RESOLUTION PREDICATE HW – 5.3

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EX – 5.3

5. Check whether the following formulas are theorems or not using predicate resolution.

3. $U_3 = (\forall x)(\forall y)P(x,y) \boxed{\leftrightarrow} (\exists x)(\forall y)P(x,y)$

THEORETICAL RESULTS
RESOLUTION PROOF METHOD

It's basic aim is to check the **consistency/inconsistency** of a set of clauses.

It is based on syntactic considerations => **syntactic method**

The **validity** of a formula **is proved by contradiction=>** => **refutation method**

Theorem (resolution - a refutation proof method)

Let $U_1, U_2, \ldots, U_n, V$ be first-order formulas.

- **>** $\vdash V$ ($\models V$) if and only if $(\neg V)^c \vdash^{Pr}_{Res} \square$

> $U_1, U_2, \ldots, U_n \vdash V$ if and only if $\{U_1^C, U_2^C, \ldots, U_n^C, (\neg V)^C\} \vdash^{Pr}_{Res} \square$

Remarks:

> All the refinements and strategies of propositional resolution can be used in predicate logic.

> The resolution algorithm for predicate logic is a **semi-decision procedure**.

Theorem:

A predicate formula admits a logical equivalent conjunctive prenex normal form.

The **prenex normal form** is obtained by applying transformations which preserve the logical equivalence, according to the following steps:

Step 1: The connectives '→' and '$\boxed{\leftrightarrow}$' are replaced using the connectives: ¬, ^, ∨

Step 2: The bound variables are renamed such that they will be distinct.

Step 3: Application of infinitary DeMorgan's laws.

Step 4: The extraction of quantifiers in front of the formula.

Step 5: The matrix is transformed into CNF using DeMorgan's laws and the distributive laws.

PRENEX NORMAL FORM

Definition

A predicate formula U is in **prenex normal form** if it has the form: $(Q_1,X_1)\ldots(Q_nX_n)M$, where [here] are quantifiers, and AI is quantifier-free. The sequence $(Q_1,X_1)\ldots(Q_nX_n)$ is called the prefix of the formula U and M is called the **matrix** of the formula U. A predicate formula is in **conjunctive prenex normal** form if it is in prenex normal form and the matrix is in CNF.

Theorem:

A predicate formula admits a logical equivalent conjunctive prenex normal form. **The prenex normal form** is obtained by applying transformations which preserve the logical equivalence, according to the following steps:

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Step 2: The bound variables are renamed such that they will be distinct.

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SKOLEM AND CLAUSAL NORMAL FORMS

Definitions:

Let U be a first-order formula, and $U^P = (Q_1, X_1)\ldots(Q_n X_n)M$ be one of its conjunctive prenex normal form.
A formula in **Skolem normal form**, denoted by U^S corresponds to U and it is obtained as follows:

> For each existential quantifier Q_r from the prefix we apply the transformation:

> if on the left side of Q_r there are no universal quantifiers, then we introduce a new constant **a**, and we replace in M all the occurrences of X_r by **a,** $(Q_r X_r)$ is deleted from the prefix.

> if $Q_{s_1},\ldots,Q_{s_m}$, $1 \leq S_1 < \ldots < S_m < r$, are all the universal quantifiers on the left side of Q_r in the prefix, then we introduce a new m -place function symbol, $\boldsymbol{f}$, and we replace in M all the occurrences of X_r by $\boldsymbol{f}(X_{s_1},\ldots,X_{s_m}).(Q_r X_r)$ is deleted from the prefix.

> The constants and functions used to replace the existentially quantified variables are called **Skolem constants and Skolem functions**. The prefix of the formula U^S contains only universal quantifiers, and the matrix is in conjunctive normal form.

A formula in clausal normal form denoted by U^C is obtained by deleting the prefix of U^S .

SOLUTION

1. Checking if theorem: $U3 = (\forall x)(\forall y)P(x,y) \boxed{\leftrightarrow} (\exists x)(\forall y)P(x,y)$

$|\!-U3$, if and only if: $(\neg U3)^C |\!-^{Pr}_{Res} \square$

2. We are going to study $\neg U3$, and replace $p \boxed{\leftrightarrow} q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\boldsymbol{\neg U3 = \neg((\forall x)(\forall y)P(x,y) \wedge (\exists x)(\forall y)P(x,y)) \wedge \neg(\neg((\forall x)(\forall y)P(x,y)) \wedge \neg((\exists x)(\forall y)P(x,y)))}$

3. Applying DeMorgan's Law: $\neg(\forall x)A(x) = (\exists x)\neg A(x)$

$\boldsymbol{\neg U3 = ((\exists x)(\exists y)\neg P(x,y) \vee (\forall x)(\exists y)\neg P(x,y)) \wedge ((\forall x)(\forall y)\neg P(x,y) \vee (\exists x)(\forall y)\neg P(x,y))}$

4. Replace the variables:

$\boldsymbol{\neg U3 = ((\exists x)(\exists y)\neg P(x,y) \vee (\forall a)(\exists b)\neg P(a,b)) \wedge ((\forall c)(\forall d)\neg P(c,d) \vee (\exists e)(\forall f)\neg P(e,f))}$

5. Extract the variables:

$$\neg U3 = (\exists x)(\exists y)(\exists b)(\exists e)(\forall a)(\forall c)(\forall d)(\forall f)((\neg P(x,y)) \vee \neg P(a,b)) \wedge (P(c,d) \wedge P(e,f)))$$

6. Using the Dremex Form:

$$(\neg U3)^P = (\exists x)(\exists y)(\exists b)(\exists e)(\forall a)(\forall c)(\forall d)(\forall f)((\neg P(x,y)) \vee \neg P(a,b)) \wedge (P(c,d) \wedge P(e,f)))$$

7. Using the skolem form ($x \leftarrow m$, $y \leftarrow n$, $b \leftarrow r$, $e \leftarrow p$):

$$(\neg U3)^S = (\forall a)(\forall c)(\forall d)(\forall f)((\neg f(m,n) \vee \neg f(a,r)) \wedge (f(c,d) \wedge f(p,f))$$

8. Using the Clausal Normal Form:

$$(\neg U3)^C = (\neg f(m,n) \vee \neg f(a,r)) \wedge (f(c,d) \wedge f(p,f))$$

We get the set of clauses: $S_3 = \{C_1 = \neg f(m,n) \vee \neg f(a,r);\ C_2 = f(c,d);\ C_3 = f(p,f)\}$

We get the reolvents: $C_4 = Res^{Pr}_{[c,d \leftarrow m,n]}(C_1, C_2) = \neg f\,(a,r)$

$C_5 = Res^{Pr}_{[a,f \leftarrow p,r]}(C_1, C_3) = \square$

$\Rightarrow (\neg U_3)^C = |\text{-}^{Pr}_{Res}\ \square$, therefore $|\text{-} U_3$

Conclusion:

U_3 is a theorem