Lecture 1

The real numbers: some basic concepts

We use the following notation for numerical sets:

$\mathbb{N} = \{1, 2, \dots\}$ – the set of natural numbers;
$\mathbb{N}_0 = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$ – the set of natural numbers including 0;
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ – the set of integers;
$\mathbb{Q} = \left\{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\right\}$ – the set of rational numbers;

$\mathbb{R}$ – the set of real numbers;
$\mathbb{R} \setminus \mathbb{Q}$ – the set of irrational numbers.

In the sequel we consider the usual ordering on $\mathbb{R}$.

Definition 1. Let $A \subseteq \mathbb{R}$. We consider the following possibly empty sets:

$\mathrm{ub}(A) = \{x \in \mathbb{R} : x \geq a, \forall a \in A\}$ – the *set of upper bounds* of A;
$\mathrm{lb}(A) = \{x \in \mathbb{R} : x \leq a, \forall a \in A\}$ – the *set of lower bounds* of A.

A number $x \in \mathbb{R}$ is said to be

- an *upper (lower) bound* of A if $x \in \mathrm{ub}(A)$ ($x \in \mathrm{lb}(A)$).
- a *maximum* (or *greatest element*) of A if $x \in A \cap \mathrm{ub}(A)$.
- a *minimum* (or *least element*) of A if $x \in A \cap \mathrm{lb}(A)$.

Remark 1. (i) Any $A \subseteq \mathbb{R}$ has at most one maximum (minimum) and, if it exists, we denote it by $\max A$ ($\min A$).

(ii) If a set has one upper (lower) bound, then it has infinitely many upper (lower) bounds.

(iii) $\mathrm{ub}(\emptyset) = \mathrm{lb}(\emptyset) = \mathbb{R}$.

Definition 2. A subset A of $\mathbb{R}$ is said to be

- *bounded above (below)* if $\mathrm{ub}(A) \neq \emptyset$ ($\mathrm{lb}(A) \neq \emptyset$).
- *bounded* if it is both bounded above and below.
- *unbounded* if it is not bounded.

Example 1. (i) $A = \{a \in \mathbb{R} \mid a \geq 2\}$:

(ii) $A = \{a \in \mathbb{R} \mid 0 < a < 1\}$:

(iii) $A = \left\{ \dfrac{1}{n+1} \mid n \in \mathbb{N} \right\}$:

(iv) Every nonempty finite set has a minimum and a maximum.

Definition 3. Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.

- If $\mathrm{ub}(A) \neq \emptyset$, x is called a *supremum* (or *least upper bound*) of A if $x = \min(\mathrm{ub}(A))$.
- If $\mathrm{lb}(A) \neq \emptyset$, x is called an *infimum* (or *greatest lower bound*) of A if $x = \max(\mathrm{lb}(A))$.

Remark 2. Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.

(i) $x = \sup A \iff \begin{cases} x \in \mathrm{ub}(A); \\ x \leq x' \text{ for all } x' \in \mathrm{ub}(A). \end{cases}$

$x = \inf A \iff \begin{cases} x \in \mathrm{lb}(A); \\ x \geq x' \text{ for all } x' \in \mathrm{lb}(A). \end{cases}$

(ii) The set A has at most one supremum (infimum) and, if it exists, we denote it by $\sup A$ ($\inf A$).

(iii) If the maximum (minimum) of A exists, then it is also the supremum (infimum). Conversely, if the supremum (infimum) of A exists and is contained in A, then it is also the maximum (minimum) of A.

Example 2. (i) $A = \{a \in \mathbb{Z} \mid -1/2 \leq a \leq \sqrt{2}\}$:

(ii) $A = \{a \in \mathbb{R} \mid 0 < a \leq 1\}$:

Supremum Property (SP): Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.

Remark 3. Using the SP, one can prove that every nonempty subset of $\mathbb{R}$ which is bounded below has an infimum in $\mathbb{R}$.

Some additional conventions

We attach to the set $\mathbb{R}$ two new elements $-\infty$ and $\infty (= +\infty)$ such that $\forall x \in \mathbb{R}$, $-\infty < x$ and $x < \infty$ (of course $-\infty < \infty$). By $\overline{\mathbb{R}}$ or $[-\infty, \infty]$ we denote the set $\mathbb{R} \cup \{-\infty, \infty\}$ called the *extended set of real numbers*.
If $A \subseteq \mathbb{R}$ is not bounded above (below), then we set $\sup A = \infty$ ($\inf A = -\infty$). Moreover, we set $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$ (any real number is both an upper and a lower bound of $\emptyset$).

Interval notation

Let $a, b \in \mathbb{R}$.

If $a \leq b$, $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ – the closed interval (with endpoints a and b);

If $a < b$, $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ – the open interval;

$$\left.\begin{array}{l} [a,b) = \{x \in \mathbb{R} : a \leq x < b\} \\ (a,b] = \{x \in \mathbb{R} : a < x \leq b\} \end{array}\right\} \text{ – the half-open (and half-closed) intervals;}$$

Unbounded intervals: $[a,\infty), (-\infty,a]$ – infinite closed;

$(a,\infty), (-\infty,a)$ – infinite open;

$(-\infty,\infty) = \mathbb{R}$.

Consequences of the Supremum Property

Nested Interval Property (NIP): For $n \in \mathbb{N}$, consider the closed intervals $I_n = [a_n, b_n]$, where $a_n \leq b_n$. If $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, i.e.,

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$$

is a nested sequence of closed intervals, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ (i.e., there exists $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $x \in I_n$).

Archimedean Property (AP): Let $x \in \mathbb{R}$. Then there exists $n \in \mathbb{N}$ such that $n > x$.

Example 1.(iii) (revisited) $A = \left\{ \dfrac{1}{n+1} \mid n \in \mathbb{N} \right\}$:

Consequence of the Archimedean Property

Density Property of $\mathbb{Q}$ in $\mathbb{R}$: Let $x, y \in \mathbb{R}$ with $x < y$. Then there exists $q \in \mathbb{Q}$ such that $x < q < y$.

Definition 4. A subset V of $\mathbb{R}$ is said to be

- a *neighborhood of* $x \in \mathbb{R}$ if there exists a real number $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq V$.
- a *neighborhood of* ∞ if there exists $a \in \mathbb{R}$ such that $(a, \infty) \subseteq V$.
- a *neighborhood of* $-\infty$ if there exists $a \in \mathbb{R}$ such that $(-\infty, a) \subseteq V$.

For $x \in \overline{\mathbb{R}}$, we denote by $\mathcal{V}(x)$ the family of all neighborhoods of x.

Example 3. (i) $(0, 1) \in \mathcal{V}(x)$ for all $x \in (0, 1)$.

(ii) $[0, 1) \notin \mathcal{V}(0)$.

(iii) $[1/2, 3) \cup \{7\} \in \mathcal{V}(x)$ for all $x \in (1/2, 3)$.

Proposition 1. *Let $x \in \overline{\mathbb{R}}$. Then:*

(i) if $x \in \mathbb{R}$ and $V \in \mathcal{V}(x)$, then $x \in V$;

(ii) if $V \in \mathcal{V}(x)$ and $U \subseteq \mathbb{R}$ such that $V \subseteq U$, then $U \in \mathcal{V}(x)$;

(iii) if $U, V \in \mathcal{V}(x)$, then $U \cap V \in \mathcal{V}(x)$.