Title: Homework, Week 10
Author: Cretu Cristian, 913

0.1 Exercise 1

The tangent plane to the unit sphere:

$$x^2 + y^2 + z^2 = 1$$

We can take $f(x, y, z) = x^2 + y^2 + z^2$ which implies that the gradient $\nabla f = (2x, 2y, 2z)$.

At any point (x_0, y_0, z_0), the tangent plane is given by:

$$(x - x_0)2x_0 + (y - y_0)2y_0 + (z - z_0)2z_0 = 0$$

Simplifying, we get:

$$x_0x + y_0y + z_0z - (x_0^2 + y_0^2 + z_0^2) = 0$$

But $x_0^2 + y_0^2 + z_0^2 = 1$ because the point lies on the unit sphere $\Rightarrow$

$$x_0x + y_0y + z_0z = 1$$

0.2 Exercise 2

Given $f(x, y) = \frac{1}{2}(x^2 + by^2)$

We replace $\nabla f(x, y)$ in our initial equation:

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - s_k \nabla f(x_k, y_k),$$

which results in:

$$(x_{k+1}, y_{k+1}) = (x_k - s_k x_k, y_k - s_k b y_k)$$

We note with $\phi \Rightarrow \phi(s_k) = f(x_{k+1}, y_{k+1})$

Since

$$\frac{d}{ds_k}\phi(s_k) = 0$$

this means that:

$$0 = -x_k(x_k - s_k x_k) - by_k(y_k - s_k b y_k)$$

We move s_k to the other side, and we get:

$$s_k = \frac{x_k^2 + by_k^2}{x_k^2 + b^2 y_k^2}$$

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

[3]:
```
def f(x, y, b):
    return 0.5*(x**2+b*y**2)
```

[11]:
```
def df(x, y, b):
    return np.array([x, b*y])
```

[35]:
```
def sk(x, y, b):
    if b**3 * y**2 + x**2 == 0:
        return 0.0
    return (b**2 * y**2 + x**2) / (b**3 * y**2 + x**2)
```

[5]:
```
b_val = [1, 2 ** -1, 5 ** -1, 10 ** -1]
```

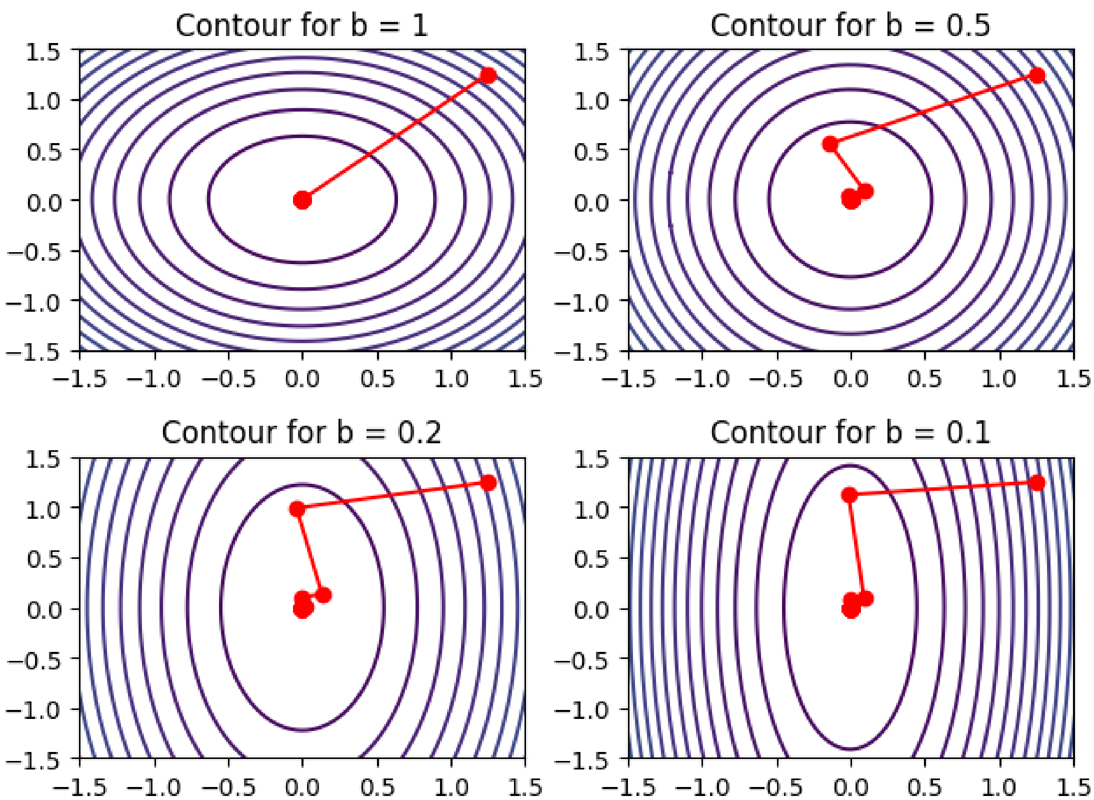
[27]:
```
plt.figure(figsize=(12,8))
x_values = np.linspace(-1.5, 1.5, 400)
y_values = np.linspace(-1.5, 1.5, 400)
x, y = np.meshgrid(x_values, y_values)
```

```
<Figure size 1200x800 with 0 Axes>
```

[20]:
```
def gradient_descent(x, y, b, iterations=100):
    points = [(x,y)]
    for _ in range(iterations):
        grad = df(x,y,b)
        s_k = sk(x,y,b)
        x, y = x - s_k * grad[0], y - s_k * grad[1]
        points.append((x,y))
    return np.array(points)
```

[43]:
```
for i, b in enumerate(b_val, 1):
    z = f(x, y, b)

    x_start = 1.25
    y_start = 1.25

    points = gradient_descent(x_start,y_start,b)

    # 4 plots
    plt.subplot(2,2,i)
    plt.contour(x,y,z, levels=50)
    plt.plot(points[:, 0], points[:, 1], 'ro-')
    plt.title(f'Contour for b = {b}')
    plt.xlim(-1.5, 1.5)
    plt.ylim(-1.5, 1.5)
```

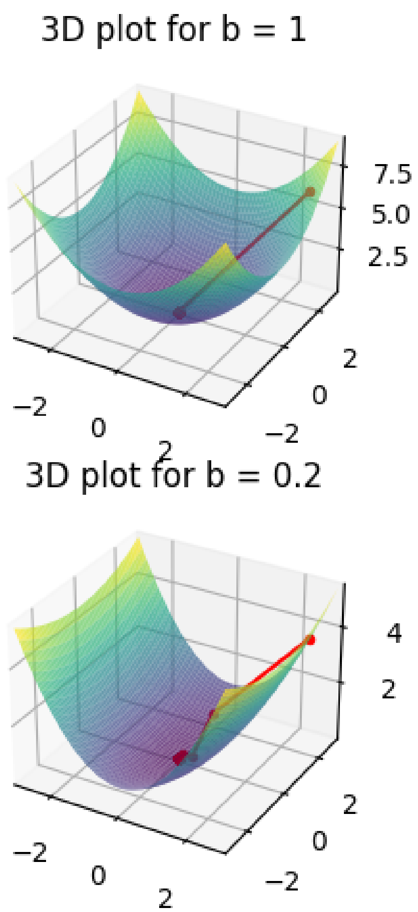
```
plt.tight_layout()
plt.show()
```



[40]:

```
from mpl_toolkits.mplot3d import Axes3D

for i, b in enumerate(b_val, 1):
    ax = plt.subplot(2, 2, i, projection='3d')

    x, y = np.meshgrid(x_values, y_values)
    z = f(x, y, b)

    ax.plot_surface(x, y, z, cmap='viridis', alpha=0.7)

    x_start = 2.5
    y_start = 2.5
    points = gradient_descent(x_start, y_start, b)

    z_points = f(points[:, 0], points[:, 1], b)
    ax.plot(points[:, 0], points[:, 1], z_points, 'r.-')
```

```
    ax.set_title(f'3D plot for b = {b}')
    ax.set_xlim(-3, 3)
    ax.set_ylim(-3, 3)
    ax.set_zlim(z.min(), z.max())

plt.tight_layout()
plt.show()
```



As b gets smaller, we see that the GD takes more steps to reach the minimum point of the function