Lecture 12

Multiple integrals

Definition 1. A set $A \subseteq \mathbb{R}^n$ is called a *nondegenerate compact interval* in $\mathbb{R}^n$ if there exist $a_i, b_i \in \mathbb{R}$, $a_i < b_i$, $i = 1, \ldots, n$ such that $A = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$.
$n = 2$: axis-aligned rectangles
$n = 3$: axis-aligned cuboids

In the following we consider $A = [a_1, b_1] \times [a_2, b_2]$ a nondegenerate compact interval in $\mathbb{R}^2$.

Definition 2. If $(x_0, x_1, \ldots, x_p)$ and $(y_0, y_1, \ldots, y_q)$ are partitions of $[a_1, b_1]$ and $[a_2, b_2]$, respectively, then

$$P = \{A_{ij} \mid i = 1, \ldots, p,\ j = 1, \ldots, q\},$$

where $A_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, is a *partition* of A.
The *norm* of P is

$$\|P\| = \max\left\{\max_{i=\overline{1,p}}\{x_i - x_{i-1}\}, \max_{j=\overline{1,q}}\{y_j - y_{j-1}\}\right\}$$

(i.e., the length of the largest width or height of any $A_{i,j}$, $i = \overline{1,p}$, $j = \overline{1,q}$).
Suppose that, for each $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, q\}$, (x_{ij}^*, y_{ij}^*) has been chosen in each A_{ij} and denote $\xi = \big((x_{ij}^*, y_{ij}^*)\big)_{\substack{1 \le i \le p \\ 1 \le j \le q}}$. Then (P, ξ) is called a *tagged partition* of A.

Definition 3. Let $f : A \to \mathbb{R}$ and (P, ξ) a tagged partition of A. Then the sum

$$\sigma(f, P, \xi) = \sum_{i=1}^{p} \sum_{j=1}^{q} f(x_{ij}^*, y_{ij}^*)(x_i - x_{i-1})(y_j - y_{j-1})$$

is called the *Riemann sum* of f w.r.t. the tagged partition (P, ξ).

Remark 1. If $f : A \to [0, \infty)$, then $f(x_{ij}^*, y_{ij}^*)(x_i - x_{i-1})(y_j - y_{j-1})$ is the volume of the cuboid with base A_{ij} and height $f(x_{ij}^*, y_{ij}^*)$. Thus, $\sigma(f, P, \xi)$ is the sum of volumes of such cuboids and provides an approximation of the volume of the solid S located under the graph of f and above A.

Definition 4. Let $f : A \to \mathbb{R}$. We say that f is *Riemann integrable* on A if there exists $I \in \mathbb{R}$ such that

$$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0 \text{ s.t. } \forall (P, \xi) \text{ tagged partition of } A \text{ with } \|P\| < \delta,\ |\sigma(f, P, \xi) - I| < \varepsilon.$$

The family of all Riemann integrable functions on A is denoted by $\mathcal{R}(A)$.
If $f \in \mathcal{R}(A)$, then $I \in \mathbb{R}$ (given above) is uniquely determined and called the *Riemann integral* (or *double integral*) of f on A.
Notation: $\iint_A f(x,y)\,dx\,dy = I$.

Remark 2. (i) If $f : A \to [0,\infty)$ and $f \in \mathcal{R}(A)$, then $V = \iint_A f(x,y)\,dx\,dy$ is the volume of the solid lying under the graph of f and above A.

(ii) $\iint_A dx\,dy$ is the area of A.

(iii) If $f \in \mathcal{R}(A)$, then f is bounded.

Theorem 1. *Let $f, g \in \mathcal{R}(A)$ and $\alpha \in \mathbb{R}$. Then*

(i) $f + g \in \mathcal{R}(A)$ *and* $\iint_A (f(x,y) + g(x,y))\,dx\,dy = \iint_A f(x,y)\,dx\,dy + \iint_A g(x,y)\,dx\,dy.$

(ii) $(\alpha f) \in \mathcal{R}(A)$ *and* $\iint_A (\alpha f(x,y))\,dx\,dy = \alpha \iint_A f(x,y)\,dx\,dy.$

(iii) $(f \cdot g) \in \mathcal{R}(A)$.

(iv) $|f| \in \mathcal{R}(A)$.

(v) If $f(x,y) \leq g(x,y)$ *for every* $(x,y) \in A$, *then* $\iint_A f(x,y)\,dx\,dy \leq \iint_A g(x,y)\,dx\,dy.$

Theorem 2. *Suppose A is divided into two nondegenerate compact intervals, A_1 and A_2 such that $A = A_1 \cup A_2$, $\operatorname{int}A_1 \cap \operatorname{int}A_2 = \emptyset$. Then*

$$f \in \mathcal{R}(A) \quad \Longleftrightarrow \quad f|_{A_1} \in \mathcal{R}(A_1) \text{ and } f|_{A_2} \in \mathcal{R}(A_2).$$

In this case, $\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy.$

Remark 3. (i) Let A be a nondegenerate compact interval in $\mathbb{R}^n$ and $f : A \to \mathbb{R}$. In a similar way as above, one can define the *Riemann integral* (or *multiple integral*) of f on A, denoted by $\int \cdots \int_A f(x_1, \ldots, x_n)\,dx_1 \ldots dx_n$.

If $n = 3$, we have a *triple integral* and we write $\iiint_A f(x,y,z)\,dx\,dy\,dz$.

(ii) If we want to integrate a given function over bounded regions of more general shape, we enclose this region inside a nondegenerate compact interval and extend the function in the following way: suppose $M \subseteq \mathbb{R}^n$ is nonempty and bounded, A is a nondegenerate compact interval in $\mathbb{R}^n$ with $M \subseteq A$ and $f : M \to \mathbb{R}$. We consider the following extension $\overline{f} : A \to \mathbb{R}$ of f on A:

$$\overline{f}(x) = \begin{cases} f(x), & \text{if } x \in M, \\ 0, & \text{if } x \in A \setminus M. \end{cases}$$

If $\overline{f} \in \mathcal{R}(A)$, then we say that f is *Riemann integrable* on M and we define the *Riemann integral* (or *multiple integral*) of f on M by

$$\int \cdots \int_M f(x_1, \ldots, x_n)\,dx_1 \ldots dx_n = \int \cdots \int_A \overline{f}(x_1, \ldots, x_n)\,dx_1 \ldots dx_n.$$

Note that this definition does not depend on the choice of A.

Computing multiple integrals

Theorem 3. *Let* $A = [a_1, b_1] \times \ldots \times [a_n, b_n]$ *be a nondegenerate compact interval in* $\mathbb{R}^n$ *and* $f : A \to \mathbb{R}$ *be continuous. Then*

(i) $f \in \mathcal{R}(A)$.

(ii) $\displaystyle\int \cdots \int_A f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \cdots \left(\int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n \right) \ldots dx_2 \right) dx_1.$

Remark 4. (i) The integral on the right hand side of the above formula is called an *iterated integral* and consists of ordinary integrals w.r.t. a single variable evaluated one at a time in the following way: we integrate $f(x_1, \ldots, x_n)$ w.r.t. x_n on $[a_n, b_n]$ regarding $x_1, \ldots, x_{n-1}$ as constants. The value of this integral is a function of variables $x_1, \ldots, x_{n-1}$ which we integrate w.r.t. x_{n-1} on $[a_{n-1}, b_{n-1}]$ regarding $x_1, \ldots, x_{n-2}$ as constants. Continuing in this way, we finally integrate w.r.t. x_1 on $[a_1, b_1]$.

(ii) Usually, we drop the brackets from the above iterated integral and simply write

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n \ldots dx_2\, dx_1.$$

Alternative notation: $\displaystyle\int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n.$

(iii) Similarly one defines other iterated integrals depending on the order of integration (there are $n!$ different orders of integration). A very important fact is that in Theorem 3.(ii), the order of integration can be chosen arbitrary and so one can consider any of the iterated integrals when computing the multiple integral. Note that it may happen that one specific iterated integral is easier to evaluate than another one.

Example 1. (i) Let $A = [0, 1] \times [0, 1]$. Compute $\displaystyle\iint_A (1 - x^2 - y^2)\, dx\, dy.$

(ii) Let $A = [0, 1] \times [0, 1]$. Compute $\displaystyle\iint_A \frac{x}{(1 + x^2 + y^2)^2}\, dx\, dy.$

(iii) Let $A = [0,1] \times [0,2] \times [0,3]$. Compute $\displaystyle\iiint_A xy^2z^3\,dx\,dy\,dz$.

Definition 5. A set $M \subseteq \mathbb{R}^2$ is called

- *simple w.r.t. the y-axis*: if there exist $a, b \in \mathbb{R}$, $a < b$ and $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ continuous functions with $\varphi_1 \leq \varphi_2$ such that $M = \big\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\big\}$.
- *simple w.r.t. the x-axis*: if there exist $c, d \in \mathbb{R}$, $c < d$ and $\psi_1, \psi_2 : [c,d] \to \mathbb{R}$ continuous functions with $\psi_1 \leq \psi_2$ such that $M = \big\{(x,y) \in \mathbb{R}^2 \mid c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\big\}$.

In the subsequent result we use the notation from Definition 5.

Theorem 4. *Let $M \subseteq \mathbb{R}^2$ and $f : M \to \mathbb{R}$ a continuous function.*
If M is simple w.r.t. the y-axis, then f is Riemann integrable on M and

$$\iint_M f(x,y)\,dx\,dy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx.$$

If M is simple w.r.t. the x-axis, then f is Riemann integrable on M and

$$\iint_M f(x,y)\,dx\,dy = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy.$$

Remark 5. Sometimes we need to exchange the order of integration to obtain an integral that we can evaluate.

Example 2. Let $M = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq 4,\ \sqrt{x} \leq y \leq \min\{x, 2\}\}$. Compute $\displaystyle\iint_M \frac{e^y}{y}\,dx\,dy$.