Geometry
Problem booklet

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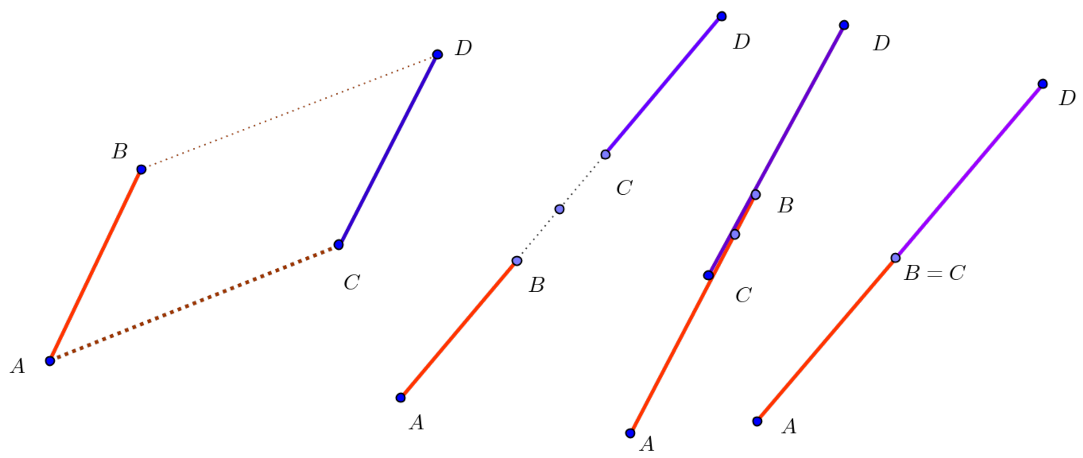
1 Week 1: Vector algebra

1.1 Free vectors

Vectors Let $\mathcal{P}$ be the three dimensional physical space in which we can talk about points, lines, planes and various relations among them. If $(A, B) \in \mathcal{P} \times \mathcal{P}$ is an ordered pair, then A is called the *original point* or the *origin* and B is called the *terminal point* or the *extremity* of (A, B).

Definition 1.1. The ordered pairs (A, B), (C, D) are said to be equipollent, written $(A, B) \sim (C, D)$, if the segments $[AD]$ and $[BC]$ have the same midpoint.



Pairs of equipollent points $(A, B) \sim (C, D)$

Remark 1.1. If the points A, B, C, $D \in \mathcal{P}$ are not collinear, then $(A, B) \sim (C, D)$ if and only if $ABDC$ is a parallelogram. In fact the length of the segments $[AB]$ and $[CD]$ is the same whenever $(A, B) \sim (C, D)$.

Proposition 1.1. *If (A, B) is an ordered pair and $O \in \mathcal{P}$ is a given point, then there exists a unique point X such that $(A,B) \sim (O, X)$.*

Proposition 1.2. *The equipollence relation is an equivalence relation on $\mathcal{P} \times \mathcal{P}$.*

Definition 1.2. The equivalence classes with respect to the equipollence relation are called *(free) vectors*.

Denote by $\overrightarrow{AB}$ the equivalence class of the ordered pair (A, B), that is $\overrightarrow{AB} = \{(X, Y) \in \mathcal{P} \times \mathcal{P} \mid (X, Y) \sim (A, B)\}$ and let $\mathcal{V} = \mathcal{P} \times \mathcal{P}\big/_{\sim} = \{\overrightarrow{AB} \mid (A, B) \in \mathcal{P} \times \mathcal{P}\}$ be the set of (free) vectors. The *length* or the *magnitude* of the vector $\overrightarrow{AB}$, denoted by $\| \overrightarrow{AB} \|$ or by $| \overrightarrow{AB} |$, is the length of the segment $[AB]$.

Remark 1.2. If two ordered pairs (A, B) and (C, D) are equippllent, i.e. the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal, then they have the same length, the same direction and the same sense. In fact a vector is determined by these three items.

Proposition 1.3. *1.* $\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \overrightarrow{AC} = \overrightarrow{BD}$.

2. $\forall A, B, O \in \mathcal{P}$, $\exists ! X \in \mathcal{P}$ *such that* $\overrightarrow{AB} = \overrightarrow{OX}$.

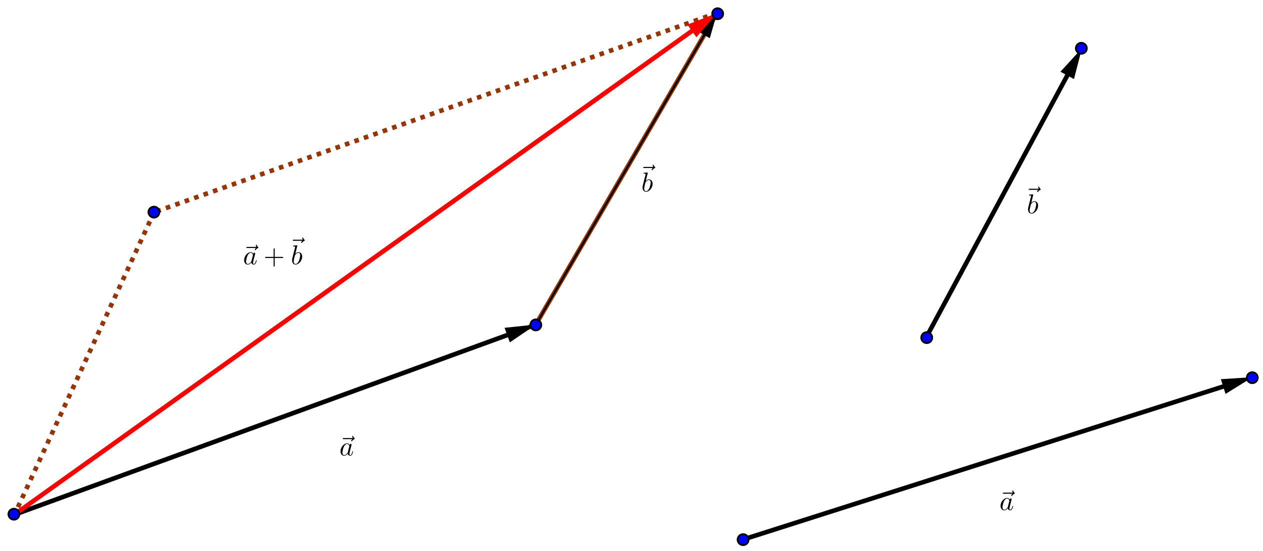
3. $\overrightarrow{AB} = \overrightarrow{A'B'}$, $\overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$.

Definition 1.3. If O, $M \in \mathcal{P}$, the the vector $\overrightarrow{OM}$ is denoted by $\overrightarrow{r}_M$ and is called the *position vector* of M with respect to O.

Corollary 1.4. *The map* $\varphi_O : \mathcal{P} \to \mathcal{V}$, $\varphi_O(M) = \overrightarrow{r}_M$ *is one-to-one and onto, i.e. bijective.*

1.1.1 Operations with vectors

• **The addition of vectors** Let $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ and $O \in \mathcal{P}$ be such that $\overrightarrow{a} = \overrightarrow{OA}$, $\overrightarrow{b} = \overrightarrow{AB}$. The vector $\overrightarrow{OB}$ is called the *sum* of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ and is written $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{a} + \overrightarrow{b}$.



Let O' be another point and A', $B' \in \mathcal{P}$ be such that $\overrightarrow{O'A'} = \overrightarrow{a}$, $\overrightarrow{A'B'} = \overrightarrow{b}$. Since $\overrightarrow{OA} = \overrightarrow{O'A'}$ and $\overrightarrow{AB} = \overrightarrow{A'B'}$ it follows, according to Proposition 1.3(3), that $\overrightarrow{OB} = \overrightarrow{O'B'}$. Therefore the vector $\overrightarrow{a} + \overrightarrow{b}$ is independent on the choice of the point O.

Proposition 1.5. *The set* $\mathcal{V}$ *endowed to the binary operation* $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, $(\overrightarrow{a}, \overrightarrow{b}) \mapsto \overrightarrow{a} + \overrightarrow{b}$, *is an abelian group whose zero element is the vector* $\overrightarrow{AA} = \overrightarrow{BB} = \overrightarrow{0}$ *and the opposite of* $\overrightarrow{AB}$, *denoted by* $-\overrightarrow{AB}$, *is the vector* $\overrightarrow{BA}$.

In particular the addition operation is associative and the vector

$$(\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c})$$

is usually denoted by $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}$. Moreover the expression

$$((\cdots(\overrightarrow{a}_1 + \overrightarrow{a}_2) + \overrightarrow{a}_3 + \cdots + \overrightarrow{a}_n)\cdots), \tag{1.1}$$

is independent of the distribution of paranthesis and it is usually denoted by

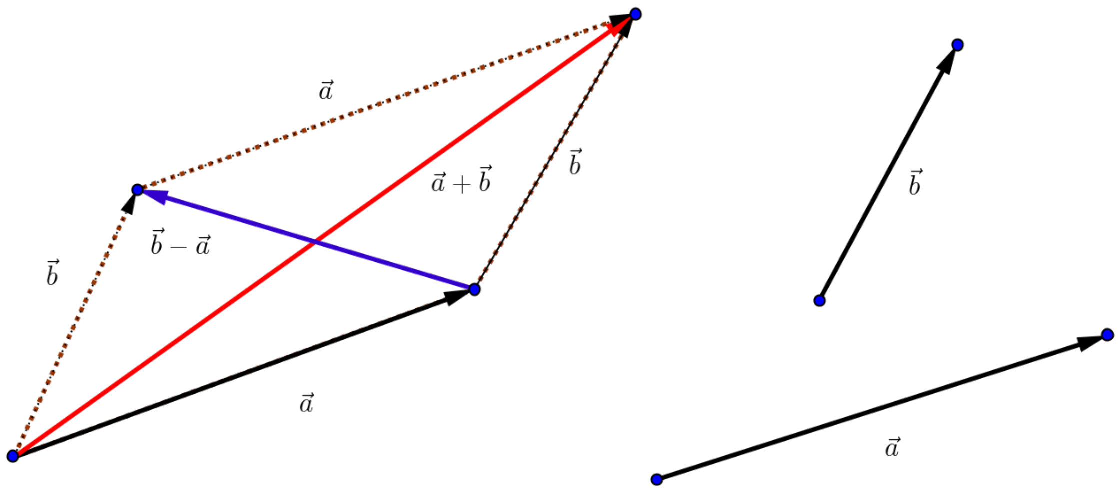
$$\overrightarrow{a}_1 + \overrightarrow{a}_2 + \cdots + \overrightarrow{a}_n .$$

Example 1.1. If $A_1, A_2, A_3, \ldots, A_n \in \mathcal{P}$ are some given points, then

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{A_1A_n} .$$

This shows that $\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1} = \overrightarrow{0}$, namely the sum of vectors constructed on the edges of a closed broken line is zero.

Corollary 1.6. *If $\overrightarrow{a} = \overrightarrow{OA}$, $\overrightarrow{b} = \overrightarrow{OB}$ are given vectors, there exists a unique vector $\overrightarrow{x} \in \mathcal{V}$ such that $\overrightarrow{a} + \overrightarrow{x} = \overrightarrow{b}$. In fact $\overrightarrow{x} = \overrightarrow{b} + (-\overrightarrow{a}) = \overrightarrow{AB}$ and is denoted by $\overrightarrow{b} - \overrightarrow{a}$.*



• **The multiplication of vectors with scalars**

Let $\alpha \in \mathbb{R}$ be a scalar and $\overrightarrow{a} = \overrightarrow{OA} \in \mathcal{V}$ be a vector. We define the vector $\alpha \cdot \overrightarrow{a}$ as follows: $\alpha \cdot \overrightarrow{a} = \overrightarrow{0}$ if $\alpha = 0$ or $\overrightarrow{a} = \overrightarrow{0}$; if $\overrightarrow{a} \neq \overrightarrow{0}$ and $\alpha > 0$, there exists a unique point on the half line $]OA$ such that $||OB|| = \alpha \cdot ||OA||$ and define $\alpha \cdot \overrightarrow{a} = \overrightarrow{OB}$; if $\alpha < 0$ we define $\alpha \cdot \overrightarrow{a} = -(|\alpha| \cdot \overrightarrow{a})$. The external binary operation

$$\mathbb{R} \times \mathcal{V} \to \mathcal{V}, \ (\alpha, \overrightarrow{a}) \mapsto \alpha \cdot \overrightarrow{a}$$

is called the *multiplication of vectors with scalars*.

Proposition 1.7. *The following properties hold:*

$(v1)$ $(\alpha+\beta)\cdot \vec{a} = \alpha\cdot \vec{a} + \beta\cdot \vec{a}$, $\forall\, \alpha,\beta\in\mathbb{R}$, $\vec{a}\in\mathcal{V}$.

$(v2)$ $\alpha\cdot(\vec{a}+\vec{b}) = \alpha\cdot \vec{a} + \alpha\cdot \vec{b}$, $\forall\, \alpha\in\mathbb{R}$, $\vec{a}, \vec{b}\in\mathcal{V}$.

$(v3)$ $\alpha\cdot(\beta\cdot \vec{a}) = (\alpha\beta)\cdot \vec{a}$, $\forall\, \alpha,\beta\in\mathbb{R}$.

$(v4)$ $1\cdot \vec{a} = \vec{a}$, $\forall\, \vec{a}\in\mathcal{V}$.

Application 1.1. Consider two parallelograms, $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ in $\mathcal{P}$, and M_1, M_2, M_3, M_4 the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$, $[A_4B_4]$ respectively. Then:

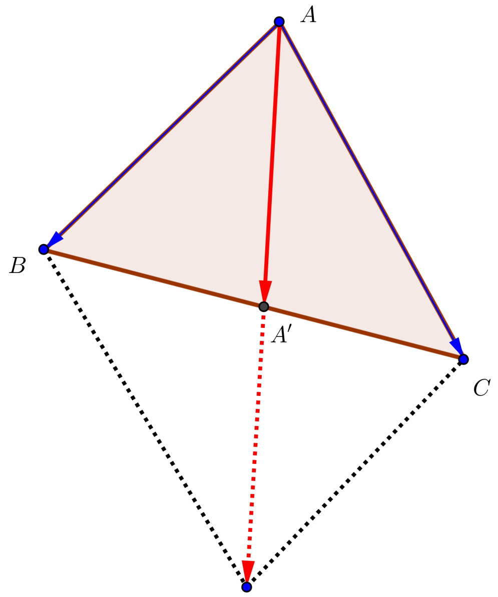
- $2\,\overrightarrow{M_1M_2} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2}$ and $2\,\overrightarrow{M_3M_4} = \overrightarrow{A_3A_4} + \overrightarrow{B_3B_4}$.
- M_1, M_2, M_3, M_4 are the vertices of a parallelogram.

1.1.2 The vector structure on the set of vectors

Theorem 1.8. *The set of (free) vectors endowed with the addition binary operation of vectors and the external binary operation of multiplication of vectors with scalars is a real vector space.*

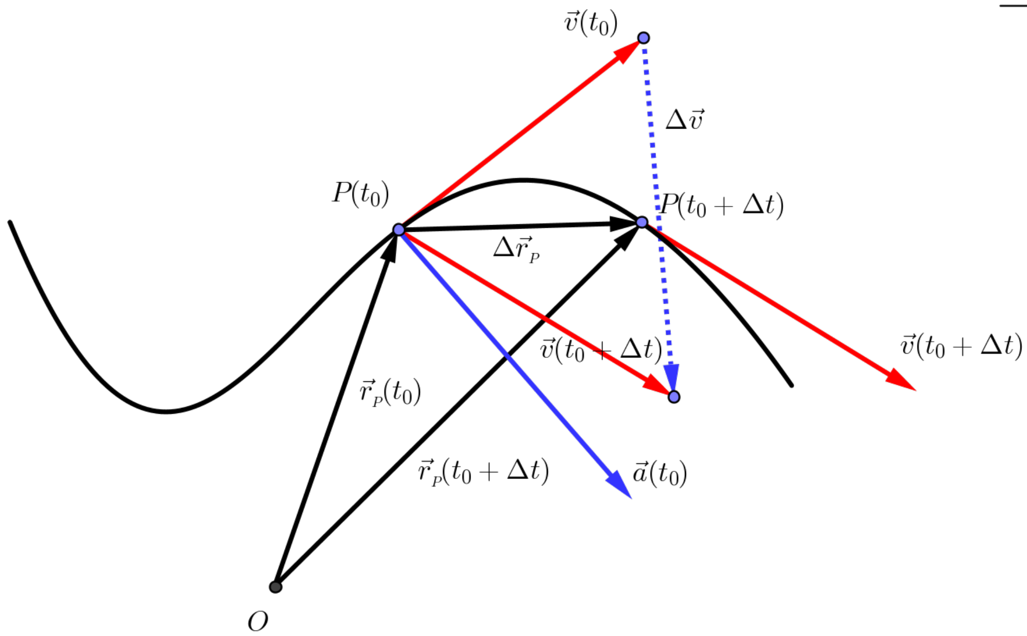
Example 1.2. If A' is the midpoint of the egde $[BC]$ of the triangle ABC, then

$$\overrightarrow{AA'} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right).$$



A few vector quantities:

1. The force, usually denoted by $\overrightarrow{F}$.
2. The velocity $\frac{d\overrightarrow{r}_P}{dt}$ of a moving particle P, is usually denoted by $\overrightarrow{v}_P$ or simply by $\overrightarrow{v}$.
3. The acceleration $\frac{d\overrightarrow{v}_P}{dt}$ of a moving particle P, is usually denoted by $\overrightarrow{a}_P$ or simply by $\overrightarrow{a}$.



• **Newton's law of gravitation**, statement that any particle of matter in the universe attracts any other with a force varying directly as the product of the masses and inversely as the square of the distance between them. In symbols, the magnitude of the attractive force F is equal to G (the gravitational constant, a number the size of which depends on the system of units used and which is a universal constant) multiplied by the product of the masses (m_1 and m_2) and divided by the square of the distance R: $F = G(m_1 m_2)/R^2$. (Encyclopdia

Britannica)
• **Newton's second law** is a quantitative description of the changes that a force can produce on the motion of a body. It states that the time rate of change of the momentum of a body is equal in both magnitude and direction to the force imposed on it. The momentum of a body is equal to the product of its mass and its velocity. Momentum, like velocity, is a vector quantity, having both magnitude and direction. A force applied to a body can change the magnitude of the momentum, or its direction, or both. Newtons second law is one of the most important in all of physics. For a body whose mass m is constant, it can be written in the form $F = ma$, where F (force) and a (acceleration) are both vector quantities. If a body has a net force acting on it, it is accelerated in accordance with the equation. Conversely, if a body is not accelerated, there is no net force acting on it. (Encyclopdia Britannica)

1.2 Problems

1. Consider a tetrahedron $ABCD$. Find the the following sums of vectors:

   (a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$.

   (b) $\overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{DC}$.

   (c) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{CD}$.

   Solution.

2. ([4, Problem 3, p. 1]) Let $OABCDE$ be a regular hexagon in which $\overrightarrow{OA}=\overrightarrow{a}$ and $\overrightarrow{OE}=\overrightarrow{b}$. Express the vectors $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$. Show that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OE} = 3\,\overrightarrow{OC}$.

   Solution.

3. Consider a pyramid with the vertex at S and the basis a parallelogram $ABCD$ whose diagonals are concurrent at O. Show the equality $\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\,\overrightarrow{SO}$.
Solution.

4. Let E and F be the midpoints of the diagonals of a quadrilateral $ABCD$. Show that

$$\overrightarrow{EF}=\frac{1}{2}\left(\overrightarrow{AB}+\overrightarrow{CD}\right)=\frac{1}{2}\left(\overrightarrow{AD}+\overrightarrow{CB}\right).$$

Solution.

5. In a triangle ABC we consider the height AD from the vertex A ($D \in BC$). Find the decomposition of the vector AD in terms of the vectors $\vec{c} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AC}$.
Solution.

6. ([4, Problem 12, p. 3]) Let M, N be the midpoints of two opposite edges of a given quadrilateral $ABCD$ and P be the midpoint of $[MN]$. Show that

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0$$

Solution.

7. ([4, Problem 12, p. 7]) Consider two perpendicular chords AB and CD of a given circle and $\{M\} = AB \cap CD$. Show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2\,\overrightarrow{OM}\,.$$

Solution.

8. ([4, Problem 13, p. 3]) If G is the centroid of a tringle ABC and O is a given point, show that
$$\overrightarrow{OG}=\frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}.$$

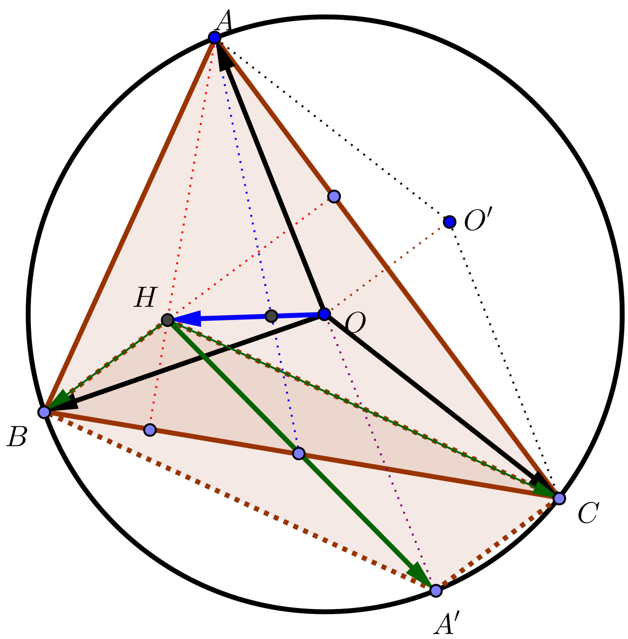
Solution.

9. ([4, Problem 14, p. 4]) Consider the triangle ABC alongside its orthocenter H, its circumcenter O and the diametrically opposed point A' of A on the latter circle. Show that:

   (a) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$.

   (b) $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$.

   (c) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\,\overrightarrow{HO}$.



Solution.

10. ([4, Problem 15, p. 4]) Consider the triangle ABC alongside its centroid G, its orthocenter H and its circumcenter O. Show that O, G, H are collinear and $3\overrightarrow{HG} = 2\overrightarrow{HO}$.
Solution.

11. ([4, Problem 27, p. 13]) Consider a tetrahedron $A_1A_2A_3A_4$ and the midpoints A_{ij} of the edges A_iA_j, $i \neq j$. Show that:

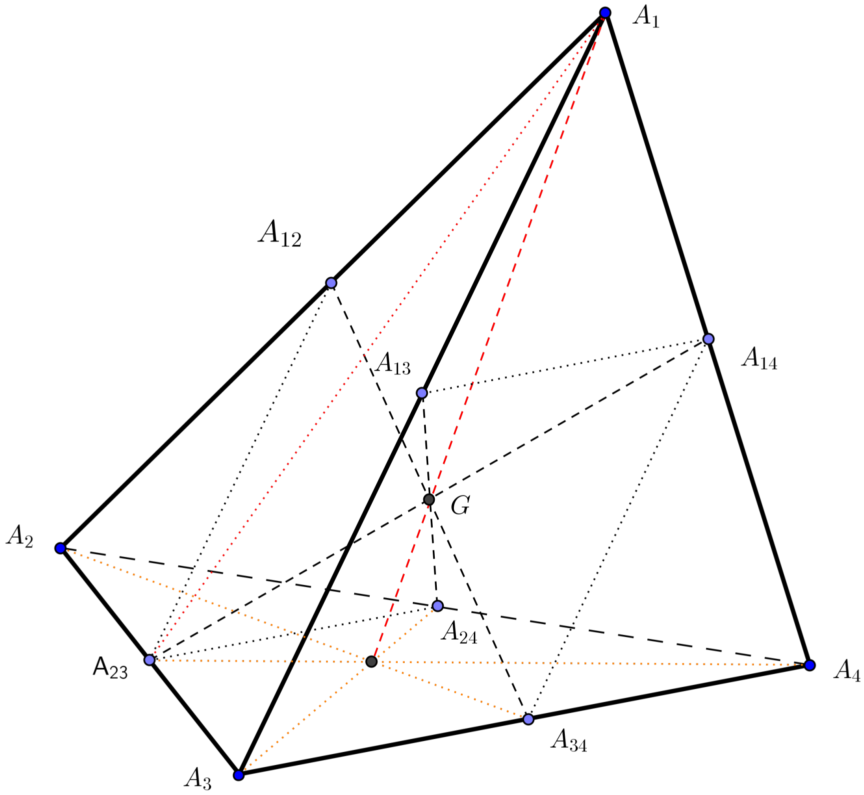
   (a) The lines $A_{12}A_{34}$, $A_{13}A_{24}$ and $A_{14}A_{23}$ are concurrent in a point G.

   (b) The medians of the tetrahedron (the lines passing through the vertices and the centroids of the opposite faces) are also concurrent at G.

   (c) Determine the ratio in which the point G divides each median.

   (d) Show that $\overrightarrow{GA_1} + \overrightarrow{GA_2} + \overrightarrow{GA_3} + \overrightarrow{GA_4} = \overrightarrow{0}$.

   (e) If M is an arbitrary point, show that $\overrightarrow{MA_1} + \overrightarrow{MA_2} + \overrightarrow{MA_3} + \overrightarrow{MA_4} = 4\,\overrightarrow{MG}$.



Solution.

12. In a triangle ABC consider the points M, L on the side AB and N, T on the side AC such that $3\overrightarrow{AL} = 2\overrightarrow{AM} = \overrightarrow{AB}$ and $3\overrightarrow{AT} = 2\overrightarrow{AN} = \overrightarrow{AC}$. Show that $\overrightarrow{AB} + \overrightarrow{AC} = 5\overrightarrow{AS}$, where $\{S\} = MT \cap LN$.

    Solution.

13. Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$, not necesarily in the same plane, alongside their centroids G_1, G_2. Show that $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3\ \overrightarrow{G_1G_2}$.

Solution.

2 Week 2: Straight lines and planes

2.1 Linear dependence and linear independence of vectors

Definition 2.1. 1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *collinear* if the points O, A, B are collinear. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *noncollinear*.

2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}, \overrightarrow{OC}$ are said to be *coplanar* if the points O, A, B, C are coplanar. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are *noncoplanar*.

Remark 2.1. 1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are linearly (in)dependent if and only if they are (non)collinear.

2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are linearly (in)dependent if and only if they are (non)coplanar.

Proposition 2.1. *The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ form a basis of $\mathcal{V}$ if and only if they are noncoplanar.*

Corollary 2.2. *The dimension of the vector space of free vectors $\mathcal{V}$ is three.*

Proposition 2.3. *Let Δ be a straight line and let $A \in \Delta$ be a given point. The set*

$$\overrightarrow{\Delta} = \{\overrightarrow{AM} \mid M \in \Delta\}$$

is an one dimensional subspace of $\mathcal{V}$. It is independent on the choice of $A \in \Delta$ and is called the director subspace *of Δ or the* direction *of Δ.*

Remark 2.2. The straight lines $\Delta,\ \Delta'$ are parallel if and only if $\overrightarrow{\Delta} = \overrightarrow{\Delta}'$

Definition 2.2. *We call* director vector *of the straigh line Δ every nonzero vector $\overrightarrow{d} \in \overrightarrow{\Delta}$.*

If $\overrightarrow{d} \in \mathcal{V}$ is a nonzero vector and $A \in \mathcal{P}$ is a given point, then there exits a unique straight line which passes through A and has the direction $\langle \overrightarrow{d} \rangle$. This stright line is

$$\Delta = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d} \rangle\}.$$

Δ is called the straight line which passes through O and is parallel to the vector $\overrightarrow{d}$.

Proposition 2.4. *Let π be a plane and let $A \in \pi$ be a given point. The set $\overrightarrow{\pi} = \{\overrightarrow{AM} \in \mathcal{V} \mid M \in \pi\}$ is a two dimensional subspace of $\mathcal{V}$. It is independent on the position of A inside π and is called the* director subspace, the director plane *or the* direction *of the plane π.*

Remark 2.3. • *The planes π, π' are parallel if and only if $\overrightarrow{\pi} = \overrightarrow{\pi}'$.*

• If $\overrightarrow{d}_1, \overrightarrow{d}_2$ are two linearly independent vectors and $A \in \mathcal{P}$ is a fixed point, then there exists a unique plane through A whose direction is $\langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle$. This plane is

$$\pi = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle\}.$$

We say that π is the plane which passes through the point A and is parallel to the vectors $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$.

Remark 2.4. Let $\Delta \subset \mathcal{P}$ be a straight line and $\pi \subset \mathcal{P}$ be given plane.

1. If $A \in \Delta$ is a given point, then $\varphi_O(\Delta) = \overrightarrow{r}_A + \overrightarrow{\Delta}$.

2. If $B \in \Delta$ is a given point, then $\varphi_O(\pi) = \overrightarrow{r}_B + \overrightarrow{\pi}$.

Generally speaking, a subset X of a vector space is called *linear variety* if either $X = \emptyset$ or there exists $a \in V$ and a vector subspace U of V, such that $X = a + U$.

$$\dim(X) = \begin{cases} -1 & \text{dacă } X = \emptyset \\ \dim(U) & \text{dacă } X = a + U, \end{cases}$$

Proposition 2.5. *The bijection φ_O transforms the straight lines and the planes of the affine space $\mathcal{P}$ into the one and two dimnensional linear varieties of the vector space $\mathcal{V}$ respectively.*

2.2 The vector ecuations of the straight lines and planes

Proposition 2.6. *Let Δ be a straight line, let π be a plane, $\{\overrightarrow{d}\}$ be a basis of $\overrightarrow{\Delta}$ and let $[\overrightarrow{d}_1, \overrightarrow{d}_2]$ be an ordered basis of $\overrightarrow{\pi}$.*

1. *The points $M \in \Delta$ are characterized by the* vector equation of Δ

$$\overrightarrow{r}_M = \overrightarrow{r}_A + \lambda \overrightarrow{d}, \ \lambda \in \mathbb{R} \tag{2.1}$$

where $A \in \Delta$ is a given point.

2. *The points $M \in \pi$ are characterized by the* vector equation of π

$$\overrightarrow{r}_M = \overrightarrow{r}_A + \lambda_1 \overrightarrow{d}_1 + \lambda_2 \overrightarrow{d}_2, \ \lambda_1, \lambda_2 \in \mathbb{R}, \tag{2.2}$$

where $A \in \pi$ is a given point.

PROOF.

□

Corollary 2.7. *If $A, B \in \mathcal{P}$ are different points, then the vector equation of the line AB is*

$$\vec{r}_M = (1-\lambda)\,\vec{r}_A + \lambda\,\vec{r}_B,\ \lambda \in \mathbb{R}. \tag{2.3}$$

PROOF.

□

Corollary 2.8. *If A, B, $C \in \mathcal{P}$ are three noncolinear points, then the vector equation of the plane (ABC) is*

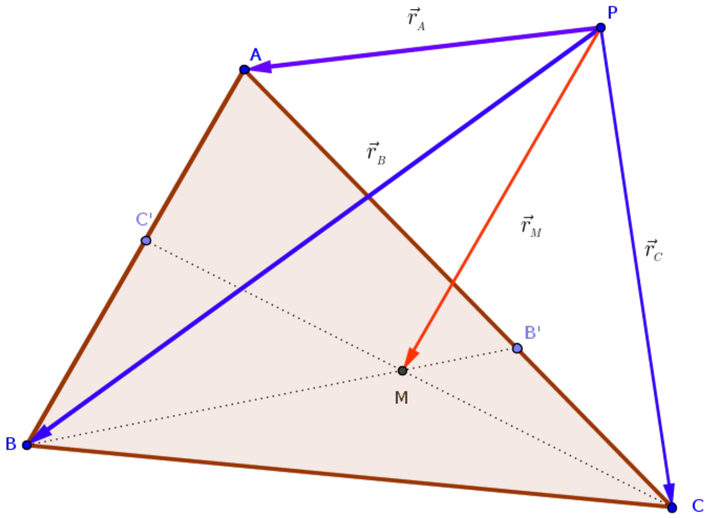
$$\vec{r}_M = (1-\lambda_1-\lambda_2)\,\vec{r}_A + \lambda_1\,\vec{r}_B + \lambda_2\,\vec{r}_C,\ \lambda_1,\ \lambda_2 \in \mathbb{R}. \tag{2.4}$$

PROOF.

□

Example 2.1. Consider the points C' and B' on the sides AB and AC of the triangle ABC such that $\overrightarrow{AC'} = \lambda\,\overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\,\overrightarrow{CB'}$. The lines BB' and CC' meet at M. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$, $\vec{r}_C = \overrightarrow{PC}$ are the position vectors, with respect to P, of the vertices A, B, C respectively, show that

$$\vec{r}_M = \frac{\vec{r}_A - \lambda\,\vec{r}_B - \mu\,\vec{r}_C}{1-\lambda-\mu}. \tag{2.5}$$



SOLUTION.

□

2.3 Problems

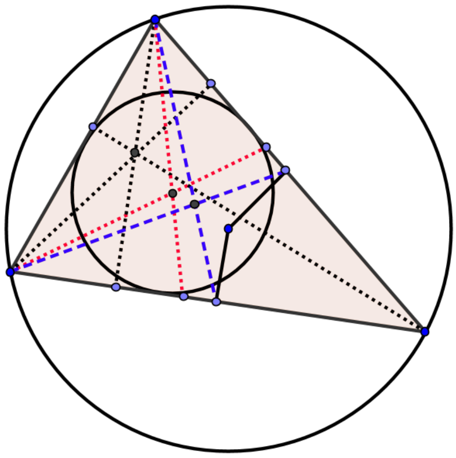
1. ([4, Problem 17, p. 5]) Consider the triangle ABC, its centroid G, its orthocenter H, its incenter I and its circumcenter O. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$, $\vec{r}_C = \overrightarrow{PC}$ are the position vectors with respect to P of the vertices A, B, C respectively, show that:

$$\vec{r}_G := \overrightarrow{PG} = \frac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}.$$

$$\vec{r}_I := \overrightarrow{PI} = \frac{a\,\vec{r}_A + b\,\vec{r}_B + c\,\vec{r}_C}{a+b+c}.$$

$$\vec{r}_H := \overrightarrow{PH} = \frac{(\tan A)\,\vec{r}_A + (\tan B)\,\vec{r}_B + (\tan C)\,\vec{r}_C}{\tan A + \tan B + \tan C}.$$

$$\vec{r}_O := \overrightarrow{PO} = \frac{(\sin 2A)\,\vec{r}_A + (\sin 2B)\,\vec{r}_B + (\sin 2C)\,\vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}.$$



Solution.

2. Consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

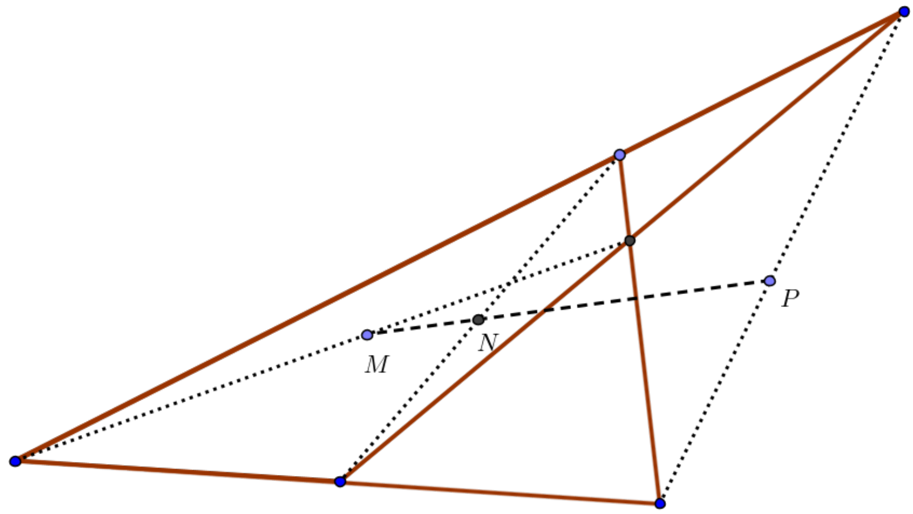
$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\,\overrightarrow{OA} \quad + \quad n\frac{1-m}{1-mn}\,\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\,\overrightarrow{OA} \quad + \quad n\frac{m-1}{m-n}\,\overrightarrow{OA'}\,.$$

where $\{M\} = AB' \cap A'B$, $\{N\} = AA' \cap BB'$, $\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\,\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\,\overrightarrow{OA'}$.

Solution.

3. Show that the midpoints of the diagonals of a complet quadrilateral are collinear (Newton’s theorem).



Solution.

4. Let d, d' be concurrent straight lines and $A, B, C \in d$, $A', B', C' \in d'$. If the following relations $AB' \nparallel A'B$, $AC' \nparallel A'C$, $BC' \nparallel B'C$ hold, show that the points $\{M\} := AB' \cap A'B$, $\{N\} := AC' \cap A'C$, $\{P\} := BC' \cap B'C$ are collinear (Pappus' theorem).

   SOLUTION.

5. Let d, d' be two straight lines and $A, B, C \in d$, $A', B', C' \in d'$ three points on each line such that $AB' \| BA'$, $AC' \| CA'$. Show that $BC' \| CB'$ (the affine Pappus' theorem).

   SOLUTION.

6. Let us consider two triangles ABC and $A'B'C'$ such that the lines AA', BB', CC' are concurrent at a point O and $AB \nparallel A'B'$, $BC \nparallel B'C'$ and $CA \nparallel C'A'$. Show that the points $\{M\} = AB \cap A'B'$, $\{N\} = BC \cap B'C'$ and $\{P\} = CA \cap C'A'$ are collinear (Desargues).

   SOLUTION.

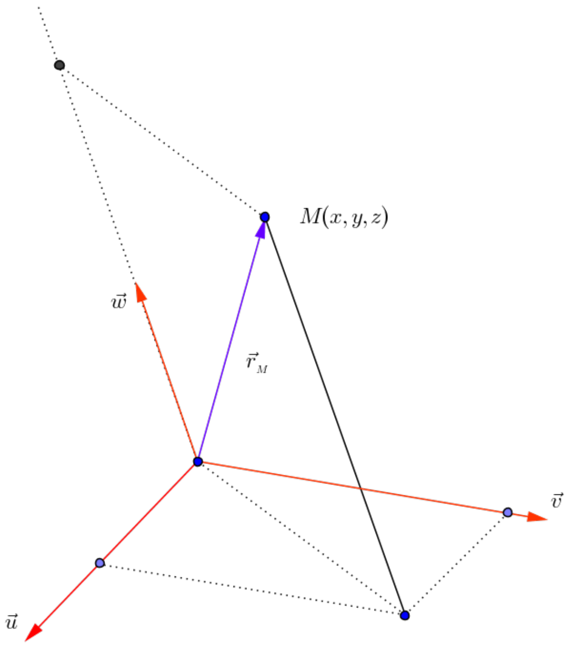
3 Week 3: Cartesian equations of lines and planes

3.1 Cartesian and affine reference systems

If $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is an ordered basis of $\mathcal{V}$ and $\overrightarrow{x} \in \mathcal{V}$, recall that the column vector of the coordinates of $\overrightarrow{x}$ with respect to b is denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

whenever $\overrightarrow{x} = x_1 \overrightarrow{u} + x_2 \overrightarrow{v} + x_3 \overrightarrow{w}$. To emphasize the coordinates of $\overrightarrow{x}$ with respect to b, we shall use the notation $\overrightarrow{x}\,(x_1, x_2, x_3)$.



Definition 3.1. A *cartesian reference system* $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ of the space $\mathcal{P}$, consists in a point $O \in \mathcal{P}$ called the *origin* of the reference system and an ordered basis $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ of the vector space $\mathcal{V}$.

Denote by E_1, E_2, E_3 the points for which $\overrightarrow{u} = \overrightarrow{OE_1}$, $\overrightarrow{v} = \overrightarrow{OE_2}$, $\overrightarrow{w} = \overrightarrow{OE_3}$.

Definition 3.2. The system of points (O, E_1, E_2, E_3) is called *the affine reference system associated to the cartesian reference system* $R = (O, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$.

The straight lines OE_i, $i \in \{1, 2, 3\}$, oriented from O to E_i are called *the coordinate axes*. The coordinates x, y, z of the position vector $\overrightarrow{r}_M = \overrightarrow{OM}$ with respect to the basis $[\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y, z)$. Also, for the column matrix of coordinates of the vector $\overrightarrow{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\overrightarrow{r}_M = x \overrightarrow{u} + y \overrightarrow{v} + z \overrightarrow{w}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Remark 3.1. If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are two points, then

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= x_B \overrightarrow{u} + y_B \overrightarrow{v} + z_B \overrightarrow{w} - (x_A \overrightarrow{u} + y_A \overrightarrow{v} + z_A \overrightarrow{w})\\ &= (x_B - x_A) \overrightarrow{u} + (y_B - y_A) \overrightarrow{v} + (z_B - z_A) \overrightarrow{w},\end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

Remark 3.2. If $R = (O, b)$ is a cartesian reference system, where $b = [\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$ is an ordered basis of $\mathcal{V}$, recall that $\varphi_O : \mathcal{P} \longrightarrow \mathcal{V}$, $\varphi_O(M) = \overrightarrow{OM}$ is bijective and $\psi_b : \mathbb{R}^3 \longrightarrow \mathcal{V}$, $\psi_b(x, y, z) = x \overrightarrow{u} + y \overrightarrow{v} + z \overrightarrow{w}$ is a linear isomorphism. The bijection φ_O defines a unique vector structure over $\mathcal{P}$ such that φ_O becomes an isomorphism. This vector structure depends on the choice of $O \in \mathcal{P}$. Therefore a point $M \in \mathcal{P}$ could be identified either with its position vector $\overrightarrow{r}_M = \varphi_O(M)$, or, with the triplet $(\psi_b^{-1} \circ \varphi_O)(M) \in \mathbb{R}^3$ of its coordinates with respect to the reference system R. If $f : X \longrightarrow \mathbb{R}^3$ is a given application, then $\varphi_O^{-1} \circ \psi_b \circ f : X \longrightarrow \mathcal{P}$ will be denoted by M_f. A similar discussion can be done for a cartesian reference system $R' = (O', b')$ of a plane π, where $b' = [\overrightarrow{u}', \overrightarrow{v}']$ is an ordered basis of $\overrightarrow{\pi}$.

Example 3.1 (Homework)**.** Consider the tetrahedron $ABCD$, where $A(1, -1, 1)$, $B(-1, 1, -1)$, $C(2, 1, -1)$ and $D(1, 1, 2)$. Find the coordinates of:

1. the centroids G_A, G_B, G_C, G_D of the triangles BCD, ACD, ABD and ABC[1] respectively.
2. the midpoints M, N, P, Q, R and S of its edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[CD]$ and $[DB]$ respectively.

SOLUTION.

[1] The centroids of its faces

3.2 The Cylindrical Coordinate System

In order to have a valid coordinate system in the 3-dimensional case, each point of the space must be associated with a unique triple of real numbers (the coordinates of the point) and each triple of real numbers must determine a unique point, as in the case of the Cartesian system of coordinates.

Let $P(x, y, z)$ be a point in a Cartesian system of coordinates $Oxyz$ and P' be the orthogonal projection of P on the plane xOy. One can associate to the point P the triple (r, θ, z), where (r, θ) are the polar coordinates of P' (see Figure 1). The polar coordinates of P' can be obtained by specifying the distance ρ from O to P' and the angle θ (measured in radians), whose "initial" side is the polar axis, i.e. the x-axis, and whose "terminal" side is the ray OP. The *polar coordinates* of the point P are (ρ, θ) (See also section (3.6.2) of the Appendix). The triple (r, θ, z) gives the *cylindrical coordinates* of the point P. There is the bijection

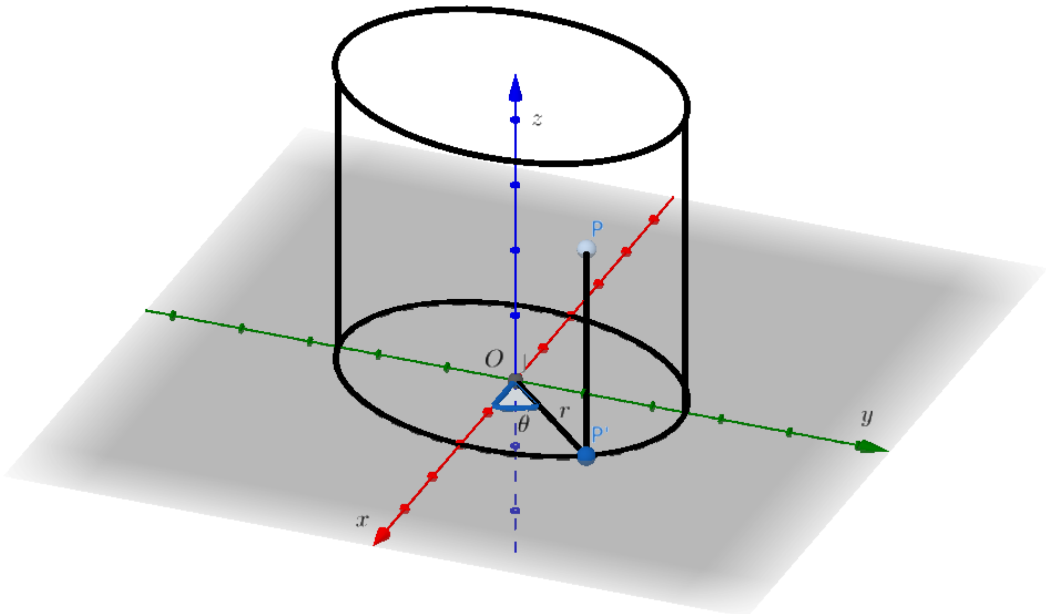


Figure 1: cylindrical coordinates

$$h_1 : \mathcal{P} \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R}, \ P \to (r, \theta, z)$$

and one obtains a new coordinate system, named the *cylindrical coordinate system* in $\mathcal{P}$. For the conversion formulas between the cylindrical coordinates and the Cartesian coordinates we refere the reader to [1, p. 19]. Note however that once we have the cylindrical coordinates (r, θ, z) of a point P, then its Cartesian coordinates are

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}.$$

3.3 The Spherical Coordinate System

Another way to associate to each point P in $\mathcal{P}$ a triple of real numbers is illustrated in Figure 2. If $P(x, y, z)$ is a point in a rectangular system of coordinates $Oxyz$ and P' its orthogonal projection on Oxy, let ρ be the length of the segment $[OP]$, θ be the oriented angle determined by $[Ox$ and $[OP'$ and φ be the oriented angle between $[Oz$ and $[OP$. The triple

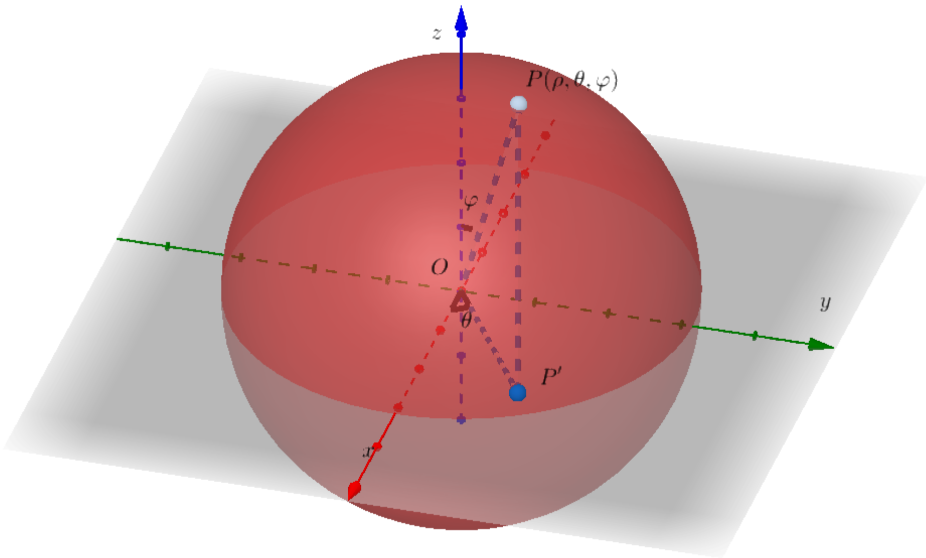


Figure 2: spherical coordinates

(ρ, θ, φ) gives the *spherical coordinates* of the point P. This way, one obtains the bijection

$$h_2 : \mathcal{P} \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi], P \to (\rho, \theta, \varphi),$$

which defines a new coordinate system in $\mathcal{P}$, called the *spherical coordinate system* . For the conversion formulas between the spherical coordinate system and the Cartesian coordinate system we refere the reader to [1, p. 20]. Note however that once we have the cylindrical coordinates (ρ, θ, φ) of a point P, then its Cartesian coordinates are

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}.$$

3.4 The cartesian equations of the straight lines

Let Δ be the straight line passing through the point $A_0(x_0, y_0, z_0)$ which is parallel to the vector $\overrightarrow{d}(p, q, r)$. Its vector equation is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda\overrightarrow{d},\ \lambda \in \mathbb{R}. \tag{3.1}$$

Denoting by x, y, z the coordinates of the generic point M of the straight line Δ, its vector equation (3.1) is equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases},\ \lambda \in \mathbb{R} \tag{3.2}$$

Indeed, the vector equation of Δ can be written, in terms of the coordinates of the vectors $\vec{r}_M$, $\vec{r}_{A_0}$ and $\vec{d}$, as follows:

$$x\vec{u} + y\vec{v} + z\vec{w} = x_0\vec{u} + y_0\vec{v} + z_0\vec{w} + \lambda(p\vec{u} + q\vec{v} + r\vec{w})$$
$$\Longleftrightarrow x\vec{u} + y\vec{v} + z\vec{w} = (x_0 + p\lambda)\vec{u} + (y_0 + q\lambda)\vec{v} + (z_0 + r\lambda)\vec{w},\ \lambda \in \mathbb{R}$$

which is obviously equivalent to (3.2). The relations (3.2) are called the *parametric equations* of the straight line Δ and they are equivalent to the following relations

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r} \tag{3.3}$$

If $r = 0$, for instance, the canonical equations of the straight line Δ are

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} \ \wedge\ z = z_0.$$

If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are different points of the line Δ, then

$$\overrightarrow{AB}\,(x_B - x_A, y_B - y_A, z_B - z_A)$$

is a director vector of Δ, its canonical equations having, in this case, the form

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}. \tag{3.4}$$

Example 3.2. Consider the tetrahedrom $ABCD$, where $A(1,-1,1)$, $B(-1,1,-1)$, $C(2,1,-1)$ and $D(1,1,2)$, as well as the centroids G_A, G_B, G_C, G_D of the triangles BCD, ACD, ABD and ABC[2] respectively. Show that the medians AG_A, BG_B, CG_C and DG_D are concurrent and find the coordinates of their intersection point.

SOLUTION. One can easily see that the coordinates of the centroids G_A, G_B, G_C, G_D are $(2/3,1,0)$, $(4/3,1/3,2/3)$, $(1/3,1/3,2/3)$ and $(2/3,1/3,-1/3)$ respectively. The equations of the medians AG_A and BG_B are

$$(AG_A)\ \frac{x-1}{2/3-1} = \frac{y+1}{1-(-1)} = \frac{z-1}{0-1} \Longleftrightarrow \frac{x-1}{-1/3} = \frac{y+1}{2} = \frac{z-1}{-1}$$
$$(BG_B)\ \frac{x+1}{4/3+1} = \frac{y-1}{1/3-1} = \frac{z+1}{2/3+1} \Longleftrightarrow \frac{x+1}{7/3} = \frac{y-1}{-2/3} = \frac{z+1}{5/3}.$$

Thus, the director space of the median AG_A is $\left\langle\left(-\frac{1}{3}, 2, -1\right)\right\rangle = \langle(-1,6,-3)\rangle$ and the director space of the median BG_B is $\left\langle\left(\frac{7}{3}, -\frac{2}{3}, \frac{5}{3}\right)\right\rangle = \langle(7,-2,5)\rangle$. Consequently, the parametric equations of the medians AG_A and BG_B are

$$(AG_A)\begin{cases} x = 1 - t \\ y = -1 + 6t \\ z = 1 - 3t \end{cases},\ t \in \mathbb{R} \text{ and } (BG_B)\begin{cases} x = -1 + 7s \\ y = 1 - 2s \\ z = -1 + 5s \end{cases},\ s \in \mathbb{R}.$$

Thus, the two medians AG_A and BG_B are concurrent if and only if there exist $s, t \in \mathbb{R}$ such that

$$\begin{cases} 1 - t = -1 + 7s \\ -1 + 6t = 1 - 2s \\ 1 - 3t = -1 + 5s \end{cases} \Longleftrightarrow \begin{cases} 7s + t = 2 \\ 2s + 6t = 2 \\ 5s + 3t = 2 \end{cases} \Longleftrightarrow \begin{cases} 7s + t = 2 \\ s + 3t = 1 \\ 5s + 3t = 2. \end{cases}$$

[2]The centroids of its faces

This system is compatible and has the unique solution $s = t = \frac{1}{4}$, which shows that the two medians AG_A and BG_B are concurrent and

$$AG_A \cap BG_B = \left\{ G\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right) \right\}.$$

One can similarly show that $BG_B \cap CG_C = CG_C \cap AG_A = \left\{ G\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right) \right\}$.

Example 3.3 (Homework)**.** Consider the tetrahedrom $ABCD$, where $A(1,-1,1)$, $B(-1,1,-1)$, $C(2,1,-1)$ and $D(1,1,2)$, as well as the midpoints M, N, P, Q, R and S of its edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[CD]$ and $[DB]$ respectively. Show that the lines MR, PQ and NS are concurrent and find the coordinates of their intersection point.

SOLUTION.

3.5 The cartesian equations of the planes

Let $A_0(x_0, y_0, z_0) \in \mathcal{P}$ and $\overrightarrow{d}_1\ (p_1, q_1, r_1)$, $\overrightarrow{d}_2\ (p_2, q_2, r_2) \in \mathcal{V}$ be linearly independent vectors, that is

$$\operatorname{rank}\begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix} = 2.$$

The vector equation of the plane π passing through A_0 which is parallel to the vectors $\overrightarrow{d}_1$ (p_1, q_1, r_1), $\overrightarrow{d}_2\ (p_2, q_2, r_2)$ is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda_1 \overrightarrow{d}_1 + \lambda_2 \overrightarrow{d}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.5}$$

If we denote by x, y, z the coordinates of the generic point M of the plane π, then the vector equation (3.5) is the equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda_1 p_1 + \lambda_2 p_2 \\ y = y_0 + \lambda_1 q_1 + \lambda_2 q_2 \\ z = z_0 + \lambda_1 r_1 + \lambda_2 r_2 \end{cases},\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.6}$$

Indeed, the vector equation of π can be written, in terms of the coordinates of the vectors $\vec{r}_M$, $\vec{r}_{A_0}$, $\vec{d}_1$ and $\vec{d}_2$, as follows:

$$x\vec{u}+y\vec{v}+z\vec{w}=x_0\vec{u}+y_0\vec{v}+z_0\vec{w}+\lambda_1(p_1\vec{u}+q_1\vec{v}+r_1\vec{w})+\lambda_2(p_2\vec{u}+q_2\vec{v}+r_2\vec{w})$$
$$\Longleftrightarrow x\vec{u}+y\vec{v}+z\vec{w}=(x_0+\lambda_1p_1+\lambda_2p_2)\vec{u}+(y_0+\lambda_1q_1+\lambda_2q_2)\vec{v}+(z_0+\lambda_1r_1+\lambda_2r_2)\vec{w},$$
$$\lambda_1,\lambda_2\in\mathbb{R},$$

which is obviously equivalent to (3.6). The relations (3.6) characterize the points of the plane π and are called the *parametric equations* of the plane π. More precisely, the compatibility of the linear system (3.6) with the unknowns λ_1,λ_2 is a necessary and sufficient condition for the point $M(x,y,z)$ to be contained within the plane π. On the other hand the compatibility of the linear system (3.6) is equivalent to

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}=0, \tag{3.7}$$

which expresses the equality between the rank of the coefficient matrix of the system and the rank of the extended matrix of the system. The equation (3.7) is a characterization of the points of the plane π in terms of the Cartesian coordinates of the generic point M and is called the *cartesian equation* of the plane π. On can put the equation (3.7) in the form

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \text{ or} \tag{3.8}$$

$$Ax+By+Cz+D=0, \tag{3.9}$$

where the coefficients A,B,C satisfy the relation $A^2+B^2+C^2>0$. It is also easy to show that every equation of the form (3.9) represents the equation of a plane. Indeed, if $A\neq 0$, then the equation (3.9) is equivalent to

$$\begin{vmatrix} x+\frac{D}{A} & y & z \\ B & -A & 0 \\ C & 0 & -A \end{vmatrix}=0.$$

We observe that one can put the equation (3.8) in the form

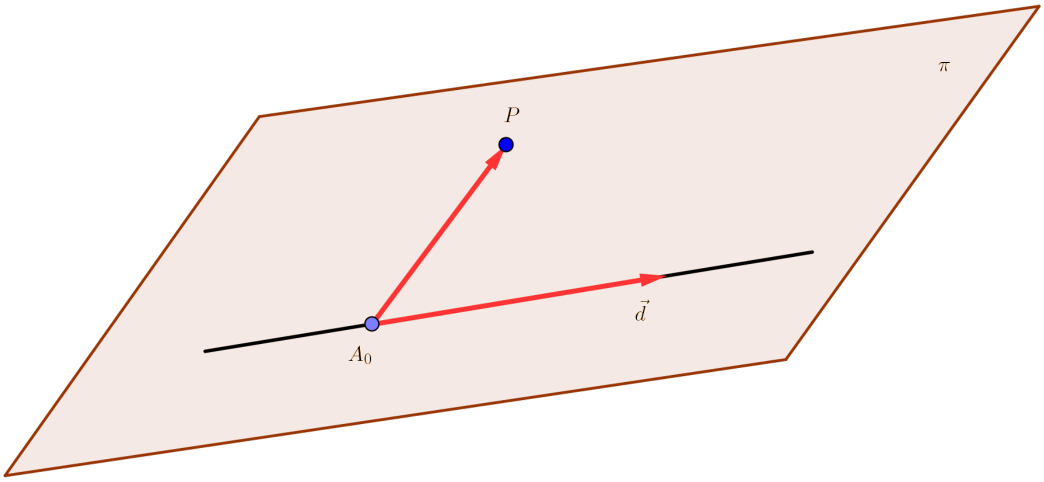
$$AX+BY+CZ=0 \tag{3.10}$$

where $X=x-x_0$, $Y=y-y_0$, $Z=z-z_0$ are the coordinates of the vector $\overrightarrow{A_0M}$.

Example 3.4. Write the equation of the plane determined by the point $P(-1,1,2)$ and the line $(\Delta)\ \frac{x-1}{3}=\frac{y}{2}=\frac{z+1}{-1}$.

SOLUTION. Note that $P\notin\Delta$, as $\frac{-1-1}{3}\neq\frac{1}{2}\neq -3=\frac{2+1}{-1}$, i.e. the point P and the line Δ determine, indeed, a plane, say π. One can regard π as the plane through the point $A_0(1,0,-1)$ which is parallel to the vectors $\overrightarrow{A_0P}\ (-1-1,1-0,2-(-1))=\overrightarrow{A_0P}\ (-2,1,3)$ and $\vec{d}\ (3,2,-1)$. Thus, the equation of π is

$$\begin{vmatrix} x-1 & y & z+1 \\ -2 & 1 & 3 \\ 3 & 2 & -1 \end{vmatrix}=0\Longleftrightarrow x-y+z=0.$$



Example 3.5 (Homework)**.** Generalize Example 3.4: Write the equation of the plane determined by the line $(\Delta)\ \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$ and the point $M(x_M, y_M, z_M) \notin \Delta$.

SOLUTION.

Remark 3.3. If $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ are noncollinear points, then the plane (ABC) determined by the three points can be viewed as the plane passing through the point A which is parallel to the vectors $\vec{d}_1 = \overrightarrow{AB}$, $\vec{d}_2 = \overrightarrow{AC}$. The coordinates of the vectors $\vec{d}_1$ şi $\vec{d}_2$ are

$$(x_B - x_A, y_B - y_A, z_B - z_A) \text{ and } (x_C - x_A, y_C - y_A, z_C - z_A) \text{ respectively.}$$

Thus, the equation of the plane (ABC) is

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0, \tag{3.11}$$

or, echivalently

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0. \tag{3.12}$$

Thus, four points $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ and $D(x_D, y_D, z_D)$ are coplanar if and ony if

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \tag{3.13}$$

Example 3.6 (Homework)**.** Write the equation of the plane determined by the points $M_1(3,-2,1)$, $M_2(5,4,1)$ and $M_3(-1,-2,3)$.

SOLUTION.

Remark 3.4. If $A(a,0,0),\ B(0,b,0),\ C(0,0,c)$ are three points ($abc \neq 0$), then for the equation of the plane (ABC) we have successively:

$$\begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & z-c & 1 \\ a & 0 & -c & 1 \\ 0 & b & -c & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & z-c \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = 0$$

$$\Longleftrightarrow ab(z-c) + bcx + acy = 0 \Longleftrightarrow bcx + acy + abz = abc$$

$$\Longleftrightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{3.14}$$

The equation (3.14) of the plane (ABC) is said to be in *intercept form* and the x,y,z-intercepts of the plane (ABC) are a,b,c respectively.

Example 3.7 (Homework)**.** Write the equation of the plane $(\pi)\ 3x-4y+6z-24=0$ in intercept form.

SOLUTION.

3.6 Appendix: The Cartesian equations of lines in the two dimensional setting

3.6.1 Cartesian and affine reference systems

If $b = [\overrightarrow{e}, \overrightarrow{f}]$ is an ordered basis of the director subspace $\overrightarrow{\pi}$ of the plane π and $\overrightarrow{x} \in \overrightarrow{\pi}$, recall that the column vector of $\overrightarrow{x}$ with respect to b is being denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

whenever $\overrightarrow{x}= x_1 \overrightarrow{e} + x_2 \overrightarrow{f}$.

Definition 3.3. A *cartesian reference system* of the plane π, is a system $R = (O, \overrightarrow{e}, \overrightarrow{f})$, where O is a point from π called the *origin* of the reference system and $b = [\overrightarrow{e}, \overrightarrow{f}]$ is a basis of the vector space $\overrightarrow{\pi}$.

Denote by E, F the points for which $\overrightarrow{e}=\overrightarrow{OE}$, $\overrightarrow{f}=\overrightarrow{OF}$.

Definition 3.4. The system of points (O, E, F) is called *the affine reference system associated to the cartesian reference system* $R = (O, \overrightarrow{e}, \overrightarrow{f})$.

The straight lines OE, OF, oriented from O to E and from O to F respectively, are called *the coordinate axes*. The coordinates x, y of the position vector $\overrightarrow{r}_M=\overrightarrow{OM}$ with respect to the basis $[\overrightarrow{e}, \overrightarrow{f}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y)$. Also, for the column matrix of coordinates of the vector $\overrightarrow{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\overrightarrow{r}_M= x \overrightarrow{e} + y \overrightarrow{f}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Remark 3.5. If $A(x_A, y_A)$, $B(x_B, y_B)$ are two points, then

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}= x_B \overrightarrow{e} + y_B \overrightarrow{f} - (x_A \overrightarrow{e} + y_A \overrightarrow{f}) \\ &= (x_B - x_A) \overrightarrow{e} + (y_B - y_A) \overrightarrow{f}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

3.6.2 The Polar Coordinate System [1, p. 17]

As an alternative to a Cartesian coordinate system (RS) one considers in the plane π a fixed point O, called *pole* and a half-line directed to the right of O, called *polar axis* (see Figure 3). By specifying the distance ρ from O to a point P and an angle θ (measured in radians), whose

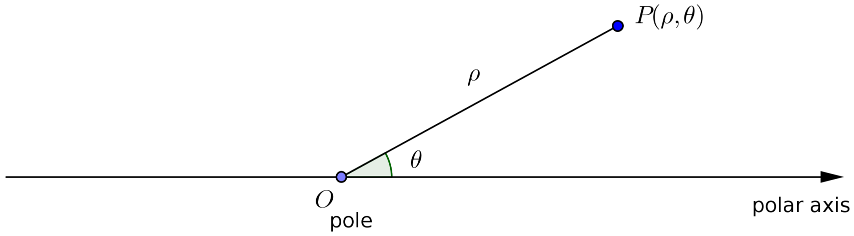


Figure 3: The pole and the polar axis related to a polar coordinate system

"initial" side is the polar axis and whose "terminal" side is the ray OP, the *polar coordinates* of the point P are (ρ, θ). One obtains a bijection

$$\pi \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi), \quad P \to (\rho, \theta)$$

which associates to any point P in $\pi \setminus \{O\}$ the pair (ρ, θ) (suppose that $O(0,0)$). The positive real number ρ is called the *polar ray* of P and θ is called the *polar angle* of P.

Consider the Cartesian coordinate system in π, whose origin O is the pole and whose positive half-axis Ox is the polar axis (see Figure 4). The following transformation formulas give the connection between the coordinates of an arbitrary point in the two systems of coordinates.

Let P be a given point of polar coordinates (ρ, θ) . Its Cartesian coordinates are

$$\begin{cases} x_P = \rho\cos\theta \\ y_P = \rho\sin\theta \end{cases} . \tag{3.15}$$

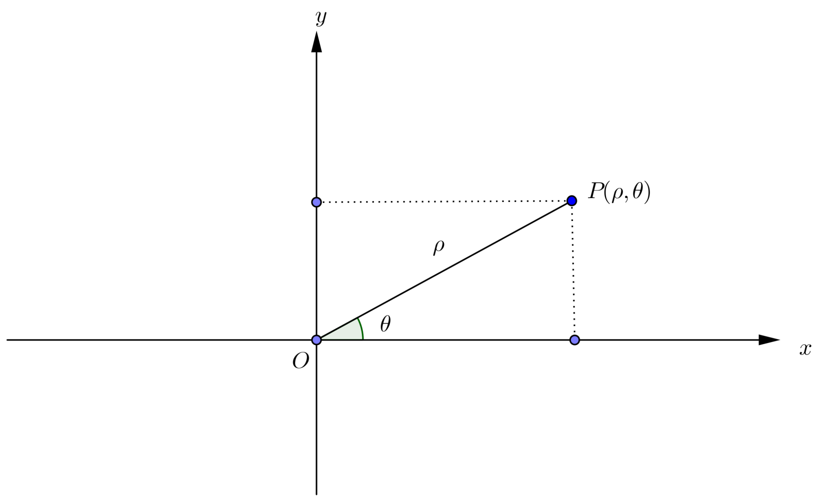


Figure 4: Polar coordinates

Let P be a point of Cartesian coordinates (x, y). It is clear that the polar ray of P is given by the formula

$$\rho = \sqrt{x^2 + y^2}. \tag{3.16}$$

In order to obtain the polar angle of P, it must be considered the quadrant where P is situated. One obtains the following formulas:

Case 1. If $x \neq 0$, then using $\tan\theta = \dfrac{y}{x}$, one has

$$\theta = \arctan\frac{y}{x} + k\pi, \quad \text{where} \quad k = \begin{cases} 0 \text{ if } P \in I \cup]Ox \\ 1 \text{ if } P \in II \cup III \cup]Ox' \\ 2 \text{ if } P \in IV; \end{cases} ;$$

Case 2. If $x = 0$ and $y \neq 0$, then

$$\theta = \begin{cases} \dfrac{\pi}{2} \text{ when } P \in]Oy \\ \dfrac{3\pi}{2} \text{ when } P \in]Oy'; \end{cases}$$

Case 3. If $x = 0$ and $y = 0$, then $\theta = 0$.

3.6.3 Parametric and Cartesian equations of Lines

Let Δ be a line passing through the point $A_0(x_0, y_0) \in \pi$ which is parallel to the vector $\overrightarrow{d}\,(p,q)$. Its vector equation is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + t\,\overrightarrow{d},\ t \in \mathbb{R}. \tag{3.17}$$

If (x,y) are the coordinates of a generic point $M \in \Delta$, then its vector equation (3.17) is equivalent to the following system

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \end{cases},\ t \in \mathbb{R}. \tag{3.18}$$

The relations are called the *parametric equations* of the line Δ and they are equivalent to the following equation

$$\frac{x - x_0}{p} = \frac{y - y_0}{q}, \tag{3.19}$$

called the *canonical equation*of Δ. If $q = 0$, then the equation (3.19) becomes $y = y_0$.

If $A(x_A, y_A)$ are two different points of the plane π, then $\overrightarrow{AB}\ (x_B - x_A, y_B - y_A)$ is a director vector of the line AB and the canonical equation of the line AB is

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}. \tag{3.20}$$

The equation (3.20) is equivalent to

$$\begin{vmatrix} x - x_A & y - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x - x_A & y - y_A & 1 \\ x_B - x_A & y_B - y_A & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} = 0.$$

Thus, three poins $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{3.21}$$

3.6.4 General Equations of Lines

We can put the equation (3.19) in the form

$$ax + by + c = 0, \qquad \text{with} \quad a^2 + a^2 > 0, \tag{3.22}$$

which means that any line from π is characterized by a first degree equation. Conversely, such of an equation represents a line, since the formula (3.22) is equivalent to

$$\frac{x + \frac{c}{a}}{-\frac{b}{a}} = \frac{y}{1}$$

and this is the *symmetric equation* of the line passing through $P_0\left(-\frac{c}{a}, 0\right)$ and parallel to $\overline{v}\left(-\frac{b}{a}, 1\right)$. The equation (3.22) is called *general equation* of the line.

Remark 3.6. The lines

$$(d)\ ax+by+c=0 \text{ and } (\Delta)\ \frac{x-x_0}{p}=\frac{x-x_0}{q}$$

are parallel if and only if $ap+bq=0$. Indeed, we have:

$$\begin{aligned} d\|\Delta \ &\Longleftrightarrow \overrightarrow{d}=\overrightarrow{\Delta} \Longleftrightarrow \langle \overrightarrow{u}\ (p,q)\rangle = \langle \overrightarrow{v}\ \left(-\tfrac{b}{a},1\right)\rangle \Longleftrightarrow \exists t\in\mathbb{R} \text{ s.t. } \overrightarrow{u}\ (p,q)=t\ \overrightarrow{v}\ \left(-\tfrac{b}{a},1\right) \\ &\Longleftrightarrow \exists t\in\mathbb{R} \text{ s.t. } =-t\tfrac{b}{a} \text{ and } q=t \Longleftrightarrow ap+bq=0. \end{aligned}$$

3.6.5 Reduced Equations of Lines

Consider a line given by its general equation $Ax+By+C=0$, where at least one of the coefficients A and B is nonzero. One may suppose that $B\neq 0$, so that the equation can be divided by B. One obtains

$$y=mx+n \tag{3.23}$$

which is said to be the *reduced equation* of the line.

Remark: If $B=0$, (3.22) becomes $Ax+C=0$, or $x=-\dfrac{C}{A}$, a line parallel to Oy. (In the same way, if $A=0$, one obtains the equation of a line parallel to Ox).

Let d be a line of equation $y=mx+n$ in a Cartesian system of coordinates and suppose that the line is not parallel to Oy. Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be two different points on d and φ be the angle determined by d and Ox (see Figure 5); $\varphi\in[0,\pi]\setminus\{\pi/2\}$. The points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ belong to d, hence

$$\begin{cases} y_1=mx_1+n \\ y_2=mx_2+n, \end{cases}$$

and $x_2\neq x_1$, since d is not parallel to Oy. Then,

$$m=\frac{y_2-y_1}{x_2-x_1}=\tan\varphi. \tag{3.24}$$

The number $m=\tan\varphi$ is called the *angular coefficient* of the line d. It is immediate that the equation of the line passing through the point $P_0(x_0,y_0)$ and of the given angular coefficient m is

$$y-y_0=m(x-x_0). \tag{3.25}$$

3.6.6 Intersection of Two Lines

Let $d_1: a_1x+b_1y+c_1=0$ and $d_2: a_2x+b_2y+c_2=0$ be two lines in $\mathcal{E}_2$. The solution of the system of equation

$$\begin{cases} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{cases}$$

will give the set of the intersection points of d_1 and d_2.

1) If $\dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2}$, the system has a unique solution (x_0,y_0) and the lines have a unique intersection point $P_0(x_0,y_0)$. They are *secant*.

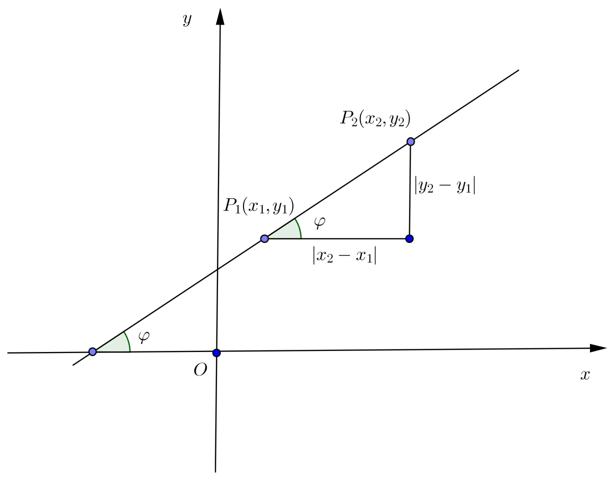


Figure 5:

2) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$, the system is not compatible, and the lines have no points in common. They are *parallel*.

3) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, the system has an infinity of solutions, and the lines coincide. They are *identical*.

If $d_i : a_i x + b_i y + c_i = 0$, $i = \overline{1,3}$ are three lines in $\mathcal{E}_2$, then they are concurrent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{3.26}$$

3.6.7 Bundles of Lines ([1])

The set of all the lines passing through a given point P_0 is said to be a *bundle* of lines. The point P_0 is called the *vertex* of the bundle.

If the point P_0 is of coordinates $P_0(x_0, y_0)$, then the equation of the bundle of vertex P_0 is

$$r(x - x_0) + s(y - y_0) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \tag{3.27}$$

Remark: Tthe *reduced bundle* of line through P_0 is,

$$y - y_0 = m(x - x_0), \qquad m \in \mathbb{R}, \tag{3.28}$$

and covers the bundle of lines through P_0, except the line $x = x_0$. Similarly, the family of lines

$$x - x_0 = k(y - y_0), \qquad k \in \mathbb{R}, \tag{3.29}$$

covers the bundle of lines through P_0, exceptthe line $y = y_0$.

If the point P_0 is given as the intersection of two lines, then its coordinates are the solution of the system

$$\begin{cases} d_1 : a_1 x + b_1 y + c_1 = 0 \\ d_2 : a_2 x + b_2 y + c_2 = 0 \end{cases},$$

assumed to be compatible. The equation of the bundle of lines through P_0 is

$$r(a_1x + b_1y + c_1) + s(a_2x + b_2y + c_2) = 0, \qquad (r,s) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{3.30}$$

Remark: As before, if $r \neq 0$ (or $s \neq 0$), one obtains the reduced equation of the bundle, containing all the lines through P_0, except d_1 (respectively d_2).

3.6.8 The Angle of Two Lines ([1])

Let d_1 and d_2 be two concurrent lines, given by their reduced equations:

$$d_1 : y = m_1x + n_1 \quad \text{and} \quad d_2 : y = m_2x + n_2.$$

The angular coefficients of d_1 and d_2 are $m_1 = \tan\varphi_1$ and $m_2 = \tan\varphi_2$ (see Figure 6). One may suppose that $\varphi_1 \neq \frac{\pi}{2}$, $\varphi_2 \neq \frac{\pi}{2}$, $\varphi_2 \geq \varphi_1$, such that $\varphi = \varphi_2 - \varphi_1 \in [0,\pi] \setminus \{\frac{\pi}{2}\}$.

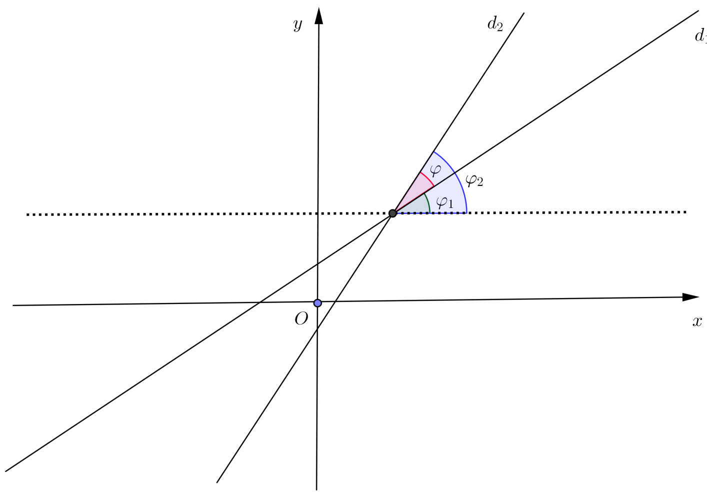


Figure 6:

The angle determined by d_1 and d_2 is given by

$$\tan\varphi = \tan(\varphi_2 - \varphi_1) = \frac{\tan\varphi_2 - \tan\varphi_1}{1 + \tan\varphi_1 \tan\varphi_2},$$

hence

$$\tan\varphi = \frac{m_2 - m_2}{1 + m_1m_2}. \tag{3.31}$$

1) The lines d_1 and d_2 are parallel if and only if $\tan\varphi = 0$, therefore

$$d_1 \parallel d_2 \iff m_1 = m_2. \tag{3.32}$$

2) The lines d_1 and d_2 are orthogonal if and only if they determine an angle of $\frac{\pi}{2}$, hence

$$d_1 \perp d_2 \iff m_1m_2 + 1 = 0. \tag{3.33}$$

3.7 Problems

1. Write the equation of the plane which passes through $M_0(1,-2,3)$ and is parallel to the vectors $\overrightarrow{v}_1\ (1,-1,0)$ and $\overrightarrow{v}_2\ (-3,2,4)$.

   HINT.
$$\begin{vmatrix} x-0 & y+2 & z-3 \\ 1 & -1 & 0 \\ -3 & 2 & 4 \end{vmatrix} = 0.$$

   vspace*2cm

2. Write the equation of the line which passes through $A(1,-2,6)$ and is parallel to

   (a) The x-axis;

   (b) The line $(d_1)\ \dfrac{x-1}{2} = \dfrac{y+5}{-3} = \dfrac{z-1}{4}$.

   (c) The vector $\overrightarrow{v}\ (1,0,2)$.

   SOLUTION.

3. Write the equation of the plane which contains the line

$$(d_1)\ \frac{x-3}{2}=\frac{y+4}{1}=\frac{z-2}{-3}$$

and is parallel to the line

$$(d_2)\ \frac{x+5}{2}=\frac{y-2}{2}=\frac{z-1}{2}.$$

HINT.

$$\begin{vmatrix} x-3 & y+4 & z-2 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{vmatrix}=0.$$

4. Consider the points $A(\alpha,0,0)$, $B(0,\beta,0)$ and $C(0,0,\gamma)$ such that

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{1}{a} \text{ where } a \text{ is a constatnt.}$$

Show that the plane (A,B,C) passes through a fixed point.

SOLUTION. The equation of the plane (ABC) can be written in intercept form, namely

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1.$$

The given relation shows that the point $P(a,a,a)\in(ABC)$ whenever α,β,γ verifies the given relation.

5. Write the equation of the line which passes through the point $M(1,0,7)$, is parallel to the plane (π) $3x-y+2z-15=0$ and intersects the line

$$(d)\ \frac{x-1}{4}=\frac{y-3}{2}=\frac{z}{1}.$$

Solution.

6. Write the equation of the plane which passes through $M_0(1,-2,3)$ and cuts the positive coordinate axes through equal intercepts.

   SOLUTION. The general equation of such a plane is $x+y+z=a$. In this particular case $a=1+(-2)+3=2$ and the equation of the required plane is $x+y+z=2$.

7. Write the equation of the plane which passes through $A(1,2,1)$ and is parallel to the straight lines

$$(d_1)\left\{\begin{array}{ccccccccc} x & + & 2y & - & z & + & 1 & = 0 \\ x & - & y & + & z & - & 1 & = 0 \end{array}\right. \quad (d_2)\left\{\begin{array}{ccccccc} 2x & - & y & + & z & = 1 \\ x & - & y & + & z & = 0. \end{array}\right.$$

   SOLUTION. We need to find some director parameters of the lines (d_1) and (d_2). In this respect we may solve the two systems. The general solution of the first system is

$$\left\{\begin{array}{l} x=-\frac{1}{3}t+\frac{1}{3} \\ y=\frac{2}{3}t-\frac{2}{3} \\ z=t \end{array}\right., \; t\in\mathbb{R}$$

   and the general solution of the second system is

$$\left\{\begin{array}{l} x=1 \\ y=t+1 \\ z=t \end{array}\right., \; t\in\mathbb{R}$$

   and these are the parametric equations of the lines (d_1) and (d_2). Thus, the direction of the line (d_1) is the 1-dimensional subspace

$$\left\langle\left(-\frac{1}{3},\frac{2}{3},1\right)\right\rangle=\langle(-1,2,3)\rangle,$$

   and the direction of the line (d_2) is the 1-dimensional subspace $\langle(0,1,1)\rangle$.

   Consequently, some director parameters of the line (d_1) are $p_1=-1, q_1=2, r_1=3$ and some director parameters of the line (d_2) are $p_2=0, q_2=r_2=1$. Finaly, the equation of the required plane is

$$\begin{vmatrix} x-1 & y-2 & z-1 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}=0.$$

   The computation of the determinant is left to the reader.

   A few questions in the two dimensional setting ([1])

8. The sides $[BC]$, $[CA]$, $[AB]$ of the triangle ΔABC are divided by the points M, N respectively P into the same ratio k. Prove that the triangles ΔABC and ΔMNP have the same center of gravity.

   SOLUTION.

9. Sketch the graph of $x^2 - 4xy + 3y^2 = 0$.

   SOLUTION.

10. Find the equation of the line passing through the intersection point of the lines

$$d_1 : 2x - 5y - 1 = 0, \quad d_2 : x + 4y - 7 = 0$$

and through a point M which divides the segment $[AB]$, $A(4, -3)$, $B(-1, 2)$, into the ratio $k = \frac{2}{3}$.

SOLUTION.

11. Let A be a mobile point on the Ox axis and B a mobile point on Oy, so that $\dfrac{1}{OA}+\dfrac{1}{OB}=k$ (constant). Prove that the lines AB passes through a fixed point.

    SOLUTION.

12. Find the equation of the line passing through the intersection point of

$$d_1 : 3x - 2y + 5 = 0, \quad d_2 : 4x + 3y - 1 = 0$$

and crossing the positive half axis of Oy at the point A with $OA = 3$.
SOLUTION.

13. Find the parametric equations of the line through P_1 and P_2, when

    (a) $P_1(3,-2)$, $P_2(5,1)$;

    (b) $P_1(4,1)$, $P_2(4,3)$.

    SOLUTION.

14. Find the parametric equations of the line through $P(-5,2)$ and parallel to $\overline{v}(2,3)$.
    SOLUTION.

15. Show that the equations

$$x = 3 - t, y = 1 + 2t \qquad \text{and} \qquad x = -1 + 3t, y = 9 - 6t$$

represent the same line.

SOLUTION.

16. Find the vector equation of the line P_1P_2, where
    (a) $P_1(2,-1)$, $P_2(-5,3)$;
    (b) $P_1(0,3)$, $P_2(4,3)$.

    SOLUTION.

17. Given the line $d : 2x + 3y + 4 = 0$, find the equation of a line d_1 through the point $M_0(2, 1)$, in the following situations:

    (a) d_1 is parallel with d;

    (b) d_1 is orthogonal on d;

    (c) the angle determined by d and d_1 is $\varphi = \dfrac{\pi}{4}$.

    SOLUTION.

18. The vertices of the triangle ΔABC are the intersection points of the lines

$$d_1 : 4x + 3y - 5 = 0, \quad d_2 : x - 3y + 10 = 0, \quad d_3 : x - 2 = 0.$$

   (a) Find the coordinates of A, B, C.
   (b) Find the equations of the median lines of the triangle.
   (c) Find the equations of the heights of the triangle.

   SOLUTION.

References

[1] Andrica, D., Ţopan, L., Analytic geometry, Cluj University Press, 2004.

[2] Galbură Gh., Radó, F., Geometrie, Editura didactică şi pedagogică-Bucureşti, 1979.

[3] Pintea, C. Geometrie. Elemente de geometrie analitică. Elemente de geometrie diferenţială a curbelor şi suprafeţelor, Presa Universitară Clujeană, 2001.

[4] Radó, F., Orban, B., Groze, V., Vasiu, A., Culegere de Probleme de Geometrie, Lit. Univ. ”Babeş-Bolyai”, Cluj-Napoca, 1979.

