Questions

Q1.

In a geometric sequence u_1, u_2, u_3, ...

- the common ratio is r
- $u_2 + u_3 = 6$
- $u_4 = 8$

(a) Show that r satisfies

$$3r^2 - 4r - 4 = 0$$

(3)

Given that the geometric sequence has a sum to infinity,

(b) find u_1

(3)

(c) find S_∞

(2)

(Total for question = 8 marks)

Q2.

The first term of a geometric series is 20 and the common ratio is $\frac{7}{8}$. The sum to infinity of the series is S_∞

(a) Find the value of S_∞

(2)

The sum to N terms of the series is S_N

(b) Find, to 1 decimal place, the value of S_{12}

(2)

(c) Find the smallest value of N, for which $S_\infty - S_N < 0.5$

(4)

(Total for question = 8 marks)

Q3.

(i) A geometric sequence has first term 4 and common ratio 6

Given that the n^{th} term is greater than 10^{100}, find the minimum possible value of n.

(3)

(ii) A different geometric sequence has first term a and common ratio r.

Given that

- the second term of the sequence is −6
- the sum to infinity of the series is 25

(a) show that

$$25r^2 - 25r - 6 = 0$$

(3)

(b) Write down the solutions of

$$25r^2 - 25r - 6 = 0$$

(1)

Hence,

(c) state the value of r, giving a reason for your answer,

(1)

(d) find the sum of the first 4 terms of the series.

(2)

(Total for question = 10 marks)

Q4.

(a) Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 3y\tan x = \mathrm{e}^{4x}\sec^3 x$$

giving your answer in the form $y = \mathrm{f}(x)$

(5)

(b) Determine the particular solution for which $y = 4$ at $x = 0$

(2)

(Total for question = 7 marks)

Q5.

(a) Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 2\sin x \qquad \text{(I)}$$

(8)

Given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$

(b) find the particular solution of differential equation (I).

(5)

(Total for question = 13 marks)

Q6.

(a) Determine the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 6\cos x$$

(7)

(b) Find the particular solution for which $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$

(5)

(Total for question = 12 marks)

Q7.

(a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 2x^2 + x$$

(8)

(b) Find the particular solution of this differential equation for which $y = 1$ and

$\frac{dy}{dx} = 0$ when $x = 0$

(5)

(Total for question = 13 marks)

Q8.

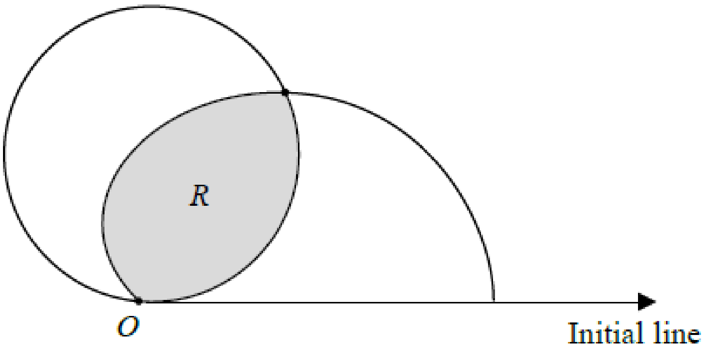

Figure 1

Figure 1 shows the two curves given by the polar equations

$$r = \sqrt{3}\ \sin\theta, \quad 0 \leqslant \theta \leqslant \pi$$
$$r = 1 + \cos\theta, \quad 0 \leqslant \theta \leqslant \pi$$

(a) Verify that the curves intersect at the point *P* with polar coordinates $\left(\frac{3}{2}, \frac{\pi}{3}\right)$.

(2)

The region *R*, bounded by the two curves, is shown shaded in Figure 1.

(b) Use calculus to find the exact area of *R*, giving your answer in the form $a\left(\pi - \sqrt{3}\right)$, where *a* is a constant to be found.

(6)

(Total for question = 8 marks)

Q9.

The curve *C*, with pole *O*, has polar equation

$$r = 1 + \cos\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

At the point *A* on *C*, the tangent to *C* is parallel to the initial line.

(a) Find the polar coordinates of *A*.

(4)

(b) Find the finite area enclosed by the initial line, the line *OA* and the curve *C*, giving your answer in the form $a\pi + b\sqrt{3}$,
where *a* and *b* are rational constants to be found.

(6)

(Total for question = 10 marks)

Q10.

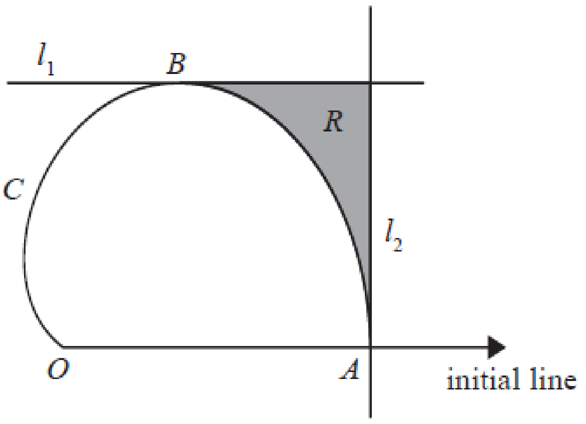

Figure 1

Figure 1 shows a sketch of the curve *C* with equation

$$r = 6(1 + \cos\theta) \qquad 0 \leqslant \theta \leqslant \pi$$

Given that *C* meets the initial line at the point *A*, as shown in Figure 1,

(a) write down the polar coordinates of *A*.

(1)

The line l_1, also shown in Figure 1, is the tangent to *C* at the point *B* and is parallel to the initial line.

(b) Use calculus to determine the polar coordinates of *B*.

(4)

The line l_2, also shown in Figure 1, is the tangent to *C* at *A* and is perpendicular to the initial line.

The region *R*, shown shaded in Figure 1, is bounded by *C*, l_1 and l_2

(c) Use algebraic integration to find the exact area of *R*, giving your answer in the form $p\sqrt{3} + q\pi$ where *p* and *q* are constants to be determined.

(8)

(Total for question = 13 marks)

Q11.

$$f(z) = 2z^3 - z^2 + az + b$$

where *a* and *b* are integers.

The complex number $-1 - 3i$ is a root of the equation $f(z) = 0$

(a) Write down another complex root of this equation.

(1)

(b) Determine the value of a and the value of *b*.

(4)

(c) Show all the roots of the equation $f(z) = 0$ on a single Argand diagram.

(2)

(Total for question = 7 marks)

Q12.

$$z_1 = 3 + 3i \qquad z_2 = p + qi \qquad p, q \in \mathbb{R}$$

Given that $|z_1 z_2| = 15\sqrt{2}$

(a) determine $|z_2|$

(2)

Given also that $p = -4$

(b) determine the possible values of *q*

(2)

(c) Show z_1 and the possible positions for z_2 on the same Argand diagram.

(2)

(Total for question = 6 marks)

Q13.

$$f(z) = 2z^4 - 19z^3 + Az^2 + Bz - 156$$

where A and B are constants.

The complex number $5 - i$ is a root of the equation $f(z) = 0$

(a) Write down another complex root of this equation.

(1)

(b) Solve the equation $f(z) = 0$ completely.

(5)

(c) Determine the value of A and the value of B.

(2)

(Total for question = 8 marks)

Q14.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Use de Moivre's theorem to show that

$$\cos 5x \equiv \cos x\left(a\sin^4 x + b\sin^2 x + c\right)$$

where a, b and c are integers to be determined.

(4)

(b) Hence solve, for $0 < \theta < \dfrac{\pi}{2}$

$$\cos 5\theta = \sin 2\theta \sin \theta - \cos \theta$$

giving your answers to 3 decimal places.

(4)

(Total for question = 8 marks)

Q15.

Given that $z = e^{i\theta}$

(a) show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$

where n is a positive integer.

(2)

(b) Show that

$$\cos^6\theta = \frac{1}{32}\left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right)$$

(5)

(c) Hence solve the equation

$$\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta = 0 \qquad 0 \leqslant \theta \leqslant \pi$$

Give your answers to 3 significant figures.

(4)

(d) Use calculus to determine the exact value of

$$\int_0^{\frac{\pi}{3}} \left(32\cos^6\theta - 4\cos^2\theta\right) d\theta$$

Solutions relying entirely on calculator technology are not acceptable.

(5)

(Total for question = 16 marks)

Q16.

Solve the equation

$z^5 - 32\text{i} = 0$

giving each answer in the form $r\text{e}^{\text{i}\theta}$ where $0 < \theta < 2\pi$

(4)

(Total for question = 4 marks)

Q17.

(a) Use de Moivre's theorem to show that

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

(5)

(b) Hence determine the five distinct solutions of the equation

$$16x^5 - 20x^3 + 5x + \frac{1}{5} = 0$$

giving your answers to 3 decimal places.

(c) Use the identity given in part (a) to show that

$$\int_0^{\frac{\pi}{4}} (4\sin^5\theta - 5\sin^3\theta - 6\sin\theta)\,\text{d}\theta = a\sqrt{2} + b$$

where a and b are rational numbers to be determined.

(4)

(Total for question = 9 marks)

Q18.

The complex number z_1 is defined as

$$z_1 = \frac{\left(\cos\frac{5\pi}{12} + \text{i}\sin\frac{5\pi}{12}\right)^4}{\left(\cos\frac{\pi}{3} - \text{i}\sin\frac{\pi}{3}\right)^3}$$

(a) Without using your calculator show that

$$z_1 = \cos\frac{2\pi}{3} + \text{i}\sin\frac{2\pi}{3}$$

(4)

(b) Shade, on a single Argand diagram, the region R defined by

$$|z - z_1| \leqslant 1 \qquad \textbf{and} \qquad 0 \leqslant \arg(z - z_1) \leqslant \frac{3\pi}{4}$$

(4)

Given that the complex number z lies in R

(c) determine the smallest possible positive value of arg z

(2)

(Total for question = 10 marks)

Q19.

$$\mathbf{M} = \begin{pmatrix} 3x & 7 \\ 4x+1 & 2-x \end{pmatrix}$$

Find the range of values of x for which the determinant of the matrix **M** is positive.

(Total for question = 5 marks)

Q20.

The matrix **M** is defined by

$$\mathbf{M} = \begin{pmatrix} k+5 & -2 \\ -3 & k \end{pmatrix}$$

(a) Determine the values of k for which **M** is singular.

(2)

Given that **M** is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of k.

(2)

(Total for question = 4 marks)

Q21.

Given that $y = \ln(5 + 3x)$

(a) determine, in simplest form, $\dfrac{d^3y}{dx^3}$

(3)

(b) Hence determine the Maclaurin series expansion of $\ln(5 + 3x)$, in ascending powers of x up to and including
the term in x^3, giving each coefficient in simplest form.

(2)

(c) Hence write down the Maclaurin series expansion of $\ln(5 - 3x)$, in ascending powers of x up to and including the term in x^3, giving each coefficient in simplest form.

(1)

(d) Use the answers to parts (b) and (c) to determine the first 2 non-zero terms, in ascending powers of x, of the Maclaurin series expansion of

$$\ln\left(\frac{5+3x}{5-3x}\right)$$

(2)

(Total for question = 8 marks)

Q22.

Given that

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ -2 & 3 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & k \\ 0 & -3 \\ 2k & 2 \end{pmatrix}$$

where k is a non-zero constant,

(a) determine the matrix **AB**

(2)

(b) determine the value of k for which det(**AB**) = 0

(3)

(Total for question = 5 marks)

Q23.

(a) Show that

$$\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3 = z^6 - \frac{1}{z^6} - k\left(z^2 - \frac{1}{z^2}\right)$$

where k is a constant to be found.

(3)

Given that $z = \cos\theta + i\sin\theta$, where θ is real,

(b) show that

(i) $z^n + \frac{1}{z^n} = 2\cos n\theta$

(ii) $z^n - \frac{1}{z^n} = 2i\sin n\theta$

(3)

(c) Hence show that

$$\cos^3\theta \sin^3\theta = \frac{1}{32} (3\sin 2\theta - \sin 6\theta)$$

(4)

(d) Find the exact value of

$$\int_0^{\frac{\pi}{8}} \cos^3\theta \sin^3\theta \, d\theta$$

(4)

(Total for question = 14 marks)

Q24.

(a) Use de Moivre's theorem to show that

$$\tan 4\theta \equiv \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

(6)

(b) Use the identity given in part (a) to find the 2 positive roots of

$$x^4 + 2x^3 - 6x^2 - 2x + 1 = 0$$

giving your answers to 3 significant figures.

(3)

(Total for question = 9 marks)

Q25.

(a) Express

$$\frac{1}{(2n-1)(2n+1)(2n+3)}$$

in partial fractions.

(2)

(b) Hence, using the method of differences, show that for all integer values of n,

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)(2r+3)} = \frac{n(n+2)}{a(2n+b)(2n+c)}$$

where a, b and c are integers to be determined.

(4)

(Total for question = 6 marks)

Q26.

(a) Determine the general solution of the differential equation

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} - xy = \mathrm{e}^{3x} \qquad x > -1$$

giving your answer in the form $y = \mathrm{f}(x)$.

(7)

(b) Determine the particular solution of the differential equation for which $y = 5$ when $x = 0$

(2)

(Total for question = 9 marks)

Q27.

$$2z + z^* = \frac{3 + 4\text{i}}{7 + \text{i}}$$

Find z, giving your answer in the form $a + bi$, where a and b are real constants. You must show all your working.

(Total for question = 5 marks)

Q28.

$$f(z) = z^4 + 6z^3 + 76z^2 + az + b$$

where a and b are real constants.

Given that $-3 + 8\text{i}$ is a complex root of the equation $f(z) = 0$

(a) write down another complex root of this equation.

(1)

(b) Hence, or otherwise, find the other roots of the equation $f(z) = 0$

(6)

(c) Show on a single Argand diagram all four roots of the equation $f(z) = 0$

(2)

(Total for question = 9 marks)

Q29.

The complex numbers z_1, z_2 and z_3 are given by

$$z_1 = 2 - \text{i} \qquad z_2 = p - \text{i} \qquad z_3 = p + \text{i}$$

where p is a real number.

(a) Find $\frac{z_2 z_3}{z_1}$ in the form $a + b\text{i}$ where a and b are real. Give your answer in its simplest form in terms of p.

(3)

Given that $\left|\frac{z_2 z_3}{z_1}\right| = 2\sqrt{5}$

(b) find the possible values of p.

(4)

(Total for question = 7 marks)

Q30.

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Given that $x = 2 + 3\text{i}$ is a root of the equation

$$2x^4 - 8x^3 + 29x^2 - 12x + 39 = 0$$

(a) write down another complex root of this equation.

(1)

(b) Use algebra to determine the other 2 roots of the equation.

(4)

(c) Show all 4 roots on a single Argand diagram.

(2)

(Total for question = 7 marks)

Q31.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

$$z_1 = 3 + 2\text{i} \qquad z_2 = 2 + 3\text{i} \qquad z_3 = a + b\text{i} \qquad a, b \in \mathbb{R}$$

(a) Determine the exact value of $|z_1 + z_2|$

(2)

Given that $w = \dfrac{z_2 z_3}{z_1}$

(b) determine *w* in terms of *a* and *b*, giving your answer in the form *x* + i*y*, where *x*, *y*∈ℝ

(4)

Given also that $w = \dfrac{4}{13} + \dfrac{58}{13}\text{i}$

(c) determine the value of *a* and the value of *b*

(2)

(d) determine arg *w*, giving your answer in radians to 4 significant figures.

(2)

(Total for question = 10 marks)

Q32.

Use the standard results for $\sum_{r=1}^{n} r$ and for $\sum_{r=1}^{n} r^3$ to show that, for all positive integers *n*,

$$\sum_{r=1}^{n} r(r^2 - 3) = \frac{n}{4}(n + a)(n + b)(n + c)$$

where *a*, *b* and *c* are integers to be found.

(Total for question = 4 marks)

Q33.

Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ to show that, for all positive integers *n*

$$\sum_{r=1}^{n} r^2(r + 2) = \frac{1}{12}n(n + 1)(an^2 + bn + c)$$

where *a*, *b* and *c* are integers to be determined.

(Total for question = 4 marks)

Q34.

(a) Express $\frac{1}{(r + 6)(r + 8)}$ in partial fractions.

(1)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{2}{(r + 6)(r + 8)} = \frac{n(an + b)}{56(n + 7)(n + 8)}$$

where *a* and *b* are integers to be found.

(4)

(Total for question = 5 marks)

Q35.

(a) Show that, for $r > 0$

$$\frac{r+2}{r(r+1)} - \frac{r+3}{(r+1)(r+2)} = \frac{r+4}{r(r+1)(r+2)}$$

(2)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = \frac{n(an+b)}{c(n+1)(n+2)}$$

where a, b and c are integers to be determined.

(4)

(Total for question = 6 marks)

Q36.

(a) Express $\dfrac{2}{r(r^2-1)}$ in partial fractions.

(3)

(b) Hence find, in terms of n,

$$\sum_{r=2}^{n} \frac{1}{r(r^2-1)}$$

Give your answer in the form

$$\frac{n^2 + An + B}{Cn(n+1)}$$

where A, B and C are constants to be found.

(5)

(Total for question = 8 marks)

Q37.

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Show that, for $r \geq 2$

$$\frac{2}{\sqrt{r}+\sqrt{r-2}} = \sqrt{r} - \sqrt{r-2}$$

(2)

(b) Hence use the method of differences to determine

$$\sum_{r=2}^{n} \frac{2}{\sqrt{r}+\sqrt{r-2}}$$

giving your answer in simplest form.

(3)

(c) Hence show that

$$\sum_{r=4}^{50} \frac{2}{\sqrt{r}+\sqrt{r-2}} = A + B\sqrt{2} + C\sqrt{3}$$

where *A*, *B* and *C* are integers to be determined.

(2)

(Total for question = 7 marks)

Q38.

(i)

$$\mathbf{P} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

The matrix **P** represents a geometrical transformation *U*

(a) Describe *U* fully as a single geometrical transformation.

(2)

The transformation *V*, represented by the 2 × 2 matrix **Q**, is a rotation through 240° anticlockwise about the origin followed by an enlargement about (0, 0) with scale factor 6

(b) Determine the matrix **Q**, giving each entry in exact numerical form.

(2)

Given that *U* followed by *V* is the transformation *T*, which is represented by the matrix **R**

(c) determine the matrix **R**

(2)

(ii) The transformation W is represented by the matrix

$$\begin{pmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix}$$

Show that there is a real number λ for which *W* maps the point (λ, 1) onto the point (4λ, 4), giving the exact value of λ

(5)

(Total for question = 11 marks)

Q39.

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

The matrix **P** represents the transformation *U*

(a) Give a full description of *U* as a single geometrical transformation.

(2)

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line $y = -x$

(b) Write down the matrix **Q**

(1)

The transformation U followed by the transformation V is represented by the matrix **R**

(c) Determine the matrix **R**

(2)

The transformation W is represented by the matrix 3**R**

The transformation W maps a triangle T to a triangle T'

The transformation W' maps the triangle T' back to the original triangle T

(d) Determine the matrix that represents W'

(3)

(Total for question = 8 marks)

Q40.

The complex number z on an Argand diagram is represented by the point P where

$$|z + 1 - 13i| = 3|z - 7 - 5i|$$

Given that the locus of P is a circle,

(a) determine the centre and radius of this circle.

(5)

The complex number w, on the same Argand diagram, is represented by the point Q, where

$$\arg(w - 8 - 6i) = -\frac{3\pi}{4}$$

Given that the locus of P intersects the locus of Q at the point R,

(b) determine the complex number representing R.

(4)

(Total for question = 9 marks)

Q41.

Given that

$$\mathbf{A} = \begin{pmatrix} k & 3 \\ -1 & k+2 \end{pmatrix}, \text{ where } k \text{ is a constant}$$

(a) show that det(**A**) > 0 for all real values of *k*,

(3)

(b) find $\mathbf{A}^{-1}$ in terms of *k*.

(2)

(Total for question = 5 marks)

Q42.

$$\mathbf{P} = \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation *U* represented by the matrix **P**.

(3)

The transformation *V*, represented by the 2×2 matrix **Q**, is a reflection in the line with equation $y = x$

(b) Write down the matrix **Q**.

(1)

Given that the transformation *V* followed by the transformation *U* is the transformation *T*, which is represented by the matrix **R**,

(c) find the matrix **R**.

(2)

(d) Show that there is a value of k for which the transformation T maps each point on the straight line $y = kx$ onto itself, and state the value of k.

(4)

(Total for question = 10 marks)

Q43.

The triangle T has vertices $A(2, 1)$, $B(2, 3)$ and $C(0, 1)$.

The triangle T' is the image of T under the transformation represented by the matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(a) Find the coordinates of the vertices of T'

(2)

(b) Describe fully the transformation represented by **P**

(2)

The 2 × 2 matrix **Q** represents a reflection in the x-axis and the 2 × 2 matrix **R** represents a rotation through 90° anticlockwise about the origin.

(c) Write down the matrix **Q** and the matrix **R**

(2)

(d) Find the matrix **RQ**

(2)

(e) Give a full geometrical description of the single transformation represented by the answer to part (d).

(2)

(Total for question = 10 marks)

Q44.

$$\mathbf{A} = \begin{pmatrix} 3 & a \\ -2 & -2 \end{pmatrix}$$

where a is a non-zero constant and $a \neq 3$

(a) Determine $\mathbf{A}^{-1}$ giving your answer in terms of a.

(2)

Given that $\mathbf{A} + \mathbf{A}^{-1} = \mathbf{I}$ where $\mathbf{I}$ is the 2×2 identity matrix,

(b) determine the value of a.

(3)

(Total for question = 5 marks)

Q45.

(i)

$$\mathbf{A} = \begin{pmatrix} -3 & 8 \\ -3 & k \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

The transformation represented by **A** transforms triangle T to triangle T'

The area of triangle T' is three times the area of triangle T

Determine the possible values of k

(4)

(ii) $\mathbf{B} = \begin{pmatrix} a & -4 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{BC} = \begin{pmatrix} 2 & 5 & 1 \\ 1 & 4 & 2 \end{pmatrix}$ where a is a constant

Determine, in terms of a, the matrix **C**

(4)

(Total for question = 8 marks)

Q46.

In part (i), the elements of each matrix should be expressed in exact numerical form.

(i) (a) Write down the 2 × 2 matrix that represents a rotation of 210° anticlockwise about the origin.

(1)

(b) Write down the 2 × 2 matrix that represents a stretch parallel to the *y*-axis with scale factor 5

(1)

The transformation *T* is a rotation of 210° anticlockwise about the origin followed by a stretch parallel to the *y*-axis with scale factor 5

(c) Determine the 2 × 2 matrix that represents *T*

(2)

(ii)

$$\mathbf{M} = \begin{pmatrix} k & k+3 \\ -5 & 1-k \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Find det **M**, giving your answer in simplest form in terms of *k*.

(2)

A closed shape *R* is transformed to a closed shape *R′* by the transformation represented by the matrix **M**.

Given that the area of *R* is 2 square units and that the area of *R′* is 16*k* square units,

(b) determine the possible values of *k*.

(3)

(Total for question = 9 marks)

Q47.

$$\mathbf{A} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Determine the matrix $\mathbf{A}^2$

(1)

(b) Describe fully the single geometrical transformation represented by the matrix $\mathbf{A}^2$

(2)

(c) Hence determine the smallest positive integer value of n for which $\mathbf{A}^n = \mathbf{I}$

(1)

The matrix **B** represents a stretch scale factor 4 parallel to the x-axis.

(d) Write down the matrix **B**

(1)

The transformation represented by matrix **A** followed by the transformation represented by matrix **B** is represented by the matrix **C**

(e) Determine the matrix **C**

(2)

The parallelogram P is transformed onto the parallelogram P' by the matrix **C**

(f) Given that the area of parallelogram P' is 20 square units, determine the area of parallelogram P

(2)

(Total for question = 9 marks)

Q48.

$$\mathbf{M} = \begin{pmatrix} k & k \\ 3 & 5 \end{pmatrix} \qquad \text{where } k \text{ is a non-zero constant}$$

(a) Determine $\mathbf{M}^{-1}$, giving your answer in simplest form in terms of k.

(2)

Hence, given that $\mathbf{N}^{-1} = \begin{pmatrix} k & k \\ 4 & -1 \end{pmatrix}$

(b) determine $(\mathbf{MN})^{-1}$, giving your answer in simplest form in terms of k.

(2)

(Total for question = 4 marks)

Q49.

(a) Show that

$$n^5 - (n-1)^5 \equiv 5n^4 - 10n^3 + 10n^2 - 5n + 1$$

(2)

(b) Hence, using the method of differences, show that for all integer values of n,

$$\sum_{r=1}^{n} r^4 = \frac{1}{30} n(n+1)(2n+1)(an^2 + bn + c)$$

where a, b and c are integers to be determined.

(7)

(Total for question = 9 marks)

Q50.

$$f(x) = x^2 + \frac{3}{x} - 1, \qquad x < 0$$

The only real root, α, of the equation $f(x) = 0$ lies in the interval $[-2, -1]$.

(a) Taking -1.5 as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 2 decimal places.

(5)

(b) Show that your answer to part (a) gives α correct to 2 decimal places.

(2)

(Total for question = 7 marks)

Q51.

(i)

$$f(x) = x^3 + 4x - 6$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval [1, 1.5]

(2)

(b) Taking 1.5 as a first approximation, apply the Newton Raphson process twice to $f(x)$ to obtain an approximate
value of α. Give your answer to 3 decimal places. Show your working clearly.

(4)

(ii)

$$g(x) = 4x^2 + x - \tan x$$

where x is measured in radians.

The equation $g(x) = 0$ has a single root β in the interval [1.4, 1.5]

Use linear interpolation on the values at the end points of this interval to obtain an approximation to β. Give your answer to 3 decimal places.

(4)

(Total for question = 10 marks)

Q52.

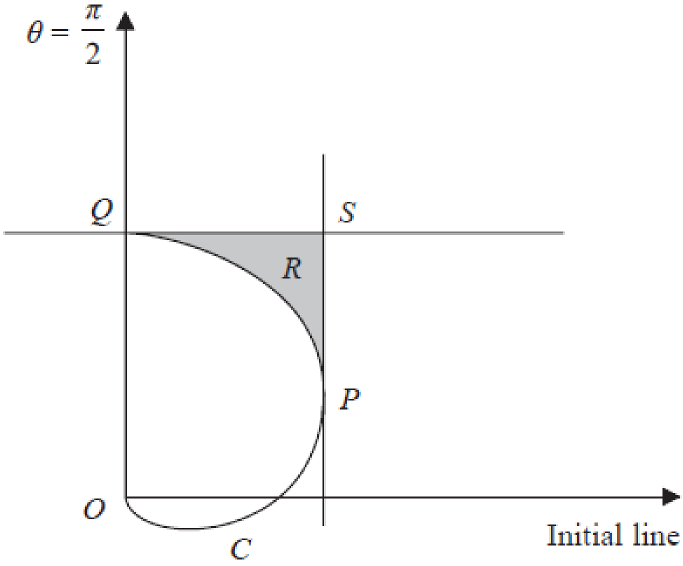

Figure 1

The curve C shown in Figure 1 has polar equation

$$r = 1 + \sin\theta \qquad -\frac{\pi}{2} < \theta \leqslant \frac{\pi}{2}$$

The point P lies on C such that the tangent to C at P is perpendicular to the initial line.

(a) Use calculus to determine the polar coordinates of P.

(5)

The tangent to C at the point Q where $\theta = \frac{\pi}{2}$ is parallel to the initial line.

The tangent to C at Q meets the tangent to C at P at the point S, as shown in Figure 1.

The finite region R, shown shaded in Figure 1, is bounded by the line segments QS, SP and the curve C.

(b) Use algebraic integration to show that the area of R is

$$\frac{1}{32}\left(a\sqrt{3} + b\pi\right)$$

where a and b are integers to be determined.

(6)

(Total for question = 11 marks)

Q53.

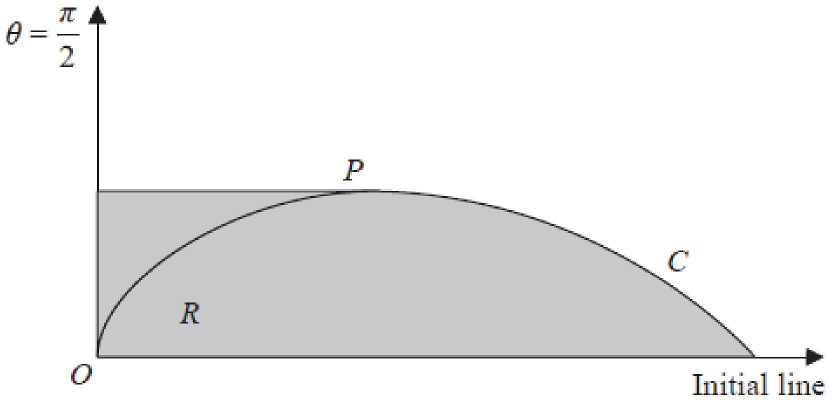

Figure 1

The curve C shown in Figure 1 has polar equation

$$r = 1 - \sin\theta \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The point P lies on C, such that the tangent to C at P is parallel to the initial line.

(a) Use calculus to determine the polar coordinates of P

(4)

The finite region R, shown shaded in Figure 1, is bounded by

- the line with equation $\theta = \frac{\pi}{2}$
- the tangent to C at P
- part of the curve C
- the initial line

(b) Use algebraic integration to show that the area of R is

$$\frac{1}{32}\left(a\pi + b\sqrt{3} + c\right)$$

where a, b and c are integers to be determined.

(6)

(Total for question = 10 marks)

Q54.

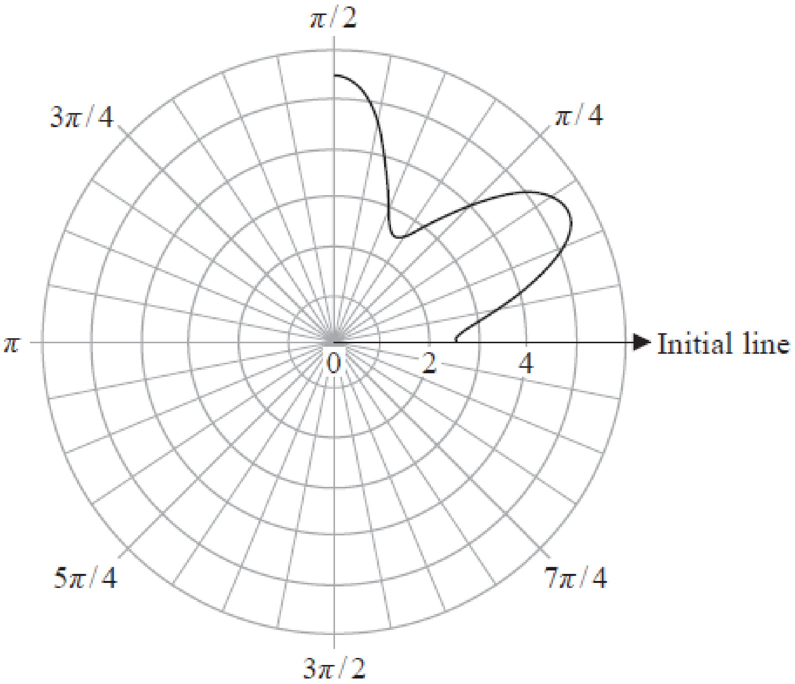

Figure 2

Figure 2 shows part of the curve with polar equation

$$r = 4 - \frac{3}{2}\cos 6\theta \qquad 0 \leq \theta < 2\pi$$

(a) Sketch, on the polar grid in Figure 2,

(i) the rest of the curve with equation $r = 4 - \frac{3}{2}\cos 6\theta \qquad 0 \leq \theta < 2\pi$

(ii) the polar curve with equation $r = 1 \qquad 0 \leq \theta < 2\pi$

(3)

In part (b) you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(b) Determine the exact area enclosed between the two curves defined in part (a).

(7)

(Total for question = 10 marks)

Q55.

(i) A sequence of positive numbers is defined by

$$u_1 = 5$$
$$u_{n+1} = 3u_n + 2, \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 2 \times (3)^n - 1$$

(5)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} \frac{4r}{3^r} = 3 - \frac{(3 + 2n)}{3^n}$$

(6)

(Total for question = 11 marks)

Q56.

(i) A sequence of numbers u_1, u_2, u_3,... is defined by

$$u_{n+1} = \frac{1}{3}(2u_n - 1) \qquad u_1 = 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 3\left(\frac{2}{3}\right)^n - 1$$

(6)

(ii) $f(n) = 2^{n+2} + 3^{2n+1}$

Prove by induction that, for $n \in \mathbb{Z}^+$, $f(n)$ is a multiple of 7

(6)

(Total for question = 12 marks)

Q57.

(a) Prove by induction that for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

(5)

(b) Hence show that

$$\sum_{r=1}^{n} (r^2 + 2) = \frac{n}{6}(an^2 + bn + c)$$

where *a*, *b* and *c* are integers to be found.

(4)

(c) Using your answers to part (b), find the value of

$$\sum_{r=10}^{25} (r^2 + 2)$$

(2)

(Total for question = 11 marks)

Q58.

Prove by induction that $4^{n+2} + 5^{2n+1}$ is divisible by 21 for all positive integers n.

(Total for question = 6 marks)

Q59.

(i) A sequence of numbers is defined by

$$u_1 = 0 \qquad u_2 = -6$$

$$u_{n+2} = 5u_{n+1} - 6u_n \qquad n \geq 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 3 \times 2^n - 2 \times 3^n$$

(5)

(ii) Prove by induction that, for all positive integers n,

$$f(n) = 3^{3n-2} + 2^{4n-1}$$

is divisible by 11

(5)

(Total for question = 10 marks)

Q60.

Prove, by induction, that for $n \in \mathbb{Z}$, $n \geq 2$

$$4^n + 6n - 10$$

is divisible by 18

(Total for question = 5 marks)

Q61.

(a) Prove by induction that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(5)

(b) Using the standard summation formulae, show that

$$\sum_{r=1}^{n} r(r+1)(r-1) = \frac{1}{4}n(n+A)(n+B)(n+C)$$

where A, B and C are constants to be determined.

(4)

(c) Determine the value of n for which

$$3\sum_{r=1}^{n} r(r+1)(r-1) = 17\sum_{r=n}^{2n} r^2$$

(5)

(Total for question = 14 marks)

Q62.

The equation $2x^2 + 5x + 7 = 0$ has roots α and β

Without solving the equation

(a) determine the exact value of $\alpha^3 + \beta^3$

(3)

(b) form a quadratic equation, with integer coefficients, which has roots

$$\frac{\alpha^2}{\beta} \text{ and } \frac{\beta^2}{\alpha}$$

(5)

(Total for question = 8 marks)

Q63.

The quadratic equation

$$4x^2 + 3x + k = 0$$

where k is an integer, has roots α and β

(a) Write down, in terms of k where appropriate, the value of $\alpha + \beta$ and the value of $\alpha\beta$

(2)

(b) Determine, in simplest form in terms of k, the value of $\frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2}$

(4)

(c) Determine a quadratic equation which has roots

$$\frac{\alpha}{\beta^2} \text{ and } \frac{\beta}{\alpha^2}$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integer values in terms of k

(3)

(Total for question = 9 marks)

Q64.

The quadratic equation

$$2x^2 + 4x - 3 = 0$$

has roots α and β.

Without solving the quadratic equation,

(a) find the exact value of

(i) $\alpha^2 + \beta^2$

(ii) $\alpha^3 + \beta^3$

(5)

(b) Find a quadratic equation which has roots $(\alpha^2 + \beta)$ and $(\beta^2 + \alpha)$, giving your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers.

(4)

(Total for question = 9 marks)

Q65.

$$f(x) = x^2 - 6x + 3$$

The equation $f(x) = 0$ has roots α and β

Without solving the equation,

(a) determine the value of

$$(\alpha^2 + 1)(\beta^2 + 1)$$

(4)

(b) find a quadratic equation which has roots

$$\frac{\alpha}{(\alpha^2 + 1)} \text{ and } \frac{\beta}{(\beta^2 + 1)}$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers to be determined.

(6)

(Total for question = 10 marks)

Q66.

The equation

$$x^4 + Ax^3 + Bx^2 + Cx + 225 = 0$$

where A, B and C are real constants, has

- a complex root 4 + 3i
- a repeated positive real root

(a) Write down the other complex root of this equation.

(1)

(b) Hence determine a quadratic factor of $x^4 + Ax^3 + Bx^2 + Cx + 225$

(2)

(c) Deduce the real root of the equation.

(2)

(d) Hence determine the value of each of the constants A, B and C

(3)

(Total for question = 8 marks)

Q67.

The complex number z is defined by

$$z = -\lambda + 3i \text{ where } \lambda \text{ is a positive real constant}$$

Given that the modulus of z is 5

(a) write down the value of λ

(1)

(b) determine the argument of z, giving your answer in radians to one decimal place.

(2)

In part (c) you must show detailed reasoning.

Solutions relying on calculator technology are not acceptable.

(c) Express in the form $a + ib$ where a and b are real,

(i) $\dfrac{z + 3i}{2 - 4i}$

(ii) z^2

(5)

(d) Show on a single Argand diagram the points A, B, C and D that represent the complex numbers

$$z,\ z^*,\ \frac{z+3\text{i}}{2-4\text{i}} \text{ and } z^2$$

(3)

(Total for question = 11 marks)

Q68.

Given that $x = \frac{3}{8} + \frac{\sqrt{71}}{8}\text{i}$ is a root of the equation

$$4x^3 - 19x^2 + px + q = 0$$

(a) write down the other complex root of the equation.

(1)

Given that $x = 4$ is also a root of the equation,

(b) find the value of p and the value of q.

(4)

(Total for question = 5 marks)

Q69.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

$$f(z) = 4z^3 + pz^2 - 24z + 108$$

where p is a constant.

Given that -3 is a root of the equation $f(z) = 0$

(a) determine the value of p

(2)

(b) using algebra, solve $f(z) = 0$ completely, giving the roots in simplest form,

(4)

(c) determine the modulus of the complex roots of $f(z) = 0$

(2)

(d) show the roots of $f(z) = 0$ on a single Argand diagram.

(2)

(Total for question = 10 marks)

Q70.

The quadratic equation

$$Ax^2 + 5x - 12 = 0$$

where A is a constant, has roots α and β

(a) Write down an expression in terms of A for

(i) $\alpha + \beta$

(ii) $\alpha\beta$

(2)

The equation

$$4x^2 - 5x + B = 0$$

where B is a constant, has roots $\alpha - \frac{3}{\beta}$ and $\beta - \frac{3}{\alpha}$

(b) Determine the value of A

(3)

(c) Determine the value of B

(3)

(Total for question = 8 marks)

Q71.

The quadratic equation

$$2x^2 – 5x + 7 = 0$$

has roots α and β

Without solving the equation,

(a) write down the value of $(\alpha + \beta)$ and the value of $\alpha\beta$

(1)

(b) determine, giving each answer as a simplified fraction, the value of

(i) $\alpha^2 + \beta^2$

(ii) $\alpha^3 + \beta^3$

(4)

(c) find a quadratic equation that has roots

$$\frac{1}{\alpha^2 + \beta} \text{ and } \frac{1}{\beta^2 + \alpha}$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers to be determined.

(4)

(Total for question = 9 marks)

Q72.

The quadratic equation

$$2x^2 – 3x + 5 = 0$$

has roots α and β

Without solving the equation,

(a) write down the value of ($\alpha + \beta$) and the value of $\alpha\beta$

(1)

(b) determine the value of

(i) $\alpha^2 + \beta^2$

(ii) $\alpha^3 + \beta^3$

(4)

(c) find a quadratic equation which has roots

$$(\alpha^3 - \beta) \text{ and } (\beta^3 - \alpha)$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers to be determined.

(5)

(Total for question = 10 marks)

Q73.

Determine the general solution of the differential equation

$$2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 2e^{3x}$$

(Total for question = 6 marks)

Q74.

(a) Determine the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 48x^2 - 34$$

(5)

Given that $y = 4$ and $\frac{dy}{dx} = 21$ at $x = 0$

(b) determine the particular solution of the differential equation.

(4)

(c) Hence find the value of y at $x = -2$, giving your answer in the form $pe^q + r$ where p, q and r are integers to be determined.

(2)

(Total for question = 11 marks)

Q75.

(a) Use the standard results for $\sum_{r=1}^{n} r^3$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that for all positive integers n,

$$\sum_{r=1}^{n} r(r-1)(r-3) = \frac{1}{12}n(n+1)(n-1)(3n-10)$$

(5)

(b) Hence show that

$$\sum_{r=n+1}^{2n+1} r(r-1)(r-3) = \frac{1}{12}n(n+1)(an^2 + bn + c)$$

where a, b and c are integers to be determined.

(3)

(Total for question = 8 marks)

Q76.

(a) Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that for all positive integers n

$$\sum_{r=0}^{n} (r+1)(r+2) = \frac{1}{3}(n+1)(n+2)(n+3)$$

(5)

(b) Hence determine the value of

10 × 11 + 11 × 12 + 12 × 13 + ... + 100 × 101

(3)

(Total for question = 8 marks)

Q77.

Given that

$$\frac{2n+1}{n^2(n+1)^2} \equiv \frac{A}{n^2} + \frac{B}{(n+1)^2}$$

(a) determine the value of A and the value of B

(1)

(b) Hence show that, for $n \geq 5$

$$\sum_{r=5}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{n^2 + an + b}{c(n+1)^2}$$

where a, b and c are integers to be determined.

(4)

(Total for question = 5 marks)

Q78.

A complex number z is represented by the point P in an Argand diagram.

Given that

$$|z - 2\mathrm{i}| = |z - 3|$$

(a) sketch the locus of P. You do **not** need to find the coordinates of any intercepts.

(2)

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{\mathrm{i}z}{z - 2\mathrm{i}} \qquad z \neq 2\mathrm{i}$$

Given that T maps $|z - 2\mathrm{i}| = |z - 3|$ to a circle C in the w-plane,

(b) find the equation of C, giving your answer in the form

$$|w - (p + q\mathrm{i})| = r$$

where p, q and r are real numbers to be determined.

(6)

(Total for question = 8 marks)

Q79.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that for all positive integers n

$$\sum_{r=1}^{n} (7r - 5)^2 = \frac{n}{6}(7n + 1)(An + B)$$

where A and B are integers to be determined.

(Total for question = 6 marks)

Q80.

(a) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that

$$\sum_{r=1}^{n}(r+1)(r+5) = \frac{n}{6}(n+7)(2n+7)$$

for all positive integers *n*.

(5)

(b) Hence show that

$$\sum_{r=n+1}^{2n}(r+1)(r+5) = \frac{7n}{6}(n+1)(an+b)$$

where *a* and *b* are integers to be determined.

(2)

(Total for question = 7 marks)

Q81.

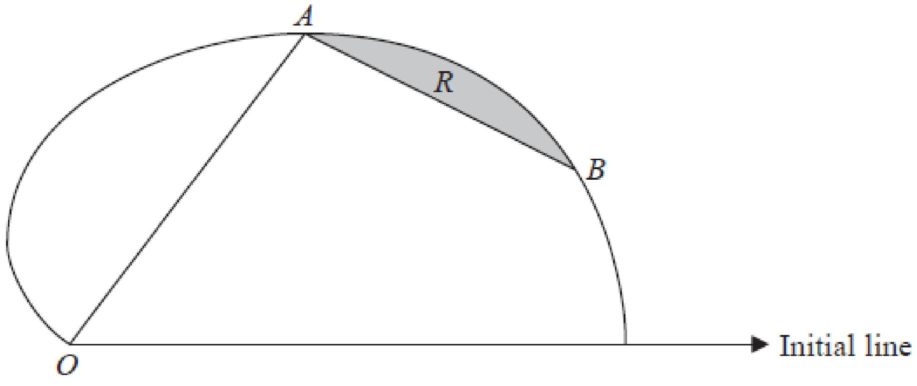

Figure 1

The curve shown in Figure 1 has polar equation

$$r = 4a(1 + \cos\theta) \qquad 0 \leq \theta < \pi$$

where a is a positive constant.

The tangent to the curve at the point A is parallel to the initial line.

(a) Show that the polar coordinates of A are $\left(6a, \frac{\pi}{3}\right)$

(6)

The point B lies on the curve such that angle $AOB = \frac{\pi}{6}$

The finite region R, shown shaded in Figure 1, is bounded by the line AB and the curve.

(b) Use calculus to determine the area of the shaded region R, giving your answer in the form $a^2(n\pi + p\sqrt{3} + q)$, where n, p and q are integers.

(7)

(Total for question = 13 marks)

Q82.

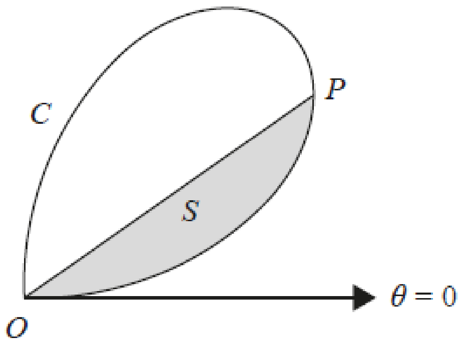

Figure 1

Figure 1 shows a sketch of curve C with polar equation

$$r = 3\sin 2\theta \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

The point P on C has polar coordinates (R, φ). The tangent to C at P is perpendicular to the initial line.

(a) Show that $\tan\phi = \frac{1}{\sqrt{2}}$

(4)

(b) Determine the exact value of *R*.

(2)

The region *S*, shown shaded in Figure 1, is bounded by *C* and the line *OP*, where *O* is the pole.

(c) Use calculus to show that the exact area of *S* is

$$p \arctan \frac{1}{\sqrt{2}} + q\sqrt{2}$$

where *p* and *q* are constants to be determined.

Solutions relying entirely on calculator technology are not acceptable.

(7)

(Total for question = 13 marks)

Q83.

(a) Given that $x = t^{\frac{1}{2}}$, determine, in terms of *y* and *t*,

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(5)

(b) Hence show that the transformation , where *t* > 0, transforms the differential equation

$$x\frac{d^2y}{dx^2} - \left(6x^2 + 1\right)\frac{dy}{dx} + 9x^3y = x^5 \qquad \text{(I)}$$

into the differential equation

$$4\frac{d^2y}{dt^2} - 12\frac{dy}{dt} + 9y = t \qquad \text{(II)}$$

(2)

(c) Solve differential equation (II) to determine a general solution for *y* in terms of *t*.

(5)

(d) Hence determine the general solution of differential equation (I).

(1)

(Total for question = 13 marks)

Q84.

The matrix **A** is defined by

$$\mathbf{A} = \begin{pmatrix} 4 & -5 \\ -3 & 2 \end{pmatrix}$$

The transformation represented by **A** maps triangle *T* onto triangle *T*'

Given that the area of triangle *T* is 23 cm^2

(a) determine the area of triangle *T*'

(2)

The point *P* has coordinates $(3p + 2, 2p - 1)$ where *p* is a constant. The transformation represented by **A** maps *P* onto the point *P*' with coordinates (17, –18)

(b) Determine the value of *p*.

(2)

Given that

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(c) describe fully the single geometrical transformation represented by matrix **B**

(2)

The transformation represented by matrix **A** followed by the transformation represented by matrix **C** is equivalent to the transformation represented by matrix **B**

(d) Determine **C**

(3)

(Total for question = 9 marks)

Q85.

(a) Show that the transformation $y = \frac{1}{z}$ transforms the differential equation

$$x^2 \frac{dy}{dx} + xy = 2y^2 \qquad \text{(I)}$$

into the differential equation

$$\frac{dz}{dx} - \frac{z}{x} = -\frac{2}{x^2} \qquad \text{(II)}$$

(3)

(b) Solve differential equation (II) to determine z in terms of x.

(4)

(c) Hence determine the particular solution of differential equation (I) for which $y = -\frac{3}{8}$

at $x = 3$

Give your answer in the form $y = f(x)$.

(2)

(Total for question = 9 marks)