P_0(x_0, y_0)$. They are *secant*.

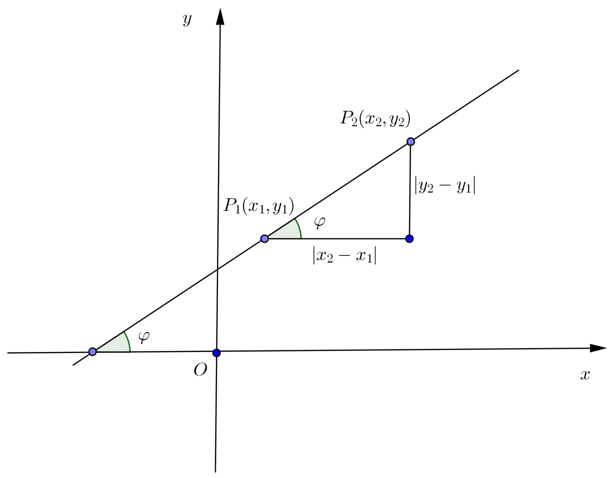


Figure 5:

2) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$, the system is not compatible, and the lines have no points in common. They are *parallel*.

3) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, the system has an infinity of solutions, and the lines coincide. They are *identical*.

If $d_i : a_i x + b_i y + c_i = 0$, $i = \overline{1,3}$ are three lines in $\mathcal{E}_2$, then they are concurrent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{3.26}$$

3.6.7 Bundles of Lines ([1])

The set of all the lines passing through a given point P_0 is said to be a *bundle* of lines. The point P_0 is called the *vertex* of the bundle.

If the point P_0 is of coordinates $P_0(x_0, y_0)$, then the equation of the bundle of vertex P_0 is

$$r(x - x_0) + s(y - y_0) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{3.27}$$

Remark: Tthe *reduced bundle* of line through P_0 is,

$$y - y_0 = m(x - x_0), \qquad m \in \mathbb{R}, \tag{3.28}$$

and covers the bundle of lines through P_0, except the line $x = x_0$. Similarly, the family of lines

$$x - x_0 = k(y - y_0), \qquad k \in \mathbb{R}, \tag{3.29}$$

covers the bundle of lines through P_0, exceptthe line $y = y_0$.

If the point P_0 is given as the intersection of two lines, then its coordinates are the solution of the system

$$\begin{cases} d_1 : a_1 x + b_1 y + c_1 = 0 \\ d_2 : a_2 x + b_2 y + c_2 = 0 \end{cases},$$

assumed to be compatible. The equation of the bundle of lines through P_0 is

$$r(a_1x + b_1y + c_1) + s(a_2x + b_2y + c_2) = 0, \qquad (r,s) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{3.30}$$

Remark: As before, if $r \neq 0$ (or $s \neq 0$), one obtains the reduced equation of the bundle, containing all the lines through P_0, except d_1 (respectively d_2).

3.6.8 The Angle of Two Lines ([1])

Let d_1 and d_2 be two concurrent lines, given by their reduced equations:

$$d_1 : y = m_1x + n_1 \quad \text{and} \quad d_2 : y = m_2x + n_2.$$

The angular coefficients of d_1 and d_2 are $m_1 = \tan\varphi_1$ and $m_2 = \tan\varphi_2$ (see Figure 6). One may suppose that $\varphi_1 \neq \frac{\pi}{2}$, $\varphi_2 \neq \frac{\pi}{2}$, $\varphi_2 \geq \varphi_1$, such that $\varphi = \varphi_2 - \varphi_1 \in [0,\pi] \setminus \{\frac{\pi}{2}\}$.

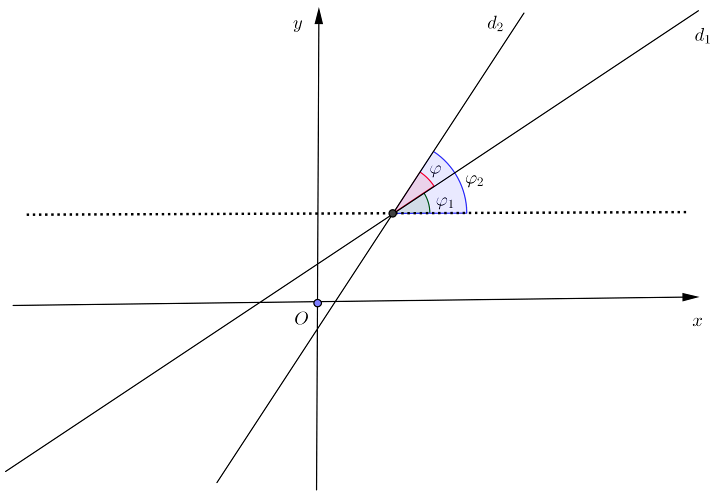


Figure 6:

The angle determined by d_1 and d_2 is given by

$$\tan\varphi = \tan(\varphi_2 - \varphi_1) = \frac{\tan\varphi_2 - \tan\varphi_1}{1 + \tan\varphi_1 \tan\varphi_2},$$

hence

$$\tan\varphi = \frac{m_2 - m_2}{1 + m_1m_2}. \tag{3.31}$$

1) The lines d_1 and d_2 are parallel if and only if $\tan\varphi = 0$, therefore

$$d_1 \parallel d_2 \iff m_1 = m_2. \tag{3.32}$$

2) The lines d_1 and d_2 are orthogonal if and only if they determine an angle of $\frac{\pi}{2}$, hence

$$d_1 \perp d_2 \iff m_1m_2 + 1 = 0. \tag{3.33}$$

3.7 Problems

1. Write the equation of the plane which passes through $M_0(1,-2,3)$ and is parallel to the vectors $\overrightarrow{v}_1\,(1,-1,0)$ and $\overrightarrow{v}_2\,(-3,2,4)$.

   HINT.
$$\begin{vmatrix} x-0 & y+2 & z-3 \\ 1 & -1 & 0 \\ -3 & 2 & 4 \end{vmatrix} = 0.$$

   vspace*2cm

2. Write the equation of the line which passes through $A(1,-2,6)$ and is parallel to

   (a) The x-axis;

   (b) The line (d_1) $\dfrac{x-1}{2} = \dfrac{y+5}{-3} = \dfrac{z-1}{4}$.

   (c) The vector $\overrightarrow{v}\,(1,0,2)$.

   SOLUTION.

3. Write the equation of the plane which contains the line

$$(d_1)\ \frac{x-3}{2}=\frac{y+4}{1}=\frac{z-2}{-3}$$

and is parallel to the line

$$(d_2)\ \frac{x+5}{2}=\frac{y-2}{2}=\frac{z-1}{2}.$$

HINT.

$$\begin{vmatrix} x-3 & y+4 & z-2 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{vmatrix}=0.$$

4. Consider the points $A(\alpha,0,0)$, $B(0,\beta,0)$ and $C(0,0,\gamma)$ such that

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{1}{a} \text{ where } a \text{ is a constatnt.}$$

Show that the plane (A,B,C) passes through a fixed point.

SOLUTION. The equation of the plane (ABC) can be written in intercept form, namely

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1.$$

The given relation shows that the point $P(a,a,a) \in (ABC)$ whenever α,β,γ verifies the given relation.

5. Write the equation of the line which passes through the point $M(1,0,7)$, is parallel to the plane (π) $3x-y+2z-15=0$ and intersects the line

$$(d)\ \frac{x-1}{4}=\frac{y-3}{2}=\frac{z}{1}.$$

Solution.

6. Write the equation of the plane which passes through $M_0(1,-2,3)$ and cuts the positive coordinate axes through equal intercepts.

   SOLUTION. The general equation of such a plane is $x+y+z=a$. In this particular case $a=1+(-2)+3=2$ and the equation of the required plane is $x+y+z=2$.

7. Write the equation of the plane which passes through $A(1,2,1)$ and is parallel to the straight lines

$$(d_1)\left\{\begin{array}{ccccccccc} x & + & 2y & - & z & + & 1 & = 0 \\ x & - & y & + & z & - & 1 & = 0 \end{array}\right. \quad (d_2)\left\{\begin{array}{cccccc} 2x & - & y & + & z & = 1 \\ x & - & y & + & z & = 0. \end{array}\right.$$

   SOLUTION. We need to find some director parameters of the lines (d_1) and (d_2). In this respect we may solve the two systems. The general solution of the first system is

$$\left\{\begin{array}{l} x=-\frac{1}{3}t+\frac{1}{3} \\ y=\frac{2}{3}t-\frac{2}{3} \\ z=t \end{array}\right. , \ t\in\mathbb{R}$$

   and the general solution of the second system is

$$\left\{\begin{array}{l} x=1 \\ y=t+1 \\ z=t \end{array}\right. , \ t\in\mathbb{R}$$

   and these are the parametric equations of the lines (d_1) and (d_2). Thus, the direction of the line (d_1) is the 1-dimensional subspace

$$\left\langle\left(-\frac{1}{3},\frac{2}{3},1\right)\right\rangle = \langle(-1,2,3)\rangle,$$

   and the direction of the line (d_2) is the 1-dimensional subspace $\langle(0,1,1)\rangle$.

   Consequently, some director parameters of the line (d_1) are $p_1=-1, q_1=2, r_1=3$ and some director parameters of the line (d_2) are $p_2=0, q_2=r_2=1$. Finaly, the equation of the required plane is

$$\begin{vmatrix} x-1 & y-2 & z-1 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = 0.$$

   The computation of the determinant is left to the reader.

   A few questions in the two dimensional setting ([1])

8. The sides $[BC]$, $[CA]$, $[AB]$ of the triangle ΔABC are divided by the points M, N respectively P into the same ratio k. Prove that the triangles ΔABC and ΔMNP have the same center of gravity.

   SOLUTION.

9. Sketch the graph of $x^2 - 4xy + 3y^2 = 0$.

   SOLUTION.

10. Find the equation of the line passing through the intersection point of the lines

$$d_1 : 2x - 5y - 1 = 0, \quad d_2 : x + 4y - 7 = 0$$

and through a point M which divides the segment $[AB]$, $A(4, -3)$, $B(-1, 2)$, into the ratio $k = \frac{2}{3}$.

SOLUTION.

11. Let A be a mobile point on the Ox axis and B a mobile point on Oy, so that $\dfrac{1}{OA}+\dfrac{1}{OB}=k$ (constant). Prove that the lines AB passes through a fixed point.

    SOLUTION.

12. Find the equation of the line passing through the intersection point of

$$d_1 : 3x - 2y + 5 = 0, \quad d_2 : 4x + 3y - 1 = 0$$

and crossing the positive half axis of Oy at the point A with $OA = 3$.

SOLUTION.

13. Find the parametric equations of the line through P_1 and P_2, when

    (a) $P_1(3,-2)$, $P_2(5,1)$;

    (b) $P_1(4,1)$, $P_2(4,3)$.

    SOLUTION.

14. Find the parametric equations of the line through $P(-5,2)$ and parallel to $\overline{v}(2,3)$.
SOLUTION.

15. Show that the equations

$$x = 3 - t, y = 1 + 2t \qquad \text{and} \qquad x = -1 + 3t, y = 9 - 6t$$

represent the same line.

SOLUTION.

16. Find the vector equation of the line P_1P_2, where
    (a) $P_1(2,-1)$, $P_2(-5,3)$;
    (b) $P_1(0,3)$, $P_2(4,3)$.

    SOLUTION.

17. Given the line $d : 2x + 3y + 4 = 0$, find the equation of a line d_1 through the point $M_0(2, 1)$, in the following situations:

    (a) d_1 is parallel with d;

    (b) d_1 is orthogonal on d;

    (c) the angle determined by d and d_1 is $\varphi = \frac{\pi}{4}$.

    SOLUTION.

18. The vertices of the triangle ΔABC are the intersection points of the lines

$$d_1 : 4x + 3y - 5 = 0, \quad d_2 : x - 3y + 10 = 0, \quad d_3 : x - 2 = 0.$$

(a) Find the coordinates of A, B, C.

(b) Find the equations of the median lines of the triangle.

(c) Find the equations of the heights of the triangle.

SOLUTION.

4 Week 4

4.1 Analytic conditions of parallelism and nonparallelism

4.1.1 The parallelism between a line and a plane

Proposition 4.1. *The equation of the director subspace $\overrightarrow{\pi}$, of the plane $\pi : Ax + By + Cz + D = 0$ is $AX + BY + CZ = 0$.*

Proof. We first recall that

$$\overrightarrow{\pi} = \{\overrightarrow{A_0M} \mid M \in \pi\}, \tag{4.1}$$

where $A_0 \in \pi$ is an arbitrary point, and the representation (4.1) of $\overrightarrow{\pi}$ is independent on the choice of $A_0 \in \pi$. According to equation (3.8), the equation of a plane π can be written in the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

where $A_0(x_0, y_0, z_0)$ is a point in π. In other words,

$$M(x, y, z) \in \pi \Longleftrightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

which shows that

$$\begin{aligned}\overrightarrow{\pi} &= \{\overrightarrow{A_0M}\ (x - x_0, y - y_0, z - z_0) \mid M(x, y, z) \in \pi\}\\ &= \{\overrightarrow{A_0M}\ (x - x_0, y - y_0, z - z_0) \mid A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\}\\ &= \{\overrightarrow{v}\ (X, Y, Z) \in \mathcal{V} \mid AX + BY + CZ = 0\}.\end{aligned}$$

Thus, the equation $AX + BY + CZ = 0$ is a necessary and sufficient condition for the vector $\overrightarrow{v}\ (X, Y, Z)$ to be contained within the direction of the plane

$$\pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

In other words, the *equation of the director subspace* $\overrightarrow{\pi}$ is $AX + BY + CZ = 0$. □

Corollary 4.2. *The straight line*

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

is parallel to the plane $\pi : Ax + By + Cz + D = 0$ if and only if

$$Ap + Bq + Cr = 0 \tag{4.2}$$

Proof. Indeed,

$$\begin{aligned}\Delta \| \pi &\Longleftrightarrow \overrightarrow{\Delta} \subseteq \overrightarrow{\pi} \Longleftrightarrow \langle (p, q, r) \rangle \subseteq \overrightarrow{\pi}\\ &\Longleftrightarrow \overrightarrow{d}\ (p, q, r) \in \overrightarrow{\pi} \Longleftrightarrow Ap + Bq + Cr = 0.\end{aligned}$$

□

4.1.2 The intersection point of a straight line and a plane

Proposition 4.3. *Consider a straight line*

$$d : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

and a plane $\pi : Ax + By + Cz + D = 0$ *which are not parallel to each other, i.e.*

$$Ap + Bq + Cr \neq 0.$$

The coordinates of the intersection point $d \cap \pi$ *are*

$$\begin{cases} x_0 - p\dfrac{F(x_0, y_0, z_0)}{Ap+Bq+Cr} \\ y_0 - q\dfrac{F(x_0, y_0, z_0)}{Ap+Bq+Cr} \\ z_0 - r\dfrac{F(x_0, y_0, z_0)}{Ap+Bq+Cr}, \end{cases} \tag{4.3}$$

where $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $F(x, y, z) = Ax + By + Cz + D$.

Proof. The parametric equations of (d) are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \ t \in \mathbb{R}. \tag{4.4}$$

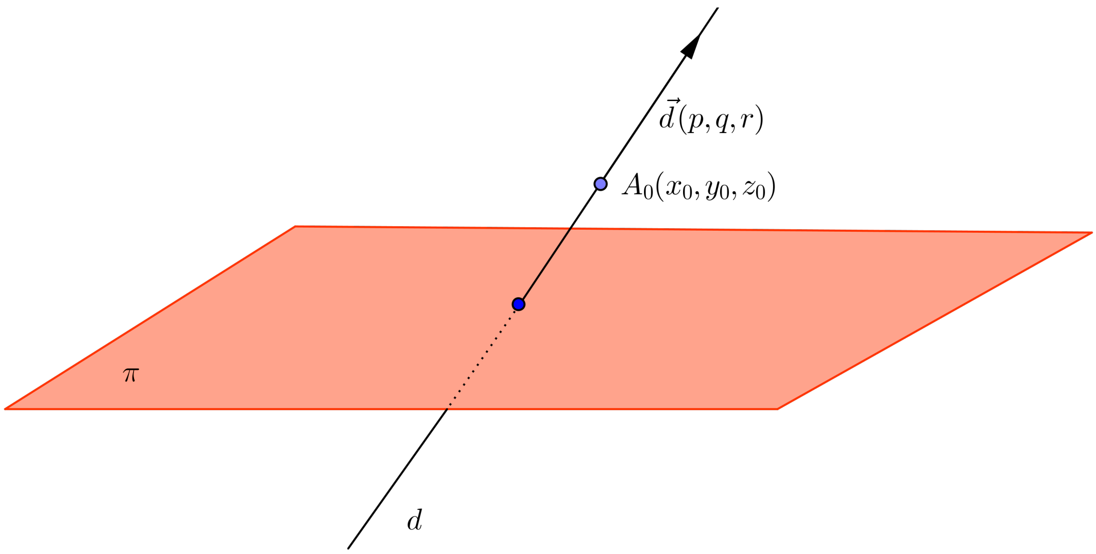
The unique value of $t \in \mathbb{R}$, which corresponds to the intersection point $d \cap \pi$, can be found by solving the equation

$$A(x_0 + pt) + B(y_0 + qt) + C(z_0 + rt) + D = 0.$$

Its unique solution is

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr} = -\frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr}$$

and can be used to obtain the required coordinates (4.3) by replacing this value in (4.4). □



Example 4.1 (Homework)**.** Decide whether the line d and the plane π are parallel or concurrent and find the coordinates of the intersection point of of Δ and π whenever $\Delta \nparallel \pi$:

1. $d : \frac{x+2}{1} = \frac{y-1}{3} = \frac{z-3}{1}$ and $\pi : x - y + 2z = 1$.
2. $d : \frac{x-3}{1} = \frac{y+1}{-2} = \frac{z-2}{-1}$ and $\pi : 2x - y + 3z - 1 = 0$.

SOLUTION.

4.1.3 Parallelism of two planes

Proposition 4.4. *Consider the planes*

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0,\ (\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0.$$

Then $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) \in \{1, 2\}$ *and the following statemenets are equivalent*

1. $\pi_1 \| \pi_2$.
2. $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) = 2$, *i.e.* $\overrightarrow{\pi}_1 = \overrightarrow{\pi}_2$.
3. *rank* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1$.
4. *The vectors* (A_1, B_1, C_1), $(A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly dependent.*

Remark 4.1. Note that

$$\text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1 \Leftrightarrow \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = 0$$
$$\Leftrightarrow A_1B_2 - A_2B_1 = A_1C_2 - A_2C_1 = B_1C_2 - C_2B_1 = 0. \tag{4.5}$$

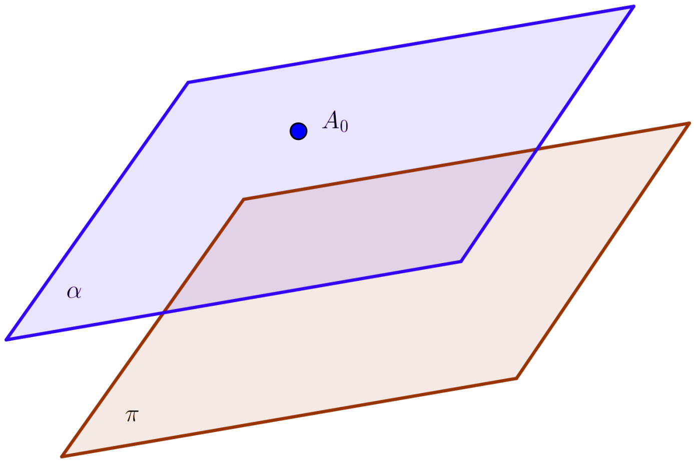
The relations (4.5) are often written in the form

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}, \tag{4.6}$$

although at most two of the coefficients A_2, B_2 or C_2 might be zero. In fact relations (4.6) should be understood in terms of linear dependence of the vectors (A_1, B_1, C_1), $(A_2, B_2, C_2) \in \mathbb{R}^3$, i.e. $(A_1, B_1, C_1) = k(A_2, B_2, C_2)$, where $k \in \mathbb{R}$ is the common value of those ratios (4.6) which do not involve any zero coefficients. Let us finally mention that the equivalences (4.5) prove the equivalence (3) $\Longleftrightarrow$ (4) of Proposition 4.4.

Example 4.2. The equation of the plane α passing through the point $A_0(x_0, y_0, z_0)$, which is parallel to the plane $\pi: \ Ax + By + Cz + D = 0$ is

$$\alpha: \ A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$



4.1.4 Straight lines as intersections of planes

Corollary 4.5. *Consider the planes*

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0,\ (\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0.$$

The following statements are equivalent

1. $\pi_1 \nparallel \pi_2$.
2. $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 1$.
3. $rank \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2.$
4. *The vectors* $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly independent.*

By using the characterization of parallelism between a line and a plane, given by Proposition 4.2, we shall find the direction of a straight line which is given as the intersection of two planes. Consider the planes $(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0$, $(\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0$ such that

$$\text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2,$$

alongside their intersection straight line $\Delta = \pi_1 \cap \pi_2$ of equations

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_1 = 0. \end{cases}$$

Thus, $\vec{\Delta} = \vec{\pi}_1 \cap \vec{\pi}_2$ and therefore, by means of some previous Proposition, it follows that the equations of $\vec{\Delta}$ are

$$(\vec{\Delta}) \begin{cases} A_1X + B_1Y + C_1Z = 0 \\ A_2X + B_2Y + C_2Z = 0. \end{cases} \tag{4.7}$$

By solving the system (4.7) one can therefore deduce that $\overrightarrow{d}\ (p,q,r) \in \overrightarrow{\Delta} \Leftrightarrow\ \exists\lambda \in \mathbb{R}$ such that

$$(p,q,r) = \lambda\left(\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}\right). \tag{4.8}$$

The relation is usually (4.8) written in the form

$$\frac{p}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{q}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}} = \frac{r}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \tag{4.9}$$

Let us finaly mention that we usually choose the values

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \text{ şi } \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \tag{4.10}$$

for the director parameters (p,q,r) of Δ.

Example 4.3. Write the equations of the plane through $P(4,-3,1)$ which is parallel to the lines

$$(\Delta_1)\left\{\begin{array}{rcrcl} 2x & & - z & + & 1 = 0 \\ & 3y & + 2z & - & 2 = 0. \end{array}\right. \text{ and } (\Delta_2)\left\{\begin{array}{rcrcrcl} x & + & y & + & z & = 0 \\ 2x & - & y & + & 3z & = 0. \end{array}\right.$$

SOLUTION. One can see the required plane as the one through $P(4,-3,1)$ which is parallel to the director vectors $\overrightarrow{d}_1\ (p_1,q_1,r_1)$ and $\overrightarrow{d}_2\ (p_2,q_2,r_2)$ of Δ_1 and Δ_2 respectively. One can choose

$$\begin{array}{lll} p_1 = \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3 & & p_2 = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4 \\ q_1 = \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} = -4 & \text{and} & q_2 = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1 \\ r_1 = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 & & r_2 = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3. \end{array}$$

Thus, the equation of the required plane is

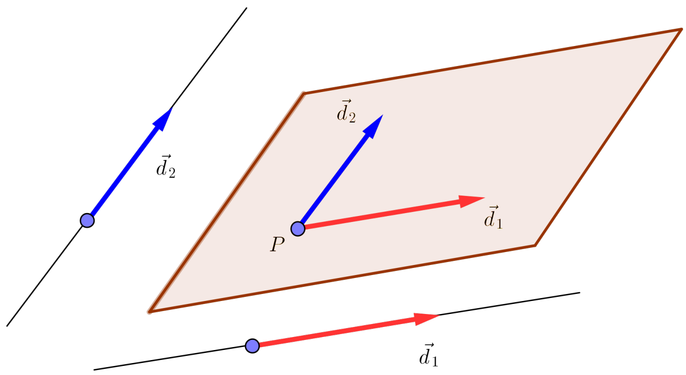


Figure 7:

$$\begin{vmatrix} x-4 & y+3 & z-1 \\ 3 & -4 & 6 \\ 4 & -1 & -3 \end{vmatrix} = 0 \iff 12(x-4) - 3(z-1) + 24(y+3) + 16(z-1) + 6(x-4) + 9(y+3) = 0$$
$$\iff 18(x-4) + 33(y+3) + 13(z-1) = 0$$
$$\iff 18x + 33y + 13z - 72 + 99 - 13 = 0$$
$$\iff 18x + 33y + 13z + 14 = 0.$$

4.2 Pencils of planes

Definition 4.1. The collection of all planes containing a given straight line

$$(\Delta)\begin{cases} A_1x+B_1y+C_1z+D_1=0\\ A_2x+B_2y+C_2z+D_2=0\end{cases}$$

is called the *pencil* or the *bundle* of planes through Δ.

Proposition 4.6. *The plane π belongs to the pencil of planes through the straight line Δ if and only if the equation of the plane π is*

$$\lambda(A_1x+B_1y+C_1z+D_1)+\mu(A_2x+B_2y+C_2z+D_2)=0. \tag{4.11}$$

for some $\lambda,\mu\in\mathbb{R}$ such that $\lambda^2+\mu^2>0$.

Proof. Every plane in the family (4.11) obviously contains the line Δ.

Conversely, assume that π is a plane through the line Δ. Consider a point $M\in\pi\setminus\Delta$ and recall that π is completely determined by Δ and M. On the other hand M and Δ are obviously contained in the plane $F_1(x_M,y_M,z_M)F_2(x,y,z)-F_2(x_M,y_M,z_M)F_1(x,y,z)=0$ of the family (4.11), where $F_1,F_2:\mathbb{R}^3\longrightarrow\mathbb{R}$, $F_i(x,y,z)=A_ix+B_iy+C_iz+D_i$, for $i=1,2$. Thus the plane π belongs to the family (4.11) and its equation is

$$F_1(x_M,y_M,z_M)F_2(x,y,z)-F_2(x_M,y_M,z_M)F_1(x,y,z)=0.$$

□

Remark 4.2. The family of planes $A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0$, where λ covers the whole real line $\mathbb{R}$, is the so called *reduced pencil of planes* through Δ and it consists in all planes through Δ except the plane of equation $A_2x+B_2y+C_2z+D_2=0$.

Example 4.4. Write the equations of the plane parallel to the line $d:\ x=2y=3z$ passing through the line

$$\Delta:\begin{cases} x+y+z=0\\ 2x-y+3z=0.\end{cases}$$

SOLUTION. Note that none of the planes $x+y+z=0$ and $x-y+3z=0$, passing through (Δ), is parallel to (d), as $1\cdot1+1\cdot\frac{1}{2}+1\cdot\frac{1}{3}\neq0$ and $2\cdot1+(-1)\cdot\frac{1}{2}+3\cdot\frac{1}{3}\neq0$. Thus, the required plane is in a reduced pencil of planes, such as the family $\pi_\lambda: x+y+z+\lambda(2x-y+3z)=0,\ \lambda\in\mathbb{R}$. The parallelism relation between (d) and $\pi_\lambda:(2\lambda+1)x+(1-\lambda)y+(3\lambda+1)z=0$ is

$$(2\lambda+1)\cdot1+(1-\lambda)\cdot\frac{1}{2}+(3\lambda+1)\cdot\frac{1}{3}=0\Longleftrightarrow 12\lambda+6+3-3\lambda+6\lambda+2=0\Longleftrightarrow\lambda=-\frac{11}{15}.$$

Thus, the required plane is

$$\pi_{-11/15}:\left(-2\frac{11}{15}+1\right)x+\left(1+\frac{11}{15}\right)y+\left(-3\frac{11}{15}+1\right)z=0\Longleftrightarrow -7x+26y-18z=0.$$

Appendix

4.3 Projections and symmetries

4.3.1 The projection on a plane parallel with a given line

Consider a straight line

$$d:\frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$$

and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{\pi,d} : \mathcal{P} \longrightarrow \pi$ of $\mathcal{P}$ on π parallel to d, whose value $p_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the intersection point between π and the line through M which is parallel to d. Due to relations (4.3), the coordinates of $p_{\pi,d}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_M - p\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ y_M - q\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ z_M - r\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr}, \end{cases} \tag{4.12}$$

where $F(x, y, z) = Ax + By + Cz + D$.

Consequently, the position vector of $p_{\pi,d}(M)$ is

$$\overrightarrow{Op_{\pi,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{Ap + Bq + Cr} \overrightarrow{d} . \tag{4.13}$$

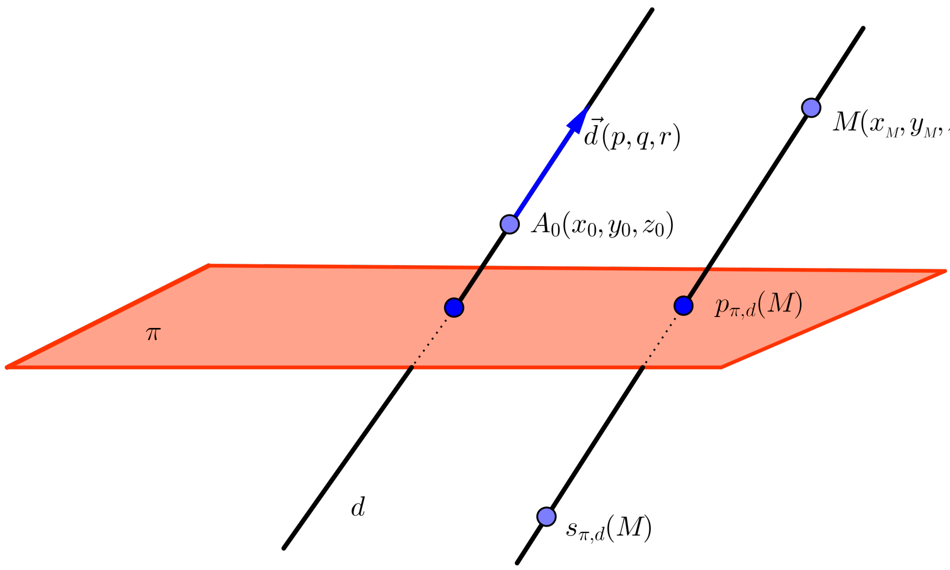
Proposition 4.7. *If $R = (O, b)$ is the Cartesian reference system behind the equations of the line*

$$(d)\ \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and the plane $(\pi)\ Ax + By + Cz + D = 0$, concurrent with (d), then

$$[p_{\pi,d}(M)]_R = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix} [M]_R - \frac{D}{Ap + Bq + Cr} [\overrightarrow{d}]_b,$$

where $\overrightarrow{d}\ (p, q, r)$ stands for the director vector of the line (d).



4.3.2 The symmetry with respect to a plane parallel with a given line

We call the function $s_{\pi,d} : \mathcal{P} \longrightarrow \mathcal{P}$, whose value $s_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $p_{\pi,d}(M)$ *the symmetry of $\mathcal{P}$ with respect to π parallel to d*. The direction of d is equally called the *direction* of the symmetry and π is called the *axis* of the symmetry. For the position vector of $s_{\pi,d}(M)$ we have

$$\overrightarrow{Op_{\pi,d}(M)}=\frac{\overrightarrow{OM}+\overrightarrow{Os_{\pi,d}(M)}}{2}, \text{ i.e.} \tag{4.14}$$

$$\overrightarrow{Os_{\pi,d}(M)}=2\,\overrightarrow{Op_{\pi,d}(M)}-\overrightarrow{OM}=\overrightarrow{OM}-2\frac{F(M)}{Ap+Bq+Cr}\overrightarrow{d}\,. \tag{4.15}$$

Proposition 4.8. *If $R=(O,b)$ is the Cartesian reference system behind the equations of the line*

$$(d)\ \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$$

and the plane $(\pi)\ Ax+By+Cz+D=0$, concurrent with (d), then

$$(Ap+Bq+Cr)[s_{\pi,d}(M)]_R=\begin{pmatrix} -Ap+Bq+Cr & -2Bp & -2Cp \\ -2Aq & Ap-Bq+Cr & -2Cq \\ -2Ar & -2Br & Ap+Bq-Cr \end{pmatrix}[M]_R-2D[\overrightarrow{d}]_b, \tag{4.16}$$

where $\overrightarrow{d}\,(p,q,r)$ stands for the director vector of the line (d).

4.3.3 The projection on a straight line parallel with a given plane

Consider a straight line

$$d:\frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$$

and a plane $\pi: Ax+By+Cz+D=0$ which are not parallel to each other, i.e.

$$Ap+Bq+Cr\neq 0.$$

For these given data we may define the projection $p_{d,\pi}:\mathcal{P}\longrightarrow d$ of $\mathcal{P}$ on d, whose value $p_{d,\pi}(M)$ at $M\in\mathcal{P}$ is the intersection point between d and the plane through M which is parallel to π. Due to relations (4.3), the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_0-p\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr} \\ y_0-q\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr} \\ z_0-r\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr}, \end{cases} \tag{4.17}$$

where $G_M(x,y,z)=A(x-x_M)+B(y-y_M)+C(z-z_M)$. Consequently, the position vector of $p_{d,\pi}(M)$ is

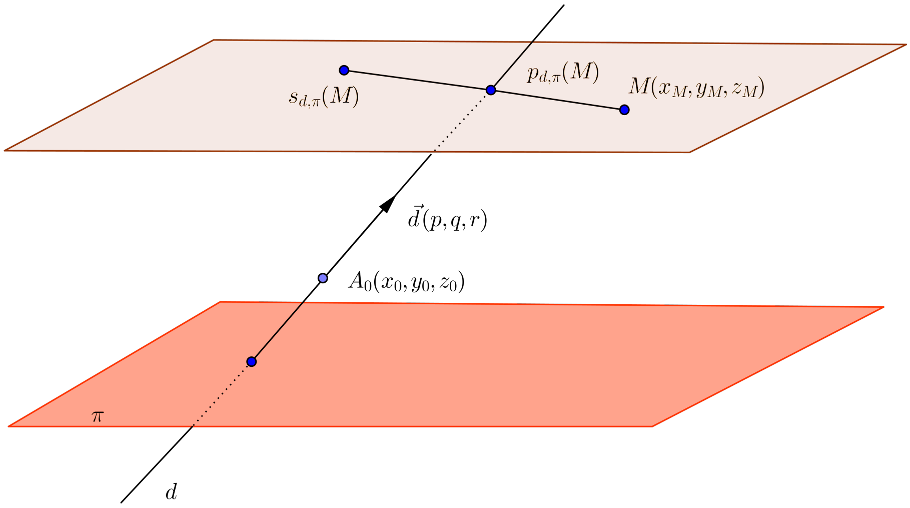
$$\overrightarrow{Op_{d,\pi}(M)}=\overrightarrow{OA_0}-\frac{G_M(A_0)}{Ap+Bq+Cr}\overrightarrow{d}, \text{ where } A_0(x_0,y_0,z_0). \tag{4.18}$$

Note that $G_M(A_0)=A(x_0-x_M)+B(y_0-y_M)+C(z_0-z_M)=F(A_0)-F(M)$, where $F(x,y,z)=Ax+By+Cz+D$. Consequently the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_0+p\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ y_0+q\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ z_0+r\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr}, \end{cases} \tag{4.19}$$

and the position vector of $p_{d,\pi}(M)$ is

$$\overrightarrow{Op_{d,\pi}(M)}=\overrightarrow{OA_0}+\frac{F(M)-F(A_0)}{Ap+Bq+Cr}\vec{d}, \text{ where } A_0(x_0,y_0,z_0). \tag{4.20}$$



4.3.4 The symmetry with respect to a line parallel with a plane

We call the function $s_{d,\pi}:\mathcal{P}\longrightarrow\mathcal{P}$, whose value $s_{d,\pi}(M)$ at $M\in\mathcal{P}$ is the symmetric point of M with respect to $p_{d,\pi}(M)$, *the symmetry of $\mathcal{P}$ with respect to d parallel to π*. The direction of π is equally called the *direction* of the symmetry and d is called the *axis* of the symmetry. For the position vector of $s_{d,\pi}(M)$ we have

$$\overrightarrow{Op_{d,\pi}(M)}=\frac{\overrightarrow{OM}+\overrightarrow{Os_{d,\pi}(M)}}{2}, \text{ i.e.} \tag{4.21}$$

$$\begin{aligned}\overrightarrow{Os_{d,\pi}(M)} &= 2\,\overrightarrow{Op_{d,\pi}(M)}-\overrightarrow{OM}\\ &= 2\,\overrightarrow{OA_0}-\overrightarrow{OM}+2\frac{F(M)-F(A_0)}{Ap+Bq+Cr}\vec{d}\,.\end{aligned} \tag{4.22}$$

4.4 Projections and symmetries in the two dimensional setting

4.4.1 The intersection point of two concurrent lines

Consider two lines

$$d:\frac{x-x_0}{p}=\frac{y-y_0}{q}$$

şi $\Delta: ax+by+c=0$ which are not parallel to each other, i.e.

$$ap+bq\neq 0.$$

The parametric equations of d are:

$$\begin{cases} x=x_0+pt\\ y=y_0+qt\end{cases},t\in\mathbb{R} \tag{4.23}$$

The value of $t\in\mathbb{R}$ for which this line (4.23) punctures the line Δ can be determined by imposing the condition on the point of coordinates

$$(x_0+pt,y_0+qt)$$

to verify the equationof the line Δ, namely

$$a(x_0 + pt) + b(y_0 + qt) + c = 0.$$

Thus

$$t = -\frac{ax_0 + by_0 + c}{ap + bq} = -\frac{F(x_0, y_0)}{ap + bq},$$

where $F(x, y) = ax + by + c$.

The coordinates of the intersection point $d \cap \Delta$ are:

$$\begin{aligned} &x_0 - p\frac{F(x_0, y_0)}{ap + bq} \\ &y_0 - q\frac{F(x_0, y_0)}{ap + bq}. \end{aligned} \tag{4.24}$$

4.4.2 The projection on a line parallel with another given line

Consider two straight non-parallel lines

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

and $\Delta : ax + by + c = 0$ which are not parallel to each other, i.e. $ap + bq \neq 0$. For these given data we may define the projection $p_{\Delta,d} : \pi \to \Delta$ of π on Δ parallel cu d, whose value $p_{\Delta,d}$ at $M \in \pi$ is the intersection point between Δ and the line through M which is parallelt to d. Due to relations (4.24), the coordinates of $p_{\Delta,d}(M)$, in terms of the coordinates of M are:

$$\begin{aligned} &x_M - p\frac{F(x_M, y_M)}{ap + bq} \\ &y_M - q\frac{F(x_M, y_M)}{ap + bq}, \end{aligned}$$

where $F(x, y) = ax + by + c$.

Consequently, the position vector of $p_{\Delta,d}(M)$ is

$$\overrightarrow{Op_{\Delta,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{ap + bq}\overrightarrow{d},$$

where $\overrightarrow{d} = p\overrightarrow{e} + g\overrightarrow{f}$.

Proposition 4.9. *If R is the Cartesian reference system of the plane π behind the equations of the concurrent lines*

$$\Delta : ax + by + c = 0 \text{ and } d : \frac{x - x_0}{p} = \frac{y - y_0}{q},$$

then

$$[p_{\Delta,d}(M)]_R = \frac{1}{ap + bq}\begin{pmatrix} bq & -bp \\ -aq & ap \end{pmatrix}[M]_R - \frac{c}{ap + bq}[\overrightarrow{d}]_b. \tag{4.25}$$

4.4.3 The symmetry with respect to a line parallel with another line

We call the function $s_{\Delta,d} : \pi \to \pi$, whose value $s_{\Delta,d}$ at $M \in \pi$ is the symmetric point of M with respect to $p_{\Delta,d}(M)$, the *symmetry of π with respect to Δ parallel to d*. The direction of d is equally called the direction of the symmetry and π is called the *axis of the symmetry*. For the position vector of $s_{\Delta,d}(M)$ we have

$$\overrightarrow{Op_{\Delta,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\Delta,d}(M)}}{2}, \text{ i.e.}$$

$$\overrightarrow{Os_{\Delta,d}(M)} = 2\overrightarrow{Op_{\Delta,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{ap+bq}\overrightarrow{d},$$

where $F(x,y) = ax + by + c$. Thus, the coordinates of $s_{\Delta,d}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_M - 2p\dfrac{F(x_M, y_M)}{ap+bq} \\ y_M - 2q\dfrac{F(x_M, y_M)}{ap+bq}. \end{cases}$$

Proposition 4.10. *If R is the Cartesian reference system of the plane π behind the equations of the concurrent lines*

$$\Delta : ax + by + c = 0 \text{ and } d : \frac{x - x_0}{p} = \frac{y - y_0}{q},$$

then

$$[s_{\Delta,d}(M)]_R = \frac{1}{ap+bq}\begin{pmatrix} -ap+bq & -2bp \\ -2aq & ap-bq \end{pmatrix}[M]_R - \frac{2c}{ap+bq}[\overrightarrow{d}]_b. \tag{4.26}$$

4.5 Problems

1. Write the equation of the plane determined by the line

$$(d)\ \begin{cases} x - 2y + 3z = 0 \\ 2x + z - 3 = 0 \end{cases}$$

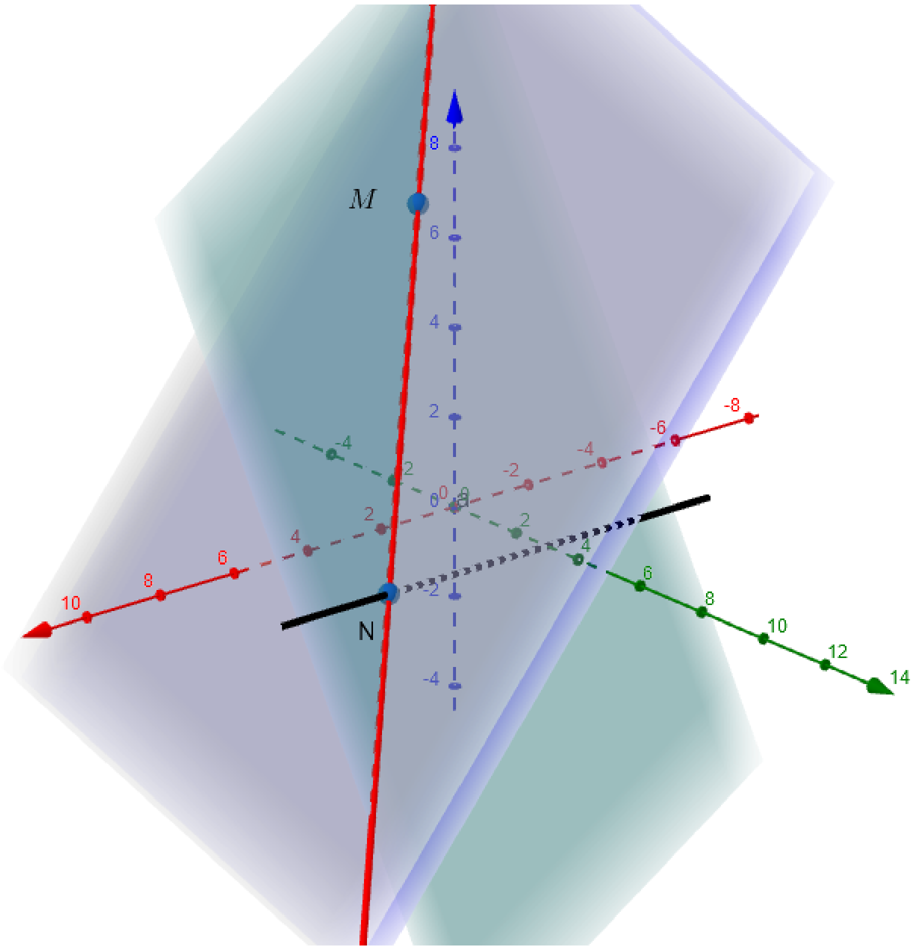
and the point $A(-1, 2, 6)$.

SOLUTION.

2. Write the equation of the line which passes through the point $M(1,0,7)$, is parallel to the plane $(\pi)\ 3x - y + 2z - 15 = 0$ and intersects the line

$$(d)\ \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

SOLUTION 1. The equation of the plane α passing through the point $M(1,0,7)$, which is parallel to the plane $(\pi)\ 3x - y + 2z - 15 = 0$, is $(\alpha)\ 3(x-1) - (y-0) + 2(z-7) = 0$, i.e. $(\alpha)\ 3x - y + 2z - 17 = 0$.



The parametric equations of the line d are

$$\begin{cases} x = 1 + 4t \\ y = 3 + 2t \\ z = t \end{cases},\ t \in \mathbb{R}.$$

The coordinates of the intersection point N between the line (d) and the plane α can be obtained by solving the equation $3((1+4t) - (3+2t) + 2t - 17 = 0$. The required line is MN.

SOLUTION 2. The required line can be equally regarded as the intersection line between the plane α (passing through the point $M(1,0,7)$, which is parallel to the plane (π)) and the plane determined by the given line (d) and the point M. While the equation $3x - y + 2z - 17 = 0$ of α was already used above, the equation of the plane determined by the line (d) and the point M can be determined via the pencil of planes through

$$(d)\ \begin{cases} \dfrac{x-1}{4} = \dfrac{y-3}{2} \\ \dfrac{y-3}{2} = \dfrac{z}{1} \end{cases} \Leftrightarrow (d)\ \begin{cases} x - 2y + 5 = 0 \\ y - 2z - 3 = 0. \end{cases}$$

Note that none of the planes $x - 2y + 5 = 0$ or $y - 2z - 3 = 0$ passes through M, which means that the plane determined by d and M is in the reduced pencil of planes

$$(\pi_\lambda)\ x - 2y + 5 = 0 + \lambda(y - 2z - 3) = 0.$$

The plane determined by d and M can be found by imposing on the coordinates of M to verify the equation of π_λ.

3. Write the equations of the projection of the line

$$(d)\ \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

on the plane $\pi : x + 2y - z = 0$ parallel to the direction $\overrightarrow{u}\,(1, 1, -2)$. Write the equations of the symmetry of the line d with respect to the plane π parallel to the direction $\overrightarrow{u}\,(1, 1, -2)$.

SOLUTION.

4. Prove Proposition 4.7

   SOLUTION.

5. Prove Proposition 5.6

   SOLUTION.

6. Show that two different parallel lines are either projected onto parallel lines or on two points by a projection $p_{\pi,d}$, where

$$\pi : Ax + By + Cz + D = 0, \;\; d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and $\pi \nparallel d$.

SOLUTION.

7. Show that two different parallel lines are mapped onto parallel lines by a symmetry $s_{\pi,d}$, where

$$\pi : Ax + By + Cz + D = 0,\ \ d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and$\pi \not\parallel d$.

SOLUTION.

8. Assume that $R = (O, b)$ ($b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$) is the Cartesian reference system behind the equations of a plane $\pi : Ax + By + Cz + D = 0$ and a line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$

If $\pi \not\parallel d$, show that

(a) $\overrightarrow{p_{\pi,d}(M)p_{\pi,d}(N)} = p(\overrightarrow{MN})$, for all $M, N \in \mathcal{V}$, where $p : \mathcal{V} \longrightarrow \mathcal{V}$ is the linear transformation whose matrix representation is

$$[p]_b = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix}.$$

SOLUTION.

(b) $\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)} = s(\overrightarrow{MN})$, for all $M, N \in \mathcal{V}$, where $s : \mathcal{V} \longrightarrow \mathcal{V}$ is the linear transformation whose matrix representation is

$$[s]_b = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} -Ap + Bq + Cr & -2Bp & -2Cp \\ -2Aq & Ap - Bq + Cr & -2Cq \\ -2Ar & -2Br & Ap + Bq - Cr \end{pmatrix}.$$

SOLUTION.

9. Consider a plane $\pi : Ax + By + Cz + D = 0$ and a line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$

If $\pi \nparallel d$, show that

(a) $p_{\pi,d} \circ p_{\pi,d} = p_{\pi,d}$.

(b) $s_{\pi,d} \circ s_{\pi,d} = id_{\mathcal{P}}$.

SOLUTION.

10. Prove Proposition 4.9.

    SOLUTION.

11. Prove Proposition 4.10.

    SOLUTION.

5 Week 5: Products of vectors

5.1 The dot product

Definition 5.1. The real number

$$\vec{a}\cdot\vec{b}=\begin{cases} 0 \text{ if } \vec{a}=0 \text{ or } \vec{b}=0 \\ \|\vec{a}\|\cdot\|\vec{b}\|cos(\widehat{\vec{a},\vec{b}}) \text{ if } \vec{a}\neq 0 \text{ and } \vec{b}\neq 0 \end{cases} \tag{5.1}$$

is called the *dot product* of the vectors $\vec{a},\vec{b}$.

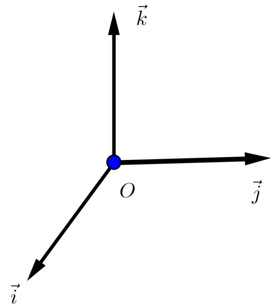
Remark 5.1. 1. $\vec{a}\perp\vec{b}\Leftrightarrow\vec{a}\cdot\vec{b}=0$.

2. $\vec{a}\cdot\vec{a}=\|\vec{a}\|\cdot\|\vec{a}\|\cos 0=\|\vec{a}\|^2$.

Proposition 5.1. *The dot product has the following properties:*

1. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a},\ \forall\ \vec{a},\vec{b}\in\mathcal{V}$.
2. $\vec{a}\cdot(\lambda\vec{b})=\lambda(\vec{a}\cdot\vec{b}),\ \forall\lambda\in\mathbb{R},\vec{a},\vec{b}\in\mathcal{V}$.
3. $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c},\ \forall\ \vec{a},\vec{b},\vec{c}\in\mathcal{V}$.
4. $\vec{a}\cdot\vec{a}\geq 0,\ \forall\ \vec{a}\in\mathcal{V}$.
5. $\vec{a}\cdot\vec{a}=0\Leftrightarrow\vec{a}=\vec{0}$.

Definition 5.2. A basis of the vector space $\mathcal{V}$ is said to be *orthonormal*, if $\|\vec{i}\|=\|\vec{j}\|=\|\vec{k}\|=1$, $\vec{i}\perp\vec{j},\ \vec{j}\perp\vec{k},\ \vec{k}\perp\vec{i}$ $(\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1,\ \vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=0)$. A Cartesian reference system $R=(O,\vec{i},\vec{j},\vec{k})$ is said to be *orthonormal* if the basis $[\vec{i},\vec{j},\vec{k}]$ is orthonormal.



orthonormal basis

Proposition 5.2. *Let* $[\vec{i},\vec{j},\vec{k}]$ *be an orthonormal basis and* $\vec{a},\vec{b}\in\mathcal{V}$. *If* $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$, *then*

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3 \tag{5.2}$$

Proof. Indeed,

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\cdot(b_1\vec{i}+b_2\vec{j}+b_3\vec{k})\\ &= a_1b_1\vec{i}\cdot\vec{i}+a_1b_2\vec{i}\cdot\vec{j}+a_1b_3\vec{i}\cdot\vec{k}\\ &+a_2b_1\vec{j}\cdot\vec{i}+a_2b_2\vec{j}\cdot\vec{j}+a_2b_3\vec{j}\cdot\vec{k}\\ &+a_3b_1\vec{k}\cdot\vec{i}+a_3b_2\vec{k}\cdot\vec{j}+a_3b_3\vec{k}\cdot\vec{k}\\ &= a_1b_1+a_2b_2+a_3b_3.\end{aligned}$$

□

Remark 5.2. Let $[\vec{i}, \vec{j}, \vec{k}]$ be an orthonormal basis and $\vec{a}, \vec{b} \in \mathcal{V}$. If $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ and $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, then

1. $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2$ and we conclude that $|| \vec{a} || = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

2.
$$\begin{aligned} \cos(\widehat{\vec{a}, \vec{b}}) &= \frac{\vec{a} \cdot \vec{b}}{|| \vec{a} || \cdot || \vec{b} ||} \\ &= \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}. \end{aligned} \tag{5.3}$$

   In particular

$$\cos(\widehat{\vec{a}, \vec{i}}) = \frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}};$$
$$\cos(\widehat{\vec{a}, \vec{j}}) = \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}};$$
$$\cos(\widehat{\vec{a}, \vec{k}}) = \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}.$$

3. $\vec{a} \perp \vec{b} \Leftrightarrow a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

5.1.1 Applications of the dot product

◊ The two dimensional setting

- **The distance between two points** Consider two points $A(x_A, y_A),\ B(x_B, y_B) \in \pi$. The norm of the vector $\overrightarrow{AB}$ $(x_B - x_A, y_B - y_A$ is

$$|| \overrightarrow{AB} || = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}.$$

- **The equation of the circle**
  Recall that the circle $\mathcal{C}(O, r)$ is the locus of points M in the plane such that $\text{dist}(O, M) = r \Longleftrightarrow \| \overrightarrow{OM} \| = r$. If (a, b) are the coordinates of O and (x, y) are the coordinates of M, then

$$\begin{aligned} \| \overrightarrow{OM} \| = r \Longleftrightarrow \sqrt{(x-a)^2 + (y-b)^2} = r \Longleftrightarrow (x-a)^2 + (y-b)^2 = r^2 \\ \Longleftrightarrow x^2 + y^2 - 2ax - 2by + c = 0, \end{aligned} \tag{5.4}$$

  where $c = a^2 + b^2 - r^2$. Conversely, every equation of the form $x^2 + y^2 + 2ex + 2fy + g = 0$ is the equation of the circle centered at $(-e, -f)$ and having the radius $r = \sqrt{e^2 + f^2 - g}$, whenever $e^2 + f^2 \geq g$. One can find the equation of the circle circumscribed to the triangle ABC by imposing the requirement on the coordinates (x_A, y_A), (x_B, y_B) and (x_C, y_C) of its vertices A, B, C to verify the equation $x^2 + y^2 + 2ex + 2fy + g = 0$. A point $M(x, y)$ belongs to this circumcircle if and only if

$$\begin{cases} x^2 + y^2 + 2ex + 2fy + g = 0 \\ x_A^2 + y_A^2 + 2ex_A + 2fy_A + g = 0 \\ x_B^2 + y_B^2 + 2ex_B + 2fy_B + g = 0 \\ x_C^2 + y_C^2 + 2ex_C + 2fy_C + g = 0 \end{cases} \tag{5.5}$$

On can regard the system (5.5) as linear with the unknowns e, g, f, whose compatibility is given, via the Kronecker-Capelli theorem, by

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_A^2+y_A^2 & x_A & y_A & 1 \\ x_B^2+y_B^2 & x_B & y_B & 1 \\ x_C^2+y_C^2 & x_C & y_C & 1 \end{vmatrix} = 0,$$

which is the equation of the circumcircle of the triangle ABC.

- **The normal vector of a line** If $R = (O, b)$ is the orthonormal Cartesian reference sytem behind the equation of a line $(d)\ ax + by + c = 0$, then $\vec{n}\ (a, b)$ is a normal vector to the direction $\vec{d}$ of d. Indeed, every vector of the direction $\vec{d}$ of d has the form $\overrightarrow{PM}$, where $P(x_P, y_P)$ and $M(x, y)$ are two points on the line d. Thus, $ax_P + by_P + c = 0 = ax_M + by_M + c$, which shows that

$$a(x_M - x_P) + b(y_M - y_P) = 0,$$

namely

$$\vec{n} \cdot \overrightarrow{PM} = 0 \Longleftrightarrow \vec{n} \perp \overrightarrow{PM}\ .$$

- **The distance from a point to a line** If $(d)\ ax + by + c = 0$ is a line and $M(x_M, y_M) \in \pi$ a given point, then the distance from M to d is

$$\delta(M, d) = \frac{|ax_M + by_M + c|}{\sqrt{a^2 + b^2}}. \tag{5.6}$$

Indeed, $\delta(M, d) = |\delta|$, where δ is the real scalar with the property $\overrightarrow{PM} = \delta \frac{\vec{n}}{\|\vec{n}\|}$ and $P(x_P, y_P)$ is the orthogonal projection of $M(x_M, y_M)$ on d. Thus $\overrightarrow{PM}\ (x_M - x_P, y_M - y_P)$ andul

$$\begin{aligned} \delta(M, d) &= |\delta| = \left| \overrightarrow{PM} \cdot \frac{\vec{n}}{\| \vec{n} \|} \right| = \frac{|\overrightarrow{PM} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|a(x_M - x_P) + b(y_M - y_P)|}{\sqrt{a^2 + b^2}} \\ &= \frac{|ax_M + by_M - ax_P - by_P)|}{\sqrt{a^2 + b^2}} = \frac{|ax_M + by_M + c|}{\sqrt{a^2 + b^2}}. \end{aligned}$$

◊ **The three dimensional setting**

- **The distance between two points** Consider two points $A(x_A, y_A, z_A),\ B(x_B, y_B, z_B) \in \mathcal{P}$. The norm of the vector $\overrightarrow{AB}\ (x_B - x_A, y_B - y_A, z_B - z_A)$ is

$$\| \overrightarrow{AB} \| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

- **The equation of the sphere**

Recall that the sphere $\mathcal{S}(O, r)$ is the locus of points M in space such that $\text{dist}(O, M) = r \Longleftrightarrow \| \overrightarrow{OM} \| = r$. If (a, b, c) are the coordinates of O and (x, y, z) are the coordinates of M, then

$$\begin{aligned} \| \overrightarrow{OM} \| = r &\Longleftrightarrow \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r \Longleftrightarrow (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \\ &\Longleftrightarrow x^2 + y^2 + z^2 - 2ax - 2by - 2cz + d = 0, \end{aligned}$$

where $d = a^2 + b^2 + c^2 - r^2$. Conversely, every equation of the form

$$x^2 + y^2 + z^2 + 2ex + 2fy + 2gz + h = 0$$

is the equation of the sphere centered at $(-e,-g,-f)$ and having the radius $r=\sqrt{e^2+f^2+g^2-h}$, whenever $e^2+f^2+g^2\geq h$. One can find the equation of the sphere circumscribed to the tetrahedron $ABCD$ by imposing the requirement on the coordinates (x_A,y_A,z_A), (x_B,y_B,z_B) and (x_C,y_C,z_C) and (x_D,y_D,z_D) of its vertices A,B,C,D to verify the equation $x^2+y^2+z^2-2ax-2by-2cz+d=0$. A point $M(x,y,z)$ belongs to this circumcircle if and only if

$$\begin{cases} x^2+y^2+z^2+2ex+2fy+2gz+h=0\\ x_A^2+y_A^2+z_A^2+2ex_A+2fy_A+2gz_A+h=0\\ x_B^2+y_B^2+z_B^2+2ex_B+2fy_B+2gz_B+h=0\\ x_C^2+y_C^2+z_C^2+2ex_C+2fy_C+2gz_C+h=0\\ x_D^2+y_D^2+z_D^2+2ex_D+2fy_D+2gz_A+h=0 \end{cases} \tag{5.7}$$

On can regard the system (5.7) as linear with the unknowns e,g,f,h, whose compatibility is given, via the Kronecker-Capelli theorem, by

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1\\ x_A^2+y_A^2+z_A^2 & x_A & y_A & z_A & 1\\ x_B^2+y_B^2+z_B^2 & x_B & y_B & z_B & 1\\ x_C^2+y_C^2+z_C^2 & x_C & y_C & z_C & 1\\ x_D^2+y_D^2+z_D^2 & x_D & y_D & z_A & 1 \end{vmatrix}=0,$$

which is the equation of the circumsphere of the tetrahedron $ABCD$.

- **The normal vector of a plane.** Consider the plane $\pi\,:\,Ax+By+Cz+D=0$ and the point $P(x_0,y_0,z_0)\in\pi$. The equation of π becomes

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0. \tag{5.8}$$

If $M(x,y,z)\in\pi$, the coordinates of $\overrightarrow{PM}$ are $(x-x_0,y-y_0,z-z_0)$ and the equation (5.8) tells us that $\overrightarrow{n}\cdot\overrightarrow{PM}=0$, for every $M\in\pi$, that is $\overrightarrow{n}\perp\overrightarrow{PM}=0$, for every $M\in\pi$, which is equivalent to $\overrightarrow{n}\perp\overrightarrow{\pi}$, where $\overrightarrow{n}\,(A,B,C)$. This is the reason to call $\overrightarrow{n}\,(A,B,C)$ *the normal vector* of the plane π.

- **The distance from a point to a plane.** Consider the plane $\pi\,:\,Ax+By+Cz+D=0$, a point $P(x_P,y_P,z_P)\in\mathcal{P}$ and M the orthogonal projection of P on π. The real number δ given by $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$ is called the *oriented distance* from P to the plane π, where $\overrightarrow{n}_0=\frac{1}{||\overrightarrow{n}||}\overrightarrow{n}$ is the versor of the normal vector $\overrightarrow{n}\,(A,B,C)$. Since $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$, it follows that $\delta(P,M)=||\overrightarrow{MP}||=|\delta|$, where $\delta(P,M)$ stands for the distance from P to π. We shall show that

$$\delta=\frac{Ax_P+By_P+Cz_P+D}{\sqrt{A^2+B^2+C^2}}.$$

Indeed, since $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$, we get successively:

$$\begin{aligned}\delta &= \overrightarrow{n}_0\cdot\overrightarrow{MP}=\left(\frac{1}{||\overrightarrow{n}||}\overrightarrow{n}\right)\cdot\overrightarrow{MP}=\frac{\overrightarrow{n}\cdot\overrightarrow{MP}}{||\overrightarrow{n}||}\\ &= \frac{A(x_P-x_M)+B(y_P-y_M)+C(z_P-z_M)}{\sqrt{A^2+B^2+C^2}}\\ &= \frac{Ax_P+By_P+Cz_P-(Ax_M+By_M+Cz_M)}{\sqrt{A^2+B^2+C^2}}\\ &= \frac{Ax_P+By_P+Cz_P+D}{\sqrt{A^2+B^2+C^2}}.\end{aligned}$$

Consequently, the distance from P to the plane π is

$$\delta(P,\pi)=||\overrightarrow{MP}||=|\delta|=\frac{|Ax_P+By_P+Cz_P+D|}{\sqrt{A^2+B^2+C^2}}.$$

Example 5.1. Compute the distance from the point $A(3,1,-1)$ to the plane

$$\pi : 22x + 4y - 20z - 45 = 0.$$

SOLUTION.

$$\delta(A,\pi) = \frac{|22\cdot 3 + 4\cdot 1 - 20\cdot(-1) - 45|}{\sqrt{22^2 + 4^2 + (-20)^2}} = \frac{45}{\sqrt{900}} = \frac{45}{30} = \frac{3}{2}.$$

5.2 Appendix: Orthogonal projections and reflections

5.2.1 The two dimensional setting

Asssume that $R = (O, \overrightarrow{i}, \overrightarrow{j})$ is the orthonormal Cartesian system of a plane π behind the equation of the line $\Delta : ax + by + c = 0$.

• **The orthogonal projection of a point on a line.** We define the projection of the ambient plane $p_\Delta : \pi \to \Delta$ on Δ, whose value p_Δ at $M \in \pi$ is the intersection point between Δ and the line through M perpendicular to Δ. Due to relations (4.24), the coordinates of $p_\Delta(M)$, in terms of the coordinates of M are:

$$x_M - p\frac{F(x_M, y_M)}{a^2+b^2}$$
$$y_M - q\frac{F(x_M, y_M)}{a^2+b^2},$$

where $F(x,y) = ax + by + c$. Consequently, the position vector of $p_\Delta(M)$ is

$$\overrightarrow{Op_\Delta(M)} = \overrightarrow{OM} - \frac{F(M)}{a^2+b^2}\overrightarrow{n}_\Delta,$$

where $\overrightarrow{n}_\Delta = a\overrightarrow{i} + b\overrightarrow{j}$.

Proposition 5.3. *If $R = (O, \overrightarrow{i}, \overrightarrow{j})$ is the orthonormal Cartesian reference system of the plane π behind the equations of the line*

$$\Delta : ax + by + c = 0,$$

then

$$[p_\Delta(M)]_R = \frac{1}{a^2+b^2}\begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix}[M]_R - \frac{c}{a^2+b^2}[\overrightarrow{n}_\Delta]_b, \tag{5.9}$$

where b stands for the orthonormal basis $[\overrightarrow{i}, \overrightarrow{j}]$ of $\overrightarrow{\pi}$.

• **The reflection of the plane about a line.** We call the function $r_\Delta : \pi \to \pi$, whose value r_Δ at $M \in \pi$ is the symmetric point of M with respect to $p_\Delta(M)$, the *reflection of π about Δ*. For the position vector of $r_\Delta(M)$ we have

$$\overrightarrow{Op_\Delta(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Or_\Delta(M)}}{2}, \text{ i.e.}$$

$$\overrightarrow{Or_\Delta(M)} = 2\overrightarrow{Op_\Delta(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{a^2+b^2}\overrightarrow{n}_\Delta,$$

where $F(x,y) = ax + by + c$ and $\overrightarrow{n}_\Delta = a\overrightarrow{i} + b\overrightarrow{j}$. Thus, the coordinates of $s_{\Delta,d}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_M - 2p\dfrac{F(x_M, y_M)}{a^2+b^2} \\ y_M - 2q\dfrac{F(x_M, y_M)}{a^2+b^2}. \end{cases}$$

Proposition 5.4. *If $R = (O, \overrightarrow{i}, \overrightarrow{j})$ is the orthonormal Cartesian reference system of the plane π behind the equations of the line*

$$\Delta : ax + by + c = 0,$$

then

$$[r_\Delta(M)]_R = \frac{1}{a^2+b^2}\begin{pmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix}[M]_R - \frac{2c}{a^2+b^2}\left[\overrightarrow{n}_\Delta\right]_b, \tag{5.10}$$

where b stands for the orthonormal basis $[\overrightarrow{i}, \overrightarrow{j}]$ of $\overrightarrow{\pi}$.

Example 5.2. Find the coordinates of the reflected point of $P(-5, 13)$ with respect to the line

$$d : 2x - 3y - 3 = 0,$$

knowing that the Cartesian reference system R behind the coordinates of A and the equation of (d) is orthonormal.

HINT. According to 5.11 it follows that

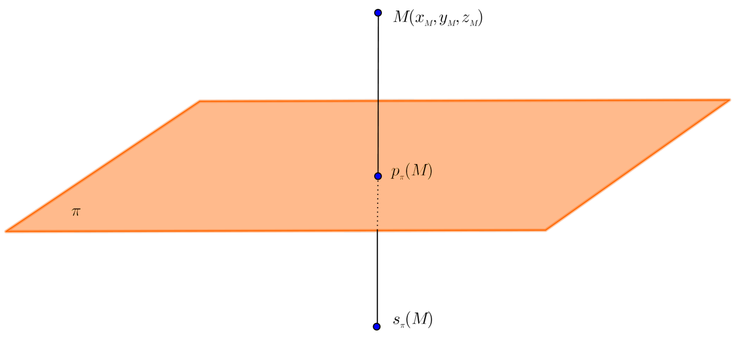
$$[r_d(P)]_R = \frac{1}{2^2+(-3)^2}\begin{pmatrix} -2^2+(-3)^2 & -2\cdot 2\cdot(-3) \\ -2\cdot 2\cdot(-3) & 2^2-(-3)^2 \end{pmatrix}\begin{bmatrix} -5 \\ 13 \end{bmatrix} - \frac{2\cdot(-3)}{2^2+(-3)^2}\begin{bmatrix} 2 \\ -3 \end{bmatrix}. \tag{5.11}$$

5.2.2 The three dimensional setting

• **The orthogonal projection of a point on a plane.** For a given plane

$$\pi : Ax + By + Cz + D = 0$$

and a given point $M(x_M, y_M, z_M)$, we shall determine the coordinates of its orthogonal projection on the plane π, as well as the coordinates of its (orthogonal) symmetric with respect to π. The equation of the plane and the coordinates of M are considered with respect to some cartezian coordinate system $R = (O, \vec{i}, \vec{j}, \vec{k})$. In this respect we consider the orthogonal line on π which passes through M.



Its parametric equations are

$$\begin{cases} x = x_M + At \\ y = y_M + Bt \\ z = z_M + Ct \end{cases}, \ t \in \mathbb{R}. \tag{5.12}$$

The orthogonal projection $p_\pi(M)$ of M on the plane π is at its intersection point with the orthogonal line (5.12) and the value of $t \in \mathbb{R}$ for which this orthogonal line (5.12) puncture the plane π can be determined by imposing the condition on the point of coordinates $(x_M + At, y_M + Bt, z_M + Ct)$ to verify the equation of the plane, namely $A(x_M + At) + B(y_M + Bt) + C(z_M + Ct) + D = 0$. Thus

$$t = -\frac{Ax_M + By_M + Cz_M + D}{A^2 + B^2 + C^2} = -\frac{F(x_M, y_M, z_M)}{\| \overrightarrow{n}_\pi \|^2},$$

where $F(x, y, z) = Ax + By + Cz + D$ şi $\overrightarrow{n}_\pi = A \overrightarrow{i} + B \overrightarrow{j} + C \overrightarrow{k}$ is the normal vector of the plasne π.

• **The orthogonal projection of the space on a plane.**

The coordinates of the orthogonal projection $p_\pi(M)$ of M on th eplane π are

$$\begin{cases} x_M - A\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ y_M - B\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ z_M - C\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2}. \end{cases}$$

Therefore, the position vector of the orthogonal projection $p_\pi(M)$ is

$$\overrightarrow{Op_\pi(M)} = \overrightarrow{OM} - \frac{F(M)}{\| \overrightarrow{n}_\pi \|^2} \overrightarrow{n}_\pi . \tag{5.13}$$

Proposition 5.5. *If $R = (O, b)$ is the orthonormal Cartesian reference system behind the equation of the plane (π) $Ax + By + Cz + D = 0$, then*

$$(A^2 + B^2 + C^2)[p_\pi(M)]_R = \begin{pmatrix} B^2 + C^2 & -AB & -AC \\ -AB & A^2 + C^2 & -BC \\ -AC & -BC & A^2 + B^2 \end{pmatrix} [M]_R - D[\overrightarrow{n}_\pi]_b. \tag{5.14}$$

Remark 5.3. The distance from the point $M(x_M, y_M, z_M)$ to the plane $\pi : Ax + By + Cz + D = 0$ can be equally computed by means of (5.13). Indeed,

$$\begin{aligned} \delta(M, \pi) &= \| \overrightarrow{Mp_\pi(M)} \| = \| \overrightarrow{Op_\pi(M)} - \overrightarrow{OM} \| \\ &= \left| -\tfrac{F(M)}{\|\overrightarrow{n}_\pi\|^2} \right| \cdot \| \overrightarrow{n}_\pi \| = \frac{|F(M)|}{\| \overrightarrow{n}_\pi \|}. \end{aligned}$$

• **The reflection of the space about a plane.** In order to find the position vector of the orthogonally symmetric point $r_\pi(M)$ of M w.r.t. π, we use the relation

$$\overrightarrow{Op_\pi(M)} = \frac{1}{2}\left(\overrightarrow{OM} + \overrightarrow{Or_\pi(M)} \right),$$

namely

$$\overrightarrow{Or_\pi(M)} = 2\, \overrightarrow{Op_\pi(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{\| \overrightarrow{n}_\pi \|^2} \overrightarrow{n}_\pi .$$

The correspondence which associate to some point M its orthogonally symmetric point w.r.t. π, is called the *reflection* in the plane π and is denoted by r_π.

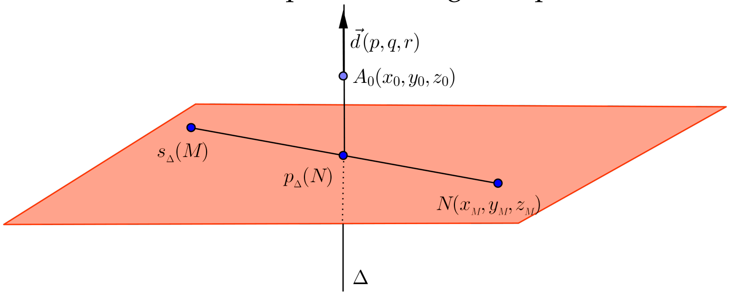
Proposition 5.6. *If $R = (O, b)$ is the orthonormal Cartesian reference system behind the equation of the plane (π) $Ax + By + Cz + D = 0$, then*

$$(A^2+B^2+C^2)[r_\pi(M)]_R = \begin{pmatrix} -A^2+B^2+C^2 & -2AB & -2AC \\ -2AB & A^2-B^2+C^2 & -2BC \\ -2AC & -2BC & A^2+B^2-C^2 \end{pmatrix} [M]_R - 2D[\overrightarrow{n}_\pi]_b. \tag{5.15}$$

• **The orthogonal projection of the space on a line.** For a given line

$$\Delta : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

and a point $N(x_N, y_N, z_N)$, we shall find the coordinates of its orthogonal projection on the line Δ, as well as the coordinates of the orthogonally symmetric point M with respect to Δ. The equations of the line and the coordinates of the point N are considered with respect to an orthonormal coordinate system $R = (O, \vec{i}, \vec{j}, \vec{k})$. In this respect we consider the plane $p(x-x_N)+q(y-y_N)+r(z-z_N)=0$ orthogonal on the line Δ which passes through the point N.



The parametric equations of the line Δ are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \ t \in \mathbb{R}. \tag{5.16}$$

The orthogonal projection of N on the line Δ is at its intersection point with the plane

$$p(x-x_N)+q(y-y_N)+r(z-z_N)=0,$$

and the value of $t \in \mathbb{R}$ for which the line Δ puncture the orthogonal plane $p(x-x_N)+q(y-y_N)+r(z-z_N)=0$ can be found by imposing the condition on the point of coordinate (x_0+pt, y_0+qt, z_0+rt) to verify the equation of the plane, namely $p(x_0+pt-x_N)+q(y_0+qt-y_N)+r(z_0+rt-z_N)=0$. Thus

$$t = -\frac{p(x_0-x_N)+q(y_0-y_N)+r(z_0-z_N)}{p^2+q^2+r^2} = -\frac{G(x_0,y_0,z_0)}{\|\vec{d}_\Delta\|^2},$$

where $G(x,y,z) = p(x-x_N)+q(y-y_N)+r(z-z_N)$ and $\vec{d}_\pi = p\,\vec{i} + q\,\vec{j} + r\,\vec{k}$ is the director vectoir of the line Δ. Ths coordinates of the orthogonal projection $p_\Delta(N)$ of N on the line Δ are therefore

$$\begin{cases} x_0 - p\dfrac{G(x_0,y_0,z_0)}{p^2+q^2+r^2} \\ y_0 - q\dfrac{G(x_0,y_0,z_0)}{p^2+q^2+r^2} \\ z_0 - r\dfrac{G(x_0,y_0,z_0)}{p^2+q^2+r^2}. \end{cases}$$

Thus, the position vector of the orthogonal projection $p_\Delta(N)$ is

$$\overrightarrow{Op_\Delta(N)} = \overrightarrow{OA_0} - \frac{G(A_0)}{\|\vec{d}_\Delta\|^2}\vec{d}_\Delta, \tag{5.17}$$

where $A_0(x_0,y_0,z_0) \in \Delta$.

• **The reflection of the space about a line.** In order to find the position vector of the orthogonally symmetric point $r_\Delta(N)$ of N with respect to the line Δ we use the relation

$$\overrightarrow{Op_\Delta(N)} = \frac{1}{2}\left(\overrightarrow{ON} + \overrightarrow{Or_\Delta(N)}\right)$$

i.e.

$$\overrightarrow{Os_{\Delta}(N)} = 2\,\overrightarrow{Op_{\Delta}(N)} - \overrightarrow{ON} = 2\,\overrightarrow{OA_0} - 2\frac{G(A_0)}{\| \overrightarrow{d}_{\Delta} \|^2}\,\overrightarrow{d}_{\Delta} - \overrightarrow{ON}\,.$$

The correspondence which associate to some point M its orthogonally symmetric point w.r.t. δ, is called the *reflection* in the line δ and is denoted by r_{δ}.

5.3 Problems

1. **(2p)** Consider the triangle ABC and the midpoint A' of the side $[BC]$. Show that

$$4\,\overrightarrow{AA'}^{\,2} - \overrightarrow{BC}^{\,2} = 4\,\overrightarrow{AB} \cdot \overrightarrow{AC}\,.$$

Solution.

2. (**2p**) Consider the rectangle $ABCD$ and the arbitrary point M witin the space. Show that

   (a) $\overrightarrow{MA} \cdot \overrightarrow{MC} = \overrightarrow{MB} \cdot \overrightarrow{MD}$.

   (b) $\overrightarrow{MA}^2 + \overrightarrow{MC}^2 = \overrightarrow{MB}^2 + \overrightarrow{MD}^2$.

   Solution.

3. (**3p**) Find the angle between:

   (a) the straight lines

$$(d_1)\left\{\begin{array}{cccccccccc} x & + & 2y & + & z & - & 1 & = 0 \\ x & - & 2y & + & z & + & 1 & = 0 \end{array}\right. \quad (d_2)\left\{\begin{array}{cccccccccc} x & - & y & - & z & - & 1 & = 0 \\ x & - & y & + & 2z & + & 1 & = 0. \end{array}\right.$$

   (b) the planes

$$\pi_1 : \ x + 3y + 2z + 1 = 0 \text{ and } \pi_2 : 3x + 2y - z = 6.$$

   (c) the plane xOy and the straight line M_1M_2, where $M_1(1,2,3)$ and $M_2(-2,1,4)$.

Solution.

4. (**3p**) Consider the noncoplanar vectors $\overrightarrow{OA}\ (1,-1,-2)$, $\overrightarrow{OB}\ (1,0,-1)$, $\overrightarrow{OC}\ (2,2,-1)$ related to an orthonormal basis $\vec{i}, \vec{j}, \vec{k}$. Let H be the foot of the perpendicular through O on the plane ABC. Determine the components of the vectors $\overrightarrow{OH}$.

   Solution.

5. (**2p**) Find the points on the z-axis which are equidistant with respect to the planes

$$\pi_1 : \ 12x + 9y - 20z - 19 = 0 \text{ and } \pi_2 : \ 16x + 12y + 15z - 9 = 0.$$

Solution.

6. (**2p**) Consider two planes

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0$$
$$(\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0$$

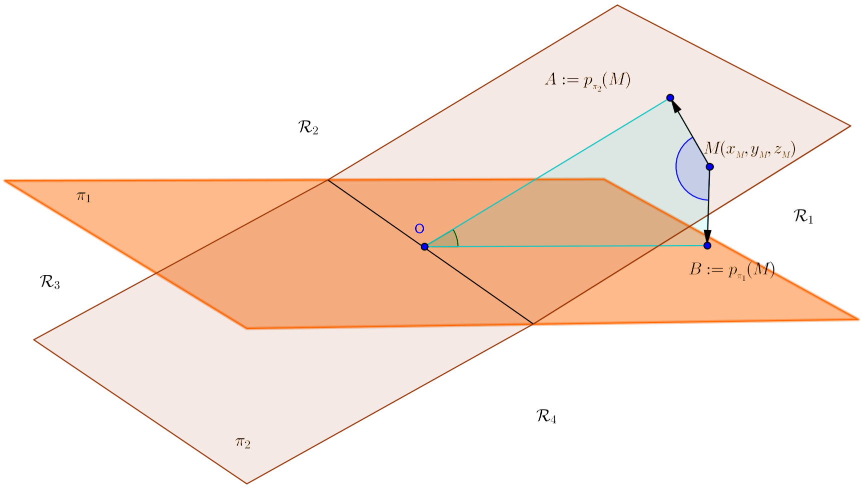
which are not parallel and not perpendicular as well. The two planes π_1, π_2 devide the space into four regions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$, two of which, say $\mathcal{R}_1$ and $\mathcal{R}_3$, correspond to the acute dihedral angle of the two planes. Show that $M(x,y,z) \in \mathcal{R}_1 \cup \mathcal{R}_3$, if and only if

$$F_1(x,y,z) \cdot F_2(x,y,z)(A_1A_2 + B_1B_2 + C_1C_2) < 0,$$

where $F_1(x,y,z) = A_1x + B_1y + C_1z + D_1$ and $F_2(x,y,z) = A_2x + B_2y + C_2z + D_2$.

Hint. The non-parallellism relation between the two planes is equivalent with the condition

$$\text{rank}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2.$$



The point M belongs to the union $\mathcal{R}_1 \cup \mathcal{R}_3$ if and only if the angle of the vectors $\overrightarrow{Mp_{\pi_1}(M)}$ and $\overrightarrow{Mp_{\pi_2}(M)}$ is at leat 90°, as the quadrilateral $OAMB$ is inscriptible. More formally

$$\begin{aligned} M(x,y,z) \in \mathcal{R}_1 \cup \mathcal{R}_3 &\Leftrightarrow m(\widehat{\overrightarrow{Mp_{\pi_1}(M)}, \overrightarrow{Mp_{\pi_2}(M)}}) > 90^\circ \\ &\Leftrightarrow \overrightarrow{Mp_{\pi_1}(M)} \cdot \overrightarrow{Mp_{\pi_2}(M)} < 0, \end{aligned}$$

where $p_{\pi_1}(M), p_{\pi_2}(M)$ are the orthogonal projections of M on the planes π_1 and π_2 respectively.

Solution.

7. (**3p**) Consider the planes $(\pi_1)\ 2x + y - 3z - 5 = 0$, $(\pi_2)\ x + 3y + 2z + 1 = 0$. Find the equations of the bisector planes of the dihedral angles formed by the planes π_1 and π_2 and select the one contained into the acute regions of the dihedral angles formed by the two planes.

   Solution.

8. (**3p**) Let a, b be two real numbers such that $a^2 \neq b^2$. Consider the planes:

$$(\alpha_1) ax + by - (a+b)z = 0$$

$$(\alpha_2) ax - by - (a-b)z = 0$$

and the quadric $(\mathcal{C}) : a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 - 2a^2xz + 2b^2yz - a^2b^2 = 0$. If $a^2 < b^2$, show that the quadric $\mathcal{C}$ is contained in the acute regions of the dihedral angles formed by the two planes. If, on the contrary, $a^2 > b^2$, show that the quadric $\mathcal{C}$ is contained in the obtuse regions of the dihedral angles formed by the two planes.

Solution.

9. If two pairs of opposite edges of the tetrahedron $ABCD$ are perpendicular ($AB \perp CD$, $AD \perp BC$), show that
   (a) The third pair of opposite edges are perpendicular too ($AC \perp BD$).
   (b) $AB^2 + CD^2 = AC^2 + BD^2 = BC^2 + AD^2$.
   (c) The heights of the tetrahedron are concurrent.
   (Such a tetrahedron is said to be orthocentric)

Solution. Denote by $\overrightarrow{AB}=\vec{b}$, $\overrightarrow{AC}=\vec{c}$ and $\overrightarrow{AD}=\vec{d}$.

(a) $AB \perp CD \implies \vec{b}\,(\vec{d} - \vec{c}) = 0 \implies \vec{b}\,\vec{d}=\vec{b}\,\vec{c}= k$
$AD \perp BC \implies \vec{d}\,(\vec{c} - \vec{b}) = 0 \implies \vec{c}\,\vec{d}=\vec{b}\,\vec{d}= k,$
deci
$\vec{c}\,\vec{b}=\vec{c}\,\vec{d} \implies \vec{c}\,(\vec{b} - \vec{d}) = 0 \implies AC \perp BD.$

(b) $AB^2 + CD^2 = \vec{b}^{\,2} + (\vec{d} - \vec{c})^2 = \vec{b}^{\,2} + \vec{d}^{\,2} + \vec{c}^{\,2} - 2k;$
$AC^2 + BD^2 = \vec{c}^{\,2} + (\vec{d} - \vec{b})^2 = \vec{b}^{\,2} + \vec{c}^{\,2} + \vec{d}^{\,2} - 2k;$
$BC^2 + AD^2 = \vec{d}^{\,2} + (\vec{c} - \vec{b})^2 = \vec{b}^{\,2} + \vec{c}^{\,2} + \vec{d}^{\,2} - 2k.$

(c) We shall show that there exists a point H such that $AH \perp (DBC)$, $BH \perp (ACD)$, $CH \perp (ABD)$, $DH \perp (ABC)$. Let $\overrightarrow{h} = \overrightarrow{AH} = m\,\overrightarrow{a} + n\,\vec{b} + p\,\vec{c}$. Writing the conditions $\overrightarrow{AH} \perp \overrightarrow{BC}$, $\overrightarrow{CD}$; $\overrightarrow{BH} \perp \overrightarrow{AC}, \overrightarrow{AD}$; $\overrightarrow{CH} \perp \overrightarrow{AB}, \overrightarrow{AD}$; $\overrightarrow{DH} \perp \overrightarrow{AB}, \overrightarrow{AC}$ we obtain a consistent system with one single solution:

$$\begin{cases} b^2 m + kn + kp = k \\ km + c^2 n + kp = k \\ km + kn + d^2 p = k. \end{cases} \tag{5.18}$$

Indeed the the matrix of the system is

$$A = \begin{pmatrix} b^2 & k & k \\ k & c^2 & k \\ k & k & d^2 \end{pmatrix}$$

and for its determinant we have successively

$$\begin{aligned} \det(A) &= \begin{vmatrix} b^2 & k & k \\ k & c^2 & k \\ k & k & d^2 \end{vmatrix} = \begin{vmatrix} b\cdot b & b\cdot c & b\cdot c \\ c\cdot b & c\cdot c & c\cdot d \\ d\cdot b & d\cdot c & d\cdot d \end{vmatrix} \\ &= \begin{vmatrix} b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 & b_1d_1 + b_2d_2 + b_3d_3 \\ c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 & c_1d_1 + c_2d_2 + c_3d_3 \\ d_1b_1 + d_2b_2 + d_3b_3 & d_1c_1 + d_2c_2 + d_3c_3 & d_1^2 + d_2^2 + d_3^2 \end{vmatrix} \end{aligned}$$

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 & d_1 \\ b_1 & c_2 & d_2 \\ b_1 & c_3 & d_3 \end{vmatrix} = (\vec{b}, \vec{c}, \vec{d}) \cdot \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = (\vec{b}, \vec{c}, \vec{d})^2.$$

The linear independence of the vectors $\vec{b}, \vec{c}, \vec{d}$ ensure that $(\vec{b}, \vec{c}, \vec{d}) \neq 0$ and shows that the linear system (5.18) is consistent and has one single solution.

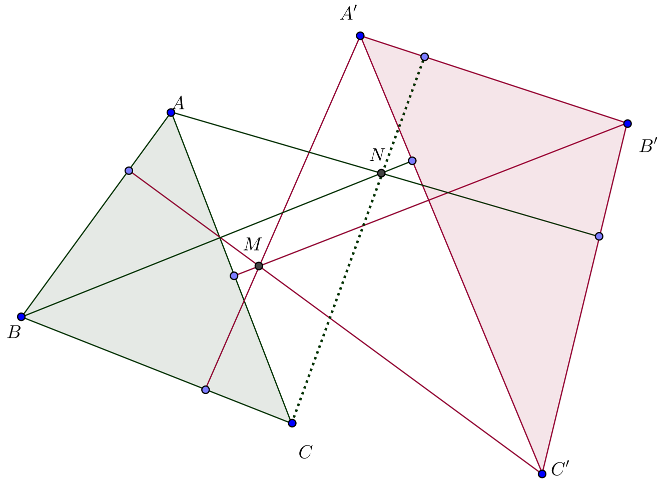
10. Two triangles ABC şi $A'B'C'$ are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices A', B', C' on the sides BC, CA, AB are concurrent. Show

that, in this case, the perpendicular lines from the vertices A, B, C on the sides $B'C'$, $C'A'$, $A'B'$ are concurrent too.

Solution Due to the given hypothesis, we have

$$\overrightarrow{MA'} \cdot \overrightarrow{BC} = \overrightarrow{MB'} \cdot \overrightarrow{CA} = \overrightarrow{MC'} \cdot \overrightarrow{AB} = 0 \tag{5.19}$$

We now consider the perpendicular lines from the vertices A and B on the edges $B'C'$ and $C'A'$ and denote by N their intersection point.



Thus

$$\overrightarrow{NA} \cdot \overrightarrow{B'C'} = \overrightarrow{NB} \cdot \overrightarrow{C'A'} = 0.$$

By using the relations (5.19) we obtain

$$\begin{aligned}
& \overrightarrow{MA'} \cdot \overrightarrow{BC} + \overrightarrow{MB'} \cdot \overrightarrow{CA} + \overrightarrow{MC'} \cdot \overrightarrow{AB} = 0 \\
\Leftrightarrow\ & \overrightarrow{MA'} \cdot (\overrightarrow{NC} - \overrightarrow{NB}) + \overrightarrow{MB'} \cdot (\overrightarrow{NA} - \overrightarrow{NC}) + \overrightarrow{MC'} \cdot (\overrightarrow{NB} - \overrightarrow{NA}) = 0 \\
\Leftrightarrow\ & (\overrightarrow{MB'} - \overrightarrow{MC'}) \cdot \overrightarrow{NA} + (\overrightarrow{MC'} - \overrightarrow{MA'}) \cdot \overrightarrow{NB} + (\overrightarrow{MA'} - \overrightarrow{MB'}) \cdot \overrightarrow{NC} = 0 \\
\Leftrightarrow\ & \overrightarrow{C'B'} \cdot \overrightarrow{NA} + \overrightarrow{A'C'} \cdot \overrightarrow{NB} + \overrightarrow{B'A'} \cdot \overrightarrow{NC} = 0 \\
\Leftrightarrow\ & \overrightarrow{B'A'} \cdot \overrightarrow{NC} = 0 \Leftrightarrow NC \perp A'B'.
\end{aligned}$$

11. (**2p**) Find the orthogonal projection

    (a) of the point $A(1,2,1)$ on the plane $\pi:\ x+y+3z+5=0$.

    (b) of the point $B(5,0,-2)$ on the straight line $(d)\ \dfrac{x-2}{3} = \dfrac{y-1}{2} = \dfrac{z-3}{4}$.

Solution.

A few questions in the two dimensional setting

12. (**3p**) Find the coordinates of the point P on the line $d : 2x - y - 5 = 0$ for which the sum $AP + PB$ is minimum, when $A(-7, 1)$ and $B(-5, 5)$.

13. (**2p**) Find the coordinates of the circumcenter (the center of the circumscribed circle) of the triangle determined by the lines $4x - y + 2 = 0$, $x - 4y - 8 = 0$ and $x + 4y - 8 = 0$.

    Solution.

14. (**3p**) Given the bundle of lines of equations $(1-t)x+(2-t)y+t-3=0$, $t \in \mathbb{R}$ and $x+y-1=0$, find:

    (a) the coordinates of the vertex of the bundle;

    (b) the equation of the line in the bundle which cuts Ox and Oy in M respectively N, such that $OM^2 \cdot ON^2 = 4(OM^2 + ON^2)$.

    Solution.

15. (**2p**) Let $\mathcal{B}$ be the bundle of lines of vertex $M_0(5,0)$. An arbitrary line from $\mathcal{B}$ intersects the lines $d_1 : y-2=0$ and $d_2 : y-3=0$ in M_1 respectively M_2. Prove that the line passing through M_1 and parallel to OM_2 passes through a fixed point.

16. (**3p**) The vertices of the quadrilateral $ABCD$ are $A(4,3)$, $B(5,-4)$, $C(-1,-3)$ and $D((-3,-1)$.

    (a) Find the coordinates of the intersection points $\{E\} = AB \cap CD$ and $\{F\} = BC \cap AD$;

    (b) Prove that the midpoints of the segments $[AC]$, $[BD]$ and $[EF]$ are collinear.

    Solution.

17. (**3p**) Let M be a point whose coordinates satisfy

$$\frac{4x+2y+8}{3x-y+1}=\frac{5}{2}.$$

   (a) Prove that M belongs to a fixed line (d);
   (b) Find the minimum of x^2+y^2, when $M \in d \setminus \{M_0(-1,-2)\}$.

Solution.

18. (**3p**) Find the locus of the points whose distances to two orthogonal lines have a constant ratio.
Solution.

6 Week 6:

6.1 The vector product

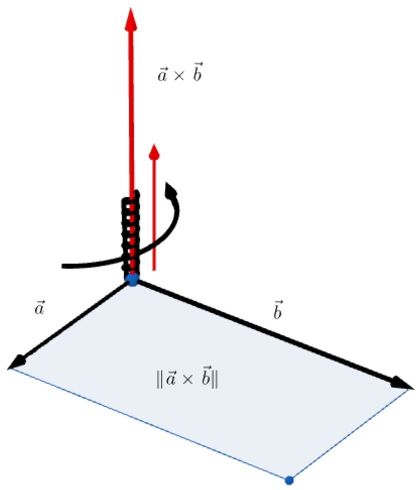
Definition 6.1. The *vector product* or the *cross product* of the vectors $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ is a vector, denoted by $\overrightarrow{a} \times \overrightarrow{b}$, which is defined to be zero if $\overrightarrow{a}, \overrightarrow{b}$ are linearly dependent (collinear), and if $\overrightarrow{a}, \overrightarrow{b}$ are linearly independent (noncollinear), then it is defined by the following data:

1. $\overrightarrow{a} \times \overrightarrow{b}$ is a vector orthogonal on the two-dimensional subspace $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$ of $\mathcal{V}$;

2. if $\overrightarrow{a} = \overrightarrow{OA}$, $\overrightarrow{b} = \overrightarrow{OB}$, then the sense of $\overrightarrow{a} \times \overrightarrow{b}$ is the one in which a right-handed screw, placed along the line passing through O orthogonal to the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, advances when it is being rotated simultaneously with the vector $\overrightarrow{a}$ from $\overrightarrow{a}$ towards $\overrightarrow{b}$ within the vector subspace $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$ and the support half line of $\overrightarrow{a}$ sweeps the interior of the angle $\widehat{AOB}$ (Screw rule).

3. the *norm* (*magnitude* or *length*) of $\overrightarrow{a} \times \overrightarrow{b}$ is defined by

$$|| \overrightarrow{a} \times \overrightarrow{b} || = || \overrightarrow{a} || \cdot || \overrightarrow{b} || \sin(\widehat{\overrightarrow{a}, \overrightarrow{b}}).$$



Remark 6.1. 1. *The* norm *(*magnitude *or* length*) of the vector $\overrightarrow{a} \times \overrightarrow{b}$ is actually the area of the parallelogram constructed on the vectors $\overrightarrow{a}, \overrightarrow{b}$.*

2. *The vectors $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ are linearly dependent (collinear) if and only if $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$.*

Proposition 6.1. *The vector product has the following properties:*

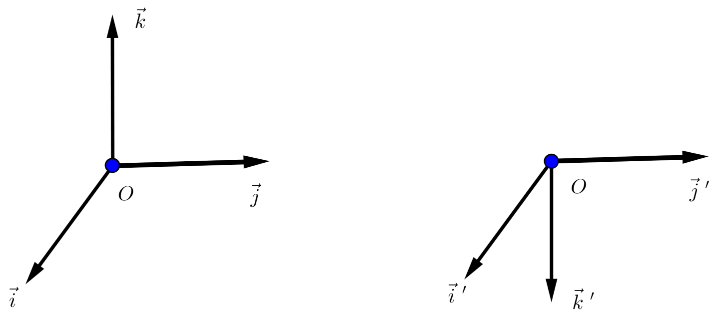
1. $\overrightarrow{a} \times \overrightarrow{b} = - \overrightarrow{b} \times \overrightarrow{a}, \forall \overrightarrow{a}, \overrightarrow{b} \in \mathcal{V};$

2. $(\lambda \overrightarrow{a}) \times \overrightarrow{b} = \overrightarrow{a} \times (\lambda \overrightarrow{b}) = \lambda(\overrightarrow{a} \times \overrightarrow{b}), \forall \lambda \in \mathbb{R}, \overrightarrow{a}, \overrightarrow{b} \in \mathcal{V};$

3. $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{c}, \forall \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in \mathcal{V}.$

6.2 The vector product in terms of coordinates

If $[\vec{i},\vec{j},\vec{k}]$ is an orthonormal basis, observe that $\vec{i}\times\vec{j}\in\{-\vec{k},\vec{k}\}$. We say that the orthonormal basis $[\vec{i},\vec{j},\vec{k}]$ is *direct* if $\vec{i}\times\vec{j}=\vec{k}$. If, on the contrary, $\vec{i}\times\vec{j}=-\vec{k}$, we say that the orthonormal basis $[\vec{i},\vec{j},\vec{k}]$ is *inverse*.



direct orthonormal basis inverse orthonormal basis

Therefore, if $[\vec{i},\vec{j},\vec{k}]$ is a direct orthonormal basis, then $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, $\vec{k}\times\vec{i}=\vec{j}$ and obviously $\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=\vec{0}$.

Proposition 6.2. *If $[\vec{i},\vec{j},\vec{k}]$ is a direct orthonormal basis and $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$, then*

$$\vec{a}\times\vec{b}=(a_2b_3-a_3b_2)\vec{i}+(a_3b_1-a_1b_3)\vec{j}+(a_1b_2-a_2b_1)\vec{k}, \tag{6.1}$$

or, equivalently,

$$\vec{a}\times\vec{b}=\begin{vmatrix} a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}-\begin{vmatrix} a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{j}+\begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k} \tag{6.2}$$

Proof. Indeed,

$$\begin{aligned}\vec{a}\times\vec{b} &= (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\times(b_1\vec{i}+b_2\vec{j}+b_3\vec{k})\\ &= a_1b_1\vec{i}\times\vec{i}+a_1b_2\vec{i}\times\vec{j}+a_1b_3\vec{i}\times\vec{k}\\ &+a_2b_1\vec{j}\times\vec{i}+a_2b_2\vec{j}\times\vec{j}+a_2b_3\vec{i}\times\vec{k}\\ &+a_3b_1\vec{k}\times\vec{i}+a_3b_2\vec{k}\times\vec{j}+a_3b_3\vec{k}\times\vec{k}\\ &= a_1b_2\vec{k}-a_1b_3\vec{j}-a_2b_1\vec{k}+a_2b_3\vec{i}+a_3b_1\vec{j}-a_3b_2\vec{i}\\ &= (a_2b_3-a_3b_2)\vec{i}+(a_3b_1-a_1b_3)\vec{j}+(a_1b_2-a_2b_1)\vec{k}\end{aligned}$$

□

One can rewrite formula (6.1) in the form

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} \tag{6.3}$$

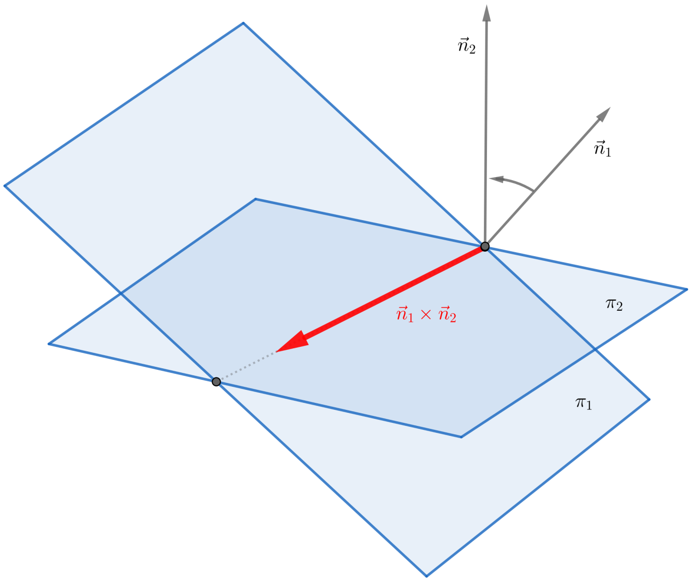
the right hand side determinant being understood in the sense of its cofactor expansion along the first line.

Remark 6.2. If $R = (O, \vec{i}, \vec{j}, \vec{k})$ is the direct Cartesian orthonormal reference system behind the equations of the line

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_1 = 0, \end{cases}$$

then we can recover the director parameters (4.10) of Δ, in this particular case of orthonormal Cartesian reference systems, by observing that $\vec{n}_1 \times \vec{n}_2$ is a director vector of Δ, where

$$\vec{n}_1 = A_1 \vec{i} + B_1 \vec{j} + C_1 \vec{k}$$
$$\vec{n}_2 = A_2 \vec{i} + B_2 \vec{j} + C_2 \vec{k}.$$



Recall that

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \vec{i} + \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \vec{j} + \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \vec{k}.$$

Note however that the director parameters were obtained before for arbitrary Cartesian reference systems (See (4.10)).

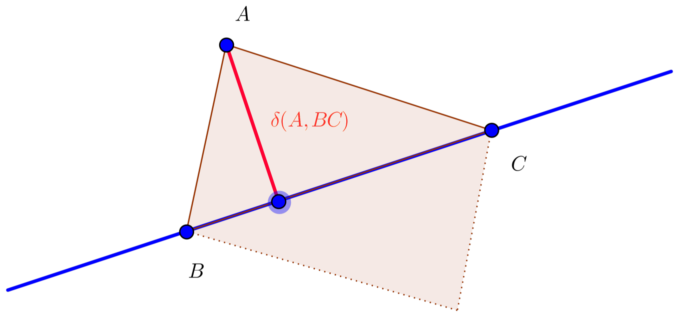
6.3 Applications of the vector product

- **The area of the triangle ABC.** $S_{ABC} = \frac{1}{2}|| \overrightarrow{AB} || \cdot || \overrightarrow{AC} || \sin \widehat{BAC} = \frac{1}{2}|| \overrightarrow{AB} \times \overrightarrow{AC} ||$. On the other hand

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_B - x_A & y_B - x_A & z_B - z_A \\ x_C - x_A & y_C - x_A & z_C - z_A \end{vmatrix},$$

  as the coordinates of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are $(x_B - x_A, y_B - x_A, z_B - z_A)$ and $(x_C - x_A, y_C - x_A, z_C - z_A)$ respectively. Thus,

$$4S^2_{ABC} = \begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2.$$



- **The distance from one point to a straight line.**

(a) *The distance $\delta(A, BC)$ from the point $A(x_A, y_A, z_A)$ to the straight line BC*, where $B(x_B, y_B, z_B)$ and $C(x_C, y_C, z_C)$. Since

$$S_{ABC} = \frac{\| \overrightarrow{BC} \| \cdot \delta(A, BC)}{2}$$

it follos that

$$\delta^2(A, BC) = \frac{4S^2_{ABC}}{\| \overrightarrow{BC} \|^2}.$$

Thus, we obtain

$$\delta^2(A, BC) = \frac{\begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2}{(x_C - x_B)^2 + (y_C - y_B)^2 + (z_C - z_B)^2}.$$

(b) The distance from $\delta(A, d)$ from one point $A(A_A, y_A, z_A)$ to the straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$

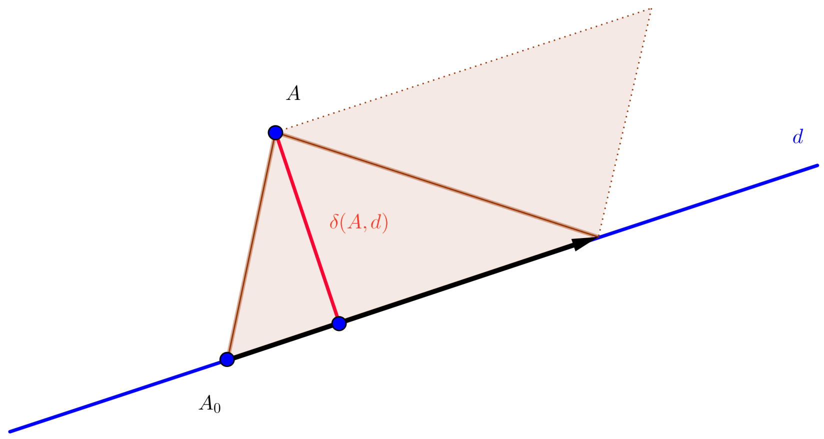
$$\delta(A, d) = \frac{\| \vec{d} \times \overrightarrow{A_0A} \|}{\|\vec{d}\|}, \tag{6.4}$$

where $A_0(x_0, y_0, z_0) \in \delta$.
Since

$$\begin{aligned} \vec{d} \times \overrightarrow{A_0A} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ x_A - x_0 & y_A - y_0 & z_A - z_0 \end{vmatrix} \\ &= \begin{vmatrix} q & r \\ y_A - y_0 & z_A - z_0 \end{vmatrix} \vec{i} + \begin{vmatrix} r & p \\ z_A - z_0 & x_A - x_0 \end{vmatrix} \vec{j} + \begin{vmatrix} p & q \\ x_A - x_0 & y_A - y_0 \end{vmatrix} \vec{k} \end{aligned}$$

it follows that

$$\delta(A, d) = \frac{\sqrt{\begin{vmatrix} q & r \\ y_A - y_0 & z_A - z_0 \end{vmatrix}^2 + \begin{vmatrix} r & p \\ z_A - z_0 & x_A - x_0 \end{vmatrix}^2 + \begin{vmatrix} p & q \\ x_A - x_0 & y_A - y_0 \end{vmatrix}^2}}{\sqrt{p^2 + q^2 + r^2}}.$$



6.4 The double vector (cross) product

The *double vector (cross) product* of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the vector $\vec{a} \times (\vec{b} \times \vec{c})$

Proposition 6.3.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c} = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \end{vmatrix}, \quad \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}. \tag{6.5}$$

Proof. (Sketch) If the vectors $\vec{b}$ and $\vec{c}$ are linearly dependent, then both sides are obviously zero. Otherwise one can choose an orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$, related to the vectors $\vec{a}, \vec{b}$ and $\vec{c}$, such that

$$\vec{b} = b_1\,\vec{i},\ \vec{c} = c_1\,\vec{i} + c_2\,\vec{j},\ \vec{a} = a_1\,\vec{i} + a_2\,\vec{j} + a_3\,\vec{k}.$$

For example one can choose $\vec{i}$ to be $\vec{b}\ /\|\ \vec{b}\ \|$ and $\vec{j}$ a unit vector in the subspace $\langle \vec{b}, \vec{c} \rangle$ which is perpendicular on $\vec{b}$. Finally, one can choose $\vec{k} = \vec{i} \times \vec{j}$. By computing the two sides of the equality 6.5, in terms of coordinates and the vectors $\vec{i}, \vec{j}, \vec{k}$, one gets the same result. □

Corollary 6.4. *1.* $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{b} \cdot \vec{c})\,\vec{a} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{a} \end{vmatrix}, \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$;

2. $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0},\ \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$ *(Jacobi's identity).*

Proof. While the first identity follows immediately via 6.5, for the Jacobi's identity we get successively:

$$\begin{aligned} &\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) \\ &= (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c} + (\vec{b} \cdot \vec{a})\,\vec{c} - (\vec{b} \cdot \vec{c})\,\vec{a} + (\vec{c} \cdot \vec{b})\,\vec{a} - (\vec{c} \cdot \vec{a})\,\vec{b} = \vec{0}. \end{aligned}$$

□

6.5 Problems

1. **(2p)** Show that $\|\ \vec{a} \times \vec{b}\ \| \leq \|\ \vec{a}\ \| \cdot \|\ \vec{b}\ \|,\ \forall\ \vec{a}, \vec{b}, \in \mathcal{V}$.

   Solution.

2. **(3p)** Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be pairwise noncollinear vectors. Show that the necesssary and sufficient condition for the existence of a triangle ABC with the properties $\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$, $\overrightarrow{AB}=\vec{c}$ is

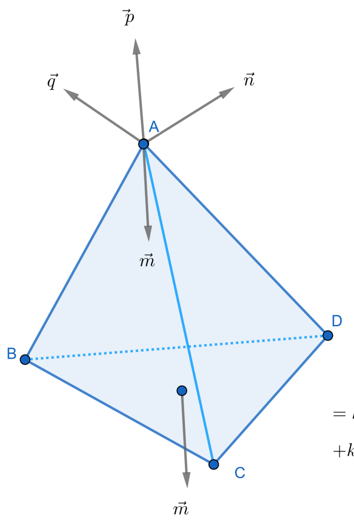
$$\vec{a} \times \vec{b}=\vec{b} \times \vec{c}=\vec{c} \times \vec{a} .$$

From the equalities of the norms deduce the low of sines.

Solution.

3. **(3p)** Show that the sum of some outer-pointing vectors perpendicular on the faces of a tetrahedron which are proportional to the areas of the faces is the zero vector.

   Solution.



The proportionality of $\vec{m}, \vec{n}, \vec{p}, \vec{q}$ with the areas of the corresponding faces of the tetrahedron show that

$\vec{m} = k\overrightarrow{BD} \times \overrightarrow{BC},\ \vec{n} = k\overrightarrow{AC} \times \overrightarrow{AD}$

$\vec{p} = k\overrightarrow{AD} \times \overrightarrow{AB},\ \vec{q} = k\overrightarrow{AB} \times \overrightarrow{AC}$

Thus, $\vec{m} + \vec{n} + \vec{p} + \vec{q}$

$= k\overrightarrow{BD} \times \overrightarrow{BC} + k\overrightarrow{AC} \times \overrightarrow{AD} +$

$+ k\overrightarrow{AD} \times \overrightarrow{AB} + k\overrightarrow{AB} \times \overrightarrow{AC}$

$= k(\overrightarrow{AD} - \overrightarrow{AB}) \times (\overrightarrow{AC} - \overrightarrow{AB}) + k\overrightarrow{AC} \times \overrightarrow{AD}$

$+ k\overrightarrow{AD} \times \overrightarrow{AB} + k\overrightarrow{AB} \times \overrightarrow{AC} =$

$= k\overrightarrow{AD} \times \overrightarrow{AC} - k\overrightarrow{AD} \times \overrightarrow{AB} - k\overrightarrow{AB} \times \overrightarrow{AC} + k\overrightarrow{AB} \times \overrightarrow{AB} =$

$+ k\overrightarrow{AC} \times \overrightarrow{AD} + k\overrightarrow{AD} \times \overrightarrow{AB} + k\overrightarrow{AB} \times \overrightarrow{AC} = \vec{0}.$

4. **(2p)** Find the distance from the point $P(1, 2, -1)$ to the straight line $(d)\ x = y = z$.

   Solution.

5. **(3p)** Find the area of the triangle ABC and the lengths of its hights, where $A(-1,1,2)$, $B(2,-1,1)$ and $C(2,-3,-2)$.

   Solution.

6. **(3p)** Let d_1, d_2, d_3, d_4 be pairwise skew straight lines. Assuming that $d_{12} \perp d_{34}$ and $d_{13} \perp d_{24}$, show that $d_{14} \perp d_{23}$, where d_{ik} is the common perpendicular of the lines d_i and d_k.

   Solution. A director vector of the common perpendicular d_{ij} is $\vec{d}_i \times \vec{d}_j$, where $\vec{d}_r$ stands for a director vector of d_r. Therefore we have successively:

$$\begin{aligned} d_{12} \perp d_{34} \quad & \Leftrightarrow \vec{d}_1 \times \vec{d}_2 \perp \vec{d}_3 \times \vec{d}_4 \Leftrightarrow (\vec{d}_1 \times \vec{d}_2) \cdot (\vec{d}_3 \times \vec{d}_4) = 0 \\ & \Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_3 & \vec{d}_1 \cdot \vec{d}_4 \\ \vec{d}_2 \cdot \vec{d}_3 & \vec{d}_2 \cdot \vec{d}_4 \end{vmatrix} = 0 \Leftrightarrow (\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_2 \cdot \vec{d}_3). \end{aligned}$$

   Similalry

$$\begin{aligned} d_{13} \perp d_{24} \quad & \Leftrightarrow \vec{d}_1 \times \vec{d}_3 \perp \vec{d}_2 \times \vec{d}_4 \Leftrightarrow (\vec{d}_1 \times \vec{d}_3) \cdot (\vec{d}_2 \times \vec{d}_4) = 0 \\ & \Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_2 & \vec{d}_1 \cdot \vec{d}_4 \\ \vec{d}_3 \cdot \vec{d}_2 & \vec{d}_3 \cdot \vec{d}_4 \end{vmatrix} = 0 \Leftrightarrow (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_3 \cdot \vec{d}_2). \end{aligned}$$

   Therefore we have

$$(\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_2 \cdot \vec{d}_3) = (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4),$$

   which shows that

$$(\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) - (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4) = 0 \Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_2 & \vec{d}_1 \cdot \vec{d}_3 \\ \vec{d}_4 \cdot \vec{d}_2 & \vec{d}_4 \cdot \vec{d}_3 \end{vmatrix} = 0 \Leftrightarrow d_{14} \perp d_{23}.$$

7 Week 7: The triple scalar product

The *triple scalar product* $(\vec{a}, \vec{b}, \vec{c})$ of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the real number $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

Proposition 7.1. *If* $[\vec{i}, \vec{j}, \vec{k}]$ *is a direct orthonormal basis and*

$$\begin{aligned}\vec{a} &= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}\\ \vec{b} &= b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}\\ \vec{c} &= c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}\end{aligned}$$

then

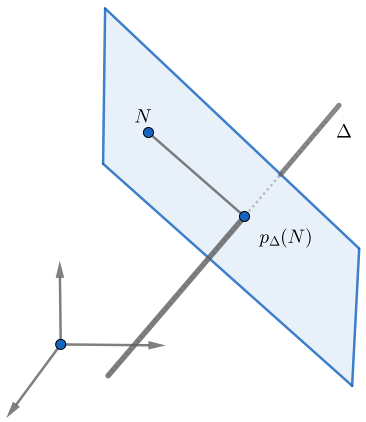
$$(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{7.1}$$

Proof. Indeed, we have successively:

$$\begin{aligned}(\vec{a}, \vec{b}, \vec{c}) &= (\vec{a} \times \vec{b}) \cdot \vec{c}\\ &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \right) \cdot (c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k})\\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.\end{aligned}$$

□

Remark 7.1. Taking into account the formula (7.2) for the distance $\delta(N, \Delta)$ from the point $N(x_N, y_N, z_N)$ to the straight line $\Delta : \dfrac{x - x_0}{p} = \dfrac{y - y_0}{q} = \dfrac{z - z_0}{r}$ as well as Proposition 6.3 we deduce that



$$\begin{aligned}\delta(N, \Delta) &= \| \overrightarrow{Np_{\Delta}(N)} \| \\ &= \| \overrightarrow{NO} + \overrightarrow{Op_{\Delta}(N)} \| = \left\| \overrightarrow{NA_0} - \frac{\vec{d}_{\Delta} \cdot \overrightarrow{NA_0}}{\| \vec{d}_{\Delta} \|^2} \vec{d}_{\Delta} \right\|\end{aligned} \tag{7.2}$$

$$= \frac{\| \left(\vec{d}_\Delta \cdot \vec{d}_\Delta\right) \overrightarrow{NA_0} - \left(\vec{d}_\Delta \cdot \overrightarrow{NA_0}\right) \vec{d}_\Delta \|}{\| \vec{d}_\Delta \|^2}$$
$$= \frac{\| \vec{d}_\Delta \times (\overrightarrow{NA_0} \times \vec{d}_\Delta) \|}{\| \vec{d}_\Delta \|^2} = \frac{\| \overrightarrow{NA_0} \times \vec{d}_\Delta \|}{\| \vec{d}_\Delta \|}.$$

Thus, we recovered the distance formula from one point to one straight line (see formula 6.4) by using different arguments.

Corollary 7.2. 1. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent (collinear) iff $(\vec{a}, \vec{b}, \vec{c}) = 0$*

2. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly independent (noncollinear) if and only if $(\vec{a}, \vec{b}, \vec{c}) \neq 0$*

3. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ form a basis of the space $\mathcal{V}$ if and only if $(\vec{a}, \vec{b}, \vec{c}) \neq 0$.*

4. *The correspondence $F : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$, $F(\vec{a}, \vec{b}, \vec{c}) = (\vec{a}, \vec{b}, \vec{c})$ is trilinear and skew-symmetric, i.e.*

$$\begin{array}{l} (\alpha \vec{a} + \alpha' \vec{a}\,', \vec{b}, \vec{c}) = \alpha(\vec{a}, \vec{b}, \vec{c}) + \alpha'(\vec{a}\,', \vec{b}, \vec{c}) \\ (\vec{a}, \beta \vec{b} + \beta' \vec{b}\,', \vec{c}) = \beta(\vec{a}, \vec{b}, \vec{c}) + \beta'(\vec{a}, \vec{b}\,', \vec{c}) \\ (\vec{a}, \vec{b}, \gamma \vec{c} + \gamma' \vec{c}\,') = \gamma(\vec{a}, \vec{b}, \vec{c}) + \gamma'(\vec{a}, \vec{b}, \vec{c}\,') \end{array} \tag{7.3}$$

$\forall\, \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{R}$, $\forall\ \vec{a}, \vec{b}, \vec{c}, \vec{a}\,', \vec{b}\,', \vec{c}\,' \in \mathcal{V}$ *şi*

$$(\vec{a}_1, \vec{a}_2, \vec{a}_3) = sgn(\sigma)(\vec{a}_{\sigma(1)}, \vec{a}_{\sigma(2)}, \vec{a}_{\sigma(3)}),\ \forall\ \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V} \text{ şi } \forall\, \sigma \in S_3 \tag{7.4}$$

Remark 7.2. *One can rewrite the relations (7.4) as follows:*

$$\begin{array}{rcl} (\vec{a}_1, \vec{a}_2, \vec{a}_3) & = & (\vec{a}_2, \vec{a}_3, \vec{a}_1) = (\vec{a}_3, \vec{a}_1, \vec{a}_2) \\ & = & -(\vec{a}_2, \vec{a}_1, \vec{a}_3) = -(\vec{a}_1, \vec{a}_3, \vec{a}_2) = -(\vec{a}_3, \vec{a}_2, \vec{a}_1), \end{array}$$

$\forall\ \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V}$

Corollary 7.3. 1. $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})\ \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.

2. *For every $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathcal{V}$ the* Laplace formula holds:

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}.$$

Proof. While the first identity is obvious, for the Laplace formula we have successively:

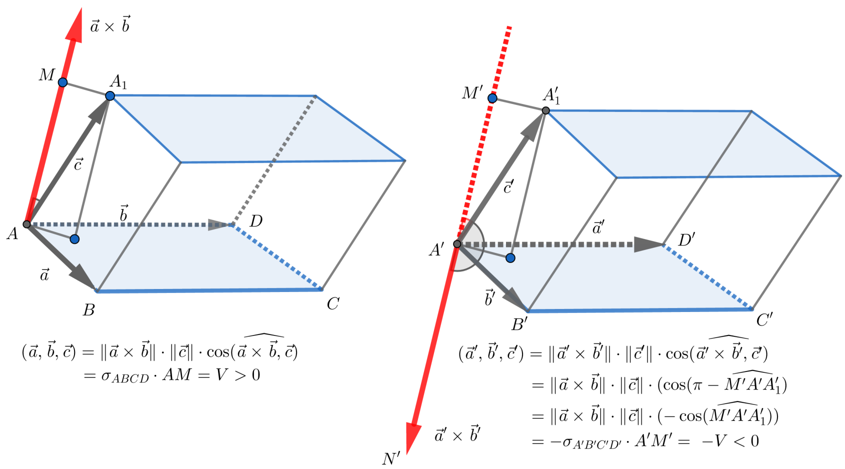
$$\begin{aligned} (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= \left(\vec{a}, \vec{b}, \vec{c} \times \vec{d}\right) = \left(\vec{c} \times \vec{d}, \vec{a}, \vec{b}\right) \\ &= \left[(\vec{c} \times \vec{d}) \times \vec{a}\right] \cdot \vec{b} = -\left[(\vec{a} \cdot \vec{d})\, \vec{c} - (\vec{a} \cdot \vec{c})\, \vec{d}\right] \cdot \vec{b} \\ &= -(\vec{a} \cdot \vec{d})(\vec{c} \cdot \vec{b}) + (\vec{a} \cdot \vec{c})(\vec{d} \cdot \vec{b}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}. \end{aligned}$$

□

Definition 7.1. The basis $[\vec{a}, \vec{b}, \vec{c}]$ of the space $\mathcal{V}$ is said to be *directe* if $(\vec{a}, \vec{b}, \vec{c}) > 0$. If, on the contrary, $(\vec{a}, \vec{b}, \vec{c}) < 0$, we say that the basis $[\vec{a}, \vec{b}, \vec{c}]$ is *inverse*

Definition 7.2. The *oriented* volume of the parallelepiped constructed on the noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ is $\varepsilon \cdot V$, where V is the volume of this parallelepiped and $\varepsilon = +1$ or -1 insomuch as the basis $[\vec{a}, \vec{b}, \vec{c}]$ is directe or inverse respectively.

Proposition 7.4. *The triple scalar product $(\vec{a}, \vec{b}, \vec{c})$ of the noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ is equal with the oriented volume of the parallelepiped constructed on these vectors.*



7.1 Applications of the triple scalar product

7.1.1 The distance between two straight lines

If d_1, d_2 are two straight lines, then the distance between them, denoted by $\delta(d_1, d_2)$, is being defined as

$$\min\{|| \overrightarrow{M_1M_2} || \mid M_1 \in d_1,\ M_2 \in d_2\}.$$

1. If $d_1 \cap d_2 \neq \emptyset$, then $\delta(d_1, d_2) = 0$.

2. If $d_1 || d_2$, then $\delta(d_1, d_2) = || \overrightarrow{MN} ||$ where $\{M\} = d \cap d_1$, $\{N\} = d \cap d_2$ and d is a straight line perpendicular to the lines d_1 and d_2. Obviously $|| \overrightarrow{MN} ||$ is independent on the choice of the line d.

3. We now assume that the straight lines d_1, d_2 are noncoplanar (skew lines). In this case there exits a unique straight line d such that $d \perp d_1, d_2$ and $d \cap d_1 = \{M_1\}$, $d \cap d_2 = \{M_2\}$. The straight line d is called the *common perpendicular* of the lines d_1, d_2 and obviously $\delta(d_1, d_2) = || \overrightarrow{M_1M_2} ||$.

Assume that the straight lines d_1, d_2 are given by their points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$ and their vectors şi au vectorii directori $\vec{d}_1\ (p_1, q_1, r_1)$ $\vec{d}_2\ (p_2, q_2, r_2)$, that is, thei equations are

$$d_1 : \frac{x - x_1}{p_1} = \frac{y - y_1}{q_1} = \frac{z - z_1}{r_1}$$

$$d_2 : \frac{x - x_2}{p_2} = \frac{y - y_2}{q_2} = \frac{z - z_2}{r_2}.$$

The common perpendicular of the lines d_1, d_2 is the intersection line between the plane containing the line d_1 which is parallel to the vector $\vec{d}_1 \times \vec{d}_2$, and the plane containing the line d_2 which is parallel to $\vec{d}_1 \times \vec{d}_2$. Since

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} \vec{i} + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} \vec{j} + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \vec{k}$$

it follows that the equations of the common perpendicular are

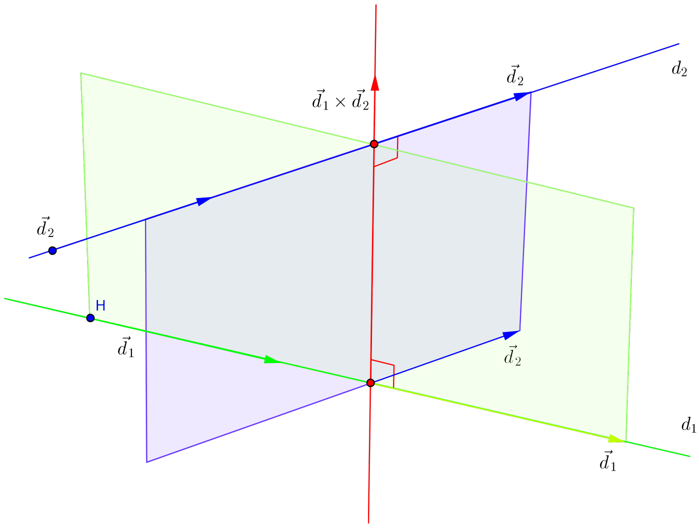


Figure 8: Perpendiculara comună a dreptelor d_1 şi d_2

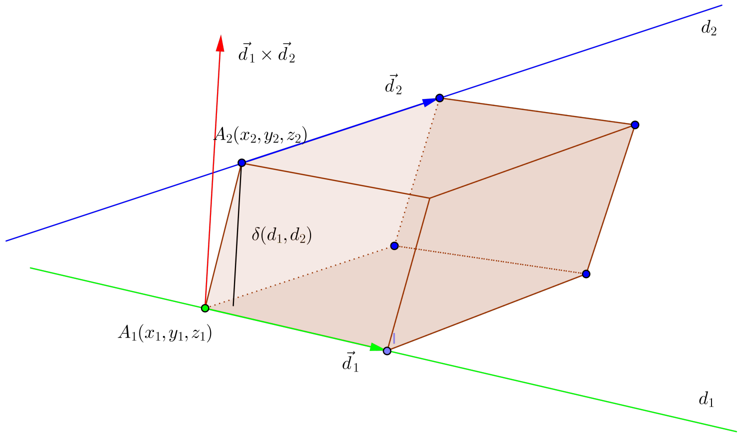
$$\begin{cases} \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ p_1 & q_1 & r_1 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0 \\ \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ p_2 & q_2 & r_2 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0. \end{cases} \tag{7.5}$$

The distance between the straight lines d_1, d_2 can be also regarded as the height of the parallelogram constructed on the vectors $\vec{d}_1, \vec{d}_2, \vec{d}_1 \times \vec{d}_2$. Thus

$$\delta(d_1, d_2) = \frac{|(\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2)|}{|| \vec{d}_1 \times \vec{d}_2 ||}. \tag{7.6}$$

Therefore we obtain

$$\delta(d_1, d_2) = \frac{\left| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} \right|}{\sqrt{\begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}^2 + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}^2 + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}^2}} \tag{7.7}$$



7.1.2 The coplanarity condition of two straight lines

Using the notations of the previous section, observe that the straight lines d_1, d_2 are coplanar if and only if the vectors $\overrightarrow{A_1A_2}$, $\vec{d}_1$, $\vec{d}_2$ are linearly dependent (coplanar), or equivalently $(\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2) = 0$. Consequently the stright lines d_1, d_2 are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0 \tag{7.8}$$

7.2 Problems

1. **(2p)** Show that

   (a) $|(\vec{a}, \vec{b}, \vec{c})| \leq \| \vec{a} \| \cdot \| \vec{b} \| \cdot \| \vec{c} \|$;
   Solution.

(b) **(2p)** $(\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a})=2(\vec{a},\vec{b},\vec{c})$.
Solution.

2. **(3p)** Prove the following identity:

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=(\vec{a},\vec{c},\vec{d})\,\vec{b}-(\vec{b},\vec{c},\vec{d})\,\vec{a}=(\vec{a},\vec{b},\vec{d})\,\vec{c}-(\vec{a},\vec{b},\vec{c})\,\vec{d}\,.$$

Solution. By using the identity $\vec{u}\times(\vec{v}\times\vec{w})=(\vec{u}\cdot\vec{w})\,\vec{v}-(\vec{u}\cdot\vec{v})\,\vec{v}$ for $\vec{u}=\vec{a}\times\vec{b}$, $\vec{v}=\vec{c}$ and $\vec{w}=\vec{d}$ we obtain

$$\begin{aligned}(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) &=\vec{u}\times(\vec{v}\times\vec{w})=(\vec{u}\cdot\vec{w})\,\vec{v}-(\vec{u}\cdot\vec{v})\,\vec{v}\\ &=\left[(\vec{a}\times\vec{b})\cdot\vec{d}\right]\vec{c}-\left[(\vec{a}\times\vec{b})\cdot\vec{c}\right]\vec{d}\\ &=(\vec{a},\vec{b},\vec{d})\,\vec{c}-(\vec{a},\vec{b},\vec{c})\,\vec{d}\,.\end{aligned}$$

By using the identity $(\vec{u}\times\vec{v})\times\vec{w}=(\vec{u}\cdot\vec{w})\,\vec{v}-(\vec{v}\cdot\vec{w})\,\vec{u}$ for $\vec{u}=\vec{a}$, $\vec{v}=\vec{b}$ and $\vec{w}=\vec{c}\times\vec{d}$ we obtain

$$\begin{aligned}(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) &=(\vec{u}\times\vec{v})\times\vec{w}=(\vec{u}\cdot\vec{w})\,\vec{v}-(\vec{v}\cdot\vec{w})\,\vec{u}\\ &=\left[\vec{a}\cdot(\vec{c}\times\vec{d})\right]\vec{b}-\left[\vec{b}\cdot(\vec{c}\times\vec{d})\right]\vec{a}\\ &=(\vec{a},\vec{c},\vec{d})\,\vec{b}-(\vec{b},\vec{c},\vec{d})\,\vec{a}\,.\end{aligned}$$

3. **(3p)** Prove the following identity: $(\vec{u}\times\vec{v},\vec{v}\times\vec{w},\vec{w}\times\vec{u})=(\vec{u},\vec{v},\vec{w})^2$.

Solution. We have successively:

$$\begin{aligned}(\vec{u}\times\vec{v},\vec{v}\times\vec{w},\vec{w}\times\vec{u}) &=\left[(\vec{u}\times\vec{v})\times(\vec{v}\times\vec{w})\right]\cdot(\vec{w}\times\vec{u})\\ &=\left[(\vec{u},\vec{v},\vec{w})\,\vec{v}-(\vec{u},\vec{v},\vec{v})\,\vec{w}\right]\cdot(\vec{w}\times\vec{u})\\ &=(\vec{u},\vec{v},\vec{w})\left[\vec{v}\cdot(\vec{w}\times\vec{u})\right]=(\vec{u},\vec{v},\vec{w})(\vec{v},\vec{w},\vec{u})=(\vec{u},\vec{v},\vec{w})^2.\end{aligned}$$

4. **(3p)** The *reciprocal vectors* of the noncoplanar vectors $\vec{u},\vec{v},\vec{w}$ are defined by

$$\vec{u}'=\frac{\vec{v}\times\vec{w}}{(\vec{u},\vec{v},\vec{w})},\ \vec{v}'=\frac{\vec{w}\times\vec{u}}{(\vec{u},\vec{v},\vec{w})},\ \vec{w}'=\frac{\vec{u}\times\vec{v}}{(\vec{u},\vec{v},\vec{w})}.$$

Show that:

(a)

$$\begin{aligned}\vec{a} &=(\vec{a}\cdot\vec{u}')\,\vec{u}+(\vec{a}\cdot\vec{v}')\,\vec{v}+(\vec{a}\cdot\vec{w}')\,\vec{w}\\ &=\frac{(\vec{a},\vec{v},\vec{w})}{(\vec{u},\vec{v},\vec{w})}\,\vec{u}+\frac{(\vec{u},\vec{a},\vec{w})}{(\vec{u},\vec{v},\vec{w})}\,\vec{v}+\frac{(\vec{u},\vec{v},\vec{a})}{(\vec{u},\vec{v},\vec{w})}\,\vec{w}\,.\end{aligned}$$

(b) the reciprocal vectors of $\vec{u}',\vec{v}',\vec{w}'$ are the vectors $\vec{u},\vec{v},\vec{w}$.

Solution. (4a) Obviously $\vec{a}=\alpha\,\vec{u}+\beta\,\vec{v}+\gamma\,\vec{c}$, as $\vec{u},\vec{v},\vec{w}$ are three linearly independent vectors of the three dimensional vector space $\mathcal{V}$, i.e. $\vec{u},\vec{v},\vec{w}$ form a basis of $\mathcal{V}$. Moreover we have

$$\begin{aligned}\vec{a}\cdot\vec{u}' &=\frac{\vec{a}\cdot(\vec{v}\times\vec{w})}{(\vec{u},\vec{v},\vec{w})}=\frac{(\vec{a},\vec{v},\vec{w})}{(\vec{u},\vec{v},\vec{w})}=\frac{(\alpha\,\vec{u}+\beta\,\vec{v}+\gamma\,\vec{c},\vec{v},\vec{w})}{(\vec{u},\vec{v},\vec{w})}\\ &=\frac{\alpha(\vec{u},\vec{v},\vec{w})+\beta(\vec{v},\vec{v},\vec{w})+\gamma(\vec{w},\vec{v},\vec{w})}{(\vec{u},\vec{v},\vec{w})}=\alpha.\end{aligned}$$

One can similalry show that

$$\vec{a}\cdot\vec{v}' = \frac{(\vec{u},\vec{a},\vec{w})}{(\vec{u},\vec{v},\vec{w})} = \beta \text{ and } \vec{a}\cdot\vec{w}' = \frac{(\vec{u},\vec{v},\vec{a})}{(\vec{u},\vec{v},\vec{w})} = \gamma.$$

(4b) Let us first observe that

$$(\vec{u}',\vec{v}',\vec{w}') = (\vec{w}',\vec{u}',\vec{v}') = \frac{(\vec{u}\times\vec{v},\vec{v}\times\vec{w},\vec{w}\times\vec{u})}{(\vec{u},\vec{v},\vec{w})^3} = \frac{(\vec{u},\vec{v},\vec{w})^2}{(\vec{u},\vec{v},\vec{w})^3} = \frac{1}{(\vec{u},\vec{v},\vec{w})}.$$

On the other hand we have:

$$\frac{\vec{v}'\times\vec{w}'}{(\vec{u}',\vec{v}',\vec{w}')} = (\vec{u},\vec{v},\vec{w})(\vec{v}'\times\vec{w}') = (\vec{u},\vec{v},\vec{w})\frac{(\vec{w}\times\vec{u})\times(\vec{u}\times\vec{v})}{(\vec{u},\vec{v},\vec{w})^2} = \frac{(\vec{w},\vec{u},\vec{v})\,\vec{u} - (\vec{w},\vec{u},\vec{u})\,\vec{v}}{(\vec{u},\vec{v},\vec{w})} = \vec{u}.$$

One can similarly show that

$$\frac{\vec{w}'\times\vec{u}'}{(\vec{u}',\vec{v}',\vec{w}')} = \vec{v} \text{ and } \frac{\vec{u}'\times\vec{v}'}{(\vec{u}',\vec{v}',\vec{w}')} = \vec{w}.$$

5. **(2p)** Find the value of the parameter α for which the pencil of planes through the straight line AB has a common plane with the pencil of planes through the straight line CD, where $A(1, 2\alpha, \alpha)$, $B(3, 2, 1)$, $C(-\alpha, 0, \alpha)$ and $D(-1, 3, -3)$.

   Solution.

6. **(2p)** Find the value of the parameter λ for which the straight lines

$$(d_1)\ \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1},\ (d_2)\ \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{\lambda}$$

are coplanar. Find the coordinates of their intersection point in that case.

Solution.

7. **(2p)** Find the distance between the straight lines

$$(d_1)\ \frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{1},\ (d_2)\ \frac{x+1}{3}=\frac{y}{4}=\frac{z-1}{3}$$

as well as the equations of the common perpendicular.

Solution.

8. **(2p)** Find the distance between the straight lines M_1M_2 and d, where $M_1(-1,0,1)$, $M_2(-2,1,0)$ and

$$(d)\left\{\begin{array}{rcrcrcl} x & + & y & + & z & = & 1 \\ 2x & - & y & - & 5z & = & 0. \end{array}\right.$$

as well as the equations of the common perpendicular.

Solution.

References

[1] Andrica, D., Ţopan, L., Analytic geometry, Cluj University Press, 2004.

[2] Galbură Gh., Radó, F., Geometrie, Editura didactică şi pedagogică-Bucureşti, 1979.

[3] Pintea, C. Geometrie. Elemente de geometrie analitică. Elemente de geometrie diferenţială a curbelor şi suprafeţelor, Presa Universitară Clujeană, 2001.

[4] Radó, F., Orban, B., Groze, V., Vasiu, A., Culegere de Probleme de Geometrie, Lit. Univ. "Babeş-Bolyai", Cluj-Napoca, 1979.

