Seminar 1

Operation: $* : A \times A \to A$ with $x, y \in A \Rightarrow x * y \in A$.
Grupoid: $(A, *)$
Semigroup: $(A, *)$ grupoid + associativity
Monoid: $(A, *)$ semigroup + identity element
Group: $(A, *)$ monoid + all elements have a simmetric
Abelian group: $(A, *)$ group + commutativity
Subgroupoid = stable part: $\forall a, b \in A \Rightarrow a * b \in A$
Subgroup: $H \leq (G, *)$ if H is a sable part in G $(H \subseteq G)$ and $(H, *)$ is a group.

1. Addition: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

   Substraction: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

   Multiplication: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$

   Division: $\mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$

2. i 3 elements in 3 spaces $\Rightarrow 3^9$

	a	b	c
a			
b			
c			

ii 3^3 (3 elements in 3 free spaces) and 3^3 (3 commutative elements in 3 spaces) $\Rightarrow 3^6$.

	a	b	c
a		c	b
b	c		a
c	b	a	

iii 3^4 (3 elemnts in 4 free spaces) and 3 elements, which can be e $\Rightarrow 3^5$

	e	b	c
e	e	b	c
b	b		
c	c		

Generalization:

i n^{n^2}

ii $n^n \cdot n^{\frac{n(n-1)}{2}}$

iii $n^{(n-1)^2+1}$

3. $(\mathbb{Z},+), (\mathbb{Q},+), (\mathbb{R},+), (\mathbb{C},+)$ and $(\mathbb{Q}^*,\cdot), (\mathbb{R}^*,\cdot), (\mathbb{C}^*,\cdot)$.

4. i Stable part: $\forall x, y \in \mathbb{R} \Rightarrow x*y = x+y+xy = (x+1)(y+1)-1 \in \mathbb{R}$
   Associativity: $\forall x, y \in \mathbb{R} \Rightarrow (x*y)*z = x*(y*z)$
   Identity element: $\exists e \in \mathbb{R}$ such that $\forall x \in \mathbb{R} \Rightarrow x*e = e*x = x$
   Commutativity: $\forall x, y \in \mathbb{R} \Rightarrow x*y = y*x$.

   ii Let A be our interval. Then A is a stable subset of $(\mathbb{R},*)$ $\iff$ $\forall x, y \in A \Rightarrow x*y \in A$.
   $x, y \in A \Rightarrow -1 \leq x, -1 \leq y \Rightarrow 0 \leq x+1, 0 \leq y+1 \Rightarrow 0 \leq (x+1)(y+1) \Rightarrow -1 \leq (x+1)(y+1)-1 \Rightarrow x*y \in A$

5. i Here is interesting to see the associativity: $\forall x, y, z \in \mathbb{N} \Rightarrow (x*y)*z = gcd(x,y)*z = gcd(gcd(x,y),z) = \alpha \Rightarrow \alpha \mid gcd(x,y)$ and $\alpha \mid z$. From $gcd(x,y) = d \Rightarrow x = dx_1$ and $y = dy_1$, but $\alpha \mid d \Rightarrow \alpha \mid x$ and $\alpha \mid y \Rightarrow \alpha \mid x,y,z \Rightarrow \alpha \mid gcd(y,z) \Rightarrow \alpha \mid gcd(x, gcd(y,z))$. Analogus for $gcd(x, gcd(y,z)) \mid \alpha$.

   ii $\forall x, y \in D_n \Rightarrow x \mid n$ and $y \mid n \Rightarrow n = xd_1$ and $n = yd_2$. We compute $x*y = gcd(x,y) = \alpha \Rightarrow x = \alpha x_1$ and $y = \alpha y_1 \Rightarrow n = \alpha x_1 d_1$ and $n = \alpha y_1 d_2 \Rightarrow \alpha \mid n \Rightarrow gcd(x,y) \mid n \Rightarrow x*y \in D_n$. Associativity, commutativity and identity element are easy to prove.

   iii $D_6 = \{1, 2, 3, 6\}$

	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

6. $H \subseteq \mathbb{Z}$ and H stable part of $\mathbb{Z} \Rightarrow \exists x \in H \Rightarrow \forall n \in \mathbb{N}^*$ we have $x^n \in H$, but H is finite $\Rightarrow \exists n \in \mathbb{N}^*$such that $x^i = x^j, i, j \in \mathbb{N}^*$ and $0 < i < j \Rightarrow x \in \{-1, 0, 1\} \Rightarrow H$ can be $\emptyset, \{0\}, \{1\}, \{0, 1\}, \{-1, 1\}, \{-1, 0, 1\}$.

7. (i) $\Rightarrow$ If G is abelian, then $xy = yx \Rightarrow (xy)^2 = xyxy = xxyy = x^2y^2$.
   $\Leftarrow \forall x, y \in G : (xy)^2 = x^2y^2 = xxyy$. But $(xy)^2 = xyxy$. So $xyxy = xxyy$. As G is a group, $\exists x^{-1}, y^{-1} \in G$, hence, we multiply with x^{-1} on the left and with y^{-1} on the right and we obtain $yx = xy \Rightarrow G$ is abelian.

   (ii) $\forall x, y \in G : x^2 = 1$ and $y^2 = 1 \Rightarrow x = x^{-1}$ and $y = y^{-1}$, so $xy = x^{-1}y^{-1}$. x * x = 1 , 1 = element neutru
   Also, $(xy)^2 = 1 \Rightarrow xy = (xy)^{-1} = y^{-1}x^{-1}$. But $(yx)^2 = 1 \Rightarrow yx = (yx)^{-1} = x^{-1}y^{-1}$.
   Hence, $x^{-1}y^{-1} = y^{-1}x^{-1} \iff xy = yx, \forall x, y \in G$.

8. (i) If $(A, \cdot)$ is a monoid, then $\cdot$ is associative and it has an identity element, let's say $e \in A$.
   Take $X, Y, Z \in P(A) : (X * Y) * Z = \{(xy)z \mid x \in X, y \in Y, z \in Z\} = \{x(yz) \mid x, \in X, y \in Y, z \in Z\} = X * (Y * Z)$. So, $*$ is associative.
   Take $X \in P(A) : X * \{e\} = \{xe \mid x \in X\} = \{ex \mid x \in X\} = \{e\} * X = \{x \mid x \in X\} = X$. So, $*$ has the identity element $\{e\}$.

   (ii) This is easy to see with a counter example.
   If $A = \emptyset$, then $P(A)$ is a group.
   If $A \neq \emptyset$, take $A = \{e\} \Rightarrow P(A) = \{\emptyset, e\}$. We know that the identity element is its own inverse, but $\emptyset$ has no inverse. Hence, $P(A)$ is not a group.