7.1.

$f: \mathbb{R} \to \mathbb{R},\ f(x) = e^x \sin x$

$f'(x) = e^x \cdot \sin x + e^x \cos x = e^x\left(\sin x + \sin\left(x + \frac{\pi}{2}\right)\right) =$

$= e^x\left(2 \cdot \sin\frac{x + x + \frac{\pi}{2}}{2} \cdot \cos\frac{x - x - \frac{\pi}{2}}{2}\right) = 2e^x\left(\sin\left(x + \frac{\pi}{4}\right) \cdot \underbrace{\cos\left(-\frac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2}}\right) =$

$= \sqrt{2}\, e^x \cdot \sin\left(x + \frac{\pi}{4}\right)$

$f''(x) = \sqrt{2}\, e^x \cdot \sin\left(x + \frac{\pi}{4}\right) + \sqrt{2}\, e^x \cdot \cos\left(x + \frac{\pi}{4}\right) = \sqrt{2}\, e^x\left(\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x + \frac{3\pi}{4}\right)\right)$

$= \sqrt{2}\, e^x\left(2 \sin\frac{x + \frac{\pi}{4} + x + \frac{3\pi}{4}}{2} \cos\frac{x + \frac{\pi}{4} - x - \frac{3\pi}{4}}{2}\right) = 2\sqrt{2}\, e^x\left(\sin\left(x + \frac{\pi}{2}\right) \cdot \underbrace{\cos\left(-\frac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2}}\right)$

$= 2e^x \cdot \sin\left(x + \frac{\pi}{2}\right)$

$f'''(x) = 2e^x \cdot \sin\left(x + \frac{\pi}{2}\right) + 2e^x \cdot \cos\left(x + \frac{\pi}{2}\right) = 2e^x\left(\sin\left(x + \frac{\pi}{2}\right) + \sin\left(x + \pi\right)\right) =$

$= 2e^x\left(2 \sin\frac{x + \frac{\pi}{2} + x + \pi}{2} \cdot \cos\frac{x + \frac{\pi}{2} - x - \pi}{2}\right) = 4e^x\left(\sin\left(x + \frac{3\pi}{4}\right) \underbrace{\cos\left(-\frac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2}}\right) =$

$= 2\sqrt{2}\, e^x \cdot \sin\left(x + \frac{3\pi}{4}\right)$

$f^{(n)}(x) = \left(\sqrt{2}\right)^n \cdot e^x \cdot \sin\left(x + n\frac{\pi}{4}\right)$

We can prove it using induction:

For $n = 1$:

$f'(x) = \sqrt{2} \cdot e^x \cdot \sin\left(x + \frac{\pi}{4}\right)$ True

We consider $f^{(k)}(x) = \left(\sqrt{2}\right)^k \cdot e^x \cdot \sin\left(x + k\frac{\pi}{4}\right)$ True and prove $f^{(k+1)}(x)$

$f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = \left(\sqrt{2}\right)^k \cdot e^x \cdot \sin\left(x + \frac{k\pi}{4}\right) + \left(\sqrt{2}\right)^k \cdot e^x \cdot \cos\left(x + \frac{k\pi}{4}\right) =$

$= \left(\sqrt{2}\right)^k \cdot e^x\left(\sin\left(x + \frac{k\pi}{4}\right) + \sin\left(x + \frac{(k+2)\pi}{4}\right)\right) = \left(\sqrt{2}\right)^k \cdot e^x\left(2 \sin\frac{x + \frac{k\pi}{4} + x + \frac{(k+2)\pi}{4}}{2}\right.$

$\left.\cdot \cos\frac{x + \frac{k\pi}{4} - x - \frac{(k+2)\pi}{4}}{2}\right) = 2 \cdot \left(\sqrt{2}\right)^k \cdot e^x \cdot \left(\sin\left(x + \frac{(2k+2)\pi}{8}\right) \cdot \underbrace{\cos\left(-\frac{2\pi}{8}\right)}_{\frac{\sqrt{2}}{2}}\right) =$

$= \left(\sqrt{2}\right)^{k+1} e^x \cdot \sin\left(x + \frac{(k+1)\pi}{4}\right)$ True $\Rightarrow f^{(n)}(x) = \left(\sqrt{2}\right)^n \cdot e^x \sin\left(x + n\frac{\pi}{4}\right)$ is True

7.2

a) $\lim\limits_{x \to \infty} \frac{x+\ln x}{x \ln x} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim\limits_{x \to \infty} \frac{1+\frac{1}{x}}{\ln x + x \cdot \frac{1}{x}} = \frac{1}{\infty} = 0$

b) $\lim\limits_{\substack{x \to 0 \\ x>0}} x \ln \sin x = \lim\limits_{\substack{x \to 0 \\ x>0}} \frac{\ln \sin x}{\frac{1}{x}} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim\limits_{\substack{x \to 0 \\ x>0}} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} =$

$= \lim\limits_{\substack{x \to 0 \\ x>0}} -x^2 \cdot \operatorname{ctg} x = \lim\limits_{\substack{x \to 0 \\ x>0}} \frac{-x^2}{\operatorname{tg} x} \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim\limits_{\substack{x \to 0 \\ x>0}} \frac{-2x}{\frac{1}{\cos^2 x}} = 0$

c) $\lim\limits_{\substack{x \to 0 \\ x>0}} (\sin x)^x = \lim\limits_{\substack{x \to 0 \\ x>0}} e^{\ln(\sin x)^x} = e^{\lim\limits_{\substack{x \to 0 \\ x>0}} x \cdot \ln \sin x} =$

$= e^0 = 1$

(com b)

7.3.

$f: \mathbb{R} \to \mathbb{R},\ f(x) = x^3 - 3x^2 + 5x + 1,\ f(1) = 1 - 3 + 5 + 1 = 4$

$f'(x) = 3x^2 - 6x + 5,\ f'(1) = 3 - 6 + 5 = 2$

$f''(x) = 6x - 6,\ f''(1) = 6 - 6 = 0$

$f'''(x) = 6,\ f'''(1) = 6$

$T_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 =$

$= 4 + 2(x-1) + 0 + 1 \cdot (x-1)^3 = x^3 - 3x^2 + 5x + 1 = f(x)$

8.1

$f:(0,\infty)\to\mathbb{R},\ f(x)=\frac{1}{x^2}$

$f'(x)=-2\frac{x}{x^4}=-2\cdot\frac{1}{x^3}$

$f''(x)=6\cdot\frac{x^2}{x^6}=6\cdot\frac{1}{x^4}$

$f'''(x)=-24\cdot\frac{x^3}{x^8}=-24\frac{1}{x^5}$

$f^{(n)}(x)=(-1)^n\cdot(n+1)!\cdot\frac{1}{x^{n+2}}$

We prove using induction that $f^{(n)}(x)=(-1)^n\cdot(n+1)!\cdot\frac{1}{x^{n+2}}$

for $n=1$:

$f'(x)=(-1)^1\cdot 2!\cdot\frac{1}{x^3}=-2\frac{1}{x^3}$ True

We consider $f^{(k)}(x)=(-1)^k\cdot(k+1)!\cdot\frac{1}{x^{k+2}}$ True and prove $f^{(k+1)}(x)$

$f^{(k+1)}(x)=\left(f^{(k)}(x)\right)'=(-1)^k\cdot(k+1)!\cdot\left(-(k+2)\cdot\frac{x^{k+1}}{x^{2(k+2)}}\right)=$

$=(-1)^{k+1}\cdot(k+2)!\cdot\frac{x^{k+1}}{x^{2k+4}}=(-1)^{k+1}\cdot(k+2)!\cdot\frac{1}{x^{k+3}}$ True $\Rightarrow$

$\Rightarrow f^{(n)}(x)=(-1)^n\cdot(n+1)!\cdot\frac{1}{x^{n+2}}$ is True, $\forall n\in\mathbb{N}_0$

$\exists c$ between 1 and x $(x\in[1,2])$ s.t. $\frac{1}{x^2}=T_n(x)+R_n(x)$ with

$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1}$

$f(1)=1,\ f'(1)=-2,\ f''(1)=6,\ \dots,\ f^{(n)}(1)=(-1)^n\cdot(n+1)!$

$T_n(x)=f(1)+\frac{f'(1)}{1!}(x-1)+\frac{f''(1)}{2!}(x-1)^2+\dots+\frac{f^{(n)}(1)}{n!}(x-1)^n=$

$=1-2(x-1)+3(x-1)^2+\dots+(-1)^n\cdot(n+1)(x-1)^n$

$$R_n(x) = \frac{(-1)^{n+1}(n+2)! \cdot \frac{1}{c^{n+3}}}{\cancel{(n+1)!}}(x-1)^{n+1} \Rightarrow |R_n(x)| = \frac{(n+2)|x-1|^{n+1}}{c^{n+3}}$$

$$1 \le x < 2 \Rightarrow 0 \le x-1 < 1 \Rightarrow 0 \le |x-1|^{n+1} < 1 \Rightarrow$$

$$\Rightarrow 0 \le (n+1)|x-1|^{n+1} \le n+1 \Rightarrow 0 \le \frac{(n+1)|x-1|^{n+1}}{c^{n+3}} < \frac{n+1}{c^{n+3}} \to 0$$

$c \ge 1$, $n \to \infty$ $\Rightarrow$ By the Squeeze theorem that $|R_n(x)| \to 0 \Rightarrow$

$$\Rightarrow R_n(x) \to 0 \Rightarrow \frac{1}{x^2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$$

8.2.

$$x, y \in \overline{B}(z, r) \Rightarrow \begin{cases} \|x-z\| \le r \\ \|y-z\| \le r \end{cases}$$

$$\|x-z\| \le r^2 \Leftrightarrow \langle x-z, x-z\rangle \le r^2 \Leftrightarrow$$

$$\Leftrightarrow \langle x, x-z\rangle - \langle z, x-z\rangle \le r^2 \Leftrightarrow \langle x, x\rangle - \langle x, z\rangle -$$

$$- \langle z, x-z\rangle \le r^2 \Leftrightarrow \langle x, x\rangle - \langle x, z\rangle - \langle z, x\rangle + \langle z, z\rangle \le r^2 \Rightarrow$$

$$\Rightarrow \langle x, x\rangle - 2\langle x, z\rangle + \langle z, z\rangle \le r^2$$

$$\|y-z\| \le r \Rightarrow \langle y, y\rangle - 2\langle y, z\rangle + \langle z, z\rangle \le r^2 \quad +$$

$$\Rightarrow \langle x, x\rangle + \langle y, y\rangle + 2\langle z, z\rangle - 2\langle x, z\rangle - 2\langle y, z\rangle \le 2r^2$$

$$\|x-y\| \ge \varepsilon r \Rightarrow \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle \ge (\varepsilon r)^2 \,|\cdot(-1) \Rightarrow$$

$$\Rightarrow -\langle x, x\rangle + 2\langle x, y\rangle - \langle y, y\rangle \le -(\varepsilon r)^2 \,|\cdot\tfrac{1}{2} \Rightarrow$$

$$\Rightarrow -\frac{\langle x, x\rangle}{2} + \langle x, y\rangle - \frac{\langle y, y\rangle}{2} \le -\frac{(\varepsilon r)^2}{2}$$

$$\langle x, x\rangle + \langle y, y\rangle + 2\langle z, z\rangle - 2\langle x, z\rangle - 2\langle y, z\rangle \le 2r^2 \quad +$$

$$\Rightarrow 2\langle z,z\rangle - 2\langle x,z\rangle - 2\langle y,z\rangle + \underbrace{\langle x,x\rangle - \frac{\langle x,x\rangle}{2}}_{\frac{\langle x,x\rangle}{2}} + \langle x,y\rangle + \underbrace{\langle y,y\rangle - \frac{\langle y,y\rangle}{2}}_{\frac{\langle y,y\rangle}{2}} \le 2n^2 - \frac{(\varepsilon n)^2}{2} \Big| \cdot \frac{1}{2} \Rightarrow$$

$$\Rightarrow \langle z,z\rangle - \langle x,z\rangle - \langle y,z\rangle + \frac{\langle x,x\rangle}{4} + \frac{\langle x,y\rangle}{2} + \frac{\langle y,y\rangle}{4} \le n^2 - \frac{(\varepsilon n)^2}{4}$$

$$\Rightarrow \sum_{i=1}^{n} z_i^2 - x_i \cdot z_i - y_i \cdot z_i + \frac{x_i^2}{4} + \frac{x_i \cdot y_i}{2} + \frac{y_i^2}{4} \le n^2\left(1 - \frac{\varepsilon^2}{4}\right) \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n} z_i^2 - z_i(x_i + y_i) + \left(\frac{x_i}{2}\right)^2 + 2 \cdot \frac{x_i}{2} \cdot \frac{y_i}{2} + \left(\frac{y_i}{2}\right)^2 \le n^2\left(1 - \frac{\varepsilon^2}{4}\right) \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n} z_i^2 - 2 \cdot z_i\left(\frac{x_i + y_i}{2}\right) + \left(\frac{x_i + y_i}{2}\right)^2 \le n^2\left(1 - \frac{\varepsilon^2}{4}\right) \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n}\left(z_i - \frac{x_i + y_i}{2}\right)^2 \le n^2\left(1 - \frac{\varepsilon^2}{4}\right) \Rightarrow$$

$$\Rightarrow \left\langle z - \frac{x+y}{2}, z - \frac{x+y}{2}\right\rangle \le n^2\left(1 - \frac{\varepsilon^2}{4}\right) \Big| \sqrt{} \Rightarrow$$

$$\Rightarrow \left\| z - \frac{x+y}{2}\right\| \le n\sqrt{1 - \frac{\varepsilon^2}{4}} \Rightarrow \text{True}$$

9.1

a) $$f(x,y) = \begin{cases} \dfrac{xy + x^2 y \ln(x^2+y^2)}{x^2+y^2}, & \text{if } (x,y) \ne 0_2 \\ 0, & \text{if } (x,y) = 0_2 \end{cases}$$

We check if $\lim_{(x,y)\to(0,0)} f(x,y) = f(0_2)$

take $a^k = \left(\frac{1}{k}, \frac{1}{k}\right)$, $k \in \mathbb{N}$

$$\lim_{k\to\infty} f(a^k) = \lim_{k\to\infty} \frac{\frac{1}{k^2} + \frac{1}{k^3} \cdot \ln\left(2 \cdot \frac{1}{k^2}\right)}{2\frac{1}{k^2}} = \frac{1}{2} + \lim_{k\to\infty} \ln\left(2 \cdot \frac{1}{k^2}\right)^{\frac{1}{2k}} =$$

$$= \frac{1}{2} + \lim_{k \to \infty} \left(\ln 2^{\frac{1}{2k}} + \ln \left(\frac{1}{k}\right)^{\frac{1}{2k}} \right) = \cancel{\frac{1}{2} + \ln 2^{0} + \ln 0^{0}} =$$

$$= \frac{1}{2} + \ln 1 \cdot \lim_{k \to \infty} \frac{\ln k}{k} =$$

$$= \frac{1}{2} + \cancel{\ln 1} + \cancel{\ln \frac{1}{1}} = \frac{1}{2} \neq f(O_2) \Rightarrow f \text{ is not cont. at } O_2$$

9.1

b) $f(x,y) = \begin{cases} \dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}, & (x,y) \neq O_2 \\ 0, & (x,y) = O_2 \end{cases}$

$$\frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} = \underbrace{\frac{1}{e^{\frac{1}{x^2+y^2}}}}_{>0} \cdot \underbrace{\frac{1}{x^4+y^4}}_{\geq 0} \geq 0, \ \forall (x,y) \in \mathbb{R}^2 \setminus \{O_2\}$$

$$x^4+y^4 \geq \frac{(x^2+y^2)^2}{2} \Leftrightarrow 2x^4+2y^4 \geq x^4+2x^2y^2+y^4 \Leftrightarrow$$

$$\Leftrightarrow x^4-2x^2y^2+y^4 \geq 0 \Leftrightarrow (x^2-y^2)^2 \geq 0 \text{ True} \Rightarrow$$

$$\Rightarrow \frac{1}{x^4+y^4} \leq \frac{2}{(x^2+y^2)^2} \Rightarrow 0 \leq \frac{1}{e^{\frac{1}{x^2+y^2}}} \cdot \frac{1}{x^4+y^4} \leq \frac{1}{e^{\frac{1}{x^2+y^2}}} \cdot \frac{2}{(x^2+y^2)^2}$$

let $k = x^2+y^2$; $(x,y) \to O_2 \Rightarrow k \to 0$

$$\lim_{k\to 0} \frac{2}{e^{\frac{1}{k}} k^2} = \cancel{\lim_{k\to 0} \frac{2e^{-\frac{1}{k}}}{k^2} \overset{L'H}{=} \lim_{k\to 0}}$$

$$= \frac{2}{\lim\limits_{k\to 0} \frac{e^{\frac{1}{k}}}{\frac{1}{k^2}}} \overset{L'H}{=} \frac{2}{\lim\limits_{k\to 0} \frac{-\frac{e^{\frac{1}{k}}}{k^2}}{-2\cdot\frac{1}{k^3}}} = \frac{4}{\lim\limits_{k\to 0} \frac{e^{\frac{1}{k}}}{\frac{1}{k}}} \overset{L'H}{=}$$

$$= \frac{4}{\lim\limits_{k\to 0} \frac{-\frac{e^{\frac{1}{k}}}{k^2}}{-\frac{1}{k^2}}} = \frac{4}{\lim\limits_{k\to 0} e^{\frac{1}{k}}} = \frac{4}{\infty} = 0 \Rightarrow$$

$\Rightarrow$ By the Squeeze Theorem that $\lim\limits_{(x,y)\to O_2} \dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} = 0 = f(0,0) \Rightarrow$

$\Rightarrow f$ is continuous at O_2

9.2.

a) $f: \mathbb{R}^2 \to \mathbb{R}, f(x,y) = \sin(x \cdot \sin y)$

$$\frac{\partial f}{\partial x} = (x \cdot \sin y)' \cdot \cos(x \cdot \sin y) = \sin y \cdot \cos(x \cdot \sin y)$$

$$\frac{\partial f}{\partial y} = (x \cdot \sin y)' \cdot \cos(x \cdot \sin y) = x \cdot \cos y \cdot \cos(x \cdot \sin y)$$

$$\frac{\partial f}{\partial x^2} = \sin y \cdot (x \sin y)' \cdot (-\sin(x \sin y)) = -\sin^2 y \cdot \sin(x \sin y)$$

$$\frac{\partial f}{\partial x \partial y} = \cos y \cdot \cos(x \cdot \sin y) + x \cos y \cdot \sin y \cdot (-\sin(x \cdot \sin y))$$

$$\frac{\partial f}{\partial y^2} = -x \sin y \cdot \cos(x \sin y) + x \cos y \left(-\sin(x \cdot \sin y)\right) \cdot x \cos y =$$

$$= -x \sin y \cdot \cos(x \cdot \sin y) - x^2 \cdot \cos^2 y \cdot \sin(x \cdot \sin y)$$

$$\frac{\partial f}{\partial y \partial x} = \cos y \cdot \cos(x \cdot \sin y) + -\sin y \cdot \sin(x \cdot \sin y) \cdot x \cdot \cos y$$

b) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = (1+x^2) y e^z = y e^z + x^2 y e^z$

$$\frac{\partial f}{\partial x} = 2xye^z \; ; \; \frac{\partial f}{\partial y} = e^z + x^2 e^z \; ; \; \frac{\partial f}{\partial z} = ye^z + x^2 y e^z$$

$$\frac{\partial f}{\partial x^2} = 2ye^z \; ; \; \frac{\partial f}{\partial x \partial y} = 2xe^z \; ; \; \frac{\partial f}{\partial x \partial z} = 2xye^z$$

$$\frac{\partial f}{\partial y^2} = 0 \; ; \; \frac{\partial f}{\partial y \partial x} = 2xe^z \; ; \; \frac{\partial f}{\partial y \partial z} = e^z + x^2 e^z$$

$$\frac{\partial f}{\partial z^2} = ye^z + x^2 y e^z \; ; \; \frac{\partial f}{\partial z \partial x} = 2xye^z \; ; \; \frac{\partial f}{\partial z \partial y} = e^z + x^2 e^z$$