Lecture 8

Lecture 7, Example 6. (revisited) The function $f : \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$, $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$, has no limit at 0_2.

Example 1. The function $f : \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$, $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$, has no limit at 0_2.

Example 2. Let $A = ((0,\infty) \times \mathbb{R}) \setminus \{(1,0)\}$, $f : A \to \mathbb{R}$, $f(x,y) = \dfrac{(x-1)^2 \ln x}{(x-1)^2 + y^2}$. Then $\lim\limits_{(x,y)\to(1,0)} f(x,y) = 0$.

Continuous functions of several variables

In the following we consider $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$.

Definition 1. Let $f : A \to \mathbb{R}$ and $c \in A$. We say that f is *continuous at* c if

$$\forall V \in \mathcal{V}(f(c)), \exists U \in \mathcal{V}(c) \text{ such that } \forall x \in U \cap A \text{ we have } f(x) \in V.$$

If B is a subset of A, we say that f is *continuous on* B if it is continuous at every point of B. If f is continuous on A, then f is simply called continuous.

Theorem 1 (Sequential characterizations of continuity, Heine). *Let $f : A \to \mathbb{R}$ and $c \in A$. Then*

$$f \text{ is continuous at } c \Longleftrightarrow \forall \text{ sequence } (x^k) \text{ in } A \text{ with } \lim_{k\to\infty} x^k = c \text{ we have } \lim_{k\to\infty} f(x^k) = f(c).$$

Remark 1. If $c \in A \cap A'$, then f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.

Theorem 2. *Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}$, $a \in A$, $f : A \to B$ and $g : B \to \mathbb{R}$. If f is continuous at $a \in A$ an g is continuous at $f(a)$, then $g \circ f : A \to \mathbb{R}$ is continuous at a.*

Remark 2. (i) Polynomial functions in n variables are continuous on $\mathbb{R}^n$.
$n = 3$: $P(x, y, z) = 4x^2y^3 + 3x^2y^2z^2 - 5x + 4z + 1$.
(ii) Rational functions (a quotient of two polynomials) are continuous on their maximal domain of definition.
$n = 2$: $f : \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\} \to \mathbb{R}$, $f(x, y) = \dfrac{x^2 + 5y}{x + y}$.
(iii) Sums, products and quotients (when defined) of continuous real-valued functions of several varables are continuous.
(iv) One can construct continuous functions of several variables by taking, for instance, g in Theorem 2 to be an elementary function:
$f : \mathbb{R}^n \to \mathbb{R}$, $f(x_1, \ldots, x_n) = (x_1)^2 + \ldots + (x_n)^2$, $g : [0, \infty) \to \mathbb{R}$, $g(u) = \sqrt{u}$. Then $g \circ f : \mathbb{R}^n \to \mathbb{R}$, $(g \circ f)(x_1, \ldots, x_n) = \sqrt{(x_1)^2 + \ldots + (x_n)^2} = \|(x_1, \ldots, x_n)\|$ is continuous on $\mathbb{R}^n$.

Partial derivatives

Definition 2. Let $A \subseteq \mathbb{R}^n$. A point $c \in A$ is called an *interior point* of A if there exists $r > 0$ such that $B(c, r) \subseteq A$. The set of all interior points of A is called the *interior* of A and is denoted by $\operatorname{int} A$. The set A is called *open* if $\operatorname{int} A = A$.

Remark 3. (i) $\operatorname{int} A \subseteq A'$.

(ii) If $c \in \operatorname{int} A$, we can move a small distance in all directions from c while not leaving the set.

Example 3. (i) $A = \{0\} \cup (1, 2]$.

(ii) $A = [0, 1] \times [0, 2] \setminus \{0_2\}$.

(iii) $A = \{(0, 2)\} \cup (\{1\} \times [0, 2])$.

(iv) $A = \{(x, y) \in \mathbb{R}^2 : x > 0, y \neq 0\}$.

(v) Any open ball in $\mathbb{R}^n$ is an open set.

In the following we consider $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$.

First order partial derivatives

Definition 3. Let $f : A \to \mathbb{R}$, $c = (c_1, \ldots, c_n) \in \operatorname{int} A$ and $j \in \{1, \ldots, n\}$. We say that f is *partially differentiable w.r.t. x_j at c* if

$$\exists \lim_{x_j \to c_j} \frac{f(c_1, \ldots, c_{j-1}, x_j, c_{j+1}, \ldots, c_n) - f(c_1, \ldots, c_n)}{x_j - c_j} \in \mathbb{R}.$$

In this case, the above limit is called the *(first order) partial derivative of f w.r.t. x_j at c* and is denoted by $\dfrac{\partial f}{\partial x_j}(c)$ (or $f'_{x_j}(c)$).
If for all $j \in \{1, \ldots, n\}$, f is partially differentiable w.r.t. all variables x_j at c, then f is called *partially differentiable at c*. In this case, the vector

$$\left(\frac{\partial f}{\partial x_1}(c), \ldots, \frac{\partial f}{\partial x_n}(c) \right) \in \mathbb{R}^n$$

is called the *gradient of f at c* and is denoted by $\nabla f(c)$.
If B is an open subset of A, we say that f is *partially differentiable w.r.t. x_j on B* if it is partially differentiable w.r.t. x_j at every point of B. In this case, the function

$$\frac{\partial f}{\partial x_j} : B \to \mathbb{R}, \quad x \in B \mapsto \frac{\partial f}{\partial x_j}(x) \in \mathbb{R}$$

is called the *(first order) partial derivative of f w.r.t. x_j on B*.
At the same tine, f is called *partially differentiable on B* if it is partially differentiable at every point of B. If A is open and f is partially differentiable on A, then f is simply called partially differentiable.

Remark 4. Partial differentiation means taking the ordinary derivative w.r.t. a single variable while keeping all other variables constant. Thus, we can apply all rules of differentiation.

Example 4. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x^3 + x \sin(yz) + y^2 e^z$.

Remark 5. Let $f : A \to \mathbb{R}$ and $c \in \operatorname{int} A$.

$$f \text{ partially differentiable at } c \quad \begin{array}{c} \not\Longrightarrow \\ \not\Longleftarrow \end{array} \quad f \text{ continuous at } c.$$

Remark 6. Let $A \subseteq \mathbb{R}^n$ open, $f : A \to \mathbb{R}$, partially differentiable.

$$\text{partial derivatives of } f \text{ are continuous} \quad \begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \quad f \text{ continuous.}$$

(Examples in this sense will be given at the seminar.)

Definition 4. If $A \subseteq \mathbb{R}^n$ is open, a function $f : A \to \mathbb{R}$ is called *continuously partially differentiable* if it is partially differentiable and all partial derivatives are continuous.
Notation: $f \in C^1(A)$.

Higher order partial derivatives

Definition 5. Let $f : A \to \mathbb{R}$, $c \in \operatorname{int} A$ and $i, j \in \{1, \dots, n\}$. We say that f is *twice partially differentiable w.r.t.* (x_i, x_j) *at* c if $\exists V \in \mathcal{V}(c)$, V open, $V \subseteq A$ such that f is partially differentiable w.r.t. x_i on V and the function

$$\frac{\partial f}{\partial x_i} : V \to \mathbb{R}, \quad x \in V \mapsto \frac{\partial f}{\partial x_i}(x) \in \mathbb{R} \tag{1}$$

is partially differentiable w.r.t. x_j at c. The partial derivative of the function (1) w.r.t. x_j at c is called the *second order partial derivative of* f *w.r.t.* (x_i, x_j) *at* c and is denoted by $\dfrac{\partial^2 f}{\partial x_j \partial x_i}(c)$ (or $f''_{x_i x_j}(c)$). If $i = j$ we use the notation $\dfrac{\partial^2 f}{\partial x_i^2}(c)$ (or $f''_{x_i^2}(c)$). If for all $i, j \in \{1, \dots, n\}$, f is twice partially differentiable w.r.t (x_i, x_j) at c, then f is called *twice partially differentiable at* c.
Inductively, one can define partial derivatives of arbitrary order.

Remark 7.
(i) $\dfrac{\partial^2 f}{\partial x_j \partial x_i}(c) = \dfrac{\partial}{\partial x_j}\left(\dfrac{\partial f}{\partial x_i}\right)(c)$, $f''_{x_i x_j}(c) = \left(f'_{x_i}\right)'_{x_j}(c)$. Note that f has n^2 second order partial derivatives.

(ii) Higher order partial derivatives w.r.t. two or more different variables are also called *mixed partial derivatives.*

(iii) As in Definition 3, one can introduce the notions of twice partial differentiability and second order partial derivative (as a function) on open sets. In particular, if A is open, then f is called twice partially differentiable if f is twice partially differentiable at every point of A.

Example 5. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = e^{xy^2}$.

Remark 8. Mixed partial derivatives of a function are not always equal. (An example in this sense will be given at the seminar.)

Definition 6. If $A \subseteq \mathbb{R}^n$ is open, a function $f : A \to \mathbb{R}$ is called *twice continuously partially differentiable* if it is twice partially differentiable and all first and second order partial derivatives are continuous.
Notation: $f \in C^2(A)$.

Theorem 3 (Schwarz)**.** *If A is open and $f \in C^2(A)$, then for every $i, j \in \{1, \dots, n\}$,*

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Definition 7. Suppose that A is open, $c \in A$ and $f : A \to \mathbb{R}$ is twice partially differentiable at c. The $n \times n$ matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(c) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(c) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(c) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(c) & \frac{\partial^2 f}{\partial x_2^2}(c) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(c) \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(c) & \frac{\partial^2 f}{\partial x_n \partial x_2}(c) & \dots & \frac{\partial^2 f}{\partial x_n^2}(c) \end{pmatrix},$$

is called the *Hessian matrix (or Hessian) of f at c* and is denoted also by $H_f(c)$ (or $\nabla^2 f(c)$).

Remark 9. If f is twice partially differentiable, then we can consider the Hessian matrix at all points of A. Note that if $f \in C^2(A)$, by Theorem 3, $H_f(c)$ is symmetric at every $c \in A$.

Example 6. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = e^{xy^2}$.