1.2.1. How many solutions have each of the following problems:

a) $x'' + t^2 x = 0,\ x(0) = 0.$

$x'' + t^2 x = 0$ is a second order linear homogenous differential equation. In order to have a single solution, there should be two conditions. However, we only get one: $x(0) = 0$.

So, this problem has an infinite number of solutions. Also, if we had $x'(0) = v$ as a condition, we would have had an ivp with different solutions as v changes. Hence the infinite solutions in the absence of this condition.

b) $x'' + t^2 x = 0,\ x(0) = 0,\ x'(0) = 0$

$x'' + t^2 x = 0$ is a second order linear homogenous differential equation. The problem has two conditions $x(0) = 0$ and $x'(0) = 0$, so we will get a unique solution.

c) $x'' + t^2 x = 0,\ x(0) = 0,\ x'(0) = 0,\ x''(0) = 1.$

$x'' + t^2 x = 0$ is a second order linear homogenous differential equation. However, the problem is not correctly defined, since it has an extra condition, $x''(0) = 1$, while the scalar differential equation is of first order.

So, this problem doesn't have any solutions.

Muraru A.

1.2.5 a) Find a particular solution of the form $x_p = ae^t$ (with $a \in \mathbb{R}$) for the equation $x' - 2x = e^t$.

$x' - 2x = e^t$ is a first-order linear non-homogenous differential equation. We apply the fundamental theorem :

$$x = x_h + x_p$$

First, we find the general solution of $x' - 2x = 0$.

$$x' = 2x$$

$$x' \to \frac{dx}{dt}$$

$$\frac{dx}{dt} = 2x \Rightarrow \int \frac{dx}{x} = \int 2dt$$

$$\Rightarrow \ln|x| = 2t + c,\ c \in \mathbb{R}$$

$$\Rightarrow |x| = e^{2t+c} \Rightarrow x = \begin{cases} \pm e^c e^{2t} \\ x = 0 \end{cases}$$

$$\Rightarrow x_h = c_1 e^{2t}$$

Now, we apply the Lagrange method :

$$x_p = \varphi(t) e^{2t}$$

$$x_p' - 2x_p = e^t$$

We know that $x_p = ae^t \Rightarrow x_p' = ae^t$

$$\Rightarrow \quad ae^t - 2ae^t = e^t$$

$$\Rightarrow \quad -ae^t = e^t \Rightarrow a = -1$$

So, a particular solution is $x_p = -e^t$.

b) Find a particular solution of the form $x_p = be^{-t}$ (with $b \in \mathbb{R}$) for the equation $x' - 2x = e^{-t}$

$$x_p' - 2x_p = e^{-t}$$

We know that $x_p = be^{-t}, b \in \mathbb{R} \Rightarrow x_p' = -be^{-t}$

$$\Rightarrow -be^{-t} - 2be^{-t} = e^{-t}$$

$$-3be^{-t} = e^{-t}$$

$$-3b = 1 \Rightarrow b = -\frac{1}{3}$$

So, a particular solution is $x_p = -\frac{1}{3}e^{-t}$.

c) Using the Superposition Principle, and a) and b), find a particular solution for the equation $x' - 2x = 5e^{t} - 3e^{-t}$.

$x' - 2x = 5e^{t} - 3e^{-t}$ is a superposition (with coefficients $c_1 = 5$ and $c_2 = -3$) of the inputs from a) and b). Therefore,

$$x_p(t) = c_1 \cdot (-e^{t}) + c_2 \cdot \left(-\frac{1}{3}e^{-t}\right) =$$
$$= -5e^{t} + e^{-t}$$

is a particular solution for the equation: $x' - 2x = 5e^{t} - 3e^{-t}$.

d) Find the general solution of $x' - 2x = 5e^{t} - 3e^{-t}$

We know from a) that the general solution of $x' - 2x = 0$ is $x = \begin{cases} \pm e^{c} e^{2t} \\ 0 \end{cases} \Rightarrow x_h = c_1 e^{2t}$.

The fundamental theorem tells us that:

$x = x_h + x_p$. We know x_p from

$\Rightarrow x = c_1 e^{2t} - 5e^{t} + e^{-t}$, $c_1 \in \mathbb{R}$. is the general solution of $x' - 2x = 5e^{t} - 3e^{-t}$.

1.3.4 Find the general solution of $x' - x = e^{t-1}$. Justify the result in two ways.

This is a first order linear non-homogeneous differential equation with constant coefficients. The non-homogeneous part is $f(t) = e^{t-1}$. We can take $I = \mathbb{R}$.

I. The integrating factor method.

$$\mu(t) = e^{\int -dt} = e^{-t}$$

We multiply the equation by the integrating factor and obtain:

$$\left.\begin{array}{l} x'e^{-t} - xe^{-t} = e^{-1} \\ \text{But: } (xe^{-t})' = x'e^{-t} + x(-e^{-t}) = e^{-t}(x' - x) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (xe^{-t})' = e^{-1}$$

We integrate with respect to t and obtain:

$$xe^{-t} = \frac{1}{e} \cdot t + c, \; c \in \mathbb{R}$$

Hence, the general solution of the given equation is:

$$x = ce^{t} + te^{t-1}$$

II. The separation of variables method + Lagrange

We write the linear homogeneous equation associated:

$$x' - x = 0.$$

For this, $x = 0$ is a solution. We look for the non-null solutions by separating the variables. We have:

$$\frac{dx}{dt} = x \Leftrightarrow \frac{dx}{x} = dt.$$

After integration, we have that:

$\ln|x| = t + c \Rightarrow x = \pm e^{c}e^{t}, \; c \in \mathbb{R}$. Recall that $x = 0$ is another solution. Then, the general solution is

$$x_h = ce^{t}, \; c \in \mathbb{R}.$$

Now we apply the Lagrange method to find a particular solution, denoted x_p, of the given equation, so we look for a function $\varphi \in C^1(\mathbb{R})$ such that

$$x_p = \varphi(t)e^t.$$

After replacing we obtain that

$$x_p' - x_p = e^{t-1}$$

$$\Leftrightarrow (\varphi(t)e^t)' - \varphi(t)e^t = e^{t-1}$$

$$\varphi'(t)e^t + \cancel{\varphi(t)e^t} - \cancel{\varphi(t)e^t} = e^{t-1}$$

$$\Rightarrow \varphi'(t) = e^{-1}$$

Such a function is $\varphi(t) = e^{-1}.t + \mathcal{C}$

Take $\varphi(t) = t \cdot e^{-1} \,/ \cdot e^t$

$$\Rightarrow x_p = te^{t-1}$$

We finally deduce that the general solution is

$$x = ce^t + te^{t-1}.$$

1.7.5. Let $t \in \mathbb{R}$. Using the Euler's formula, compute e^{it}, $e^{i\pi}$, $e^{i\pi/2}$, $e^{(-1+i)t}$ //

Euler's formula: $e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$

$e^{it} = e^{0}(\cos t + i\sin t) = \cos t + i\sin t$

$e^{i\pi} = e^{0}(\cos\pi + i\sin\pi) = -1$

$e^{i\frac{\pi}{2}} = e^{0}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i$.

$e^{(-1+i)t} = e^{-t+it} = e^{-t}(\cos t + i\sin t) = \frac{\cos t + i\sin t}{e^{t}}$

1.7.8. Find the linear homogenous differential equation of minimal order that has as solution the function $1+t(1+e^{-t})$

$1+t(1+e^{-t}) = 1+t+te^{-t}$.

We can see that 1 and t are solutions, so e^0 and $t \cdot e^0$ are solutions. This means that $r=0$ is a double root.

Also, we see that te^{-t} is a solution. This means that e^{-t} is also a solution, so $r=-1$ is a double root.

We get the following characteristic equation:

$(r+1)^2 r^2 = 0$

$\Leftrightarrow (r^2+2r+1) r^2 = 0$

$\Leftrightarrow r^4 + 2r^3 + r^2 = 0$

$\Rightarrow$ We get the following linear homogenous differential equation that has as solution the function $1+t(1+e^{-t})$:

$x^{(4)} + 2x''' + x'' = 0$, which is of minimum order, because it only contains the solutions visible in our function $1+t(1+e^{-t})$ (we didn't insert other roots in the characteristic equation than necessary)

1.7.9. Let $k, \eta \in \mathbb{R}$ be fixed parameters. Find the solution of the IVP

$$x' = k(21 - x), \quad x(0) = \eta$$

$$x' = k(21 - x)$$
$$x' + kx = 21k.$$

First, we find the general solution of $x' + kx = 0$.

$$x' = -kx$$
$$\frac{dx}{dt} = -k \cdot x$$
$$\frac{dx}{x} = -k\,dt$$
$$\int \frac{dx}{x} = \int -k\,dt$$
$$\Rightarrow \ln|x| = -kt + c.$$
$$|x| = e^{-kt+c} \qquad \begin{cases} x = \pm e^{c} e^{-kt} \\ x = 0. \end{cases}$$
$$\Rightarrow x_h = c_1 e^{-kt}$$

We note that $x_p = 21$ verifies the differential equation.

$\Rightarrow$ the general solution is:

$$x = c_1 e^{-kt} + 21$$

We know that $x(0) = \eta$

$$\Rightarrow \quad c_1 + 21 = \eta \Rightarrow c_1 = \eta - 21$$

So, the solution of the IVP is

$$x = (\eta - 21) e^{-kt} + 21.$$

1.7.10. We consider the equation $x''-x=te^{-2t}$.

a) Find a particular solution of the form $x_p(t)=(at+b)e^{-2t}$, $a,b\in\mathbb{R}$

$$x_p' = ae^{-2t}-2(at+b)e^{-2t}$$

$$x_p'' = -2ae^{-2t}-2ae^{-2t}+4(at+b)e^{-2t}$$

$$x_p''-x=te^{-2t}$$

$$\Leftrightarrow -2ae^{-2t}-2ae^{-2t}+4(at+b)e^{-2t}-(at+b)e^{-2t}=te^{-2t}$$

$$e^{-2t}(-4a+4at+4b-at-b)=te^{-2t}$$

$$e^{-2t}(3b-4a)+e^{-2t}\cdot t(3a-1)=0$$

$$\Rightarrow \begin{cases} 3b-4a=0 \\ 3a-1=0 \end{cases} \Rightarrow a=\frac{1}{3} \quad \Rightarrow b=\frac{4}{9}.$$

$\Rightarrow x_p(t)=\left(\frac{1}{3}t+\frac{4}{9}\right)e^{-2t}$ is a particular solution.

b) Find the general solution

$x''-x=te^{-2t}$ is a second order linear non-homogenous differential equation. We apply the fundamental theorem:

$$x=x_h+x_p$$

First, we find the general solution of $x''-x=0$. We write the characteristic equation:

$$r^2-1=0 \quad \begin{cases} r_1=1 \vdash e^{t} \\ r_2=-1 \vdash e^{-t} \end{cases}$$

$$\Rightarrow x=c_1e^{t}+c_2e^{-t}=x_h.$$

$\Rightarrow$ The general solution is:

$$x=x_h+x_p=c_1e^{t}+c_2e^{-t}+\left(\frac{1}{3}t+\frac{4}{9}\right)e^{-2t}.$$

c) Find the solution that satisfies the initial conditions $x(0)=0, x'(0)=0$.

$$x(0)=c_1+c_2+\frac{4}{9}=0 \Rightarrow c_1+c_2=-\frac{4}{9}$$

$$x'(t)=c_1e^t-c_2e^{-t}+\frac{1}{3}e^{-2t}-2\left(\frac{4}{9}+\frac{1}{3}t\right)e^{-2t}$$

$$x'(0)=c_1-c_2+\frac{1}{3}-\frac{8}{9}=c_1-c_2-\frac{5}{9}=0 \Rightarrow c_1-c_2=\frac{5}{9}$$

$$\begin{cases} c_1+c_2=-\frac{4}{9} \\ c_1-c_2=\frac{5}{9} \end{cases} \quad (+)$$

$$2c_1=\frac{1}{9} \Rightarrow c_1=\frac{1}{18}$$

$$\frac{1}{18}-c_2=\frac{5}{9} \Rightarrow c_2=\frac{1}{18}-\frac{10}{18}=-\frac{9}{18}=-\frac{1}{2}$$

$\Rightarrow$ The solution that satisfies this IVP is:

$$x=\frac{1}{18}e^t-\frac{1}{2}e^{-t}+\left(\frac{1}{3}t+\frac{4}{9}\right)e^{-2t}.$$

1.7.12. Let $\mathcal{L}: C^2(\mathbb{R}) \to C(\mathbb{R})$ be defined by
$\mathcal{L}(x) = x'' - 2x' + x, \ \forall x \in C^2(\mathbb{R})$.

a) Prove that $\mathcal{L}$ is a linear map. What is the dimension of its kernel?

b) Find the general solution of the equation $x'' - 2x' + x = \cos 2t$ knowing that it has a particular solution of the form $a\cos 2t + b\sin 2t$, for some $a, b \in \mathbb{R}$.

c) Let $f_1(t) = e^{2t}$ and $f_2(t) = e^{-2t}$ for all $t \in \mathbb{R}$. Find a particular solution of the equation $\mathcal{L}(x) = 3f_1 + 5f_2$.

---

a) $\mathcal{L}: C^2(\mathbb{R}) \to C(\mathbb{R})$ is a map, $C^2(\mathbb{R})$ and $C(\mathbb{R})$ are linear spaces with the usual addition operation and multiplication with reals. In order to show it is linear, we prove that
$\mathcal{L}(\alpha x + \beta y) = \alpha \mathcal{L}(x) + \beta \mathcal{L}(y), \ \forall x, y \in C^2(\mathbb{R}), \forall \alpha, \beta \in \mathbb{R}$

$$\mathcal{L}(\alpha x + \beta y) = (\alpha x + \beta y)'' - 2(\alpha x + \beta y)' + (\alpha x + \beta y) =$$

$$= \alpha x'' + \beta y'' - 2\alpha x' - 2\beta y' + \alpha x + \beta y =$$

$$= \alpha(x'' - 2x' + x) + \beta(y'' - 2y' + y) =$$

$$= \alpha \mathcal{L}(x) + \beta \mathcal{L}(y)$$

$\Rightarrow \mathcal{L}$ is a linear map.

Now, we will find the dimension of its kernel:

$\mathbb{R}^2$ is a vector space of dimension 2. Fix $t_0 \in \mathbb{R}$.

Let $T: \ker \mathcal{L} \to \mathbb{R}^2$ be defined by $T(\varphi) = \begin{pmatrix} \varphi(t_0) \\ \varphi'(t_0) \end{pmatrix}$

$\forall \varphi \in \ker \mathcal{L}$.

T bijective $\Leftrightarrow \forall \eta \in \mathbb{R}^2 \ \exists! \ \varphi \in \ker \mathcal{L}$ s.t. $T(\varphi) = \eta$.

$\Leftrightarrow \forall \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^n \ \exists! \ \varphi \in C^2(\mathbb{R})$ s.t. $\mathcal{L}(\varphi) = 0$.

$$\Leftrightarrow \begin{cases} \varphi(t_0) = \eta_1 \\ \varphi'(t_0) = \eta_2 \end{cases}$$

This is true by the existence and uniqueness theorem

T is linear $\iff T(\alpha_1 \varphi_1 + \alpha_2 \varphi_2) = \alpha_1 T(\varphi_1) + \alpha_2 T(\varphi_2)$, $\forall \alpha_1, \alpha_2 \in \mathbb{R}$, $\forall \varphi_1, \varphi_2 \in \ker \mathcal{L}$.

$$T(\alpha_1 \varphi_1 + \alpha_2 \varphi_2) = \begin{pmatrix} (\alpha_1\varphi_1+\alpha_2\varphi_2)(t_0) \\ (\alpha_1\varphi_1+\alpha_2\varphi_2)'(t_0) \end{pmatrix} =$$

$$= \alpha_1 T(\varphi_1) + \alpha_2 T(\varphi_2).$$

T bijective and linear $\Rightarrow$ T is an isomorphism between $\ker \mathcal{L}$ and $\mathbb{R}^2$ $\Rightarrow$ dimension of $\ker \mathcal{L}$ is 2.

b) $x'' - 2x' + x = \cos 2t$.

$x_p = a\cos 2t + b\sin 2t$.

First, we find the general solution of:

$x'' - 2x' + x = 0$.

We write the characteristic equation:

$r^2 - 2r + 1 = 0$.

$(r-1)^2 = 0 \Rightarrow \underset{\text{double-root}}{r_{1,2} = 1} \vdash e^t, te^t$.

$\Rightarrow x = c_1 e^t + c_2 t e^t = x_h$.

Now, we calculate a particular solution:

$x_p'' - 2x_p' + x_p = \cos 2t \quad (1)$

$x_p = a\cos 2t + b\sin 2t \Rightarrow x_p' = -2a\sin 2t + 2b\cos 2t$

$x_p'' = -4a\cos 2t - 4b\sin 2t$.

$\overset{(1)}{\Rightarrow} -4a\cos 2t - 4b\sin 2t + 4a\sin 2t - 4b\cos 2t + a\cos 2t + b\sin 2t = \cos 2t$

$\cos 2t(-4a - 4b + a - 1) + \sin 2t(-4b + 4a + b) = 0$.

$\cos 2t(-3a - 4b - 1) + \sin 2t(-3b + 4a) = 0$

$$\begin{cases} -3a - 4b - 1 = 0 \;/\cdot 4 \\ -3b + 4a = 0 \;/\cdot 3 \end{cases} \iff \begin{cases} -12a - 16b - 4 = 0 \\ -9b + 12a = 0 \end{cases} \quad (+)$$

$-25b - 4 = 0 \Rightarrow b = \frac{-4}{25}$

$\frac{36}{25} + 12a = 0 \Rightarrow a = -\frac{36}{25 \cdot 12} = -\frac{3}{25}$

$\Rightarrow x_p = -\frac{3}{25}\cos 2t - \frac{4}{25}\sin 2t$

$\Rightarrow x = x_h + x_p = c_1 e^t + c_2 t e^t - \frac{3}{25}\cos 2t - \frac{4}{25}\sin 2t$

c) $\mathcal{L}(x) = 3e^{2t} + 5e^{-2t}$

$\Leftrightarrow x'' - 2x' + x = 3e^{2t} + 5e^{-2t}$

We will use the undetermined coefficients method. We denote the characteristic polynomial:

$l(r) = r^2 - 2r + 1.$

I $l(r) = f_1.$

$\Leftrightarrow r^2 - 2r + 1 = 3e^{2t}.$

We have to check whether $r = 2$ is a root of $l(r)$. It is not a root. Then we look for $x_p = ae^{2t}$, where $a \in \mathbb{R}$ has to be determined.

$x_p'' - 2x_p' + x_p = 3e^{2t}$

~~$ae^{2t} = 3e^{2t} \Rightarrow a = 3 \Rightarrow x_p = 3e^{2t}.$~~

$x_p = ae^{2t}, \; x_p' = 2ae^{2t}, \; x_p'' = 4ae^{2t}.$

$\Rightarrow \cancel{4ae^{2t}} - \cancel{4ae^{2t}} + ae^{2t} = 3e^{2t} \Rightarrow a = 3 \Rightarrow x_p = 3e^{2t}.$

II $l(r) = f_2.$

$\Leftrightarrow r^2 - 2r + 1 = 5e^{-2t}.$

We have to check whether $r = -2$ is a root of $l(r)$. It is not a root. Then we look for $x_p = ae^{-2t}$, where $a \in \mathbb{R}$ has to be determined.

$x_p'' - 2x_p' + x_p = 5e^{-2t}$

$x_p = ae^{-2t}, \; x_p' = -2ae^{-2t}, \; x_p'' = 4ae^{-2t}$

$\Rightarrow 4ae^{-2t} + 4ae^{-2t} + ae^{-2t} = 5e^{-2t}$

$9ae^{-2t} = 5e^{-2t} \Rightarrow a = \frac{5}{9} \Rightarrow x_p = \frac{5}{9}e^{-2t}.$

Denote $f = f_1 + f_2 \in C(\mathbb{R})$.

$\mathcal{L}(x) = f_1$ has the particular solution $x_{p_1} = 3e^{2t}$

$\mathcal{L}(x) = f_2$ has the particular solution $x_{p_2} = \frac{5}{9}e^{-2t}$

$\Rightarrow$ A particular solution of $\mathcal{L}(x) = f$ is

$x_p = x_{p_1} + x_{p_2} = 3e^{2t} + \frac{5}{9}e^{-2t}$, by the superposition principle.

1.7.19 We consider the differential equation

$$x'' + 4x = \cos 2t.$$

a) Find a solution of the form $x_p = t(a\cos 2t + b\sin 2t)$, with $a, b \in \mathbb{R}$.

b) Find its general solution.

c) Describe the motion of a spring-mass system governed by this equation.

a) $x_p = t(a\cos 2t + b\sin 2t)$

$x_p' = a\cos 2t + b\sin 2t + t(-2a\sin 2t + 2b\cos 2t)$

$x_p'' = -2a\sin 2t + 2b\cos 2t - 2a\sin 2t + 2b\cos 2t$
$\quad -4ta\cos 2t - 4tb\sin 2t$

$x'' + 4x = \cos 2t$

$x_p'' + 4x_p = \cos 2t$

$-2a\sin 2t + 2b\cos 2t - 2a\sin 2t + 2b\cos 2t - \cancel{4at\cos 2t} - \cancel{4bt\sin 2t}$
$+ 4at\cancel{\cos 2t} + 4b\cancel{t\sin 2t} = \cos 2t.$

$\sin 2t(-2a-2a) + \cos 2t(2b+2b-1) = 0.$

$$\Rightarrow \begin{cases} -4a = 0 \\ 4b - 1 = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = \frac{1}{4} \end{cases}$$

$\Rightarrow x_p = t \cdot \frac{1}{4}\sin 2t.$

b) We find the general solution of the equation: $x'' + 4x = 0$.

We write the characteristic equation:

$r^2 + 4 = 0 \Rightarrow r_1 = 2i, r_2 = -2i \vdash \cos 2t, \sin 2t$

$\Rightarrow x = c_1\cos 2t + c_2\sin 2t = x_h$

By the Fundamental Theorem for linear non-homogenous differential equations, we have that

$x = x_h + x_p$

$\Rightarrow x = c_1\cos 2t + c_2\sin 2t + \frac{t}{4}\sin 2t.$

c) $x'' + 4x = \cos 2t$

This is the equation of an undamped motion with external force.

$x'' + \frac{k}{m} x = A \cos \omega t$ → general form

Here, $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{4} = 2 = \omega_0$

⇒ A particular solution is $x_p = \frac{1}{4} t \sin 2t$ (as we've seen at a)).

The general solution is:

$x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4} t \sin 2t$.

Any function of the above form is unbounded. In this case oscillations occur with an amplitude that increases to ∞ (we have resonance).

1.7.24. We say that a differential equation exhibits resonance when all its solutions are unbounded. For what values of the mass m will

$mx'' + 25x = 12\cos(36\pi t)$ exhibit resonance?

Assuming the mass is not null (which is true), we divide the differential equation by m and obtain:

$$x'' + \frac{25}{m}x = \frac{12}{m}\cos(36\pi t).$$

In order to have resonance, we need $\omega_0 = \omega$.

$$\left.\begin{array}{l} \omega = 36\pi. \\ \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{25}{m}} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \sqrt{\frac{25}{m}} = 36\pi \,\Big|^2$$

$$\frac{25}{m} = 1296\pi^2 \Rightarrow m = \frac{25}{1296\pi^2}$$

For this value of the mass, we will get resonance. The general solution will be:

$$x = c_1\cos(36\pi t) + c_2\sin(36\pi t) + \frac{1}{72\pi}t\sin(36\pi t).$$

1.7.25. Find the general solution of $\ddot{\theta}+\dot{\theta}+\theta=0$. Prove that $\lim_{t\to\infty}\theta(t)=0$ for any solution θ of this differential equation.

//

$\ddot{\theta}+\dot{\theta}+\theta=0$.

We write the characteristic equation:

$r^2+r+1=0$.

$r_{1,2}=-\frac{1}{2}\pm\frac{i\sqrt{3}}{2} \vdash e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t,\ e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

$\Rightarrow x=c_1e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t+c_2e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

$\lim_{t\to\infty}x=\lim_{t\to\infty}\left(c_1e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t+c_2e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t\right)=$

$=c_1\lim_{t\to\infty}\left(\underbrace{e^{-\frac{1}{2}t}}_{\to 0}\underbrace{\cos\frac{\sqrt{3}}{2}t}_{\in[-1,1]}\right)+c_2\lim_{t\to\infty}\left(\underbrace{e^{-\frac{1}{2}t}}_{\to 0}\underbrace{\sin\frac{\sqrt{3}}{2}t}_{\in[-1,1]}\right)=$

$=0+0=0$

$\Rightarrow \lim_{t\to\infty}\theta(t)=0$, for any solution θ of this differential equation.

1.7.29. We consider the differential equation

$$t^2x'' + 2tx' - 2x = 0,\ t \in (0,\infty)$$

a) Find solutions of the form $x(t) = t^r$, where $r \in \mathbb{R}$ has to be determined

b) Specify its type and find its general solution

c) Find the solution of the IVP

$$t^2x'' + 2tx' - 2x = 0,\ x(1) = 0,\ x'(1) = 1$$

---

a) $x(t) = t^r$

$x'(t) = rt^{r-1}$

$x''(t) = (r^2 - r)t^{r-2}$

$t^2x'' + 2tx' - 2x = 0$

$\Leftrightarrow t^2(r^2 - r)t^{r-2} + 2trt^{r-1} - 2t^r = 0$

$(r^2 - r)t^r + 2rt^r - 2t^r = 0$

$t^r(r^2 - r + 2r - 2) = 0$

$t^r(r^2 + r - 2) = 0$

$\Rightarrow r^2 + r - 2 = 0$

$\Delta = 9$

$r_{1,2} = \frac{-1 \pm 3}{2}$ $\begin{cases} r_1 = -2 \Rightarrow x_1 = t^{-2} \rightarrow \text{solution} \\ r_2 = 1 \Rightarrow x_2 = t \rightarrow \text{solution} \end{cases}$

b) $t^2x'' + 2tx' - 2x = 0$ is a linear homogeneous differential equation of second order with non-constant coefficients.

We divide by t^2 and obtain: $(t \neq 0)$

$x'' + \frac{2}{t}x' - \frac{2}{t^2}x = 0.$ (1)

We make the substitution $x = ty$.

$x' = y + ty'$

$x'' = y' + y' + ty'' = 2y' + ty''$.

We introduce these in equation (1) and obtain:

$$2y' + ty'' + \frac{2}{t}(y + ty') - \frac{2}{t^2}\cdot ty = 0.$$

$$\Leftrightarrow 2y' + ty'' + \cancel{\frac{2}{t}y} + 2y' - \cancel{\frac{2}{t}\cdot y} = 0.$$

$$ty'' + 4y' = 0.$$

We make a second change of variable:

$$u = y'$$

and obtain:

$$tu' + 4u = 0$$

$$t\cdot\frac{du}{dt} = -4u$$

$$\frac{du}{u} = -\frac{4}{t}\,dt$$

$$\int\frac{du}{u} = -4\int\frac{dt}{t}$$

$$\ln|u| = -4\ln|t| \Rightarrow \begin{cases} u = \pm e^{-4\ln t} \\ u = 0 \end{cases}$$

$\Rightarrow$ The general solution of $tu' + 4u = 0$ is

$$u = c_1 e^{-4\ln t} = \frac{c_1}{t^4},\ c_1 \in \mathbb{R}$$

Since $u = y'$, we find y by integrating u.

Hence:

$$y = -\frac{c_1}{3}t^{-3} + c_2,\ c_1, c_2 \in \mathbb{R}.$$

Using $x = t\,y$ we find

$$x = -\frac{c_1}{3}\cdot t^{-2} + t\,c_2 =$$

$$= k_1 t^{-2} + k_2 t,\ k_1, k_2 \in \mathbb{R}$$

c) Find the solution of the IVP:

$$t^2x'' + 2tx' - 2x = 0, \quad x(1) = 0, x'(1) = 1.$$

The general solution of this equation, computed at b), is:

$$x = k_1 t^{-2} + k_2 t$$

$$x' = -\frac{2k_1}{t^3} + k_2$$

$$x(1) = 0 \Leftrightarrow k_1 + k_2 = 0 \;|\cdot 2$$

$$x'(1) = 1 \Leftrightarrow -2k_1 + k_2 = 1 \quad (+)$$

$$3k_2 = 1 \Rightarrow k_2 = \frac{1}{3} \Rightarrow k_1 = -\frac{1}{3}$$

$\Rightarrow x = -\frac{1}{3}t^{-2} + \frac{1}{3}t$ is the solution of the IVP.

1.7.34. We use the notation $\mathcal{L}(x) = x'' + 25x$.

(ii) Find the solution of the IVP

$\mathcal{L}(x) = 0,\ x(0) = 0,\ x'(0) = 1.$

Represent this integral curve and describe its long-term behavior.

We find the general solution of $\mathcal{L}(x) = 0$.

$x'' + 25x = 0.$

We write the characteristic equation:

$r^2 + 25 = 0 \Rightarrow r_{1,2} = \pm 5i \vdash \cos 5t, \sin 5t$

$\Rightarrow x = c_1 \cos 5t + c_2 \sin 5t.$

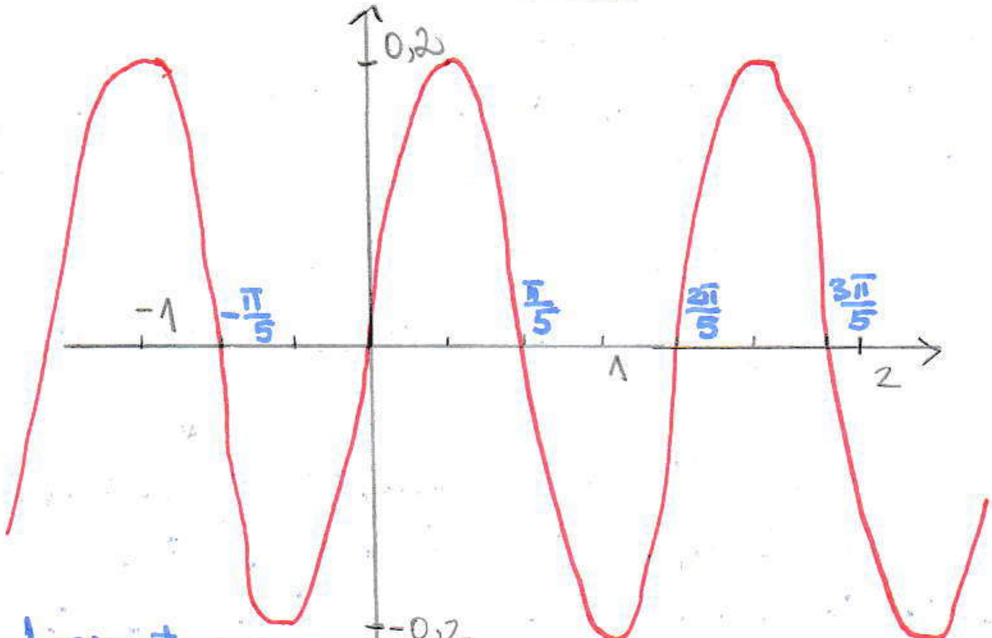
$x(0) = 0 \Leftrightarrow c_1 = 0$

$x'(0) = 1 \Leftrightarrow (-5c_1 \sin(5t) + 5c_2 \cos(5t))(0) = 1$

$\Leftrightarrow 5c_2 = 1 \Rightarrow c_2 = \frac{1}{5}$

$\Rightarrow x = \frac{1}{5} \sin 5t$ is the solution of the IVP.

We see that: $-\frac{1}{5}$ is a lower bound, $\frac{1}{5}$ is an upper bound, so x is bounded. Also, it is periodic, with period $\frac{2\pi}{5}k = 1{,}256\,k$, $k \in \mathbb{Z}$. This is its integral curve:



$Ox: \frac{1}{5} \sin 5t = 0 \Rightarrow t = \frac{\pi}{5} \cdot k$

$Oy: -1 < \sin 5t < 1$

$-\frac{1}{5} < \frac{1}{5} \sin 5t < \frac{1}{5}$

$\lim_{t \to \pm\infty} \frac{1}{5} \sin 5t$ does not exist

(ii) Let $\varphi_1(t) = t\cos(5t)$ and $\varphi_2(t) = t\sin(5t)$ for all $t \in \mathbb{R}$. Compute $\mathcal{L}(\varphi_1)$, $\mathcal{L}(5)$, $\mathcal{L}(\varphi_2)$

$$\mathcal{L}(5) = 5'' + 25 \cdot 5 = 0 + 125 = 125$$

$$\varphi_1' = \cos 5t - 5t\sin 5t$$

$$\varphi_1'' = -5\sin 5t - 25t\cos 5t - 5\sin 5t$$

$$\Rightarrow \mathcal{L}(\varphi_1) = \varphi_1'' + 25\varphi_1 = -10\sin 5t \cancel{-25t\cos 5t} + \cancel{25t\cos 5t} =$$

$$= -10\sin 5t$$

$$\varphi_2' = \sin 5t + 5t\cos 5t$$

$$\varphi_2'' = 5\cos 5t - 25t\sin 5t + 5\cos 5t$$

$$\Rightarrow \mathcal{L}(\varphi_2) = \varphi_2'' + 25\varphi_2 = 10\cos 5t - 25t\sin 5t + 25t\sin 5t$$

$$= 10\cos 5t.$$

(iii) Find a constant solution for $\mathcal{L}(x) = 5$

We notice that $x_p = \frac{1}{5}$ verifies $x'' + 25x = 5$.

$\Rightarrow$ it is a particular solution, and it is constant.

or : x constant $\Rightarrow x' = x'' = 0$

$$\left.\begin{aligned} \mathcal{L}(x) &= 25x \\ \mathcal{L}(x) &= 5 \end{aligned}\right\} \Rightarrow 25x = 5 \Rightarrow x = \frac{1}{5} \text{ constant solution}$$

(iv) Find the general solution of the differential equation $\mathcal{L}(x) = 25 - 25\sin(5t)$

We know that $x_h = c_1\cos 5t + c_2\sin 5t$ from i).

$$25 - 25\sin(5t) = \frac{125}{5} + \frac{5}{2}(-10\sin 5t) =$$

$$= \frac{1}{5}\mathcal{L}(5) + \frac{5}{2}\mathcal{L}(\varphi_1) =$$

$$= \mathcal{L}(1) + \mathcal{L}\left(\frac{5}{2}\varphi_1\right) = \mathcal{L}\left(\frac{5}{2}\varphi_1 + 1\right).$$

Therefore, a particular solution is:

$$x_p = \frac{5}{2}t\cos 5t + 1.$$

By the fundamental theorem, we get that:

$$x = x_h + x_p = c_1\cos 5t + c_2\sin 5t + \frac{5}{2}t\cos 5t + 1,$$

$$c_1, c_2 \in \mathbb{R}.$$

1.7.35 We consider the differential equation

$$x' + \frac{1}{t^2}x = 0,\ t \in (-\infty, 0).$$

a) Check that $x = e^{\frac{1}{t}}$ is a solution of this d.e.

b) Find the solution of the IVP $x' + \frac{1}{t^2}x = 0$, $x(-1) = 1$.

c) Find the general solution of $x' + \frac{1}{t^2}x = 1 + \frac{1}{t}$, $t \in (-\infty, 0)$

a) $x = e^{\frac{1}{t}} \Rightarrow x' = -\frac{1}{t^2}e^{\frac{1}{t}}$.

$x' + \frac{1}{t^2}x = -\frac{1}{t^2}e^{\frac{1}{t}} + \frac{1}{t^2}e^{\frac{1}{t}} = 0 \Rightarrow x = e^{\frac{1}{t}}$ is a solution of this d.e.

b) $x' + \frac{1}{t^2}x = 0$.

$$\frac{dx}{dt} = -\frac{x}{t^2}$$

$$\frac{dx}{x} = -\frac{dt}{t^2}$$

$$\int \frac{dx}{x} = \int -\frac{1}{t^2}\,dt$$

$\ln|x| = \frac{1}{t} + c \Rightarrow x = \pm e^c e^{\frac{1}{t}}$, $c \in \mathbb{R}$. Recall that $x = 0$ is another solution. Then, the general solution is

$$x = c_1 e^{\frac{1}{t}},\ c_1 \in \mathbb{R}.$$

$x(-1) = 1 \Leftrightarrow \frac{c_1}{e} = 1 \Rightarrow c_1 = e$

$\Rightarrow$ The solution of the IVP is $x = e^{\frac{1}{t}+1} = e^{\frac{t+1}{t}}$

c) $x_h = ce^{\frac{1}{t}}$ is the general solution of the equation $x' + \frac{1}{t^2}x = 0$, as computed at b).

We look for a particular solution $x_p = \varphi(t)e^{\frac{1}{t}}$.

We observe that $x_p = t$ is a particular solution.

$$x_p' + \frac{1}{t^2}x_p = 1 + \frac{1}{t^2}\cdot t = 1 + \frac{1}{t}$$

$\Rightarrow x = x_h + x_p = ce^{\frac{1}{t}} + t$ is the general solution

1.6.1. Let $A \in M_2(\mathbb{R})$. Using both methods that we learned, the characteristic equation method and reduction to second order equation, find the general solution of the system $X' = AX$ in each of the following situations. Also, find a fundamental matrix solution and, finally, find e^{tA}, the principal matrix solution.

j) $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ → not diagonalizable

Consider

$$(1)\begin{cases} x_1' = 2x_1 \\ x_2' = x_1 + 2x_2 \end{cases}$$ ← this is a coupled system.

I. Reduction to second order equation

$$x_1 = \frac{x_2' - 2x_2}{1} \quad (2)$$

$$x_2'' = x_1' + 2x_2' =$$
$$= 2x_1 + 2x_2' =$$
$$\Rightarrow (2) = 2(x_2' - 2x_2) + 2x_2' =$$
$$= 4x_2' - 4x_2$$
$$\Rightarrow x_2'' - 4x_2' + 4x_2 = 0.$$

This is a second-order linear homogenous differential equation.

We write the characteristic equation:

$$r^2 - 4r + 4 = 0.$$
$$(r-2)^2 = 0 \Rightarrow r_1 = 2 \mapsto e^{2t}$$
$$\underbrace{r_2 = 2}_{\text{double root}} \mapsto t e^{2t}$$

$$\Rightarrow x_2 = c_1 e^{2t} + c_2 t e^{2t} \Rightarrow x_2' = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}$$

From (2): $x_1 = x_2' - 2x_2 =$

$$= \cancel{2c_1 e^{2t}} + c_2 e^{2t} + \cancel{2c_2 t e^{2t}} - \cancel{2c_1 e^{2t}} - \cancel{2c_2 t e^{2t}}$$
$$= c_2 e^{2t}.$$

⇒ The general solution is $x_2 = c_1 e^{2t} + c_2 t e^{2t}$ and $x_1 = c_2 e^{2t}$.

II <u>Characteristic equation method</u> (not computed, matrix is not diagonalizable)

$\det(A-\lambda I_2)=0 \iff \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0 \iff (2-\lambda)^2=0.$

$\lambda_1=\lambda_2 \Rightarrow$ their eigenvectors are not linearly independent. This means that A is not diagonalizable.

$\lambda_1=\lambda_2=2.$

For these eigenvalues, we compute the eigenvector and obtain $v=\begin{pmatrix}0\\1\end{pmatrix}$

Now, we need to find a second linearly independent solution.

$A=2I_2+N,\ N=\begin{pmatrix}0&0\\1&0\end{pmatrix}$

$N^2=\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}=O_2$

$\Rightarrow e^{tN}=2I_2+tN=\begin{pmatrix}2&0\\t&2\end{pmatrix}$

The fundamental matrix is e^{tA}.

$e^{tA}=e^{t(2I_2+N)}=e^{2tI_2}e^{tN}=$

$=\begin{pmatrix}e^{2t}&0\\te^{t}&e^{2t}\end{pmatrix}=U$

$\det U=(e^{2t})^2\neq 0,\ \forall t\in\mathbb{R} \Rightarrow U$ is a fundamental matrix solution.

k) $A = \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix}$

Consider

$$(1) \begin{cases} x_1' = 4x_2 \\ x_2' = 5x_1 + x_2 \end{cases}$$ ← this is a coupled system

I Reduction to second order equation

$$x_1 = \frac{x_2' - x_2}{5} \quad (2)$$

$$x_2'' = 5x_1' + x_2' = 20x_2 + x_2'$$

$$\Rightarrow x_2'' - x_2' - 20x_2 = 0.$$

This is a second-order linear homogenous differential equation.

We write the characteristic equation:

$$r^2 - r - 20 = 0$$

$$\Delta = 81 \Rightarrow r_{1,2} = \frac{1 \pm 9}{2} \begin{cases} r_1 = -4 \vdash e^{-4t} \\ r_2 = 5 \vdash e^{5t} \end{cases}$$

$$\Rightarrow x_2 = c_1 e^{-4t} + c_2 e^{5t}$$

$$x_2' = -4c_1 e^{-4t} + 5c_2 e^{5t}$$

From (2) we have that:

$$x_1 = x_2' - x_2 = -4c_1 e^{-4t} + 5c_2 e^{5t} - c_1 e^{-4t} - c_2 e^{5t}$$

$$= -5c_1 e^{-4t} + 4c_2 e^{5t}.$$

$\Rightarrow$ The general solution is:

$$\begin{cases} x_1 = -5c_1 e^{-4t} + 4c_2 e^{5t} \\ x_2 = c_1 e^{-4t} + c_2 e^{5t} \end{cases}$$

Now, we use the second method:

II Characteristic equation method

$$\det(A-\lambda I_2)=0$$

$$\Leftrightarrow \lambda^2-\lambda-\cancel{\ }20=0.$$

$$\Delta=81 \quad \begin{cases}\lambda_1=-4 \rightarrow \text{eigenvector } v_1=\begin{pmatrix}-1\\1\end{pmatrix}\\ \lambda_2=5. \rightarrow \text{eigenvector } v_2=\begin{pmatrix}\frac{4}{5}\\1\end{pmatrix}\end{cases}$$

The roots are real and distinct, so we consider

$$\varphi_1(t)=e^{-4t}v_1$$

$$\varphi_2(t)=e^{5t}v_2$$

So, the general solution is:

$$X=c_1\varphi_1(t)+c_2\varphi_2(t)=$$

$$=c_1e^{-4t}\begin{pmatrix}-1\\1\end{pmatrix}+c_2e^{5t}\begin{pmatrix}\frac{4}{5}\\1\end{pmatrix}=$$

$$=\begin{pmatrix}-c_1e^{-4t}+\frac{4c_2e^{5t}}{5}\\ c_1e^{-4t}+c_2e^{5t}\end{pmatrix}$$

$$\Rightarrow x_1=-5c_1e^{-4t}+4c_2e^{5t}$$
$$x_2=c_1e^{-4t}+c_2e^{5t}$$
is the general solution.

The corresponding solution vectors are:

$$\vec{u_1}=e^{-4t}\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\vec{u_2}=e^{5t}\begin{pmatrix}\frac{4}{5}\\1\end{pmatrix}$$

So, our fundamental matrix is:

$$\begin{pmatrix}-e^{-4t} & \frac{4}{5}e^{5t}\\ e^{-4t} & e^{5t}\end{pmatrix}.$$

$\underline{e^{At}}$

$$D = \begin{pmatrix} -4 & 0 \\ 0 & 5 \end{pmatrix}$$

$$P = (v_1 \quad v_2) = \begin{pmatrix} -1 & \frac{4}{5} \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -\frac{5}{9} & \frac{4}{9} \\ \frac{5}{9} & \frac{5}{9} \end{pmatrix}$$

$$e^{tA} = Pe^{Dt}P^{-1} = \begin{pmatrix} -1 & \frac{4}{5} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-4t} & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} -5 & 4 \\ 5 & 5 \end{pmatrix} \frac{1}{9}$$

$$= \begin{pmatrix} -e^{-4t} & \frac{4}{5} e^{5t} \\ e^{-4t} & e^{5t} \end{pmatrix} \begin{pmatrix} -5 & 4 \\ 5 & 5 \end{pmatrix} \frac{1}{9} =$$

$$= \frac{1}{9} \begin{pmatrix} 5e^{-4t} + 4e^{5t} & -4e^{-4t} + 4e^{5t} \\ -5e^{-4t} + 5e^{5t} & 4e^{-4t} + 5e^{5t} \end{pmatrix}$$

m) $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

Consider

$$(1)\begin{cases} x_1' = -2x_2 \\ x_2' = 2x_1 \end{cases}$$ ← this is an uncoupled system

I. Reduction to second order equation

$x_1 = \frac{x_2'}{2}$ (2)

$x_2'' = 2x_1' = -4x_2$

$\Rightarrow x_2'' + 4x_2 = 0.$

This is a second-order linear homogenous differential equation.

We write the characteristic equation:

$r^2 + 4 = 0.$

$\Rightarrow r_1 = 2i \vdash \cos 2t$

$r_2 = -2i \vdash \sin 2t$

$\Rightarrow x_2 = c_1 \cos 2t + c_2 \sin 2t.$

From (2) we have that:

$x_1 = \frac{x_2'}{2} = \frac{-2c_1 \sin 2t + 2c_2 \cos 2t}{2} =$

$= -c_1 \sin 2t + c_2 \cos 2t.$

$\Rightarrow$ The general solution is:

$$\begin{cases} x_1 = -c_1 \sin 2t + c_2 \cos 2t \\ x_2 = c_1 \cos 2t + c_2 \sin 2t \end{cases}$$

II Characteristic equation method

$\det(A-\lambda I_2)=0$

$\Leftrightarrow \lambda^2+4=0 \Rightarrow \lambda_1=-2i \to$ eigenvector $\begin{pmatrix}-i\\1\end{pmatrix}$

$\lambda_2=2i \to$ eigenvector $\begin{pmatrix}i\\1\end{pmatrix}$.

The roots are not real, so we consider

$\varphi_1(t)=\cos 2t\begin{pmatrix}i\\1\end{pmatrix}-\sin 2t\begin{pmatrix}-i\\1\end{pmatrix}$

$\varphi_2(t)=\sin 2t\begin{pmatrix}i\\1\end{pmatrix}+\cos 2t\begin{pmatrix}-i\\1\end{pmatrix}$

So, the general solution is:

$X=c_1\varphi_1(t)+c_2\varphi_2(t)=$

$=\begin{pmatrix}c_1 i\cos 2t+c_1 i\sin 2t+c_2 i\sin 2t-c_2 i\cos 2t\\ c_1\cos 2t-c_1\sin 2t+c_2\sin 2t+c_2\cos 2t\end{pmatrix}$

$\Rightarrow x_1=(c_1-c_2)i\cos 2t+(c_1+c_2)i\sin 2t$ is the general solution

$x_2=(c_1+c_2)\cos 2t+(c_2-c_1)\sin 2t$

The corresponding solution vectors are:

$\vec{u_1}=e^{-2i}\begin{pmatrix}-i\\1\end{pmatrix}$

$\vec{u_2}=e^{2i}\begin{pmatrix}i\\1\end{pmatrix}$

So, our fundamental matrix is:

$\begin{pmatrix}-ie^{-2i} & ie^{2i}\\ e^{-2i} & e^{2i}\end{pmatrix}$

$\underline{e^{At}}$

$$D = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}$$

$$P = (v_1 \;\; v_2) = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

$$e^{tA} = P e^{Dt} P^{-1} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2i} & 0 \\ 0 & e^{2i} \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -ie^{-2i} & ie^{2i} \\ e^{-2i} & e^{2i} \end{pmatrix} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \cdot \frac{1}{2} =$$

$$= \frac{1}{2} \begin{pmatrix} e^{-2i} + e^{2i} & -ie^{-2i} + ie^{2i} \\ ie^{-2i} - ie^{2i} & e^{-2i} + e^{2i} \end{pmatrix}$$