4.1.

$x_1 \in (0, 1)$

$$\left.\begin{array}{l} x_{n+1} = x_n - x_n^2 \\ x_n^2 \geq 0 \end{array}\right\} \Rightarrow x_{n+1} \leq x_n \Rightarrow x_n \text{ is decreasing} \Rightarrow$$

$\Rightarrow x_n$ is bounded above by x_1

$$\left.\begin{array}{l} x_2 = x_1 - x_1^2 \\ x_1 \in (0, 1) \Rightarrow x_1^2 < x_1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 0 < x_2 < x_1 \\ x_n \text{ is decreasing} \end{array}\right\} \Rightarrow$$

$$\Rightarrow 0 < x_n < x_1 \Rightarrow \left\{\begin{array}{l} x_n \text{ is bounded below} \\ \lim\limits_{n\to\infty} x_n = \inf x_n \end{array}\right\} \Rightarrow$$

$\Rightarrow x_n$ is convergent with $\lim\limits_{n\to\infty} x_n = 0$

For $n \cdot x_n$:

let $a_n = n$ and $b_n = \frac{1}{x_n}$ $\left(\frac{a_n}{b_n} = \frac{n}{\frac{1}{x_n}} = n \cdot x_n \right)$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \lim_{n\to\infty} \frac{n+1-n}{\frac{x_n - x_{n+1}}{x_{n+1} \cdot x_n}} =$$

$$= \lim_{n\to\infty} \frac{x_{n+1} \cdot x_n}{x_n - x_{n+1}} = \lim_{n\to\infty} \frac{(x_n - x_n^2) \cdot x_n}{\cancel{x_n} - \cancel{x_n} + x_n^2} = \lim_{n\to\infty} \frac{x_n^2 - x_n^3}{x_n^2} =$$

S.C.

$$= \lim_{n\to\infty} \frac{\cancel{x_n^2}(1 - x_n)}{\cancel{x_n^2}} = 1 \Rightarrow n \cdot x_n \text{ is convergent with}$$

$$\lim_{n\to\infty} n \cdot x_n = 1$$

4.3.

a) $\sum_{n\geq 1}\left(-\frac{\pi}{4}\right)^n$

$\pi < 4 \Leftrightarrow -\frac{\pi}{4} \in (-1;1)$

$\sum_{n\geq m} q^n = \frac{q^m}{1-q}, q \in (-1;1)$

$\Rightarrow \sum_{n\geq 1}\left(-\frac{\pi}{4}\right)^n = \frac{-\frac{\pi}{4}}{1+\frac{\pi}{4}} =$

$= -\frac{\pi}{4}\cdot\frac{4}{\pi+4} = -\frac{\pi}{\pi+4} \Rightarrow$ convergent

b) $\sum_{n\geq 0}\frac{2^{3n}}{5^{n-1}} = \sum_{n\geq 0} 5\cdot\left(\frac{2^3}{5}\right)^n = 5\cdot\sum_{n\geq 0}\left(\frac{8}{5}\right)^n$

$\sum_{n\geq 0}\left(\frac{8}{5}\right)^n$

$\lim_{n\to\infty}\left(\frac{8}{5}\right)^n = \infty$

$\Rightarrow \sum_{n\geq 0}\frac{2^{3n}}{5^{n-1}} \to \infty \Rightarrow$ divergent

c) $\sum_{n\geq 1}\frac{1}{4n^2-1} = \sum_{n\geq 1}\frac{1}{(2n-1)(2n+1)}$

$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\cdot\frac{2}{(2n-1)(2n+1)} = \frac{1}{2}\cdot\frac{2n+1-(2n-1)}{(2n+1)(2n-1)} =$

$= \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \Rightarrow \cancel{\sum_{n\geq 1}}$

$\Rightarrow \sum_{n\geq 1}\frac{1}{4n^2-1} = \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\ldots} + \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right) =$

$= \frac{1}{2}\left(1 - \underbrace{\frac{1}{2n+1}}_{0}\right) \to \frac{1}{2} \Rightarrow$ convergent

d) $\sum\limits_{n\geq 1} \ln\left(1+\frac{1}{n}\right)$

$\ln\left(1+\frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n$

$\sum\limits_{n\geq 1} \ln\left(1+\frac{1}{n}\right) = \cancel{\ln 2} - \ln 1 + \cancel{\ln 3} - \cancel{\ln 2} + \dots + \ln(n+1) - \cancel{\ln n} =$

$= \ln(n+1) - 0 \longrightarrow \infty \Rightarrow$ divergent

e) $\sum\limits_{n\geq 1} \frac{3n-2}{2^n}$

$s_n = \frac{1}{2} + \frac{4}{2^2} + \frac{7}{2^3} + \dots + \frac{3n-2}{2^n}$

$\frac{1}{2} s_n = \frac{1}{2^2} + \frac{4}{2^3} + \frac{7}{2^4} + \dots + \frac{3n-5}{2^n} + \frac{3n-2}{2^{n+1}}$

$s_n - \frac{1}{2} s_n = \frac{1}{2} + \frac{3}{2^2} + \frac{3}{2^3} + \dots + \frac{3}{2^n} - \frac{3n-2}{2^{n+1}} =$

$= 3\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}\right) - 3 - \frac{3}{2} + \frac{1}{2} - \frac{3n-2}{2^{n+1}} =$

$= 3\, \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} - 4 - \frac{3n-2}{2^{n+1}} \to 6 - 4 \to 2$

$s_n - \frac{1}{2} s_n = 2 \Rightarrow \frac{1}{2} s_n = 2 \Rightarrow s_n = 4 \Rightarrow$

$\Rightarrow \sum\limits_{n\geq 1} \frac{3n-2}{2^n} = 4 \Rightarrow$ convergent

5.1.

a) $\sum_{n\ge1}\left(1-\frac{1}{n}\right)^{n}$

$\lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{-n}\right]^{-\frac{n}{n}} = e^{-1} = \frac{1}{e} \neq 0 \Rightarrow$

$\Rightarrow$ by the n-th term test $\sum_{n\ge1}\left(1-\frac{1}{n}\right)^{n}$ is divergent

b) $\sum_{n\ge1} \sin\frac{1}{n^{5/4}}$

$\sin\frac{1}{n^{5/4}} \le \frac{1}{n^{5/4}}$

$\sum_{n\ge1}\frac{1}{n^{5/4}}$ by the generalized harmonic series $\Rightarrow \sum_{n\ge1}\frac{1}{n^{5/4}}$ is convergent

$\frac{5}{4} > 1$

$\Rightarrow$ by the first comparison test $\sum_{n\ge1}\sin\frac{1}{n^{5/4}}$ is convergent

~~c)~~ $\cancel{\sum_{n\ge1}\frac{n^{\frac{1}{2}}}{n^{\frac{4}{3}}+2}}$

c) $\sum_{n\ge1}\frac{\sqrt{n}}{n\sqrt[3]{n}+2} = \sum_{n\ge1}\frac{n^{\frac{1}{2}}}{n^{\frac{4}{3}}+2}$

let $x_n = \frac{n^{\frac{1}{2}}}{n^{\frac{4}{3}}+2}$ and $y_n = n^{-\frac{5}{6}}\left(\frac{1}{n^{\frac{5}{6}}}, \frac{5}{6}\le 1 \Rightarrow \sum_{n\ge1} n^{-\frac{5}{6}} \text{ is divergent}\right)$

$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{\frac{n^{\frac{1}{2}}}{n^{\frac{4}{3}}\left(1+\frac{2}{n^{\frac{4}{3}}}\right)}}{n^{-\frac{5}{6}}} = \lim_{n\to\infty}\frac{\cancel{n^{\frac{1}{2}}}}{\cancel{n^{\frac{1}{2}}}\left(1+\underbrace{\frac{2}{n^{4/3}}}_{\to 0}\right)} = 1 \in (0,\infty) \Rightarrow$

$\Rightarrow$ by the second comparison test $\sum_{n\ge1}\frac{\sqrt{n}}{n\sqrt[3]{n}+2}$ is divergent

d) $\sum_{n\geq 1} \frac{n!}{3\cdot 5\cdot \ldots \cdot (2n+1)}$

~~$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)!}{3\cdot 5\cdot \ldots \cdot (2n+1)(2n+3)}$~~

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)\cancel{!}}{\cancel{3\cdot 5\cdot \ldots \cdot (2n+1)}(2n+3)} \cdot \frac{\cancel{3\cdot 5\cdot \ldots \cdot (2n+1)}}{\cancel{n!}} =$

$= \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2} \in (0,1) \Rightarrow$ by the ratio test, the given series is convergent

e) $\sum_{n\geq 1} \frac{n^3 \cdot 5^n}{2^{3n+1}} = \sum_{n\geq 1} \frac{2^{-3n} \cdot 5^n \cdot n^3}{2} = \frac{1}{2} \sum_{n\geq 1} \left(\frac{5}{8}\right)^n \cdot n^3$

$\sqrt[n]{\left(\frac{5}{8}\right)^n \cdot n^3} = \frac{5}{8} \cdot \sqrt[n]{n^3}$

$\lim_{n\to\infty} \sqrt[n]{n^3} = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} = 1$

$\Rightarrow \sqrt[n]{\left(\frac{5}{8}\right)^n \cdot n^3} \to \frac{5}{8} < 1 \Rightarrow$

$\Rightarrow$ the given series is convergent (by the root test)

f) $\sum_{n\geq 1} \frac{2\cdot 5\cdot \ldots \cdot (3n-1)}{3\cdot 6\cdot \ldots \cdot (3n)}$

$n\left(\frac{\cancel{2\cdot 5\cdot \ldots \cdot (3n-1)}}{\cancel{3\cdot 6\cdot \ldots \cdot (3n)}} \cdot \frac{\cancel{3\cdot 6\cdot \ldots \cdot (3n)}\cdot (3n+3)}{\cancel{2\cdot 5\cdot \ldots \cdot (3n-1)}\cdot (3n+2)} - 1\right) =$

$= n\left(\frac{3n+3}{3n+2} - 1\right) = \frac{3n^2+3n}{3n+2} - n = \frac{\cancel{3n^2} + 3n - \cancel{3n^2} - 2n}{3n+2} =$

$= \frac{n}{3n+2} \to \frac{1}{3} < 1 \Rightarrow$ by Raabe's test, the given series is divergent

5.2.

$\sum_{n\geq 1} \frac{x_n}{y_n}$ and $\sum_{n\geq 1} y_n$ are convergent

$$\left.\begin{array}{l} \lim_{n=1} \frac{x_n}{y_n} = 0 \\ y_n \text{ is convergent} \end{array}\right\} \overset{S.C.T.}{\Rightarrow} x_n \text{ is convergent}$$

$$\left.\begin{array}{l} \text{take } x_n = \frac{1}{n^2}, \ \sum_{n\geq 1} \frac{1}{n^2} \text{ is convergent} \\ \sqrt{x_n} = \frac{1}{n}, \ \sum_{n\geq 1} \frac{1}{n} \text{ is divergent} \end{array}\right\} \Rightarrow \sum_{n\geq 1}\sqrt{x_n} \text{ might not be convergent}$$

6.1.

a) $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n\sqrt{n+1}}$

$$\left|\frac{(-1)^{n+1}}{n\sqrt{n+1}}\right| = \frac{1}{n\sqrt{n+1}}$$

$$\cancel{n\left(\frac{1}{n\sqrt{n+1}}\cdot(n+1)\sqrt{n+2} - 1\right)} = \frac{n^2\sqrt{n+2} + n\sqrt{n+2}}{n\sqrt{n+1}} - n =$$

$$\cancel{= \frac{n^2\sqrt{n+2} + n\sqrt{n+2} - n^2\sqrt{n+1}}{n\sqrt{n+1}} = n^2}$$

$$n\left(\frac{(n+1)\sqrt{n+2}}{n\sqrt{n+1}} - 1\right) = n\left(\frac{\sqrt{n+1}\cdot\sqrt{n+2}}{n} - 1\right) =$$

$$= \sqrt{n+1}\cdot\sqrt{n+2} - n = \frac{(n+1)(n+2) - n^2}{\sqrt{n+1}\cdot\sqrt{n+2} + n} = \frac{\cancel{n^2} + 3n + 2 - \cancel{n^2}}{\sqrt{n^2+3n+2} + n} =$$

$$= \frac{\cancel{n}\left(3 + \frac{2}{n}\right)}{\cancel{n}\left(\sqrt{1 + \frac{3}{n} + \frac{2}{n^2}} + 1\right)} \to \frac{3}{2} > 1 \Rightarrow \frac{1}{n\sqrt{n+1}} \text{ is convergent} \Rightarrow$$

(Raabe's test)

$\Rightarrow$ the given series is ~~convergent~~ absolutely convergent

h) $\sum_{n \geq 1} \frac{n}{n^2+1} \cdot \overbrace{\cos(n\pi)}^{-1/1}$

$$\left| \frac{n}{n^2+1} \cdot \cos(n\pi) \right| = \frac{n}{n^2+1}$$

let $x_n = \frac{n}{n^2+1}$ and $y_n = \frac{1}{n}$ ($\sum_{n\geq 1} \frac{1}{n}$ is divergent) $\Rightarrow$

$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n}{n^2+1} \cdot n = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \in (0, \infty)$

$\Rightarrow x_n$ is divergent $\Rightarrow$ the given series is not absolutly convergent

We check if $\frac{n}{n^2+1}$ is decreasing

$$\frac{n+1}{(n+1)^2+1} \cdot \frac{n^2+1}{n} = \frac{n+1}{n^2+2n+2} \cdot \frac{n^2+1}{n} \overset{?}{\leq} 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{n+1}{n^2+2n+2} \leq \frac{n}{n^2+1} \Leftrightarrow n^3+n^2+n+1 \leq n^3+2n^2+2n \Leftrightarrow$$

$\Leftrightarrow 1 \leq n^2+n$, $n \geq 1$ $\Rightarrow$ True $\Rightarrow$ by the alternating series test, the given series is semi-convergent.

6.2.

$f, g : [0,1] \to \mathbb{R}$; $f(x) = g(x)$, $\forall x \in [0,1] \cap \mathbb{Q}$

Let $c \in [0,1]$. Every real number is the limit of a strictly inc. seq. of rationals or irationals.

$\exists x_n \subseteq [0,1] \cap \mathbb{Q}$, $x_n \to c$, $x_n < c$, $\forall n \in \mathbb{N}$

$\exists y_n \subseteq [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$, $y_n \to c$, $y_n < c$, $\forall n \in \mathbb{N}$

$$\left.\begin{array}{l} f(x_n) = g(x_n) \\ f, g \text{ continuous on } [0,1) \end{array}\right\} \Rightarrow f(y_n) = g(y_n),\ y_n \subseteq [0,1) \cap \mathbb{R} \setminus \mathbb{Q}$$

The same applies for the right-hand limits $\Rightarrow$

$$\Rightarrow \left.\begin{cases} \forall a \in [0,1) \cap \mathbb{Q}, & f(a) = g(a) \\ \forall b \in [0,1) \cap \mathbb{R} \setminus \mathbb{Q}, & f(b) = g(b) \end{cases}\right\} \Rightarrow f(x) = g(x),\ \forall x \in [0,1)$$