Lecture 10

Directional derivatives

In the following we consider $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$.

Definition 1. Let $f : A \to \mathbb{R}$, $c \in \operatorname{int} A$ and $v \in \mathbb{R}^n$ a unit vector (that is, $\|v\| = 1$). We say that f is *differentiable in the direction v at c* if

$$\exists \lim_{t \to 0} \frac{f(c + tv) - f(c)}{t} \in \mathbb{R}.$$

In this case, the above limit is called the *directional derivative of f in the direction v at c* and is denoted by $f'(c; v)$.

Remark 1. For $v = e^j$, $j \in \{1, \ldots, n\}$, we obtain the partial derivative of f w.r.t. x_j.

Theorem 1. *Let $f : A \to \mathbb{R}$, $c \in \operatorname{int} A$ and suppose that f is C^1 near c. Then $\forall v \in \mathbb{R}^n$ with $\|v\| = 1$, f is differentiable in the direction v at c and $f'(c; v) = \langle \nabla f(c), v \rangle$.*

Example 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2. \end{cases}$

Remark 2. Let $f : A \to \mathbb{R}$ and $c \in \operatorname{int} A$.

$$f \text{ differentiable in every direction at } c \quad \not\Longrightarrow \quad f \text{ continuous at } c.$$

The gradient (revisited)

Let $A \subseteq \mathbb{R}^2$, $A \neq \emptyset$, $f : A \to \mathbb{R}$. Take $c \in \operatorname{int} A$ and suppose that f is C^1 near c and that c is not a stationary point for f.

<u>Problem</u>: In which direction should we move away from c to get the maximal increase for f?

Riemann integrals

In the following we consider $a, b \in \mathbb{R}$ with $a < b$.

Definition 2. A *partition* of the interval $[a, b]$ is a finite ordered set $P = (x_0, x_1, \ldots, x_n)$ of real numbers such that $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. The intervals $[x_{i-1}, x_i]$ $(i = 1, \ldots, n)$ are called subintervals of the partition P.
The *norm* of P is $\|P\| = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}$ (i.e., the length of the largest subinterval of the partition P).
Suppose that, for each $i = 1, \ldots, n$, ξ_i has been chosen in each subinterval $[x_{i-1}, x_i]$ and denote $\xi = (\xi_1, \ldots, \xi_n)$. Then (P, ξ) is called a *tagged partition* of $[a, b]$.

Definition 3. Let $f : [a, b] \to \mathbb{R}$ and (P, ξ) a tagged partition of $[a, b]$. Then the sum

$$\sigma(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

is called the *Riemann sum* of f w.r.t. the tagged partition (P, ξ).

Definition 4. Let $f : [a, b] \to \mathbb{R}$. We say that f is *Riemann integrable* on $[a, b]$ if there exists $I \in \mathbb{R}$ such that

$$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0 \text{ s.t. } \forall (P, \xi) \text{ tagged partition of } [a, b] \text{ with } \|P\| < \delta,\ |\sigma(f, P, \xi) - I| < \varepsilon. \quad (1)$$

The family of all Riemann integrable functions on $[a, b]$ is denoted by $\mathcal{R}[a, b]$.
If $f \in \mathcal{R}[a, b]$, then $I \in \mathbb{R}$ satisfying (1) is uniquely determined and called the *Riemann integral* (or *definite integral*) of f on $[a, b]$.
Notation: $\displaystyle\int_a^b f(x)dx = \int_a^b f = I.$

Remark 3. (i) If $f : [a,b] \to [0,\infty)$ and $f \in \mathcal{R}[a,b]$, then $\mathcal{A} = \int_a^b f$ is the area under the graph of f (and above the Ox axis).

(ii) If $f : [a,b] \to \mathbb{R}$ is constantly equal to $M \in \mathbb{R}$, then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = M(b-a)$.

(iii) If $f : [a,b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}[a,b]$.

(iv) If $f : [a,b] \to \mathbb{R}$ is monotone, then $f \in \mathcal{R}[a,b]$.

(v) If $f \in \mathcal{R}[a,b]$, then f is bounded.

Theorem 2. *Let $a, b \in \mathbb{R}$, $a < b$, $f, g \in \mathcal{R}[a,b]$ and $\alpha \in \mathbb{R}$. Then*

(i) $f + g \in \mathcal{R}[a,b]$ *and* $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

(ii) $(\alpha f) \in \mathcal{R}[a,b]$ *and* $\int_a^b (\alpha f) = \alpha \int_a^b f$.

(iii) $(f \cdot g) \in \mathcal{R}[a,b]$.

(iv) $|f| \in \mathcal{R}[a,b]$.

(v) If $f \leq g$, *then* $\int_a^b f \leq \int_a^b g$.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ and $c \in (a,b)$. Then*

$$f \in \mathcal{R}[a,b] \quad \Longleftrightarrow \quad f|_{[a,c]} \in \mathcal{R}[a,c] \text{ and } f|_{[c,b]} \in \mathcal{R}[c,b].$$

In this case, $\int_a^b f = \int_a^c f + \int_c^b f$.

Theorem 4 (First Fundamental Theorem of Calculus)**.** *Let $f \in \mathcal{R}[a,b]$. Define $F : [a,b] \to \mathbb{R}$,*

$$F(t) = \int_a^t f.$$

Then F is continuous. Moreover, if f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.

Example 2. Take $f : [-1, 1] \to \mathbb{R}$, $f(x) = \begin{cases} -1, & \text{if } x \in [-1, 0), \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x \in (0, 1]. \end{cases}$

Theorem 5 (Second Fundamental Theorem of Calculus,)**.** *Let $f \in \mathcal{R}[a, b]$. If $F : [a, b] \to \mathbb{R}$ is an antiderivative of f (i.e., $F'(x) = f(x)$ for all $x \in [a, b]$), then*

$$\int_a^b f = F(b) - F(a) \quad \text{(the Leibniz-Newton formula)}.$$