Title: Homework, Week 8
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0.1 Exercise 2

For $x, y \in \mathbb{R}^n$ prove that the following statements are equivalent:

(a) $\langle x, y\rangle = 0$.

(b) $\|x + y\| = \|x - y\|$.

(c) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Proof:

$(a) \Rightarrow (b)$: Assume $\langle x, y\rangle = 0$, then

$$\begin{aligned}
\|x + y\|^2 &= (x + y)(x + y) \\
&= xx + 2xy + yy \\
&= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\
&= \|x\|^2 + 2 \cdot 0 + \|y\|^2 \\
&= \|x\|^2 + \|y\|^2.
\end{aligned}$$

Similarly: $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2 * 0$. Which results that $\|x + y\| = \|x - y\|$.

$(b) \Rightarrow (c)$: Assume $\|x + y\| = \|x - y\|$, then

$$\begin{aligned}
\|x + y\|^2 &= \|x - y\|^2 \\
\langle x + y, x + y\rangle &= \langle x - y, x - y\rangle \\
\|x\|^2 + 2\langle x, y\rangle + \|y\|^2 &= \|x\|^2 - 2\langle x, y\rangle + \|y\|^2.
\end{aligned}$$

Which means that $2\langle x, y\rangle = -2\langle x, y\rangle \Rightarrow \langle x, y\rangle = 0$. Similarly to (a), this suggests that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

$(c) \Rightarrow (a)$: Assume $\|x + y\|^2 = \|x\|^2 + \|y\|^2$, then

$$\begin{aligned}
\langle x + y, x + y\rangle &= \|x\|^2 + \|y\|^2 \\
\|x\|^2 + 2\langle x, y\rangle + \|y\|^2 &= \|x\|^2 + \|y\|^2.
\end{aligned}$$

Which means that $2\langle x, y\rangle = 0$, and resulting that $\langle x, y\rangle = 0$.

Since (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a), this proves that the 3 statements are equivalent.

0.2 Exercise 6

[11]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

[34]:
```
def p_norm(x, y, p):
    return np.sum(np.abs(np.array([x, y]))**p, axis=0)**(1/p)
```

[3]:

```python
p_values = [1.25, 1.5, 3.0, 8.0]
n_points = 10000
```

[63]:

```python
# init the plot
fig, axs = plt.subplots(1, len(p_values), figsize=(20, 5))

xx, yy = np.meshgrid(np.linspace(-1.5, 1.5, 400), np.linspace(-1.5, 1.5, 400))


for i, p_value in enumerate(p_values):
    # generate random points
    x = np.random.uniform(-1.25, 1.25, n_points)
    y = np.random.uniform(-1.25, 1.25, n_points)

    # calculate the p-norm
    norms = p_norm(x, y, p_value)
    is_inside = norms <= 1
    axs[i].scatter(x[is_inside], y[is_inside], s=1, c='green')

    norms_grid = p_norm(xx, yy, p_value)
    axs[i].contour(xx, yy, norms_grid, levels=[1.0], colors='b')

    axs[i].scatter(x[~is_inside], y[~is_inside], s=3, c='r', alpha=0.05)
    axs[i].set_aspect('equal', adjustable='box')
    axs[i].legend(['inside', 'outside'])
    axs[i].set_title(f'Unit Ball in R^2 for p={p_value}')
    axs[i].set_xlim([-1.5, 1.5])
    axs[i].set_ylim([-1.5, 1.5])

plt.tight_layout()
plt.show()
```