DATA STRUCTURES AND ALGORITHMS
LECTURE 12

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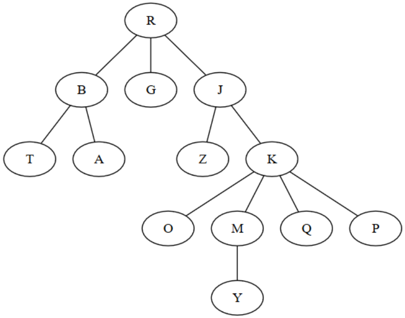
In Lecture 11...

- Hash tables - open addressing
- Trees

Today

- Trees
- Binary Trees
- Binary Search Trees

Tree - Example



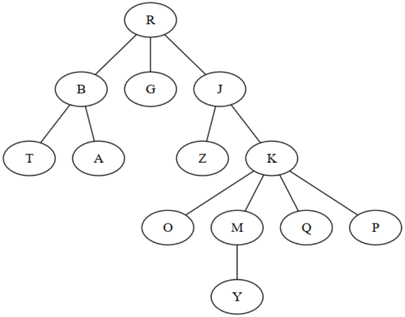
Tree traversals

- A node of a tree is said to be *visited* when the program control arrives at the node, usually with the purpose of performing some operation on the node (printing it, checking the value from the node, etc.).
- *Traversing* a tree means visiting all of its nodes.
- For a k-ary tree there are 2 possible traversals:
    - Depth-first traversal
    - Level order (breadth first) traversal

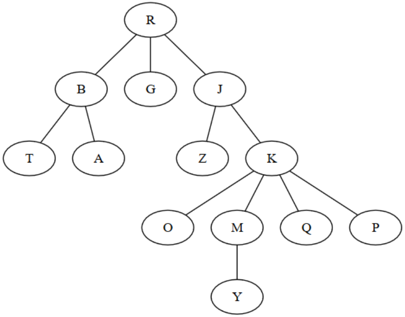
Depth first traversal

- Traversal starts from root
- From root we visit one of the children, than one child of that child, and so on. We go down (in depth) as much as possible, and continue with other children of a node only after all descendants of the "first" child were visited.
- For depth first traversal we use a stack to remember the nodes that have to be visited.

Depth first traversal example



Depth first traversal example

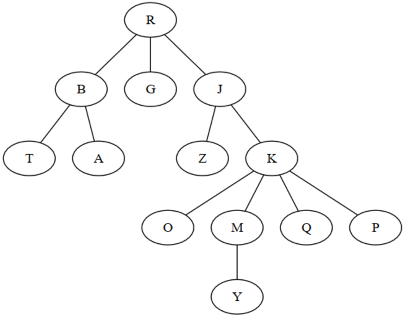


- Stack s with the root: R
- Visit R (pop from stack) and push its children: s = [B G J]
- Visit B and push its children: s = [T A G J]
- Visit T and push nothing: s = [A G J]
- Visit A and push nothing: s = [G J]
- Visit G and push nothing: s = [J]
- Visit J and push its children: s = [Z K]
- etc...

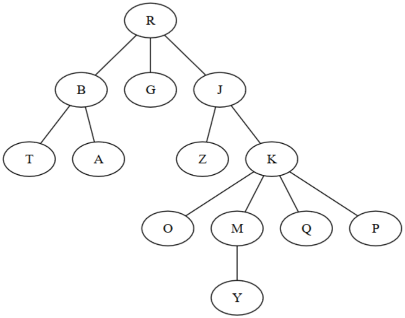
Level order traversal

- Traversal starts from root
- We visit all children of the root (one by one) and once all of them were visited we go to their children and so on. We go down one level, only when all nodes from a level were visited.
- For level order traversal we use a queue to remember the nodes that have to be visited.

Level order traversal example



Level order traversal example

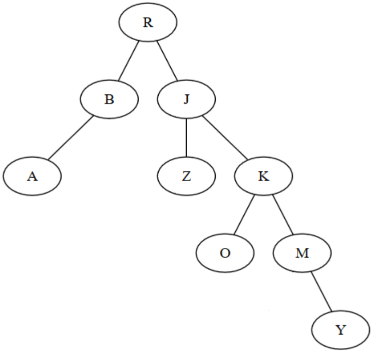


- Queue q with the root: R
- Visit R (pop from queue) and push its children: q = [B G J]
- Visit B and push its children: q = [G J T A]
- Visit G and push nothing: q = [J T A]
- Visit J and push its children: q = [T A Z K]
- Visit T and push nothing: q = [A Z K]
- Visit A and push nothing: q = [Z K]
- etc...

Binary trees

- An ordered tree in which each node has at most two children is called *binary tree*.
- In a binary tree we call the children of a node the *left child* and *right child*.
- Even if a node has only one child, we still have to know whether that is the left or the right one.

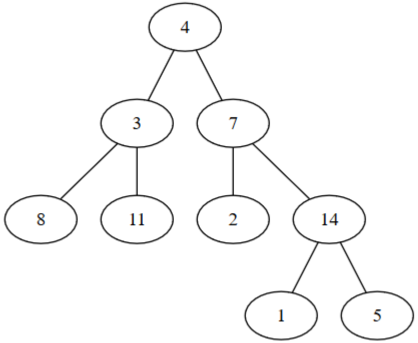
Binary tree - example



- *A* is the left child of *B*
- *Y* is the right child of *M*

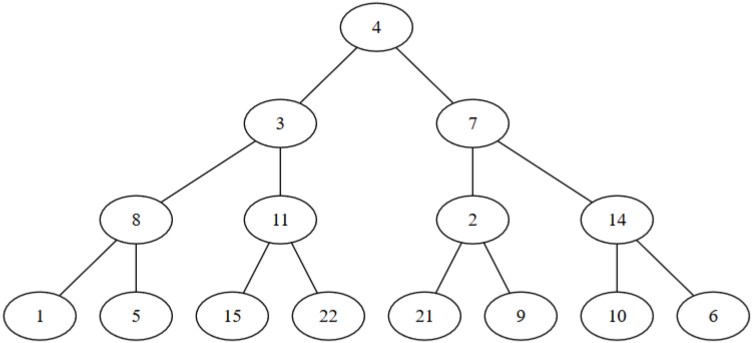
Binary tree - Terminology I

- A binary tree is called *full* if every internal node has exactly two children.



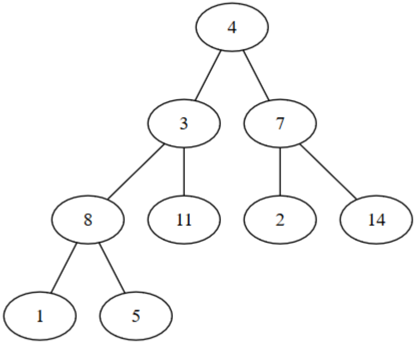
Binary tree - Terminology II

- A binary tree is called *complete* if all leaves are one the same level and all internal nodes have exactly 2 children.



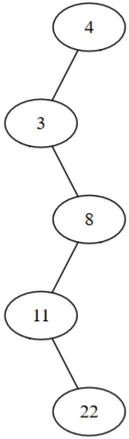
Binary tree - Terminology III

- A binary tree is called *almost complete* if it is a *complete* binary tree except for the last level, where nodes are completed from left to right (binary heap - structure).



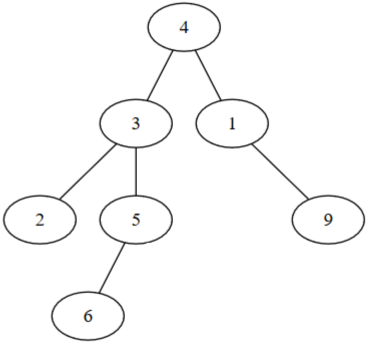
Binary tree - Terminology IV

- A binary tree is called *degenerate* if every internal node has exactly one child (it is actually a chain of nodes).



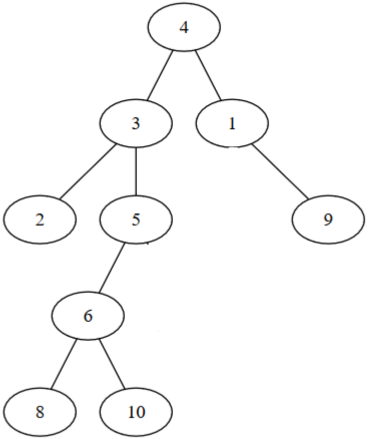
Binary tree - Terminology V

- A binary tree is called *balanced* if the difference between the height of the left and right subtrees is at most 1 (for every node from the tree).



Binary tree - Terminology VI

- Obviously, there are many binary trees that are none of the above categories, for example:



Binary tree - properties

- A binary tree with n nodes has exactly $n-1$ edges (this is true for every tree, not just binary trees)
- The number of nodes in a complete binary tree of height N is $2^{N+1}-1$ (it is $1+2+4+8+...+2^N$)
- The maximum number of nodes in a binary tree of height N is $2^{N+1}-1$ - if the tree is complete.
- The minimum number of nodes in a binary tree of height N is $N+1$ - if the tree is degenerate.
- A binary tree with N nodes has a height between $log_2 N$ and $N-1$.

ADT Binary Tree I

- Domain of ADT Binary Tree:

$$\mathcal{BT} = \{bt \mid bt \text{ binary tree with nodes containing information of type TElem}\}$$

ADT Binary Tree II

- init(bt)
  - **descr:** creates a new, empty binary tree
  - **pre:** true
  - **post:** $bt \in \mathcal{BT}$, bt is an empty binary tree

ADT Binary Tree III

- initLeaf(bt, e)
    - **descr:** creates a new binary tree, having only the root with a given value
    - **pre:** $e \in TElem$
    - **post:** $bt \in \mathcal{BT}$, bt is a binary tree with only one node (its root) which contains the value e

ADT Binary Tree IV

- initTree(bt, left, e, right)
  - **descr:** creates a new binary tree, having a given information in the root and two given binary trees as children
  - **pre:** $left, right \in \mathcal{BT}$, $e \in TElem$
  - **post:** $bt \in \mathcal{BT}$, bt is a binary tree with left child equal to $left$, right child equal to $right$ and the information from the root is e

ADT Binary Tree V

- insertLeftSubtree(bt, left)
    - **descr:** sets the left subtree of a binary tree to a given value (if the tree had a left subtree, it will be changed)
    - **pre:** $bt, left \in \mathcal{BT}$
    - **post:** $bt' \in \mathcal{BT}$, the left subtree of bt' is equal to *left*

ADT Binary Tree VI

- insertRightSubtree(bt, right)
    - **descr:** sets the right subtree of a binary tree to a given value (if the tree had a right subtree, it will be changed)
    - **pre:** $bt, right \in \mathcal{BT}$
    - **post:** $bt' \in \mathcal{BT}$, the right subtree of bt' is equal to *right*

ADT Binary Tree VII

- root(bt)
  - **descr:** returns the information from the root of a binary tree
  - **pre:** $bt \in \mathcal{BT}$, $bt \neq \Phi$
  - **post:** $root \leftarrow e$, $e \in TElem$, e is the information from the root of bt
  - **throws:** an exception if bt is empty

ADT Binary Tree VIII

- left(bt)
    - **descr:** returns the left subtree of a binary tree
    - **pre:** $bt \in \mathcal{BT}$, $bt \neq \Phi$
    - **post:** $left \leftarrow, l$, $l \in \mathcal{BT}$, l is the left subtree of bt
    - **throws:** an exception if bt is empty

ADT Binary Tree IX

- right(bt)
  - **descr:** returns the right subtree of a binary tree
  - **pre:** $bt \in \mathcal{BT}$, $bt \neq \Phi$
  - **post:** $right \leftarrow r$, $r \in \mathcal{BT}$, r is the right subtree of bt
  - **throws:** an exception if bt is empty

ADT Binary Tree X

- isEmpty(bt)
  - **descr:** checks if a binary tree is empty
  - **pre:** $bt \in \mathcal{BT}$
  - **post:**

$$empty \leftarrow \begin{cases} True, & \text{if } bt = \Phi \\ False, & \text{otherwise} \end{cases}$$

ADT Binary Tree XI

- iterator (bt, traversal, i)
  - **descr:** returns an iterator for a binary tree
  - **pre:** $bt \in \mathcal{BT}$, *traversal* represents the order in which the tree has to be traversed
  - **post:** $i \in \mathcal{I}$, i is an iterator over *bt* that iterates in the order given by *traversal*

ADT Binary Tree XII

- destroy(bt)
  - **descr:** destorys a binary tree
  - **pre:** $bt \in \mathcal{BT}$
  - **post:** bt was destroyed

ADT Binary Tree XIII

- Other possible operations:
  - change the information from the root of a binary tree
  - remove a subtree (left or right) of a binary tree
  - search for an element in a binary tree
  - return the number of elements from a binary tree

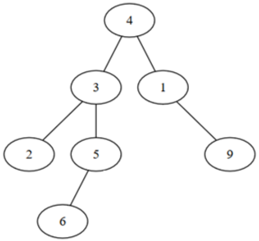
Possible representations

- If we want to implement a binary tree, what representation can we use?
- We have several options:
  - Representation using an array (similar to a binary heap)
  - Linked representation
    - with dynamic allocation
    - on an array

Possible representations I

- Representation using an array
  - Store the elements in an array
  - First position from the array is the root of the tree
  - Left child of node from position i is at position $2 * i$, right child is at position $2 * i + 1$.
  - Some special value is needed to denote the place where no element is.

Possible representations II



Pos	Elem
1	4
2	3
3	1
4	2
5	5
6	-1
7	9
8	-1
9	-1
10	6
11	-1
12	-1
13	-1
...	...

- Disadvantage: depending on the form of the tree, we might waste a lot of space.

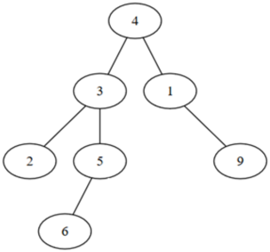
Possible representations I

- Linked representation with dynamic allocation
    - We have a structure to represent a node, containing the information, the address of the left child and the address of the right child (possibly the address of the parent as well).
    - An empty tree is denoted by the value NIL for the root.
    - We have one node for every element of the tree.

Possible representations II

- Linked representation on an array
    - Information from the nodes is placed in an array. The *address* of the left and right child is the *index* where the corresponding elements can be found in the array.
    - We can have a separate array for the parent as well.

Possible representations III



Pos	1	2	3	4	5	6	7	8
Info	4	3	2	5	6	1	9	
Left	2	3	-1	5	-1	-1	-1	
Right	6	4	-1	-1	-1	7	-1	
Parent	-1	1	2	2	4	1	6	

- We need to know that the root is at position 1 (could be any position).
- If the array is full, we can allocate a larger one.
- We have to keep a linked list of empty positions to make adding a new node easier.

Possible representations IV

- The linked list of empty positions has to be created when the empty binary tree is created. While a tree is a non-linear data structure, we can still use the left (and/or right) array to create a singly (or doubly) linked list of empty positions.
- Obviously, when we do a resize, the newly created empty positions have to be linked again.

info								
left	2	3	4	5	6	7	8	-1
right								

firstEmpty = 1

root = -1

cap = 8

Binary Tree Traversal

- A node of a (binary) tree is visited when the program control arrives at the node, usually with the purpose of performing some operation on the node (printing it, checking the value from the node, etc.).
- *Traversing* a (binary) tree means visiting all of its nodes.
- For a binary tree there are 4 possible traversals:
    - Preorder
    - Inorder
    - Postorder
    - Level order (breadth first) - the same as in case of a (non-binary) tree

Binary tree representation

- In the following, for the implementation of the traversal algorithms, we are going to use the following representation for a binary tree:

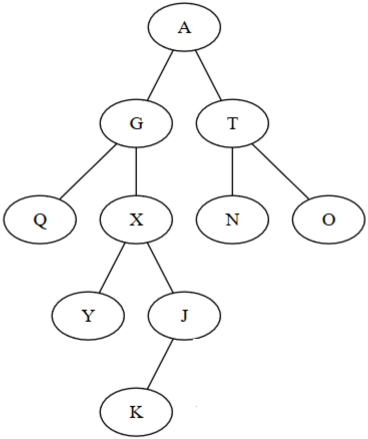
```
BTNode:
    info: TElem
    left: ↑ BTNode
    right: ↑ BTNode
```

```
BinaryTree:
    root: ↑ BTNode
```

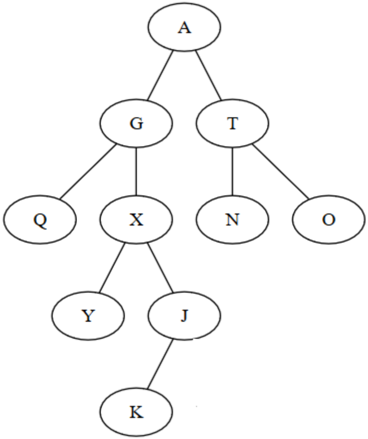
Preorder traversal

- In case of a preorder traversal:
  - Visit the *root* of the tree
  - Traverse the left subtree - if exists
  - Traverse the right subtree - if exists
- When traversing the subtrees (left or right) the same preorder traversal is applied (so, from the left subtree we visit the root first and then traverse the left subtree and then the right subtree).

Preorder traversal example



Preorder traversal example



- Preorder traversal: A, G, Q, X, Y, J, K, T, N, O

Preorder traversal - recursive implementation

- The simplest implementation for preorder traversal is with a recursive algorithm.

```
subalgorithm preorder_recursive(node) is:
//pre: node is a ↑ BTNode
   if node $\neq$ NIL then
      @visit [node].info
      preorder_recursive([node].left)
      preorder_recursive([node].right)
   end-if
end-subalgorithm
```

Preorder traversal - recursive implementation

- The *preorder_recursive* subalgorithm receives as parameter a pointer to a node, so we need a wrapper subalgorithm, one that receives a *BinaryTree* and calls the function for the root of the tree.

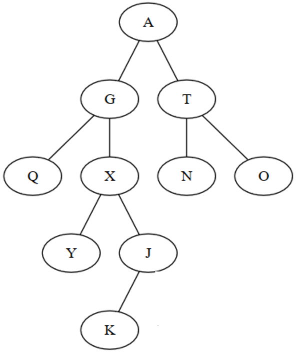
```
subalgorithm preorderRec(tree) is:
//pre: tree is a BinaryTree
  preorder_recursive(tree.root)
end-subalgorithm
```

- Assuming that visiting a node takes constant time (print the info from the node, for example), the whole traversal takes $\Theta(n)$ time for a tree with n nodes.

Preorder traversal - non-recursive implementation

- We can implement the preorder traversal algorithm without recursion, using an auxiliary *stack* to store the nodes.
    - We start with an empty stack
    - Push the root of the tree to the stack
    - While the stack is not empty:
        - Pop a node and visit it
        - Push the node's right child to the stack
        - Push the node's left child to the stack

Preorder traversal - non-recursive implementation example



- Stack: A
- Visit A, push children (Stack: T G)
- Visit G, push children (Stack: T X Q)
- Visit Q, push nothing (Stack: T X)
- Visit X, push children (Stack: T J Y)
- Visit Y, push nothing (Stack: T J)
- Visit J, push child (Stack: T K)
- Visit K, push nothing (Stack: T)
- Visit T, push children (Stack: O N)
- Visit N, push nothing (Stack: O)
- Visit O, push nothing (Stack:)
- Stack is empty, traversal is complete

Preorder traversal - non-recursive implementation

```
subalgorithm preorder(tree) is:
//pre: tree is a binary tree
    s: Stack //s is an auxiliary stack
    if tree.root $\neq$ NIL then
        push(s, tree.root)
    end-if
    while not isEmpty(s) execute
        currentNode $\leftarrow$ pop(s)
        @visit currentNode
        if [currentNode].right $\neq$ NIL then
            push(s, [currentNode].right)
        end-if
        if [currentNode].left $\neq$ NIL then
            push(s, [currentNode].left)
        end-if
    end-while
end-subalgorithm
```

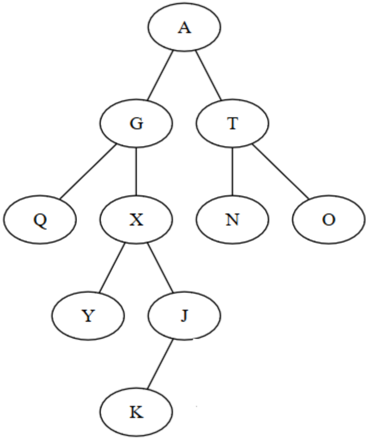
Preorder traversal - non - recursive implementation

- Time complexity of the non-recursive traversal is $\Theta(n)$, and we also need $O(n)$ extra space (the stack)
- Obs: Preorder traversal is exactly the same as *depth first traversal* (you can see it especially in the implementation), with the observation that here we need to be careful to first push the right child to the stack and then the left one (in case of *depth-first traversal* the order in which we pushed the children was not that important).

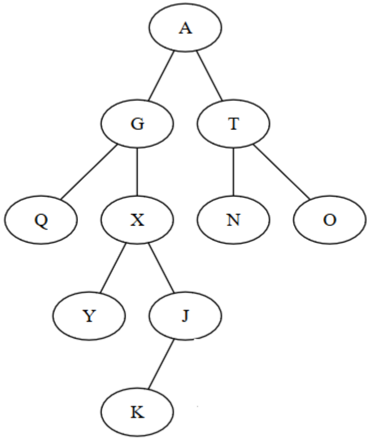
Inorder traversal

- In case of *inorder* traversal:
  - Traverse the left subtree - if exists
  - Visit the *root* of the tree
  - Traverse the right subtree - if exists
- When traversing the subtrees (left or right) the same inorder traversal is applied (so, from the left subtree we traverse the left subtree, then we visit the root and then traverse the right subtree).

Inorder traversal example



Inorder traversal example



- Inorder traversal: Q, G, Y, X, K, J, A, N, T, O

Inorder traversal - recursive implementation

- The simplest implementation for inorder traversal is with a recursive algorithm.

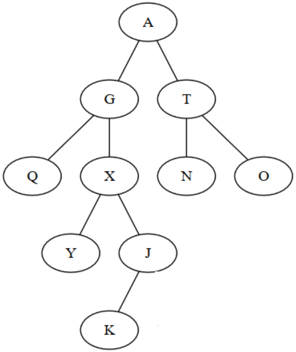
```
subalgorithm inorder_recursive(node) is:
//pre: node is a ↑ BTNode
  if node $\neq$ NIL then
    inorder_recursive([node].left)
    @visit [node].info
    inorder_recursive([node].right)
  end-if
end-subalgorithm
```

- We need again a wrapper subalgorithm to perform the first call to *inorder_recursive* with the root of the tree as parameter.
- The traversal takes $\Theta(n)$ time for a tree with n nodes.

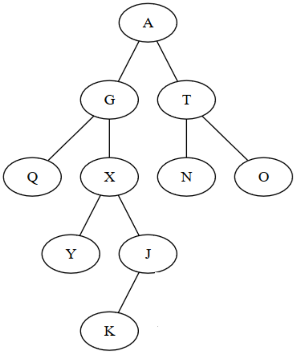
Inorder traversal - non-recursive implementation

- We can implement the inorder traversal algorithm without recursion, using an auxiliary stack to store the nodes.
    - We start with an empty stack and a current node set to the root
    - While current node is not NIL, push it to the stack and set it to its left child
    - While stack not empty
        - Pop a node and visit it
        - Set current node to the right child of the popped node
        - While current node is not NIL, push it to the stack and set it to its left child

Inorder traversal - non-recursive implementation example



Inorder traversal - non-recursive implementation example



- CurrentNode: A (Stack:)
- CurrentNode: NIL (Stack: A G Q)
- Visit Q, currentNode NIL (Stack: A G)
- Visit G, currentNode X (Stack: A)
- CurrentNode: NIL (Stack: A X Y)
- Visit Y, currentNode NIL (Stack: A X)
- Visit X, currentNode J (Stack: A)
- CurrentNode: NIL (Stack: A J K)
- Visit K, currentNode NIL (Stack: A J)
- Visit J, currentNode NIL (Stack: A)
- Visit A, currentNode T (Stack:)
- CurrentNode: NIL (Stack: T N)
- ...

Inorder traversal - non-recursive implementation

```
subalgorithm inorder(tree) is:
//pre: tree is a BinaryTree
    s: Stack //s is an auxiliary stack
    currentNode ← tree.root
    while currentNode ≠ NIL execute
        push(s, currentNode)
        currentNode ← [currentNode].left
    end-while
    while not isEmpty(s) execute
        currentNode ← pop(s)
        @visit currentNode
        currentNode ← [currentNode].right
        while currentNode ≠ NIL execute
            push(s, currentNode)
            currentNode ← [currentNode].left
        end-while
    end-while
end-subalgorithm
```

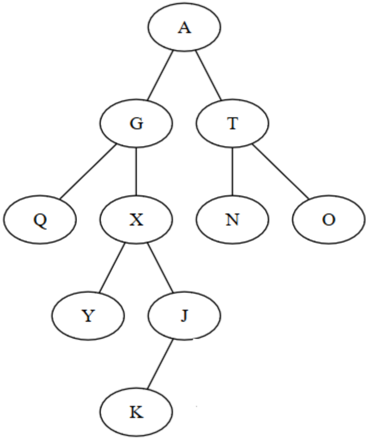
Inorder traversal - non-recursive implementation

- Time complexity $\Theta(n)$, extra space complexity $O(n)$

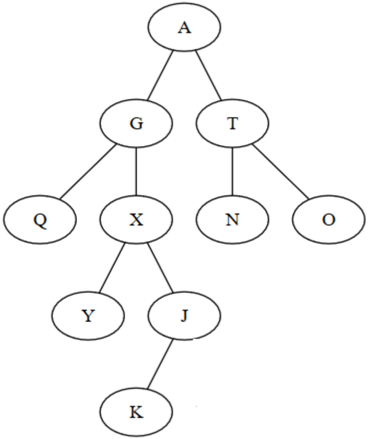
Postorder traversal

- In case of *postorder* traversal:
  - Traverse the left subtree - if exists
  - Traverse the right subtree - if exists
  - Visit the *root* of the tree
- When traversing the subtrees (left or right) the same postorder traversal is applied (so, from the left subtree we traverse the left subtree, then traverse the right subtree and then visit the root).

Postorder traversal example



Postorder traversal example



- Postorder traversal: Q, Y, K, J, X, G, N, O, T, A

Postorder traversal - recursive implementation

- The simplest implementation for postorder traversal is with a recursive algorithm.

```
subalgorithm postorder_recursive(node) is:
//pre: node is a $\uparrow$ BTNode
    if node $\neq$ NIL then
        postorder_recursive([node].left)
        postorder_recursive([node].right)
        @visit [node].info
    end-if
end-subalgorithm
```

- We need again a wrapper subalgorithm to perform the first call to *postorder_recursive* with the root of the tree as parameter.

- The traversal takes $\Theta(n)$ time for a tree with n nodes.

Postorder traversal - non-recursive implementation

- We can implement the postorder traversal without recursion, but it is slightly more complicated than preorder and inorder traversals.
- We can have an implementation that uses two stacks and there is also an implementation that uses one stack.

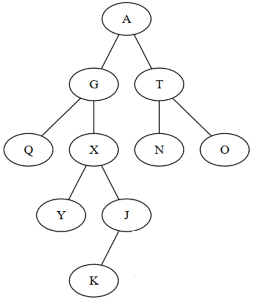
Postorder traversal with two stacks

- The main idea of postorder traversal with two stacks is to build the reverse of postorder traversal in one stack. If we have this, popping the elements from the stack until it becomes empty will give us postorder traversal.
- Building the reverse of postorder traversal is similar to building preorder traversal, except that we need to traverse the right subtree first (not the left one). The other stack will be used for this.
- The algorithm is similar to *preorder* traversal, with two modifications:
    - When a node is removed from the stack, it is added to the second stack (instead of being visited)
    - For a node taken from the stack we first push the left child and then the right child to the stack.

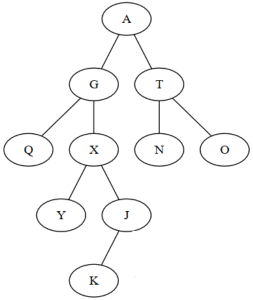
Postorder traversal with one stack

- We start with an empty stack and a current node set to the root of the tree
- While the current node is not NIL, push to the stack the right child of the current node (if exists) and the current node and then set the current node to its left child.
- While the stack is not empty
    - Pop a node from the stack (call it current node)
    - If the current node has a right child, the stack is not empty and contains the right child on top of it, pop the right child, push the current node, and set current node to the right child.
    - Otherwise, visit the current node and set it to NIL
    - While the current node is not NIL, push to the stack the right child of the current node (if exists) and the current node and then set the current node to its left child.

Postorder traversal - non-recursive implementation example



Postorder traversal - non-recursive implementation example



- Node: A (Stack:)
- Node: NIL (Stack: T A X G Q)
- Visit Q, Node NIL (Stack: T A X G)
- Node: X (Stack: T A G)
- Node: NIL (Stack: T A G J X Y)
- Visit Y, Node: NIL (Stack: T A G J X)
- Node: J (Stack: T A G X)
- Node: NIL (Stack: T A G X J K)
- Visit K, Node: NIL (Stack: T A G X J)
- Visit J, Node: NIL (Stack: T A G X)
- Visit X, Node: NIL (Stack: T A G)
- Visit G, Node: NIL (Stack: T A)
- Node: T (Stack: A)
- Node: NIL (Stack: A O T N)
- ...

Postorder traversal - non-recursive implementation

```
subalgorithm postorder(tree) is:
//pre: tree is a BinaryTree
   s: Stack //s is an auxiliary stack
   node ← tree.root
   while node $\neq$ NIL execute
      if [node].right $\neq$ NIL then
         push(s, [node].right)
      end-if
      push(s, node)
      node ← [node].left
   end-while
   while not isEmpty(s) execute
      node ← pop(s)
      if [node].right $\neq$ NIL and (not isEmpty(s)) and [node].right = top(s) th
         pop(s)
         push(s, node)
         node ← [node].right
//continued on the next slide
```

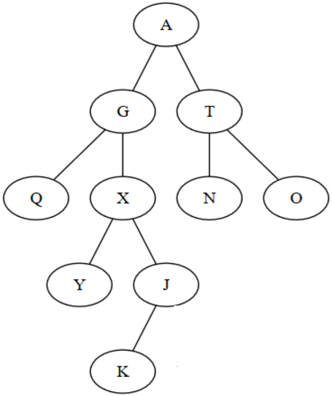
Postorder traversal - non-recursive implementation

```
        else
            @visit node
            node ← NIL
        end-if
        while node $\neq$ NIL execute
            if [node].right $\neq$ NIL then
                push(s, [node].right)
            end-if
            push(s, node)
            node ← [node].left
        end-while
    end-while
end-subalgorithm
```

- Time complexity $\Theta(n)$, extra space complexity $O(n)$

Level order traversal

- In case of level order traversal we first visit the root, then the children of the root, then the children of the children, etc.



- Level order traversal: A, G, T, Q, X, N, O, Y, J, K

Binary tree iterator

- The interface of the binary tree contains the *iterator* operation, which should return an iterator.
- This operation receives a parameter that specifies what kind of traversal we want to do with the iterator (preorder, inorder, postorder, level order)
- The traversal algorithms discussed so far, traverse all the elements of the binary tree at once, but an iterator has to do element-by-element traversal.
- For defining an iterator, we have to divide the code into the functions of an iterator: *init*, *getCurrent*, *next*, *valid*

Inorder binary tree iterator

- Assume an implementation without a parent node.
- What fields do we need to keep in the iterator structure?

```
InorderIterator:
    bt: BinaryTree
    s: Stack
    currentNode: ↑ BTNode
```

Inorder binary tree iterator - init

- What should the *init* operation do?

```
subalgorithm init (it, bt) is:
//pre: it - is an InorderIterator, bt is a BinaryTree
    it.bt ← bt
    init(it.s)
    node ← bt.root
    while node ≠ NIL execute
        push(it.s, node)
        node ← [node].left
    end-while
    if not isEmpty(it.s) then
        it.currentNode ← top(it.s)
    else
        it.currentNode ← NIL
    end-if
end-subalgorithm
```

Inorder binary tree iterator - getCurrent

- What should the *getCurrent* operation do?

```
**function** getCurrent(it) **is:**
    getCurrent $\leftarrow$ [it.currentNode].info
**end-function**
```

Inorder binary tree iterator - valid

- What should the *valid* operation do?

```
function valid(it) is:
    if it.currentNode = NIL then
        valid ← false
    else
        valid ← true
    end-if
end-function
```

Inorder binary tree iterator - next

- What should the *next* operation do?

```
subalgorithm next(it) is:
    node ← pop(it.s)
    if [node].right ≠ NIL then
        node ← [node].right
        while node ≠ NIL execute
            push(it.s, node)
            node ← [node].left
        end-while
    end-if
    if not isEmpty(it.s) then
        it.currentNode ← top(it.s)
    else
        it.currentNode ← NIL
    end-if
end-subalgorithm
```

Preorder, Inorder, Postorder

- How to remember the difference between traversals?
  - Left subtree is always traversed before the right subtree.
  - The visiting of the root is what changes:
    - PREorder - visit the root before the left and right
    - INorder - visit the root between the left and right
    - POSTorder - visit the root after the left and right

Think about it

- Assume you have a binary tree, but you do not know how it looks like, but you have the *preorder* and *inorder* traversal of the tree. Give an algorithm for building the tree based on these two traversals.

- For example:
  - Preorder: A B F G H E L M
  - Inorder: B G F H A L E M

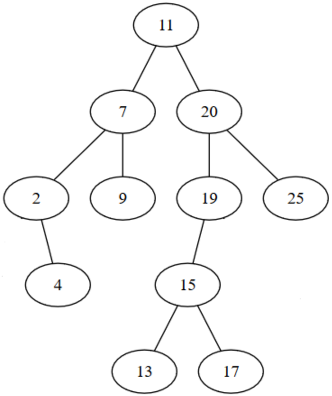
Think about it

- Can you rebuild the tree if you have the *postorder* and the *inorder* traversal?
- Can you rebuild the tree if you have the *preorder* and the *postorder* traversal?

Binary search trees

- A *Binary Search Tree* is a binary tree that satisfies the following property:
  - if x is a node of the binary search tree then:
    - For every node y from the left subtree of x, the information from y is less than or equal to the information from x
    - For every node y from the right subtree of x, the information from y is greater than or equal to the information from x
- Obviously, the relation used to order the nodes can be considered in an abstract way (instead of having "$\leq$" as in the definition).

Binary Search Tree Example



- If we do an inorder traversal of a binary search tree, we will get the elements in increasing order (according to the relation used).

Binary Search Tree

- The terminology and many properties discussed for binary tree is valid for binary search trees as well:
  - We can have a binary search tree that is full, complete, almost complete, degenerate or balanced
  - The maximum number of nodes in a binary search tree of height N is $2^{N+1} - 1$ - if the tree is complete.
  - The minimum number of nodes in a binary search tree of height N is N - if the tree is degenerate.
  - A binary search tree with N nodes has a height between $log_2 N$ and N (we will denote the height of the tree by h).

Binary Search Tree

- Binary search trees can be used as representation for sorted containers: sorted maps, sorted multimaps, priority queues, sorted sets, etc.
- In order to implement these containers on a binary search tree, we need to define the following basic operations:
  - search for an element
  - insert an element
  - remove an element
- Other operations that can be implemented for binary search trees (and can be used by the containers): get minimum/maximum element, find the successor/predecessor of an element.

Binary Search Tree - Representation

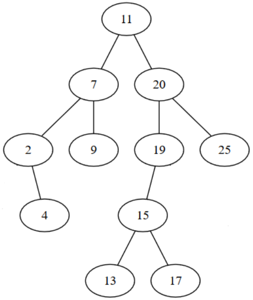
- We will use a linked representation with dynamic allocation (similar to what we used for binary trees)

```
BSTNode:
    info: TElem
    left: ↑ BSTNode
    right: ↑ BSTNode
```

```
BinarySearchTree:
    root: ↑ BSTNode
```

Binary Search Tree - search operation

- How can we search for an element in a binary search tree?



- How can we search for element 15? And for element 14?

BST - search operation - recursive implementation

- How can we implement the *search algorithm* recursively?

BST - search operation - recursive implementation

- How can we implement the *search algorithm* recursively?

```
function search_rec (node, elem) is:
//pre: node is a BSTNode and elem is the TElem we are searching for
    if node = NIL then
        search_rec ← false
    else
        if [node].info = elem then
            search_rec ← true
        else if [node].info < elem then
            search_rec ← search_rec([node].right, elem)
        else
            search_rec ← search_rec([node].left, elem)
    end-if
end-function
```

BST - search operation - recursive implementation

- Complexity of the search algorithm:

BST - search operation - recursive implementation

- Complexity of the search algorithm: $O(h)$ (which is $O(n)$)
- Since the *search* algorithm takes as parameter a node, we need a wrapper function to call it with the root of the tree.

```
**function** search (tree, e) **is:**
//pre: tree is a BinarySearchTree, e is the elem we are looking for
  search ← search_rec(tree.root, e)
**end-function**
```

BST - search operation - non-recursive implementation

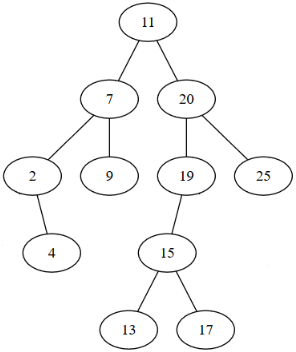
- How can we define the search operation non-recursively?

BST - search operation - non-recursive implementation

- How can we define the search operation non-recursively?

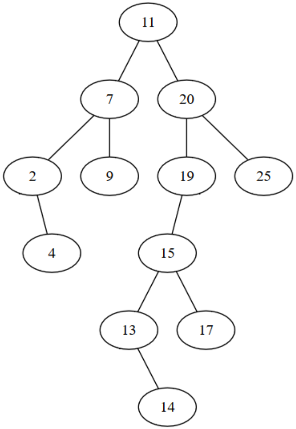
```
function search (tree, elem) is:
//pre: tree is a BinarySearchTree and elem is the TElem we are searching for
   currentNode ← tree.root
   found ← false
   while currentNode $\neq$ NIL and not found execute
      if [currentNode].info = elem then
         found ← true
      else if [currentNode].info < elem then
         currentNode ← [currentNode].right
      else
         currentNode ← [currentNode].left
      end-if
   end-while
   search ← found
end-function
```

BST - insert operation



- How/Where can we insert element 14?

BST - insert operation



BST - insert operation - recursive implementation

- How can we implement the *insert* operation?

BST - insert operation - recursive implementation

- How can we implement the *insert* operation?
- We will start with a function that creates a new node given the information to be stored in it.

```
function initNode(e) is:
//pre: e is a TComp
//post: initNode ← a node with e as information
   allocate(node)
   [node].info ← e
   [node].left ← NIL
   [node].right ← NIL
   initNode ← node
end-function
```

BST - insert operation - recursive implementation

```
function insert_rec(node, e) is:
//pre: node is a BSTNode, e is TComp
//post: a node containing e was added in the tree starting from node
    if node = NIL then
        node ← initNode(e)
    else if [node].info ≥ e then
        [node].left ← insert_rec([node].left, e)
    else
        [node].right ← insert_rec([node].right, e)
    end-if
    insert_rec ← node
end-function
```

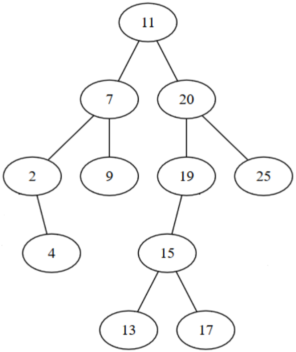
- Complexity:

BST - insert operation - recursive implementation

```
function insert_rec(node, e)  is:
//pre: node is a BSTNode, e is TComp
//post: a node containing e was added in the tree starting from node
    if node = NIL then
        node ← initNode(e)
    else if [node].info $\geq$ e then
        [node].left ← insert_rec([node].left, e)
    else
        [node].right ← insert_rec([node].right, e)
    end-if
    insert_rec ← node
end-function
```

- Complexity: O(n)
- Like in case of the *search* operation, we need a wrapper function to call *insert_rec* with the root of the tree.

BST - Finding the minimum element



- How can we find the minimum element of the binary search tree?

BST - Finding the minimum element

```
function minimum(tree) is:
//pre: tree is a BinarySearchTree
//post: minimum ← the minimum value from the tree
   currentNode ← tree.root
   if currentNode = NIL then
      @empty tree, no minimum
   else
      while [currentNode].left ≠ NIL execute
         currentNode ← [currentNode].left
      end-while
      minimum ← [currentNode].info
   end-if
end-function
```

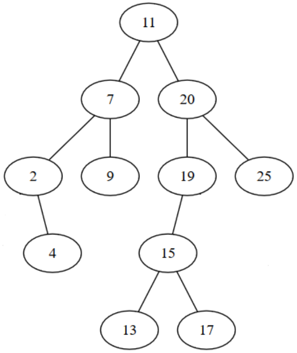
BST - Finding the minimmum element

- Complexity of the minimum operation:

BST - Finding the minimmum element

- Complexity of the minimum operation: $O(n)$
- We can have an implementation for the minimum, when we want to find the minimum element of a subtree, in this case the parameter to the function would be a node, not a tree.
- We can have an implementation where we return the node containing the minimum element, instead of just the value (depends on what we want to do with the operation)
- Maximum element of the tree can be found similarly.

Finding the parent of a node



- Given a node, how can we find the parent of the node? (assume a representation where the node has no parent field).

Finding the parent of a node

```
function parent(tree, node) is:
//pre: tree is a BinarySearchTree, node is a pointer to a BSTNode, node $\neq$ NIL
//post: returns the parent of node, or NIL if node is the root
    c $\leftarrow$ tree.root
    if c = node then //node is the root
        parent $\leftarrow$ NIL
    else
        while c $\neq$ NIL and [c].left $\neq$ node and [c].right $\neq$ node execute
            if [c].info $\geq$ [node].info then
                c $\leftarrow$ [c].left
            else
                c $\leftarrow$ [c].right
            end-if
        end-while
        parent $\leftarrow$ c
    end-if
end-function
```

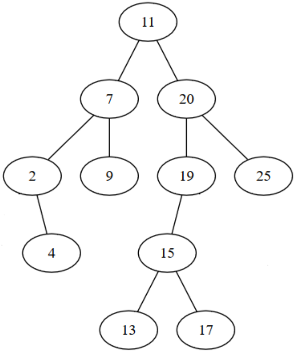
- Complexity:

Finding the parent of a node

```
function parent(tree, node) is:
//pre: tree is a BinarySearchTree, node is a pointer to a BSTNode, node $\neq$ NIL
//post: returns the parent of node, or NIL if node is the root
    c $\leftarrow$ tree.root
    if c = node then //node is the root
        parent $\leftarrow$ NIL
    else
        while c $\neq$ NIL and [c].left $\neq$ node and [c].right $\neq$ node execute
            if [c].info $\geq$ [node].info then
                c $\leftarrow$ [c].left
            else
                c $\leftarrow$ [c].right
            end-if
        end-while
        parent $\leftarrow$ c
    end-if
end-function
```

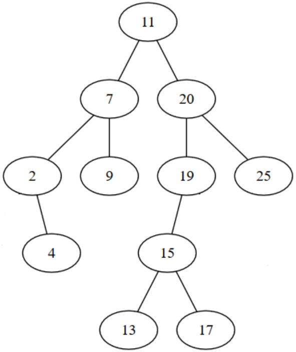
- Complexity: $O(n)$

BST - Finding the successor of a node



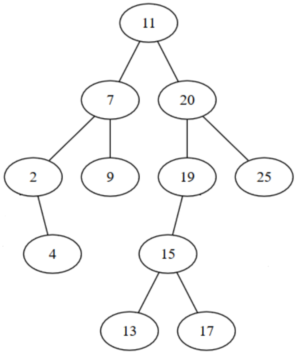
- Given a node, how can we find the node containing the next value (considering the relation used for ordering the elements)?
- How can we find the next after 11?

BST - Finding the successor of a node



- Given a node, how can we find the node containing the next value (considering the relation used for ordering the elements)?
- How can we find the next after 11? After 13?

BST - Finding the successor of a node



- Given a node, how can we find the node containing the next value (considering the relation used for ordering the elements)?
- How can we find the next after 11? After 13? After 17?

BST - Finding the successor of a node

```
function successor(tree, node) is:
//pre: tree is a BinarySearchTree, node is a pointer to a BSTNode, node $\neq$ NIL
//post: returns the node with the next value after the value from node
//or NIL if node is the maximum
    if [node].right $\neq$ NIL then
        c $\leftarrow$ [node].right
        while [c].left $\neq$ NIL execute
            c $\leftarrow$ [c].left
        end-while
        successor $\leftarrow$ c
    else
        p $\leftarrow$ parent(tree, c)
        while p $\neq$ NIL and [p].left $\neq$ c execute
            c $\leftarrow$ p
            p $\leftarrow$ parent(tree, p)
        end-while
        successor $\leftarrow$ p
    end-if
end-function
```

BST - Finding the successor of a node

- Complexity of successor:

BST - Finding the successor of a node

- Complexity of successor: depends on parent function:
    - If *parent* is $\Theta(1)$, complexity of successor is $O(n)$
    - If *parent* is $O(n)$, complexity of successor is $O(n^2)$
- What if, instead of receiving a node, successor algorithm receives as parameter a value from a node (assume unique values in the nodes)? How can we find the successor then?
- Similar to successor, we can define a predecessor function as well.

BST - Finding successor of a node

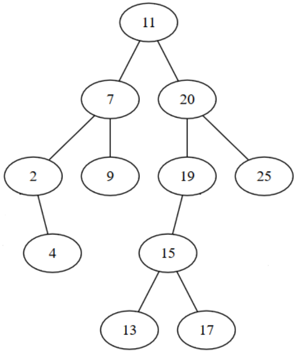
- If we do not have direct access to the parent, finding the successor of a node is $O(n^2)$.
- Can we reduce this complexity?

BST - Finding successor of a node

- If we do not have direct access to the parent, finding the successor of a node is $O(n^2)$.

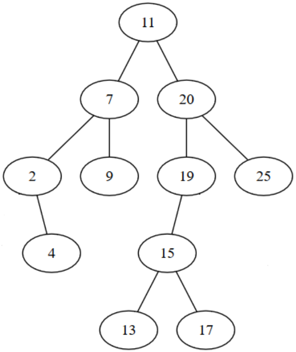
- Can we reduce this complexity?
- $O(n^2)$ is given by the potentially repeated calls for the *parent* function. But we only need the *last* parent, where we went left. We can do one single traversal from the root to our node, and every time we continue left (i.e. current node is greater than the one we are looking for) we memorize that node in a variable (and change the variable when we find a new such node). When the current node is at the one we are looking for, this variable contains its successor.

BST - Remove a node



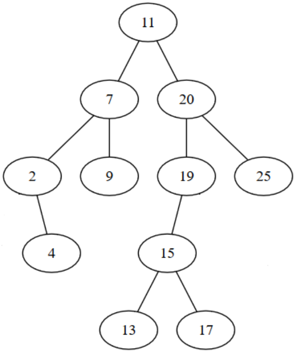
- How can we remove the value 25?

BST - Remove a node



- How can we remove the value 25? And value 2?

BST - Remove a node



- How can we remove the value 25? And value 2? And value 11?

BST - Remove a node

- When we want to remove a value (a node containing the value) from a binary search tree we have three cases:
    - The node to be removed has no descendant
        - Set the corresponding child of the parent to NIL
    - The node to be removed has one descendant
        - Set the corresponding child of the parent to the descendant
    - The node to be removed has two descendants
        - Find the maximum of the left subtree, move it to the node to be deleted, and delete the maximum
          OR
        - Find the minimum of the right subtree, move it to the node to be deleted, and delete the minimum

Binary Search Tree

- Think about it:
  - Can we define a Sorted List on a Binary Search Tree? If not, why not? If yes, how exactly? What would be the most complicated part?

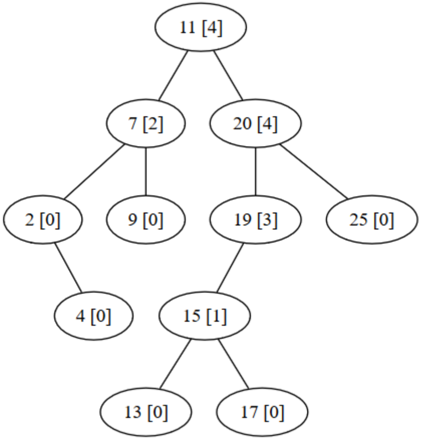
Binary Search Tree

- Think about it:
    - Can we define a Sorted List on a Binary Search Tree? If not, why not? If yes, how exactly? What would be the most complicated part?
    - Lists have positions and operations based on positions. In case of a SortedList we have an operation to remove an element from a position and to return an element from a position (but no insert to position). How can we keep track of positions in a binary search tree?

Binary Search Tree

- Think about it:
    - Can we define a Sorted List on a Binary Search Tree? If not, why not? If yes, how exactly? What would be the most complicated part?
    - Lists have positions and operations based on positions. In case of a SortedList we have an operation to remove an element from a position and to return an element from a position (but no insert to position). How can we keep track of positions in a binary search tree?
    - We can keep in every node, besides the information, the number of nodes in the left subtree. This gives us automatically the "position" of the root in the SortedList. When we have operations that are based on positions, we use these values to decide if we go left or right.

Binary Search Tree


11 [4]
7 [2]
20 [4]
2 [0]
9 [0]
19 [3]
25 [0]
4 [0]
15 [1]
13 [0]
17 [0]