

# Geometry

## Problem booklet

Assoc. Prof. Cornel Pintea

E-mail: cpintea math.ubbcluj.ro

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**Module leader:** Assoc. Prof. Cornel Pintea

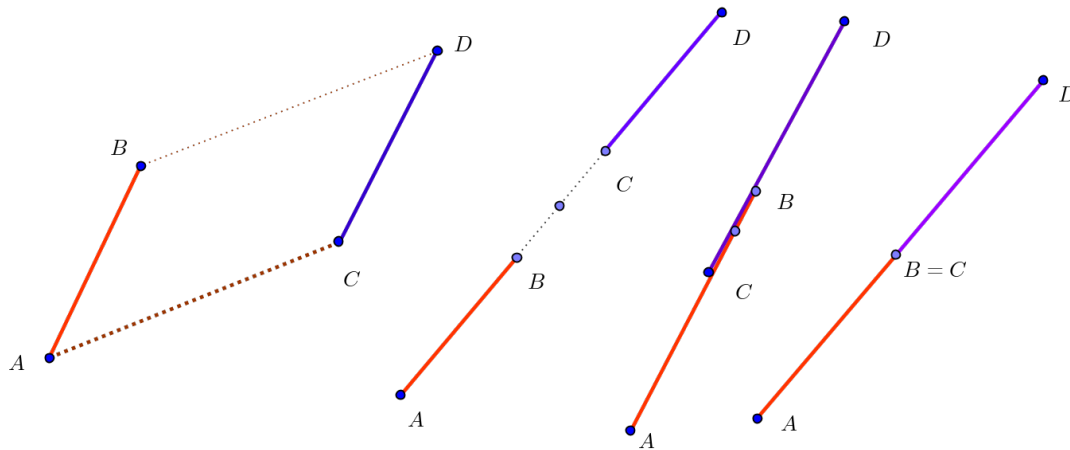
Department of Mathematics,  
 “Babeş-Bolyai” University  
 400084 M. Kogălniceanu 1,  
 Cluj-Napoca, Romania

# 1 Week 1: Vector algebra

## 1.1 Free vectors

**Vectors** Let  $\mathcal{P}$  be the three dimensional physical space in which we can talk about points, lines, planes and various relations among them. If  $(A, B) \in \mathcal{P} \times \mathcal{P}$  is an ordered pair, then  $A$  is called the *original point* or the *origin* and  $B$  is called the *terminal point* or the *extremity* of  $(A, B)$ .

**Definition 1.1.** The ordered pairs  $(A, B)$ ,  $(C, D)$  are said to be equipollent, written  $(A, B) \sim (C, D)$ , if the segments  $[AD]$  and  $[BC]$  have the same midpoint.



Pairs of equipollent points  $(A, B) \sim (C, D)$

**Remark 1.1.** If the points  $A, B, C, D \in \mathcal{P}$  are not collinear, then  $(A, B) \sim (C, D)$  if and only if  $ABDC$  is a parallelogram. In fact the length of the segments  $[AB]$  and  $[CD]$  is the same whenever  $(A, B) \sim (C, D)$ .

**Proposition 1.1.** If  $(A, B)$  is an ordered pair and  $O \in \mathcal{P}$  is a given point, then there exists a unique point  $X$  such that  $(A, B) \sim (O, X)$ .

**Proposition 1.2.** The equipollence relation is an equivalence relation on  $\mathcal{P} \times \mathcal{P}$ .

**Definition 1.2.** The equivalence classes with respect to the equipollence relation are called (*free*) *vectors*.

Denote by  $\overrightarrow{AB}$  the equivalence class of the ordered pair  $(A, B)$ , that is  $\overrightarrow{AB} = \{(X, Y) \in \mathcal{P} \times \mathcal{P} \mid (X, Y) \sim (A, B)\}$  and let  $\mathcal{V} = \mathcal{P} \times \mathcal{P} / \sim = \{\overrightarrow{AB} \mid (A, B) \in \mathcal{P} \times \mathcal{P}\}$  be the set of (*free*) vectors. The *length* or the *magnitude* of the vector  $\overrightarrow{AB}$ , denoted by  $\|\overrightarrow{AB}\|$  or by  $|\overrightarrow{AB}|$ , is the length of the segment  $[AB]$ .

**Remark 1.2.** If two ordered pairs  $(A, B)$  and  $(C, D)$  are equipollent, i.e. the vectors  $\overrightarrow{AB}$  and  $\overrightarrow{CD}$  are equal, then they have the same length, the same direction and the same sense. In fact a vector is determined by these three items.

**Proposition 1.3.** 1.  $\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \overrightarrow{AC} = \overrightarrow{BD}$ .

2.  $\forall A, B, O \in \mathcal{P}, \exists ! X \in \mathcal{P}$  such that  $\overrightarrow{AB} = \overrightarrow{OX}$ .

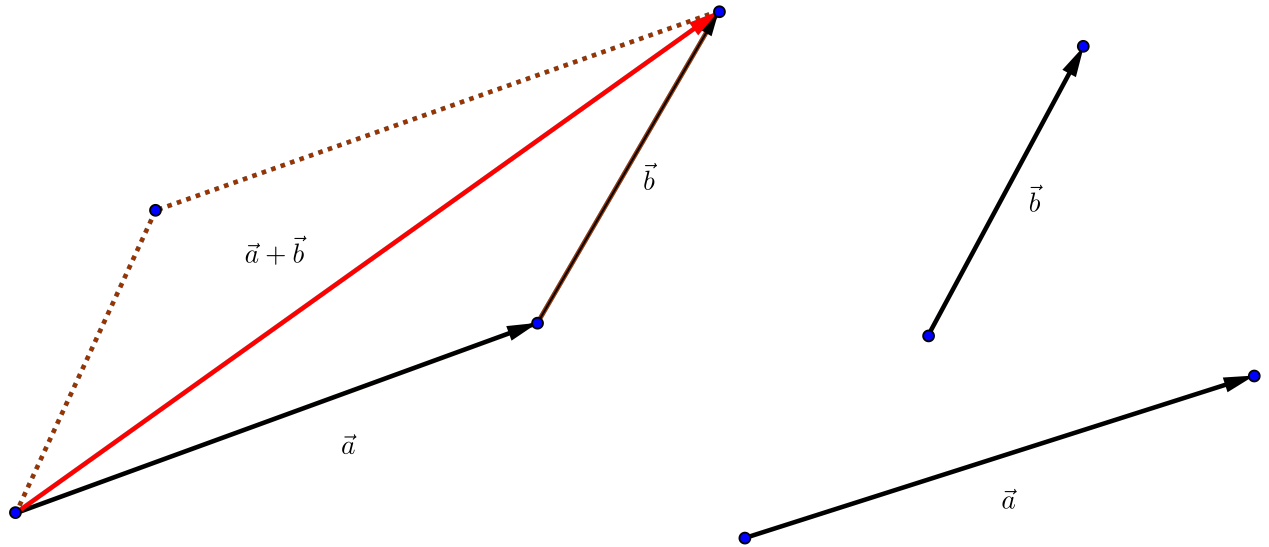
3.  $\overrightarrow{AB} = \overrightarrow{A'B'}, \overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$ .

**Definition 1.3.** If  $O, M \in \mathcal{P}$ , the vector  $\overrightarrow{OM}$  is denoted by  $\vec{r}_M$  and is called the *position vector* of  $M$  with respect to  $O$ .

**Corollary 1.4.** The map  $\varphi_O : \mathcal{P} \rightarrow \mathcal{V}, \varphi_O(M) = \vec{r}_M$  is one-to-one and onto, i.e. bijective.

### 1.1.1 Operations with vectors

• **The addition of vectors** Let  $\vec{a}, \vec{b} \in \mathcal{V}$  and  $O \in \mathcal{P}$  be such that  $\vec{a} = \overrightarrow{OA}, \vec{b} = \overrightarrow{AB}$ . The vector  $\overrightarrow{OB}$  is called the *sum* of the vectors  $\vec{a}$  and  $\vec{b}$  and is written  $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$ .



Let  $O'$  be another point and  $A', B' \in \mathcal{P}$  be such that  $\overrightarrow{O'A'} = \vec{a}, \overrightarrow{A'B'} = \vec{b}$ . Since  $\overrightarrow{OA} = \overrightarrow{O'A'}$  and  $\overrightarrow{AB} = \overrightarrow{A'B'}$  it follows, according to Proposition 1.3(3), that  $\overrightarrow{OB} = \overrightarrow{O'B'}$ . Therefore the vector  $\vec{a} + \vec{b}$  is independent on the choice of the point  $O$ .

**Proposition 1.5.** The set  $\mathcal{V}$  endowed to the binary operation  $\mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}, (\vec{a}, \vec{b}) \mapsto \vec{a} + \vec{b}$ , is an abelian group whose zero element is the vector  $\overrightarrow{AA} = \overrightarrow{BB} = \vec{0}$  and the opposite of  $\vec{a}$ , denoted by  $-\vec{a}$ , is the vector  $\overrightarrow{BA}$ .

In particular the addition operation is associative and the vector

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

is usually denoted by  $\vec{a} + \vec{b} + \vec{c}$ . Moreover the expression

$$((\cdots (\vec{a}_1 + \vec{a}_2) + \vec{a}_3 + \cdots + \vec{a}_n) \cdots), \quad (1.1)$$

is independent of the distribution of paranthesis and it is usually denoted by

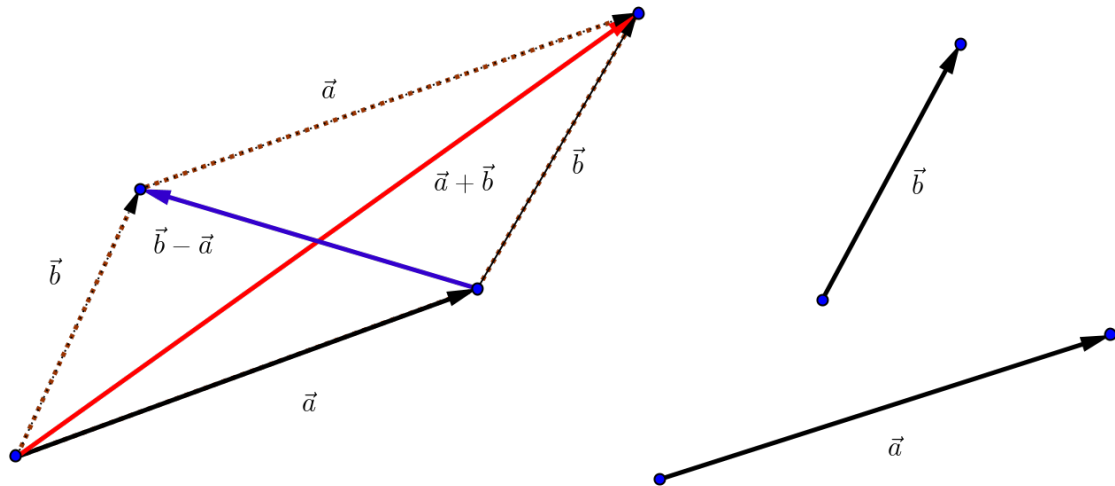
$$\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n.$$

**Example 1.1.** If  $A_1, A_2, A_3, \dots, A_n \in \mathcal{P}$  are some given points, then

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \dots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{A_1A_n}.$$

This shows that  $\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \dots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1} = \vec{0}$ , namely the sum of vectors constructed on the edges of a closed broken line is zero.

**Corollary 1.6.** If  $\vec{a} = \overrightarrow{OA}$ ,  $\vec{b} = \overrightarrow{OB}$  are given vectors, there exists a unique vector  $\vec{x} \in \mathcal{V}$  such that  $\vec{a} + \vec{x} = \vec{b}$ . In fact  $\vec{x} = \vec{b} + (-\vec{a}) = \overrightarrow{AB}$  and is denoted by  $\vec{b} - \vec{a}$ .



### • The multiplication of vectors with scalars

Let  $\alpha \in \mathbb{R}$  be a scalar and  $\vec{a} = \overrightarrow{OA} \in \mathcal{V}$  be a vector. We define the vector  $\alpha \cdot \vec{a}$  as follows:  $\alpha \cdot \vec{a} = \vec{0}$  if  $\alpha = 0$  or  $\vec{a} = \vec{0}$ ; if  $\vec{a} \neq \vec{0}$  and  $\alpha > 0$ , there exists a unique point on the half line  $]OA$  such that  $||OB|| = \alpha \cdot ||OA||$  and define  $\alpha \cdot \vec{a} = \overrightarrow{OB}$ ; if  $\alpha < 0$  we define  $\alpha \cdot \vec{a} = -(|\alpha| \cdot \vec{a})$ . The external binary operation

$$\mathbb{R} \times \mathcal{V} \rightarrow \mathcal{V}, (\alpha, \vec{a}) \mapsto \alpha \cdot \vec{a}$$

is called the *multiplication of vectors with scalars*.

**Proposition 1.7.** *The following properties hold:*

$$(v1) \quad (\alpha + \beta) \cdot \vec{a} = \alpha \cdot \vec{a} + \beta \cdot \vec{a}, \quad \forall \alpha, \beta \in \mathbb{R}, \vec{a} \in \mathcal{V}.$$

$$(v2) \quad \alpha \cdot (\vec{a} + \vec{b}) = \alpha \cdot \vec{a} + \alpha \cdot \vec{b}, \quad \forall \alpha \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}.$$

$$(v3) \quad \alpha \cdot (\beta \cdot \vec{a}) = (\alpha\beta) \cdot \vec{a}, \quad \forall \alpha, \beta \in \mathbb{R}.$$

$$(v4) \quad 1 \cdot \vec{a} = \vec{a}, \quad \forall \vec{a} \in \mathcal{V}.$$

**Application 1.1.** Consider two parallelograms,  $A_1A_2A_3A_4, B_1B_2B_3B_4$  in  $\mathcal{P}$ , and  $M_1, M_2, M_3, M_4$  the midpoints of the segments  $[A_1B_1], [A_2B_2], [A_3B_3], [A_4B_4]$  respectively. Then:

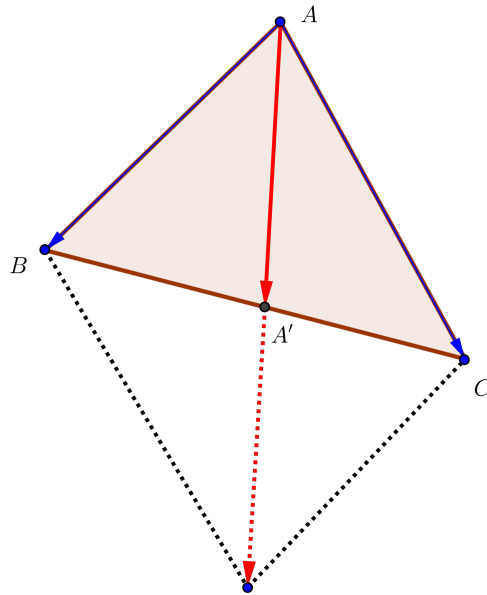
- $2 \vec{M_1M_2} = \vec{A_1A_2} + \vec{B_1B_2}$  and  $2 \vec{M_3M_4} = \vec{A_3A_4} + \vec{B_3B_4}$ .
- $M_1, M_2, M_3, M_4$  are the vertices of a parallelogram.

### 1.1.2 The vector structure on the set of vectors

**Theorem 1.8.** *The set of (free) vectors endowed with the addition binary operation of vectors and the external binary operation of multiplication of vectors with scalars is a real vector space.*

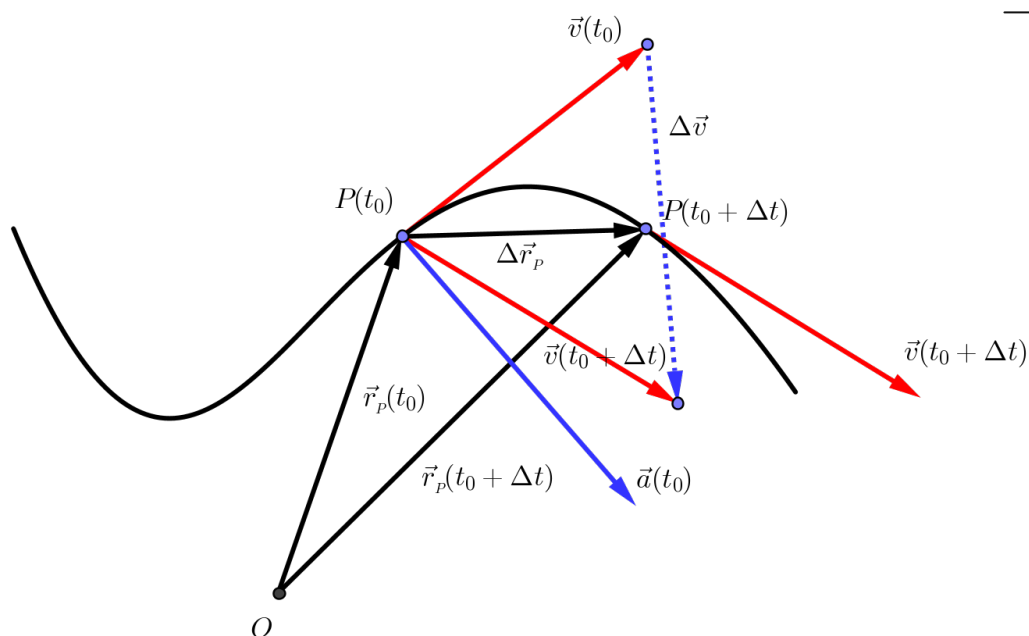
**Example 1.2.** If  $A'$  is the midpoint of the edge  $[BC]$  of the triangle  $ABC$ , then

$$\vec{AA'} = \frac{1}{2} (\vec{AB} + \vec{AC}).$$



A few vector quantities:

1. The force, usually denoted by  $\vec{F}$ .
2. The velocity  $\frac{d\vec{r}_P}{dt}$  of a moving particle  $P$ , is usually denoted by  $\vec{v}_P$  or simply by  $\vec{v}$ .
3. The acceleration  $\frac{d\vec{v}_P}{dt}$  of a moving particle  $P$ , is usually denoted by  $\vec{a}_P$  or simply by  $\vec{a}$ .



• **Newton's law of gravitation**, statement that any particle of matter in the universe attracts any other with a force varying directly as the product of the masses and inversely as the square of the distance between them. In symbols, the magnitude of the attractive force  $F$  is equal to  $G$  (the gravitational constant, a number the size of which depends on the system of units used and which is a universal constant) multiplied by the product of the masses ( $m_1$  and  $m_2$ ) and divided by the square of the distance  $R$ :  $F = G(m_1 m_2) / R^2$ . (Encyclopædia

Britannica)

• **Newton's second law** is a quantitative description of the changes that a force can produce on the motion of a body. It states that the time rate of change of the momentum of a body is equal in both magnitude and direction to the force imposed on it. The momentum of a body is equal to the product of its mass and its velocity. Momentum, like velocity, is a vector quantity, having both magnitude and direction. A force applied to a body can change the magnitude of the momentum, or its direction, or both. Newton's second law is one of the most important in all of physics. For a body whose mass  $m$  is constant, it can be written in the form  $F = ma$ , where  $F$  (force) and  $a$  (acceleration) are both vector quantities. If a body has a net force acting on it, it is accelerated in accordance with the equation. Conversely, if a body is not accelerated, there is no net force acting on it. (Encyclopdia Britannica)

## 1.2 Problems

1. Consider a tetrahedron  $ABCD$ . Find the the following sums of vectors:

(a)  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ .

(b)  $\overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{DC}$ .

(c)  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{CD}$ .

*Solution.*



2. ([4, Problem 3, p. 1]) Let  $OABCDE$  be a regular hexagon in which  $\overrightarrow{OA} = \vec{a}$  and  $\overrightarrow{OE} = \vec{b}$ . Express the vectors  $\overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD}$  in terms of the vectors  $\vec{a}$  and  $\vec{b}$ . Show that  $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OE} = 3 \overrightarrow{OC}$ .

*Solution.*

3. Consider a pyramid with the vertex at  $S$  and the basis a parallelogram  $ABCD$  whose diagonals are concurrent at  $O$ . Show the equality  $\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4 \vec{SO}$ .

*Solution.*

4. Let  $E$  and  $F$  be the midpoints of the diagonals of a quadrilateral  $ABCD$ . Show that

$$\overrightarrow{EF} = \frac{1}{2} \left( \overrightarrow{AB} + \overrightarrow{CD} \right) = \frac{1}{2} \left( \overrightarrow{AD} + \overrightarrow{CB} \right).$$

*Solution.*

5. In a triangle  $ABC$  we consider the height  $AD$  from the vertex  $A$  ( $D \in BC$ ). Find the decomposition of the vector  $AD$  in terms of the vectors  $\vec{c} = \vec{AB}$  and  $\vec{b} = \vec{AC}$ .

*Solution.*

6. ([4, Problem 12, p. 3]) Let  $M, N$  be the midpoints of two opposite edges of a given quadrilateral  $ABCD$  and  $P$  be the midpoint of  $[MN]$ . Show that

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0$$

*Solution.*

7. ([4, Problem 12, p. 7]) Consider two perpendicular chords  $AB$  and  $CD$  of a given circle and  $\{M\} = AB \cap CD$ . Show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2 \overrightarrow{OM}.$$

*Solution.*

8. ([4, Problem 13, p. 3]) If  $G$  is the centroid of a triangle  $ABC$  and  $O$  is a given point, show that

$$\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}.$$

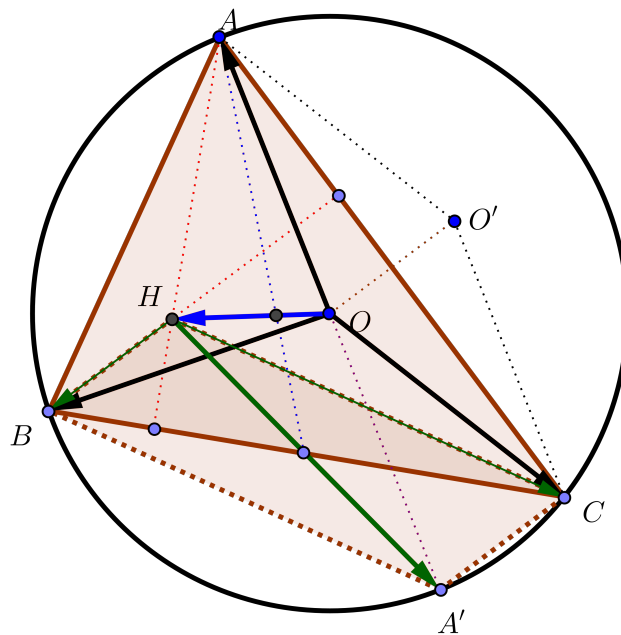
*Solution.*

9. ([4, Problem 14, p. 4]) Consider the triangle  $ABC$  alongside its orthocenter  $H$ , its circumcenter  $O$  and the diametrically opposed point  $A'$  of  $A$  on the latter circle. Show that:

(a)  $\vec{OA} + \vec{OB} + \vec{OC} = \vec{OH}$ .

(b)  $\vec{HB} + \vec{HC} = \vec{HA'}$ .

(c)  $\vec{HA} + \vec{HB} + \vec{HC} = 2\vec{HO}$ .



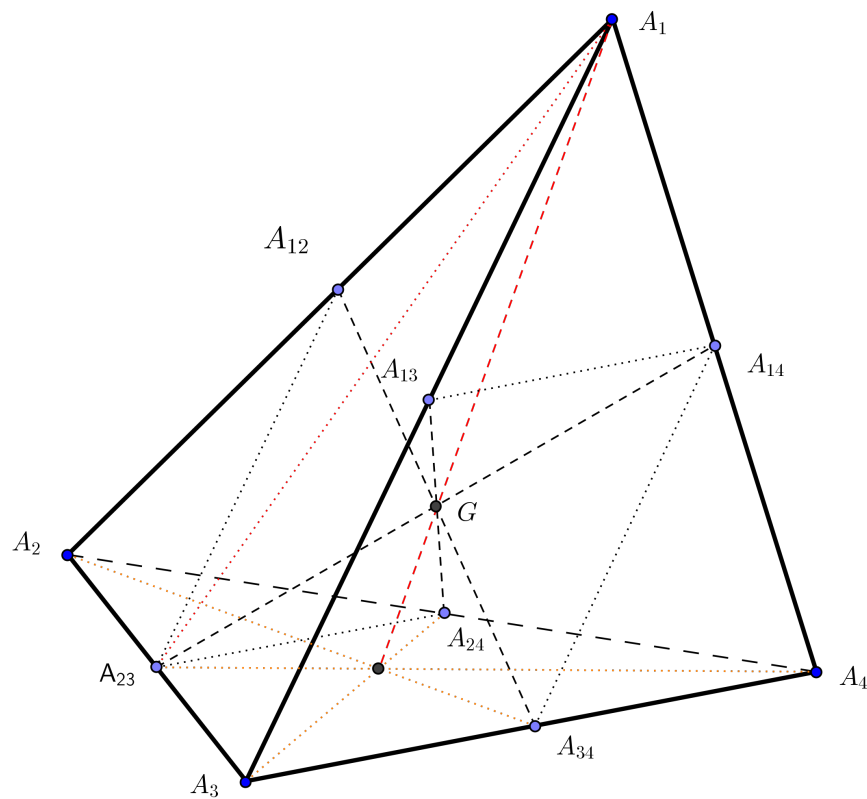
*Solution.*



10. ([4, Problem 15, p. 4]) Consider the triangle  $ABC$  alongside its centroid  $G$ , its orthocenter  $H$  and its circumcenter  $O$ . Show that  $O, G, H$  are collinear and  $3 \overrightarrow{HG} = 2 \overrightarrow{HO}$ .

*Solution.*

11. ([4, Problem 27, p. 13]) Consider a tetrahedron  $A_1A_2A_3A_4$  and the midpoints  $A_{ij}$  of the edges  $A_iA_j, i \neq j$ . Show that:
- The lines  $A_{12}A_{34}, A_{13}A_{24}$  and  $A_{14}A_{23}$  are concurrent in a point  $G$ .
  - The medians of the tetrahedron (the lines passing through the vertices and the centroids of the opposite faces) are also concurrent at  $G$ .
  - Determine the ratio in which the point  $G$  divides each median.
  - Show that  $\vec{GA_1} + \vec{GA_2} + \vec{GA_3} + \vec{GA_4} = \vec{0}$ .
  - If  $M$  is an arbitrary point, show that  $\vec{MA_1} + \vec{MA_2} + \vec{MA_3} + \vec{MA_4} = 4 \vec{MG}$ .



*Solution.*



12. In a triangle  $ABC$  consider the points  $M, L$  on the side  $AB$  and  $N, T$  on the side  $AC$  such that  $3 \overrightarrow{AL} = 2 \overrightarrow{AM} = \overrightarrow{AB}$  and  $3 \overrightarrow{AT} = 2 \overrightarrow{AN} = \overrightarrow{AC}$ . Show that  $\overrightarrow{AB} + \overrightarrow{AC} = 5 \overrightarrow{AS}$ , where  $\{S\} = MT \cap LN$ .

*Solution.*

13. Consider two triangles  $A_1B_1C_1$  and  $A_2B_2C_2$ , not necessarily in the same plane, alongside their centroids  $G_1, G_2$ . Show that  $\vec{A_1A_2} + \vec{B_1B_2} + \vec{C_1C_2} = 3 \vec{G_1G_2}$ .

*Solution.*

## 2 Week 2: Straight lines and planes

### 2.1 Linear dependence and linear independence of vectors

**Definition 2.1.** 1. The vectors  $\vec{OA}, \vec{OB}$  are said to be *collinear* if the points  $O, A, B$  are collinear. Otherwise the vectors  $\vec{OA}, \vec{OB}$  are said to be *noncollinear*.

2. The vectors  $\vec{OA}, \vec{OB}, \vec{OC}$  are said to be *coplanar* if the points  $O, A, B, C$  are coplanar. Otherwise the vectors  $\vec{OA}, \vec{OB}, \vec{OC}$  are *noncoplanar*.

**Remark 2.1.** 1. The vectors  $\vec{OA}, \vec{OB}$  are linearly (in)dependent if and only if they are (non)collinear.

2. The vectors  $\vec{OA}, \vec{OB}, \vec{OC}$  are linearly (in)dependent if and only if they are (non)coplanar.

**Proposition 2.1.** The vectors  $\vec{OA}, \vec{OB}, \vec{OC}$  form a basis of  $\mathcal{V}$  if and only if they are noncoplanar.

**Corollary 2.2.** The dimension of the vector space of free vectors  $\mathcal{V}$  is three.

**Proposition 2.3.** Let  $\Delta$  be a straight line and let  $A \in \Delta$  be a given point. The set

$$\vec{\Delta} = \{ \vec{AM} \mid M \in \Delta \}$$

is an one dimensional subspace of  $\mathcal{V}$ . It is independent on the choice of  $A \in \Delta$  and is called the director subspace of  $\Delta$  or the direction of  $\Delta$ .

**Remark 2.2.** The straight lines  $\Delta, \Delta'$  are parallel if and only if  $\vec{\Delta} = \vec{\Delta}'$

**Definition 2.2.** We call director vector of the straight line  $\Delta$  every nonzero vector  $\vec{d} \in \vec{\Delta}$ .

If  $\vec{d} \in \mathcal{V}$  is a nonzero vector and  $A \in \mathcal{P}$  is a given point, then there exists a unique straight line which passes through  $A$  and has the direction  $\langle \vec{d} \rangle$ . This straight line is

$$\Delta = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \vec{d} \rangle\}.$$

$\Delta$  is called the straight line which passes through  $O$  and is parallel to the vector  $\vec{d}$ .

**Proposition 2.4.** *Let  $\pi$  be a plane and let  $A \in \pi$  be a given point. The set  $\vec{\pi} = \{\overrightarrow{AM} \in \mathcal{V} \mid M \in \pi\}$  is a two dimensional subspace of  $\mathcal{V}$ . It is independent on the position of  $A$  inside  $\pi$  and is called the director subspace, the director plane or the direction of the plane  $\pi$ .*

**Remark 2.3.** • *The planes  $\pi, \pi'$  are parallel if and only if  $\vec{\pi} = \vec{\pi}'$ .*

• If  $\vec{d}_1, \vec{d}_2$  are two linearly independent vectors and  $A \in \mathcal{P}$  is a fixed point, then there exists a unique plane through  $A$  whose direction is  $\langle \vec{d}_1, \vec{d}_2 \rangle$ . This plane is

$$\pi = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \vec{d}_1, \vec{d}_2 \rangle\}.$$

We say that  $\pi$  is the plane which passes through the point  $A$  and is parallel to the vectors  $\vec{d}_1$  and  $\vec{d}_2$ .

**Remark 2.4.** Let  $\Delta \subset \mathcal{P}$  be a straight line and  $\pi \subset \mathcal{P}$  be given plane.

1. If  $A \in \Delta$  is a given point, then  $\varphi_O(\Delta) = \vec{r}_A + \vec{\Delta}$ .

2. If  $B \in \Delta$  is a given point, then  $\varphi_O(\pi) = \vec{r}_B + \vec{\pi}$ .

Generally speaking, a subset  $X$  of a vector space is called *linear variety* if either  $X = \emptyset$  or there exists  $a \in V$  and a vector subspace  $U$  of  $V$ , such that  $X = a + U$ .

$$\dim(X) = \begin{cases} -1 & \text{dacă } X = \emptyset \\ \dim(U) & \text{dacă } X = a + U, \end{cases}$$

**Proposition 2.5.** The bijection  $\varphi_O$  transforms the straight lines and the planes of the affine space  $\mathcal{P}$  into the one and two dimensional linear varieties of the vector space  $\mathcal{V}$  respectively.

## 2.2 The vector equations of the straight lines and planes

**Proposition 2.6.** Let  $\Delta$  be a straight line, let  $\pi$  be a plane,  $\{\vec{d}\}$  be a basis of  $\vec{\Delta}$  and let  $[\vec{d}_1, \vec{d}_2]$  be an ordered basis of  $\vec{\pi}$ .

1. The points  $M \in \Delta$  are characterized by the vector equation of  $\Delta$

$$\vec{r}_M = \vec{r}_A + \lambda \vec{d}, \lambda \in \mathbb{R} \quad (2.1)$$

where  $A \in \Delta$  is a given point.

2. The points  $M \in \pi$  are characterized by the vector equation of  $\pi$

$$\vec{r}_M = \vec{r}_A + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2, \lambda_1, \lambda_2 \in \mathbb{R}, \quad (2.2)$$

where  $A \in \pi$  is a given point.

PROOF.

□



**Corollary 2.7.** *If  $A, B \in \mathcal{P}$  are different points, then the vector equation of the line  $AB$  is*

$$\vec{r}_M = (1 - \lambda) \vec{r}_A + \lambda \vec{r}_B, \lambda \in \mathbb{R}. \quad (2.3)$$

PROOF.

□

**Corollary 2.8.** *If  $A, B, C \in \mathcal{P}$  are three noncolinear points, then the vector equation of the plane  $(ABC)$  is*

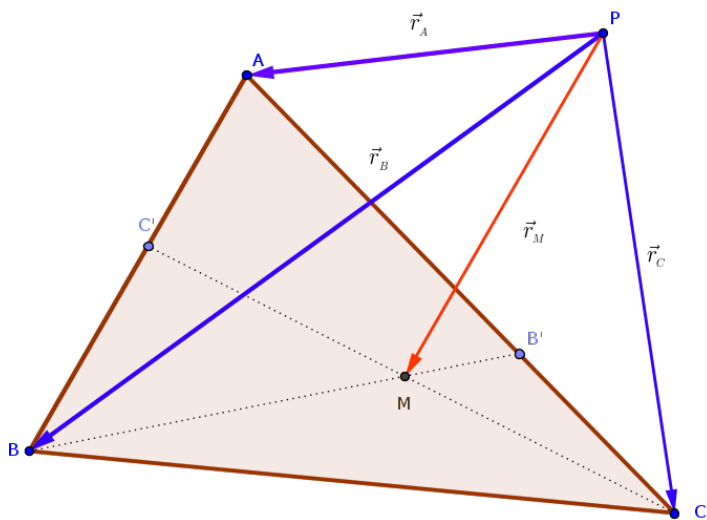
$$\vec{r}_M = (1 - \lambda_1 - \lambda_2) \vec{r}_A + \lambda_1 \vec{r}_B + \lambda_2 \vec{r}_C, \lambda_1, \lambda_2 \in \mathbb{R}. \quad (2.4)$$

PROOF.

□

**Example 2.1.** Consider the points  $C'$  and  $B'$  on the sides  $AB$  and  $AC$  of the triangle  $ABC$  such that  $\vec{AC'} = \lambda \vec{BC'}$ ,  $\vec{AB'} = \mu \vec{CB'}$ . The lines  $BB'$  and  $CC'$  meet at  $M$ . If  $P \in \mathcal{P}$  is a given point and  $\vec{r}_A = \vec{PA}$ ,  $\vec{r}_B = \vec{PB}$ ,  $\vec{r}_C = \vec{PC}$  are the position vectors, with respect to  $P$ , of the vertices  $A, B, C$  respectively, show that

$$\vec{r}_M = \frac{\vec{r}_A - \lambda \vec{r}_B - \mu \vec{r}_C}{1 - \lambda - \mu}. \quad (2.5)$$



SOLUTION.

□

## 2.3 Problems

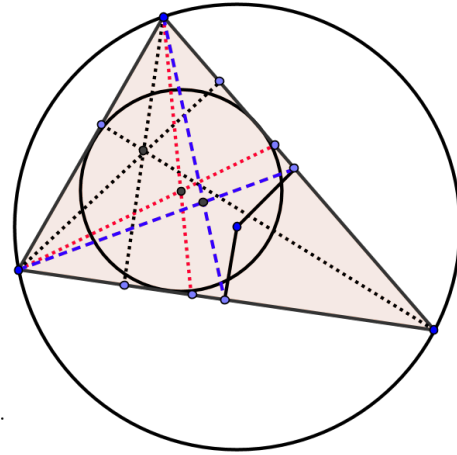
1. ([4, Problem 17, p. 5]) Consider the triangle  $ABC$ , its centroid  $G$ , its orthocenter  $H$ , its incenter  $I$  and its circumcenter  $O$ . If  $P \in \mathcal{P}$  is a given point and  $\vec{r}_A = \overrightarrow{PA}$ ,  $\vec{r}_B = \overrightarrow{PB}$ ,  $\vec{r}_C = \overrightarrow{PC}$  are the position vectors with respect to  $P$  of the vertices  $A, B, C$  respectively, show that:

$$\vec{r}_G := \overrightarrow{PG} = \frac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}.$$

$$\vec{r}_I := \overrightarrow{PI} = \frac{a \vec{r}_A + b \vec{r}_B + c \vec{r}_C}{a + b + c}.$$

$$\vec{r}_H := \overrightarrow{PH} = \frac{(\tan A) \vec{r}_A + (\tan B) \vec{r}_B + (\tan C) \vec{r}_C}{\tan A + \tan B + \tan C}.$$

$$\vec{r}_O := \overrightarrow{PO} = \frac{(\sin 2A) \vec{r}_A + (\sin 2B) \vec{r}_B + (\sin 2C) \vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}.$$



*Solution.*

2. Consider the angle  $BOB'$  and the points  $A \in [OB]$ ,  $A' \in [OB']$ . Show that

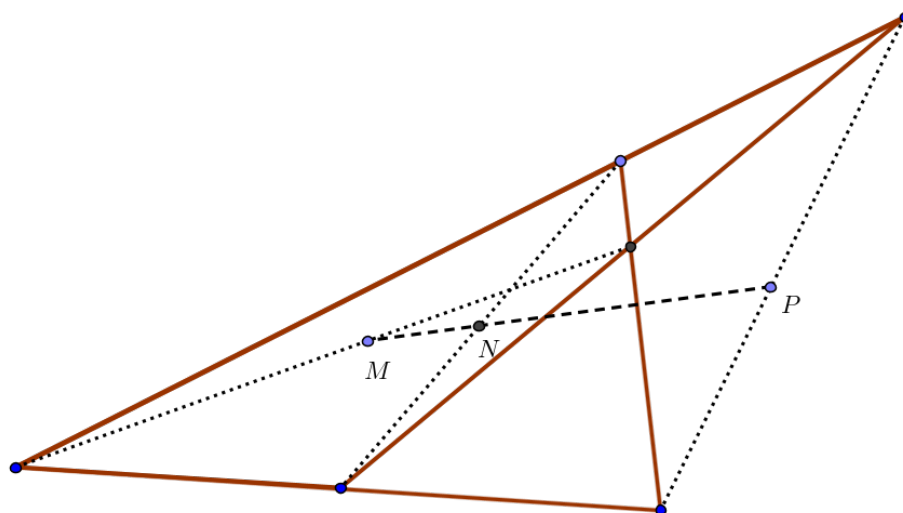
$$\overrightarrow{OM} = m \frac{1-n}{1-mn} \overrightarrow{OA} + n \frac{1-m}{1-mn} \overrightarrow{OA'}$$

$$\overrightarrow{ON} = m \frac{n-1}{n-m} \overrightarrow{OA} + n \frac{m-1}{m-n} \overrightarrow{OA'}.$$

where  $\{M\} = AB' \cap A'B$ ,  $\{N\} = AA' \cap BB'$ ,  $\vec{u} = \overrightarrow{OA}$ ,  $\vec{v} = \overrightarrow{OA'}$ ,  $\overrightarrow{OB} = m \overrightarrow{OA}$  and  $\overrightarrow{OB'} = n \overrightarrow{OA'}$ .

*Solution.*

3. Show that the midpoints of the diagonals of a complete quadrilateral are collinear (Newton's theorem).



*Solution.*

4. Let  $d, d'$  be concurrent straight lines and  $A, B, C \in d, A', B', C' \in d'$ . If the following relations  $AB' \parallel A'B, AC' \parallel A'C, BC' \parallel B'C$  hold, show that the points  $\{M\} := AB' \cap A'B, \{N\} := AC' \cap A'C, \{P\} := BC' \cap B'C$  are collinear (Pappus' theorem).

SOLUTION.

5. Let  $d, d'$  be two straight lines and  $A, B, C \in d, A', B', C' \in d'$  three points on each line such that  $AB' \parallel BA', AC' \parallel CA'$ . Show that  $BC' \parallel CB'$  (the affine Pappus' theorem).

SOLUTION.



6. Let us consider two triangles  $ABC$  and  $A'B'C'$  such that the lines  $AA'$ ,  $BB'$ ,  $CC'$  are concurrent at a point  $O$  and  $AB \parallel A'B'$ ,  $BC \parallel B'C'$  and  $CA \parallel C'A'$ . Show that the points  $\{M\} = AB \cap A'B'$ ,  $\{N\} = BC \cap B'C'$  and  $\{P\} = CA \cap C'A'$  are collinear (Desargues).

SOLUTION.

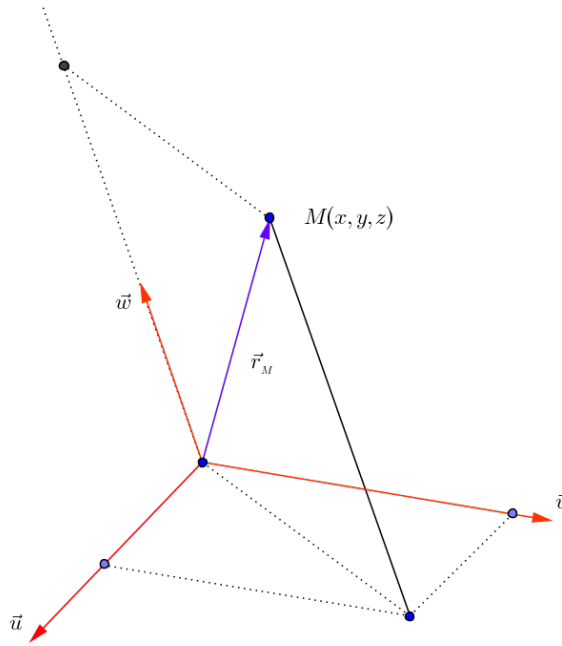
### 3 Week 3: Cartesian equations of lines and planes

#### 3.1 Cartesian and affine reference systems

If  $b = [\vec{u}, \vec{v}, \vec{w}]$  is an ordered basis of  $\mathcal{V}$  and  $\vec{x} \in \mathcal{V}$ , recall that the column vector of the coordinates of  $\vec{x}$  with respect to  $b$  is denoted by  $[\vec{x}]_b$ . In other words

$$[\vec{x}]_b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

whenever  $\vec{x} = x_1 \vec{u} + x_2 \vec{v} + x_3 \vec{w}$ . To emphasize the coordinates of  $\vec{x}$  with respect to  $b$ , we shall use the notation  $\vec{x}(x_1, x_2, x_3)$ .



**Definition 3.1.** A cartesian reference system  $R = (O, \vec{u}, \vec{v}, \vec{w})$  of the space  $\mathcal{P}$ , consists in a point  $O \in \mathcal{P}$  called the *origin* of the reference system and an ordered basis  $b = [\vec{u}, \vec{v}, \vec{w}]$  of the vector space  $\mathcal{V}$ .

Denote by  $E_1, E_2, E_3$  the points for which  $\vec{u} = \vec{OE}_1$ ,  $\vec{v} = \vec{OE}_2$ ,  $\vec{w} = \vec{OE}_3$ .

**Definition 3.2.** The system of points  $(O, E_1, E_2, E_3)$  is called *the affine reference system associated to the cartesian reference system*  $R = (O, \vec{u}, \vec{v}, \vec{w})$ .

The straight lines  $OE_i$ ,  $i \in \{1, 2, 3\}$ , oriented from  $O$  to  $E_i$  are called *the coordinate axes*. The coordinates  $x, y, z$  of the position vector  $\vec{r}_M = \vec{OM}$  with respect to the basis  $[\vec{u}, \vec{v}, \vec{w}]$  are called the coordinates of the point  $M$  with respect to the cartesian system  $R$  written  $M(x, y, z)$ . Also, for the column matrix of coordinates of the vector  $\vec{r}_M$  we are going to use the notation  $[M]_R$ . In other words, if  $\vec{r}_M = x \vec{u} + y \vec{v} + z \vec{w}$ , then

$$[M]_R = [\vec{OM}]_b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

**Remark 3.1.** If  $A(x_A, y_A, z_A)$ ,  $B(x_B, y_B, z_B)$  are two points, then

$$\begin{aligned}\vec{AB} &= \vec{OB} - \vec{OA} \\ &= x_B \vec{u} + y_B \vec{v} + z_B \vec{w} - (x_A \vec{u} + y_A \vec{v} + z_A \vec{w}) \\ &= (x_B - x_A) \vec{u} + (y_B - y_A) \vec{v} + (z_B - z_A) \vec{w},\end{aligned}$$

i.e. the coordinates of the vector  $\vec{AB}$  are being obtained by performing the differences of the coordinates of the points  $A$  and  $B$ .

**Remark 3.2.** If  $R = (O, b)$  is a cartesian reference system, where  $b = [\vec{u}, \vec{v}, \vec{w}]$  is an ordered basis of  $\mathcal{V}$ , recall that  $\varphi_O : \mathcal{P} \rightarrow \mathcal{V}$ ,  $\varphi_O(M) = \vec{OM}$  is bijective and  $\psi_b : \mathbb{R}^3 \rightarrow \mathcal{V}$ ,  $\psi_b(x, y, z) = x \vec{u} + y \vec{v} + z \vec{w}$  is a linear isomorphism. The bijection  $\varphi_O$  defines a unique vector structure over  $\mathcal{P}$  such that  $\varphi_O$  becomes an isomorphism. This vector structure depends on the choice of  $O \in \mathcal{P}$ . Therefore a point  $M \in \mathcal{P}$  could be identified either with its position vector  $\vec{r}_M = \varphi_O(M)$ , or, with the triplet  $(\psi_b^{-1} \circ \varphi_O)(M) \in \mathbb{R}^3$  of its coordinates with respect to the reference system  $R$ . If  $f : X \rightarrow \mathbb{R}^3$  is a given application, then  $\varphi_O^{-1} \circ \psi_b \circ f : X \rightarrow \mathcal{P}$  will be denoted by  $M_f$ . A similar discussion can be done for a cartesian reference system  $R' = (O', b')$  of a plane  $\pi$ , where  $b' = [\vec{u}', \vec{v}']$  is an ordered basis of  $\vec{\pi}$ .

**Example 3.1 (Homework).** Consider the tetrahedron  $ABCD$ , where  $A(1, -1, 1)$ ,  $B(-1, 1, -1)$ ,  $C(2, 1, -1)$  and  $D(1, 1, 2)$ . Find the coordinates of:

1. the centroids  $G_A, G_B, G_C, G_D$  of the triangles  $BCD, ACD, ABD$  and  $ABC$ <sup>1</sup> respectively.
2. the midpoints  $M, N, P, Q, R$  and  $S$  of its edges  $[AB], [AC], [AD], [BC], [CD]$  and  $[DB]$  respectively.

SOLUTION.

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<sup>1</sup>The centroids of its faces

### 3.2 The cylindrical coordinate System

In order to have a valid coordinate system in the 3-dimensional case, each point of the space must be associated with a unique triple of real numbers (the coordinates of the point) and each triple of real numbers must determine a unique point, as in the case of the Cartesian system of coordinates.

Let  $P(x, y, z)$  be a point in a Cartesian system of coordinates  $Oxyz$  and  $P'$  be the orthogonal projection of  $P$  on the plane  $xOy$ . One can associate to the point  $P$  the triple  $(r, \theta, z)$ , where  $(r, \theta)$  are the polar coordinates of  $P'$  (see Figure 1). The polar coordinates of  $P'$  can be obtained by specifying the distance  $\rho$  from  $O$  to  $P'$  and the angle  $\theta$  (measured in radians), whose "initial" side is the polar axis, i.e. the  $x$ -axis, and whose "terminal" side is the ray  $OP'$ . The *polar coordinates* of the point  $P$  are  $(\rho, \theta)$  (See also section (3.6.2) of the Appendix). The triple  $(r, \theta, z)$  gives the *cylindrical coordinates* of the point  $P$ . There is the bijection

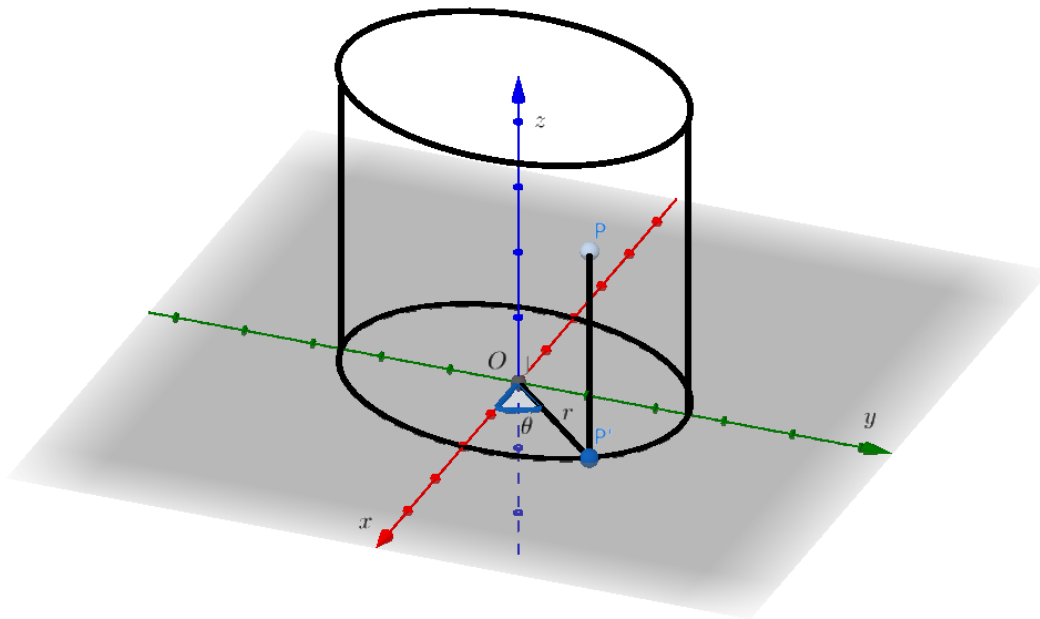


Figure 1: cylindrical coordinates

$$h_1 : \mathcal{P} \setminus \{O\} \rightarrow \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R}, P \rightarrow (r, \theta, z)$$

and one obtains a new coordinate system, named the *cylindrical coordinate system* in  $\mathcal{P}$ . For the conversion formulas between the cylindrical coordinates and the Cartesian coordinates we refer the reader to [1, p. 19]. Note however that once we have the cylindrical coordinates  $(r, \theta, z)$  of a point  $P$ , then its Cartesian coordinates are

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}.$$

### 3.3 The spherical coordinate system

Another way to associate to each point  $P$  in  $\mathcal{P}$  a triple of real numbers is illustrated in Figure 2. If  $P(x, y, z)$  is a point in a rectangular system of coordinates  $Oxyz$  and  $P'$  its or-

thogonal projection on  $Oxy$ , let  $\rho$  be the length of the segment  $[OP]$ ,  $\theta$  be the oriented angle determined by  $[Ox]$  and  $[OP']$  and  $\varphi$  be the oriented angle between  $[Oz]$  and  $[OP]$ . The triple

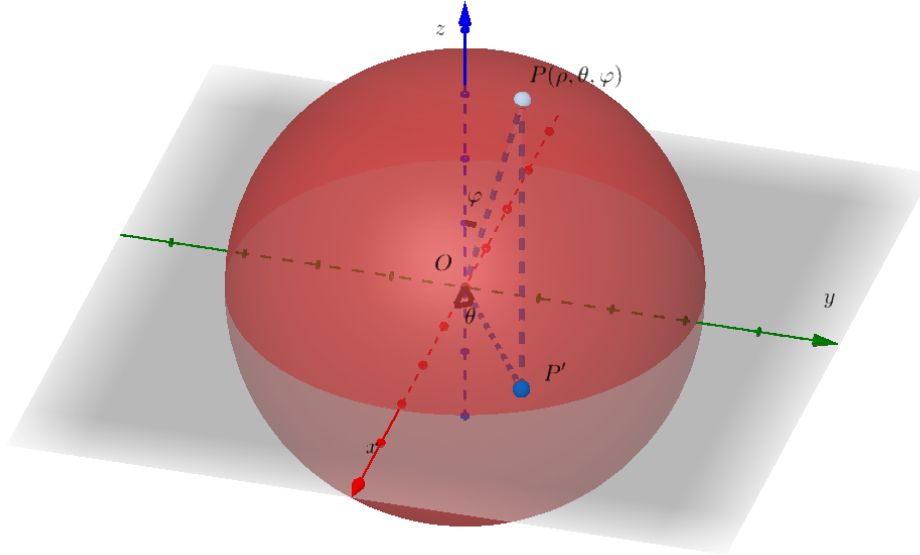


Figure 2: spherical coordinates

$(\rho, \theta, \varphi)$  gives the *spherical coordinates* of the point  $P$ . This way, one obtains the bijection

$$h_2 : \mathcal{P} \setminus \{O\} \rightarrow \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi], P \rightarrow (\rho, \theta, \varphi),$$

which defines a new coordinate system in  $\mathcal{P}$ , called the *spherical coordinate system*. For the conversion formulas between the spherical coordinate system and the Cartesian coordinate system we refer the reader to [1, p. 20]. Note however that once we have the cylindrical coordinates  $(\rho, \theta, \varphi)$  of a point  $P$ , then its Cartesian coordinates are

$$\begin{cases} x = r \cos \theta \sin \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \varphi \end{cases}.$$

### 3.4 The Cartesian equations of the straight lines

Let  $\Delta$  be the straight line passing through the point  $A_0(x_0, y_0, z_0)$  which is parallel to the vector  $\vec{d}(p, q, r)$ . Its vector equation is

$$\vec{r}_M = \vec{r}_{A_0} + \lambda \vec{d}, \lambda \in \mathbb{R}. \quad (3.1)$$

Denoting by  $x, y, z$  the coordinates of the generic point  $M$  of the straight line  $\Delta$ , its vector equation (3.1) is equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases}, \lambda \in \mathbb{R} \quad (3.2)$$

Indeed, the vector equation of  $\Delta$  can be written, in terms of the coordinates of the vectors  $\vec{r}_M$ ,  $\vec{r}_{A_0}$  and  $\vec{d}$ , as follows:

$$\begin{aligned} x \vec{u} + y \vec{v} + z \vec{w} &= x_0 \vec{u} + y_0 \vec{v} + z_0 \vec{w} + \lambda(p \vec{u} + q \vec{v} + r \vec{w}) \\ \iff x \vec{u} + y \vec{v} + z \vec{w} &= (x_0 + p\lambda) \vec{u} + (y_0 + q\lambda) \vec{v} + (z_0 + r\lambda) \vec{w}, \lambda \in \mathbb{R} \end{aligned}$$

which is obviously equivalent to (3.2). The relations (3.2) are called the *parametric equations* of the straight line  $\Delta$  and they are equivalent to the following relations

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r} \quad (3.3)$$

If  $r = 0$ , for instance, the canonical equations of the straight line  $\Delta$  are

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} \wedge z = z_0.$$

If  $A(x_A, y_A, z_A)$ ,  $B(x_B, y_B, z_B)$  are different points of the line  $\Delta$ , then

$$\vec{AB} (x_B - x_A, y_B - y_A, z_B - z_A)$$

is a director vector of  $\Delta$ , its canonical equations having, in this case, the form

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}. \quad (3.4)$$

**Example 3.2.** Consider the tetrahedron  $ABCD$ , where  $A(1, -1, 1)$ ,  $B(-1, 1, -1)$ ,  $C(2, 1, -1)$  and  $D(1, 1, 2)$ , as well as the centroids  $G_A$ ,  $G_B$ ,  $G_C$ ,  $G_D$  of the triangles  $BCD$ ,  $ACD$ ,  $ABD$  and  $ABC$ <sup>2</sup> respectively. Show that the medians  $AG_A$ ,  $BG_B$ ,  $CG_C$  and  $DG_D$  are concurrent and find the coordinates of their intersection point.

**SOLUTION.** One can easily see that the coordinates of the centroids  $G_A$ ,  $G_B$ ,  $G_C$ ,  $G_D$  are  $(2/3, 1, 0)$ ,  $(4/3, 1/3, 2/3)$ ,  $(1/3, 1/3, 2/3)$  and  $(2/3, 1/3, -1/3)$  respectively. The equations of the medians  $AG_A$  and  $BG_B$  are

$$\begin{aligned} (AG_A) \quad \frac{x - 1}{2/3 - 1} &= \frac{y + 1}{1 - (-1)} = \frac{z - 1}{0 - 1} \iff \frac{x - 1}{-1/3} = \frac{y + 1}{2} = \frac{z - 1}{-1} \\ (BG_B) \quad \frac{x + 1}{4/3 + 1} &= \frac{y - 1}{1/3 - 1} = \frac{z + 1}{2/3 + 1} \iff \frac{x + 1}{7/3} = \frac{y - 1}{-2/3} = \frac{z + 1}{5/3}. \end{aligned}$$

Thus, the director space of the median  $AG_A$  is  $\langle (-\frac{1}{3}, 2, -1) \rangle = \langle (-1, 6, -3) \rangle$  and the director space of the median  $BG_B$  is  $\langle (\frac{7}{3}, -\frac{2}{3}, \frac{5}{3}) \rangle = \langle (7, -2, 5) \rangle$ . Consequently, the parametric equations of the medians  $AG_A$  and  $BG_B$  are

$$(AG_A) \begin{cases} x = 1 - t \\ y = -1 + 6t \\ z = 1 - 3t \end{cases}, t \in \mathbb{R} \text{ and } (BG_B) \begin{cases} x = -1 + 7s \\ y = 1 - 2s \\ z = -1 + 5s \end{cases}, s \in \mathbb{R}.$$

Thus, the two medians  $AG_A$  and  $BG_B$  are concurrent if and only if there exist  $s, t \in \mathbb{R}$  such that

$$\begin{cases} 1 - t = -1 + 7s \\ -1 + 6t = 1 - 2s \\ 1 - 3t = -1 + 5s \end{cases} \iff \begin{cases} 7s + t = 2 \\ 2s + 6t = 2 \\ 5s + 3t = 2 \end{cases} \iff \begin{cases} 7s + t = 2 \\ s + 3t = 1 \\ 5s + 3t = 2. \end{cases}$$

<sup>2</sup>The centroids of its faces

This system is compatible and has the unique solution  $s = t = \frac{1}{4}$ , which shows that the two medians  $AG_A$  and  $BG_B$  are concurrent and

$$AG_A \cap BG_B = \left\{ G \left( \frac{3}{4}, \frac{1}{2}, \frac{1}{4} \right) \right\}.$$

One can similarly show that  $BG_B \cap CG_C = CG_C \cap AG_A = \left\{ G \left( \frac{3}{4}, \frac{1}{2}, \frac{1}{4} \right) \right\}$ .

**Example 3.3 (Homework).** Consider the tetrahedron  $ABCD$ , where  $A(1, -1, 1)$ ,  $B(-1, 1, -1)$ ,  $C(2, 1, -1)$  and  $D(1, 1, 2)$ , as well as the midpoints  $M, N, P, Q, R$  and  $S$  of its edges  $[AB]$ ,  $[AC]$ ,  $[AD]$ ,  $[BC]$ ,  $[CD]$  and  $[DB]$  respectively. Show that the lines  $MR$ ,  $PQ$  and  $NS$  are concurrent and find the coordinates of their intersection point.

SOLUTION.

### 3.5 The Cartesian equations of the planes

Let  $A_0(x_0, y_0, z_0) \in \mathcal{P}$  and  $\vec{d}_1(p_1, q_1, r_1), \vec{d}_2(p_2, q_2, r_2) \in \mathcal{V}$  be linearly independent vectors, that is

$$\text{rank} \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix} = 2.$$

The vector equation of the plane  $\pi$  passing through  $A_0$  which is parallel to the vectors  $\vec{d}_1(p_1, q_1, r_1), \vec{d}_2(p_2, q_2, r_2)$  is

$$\vec{r}_M = \vec{r}_{A_0} + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2, \quad \lambda_1, \lambda_2 \in \mathbb{R}. \quad (3.5)$$

If we denote by  $x, y, z$  the coordinates of the generic point  $M$  of the plane  $\pi$ , then the vector equation (3.5) is the equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda_1 p_1 + \lambda_2 p_2 \\ y = y_0 + \lambda_1 q_1 + \lambda_2 q_2 \\ z = z_0 + \lambda_1 r_1 + \lambda_2 r_2 \end{cases}, \quad \lambda_1, \lambda_2 \in \mathbb{R}. \quad (3.6)$$

Indeed, the vector equation of  $\pi$  can be written, in terms of the coordinates of the vectors  $\vec{r}_M, \vec{r}_{A_0}, \vec{d}_1$  and  $\vec{d}_2$ , as follows:

$$\begin{aligned} x \vec{u} + y \vec{v} + z \vec{w} &= x_0 \vec{u} + y_0 \vec{v} + z_0 \vec{w} + \lambda_1(p_1 \vec{u} + q_1 \vec{v} + r_1 \vec{w}) + \lambda_2(p_2 \vec{u} + q_2 \vec{v} + r_2 \vec{w}) \\ \iff x \vec{u} + y \vec{v} + z \vec{w} &= (x_0 + \lambda_1 p_1 + \lambda_2 p_2) \vec{u} + (y_0 + \lambda_1 q_1 + \lambda_2 q_2) \vec{v} + (z_0 + \lambda_1 r_1 + \lambda_2 r_2) \vec{w}, \\ \lambda_1, \lambda_2 &\in \mathbb{R}, \end{aligned}$$

which is obviously equivalent to (3.6). The relations (3.6) characterize the points of the plane  $\pi$  and are called the *parametric equations* of the plane  $\pi$ . More precisely, the compatibility of the linear system (3.6) with the unknowns  $\lambda_1, \lambda_2$  is a necessary and sufficient condition for the point  $M(x, y, z)$  to be contained within the plane  $\pi$ . On the other hand the compatibility of the linear system (3.6) is equivalent to

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0, \quad (3.7)$$

which expresses the equality between the rank of the coefficient matrix of the system and the rank of the extended matrix of the system. The equation (3.7) is a characterization of the points of the plane  $\pi$  in terms of the Cartesian coordinates of the generic point  $M$  and is called the *cartesian equation* of the plane  $\pi$ . One can put the equation (3.7) in the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \text{ or} \quad (3.8)$$

$$Ax + By + Cz + D = 0, \quad (3.9)$$

where the coefficients  $A, B, C$  satisfy the relation  $A^2 + B^2 + C^2 > 0$ . It is also easy to show that every equation of the form (3.9) represents the equation of a plane. Indeed, if  $A \neq 0$ , then the equation (3.9) is equivalent to

$$\begin{vmatrix} x + \frac{D}{A} & y & z \\ B & -A & 0 \\ C & 0 & -A \end{vmatrix} = 0.$$

We observe that one can put the equation (3.8) in the form

$$AX + BY + CZ = 0 \quad (3.10)$$

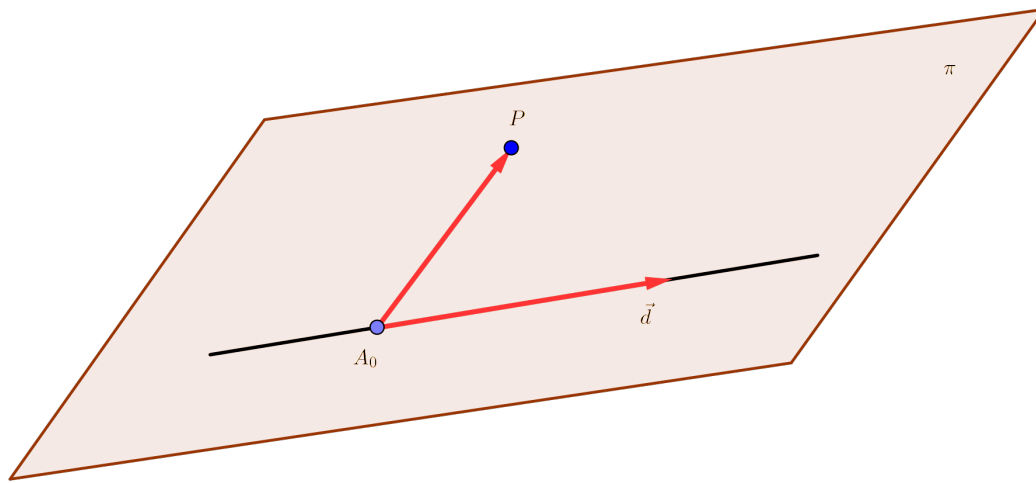
where  $X = x - x_0, Y = y - y_0, Z = z - z_0$  are the coordinates of the vector  $\vec{A_0M}$ .

**Example 3.4.** Write the equation of the plane determined by the point  $P(-1, 1, 2)$  and the line  $(\Delta) \frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{-1}$ .

**SOLUTION.** Note that  $P \notin \Delta$ , as  $\frac{-1-1}{3} \neq \frac{1}{2} \neq -3 = \frac{2+1}{-1}$ , i.e. the point  $P$  and the line  $\Delta$  determine, indeed, a plane, say  $\pi$ . One can regard  $\pi$  as the plane through the point  $A_0(1, 0, -1)$  which is parallel to the vectors  $\vec{A_0P} (-1 - 1, 1 - 0, 2 - (-1)) = \vec{A_0P} (-2, 1, 3)$  and  $\vec{d} (3, 2, -1)$ . Thus, the equation of  $\pi$  is

$$\begin{vmatrix} x - 1 & y & z + 1 \\ -2 & 1 & 3 \\ 3 & 2 & -1 \end{vmatrix} = 0 \iff x - y + z = 0.$$





**Example 3.5 (Homework).** Generalize Example 3.4: Write the equation of the plane determined by the line  $(\Delta) \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$  and the point  $M(x_M, y_M, z_M) \notin \Delta$ .

**SOLUTION.**

**Remark 3.3.** If  $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$  are noncollinear points, then the plane  $(ABC)$  determined by the three points can be viewed as the plane passing through the point  $A$  which is parallel to the vectors  $\vec{d}_1 = \vec{AB}, \vec{d}_2 = \vec{AC}$ . The coordinates of the vectors  $\vec{d}_1$  și  $\vec{d}_2$  are

$(x_B - x_A, y_B - y_A, z_B - z_A)$  and  $(x_C - x_A, y_C - y_A, z_C - z_A)$  respectively.

Thus, the equation of the plane  $(ABC)$  is

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0, \quad (3.11)$$

or, echivalently

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0. \quad (3.12)$$

Thus, four points  $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$  and  $D(x_D, y_D, z_D)$  are coplanar if and only if

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \quad (3.13)$$

**Example 3.6 (Homework).** Write the equation of the plane determined by the points  $M_1(3, -2, 1)$ ,  $M_2(5, 4, 1)$  and  $M_3(-1, -2, 3)$ .

SOLUTION.

**Remark 3.4.** If  $A(a, 0, 0)$ ,  $B(0, b, 0)$ ,  $C(0, 0, c)$  are three points ( $abc \neq 0$ ), then for the equation of the plane  $(ABC)$  we have successively:

$$\begin{aligned} \begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = 0 &\iff \begin{vmatrix} x & y & z-c & 1 \\ a & 0 & -c & 1 \\ 0 & b & -c & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0 \iff \begin{vmatrix} x & y & z-c \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = 0 \\ &\iff ab(z-c) + bcx + acy = 0 \iff bcx + acy + abz = abc \\ &\iff \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \end{aligned} \quad (3.14)$$

The equation (3.14) of the plane  $(ABC)$  is said to be in *intercept form* and the  $x, y, z$ -intercepts of the plane  $(ABC)$  are  $a, b, c$  respectively.

**Example 3.7 (Homework).** Write the equation of the plane  $(\pi)$   $3x - 4y + 6z - 24 = 0$  in intercept form.

SOLUTION.

## 3.6 Appendix: The Cartesian equations of lines in the two dimensional setting

### 3.6.1 Cartesian and affine reference systems

If  $b = [\vec{e}, \vec{f}]$  is an ordered basis of the director subspace  $\vec{\pi}$  of the plane  $\pi$  and  $\vec{x} \in \vec{\pi}$ , recall that the column vector of  $\vec{x}$  with respect to  $b$  is being denoted by  $[\vec{x}]_b$ . In other words

$$[\vec{x}]_b = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

whenever  $\vec{x} = x_1 \vec{e} + x_2 \vec{f}$ .

**Definition 3.3.** A *cartesian reference system* of the plane  $\pi$ , is a system  $R = (O, \vec{e}, \vec{f})$ , where  $O$  is a point from  $\pi$  called the *origin* of the reference system and  $b = [\vec{e}, \vec{f}]$  is a basis of the vector space  $\vec{\pi}$ .

Denote by  $E, F$  the points for which  $\vec{e} = \overrightarrow{OE}$ ,  $\vec{f} = \overrightarrow{OF}$ .

**Definition 3.4.** The system of points  $(O, E, F)$  is called *the affine reference system associated to the cartesian reference system*  $R = (O, \vec{e}, \vec{f})$ .

The straight lines  $OE$ ,  $OF$ , oriented from  $O$  to  $E$  and from  $O$  to  $F$  respectively, are called *the coordinate axes*. The coordinates  $x, y$  of the position vector  $\vec{r}_M = \overrightarrow{OM}$  with respect to the basis  $[\vec{e}, \vec{f}]$  are called the coordinates of the point  $M$  with respect to the cartesian system  $R$  written  $M(x, y)$ . Also, for the column matrix of coordinates of the vector  $\vec{r}_M$  we are going to use the notation  $[M]_R$ . In other words, if  $\vec{r}_M = x \vec{e} + y \vec{f}$ , then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \end{pmatrix}.$$

**Remark 3.5.** If  $A(x_A, y_A)$ ,  $B(x_B, y_B)$  are two points, then

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = x_B \vec{e} + y_B \vec{f} - (x_A \vec{e} + y_A \vec{f}) \\ &= (x_B - x_A) \vec{e} + (y_B - y_A) \vec{f}, \end{aligned}$$

i.e. the coordinates of the vector  $\overrightarrow{AB}$  are being obtained by performing the differences of the coordinates of the points  $A$  and  $B$ .

### 3.6.2 The Polar Coordinate System [1, p. 17]

As an alternative to a Cartesian coordinate system (RS) one considers in the plane  $\pi$  a fixed point  $O$ , called *pole* and a half-line directed to the right of  $O$ , called *polar axis* (see Figure 3). By specifying the distance  $\rho$  from  $O$  to a point  $P$  and an angle  $\theta$  (measured in radians), whose

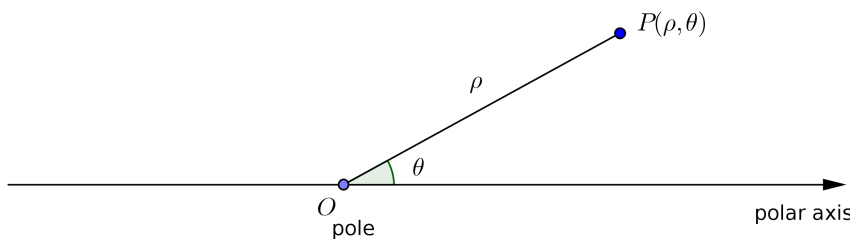


Figure 3: The pole and the polar axis related to a polar coordinate system

"initial" side is the polar axis and whose "terminal" side is the ray  $OP$ , the *polar coordinates* of the point  $P$  are  $(\rho, \theta)$ . One obtains a bijection

$$\pi \setminus \{O\} \rightarrow \mathbb{R}_+ \times [0, 2\pi), \quad P \rightarrow (\rho, \theta)$$

which associates to any point  $P$  in  $\pi \setminus \{O\}$  the pair  $(\rho, \theta)$  (suppose that  $O(0, 0)$ ). The positive real number  $\rho$  is called the *polar ray* of  $P$  and  $\theta$  is called the *polar angle* of  $P$ .

Consider the Cartesian coordinate system in  $\pi$ , whose origin  $O$  is the pole and whose positive half-axis  $Ox$  is the polar axis (see Figure 4). The following transformation formulas give the connection between the coordinates of an arbitrary point in the two systems of coordinates.

Let  $P$  be a given point of polar coordinates  $(\rho, \theta)$ . Its Cartesian coordinates are

$$\begin{cases} x_P = \rho \cos \theta \\ y_P = \rho \sin \theta \end{cases} \quad (3.15)$$

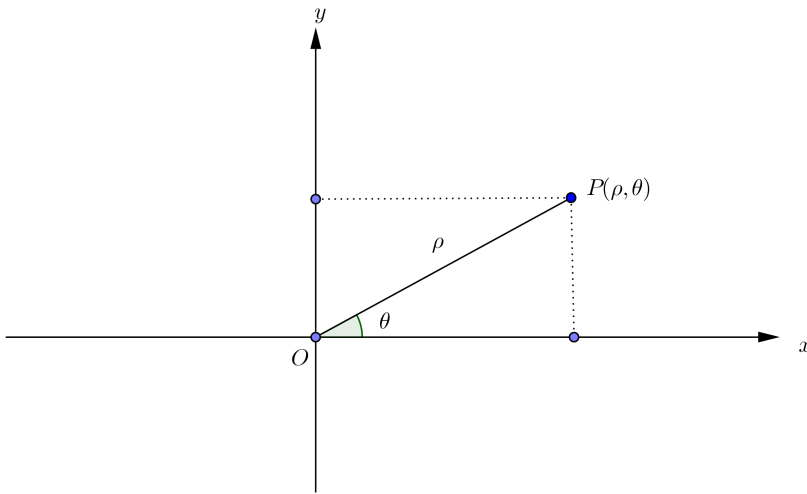


Figure 4: Polar coordinates

Let  $P$  be a point of Cartesian coordinates  $(x, y)$ . It is clear that the polar ray of  $P$  is given by the formula

$$\rho = \sqrt{x^2 + y^2}. \quad (3.16)$$

In order to obtain the polar angle of  $P$ , it must be considered the quadrant where  $P$  is situated. One obtains the following formulas:

Case 1. If  $x \neq 0$ , then using  $\tan \theta = \frac{y}{x}$ , one has

$$\theta = \arctan \frac{y}{x} + k\pi, \quad \text{where } k = \begin{cases} 0 & \text{if } P \in I \cup ]Ox \\ 1 & \text{if } P \in II \cup III \cup ]Ox' \\ 2 & \text{if } P \in IV; \end{cases}$$

Case 2. If  $x = 0$  and  $y \neq 0$ , then

$$\theta = \begin{cases} \frac{\pi}{2} & \text{when } P \in ]Oy \\ \frac{3\pi}{2} & \text{when } P \in ]Oy'; \end{cases}$$

Case 3. If  $x = 0$  and  $y = 0$ , then  $\theta = 0$ .

### 3.6.3 Parametric and Cartesian equations of Lines

Let  $\Delta$  be a line passing through the point  $A_0(x_0, y_0) \in \pi$  which is parallel to the vector  $\vec{d}(p, q)$ . Its vector equation is

$$\vec{r}_M = \vec{r}_{A_0} + t \vec{d}, \quad t \in \mathbb{R}. \quad (3.17)$$

If  $(x, y)$  are the coordinates of a generic point  $M \in \Delta$ , then its vector equation (3.17) is equivalent to the following system

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \end{cases}, \quad t \in \mathbb{R}. \quad (3.18)$$

The relations are called the *parametric equations* of the line  $\Delta$  and they are equivalent to the following equation

$$\frac{x - x_0}{p} = \frac{y - y_0}{q}, \quad (3.19)$$

called the *canonical equation* of  $\Delta$ . If  $q = 0$ , then the equation (3.19) becomes  $y = y_0$ .

If  $A(x_A, y_A)$  and  $B(x_B, y_B)$  are two different points of the plane  $\pi$ , then  $\vec{AB}(x_B - x_A, y_B - y_A)$  is a director vector of the line  $AB$  and the canonical equation of the line  $AB$  is

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}. \quad (3.20)$$

The equation (3.20) is equivalent to

$$\begin{vmatrix} x - x_A & y - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = 0 \iff \begin{vmatrix} x - x_A & y - y_A & 1 \\ x_B - x_A & y_B - y_A & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \iff \begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} = 0.$$

Thus, three points  $P_1(x_1, y_1)$ ,  $P_2(x_2, y_2)$  and  $P_3(x_3, y_3)$  are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \quad (3.21)$$

### 3.6.4 General Equations of Lines

We can put the equation (3.19) in the form

$$ax + by + c = 0, \quad \text{with } a^2 + b^2 > 0, \quad (3.22)$$

which means that any line from  $\pi$  is characterized by a first degree equation. Conversely, such of an equation represents a line, since the formula (3.22) is equivalent to

$$\frac{x + \frac{c}{a}}{-\frac{b}{a}} = \frac{y}{1}$$

and this is the *symmetric equation* of the line passing through  $P_0\left(-\frac{c}{a}, 0\right)$  and parallel to  $\vec{v}\left(-\frac{b}{a}, 1\right)$ . The equation (3.22) is called *general equation* of the line.

**Remark 3.6.** The lines

$$(d) \ ax + by + c = 0 \text{ and } (\Delta) \ \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

are parallel if and only if  $ap + bq = 0$ . Indeed, we have:

$$\begin{aligned} d \parallel \Delta &\iff \vec{d} = \vec{\Delta} \iff \langle \vec{u}(p, q) \rangle = \langle \vec{v}\left(-\frac{b}{a}, 1\right) \rangle \iff \exists t \in \mathbb{R} \text{ s.t. } \vec{u}(p, q) = t \vec{v}\left(-\frac{b}{a}, 1\right) \\ &\iff \exists t \in \mathbb{R} \text{ s.t. } p = -t\frac{b}{a} \text{ and } q = t \iff ap + bq = 0. \end{aligned}$$

### 3.6.5 Reduced Equations of Lines

Consider a line given by its general equation  $Ax + By + C = 0$ , where at least one of the coefficients  $A$  and  $B$  is nonzero. One may suppose that  $B \neq 0$ , so that the equation can be divided by  $B$ . One obtains

$$y = mx + n \quad (3.23)$$

which is said to be the *reduced equation* of the line.

*Remark:* If  $B = 0$ , (3.22) becomes  $Ax + C = 0$ , or  $x = -\frac{C}{A}$ , a line parallel to  $Oy$ . (In the same way, if  $A = 0$ , one obtains the equation of a line parallel to  $Ox$ ).

Let  $d$  be a line of equation  $y = mx + n$  in a Cartesian system of coordinates and suppose that the line is not parallel to  $Oy$ . Let  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  be two different points on  $d$  and  $\varphi$  be the angle determined by  $d$  and  $Ox$  (see Figure 5);  $\varphi \in [0, \pi] \setminus \{\pi/2\}$ . The points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  belong to  $d$ , hence

$$\begin{cases} y_1 = mx_1 + n \\ y_2 = mx_2 + n, \end{cases}$$

and  $x_2 \neq x_1$ , since  $d$  is not parallel to  $Oy$ . Then,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan \varphi. \quad (3.24)$$

The number  $m = \tan \varphi$  is called the *angular coefficient* of the line  $d$ . It is immediate that the equation of the line passing through the point  $P_0(x_0, y_0)$  and of the given angular coefficient  $m$  is

$$y - y_0 = m(x - x_0). \quad (3.25)$$

### 3.6.6 Intersection of Two Lines

Let  $d_1 : a_1x + b_1y + c_1 = 0$  and  $d_2 : a_2x + b_2y + c_2 = 0$  be two lines in  $\mathcal{E}_2$ . The solution of the system of equation

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

will give the set of the intersection points of  $d_1$  and  $d_2$ .

- 1) If  $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ , the system has a unique solution  $(x_0, y_0)$  and the lines have a unique intersection point  $P_0(x_0, y_0)$ . They are *secant*.

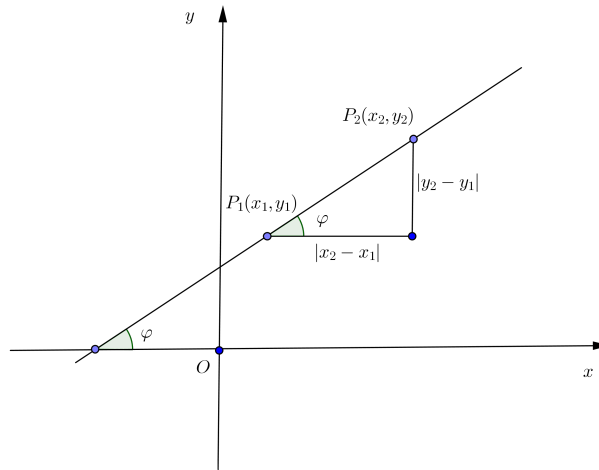


Figure 5:

- 2) If  $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ , the system is not compatible, and the lines have no points in common. They are *parallel*.
- 3) If  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ , the system has an infinity of solutions, and the lines coincide. They are *identical*.

If  $d_i : a_i x + b_i y + c_i = 0, i = \overline{1, 3}$  are three lines in  $\mathcal{E}_2$ , then they are concurrent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \quad (3.26)$$

### 3.6.7 Bundles of Lines ([1])

The set of all the lines passing through a given point  $P_0$  is said to be a *bundle* of lines. The point  $P_0$  is called the *vertex* of the bundle.

If the point  $P_0$  is of coordinates  $P_0(x_0, y_0)$ , then the equation of the bundle of vertex  $P_0$  is

$$r(x - x_0) + s(y - y_0) = 0, \quad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \quad (3.27)$$

*Remark:* The *reduced bundle* of line through  $P_0$  is,

$$y - y_0 = m(x - x_0), \quad m \in \mathbb{R}, \quad (3.28)$$

and covers the bundle of lines through  $P_0$ , except the line  $x = x_0$ . Similarly, the family of lines

$$x - x_0 = k(y - y_0), \quad k \in \mathbb{R}, \quad (3.29)$$

covers the bundle of lines through  $P_0$ , except the line  $y = y_0$ .

If the point  $P_0$  is given as the intersection of two lines, then its coordinates are the solution of the system

$$\begin{cases} d_1 : a_1 x + b_1 y + c_1 = 0 \\ d_2 : a_2 x + b_2 y + c_2 = 0 \end{cases} ,$$

assumed to be compatible. The equation of the bundle of lines through  $P_0$  is

$$r(a_1x + b_1y + c_1) + s(a_2x + b_2y + c_2) = 0, \quad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \quad (3.30)$$

*Remark:* As before, if  $r \neq 0$  (or  $s \neq 0$ ), one obtains the reduced equation of the bundle, containing all the lines through  $P_0$ , except  $d_1$  (respectively  $d_2$ ).

### 3.6.8 The Angle of Two Lines ([1])

Let  $d_1$  and  $d_2$  be two concurrent lines, given by their reduced equations:

$$d_1 : y = m_1x + n_1 \quad \text{and} \quad d_2 : y = m_2x + n_2.$$

The angular coefficients of  $d_1$  and  $d_2$  are  $m_1 = \tan \varphi_1$  and  $m_2 = \tan \varphi_2$  (see Figure 6). One may suppose that  $\varphi_1 \neq \frac{\pi}{2}$ ,  $\varphi_2 \neq \frac{\pi}{2}$ ,  $\varphi_2 \geq \varphi_1$ , such that  $\varphi = \varphi_2 - \varphi_1 \in [0, \pi] \setminus \{\frac{\pi}{2}\}$ .

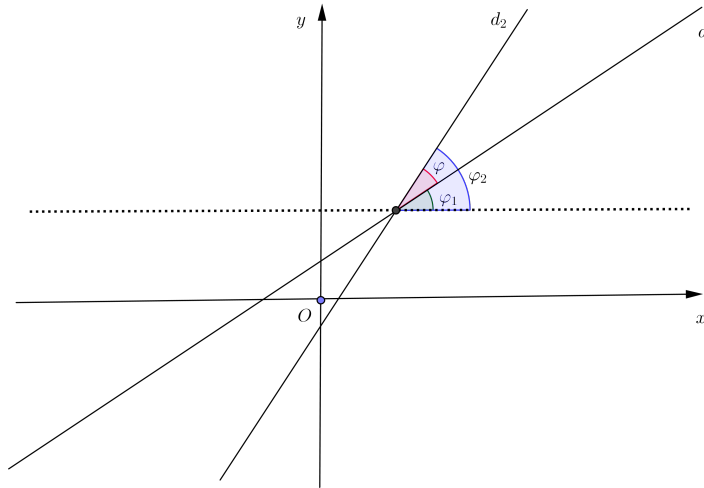


Figure 6:

The angle determined by  $d_1$  and  $d_2$  is given by

$$\tan \varphi = \tan(\varphi_2 - \varphi_1) = \frac{\tan \varphi_2 - \tan \varphi_1}{1 + \tan \varphi_1 \tan \varphi_2},$$

hence

$$\tan \varphi = \frac{m_2 - m_1}{1 + m_1 m_2}. \quad (3.31)$$

1) The lines  $d_1$  and  $d_2$  are parallel if and only if  $\tan \varphi = 0$ , therefore

$$d_1 \parallel d_2 \iff m_1 = m_2. \quad (3.32)$$

2) The lines  $d_1$  and  $d_2$  are orthogonal if and only if they determine an angle of  $\frac{\pi}{2}$ , hence

$$d_1 \perp d_2 \iff m_1 m_2 + 1 = 0. \quad (3.33)$$



### 3.7 Problems

1. Write the equation of the plane which passes through  $M_0(1, -2, 3)$  and is parallel to the vectors  $\vec{v}_1 (1, -1, 0)$  and  $\vec{v}_2 (-3, 2, 4)$ .

HINT.

$$\begin{vmatrix} x-1 & y+2 & z-3 \\ 1 & -1 & 0 \\ -3 & 2 & 4 \end{vmatrix} = 0.$$

vspace\*2cm

2. Write the equation of the line which passes through  $A(1, -2, 6)$  and is parallel to

(a) The  $x$ -axis;

(b) The line  $(d_1) \frac{x-1}{2} = \frac{y+5}{-3} = \frac{z-1}{4}$ .

(c) The vector  $\vec{v} (1, 0, 2)$ .

SOLUTION.

3. Write the equation of the plane which contains the line

$$(d_1) \frac{x-3}{2} = \frac{y+4}{1} = \frac{z-2}{-3}$$

and is parallel to the line

$$(d_2) \frac{x+5}{2} = \frac{y-2}{2} = \frac{z-1}{2}.$$

HINT.

$$\begin{vmatrix} x-3 & y+4 & z-2 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{vmatrix} = 0.$$

4. Consider the points  $A(\alpha, 0, 0)$ ,  $B(0, \beta, 0)$  and  $C(0, 0, \gamma)$  such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{a} \text{ where } a \text{ is a constant.}$$

Show that the plane  $(A, B, C)$  passes through a fixed point.

SOLUTION. The equation of the plane  $(ABC)$  can be written in intercept form, namely

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1.$$

The given relation shows that the point  $P(a, a, a) \in (ABC)$  whenever  $\alpha, \beta, \gamma$  verifies the given relation.

5. Write the equation of the line which passes through the point  $M(1, 0, 7)$ , is parallel to the plane  $(\pi) 3x - y + 2z - 15 = 0$  and intersects the line

$$(d) \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

*Solution.*

6. Write the equation of the plane which passes through  $M_0(1, -2, 3)$  and cuts the positive coordinate axes through equal intercepts.

SOLUTION. The general equation of such a plane is  $x + y + z = a$ . In this particular case  $a = 1 + (-2) + 3 = 2$  and the equation of the required plane is  $x + y + z = 2$ .

7. Write the equation of the plane which passes through  $A(1, 2, 1)$  and is parallel to the straight lines

$$(d_1) \begin{cases} x + 2y - z + 1 = 0 \\ x - y + z - 1 = 0 \end{cases} \quad (d_2) \begin{cases} 2x - y + z = 1 \\ x - y + z = 0. \end{cases}$$

SOLUTION. We need to find some director parameters of the lines  $(d_1)$  and  $(d_2)$ . In this respect we may solve the two systems. The general solution of the first system is

$$\begin{cases} x = -\frac{1}{3}t + \frac{1}{3} \\ y = \frac{2}{3}t - \frac{2}{3} \\ z = t \end{cases}, t \in \mathbb{R}$$

and the general solution of the second system is

$$\begin{cases} x = 1 \\ y = t + 1 \\ z = t \end{cases}, t \in \mathbb{R}$$

and these are the parametric equations of the lines  $(d_1)$  and  $(d_2)$ . Thus, the direction of the line  $(d_1)$  is the 1-dimensional subspace

$$\left\langle \left( -\frac{1}{3}, \frac{2}{3}, 1 \right) \right\rangle = \langle (-1, 2, 3) \rangle,$$

and the direction of the line  $(d_2)$  is the 1-dimensional subspace  $\langle (0, 1, 1) \rangle$ .

Consequently, some director parameters of the line  $(d_1)$  are  $p_1 = -1, q_1 = 2, r_1 = 3$  and some director parameters of the line  $(d_2)$  are  $p_2 = 0, q_2 = r_2 = 1$ . Finally, the equation of the required plane is

$$\begin{vmatrix} x-1 & y-2 & z-1 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = 0.$$

The computation of the determinant is left to the reader.

### A few questions in the two dimensional setting ([1])

8. The sides  $[BC]$ ,  $[CA]$ ,  $[AB]$  of the triangle  $\Delta ABC$  are divided by the points  $M$ ,  $N$  respectively  $P$  into the same ratio  $k$ . Prove that the triangles  $\Delta ABC$  and  $\Delta MNP$  have the same center of gravity.

SOLUTION.



9. Sketch the graph of  $x^2 - 4xy + 3y^2 = 0$ .

SOLUTION.

10. Find the equation of the line passing through the intersection point of the lines

$$d_1 : 2x - 5y - 1 = 0, \quad d_2 : x + 4y - 7 = 0$$

and through a point  $M$  which divides the segment  $[AB]$ ,  $A(4, -3)$ ,  $B(-1, 2)$ , into the ratio  $k = \frac{2}{3}$ .

SOLUTION.

11. Let  $A$  be a mobile point on the  $Ox$  axis and  $B$  a mobile point on  $Oy$ , so that  $\frac{1}{OA} + \frac{1}{OB} = k$  (constant). Prove that the lines  $AB$  passes through a fixed point.

SOLUTION.



12. Find the equation of the line passing through the intersection point of

$$d_1 : 3x - 2y + 5 = 0, \quad d_2 : 4x + 3y - 1 = 0$$

and crossing the positive half axis of  $Oy$  at the point  $A$  with  $OA = 3$ .

SOLUTION.

13. Find the parametric equations of the line through  $P_1$  and  $P_2$ , when

(a)  $P_1(3, -2), P_2(5, 1)$ ;

(b)  $P_1(4, 1), P_2(4, 3)$ .

SOLUTION.

14. Find the parametric equations of the line through  $P(-5, 2)$  and parallel to  $\vec{v}(2, 3)$ .

SOLUTION.

15. Show that the equations

$$x = 3 - t, y = 1 + 2t \quad \text{and} \quad x = -1 + 3t, y = 9 - 6t$$

represent the same line.

SOLUTION.

16. Find the vector equation of the line  $P_1P_2$ , where

(a)  $P_1(2, -1), P_2(-5, 3)$ ;

(b)  $P_1(0, 3), P_2(4, 3)$ .

SOLUTION.

17. Given the line  $d : 2x + 3y + 4 = 0$ , find the equation of a line  $d_1$  through the point  $M_0(2, 1)$ , in the following situations:
- (a)  $d_1$  is parallel with  $d$ ;
  - (b)  $d_1$  is orthogonal on  $d$ ;
  - (c) the angle determined by  $d$  and  $d_1$  is  $\varphi = \frac{\pi}{4}$ .

SOLUTION.

18. The vertices of the triangle  $\triangle ABC$  are the intersection points of the lines

$$d_1 : 4x + 3y - 5 = 0, \quad d_2 : x - 3y + 10 = 0, \quad d_3 : x - 2 = 0.$$

- (a) Find the coordinates of  $A, B, C$ .
- (b) Find the equations of the median lines of the triangle.
- (c) Find the equations of the heights of the triangle.

SOLUTION.

## 4 Week 4

### 4.1 Analytic conditions of parallelism and nonparallelism

#### 4.1.1 The parallelism between a line and a plane

**Proposition 4.1.** *The equation of the director subspace  $\vec{\pi}$ , of the plane  $\pi : Ax + By + Cz + D = 0$  is  $AX + BY + CZ = 0$ .*

*Proof.* We first recall that

$$\vec{\pi} = \{ \overrightarrow{A_0M} \mid M \in \pi \}, \quad (4.1)$$

where  $A_0 \in \pi$  is an arbitrary point, and the representation (4.1) of  $\vec{\pi}$  is independent on the choice of  $A_0 \in \pi$ . According to equation (3.8), the equation of a plane  $\pi$  can be written in the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

where  $A_0(x_0, y_0, z_0)$  is a point in  $\pi$ . In other words,

$$M(x, y, z) \in \pi \iff A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

which shows that

$$\begin{aligned} \vec{\pi} &= \{ \overrightarrow{A_0M} (x - x_0, y - y_0, z - z_0) \mid M(x, y, z) \in \pi \} \\ &= \{ \overrightarrow{A_0M} (x - x_0, y - y_0, z - z_0) \mid A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \} \\ &= \{ \vec{v} (X, Y, Z) \in \mathcal{V} \mid AX + BY + CZ = 0 \}. \end{aligned}$$

Thus, the equation  $AX + BY + CZ = 0$  is a necessary and sufficient condition for the vector  $\vec{v} (X, Y, Z)$  to be contained within the direction of the plane

$$\pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

In other words, the equation of the director subspace  $\vec{\pi}$  is  $AX + BY + CZ = 0$ . □

**Corollary 4.2.** *The straight line*

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

*is parallel to the plane  $\pi : Ax + By + Cz + D = 0$  if and only if*

$$Ap + Bq + Cr = 0 \quad (4.2)$$

*Proof.* Indeed,

$$\begin{aligned} \Delta \parallel \pi &\iff \vec{\Delta} \subseteq \vec{\pi} \iff \langle (p, q, r) \rangle \subseteq \vec{\pi} \\ &\iff \vec{d} (p, q, r) \in \vec{\pi} \iff Ap + Bq + Cr = 0. \end{aligned}$$

□



### 4.1.2 The intersection point of a straight line and a plane

**Proposition 4.3.** Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane  $\pi : Ax + By + Cz + D = 0$  which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

The coordinates of the intersection point  $d \cap \pi$  are

$$\begin{cases} x_0 - p \frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ y_0 - q \frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ z_0 - r \frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \end{cases} \quad (4.3)$$

where  $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $F(x, y, z) = Ax + By + Cz + D$ .

*Proof.* The parametric equations of  $(d)$  are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, t \in \mathbb{R}. \quad (4.4)$$

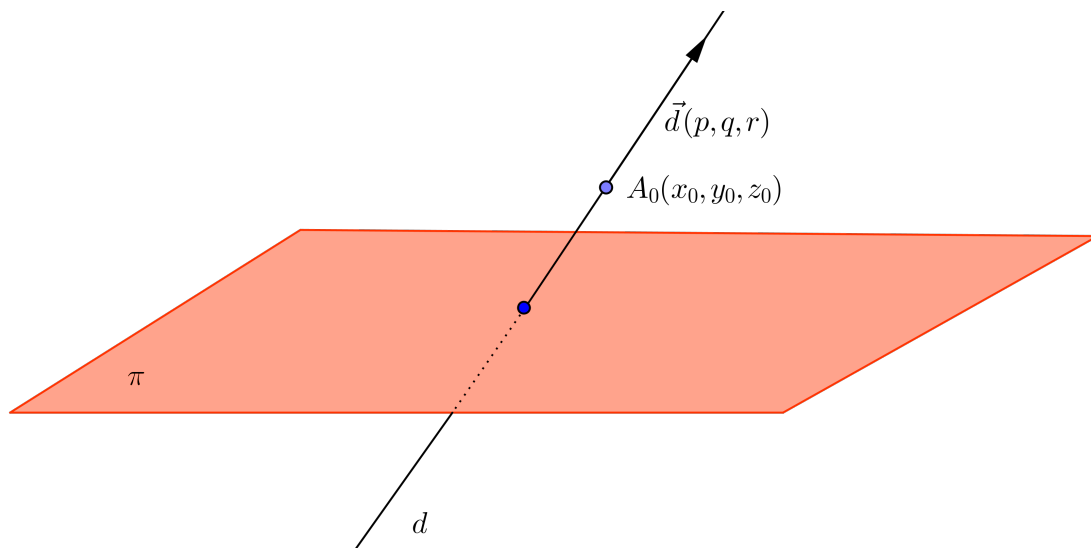
The unique value of  $t \in \mathbb{R}$ , which corresponds to the intersection point  $d \cap \pi$ , can be found by solving the equation

$$A(x_0 + pt) + B(y_0 + qt) + C(z_0 + rt) + D = 0.$$

Its unique solution is

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr} = -\frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr}$$

and can be used to obtain the required coordinates (4.3) by replacing this value in (4.4).  $\square$



**Example 4.1 (Homework).** Decide whether the line  $d$  and the plane  $\pi$  are parallel or concurrent and find the coordinates of the intersection point of  $d$  and  $\pi$  whenever  $d \not\parallel \pi$ :

1.  $d : \frac{x+2}{1} = \frac{y-1}{3} = \frac{z-3}{1}$  and  $\pi : x - y + 2z = 1$ .
2.  $d : \frac{x-3}{1} = \frac{y+1}{-2} = \frac{z-2}{-1}$  and  $\pi : 2x - y + 3z - 1 = 0$ .

SOLUTION.

### 4.1.3 Parallelism of two planes

**Proposition 4.4.** Consider the planes

$$(\pi_1) A_1x + B_1y + C_1z + D_1 = 0, (\pi_2) A_2x + B_2y + C_2z + D_2 = 0.$$

Then  $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) \in \{1, 2\}$  and the following statements are equivalent

1.  $\pi_1 \parallel \pi_2$ .
2.  $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 2$ , i.e.  $\vec{\pi}_1 = \vec{\pi}_2$ .
3.  $\text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1$ .
4. The vectors  $(A_1, B_1, C_1), (A_2, B_2, C_2) \in \mathbb{R}^3$  are linearly dependent.

**Remark 4.1.** Note that

$$\begin{aligned} \text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1 &\Leftrightarrow \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = 0 \\ &\Leftrightarrow A_1B_2 - A_2B_1 = A_1C_2 - A_2C_1 = B_1C_2 - C_2B_1 = 0. \end{aligned} \quad (4.5)$$

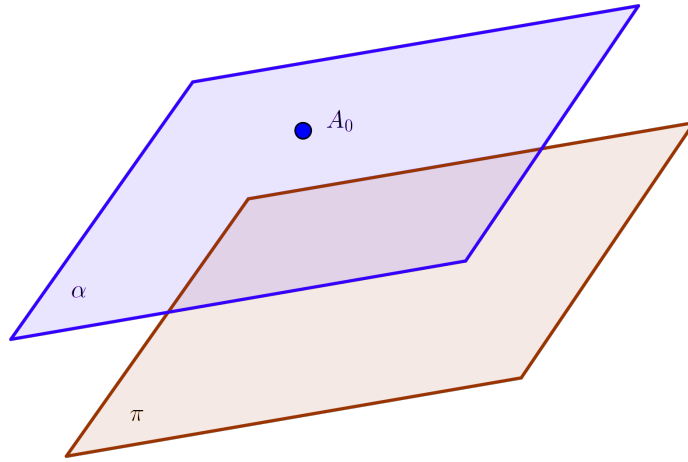
The relations (4.5) are often written in the form

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}, \quad (4.6)$$

although at most two of the coefficients  $A_2, B_2$  or  $C_2$  might be zero. In fact relations (4.6) should be understood in terms of linear dependence of the vectors  $(A_1, B_1, C_1), (A_2, B_2, C_2) \in \mathbb{R}^3$ , i.e.  $(A_1, B_1, C_1) = k(A_2, B_2, C_2)$ , where  $k \in \mathbb{R}$  is the common value of those ratios (4.6) which do not involve any zero coefficients. Let us finally mention that the equivalences (4.5) prove the equivalence (3)  $\Leftrightarrow$  (4) of Proposition 4.4.

**Example 4.2.** The equation of the plane  $\alpha$  passing through the point  $A_0(x_0, y_0, z_0)$ , which is parallel to the plane  $\pi : Ax + By + Cz + D = 0$  is

$$\alpha : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$



#### 4.1.4 Straight lines as intersections of planes

**Corollary 4.5.** Consider the planes

$$(\pi_1) A_1x + B_1y + C_1z + D_1 = 0, (\pi_2) A_2x + B_2y + C_2z + D_2 = 0.$$

The following statements are equivalent

1.  $\pi_1 \nparallel \pi_2$ .
2.  $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 1$ .
3.  $\text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$ .
4. The vectors  $(A_1, B_1, C_1), (A_2, B_2, C_2) \in \mathbb{R}^3$  are linearly independent.

By using the characterization of parallelism between a line and a plane, given by Proposition 4.2, we shall find the direction of a straight line which is given as the intersection of two planes. Consider the planes  $(\pi_1) A_1x + B_1y + C_1z + D_1 = 0, (\pi_2) A_2x + B_2y + C_2z + D_2 = 0$  such that

$$\text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2,$$

alongside their intersection straight line  $\Delta = \pi_1 \cap \pi_2$  of equations

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_1 = 0. \end{cases}$$

Thus,  $\vec{\Delta} = \vec{\pi}_1 \cap \vec{\pi}_2$  and therefore, by means of some previous Proposition, it follows that the equations of  $\vec{\Delta}$  are

$$(\vec{\Delta}) \begin{cases} A_1X + B_1Y + C_1Z = 0 \\ A_2X + B_2Y + C_2Z = 0. \end{cases} \quad (4.7)$$

By solving the system (4.7) one can therefore deduce that  $\vec{d}(p, q, r) \in \vec{\Delta} \Leftrightarrow \exists \lambda \in \mathbb{R}$  such that

$$(p, q, r) = \lambda \left( \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \right). \quad (4.8)$$

The relation is usually (4.8) written in the form

$$\frac{p}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{q}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}} = \frac{r}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \quad (4.9)$$

Let us finally mention that we usually choose the values

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \text{ şi } \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \quad (4.10)$$

for the director parameters  $(p, q, r)$  of  $\Delta$ .

**Example 4.3.** Write the equations of the plane through  $P(4, -3, 1)$  which is parallel to the lines

$$(\Delta_1) \begin{cases} 2x - z + 1 = 0 \\ 3y + 2z - 2 = 0 \end{cases} \text{ and } (\Delta_2) \begin{cases} x + y + z = 0 \\ 2x - y + 3z = 0 \end{cases}.$$

**SOLUTION.** One can see the required plane as the one through  $P(4, -3, 1)$  which is parallel to the director vectors  $\vec{d}_1(p_1, q_1, r_1)$  and  $\vec{d}_2(p_2, q_2, r_2)$  of  $\Delta_1$  and  $\Delta_2$  respectively. One can choose

$$\begin{aligned} p_1 &= \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3 & p_2 &= \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4 \\ q_1 &= \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} = -4 & \text{and } q_2 &= \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1 \\ r_1 &= \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 & r_2 &= \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3. \end{aligned}$$

Thus, the equation of the required plane is

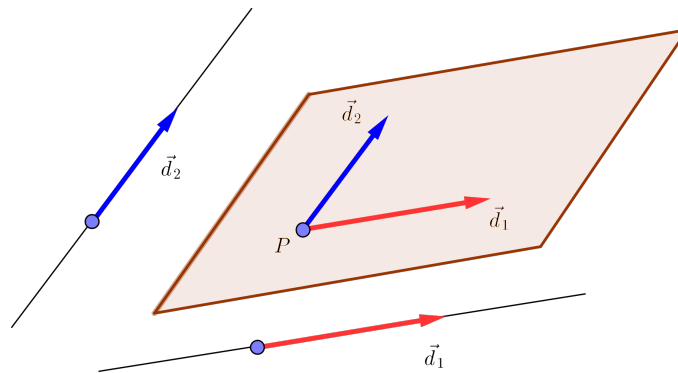


Figure 7:

$$\begin{aligned} \begin{vmatrix} x-4 & y+3 & z-1 \\ 3 & -4 & 6 \\ 4 & -1 & -3 \end{vmatrix} = 0 & \Leftrightarrow 12(x-4) - 3(z-1) + 24(y+3) + 16(z-1) + 6(x-4) + 9(y+3) = 0 \\ & \Leftrightarrow 18(x-4) + 33(y+3) + 13(z-1) = 0 \\ & \Leftrightarrow 18x + 33y + 13z - 72 + 99 - 13 = 0 \\ & \Leftrightarrow 18x + 33y + 13z + 14 = 0. \end{aligned}$$

## 4.2 Pencils of planes

**Definition 4.1.** The collection of all planes containing a given straight line

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

is called the *pencil* or the *bundle* of planes through  $\Delta$ .

**Proposition 4.6.** The plane  $\pi$  belongs to the pencil of planes through the straight line  $\Delta$  if and only if the equation of the plane  $\pi$  is

$$\lambda(A_1x + B_1y + C_1z + D_1) + \mu(A_2x + B_2y + C_2z + D_2) = 0. \quad (4.11)$$

for some  $\lambda, \mu \in \mathbb{R}$  such that  $\lambda^2 + \mu^2 > 0$ .

*Proof.* Every plane in the family (4.11) obviously contains the line  $\Delta$ .

Conversely, assume that  $\pi$  is a plane through the line  $\Delta$ . Consider a point  $M \in \pi \setminus \Delta$  and recall that  $\pi$  is completely determined by  $\Delta$  and  $M$ . On the other hand  $M$  and  $\Delta$  are obviously contained in the plane  $F_1(x_M, y_M, z_M)F_2(x, y, z) - F_2(x_M, y_M, z_M)F_1(x, y, z) = 0$  of the family (4.11), where  $F_1, F_2 : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $F_i(x, y, z) = A_ix + B_iy + C_iz + D_i$ , for  $i = 1, 2$ . Thus the plane  $\pi$  belongs to the family (4.11) and its equation is

$$F_1(x_M, y_M, z_M)F_2(x, y, z) - F_2(x_M, y_M, z_M)F_1(x, y, z) = 0.$$

□

**Remark 4.2.** The family of planes  $A_1x + B_1y + C_1z + D_1 + \lambda(A_2x + B_2y + C_2z + D_2) = 0$ , where  $\lambda$  covers the whole real line  $\mathbb{R}$ , is the so called *reduced pencil of planes* through  $\Delta$  and it consists in all planes through  $\Delta$  except the plane of equation  $A_2x + B_2y + C_2z + D_2 = 0$ .

**Example 4.4.** Write the equations of the plane parallel to the line  $d : x = 2y = 3z$  passing through the line

$$\Delta : \begin{cases} x + y + z = 0 \\ 2x - y + 3z = 0. \end{cases}$$

**SOLUTION.** Note that none of the planes  $x + y + z = 0$  and  $x - y + 3z = 0$ , passing through  $(\Delta)$ , is parallel to  $(d)$ , as  $1 \cdot 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} \neq 0$  and  $2 \cdot 1 + (-1) \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} \neq 0$ . Thus, the required plane is in a reduced pencil of planes, such as the family  $\pi_\lambda : x + y + z + \lambda(2x - y + 3z) = 0$ ,  $\lambda \in \mathbb{R}$ . The parallelism relation between  $(d)$  and  $\pi_\lambda : (2\lambda + 1)x + (1 - \lambda)y + (3\lambda + 1)z = 0$  is

$$(2\lambda + 1) \cdot 1 + (1 - \lambda) \cdot \frac{1}{2} + (3\lambda + 1) \cdot \frac{1}{3} = 0 \iff 12\lambda + 6 + 3 - 3\lambda + 6\lambda + 2 = 0 \iff \lambda = -\frac{11}{15}.$$

Thus, the required plane is

$$\pi_{-11/15} : \left(-2\frac{11}{15} + 1\right)x + \left(1 + \frac{11}{15}\right)y + \left(-3\frac{11}{15} + 1\right)z = 0 \iff -7x + 26y - 18z = 0.$$

## Appendix

### 4.3 Projections and symmetries

#### 4.3.1 The projection on a plane parallel with a given line

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane  $\pi : Ax + By + Cz + D = 0$  which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection  $p_{\pi,d} : \mathcal{P} \rightarrow \pi$  of  $\mathcal{P}$  on  $\pi$  parallel to  $d$ , whose value  $p_{\pi,d}(M)$  at  $M \in \mathcal{P}$  is the intersection point between  $\pi$  and the line through  $M$  which is parallel to  $d$ . Due to relations (4.3), the coordinates of  $p_{\pi,d}(M)$ , in terms of the coordinates of  $M$ , are

$$\begin{cases} x_M - p \frac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ y_M - q \frac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ z_M - r \frac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \end{cases} \quad (4.12)$$

where  $F(x, y, z) = Ax + By + Cz + D$ .

Consequently, the position vector of  $p_{\pi,d}(M)$  is

$$\overrightarrow{Op_{\pi,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{Ap + Bq + Cr} \vec{d}. \quad (4.13)$$

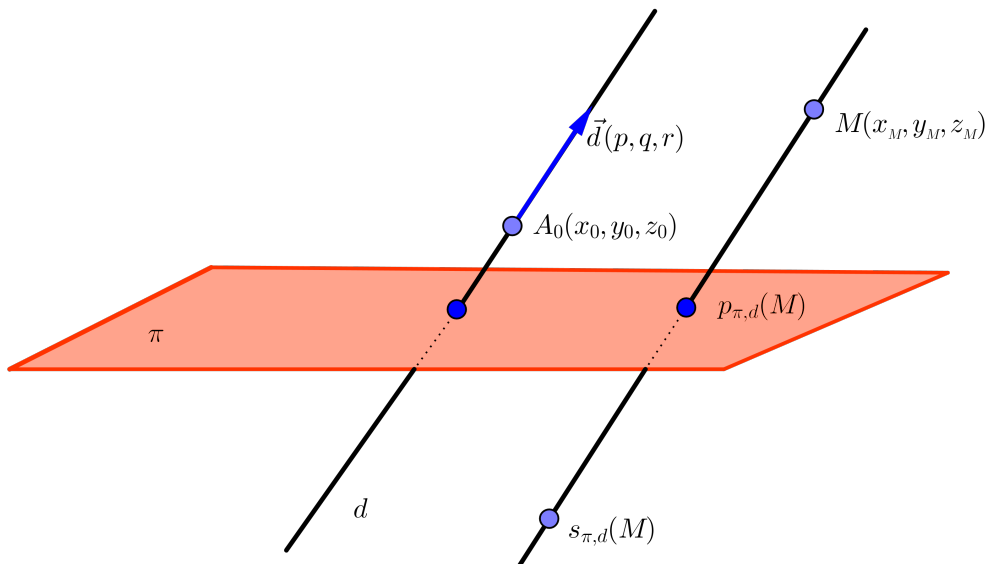
**Proposition 4.7.** If  $R = (O, b)$  is the Cartesian reference system behind the equations of the line

$$(d) \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and the plane  $(\pi) Ax + By + Cz + D = 0$ , concurrent with  $(d)$ , then

$$[p_{\pi,d}(M)]_R = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix} [M]_R - \frac{D}{Ap + Bq + Cr} [\vec{d}]_b,$$

where  $\vec{d}(p, q, r)$  stands for the director vector of the line  $(d)$ .



### 4.3.2 The symmetry with respect to a plane parallel with a given line

We call the function  $s_{\pi,d} : \mathcal{P} \rightarrow \mathcal{P}$ , whose value  $s_{\pi,d}(M)$  at  $M \in \mathcal{P}$  is the symmetric point of  $M$  with respect to  $p_{\pi,d}(M)$  the symmetry of  $\mathcal{P}$  with respect to  $\pi$  parallel to  $d$ . The direction of  $d$  is equally

called the *direction* of the symmetry and  $\pi$  is called the *axis* of the symmetry. For the position vector of  $s_{\pi,d}(M)$  we have

$$\overrightarrow{Op_{\pi,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\pi,d}(M)}}{2}, \text{ i.e.} \quad (4.14)$$

$$\overrightarrow{Os_{\pi,d}(M)} = 2 \overrightarrow{Op_{\pi,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2 \frac{F(M)}{Ap + Bq + Cr} \vec{d}. \quad (4.15)$$

**Proposition 4.8.** If  $R = (O, b)$  is the Cartesian reference system behind the equations of the line

$$(d) \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and the plane  $(\pi) Ax + By + Cz + D = 0$ , concurrent with  $(d)$ , then

$$(Ap + Bq + Cr)[s_{\pi,d}(M)]_R = \begin{pmatrix} -Ap + Bq + Cr & -2Bp & -2Cp \\ -2Aq & Ap - Bq + Cr & -2Cq \\ -2Ar & -2Br & Ap + Bq - Cr \end{pmatrix} [M]_R - 2D[\vec{d}]_b, \quad (4.16)$$

where  $\vec{d}(p, q, r)$  stands for the director vector of the line  $(d)$ .

### 4.3.3 The projection on a straight line parallel with a given plane

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane  $\pi : Ax + By + Cz + D = 0$  which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection  $p_{d,\pi} : \mathcal{P} \rightarrow d$  of  $\mathcal{P}$  on  $d$ , whose value  $p_{d,\pi}(M)$  at  $M \in \mathcal{P}$  is the intersection point between  $d$  and the plane through  $M$  which is parallel to  $\pi$ . Due to relations (4.3), the coordinates of  $p_{d,\pi}(M)$ , in terms of the coordinates of  $M$ , are

$$\begin{cases} x_0 - p \frac{G_M(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ y_0 - q \frac{G_M(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ z_0 - r \frac{G_M(x_0, y_0, z_0)}{Ap + Bq + Cr} \end{cases} \quad (4.17)$$

where  $G_M(x, y, z) = A(x - x_M) + B(y - y_M) + C(z - z_M)$ . Consequently, the position vector of  $p_{d,\pi}(M)$  is

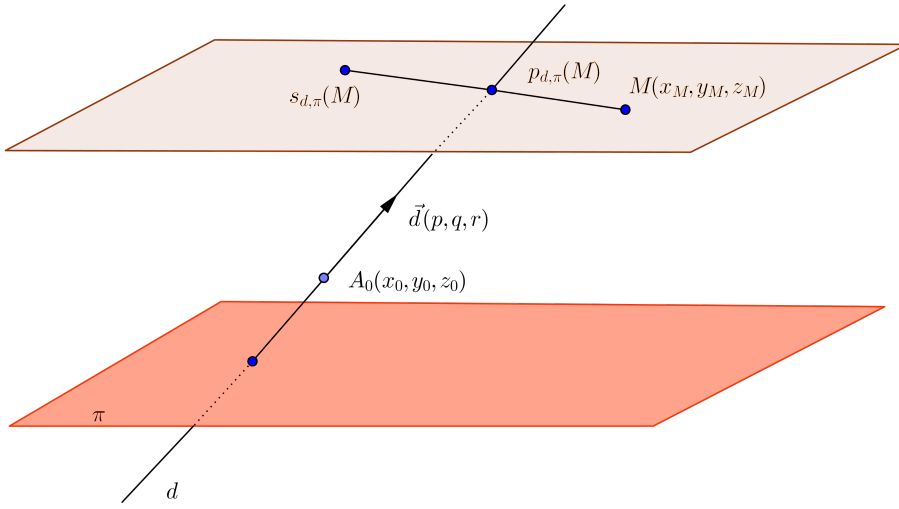
$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} - \frac{G_M(A_0)}{Ap + Bq + Cr} \vec{d}, \text{ where } A_0(x_0, y_0, z_0). \quad (4.18)$$

Note that  $G_M(A_0) = A(x_0 - x_M) + B(y_0 - y_M) + C(z_0 - z_M) = F(A_0) - F(M)$ , where  $F(x, y, z) = Ax + By + Cz + D$ . Consequently the coordinates of  $p_{d,\pi}(M)$ , in terms of the coordinates of  $M$ , are

$$\begin{cases} x_0 + p \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \\ y_0 + q \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \\ z_0 + r \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \end{cases} \quad (4.19)$$

and the position vector of  $p_{d,\pi}(M)$  is

$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} + \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \vec{d}, \text{ where } A_0(x_0, y_0, z_0). \quad (4.20)$$



#### 4.3.4 The symmetry with respect to a line parallel with a plane

We call the function  $s_{d,\pi} : \mathcal{P} \rightarrow \mathcal{P}$ , whose value  $s_{d,\pi}(M)$  at  $M \in \mathcal{P}$  is the symmetric point of  $M$  with respect to  $p_{d,\pi}(M)$ , the *symmetry of  $\mathcal{P}$  with respect to  $d$  parallel to  $\pi$* . The direction of  $\pi$  is equally called the *direction* of the symmetry and  $d$  is called the *axis* of the symmetry. For the position vector of  $s_{d,\pi}(M)$  we have

$$\overrightarrow{Op_{d,\pi}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{d,\pi}(M)}}{2}, \text{ i.e.} \quad (4.21)$$

$$\begin{aligned} \overrightarrow{Os_{d,\pi}(M)} &= 2 \overrightarrow{Op_{d,\pi}(M)} - \overrightarrow{OM} \\ &= 2 \overrightarrow{OA_0} - \overrightarrow{OM} + 2 \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \vec{d}. \end{aligned} \quad (4.22)$$

### 4.4 Projections and symmetries in the two dimensional setting

#### 4.4.1 The intersection point of two concurrent lines

Consider two lines

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

şi  $\Delta : ax + by + c = 0$  which are not parallel to each other, i.e.

$$ap + bq \neq 0.$$

The parametric equations of  $d$  are:

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \end{cases}, t \in \mathbb{R} \quad (4.23)$$

The value of  $t \in \mathbb{R}$  for which this line (4.23) punctures the line  $\Delta$  can be determined by imposing the condition on the point of coordinates

$$(x_0 + pt, y_0 + qt)$$



to verify the equation of the line  $\Delta$ , namely

$$a(x_0 + pt) + b(y_0 + qt) + c = 0.$$

Thus

$$t = -\frac{ax_0 + by_0 + c}{ap + bq} = -\frac{F(x_0, y_0)}{ap + bq},$$

where  $F(x, y) = ax + by + c$ .

The coordinates of the intersection point  $d \cap \Delta$  are:

$$\begin{aligned} x_0 - p \frac{F(x_0, y_0)}{ap + bq} \\ y_0 - q \frac{F(x_0, y_0)}{ap + bq}. \end{aligned} \quad (4.24)$$

#### 4.4.2 The projection on a line parallel with another given line

Consider two straight non-parallel lines

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

and  $\Delta : ax + by + c = 0$  which are not parallel to each other, i.e.  $ap + bq \neq 0$ . For these given data we may define the projection  $p_{\Delta, d} : \pi \rightarrow \Delta$  of  $\pi$  on  $\Delta$  parallel to  $d$ , whose value  $p_{\Delta, d}$  at  $M \in \pi$  is the intersection point between  $\Delta$  and the line through  $M$  which is parallel to  $d$ . Due to relations (4.24), the coordinates of  $p_{\Delta, d}(M)$ , in terms of the coordinates of  $M$  are:

$$\begin{aligned} x_M - p \frac{F(x_M, y_M)}{ap + bq} \\ y_M - q \frac{F(x_M, y_M)}{ap + bq}, \end{aligned}$$

where  $F(x, y) = ax + by + c$ .

Consequently, the position vector of  $p_{\Delta, d}(M)$  is

$$\overrightarrow{Op_{\Delta, d}(M)} = \overrightarrow{OM} - \frac{F(M)}{ap + bq} \vec{d},$$

where  $\vec{d} = p \vec{e} + q \vec{f}$ .

**Proposition 4.9.** If  $R$  is the Cartesian reference system of the plane  $\pi$  behind the equations of the concurrent lines

$$\Delta : ax + by + c = 0 \text{ and } d : \frac{x - x_0}{p} = \frac{y - y_0}{q},$$

then

$$[p_{\Delta, d}(M)]_R = \frac{1}{ap + bq} \begin{pmatrix} bq & -bp \\ -aq & ap \end{pmatrix} [M]_R - \frac{c}{ap + bq} [\vec{d}]_b. \quad (4.25)$$

#### 4.4.3 The symmetry with respect to a line parallel with another line

We call the function  $s_{\Delta, d} : \pi \rightarrow \pi$ , whose value  $s_{\Delta, d}$  at  $M \in \pi$  is the symmetric point of  $M$  with respect to  $p_{\Delta, d}(M)$ , the *symmetry of  $\pi$  with respect to  $\Delta$  parallel to  $d$* . The direction of  $d$  is equally called the direction of the symmetry and  $\pi$  is called the *axis of the symmetry*. For the position vector of  $s_{\Delta, d}(M)$  we have

$$\overrightarrow{Op_{\Delta,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\Delta,d}(M)}}{2}, \text{ i.e.}$$

$$\overrightarrow{Os_{\Delta,d}(M)} = 2\overrightarrow{Op_{\Delta,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{ap + bq} \vec{d},$$

where  $F(x, y) = ax + by + c$ . Thus, the coordinates of  $s_{\Delta,d}(M)$ , in terms of the coordinates of  $M$ , are

$$\begin{cases} x_M - 2p \frac{F(x_M, y_M)}{ap + bq} \\ y_M - 2q \frac{F(x_M, y_M)}{ap + bq}. \end{cases}$$

**Proposition 4.10.** *If  $R$  is the Cartesian reference system of the plane  $\pi$  behind the equations of the concurrent lines*

$$\Delta : ax + by + c = 0 \text{ and } d : \frac{x - x_0}{p} = \frac{y - y_0}{q},$$

then

$$[s_{\Delta,d}(M)]_R = \frac{1}{ap + bq} \begin{pmatrix} -ap + bq & -2bp \\ -2aq & ap - bq \end{pmatrix} [M]_R - \frac{2c}{ap + bq} [\vec{d}]_b. \quad (4.26)$$

## 4.5 Problems

1. Write the equation of the plane determined by the line

$$(d) \begin{cases} x - 2y + 3z = 0 \\ 2x + z - 3 = 0 \end{cases}$$

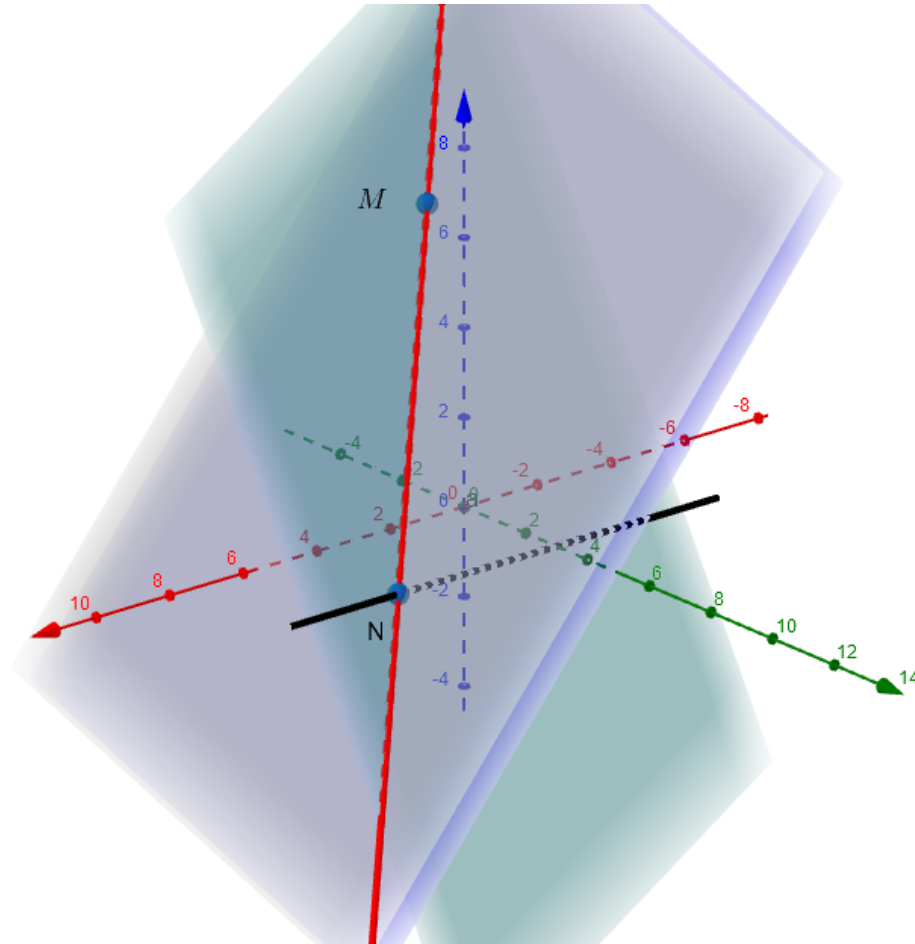
and the point  $A(-1, 2, 6)$ .

SOLUTION.

2. Write the equation of the line which passes through the point  $M(1, 0, 7)$ , is parallel to the plane  $(\pi) 3x - y + 2z - 15 = 0$  and intersects the line

$$(d) \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

SOLUTION 1. The equation of the plane  $\alpha$  passing through the point  $M(1, 0, 7)$ , which is parallel to the plane  $(\pi) 3x - y + 2z - 15 = 0$ , is  $(\alpha) 3(x-1) - (y-0) + 2(z-7) = 0$ , i.e.  $(\alpha) 3x - y + 2z - 17 = 0$ .



The parametric equations of the line  $d$  are

$$\begin{cases} x = 1 + 4t \\ y = 3 + 2t \\ z = t \end{cases}, t \in \mathbb{R}.$$

The coordinates of the intersection point  $N$  between the line  $(d)$  and the plane  $\alpha$  can be obtained by solving the equation  $3((1+4t) - (3+2t) + 2t - 17 = 0$ . The required line is  $MN$ .

SOLUTION 2. The required line can be equally regarded as the intersection line between the plane  $\alpha$  (passing through the point  $M(1, 0, 7)$ , which is parallel to the plane  $(\pi)$ ) and the plane determined by the given line  $(d)$  and the point  $M$ . While the equation  $3x - y + 2z - 17 = 0$  of  $\alpha$  was already used above, the equation of the plane determined by the line  $(d)$  and the point  $M$  can be determined via the pencil of planes through

$$(d) \left\{ \begin{array}{l} \frac{x-1}{4} = \frac{y-3}{2} \\ \frac{y-3}{2} = \frac{z}{1} \end{array} \right. \Leftrightarrow (d) \left\{ \begin{array}{l} x - 2y + 5 = 0 \\ y - 2z - 3 = 0. \end{array} \right.$$

Note that none of the planes  $x - 2y + 5 = 0$  or  $y - 2z - 3 = 0$  passes through  $M$ , which means that the plane determined by  $d$  and  $M$  is in the reduced pencil of planes

$$(\pi_\lambda) \quad x - 2y + 5 = 0 + \lambda(y - 2z - 3) = 0.$$

The plane determined by  $d$  and  $M$  can be found by imposing on the coordinates of  $M$  to verify the equation of  $\pi_\lambda$ .

3. Write the equations of the projection of the line

$$(d) \quad \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

on the plane  $\pi : x + 2y - z = 0$  parallel to the direction  $\vec{u} (1, 1, -2)$ . Write the equations of the symmetry of the line  $d$  with respect to the plane  $\pi$  parallel to the direction  $\vec{u} (1, 1, -2)$ .

SOLUTION.

## 4. Prove Proposition 4.7

SOLUTION.

## 5. Prove Proposition 4.8

SOLUTION.

6. Show that two different parallel lines are either projected onto parallel lines or on two points by a projection  $p_{\pi,d}$ , where

$$\pi : Ax + By + Cz + D = 0, \quad d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and  $\pi \nparallel d$ .

SOLUTION.

7. Show that two different parallel lines are mapped onto parallel lines by a symmetry  $s_{\pi, d}$ , where

$$\pi : Ax + By + Cz + D = 0, \quad d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and  $\pi \not\parallel d$ .

SOLUTION.



8. Assume that  $R = (O, b)$  ( $b = [\vec{u}, \vec{v}, \vec{w}]$ ) is the Cartesian reference system behind the equations of a plane  $\pi : Ax + By + Cz + D = 0$  and a line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$

If  $\pi \nparallel d$ , show that

- (a)  $\overrightarrow{p_{\pi,d}(M)p_{\pi,d}(N)} = \overrightarrow{p(MN)}$ , for all  $M, N \in \mathcal{V}$ , where  $p : \mathcal{V} \longrightarrow \mathcal{V}$  is the linear transformation whose matrix representation is

$$[p]_b = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix}.$$

SOLUTION.

- (b)  $\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)} = \overrightarrow{s(MN)}$ , for all  $M, N \in \mathcal{V}$ , where  $s : \mathcal{V} \rightarrow \mathcal{V}$  is the linear transformation whose matrix representation is

$$[s]_b = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} -Ap + Bq + Cr & -2Bp & -2Cp \\ -2Aq & Ap - Bq + Cr & -2Cq \\ -2Ar & -2Br & Ap + Bq - Cr \end{pmatrix}.$$

SOLUTION.

9. Consider a plane  $\pi : Ax + By + Cz + D = 0$  and a line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$

If  $\pi \nparallel d$ , show that

(a)  $p_{\pi,d} \circ p_{\pi,d} = p_{\pi,d}.$

(b)  $s_{\pi,d} \circ s_{\pi,d} = id_{\mathcal{P}}.$

SOLUTION.

10. Prove Proposition 4.9.

SOLUTION.

11. Prove Proposition 4.10.

SOLUTION.

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