Questions

Q1.

$$y = \ln(\tanh 2x) \qquad x > 0$$

(a) Show that

$$\frac{dy}{dx} = p \operatorname{cosech} 4x$$

where p is a constant to be determined.

(4)

(b) Hence determine, in simplest form, the exact value of x for which $\frac{dy}{dx} = 1$

(2)

(Total for question = 6 marks)

Q2.

(a) Given that $y = \operatorname{arsech}\left(\frac{x}{2}\right)$, where $0 < x \leq 2$, show that

$$\frac{dy}{dx} = \frac{p}{x\sqrt{q - x^2}}$$

where p and q are constants to be determined.

(4)

In part (b) solutions based entirely on calculator technology are not acceptable.

$$f(x) = \operatorname{artanh}(x) + \operatorname{arsech}\left(\frac{x}{2}\right) \qquad 0 < x \leq 1$$

(b) Determine, in simplest form, the exact value of x for which $f'(x) = 0$

(5)

(Total for question = 9 marks)

Q3.

(a) Differentiate x arcosh $5x$ with respect to x

(2)

(b) Hence, or otherwise, show that

$$\int_{\frac{1}{4}}^{\frac{3}{5}} \operatorname{arcosh} 5x \, dx = \frac{3}{20} - \frac{2\sqrt{2}}{5} + \ln\left(p + q\sqrt{2}\right)^k - \frac{1}{4}\ln r$$

where p, q, r and k are rational numbers to be determined.

(8)

(Total for question = 10 marks)

Q4.

$$y = \arccos\left(2\sqrt{x}\right)$$

(a) Determine $\frac{dy}{dx}$

(3)

(b) Show that

$$\int y \, dx = x \arccos\left(2\sqrt{x}\right) + \int \frac{\sqrt{x}}{\sqrt{1-4x}} \, dx$$

(2)

(c) Use the substitution

$\sqrt{x} = \frac{1}{2}\cos\theta$ to show that

$$\int_0^{\frac{1}{8}} \frac{\sqrt{x}}{\sqrt{1-4x}}\,\mathrm{d}x = \frac{1}{4}\int_a^b \cos^2\theta\,\mathrm{d}\theta$$

where *a* and *b* are limits to be determined.

(4)

(d) Hence, determine the exact value of

$$\int_0^{\frac{1}{8}} \arccos\left(2\sqrt{x}\right)\mathrm{d}x$$

(4)

(Total for question = 13 marks)

Q5.

$$y = \operatorname{artanh}\left(\frac{\cos x + a}{\cos x - a}\right)$$

where *a* is a non-zero constant.

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = k\tan x$$

where *k* is a constant to be determined.

(Total for question = 4 marks)

Q6.

The curve *C* has equation

$$y = 9 \cosh x + 3 \sinh x + 7x$$

Use differentiation to find the exact x coordinate of the stationary point of C, giving your answer as a natural logarithm.

(Total for question = 6 marks)

Q7.

(a) Using the definitions of hyperbolic functions in terms of exponentials, show that

$$1 - \tanh^2 x \equiv \operatorname{sech}^2 x$$

(3)

(b) Solve the equation

$$2 \operatorname{sech}^2 x + 3 \tanh x = 3$$

giving your answer as an exact logarithm.

(3)

(Total for question = 6 marks)

Q8.

(a) Use the definitions of hyperbolic functions in terms of exponentials to prove that

$$8 \cosh^4 x = \cosh 4x + p \cosh 2x + q$$

where p and q are constants to be determined.

(3)

(b) Hence, or otherwise, solve the equation

$$\cosh 4x - 17 \cosh 2x + 9 = 0$$

giving your answers in exact simplified form in terms of natural logarithms.

(5)

(Total for question = 8 marks)

Q9.

(a) Use the definitions of sinh x and cosh x in terms of exponentials to show that

$$\cosh A \cosh B + \sinh A \sinh B \equiv \cosh(A + B)$$

(2)

(b) Hence find the value of x for which

$$\cosh(x + \ln 2) = 5 \sinh x$$

giving your answer in the form $\frac{1}{2} \ln k$, where k is a rational number to be determined.

(5)

(Total for question = 7 marks)

Q10.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Solve the equation

$$7 \cosh x + 3 \sinh x = 2e^x + 7$$

Give your answers as simplified natural logarithms.

(Total for question = 5 marks)

Q11.

Given that y = artanh(cos x)

(a) show that

$$\frac{dy}{dx} = -\operatorname{cosec} x$$

(2)

(b) Hence find the exact value of

$$\int_0^{\frac{\pi}{6}} \cos x \operatorname{artanh}(\cos x)\, dx$$

giving your answer in the form $a\ln\left(b + c\sqrt{3}\right) + d\pi,$ where a, b, c and d are rational numbers to be found.

(5)

(Total for question = 7 marks)

Q12.

Using the substitution $x = 4\cosh\theta$ show that

$$\int \frac{1}{\left(x^2 - 16\right)^{\frac{3}{2}}}\, dx = \frac{ax}{\sqrt{x^2 - 16}} + c \qquad |x| > 4$$

where a is a constant to be determined and c is an arbitrary constant.

(6)

(Total for question = 6 marks)

Q13.

Using calculus, find the exact values of

(i)

$$\int_1^2 \frac{1}{x^2 - 4x + 5}\,\mathrm{d}x$$

(3)

(ii)

$$\int_{\sqrt{3}}^3 \frac{\sqrt{x^2 - 3}}{x^2}\,\mathrm{d}x$$

(5)

(Total for question = 8 marks)

Q14.

(i)

$$\mathrm{f}(x) = x \arccos x \qquad -1 \leq x \leq 1$$

Find the exact value of f ′(0.5).

(3)

(ii)

$$\mathrm{g}(x) = \arctan(\mathrm{e}^{2x})$$

Show that

$$\mathrm{g}''(x) = k\,\mathrm{sech}(2x)\tanh(2x)$$

where k is a constant to be found.

(5)

(Total for question = 8 marks)

Q15.

Determine

(i) $$\int \frac{1}{\sqrt{x^2 - 3x + 5}}\, \mathrm{d}x$$

(3)

(ii) $$\int \frac{1}{\sqrt{63 + 4x - 4x^2}}\, \mathrm{d}x$$

(4)

(Total for question = 7 marks)

Q16.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(i) Determine

$$\int \frac{1}{\sqrt{5 + 4x - x^2}}\, \mathrm{d}x$$

(3)

(ii) Use the substitution $x = 3 \sec \theta$ to determine the exact value of

$$\int_{2\sqrt{3}}^{6} \frac{18}{(x^2 - 9)^{\frac{3}{2}}}\, \mathrm{d}x$$

Give your answer in the form $A + B\sqrt{3}$ where A and B are constants to be found.

(6)

(Total for question = 9 marks)

Q17.

Without using a calculator, find

(a) $\displaystyle\int_{-2}^{1} \frac{1}{x^2 + 4x + 13}\,dx,$ giving your answer as a multiple of π,

(5)

(b) $\displaystyle\int_{-1}^{4} \frac{1}{\sqrt{4x^2 - 12x + 34}}\,dx,$ giving your answer in the form $p\ln\left(q + r\sqrt{2}\right),$

where p, q and r are rational numbers to be found.

(7)

(Total for question = 12 marks)

Q18.

The point P has coordinates (1, 2, 1)

The line l has Cartesian equation

$$\frac{x-3}{5} = \frac{y+1}{3} = \frac{z+5}{-8}$$

The plane Π_1 contains the point P and the line l.

(a) Show that a Cartesian equation for Π_1 is

$$6x - 2y + 3z = 5$$

(5)

The point Q has coordinates $(2, k, -7)$, where k is a constant.

(b) Show that the shortest distance between Π_1 and Q is

$\frac{2}{7}|k+7|$

(2)

The plane Π_2 has Cartesian equation $8x - 4y + z = -3$

Given that the shortest distance between Π_1 and Q is the same as the shortest distance between Π_2 and Q,

(c) determine the possible values of k.

(4)

(Total for question = 11 marks)

Q19.

The line l_1 has equation

$$\frac{x-3}{4} = \frac{y-5}{-2} = \frac{z-4}{7}$$

The plane Π has equation

$$2x + 4y - z = 1$$

The line l_1 intersects the plane Π at the point P

(a) Determine the coordinates of P

(3)

The acute angle between l_1 and Π is θ degrees.

(b) Determine, to one decimal place, the value of θ

(3)

The line l_2 lies in Π and passes through P

Given that the acute angle between l_1 and l_2 is also θ degrees,

(c) determine a vector equation for l_2

(5)

(Total for question = 11 marks)

Q20.

The plane Π has equation

$$3x + 4y - z = 17$$

The line l_1 is perpendicular to Π and passes through the point $P(-4, -5, 3)$

The line l_1 intersects Π at the point Q

(a) Determine the coordinates of Q

(4)

Given that the point $R(-1, 6, 4)$ lies on Π

(b) determine a Cartesian equation of the plane containing PQR

(4)

The line l_2 passes through P and R

The line l_3 is the reflection of l_2 in Π

(c) Determine a vector equation for l_3

(4)

(Total for question = 12 marks)

Q21.

The line l_1 has equation

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k})$$

and the line l_2 has equation

$$\mathrm{r} = 2\mathbf{i} + s\mathbf{j} + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

where s is a constant and λ and μ are scalar parameters.

Given that l_1 and l_2 both lie in a common plane Π_1

(a) show that an equation for Π_1 is $3x + y - z = 3$

(4)

(b) find the value of s.

(1)

The plane Π_2 has equation $\mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 3$

(c) Find an equation for the line of intersection of Π_1 and Π_2

(4)

(d) Find the acute angle between Π_1 and Π_2 giving your answer in degrees to 3 significant figures.

(4)

(Total for question = 13 marks)

Q22.

Given that

$$\cosh y = x \qquad \text{and} \qquad y < 0$$

use the definition of cosh y in terms of exponential functions to prove that

$$y = \ln\left(x - \sqrt{x^2 - 1}\right)$$

(6)

(Total for question = 6 marks)

Q23.

The skew lines l_1 and l_2 have equations

$$l_1 : \mathbf{r} = (\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) + \lambda(5\mathbf{i} + \mathbf{j})$$

and

$$l_2 : \mathbf{r} = (2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) + \mu(8\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

where λ and μ are scalar parameters.

(a) Determine a vector that is perpendicular to both l_1 and l_2

(2)

(b) Determine an equation of the plane parallel to l_1 that contains l_2

(i) in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$

(1)

(ii) in the form $\mathbf{r}.\mathbf{n} = p$

(2)

(c) Determine the shortest distance between l_1 and l_2

Give your answer in simplest form.

(5)

(Total for question = 10 marks)