Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Mathematical Analysis - Lecture Notes
Computer Science, Academic Year: 2020/2021

Lecture 4

Example 1. (i) *The harmonic series:* $\sum_{n\geq 1} \frac{1}{n}$ is divergent with sum ∞.

The generalized harmonic series: Let $\alpha \in \mathbb{R}$. Then

$$\sum_{n\geq 1} \frac{1}{n^\alpha} = \begin{cases} \text{convergent}, & \text{if } \alpha > 1, \\ \text{divergent}, & \text{if } \alpha \leq 1. \end{cases}$$

In particular, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(ii) $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent with sum e. Thus, Euler's number can be equivalently defined via this series.

Theorem 1 (The n^{th} Term Test)**.** *If the series $\sum_{n\geq 1} x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.*

Remark 1.

Series with nonnegative terms

Let (x_n) be a sequence of in $\mathbb{R}$. Consider the series $\sum_{n\geq 1} x_n$ and the sequence (s_n) of partial sums.

A series $\sum_{n\geq 1} x_n$ is with *nonnegative (positive) terms* if $\forall n \in \mathbb{N}, x_n \geq 0$ $(x_n > 0)$.
Assume that $\sum_{n\geq 1} x_n$ is a series with nonnegative terms.

Hence, series with nonnegative terms always have a sum in $[0, \infty) \cup \{\infty\}$:

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n = \sup_{n\in\mathbb{N}} s_n.$$

Theorem 2 (First Comparison Test)**.** *Let $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ be series with nonnegative terms satisfying*

$$\exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, x_n \leq y_n.$$

Then:

(i) if $\sum_{n\geq 1} y_n$ is convergent, then $\sum_{n\geq 1} x_n$ is convergent.

(ii) if $\sum_{n\geq 1} x_n$ is divergent, then $\sum_{n\geq 1} y_n$ is divergent.

Example 2. Let $\alpha \in \mathbb{R}$, $\alpha \leq 1$. Then $\displaystyle\sum_{n\geq 1} \frac{1}{n^{\alpha}}$ is divergent.

Theorem 3 (Second Comparison Test)**.** *Let $\sum_{n\geq 1} x_n$ be a series with nonnegative terms and $\sum_{n\geq 1} y_n$ a series with positive terms. Suppose $\exists L = \lim\limits_{n\to\infty} \frac{x_n}{y_n} \in [0,\infty)\cup\{\infty\}$. Then:*

(i) for $L \in (0,\infty)$: $\sum_{n\geq 1} x_n$ is convergent if and only if $\sum_{n\geq 1} y_n$ is convergent (equivalently, $\sum_{n\geq 1} x_n$ is divergent if and only if $\sum_{n\geq 1} y_n$ is divergent).

(ii) for $L = 0$: if $\sum_{n\geq 1} y_n$ is convergent, then $\sum_{n\geq 1} x_n$ is convergent (equivalently, if $\sum_{n\geq 1} x_n$ is divergent, then $\sum_{n\geq 1} y_n$ is divergent).

(iii) for $L = \infty$: if $\sum_{n\geq 1} x_n$ is convergent, then $\sum_{n\geq 1} y_n$ is convergent (equivalently, if $\sum_{n\geq 1} y_n$ is divergent, then $\sum_{n\geq 1} x_n$ is divergent).

Example 3. $\displaystyle\sum_{n\geq 1} \frac{1}{n^2 - \ln n + \sin n}$ is convergent.

Theorem 4 (Ratio Test, d'Alembert)**.** *Let $\sum_{n\geq 1} x_n$ be a series with positive terms. Then:*

(i) if $\exists q \in (0,1), \exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\frac{x_{n+1}}{x_n} \leq q$, then $\sum_{n\geq 1} x_n$ is convergent.

(ii) if $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\frac{x_{n+1}}{x_n} \geq 1$, then $\sum_{n\geq 1} x_n$ is divergent.

(iii) assuming $\exists L = \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} \in [0,\infty)\cup\{\infty\}$, we have:

(a) if $L < 1$, then $\sum_{n\geq 1} x_n$ is convergent.
(b) if $L > 1$, then $\sum_{n\geq 1} x_n$ is divergent.
(c) if $L = 1$, the test gives no information.

Example 4. (i) $\displaystyle\sum_{n\geq 1} \frac{(n!)^2}{(2n)!}$ is convergent.

(ii) $\displaystyle\sum_{n\geq 1} \frac{1}{n}$ is divergent, $\displaystyle\sum_{n\geq 1} \frac{1}{n^2}$ is convergent, yet $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1 = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}$.

Theorem 5 (Root Test, Cauchy)**.** *Let $\sum_{n\geq 1} x_n$ be a series with nonnegative terms. Then:*

(i) if $\exists q \in [0,1), \exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\sqrt[n]{x_n} \leq q$, then $\sum_{n\geq 1} x_n$ is convergent.

(ii) if $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\sqrt[n]{x_n} \geq 1$, then $\sum_{n\geq 1} x_n$ is divergent.

(iii) assuming $\exists L = \lim\limits_{n\to\infty} \sqrt[n]{x_n} \in [0,\infty)\cup\{\infty\}$, we have:

(a) if $L < 1$, then $\sum_{n\geq 1} x_n$ is convergent.

(b) if $L > 1$, then $\sum_{n\geq 1} x_n$ is divergent.

(c) if $L = 1$, the test gives no information.

Example 5. $\displaystyle\sum_{n\geq 1} \frac{n^\alpha}{(1+\beta)^n}$, where $\alpha \in \mathbb{N}$ and $\beta > 0$, is convergent.

Theorem 6 (Raabe's Test)**.** *Let $\sum_{n\geq 1} x_n$ be a series with positive terms. Then:*

(i) if $\exists q > 1, \exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, n\left(\dfrac{x_n}{x_{n+1}} - 1\right) \geq q$, then $\sum_{n\geq 1} x_n$ is convergent.

(ii) if $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, n\left(\dfrac{x_n}{x_{n+1}} - 1\right) \leq 1$, then $\sum_{n\geq 1} x_n$ is divergent.

(iii) assuming $\exists L = \lim\limits_{n\to\infty} n\left(\dfrac{x_n}{x_{n+1}} - 1\right) \in \overline{\mathbb{R}}$, we have:

(a) if $L > 1$, then $\sum_{n\geq 1} x_n$ is convergent.

(b) if $L < 1$, then $\sum_{n\geq 1} x_n$ is divergent.

(c) if $L = 1$, the test gives no information.

Example 6. Let $a > 0$. Then

$$\sum_{n\geq 1} \frac{n!}{a(a+1)\cdot \ldots \cdot (a+n)} = \begin{cases} \text{divergent}, & \text{if } a \in (0,1], \\ \text{convergent}, & \text{if } a > 1. \end{cases}$$

Series with arbitrary terms

Definition 1. A series $\sum_{n\geq 1} x_n$ is called *alternating* if either

$$x_n = (-1)^{n+1}|x_n|, \forall n \in \mathbb{N} : x_1 \geq 0,\ x_2 \leq 0,\ x_3 \geq 0,\ \ldots$$

or

$$x_n = (-1)^{n}|x_n|, \forall n \in \mathbb{N} : x_1 \leq 0,\ x_2 \geq 0,\ x_3 \leq 0,\ \ldots$$

Example 7. (i) $\displaystyle\sum_{n\geq 1} (-1)^{n+1}\frac{n}{n+1}$ is divergent.

(ii) $\sum_{n\geq 1} \cos(n\pi)$ is divergent.

Theorem 7 (Alternating Series Test, Leibniz)**.** *Let* $\sum_{n\geq 1} x_n$ *be an alternating series. If the sequence* $(|x_n|)$ *is decreasing, then* $\sum_{n\geq 1} x_n$ *is convergent if and only if* $\lim_{n\to\infty} x_n = 0$.

Definition 2. We say that a series $\sum_{n\geq 1} x_n$ is

- *absolutely convergent* if the series $\sum_{n\geq 1} |x_n|$ is convergent.
- *semi-convergent* (or *conditionally convergent*) if it is convergent, but not absolutely convergent.

Theorem 8. *Let* $\sum_{n\geq 1} x_n$ *be an absolutely convergent series. Then* $\sum_{n\geq 1} x_n$ *is convergent.*

Remark 2. If $\sum_{n\geq 1} x_n$ is with nonnegative terms, then absolute convergence and convergence are equivalent. However, in general, convergence does not imply absolute convergence (i.e., there exist semi-convergent series).

Example 8. *The alternating generalized harmonic series:* Let $\alpha \in \mathbb{R}$.

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^\alpha} = \begin{cases} \text{divergent,} & \text{if } \alpha \leq 0, \\ \text{semi-convergent,} & \text{if } \alpha \in (0,1], \\ \text{absolutely convergent,} & \text{if } \alpha > 1. \end{cases}$$