11.1.

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & , (x,y) \neq O_2 \\ 0 & , (x,y) = O_2 \end{cases}$$

$$\forall (x,y) \in \mathbb{R}^2 \setminus \{O_2\},\ 0 \leq |f(x,y) - f(O_2)| = \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \frac{|xy|}{\sqrt{x^2+y^2}}$$

$$|xy| \boxed{\leq} x^2 + y^2 \iff x^2y^2 \boxed{\leq} x^4 + 2x^2y^2 + y^4 \iff 0 \boxed{\leq} x^4 + x^2y^2 + y^4$$

$$\Rightarrow \sqrt{|xy|} \leq \sqrt{x^2+y^2} \Rightarrow \frac{1}{\sqrt{|xy|}} \geq \frac{1}{\sqrt{x^2+y^2}}$$

$$0 \leq \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{|xy|}{\sqrt{|xy|}} = \sqrt{|xy|} \xrightarrow[(x,y)\to(0,0)]{} 0 \Rightarrow$$

$\Rightarrow$ By the Squeeze Thm., $\lim\limits_{(x,y)\to O_2} f(x,y) = f(O_2) = 0 \Rightarrow$

$\Rightarrow f$ is continuous on $O_2 \Rightarrow f$ is continuous on $\mathbb{R}^2$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x \to 0} \frac{\frac{x \cdot 0}{\sqrt{x^2+0}} - 0}{x} = 0 \Rightarrow$$

$\Rightarrow f$ is partially differentiable w.r.t. x at O_2 $\left(\frac{\partial f}{\partial x}(0,0) = 0\right)$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y \to 0} \frac{\frac{0 \cdot y}{\sqrt{0+y^2}} - 0}{y} = 0 \Rightarrow$$

$\Rightarrow f$ is partially differentiable w.r.t. y at O_2 $\left(\frac{\partial f}{\partial y}(0,0) = 0\right)$ *

$\Rightarrow f$ is partially differentiable at O_2

11.2

$f: \mathbb{R}^3 \to \mathbb{R}, f(x) = e^{2x+y} \cdot \cos(3z)$

$$\frac{\partial f}{\partial x}(x,y,z) = \left(e^{2x+y} \cdot \cos(3z)\right)' = (2x+y)' \cdot e^{2x+y} \cdot \cos(3z) = 2e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial f}{\partial x}\left(0,0,\frac{\pi}{6}\right) = 0$$

$$\frac{\partial f}{\partial y}(x,y,z) = \left(e^{2x+y} \cdot \cos(3z)\right)' = (2x+y)' \cdot e^{2x+y} \cdot \cos(3z) = e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial f}{\partial y}\left(0,0,\frac{\pi}{6}\right) = 0$$

$$\frac{\partial f}{\partial z}(x,y,z) = \left(e^{2x+y} \cdot \cos(3z)\right)' = -3 \cdot e^{2x+y} \cdot \sin(3z)$$

$$\frac{\partial f}{\partial z}\left(0,0,\frac{\pi}{6}\right) = -3$$

$$\Rightarrow \nabla f\left(0,0,\frac{\pi}{6}\right) = (0,0,-3)$$

$$\frac{\partial f}{\partial x^2}(x,y,z) = 2 \cdot (2x+y)' \cdot e^{2x+y} \cdot \cos(3z) = 4e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial f}{\partial x^2}\left(0,0,\frac{\pi}{6}\right) = 0$$

$$\frac{\partial f}{\partial y^2}(x,y,z) = e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial f}{\partial y^2}\left(0,0,\frac{\pi}{6}\right) = 0$$

$$\frac{\partial f}{\partial z^2}(x,y,z) = -3 \cdot e^{2x+y} \cdot (\sin 3z)' = -9 \cdot e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial f}{\partial z^2}(0,0,\tfrac{\pi}{6}) = 0$$

$$\frac{\partial f}{\partial y \partial x}(x,y,z) = 2e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial f}{\partial y \partial x}(0,0,\tfrac{\pi}{6}) = 0$$

$$\frac{\partial f}{\partial z \partial x}(x,y,z) = -6e^{2x+y} \cdot \sin(3z)$$

$$\frac{\partial f}{\partial z \partial x}(0,0,\tfrac{\pi}{6}) = -6$$

$$\Rightarrow H_f(0,0,\tfrac{\pi}{6}) = \begin{pmatrix} 0 & 0 & -6 \\ 0 & 0 & -3 \\ -6 & -3 & 0 \end{pmatrix}$$

$$\frac{\partial f}{\partial x \partial y}(x,y,z) = 2e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial f}{\partial x \partial y}(0,0,\tfrac{\pi}{6}) = 0$$

$$\frac{\partial f}{\partial z \partial y}(x,y,z) = -3e^{2x+y} \cdot \sin(3z)$$

$$\frac{\partial f}{\partial z \partial y}(0,0,\tfrac{\pi}{6}) = -3$$

$$\frac{\partial f}{\partial x \partial z}(x,y,z) = -6e^{2x+y} \cdot \sin(3z)$$

$$\frac{\partial f}{\partial x \partial z}(0,0,\tfrac{\pi}{6}) = -6$$

$$\frac{\partial f}{\partial y \partial z}(x,y,z) = -3e^{2x+y} \cdot \sin(3z)$$

$$\frac{\partial f}{\partial y \partial z}(0,0,\tfrac{\pi}{6}) = -3$$

12.1

$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + y^3 - 3xy$

$$\frac{\partial f}{\partial x} = 3x^2 - 3y, \quad \frac{\partial f}{\partial y} = 3y^2 - 3x \Rightarrow \begin{cases} 3x^2 - 3y = 0 \mid :3 \\ 3y^2 - 3x = 0 \mid :3 \end{cases} \Leftrightarrow \begin{cases} y = x^2 \\ x^4 - x = 0 \end{cases}$$

$\cancel{x^4 - x = 0 \Leftrightarrow x^4 = x \Leftrightarrow}$

$$x^4 - x = 0 \Leftrightarrow x(x^3 - 1) = 0 \Leftrightarrow \begin{cases} x = 0 \Leftrightarrow y = 0 \\ x = 1 \Leftrightarrow y = 1 \end{cases} \Rightarrow$$

$\Rightarrow$ Stationary points: $(0,0)$, $(1,1)$

$$\frac{\partial f}{\partial x^2} = 6x, \quad \frac{\partial f}{\partial y^2} = 6y, \quad \frac{\partial f}{\partial y \partial x} = -3, \quad \frac{\partial f}{\partial x \partial y} = -3$$

$(0,0)$:

$H_{f(0,0)} = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$: $\Delta_2 = -9 < 0 \Rightarrow$

$\Rightarrow H_{f(0,0)}$ indefinite $\Rightarrow (0,0)$ is not a local ~~extremum~~ point

$(1,1)$:

$H_{f(1,1)} = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$: $\Delta_2 = 36 - 9 = 27 > 0$

$\Delta_1 = 6 > 0 \Rightarrow H_{f(1,1)}$ is positive definite $\Rightarrow$

$\Rightarrow (1,1)$ is a local minimum point of f

13.1

$f:[1,\infty)\to\mathbb{R}, f(x)=\dfrac{x^\alpha \arctan x}{1+x^\beta}$

for $\forall x\in[1,\infty), f(x)\geq 0$

$L=\lim\limits_{x\to\infty} x^\lambda\cdot\dfrac{x^\alpha \arctan x}{1+x^\beta}=\lim\limits_{x\to\infty}\dfrac{x^{\lambda+\alpha}\cdot\arctan x}{x^\beta\left(\frac{1}{x^\beta}+1\right)}\Rightarrow$

$\Rightarrow$ For $\lambda=\beta-\alpha$, $L=\frac{\pi}{2}\in(0,\infty)\Rightarrow$

$\Rightarrow f$ is improper integrable on $[1,\infty)\Leftrightarrow\beta-\alpha>1$