Lecture 7

Functions of several variables

The Euclidean space $\mathbb{R}^n$

Let $n \in \mathbb{N}$. Consider the set $\mathbb{R}^n = \{(x_1, x_2, \dots x_n) \mid x_1, x_2, \dots, x_n \in \mathbb{R}\}$ ($\mathbb{R}^n$ consists of all ordered n-tuples of real numbers). The elements of $\mathbb{R}^n$ are called *vectors* or *points*. If $x \in \mathbb{R}^n$, then $x = (x_1, x_2, \dots, x_n)$ and $x_1, x_2, \dots, x_n$ are called the *coordinates* (or *components*) of the vector x (for $i = 1, 2, \dots, n$, x_i is the i^{th} coordinate (or component) of x).
Vector addition (sum of two vectors): If $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, then $x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in \mathbb{R}^n$.
Scalar multiplication (multiplication of a vector by a number): If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then $\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n) \in \mathbb{R}^n$.
$\mathbb{R}^n$ together with the vector addition and the scalar multiplication

$$\forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \longmapsto x + y \in \mathbb{R}^n$$
$$\forall (\alpha, x) \in \mathbb{R} \times \mathbb{R}^n \longmapsto \alpha x \in \mathbb{R}^n$$

is a vector space over the field $\mathbb{R}$ of real numbers. The zero vector (origin) of this vector space is the point $0_n = (0, 0, \dots, 0)$ (should not be confused with the number 0) and the additive inverse of a point $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is the point denoted by $-x = (-x_1, -x_2, \dots, -x_n) = (-1)x$.
The vectors

$$\begin{aligned} e^1 &= (1, 0, 0, \dots, 0) \in \mathbb{R}^n \\ e^2 &= (0, 1, 0, \dots, 0) \in \mathbb{R}^n \\ &\vdots \\ e^n &= (0, 0, 0, \dots, 1) \in \mathbb{R}^n \end{aligned}$$

form a basis of the vector space $\mathbb{R}^n$ called the *standard (canonical) basis* of $\mathbb{R}^n$. If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then $x = x_1 e^1 + x_2 e^2 + \dots + x_n e^n$.

Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. The real number defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

is called the *scalar product of x and y*. The nonnegative number

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{(x_1)^2 + (x_2)^2 + \dots + (x_n)^2}$$

is called the *Euclidean norm of x*. The *Euclidean distance between x and y* is given by

$$\|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}.$$

Remark 1. For $x \in \mathbb{R}^n$, $\|x\|$ represents the Euclidean distance between x and 0_n.

Remark 2. A vector $x \in \mathbb{R}^n$ is called a *unit vector* if $\|x\| = 1$. For $i \in \{1, \dots, n\}$, the vectors e^i are unit vectors called the *canonical unit vectors*.

If $y \in \mathbb{R}^n \setminus \{0_n\}$, we can define the vector $\dfrac{1}{\|y\|} y$ which is a unit vector (we say that we *normalize* y).

Example 1. (i) $n = 1$: every vector $x \in \mathbb{R}$ can be identified with exactly one point on the real line. If $x, y \in \mathbb{R}$, then $\langle x, y\rangle = xy$, $\|x\| = |x|$ and $\|x - y\| = |x - y|$.

(ii) $n = 2$: every vector $(x, y) \in \mathbb{R}^2$ can be identified with exactly one point in a plane Cartesian coordinate system Oxy. If $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2) \in \mathbb{R}^2$, then, by the Pythagorean Theorem, the length of the segment $[P_1P_2]$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, which is exactly the Euclidean distance between the vectors (x_1, y_1) and (x_2, y_2).

(iii) $n = 3$: every vector $(x, y, z) \in \mathbb{R}^3$ can be identified with exactly one point in a Cartesian coordinate system $Oxyz$. Let $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$. Take $P_3 = (x_2, y_2, z_1)$. Note P_2 and P_3 are on the same vertical line, so the length of the segment $[P_2P_3]$ is $|z_1 - z_2|$. Also, P_1 and P_3 are on the same horizontal plane, so the length of the segment $[P_1P_3]$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Since the points P_1, P_2 and P_3 form a right triangle with right angle at P_3, by the Pythagorean Theorem, we have that the length of the segment $[P_1P_2]$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$, which is exactly the Euclidean distance between the vectors (x_1, y_1, z_1) and (x_2, y_2, z_2).

Proposition 1 (Properties of the scalar product in $\mathbb{R}^n$)**.**

(i) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle, \quad \forall x, y, z \in \mathbb{R}^n$.

(ii) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle, \quad \forall \alpha \in \mathbb{R}, \forall x, y \in \mathbb{R}^n$.

(iii) $\langle x, y\rangle = \langle y, x\rangle, \quad \forall x, y \in \mathbb{R}^n$.

(iv) $\langle x, x\rangle > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0_n\}$.

(v) $\langle 0_n, x\rangle = 0, \quad \forall x \in \mathbb{R}^n$.

(vi) $\langle x, x\rangle = 0 \Leftrightarrow x = 0_n$.

Proof. Follows straightforward from the definition of the scalar product. □

Proposition 2 (Cauchy-Schwarz inequality)**.**

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|, \quad \forall x, y \in \mathbb{R}^n.$$

Proposition 3 (Properties of the Euclidean norm)**.**

(i) $\|x\| = 0 \Leftrightarrow x = 0_n$.

(ii) $\|\alpha x\| = |\alpha| \cdot \|x\|, \quad \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n$.

(iii) $\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in \mathbb{R}^n$ *(the triangle inequality).*

Definition 1. Let $x_0 \in \mathbb{R}^n$ and $r > 0$. The set

- $B(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$ is the *open ball of radius r centered at x_0.*
- $\overline{B}(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ is the *closed ball of radius r centered at x_0.*

Example 2. (i) $n = 1$: let $x_0 \in \mathbb{R}$ and $r > 0$. Then $B(x_0, r) = (x_0 - r, x_0 + r)$ and $\overline{B}(x_0, r) = [x_0 - r, x_0 + r]$.

(ii) $n = 2$: let $(x_0, y_0) \in \mathbb{R}^2$ and $r > 0$. Then $B\left((x_0, y_0), r\right)$ is the open disc of radius r centered at (x_0, y_0) (excluding its enclosing circle) and $\overline{B}\left((x_0, y_0), r\right)$ is the closed disc of radius r centered at (x_0, y_0) (including its enclosing circle).

(iii) $n=3$: let $(x_0,y_0,z_0)\in\mathbb{R}^3$ and $r>0$. Then $B\left((x_0,y_0,z_0),r\right)$ consists of all points inside the sphere of radius r centered at (x_0,y_0,z_0) excluding the sphere itself and $\overline{B}\left((x_0,y_0,z_0),r\right)$ consists of all points inside the sphere of radius r centered at (x_0,y_0,z_0) including the sphere itself.

Remark 3. Let $x_0\in\mathbb{R}^n$, $r_1>r>0$. Then

(i) $x_0\in B(x_0,r)$.

(ii) $B(x_0,r)\subseteq\overline{B}(x_0,r)\subseteq B(x_0,r_1)\subseteq\overline{B}(x_0,r_1)$.

(iii) $\forall x\in B(x_0,r)$, $B\left(x,r-\|x_0-x\|\right)\subseteq B(x_0,r)$.

Definition 2. A *neighborhood* of $x\in\mathbb{R}^n$ is a set $V\subseteq\mathbb{R}^n$ for which $\exists r>0$ such that $B(x,r)\subseteq V$. We denote by $\mathcal{V}(x)$ the family of all neighborhoods of x.

Example 3. $A=[0,1]\times[0,2]\setminus\{0_2\}$.

(i) If $(x,y)\in(0,1)\times(0,2)$, then $A\in\mathcal{V}((x,y))$.

(ii) $A\notin\mathcal{V}(0_2)$.

(iii) if $(x,y)\in A$ with $x\in\{0,1\}$ or $y\in\{0,2\}$, then $A\notin\mathcal{V}((x,y))$.

Sequences in $\mathbb{R}^n$

Notation: $(x^k)_{k\geq 1}$, $(x^k)_{k\in\mathbb{N}}$, or (x^k) (we do not index this sequence by n since n is the dimension of $\mathbb{R}^n$; we use an upper index notation since lower indexes are used for vector coordinates). Written explicitly,

$$\begin{aligned} x^1 &= (x_1^1,x_2^1,\ldots,x_n^1)\in\mathbb{R}^n\\ x^2 &= (x_1^2,x_2^2,\ldots,x_n^2)\in\mathbb{R}^n\\ &\vdots\\ x^k &= (x_1^k,x_2^k,\ldots,x_n^k)\in\mathbb{R}^n\\ &\vdots \end{aligned}$$

The sequences of real numbers $(x_1^k)_{k\in\mathbb{N}}$, $(x_2^k)_{k\in\mathbb{N}}$, …, $(x_n^k)_{k\in\mathbb{N}}$ are called the *component sequences* of the sequence (x^k).

Definition 3. A sequence (x^k) in $\mathbb{R}^n$ is said to *converge* (or *tend*) to $x\in\mathbb{R}^n$ if

$$\forall V\in\mathcal{V}(x),\exists k_V\in\mathbb{N}\text{ such that }\forall k\in\mathbb{N},k\geq k_V\text{ we have }x^k\in V.$$

Remark 4. A sequence in $\mathbb{R}^n$ cannot converge to two distinct vectors in $\mathbb{R}^n$.

Definition 4. If a sequence (x^k) in $\mathbb{R}^n$ converges to some $x\in\mathbb{R}^n$, we say that (x^k) is *convergent* and the vector x is called the *limit* of (x^k).
Notation: $\lim\limits_{k\to\infty}x^k=x$ or $x^k\to x$. If (x^k) does not converge to any vector in $\mathbb{R}^n$, we say that (x^k) is *divergent*.

Proposition 4. *Let (x^k) be a sequence in $\mathbb{R}^n$ with $x^k=(x_1^k,x_2^k,\ldots,x_n^k)$ and let $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$. Then the following are equivalent:*

(i) $\lim\limits_{k\to\infty}x^k=x$.

(ii) $\forall\varepsilon>0,\exists k_\varepsilon\in\mathbb{N},\forall k\in\mathbb{N},k\geq k_\varepsilon,\|x^k-x\|<\varepsilon$.

(iii) $\forall i\in\{1,\ldots,n\},\lim\limits_{k\to\infty}x_i^k=x_i$.

Real-valued functions of several variables

Definition 5. Let $A \subseteq \mathbb{R}^n$ nonempty. A function $f : A \to \mathbb{R}$ is called a *real-valued function of n variables.*

Example 4. (i) $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|x\|$ – the Euclidean norm.

(ii) $f : (0, \infty) \times (0, \infty) \to \mathbb{R}$, $f(x) = \pi(x_1)^2 x_2$, $\forall x = (x_1, x_2) \in (0, \infty) \times (0, \infty)$
– the volume of a right circular cylinder of radius x_1 and height x_2.

Limits of functions

Definition 6. Let $A \subseteq \mathbb{R}^n$. An element $c \in \mathbb{R}^n$ is called an *accumulation point* of A if

$$\forall V \in \mathcal{V}(c), \quad V \cap (A \setminus \{c\}) \neq \emptyset.$$

The set of all accumulation points of A is called the *derived set* of A and is denoted by A'.

Example 5. $A = \{(0, 2)\} \cup (\{1\} \times [0, 2])$.

In the following we consider $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$.

Definition 7. Let $f : A \to \mathbb{R}$ and $c \in A'$. We say that f *has a limit at* c if $\exists L \in \overline{\mathbb{R}}$ such that

$$\forall V \in \mathcal{V}(L), \exists U \in \mathcal{V}(c) \text{ such that } \forall x \in U \cap (A \setminus \{c\}) \text{ we have } f(x) \in V. \tag{1}$$

Remark 5. f cannot have two distinct limits at c.

Definition 8. If $f : A \to \mathbb{R}$ has a limit at $c \in A'$, then the unique $L \in \overline{\mathbb{R}}$ satisfying (1) is called *limit of f at c.*
Notation: $\lim\limits_{x \to c} f(x) = L$.

Theorem 1 (Sequential characterization of limits, Heine)**.** *Let $f : A \to \mathbb{R}$, $c \in A'$ and $L \in \overline{\mathbb{R}}$. Then*

$$\lim_{x \to c} f(x) = L \iff \forall \textit{ sequence } (x^k) \textit{ in } A \setminus \{c\} \textit{ with } \lim_{k \to \infty} x^k = c \textit{ we have } \lim_{k \to \infty} f(x^k) = L.$$

Remark 6. (i) Since by the above result, limits of functions of several variables can be characterized using limits of sequences, limit theorems and rules for functions of several variables can be derived from corresponding ones for sequences. For instance, there exists a Squeeze Theorem for functions of several variables.

(ii) If the limit L exists, then the same value L for the limit must be obtained along all paths to c. If there are paths to c which do not yield the same value, then the limit does not exist.

Example 6. $f : \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$, $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$, has no limit at 0_2.