Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Time 1 hour 30 minutes

Paper reference **9FM0/02**

Further Mathematics

Advanced

PAPER 2: Core Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- Good luck with your examination.

Turn over

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1. Given that

$$z_1 = 3\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$$

$$z_2 = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) - \mathrm{i}\sin\left(\frac{\pi}{12}\right)\right)$$

$-\sin(\theta) = \sin(-\theta)$
$\cos(\theta) = \cos(-\theta)$

(a) write down the exact value of

(i) $|z_1 z_2|$

(ii) $\arg(z_1 z_2)$

(2)

Given that $w = z_1 z_2$ and that $\arg(w^n) = 0$, where $n \in \mathbb{Z}^+$

(b) determine

(i) the smallest positive value of n

(ii) the corresponding value of $|w^n|$

(3)

(a) $z_1 = 3[\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})]$

$z_2 = \sqrt{2}[\cos(-\frac{\pi}{12}) + i\sin(-\frac{\pi}{12})]$

(i) $|z_1 z_2| = |z_1| \times |z_2|$

$= 3 \times \sqrt{2}$

$= 3\sqrt{2}$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$= \frac{\pi}{3} + -\frac{\pi}{12}$

$= \frac{\pi}{4}$

(b) $w = 3\sqrt{2}[\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})]$

$w^n = \{3\sqrt{2}[\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})]\}^n$

$= (3\sqrt{2})^n[\cos(\frac{\pi n}{4}) + i\sin(\frac{\pi n}{4})]$

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Question 1 continued

$\arg(w^n) = 0$, THEN $\left(\frac{\pi n}{4}\right)$ IS A MULTIPLE OF 2π

$\therefore n = 8$

(ii) $|w^8| = (3\sqrt{2})^8$

$= 104\,976$

(Total for Question 1 is 5 marks)

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2. $$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}$$

The matrix **A** represents the linear transformation M.

Prove that, for the linear transformation M, there are no invariant lines.

(5)

ASUME (x, y) AND (x', y') LIE ON $y = mx + c$ ①

$$\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} 4x - 2y \\ 5x + 3y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$x' = 4x - 2y = 4x - 2(mx + c) = (4 - 2m)x - 2c$ ②

$y' = 5x + 3y = 5x + 3(mx + c) = (5 + 3m)x + 3c$ ③

i.e. $(5 + 3m)x + 3c = m[(4 - 2m)x - 2c] + c$

$(5 + 3m)x + 3c = (4m - 2m^2)x + (c - 2mc)$

COMPARE COEFFICIENTS:

x: $5 + 3m = 4m - 2m^2$ NO°: $3c = c - 2mc$

$2m^2 - m + 5 = 0$ ④ $2c = -2mc$ ⑤

FROM ④: $(-1)^2 - 4(2)(5) = -39 < 0$

$\therefore$ NO REAL VALUES OF m

$\therefore$ ⑤ HAS NO SOLUTIONS

$\therefore$ NO INVARIANT LINES

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3. $\mathrm{f}(x) = \arcsin x \qquad -1 \leqslant x \leqslant 1$

(a) Determine the first two non-zero terms, in ascending powers of x, of the Maclaurin series for $\mathrm{f}(x)$, giving each coefficient in its simplest form.

(4)

(b) Substitute $x = \frac{1}{2}$ into the answer to part (a) and hence find an approximate value for π

Give your answer in the form $\frac{p}{q}$ where p and q are integers to be determined.

(2)

(a) $f(x) = \cancel{f(0)} + xf'(0) + \cancel{\frac{x^2}{2!}f''(0)} + \frac{x^3}{3!}f'''(0) + \cdots$

$f(x) = \arcsin x$

$f(0) = \arcsin(0) = 0$

$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$

$f'(0) = (1-0^2)^{-1/2} = 1$

$f''(x) = -\frac{1}{2}(1-x^2)^{-3/2} \times -2x = x(1-x^2)^{-3/2}$

$f''(0) = 0(1-0^2)^{-3/2} = 0$

$f'''(x) = x \times -\frac{3}{2}(1-x^2)^{-5/2} \times -2x + 1 \times (1-x^2)^{-3/2}$

$= 3x^2(1-x^2)^{-5/2} + (1-x^2)^{-3/2}$

$f'''(0) = 3 \times 0^2 \times (1-0^2)^{-5/2} + (1-0^2)^{-3/2} = 1$

$\therefore f(x) = x(1) + \frac{x^3}{3!}(1)$

$\therefore \arcsin x = x + \frac{x^3}{6}$

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Question 3 continued

(b) $x = 1/2 : \arcsin(1/2) = (1/2) + \dfrac{(1/2)^3}{6}$

$\dfrac{\pi}{6} = \dfrac{25}{48}$

$\pi = 6 \times \dfrac{25}{48}$

$\therefore \ \pi \approx \dfrac{25}{8}$

$\therefore \ p = 25, \ q = 8$

(Total for Question 3 is 6 marks)

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4. In this question you may assume the results for

$$\sum_{r=1}^{n} r^3, \quad \sum_{r=1}^{n} r^2 \quad \text{and} \quad \sum_{r=1}^{n} r$$

(a) Show that the sum of the cubes of the first n positive odd numbers is

$$n^2(2n^2 - 1)$$

(5)

The sum of the cubes of 10 consecutive positive odd numbers is 99 800

(b) Use the answer to part (a) to determine the smallest of these 10 consecutive positive odd numbers.

(4)

(a) CUBE NO°: $1^3 = 1,\ 2^3 = 8,\ 3^3 = 27,\ 4^3 = 64,\ 5^3 = 125,\ 6^3 = 216, \ldots$

$1, 3, 5, 7, \ldots$ HAS r^{th} TERM $2r-1$

REQUIRE: $1^3 + 3^3 + 5^3 + 7^3 + \ldots + (2r-1)^3$

$$\therefore \sum_{r=1}^{n}(2r-1)^3 = \sum_{r=1}^{n}\left[\binom{3}{0}(2r)^3(-1)^0 + \binom{3}{1}(2r)^2(-1)^1 + \binom{3}{2}(2r)^1(-1)^3 + \binom{3}{3}(2r)^0(-1)^3\right]$$

$$\equiv \sum_{r=1}^{n}\left[8r^3 - 12r^2 + 6r - 1\right]$$

$$\equiv 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$= 8 \times \frac{1}{4} \times n^2(n+1)^2 - 12 \times \frac{1}{6} \times n(n+1)(2n+1) + 6 \times \frac{1}{2} \times n(n+1) - n$$

$$= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$$

$$= 2n^2(n^2+2n+1) - 2n(2n^2+3n+1) + 3n^2 + 3n - n$$

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Question 4 continued

$= 2n^4 + 4n^3 + 2n^2 - 2n^3 - 6n^2 - 2n + 3n^2$

$+ 3n - n$

$= 2n^4 - n^2$

$= n^2(2n^2 - 1)$

(b) $\sum_{r=1}^{10} (2r-1)^3 = 10^2 \times (2 \times 10^2 - 1) = 19800$

$\therefore$ NOT STARTING AT $r=1$ AND FINISHING AT $r=10$

LET x = POSITION OF SMALLEST OF THESE 10 CONSECUTIVE POSITIVE ODD NUMBERS

$\therefore \sum_{r=x}^{x+9} (2r-1)^3 = \sum_{r=1}^{x+9} (2r-1)^3 - \sum_{r=1}^{x-1} (2r-1)^3$

$= (x+9)^2(2(x+9)^2 - 1) - (x-1)^2(2(x-1)^2 - 1)$

$= (x^2 + 18x + 81)(2x^2 + 36x + 161) -$

$(x^2 - 2x + 1)(2x^2 - 4x + 1)$

$= (2x^4 + 36x^3 + 161x^2 + 36x^3 + 648x^2 +$

$2898x + 162x^2 + 2916x + 13041)$

$-(2x^4 - 4x^3 + x^2 - 4x^3 + 8x^3 - 2x + 2x^2$

$-4x + 1)$

$= 80x^3 + 960x^2 + 5820x + 13040$

Question 4 continued

i.e. $80x^3 + 960x^2 + 5820x + 13040 = 99800$

$80x^3 + 960x^2 + 5820x - 86760 = 0$

$x = 6, -9 \pm 9.87\ldots i$

$\therefore$ $x = 6$ IS ONLY SOLUTION

1, 3, 5, 7, 9, 11

$\therefore$ SMALLEST OF THESE 10 CONSECUTIVE POSITIVE ODD NO$^{\underline{o}}$S IS 11

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5. The curve C has equation

$$y = \arccos\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2$$

(a) Show that C has no stationary points.

(3)

The normal to C, at the point where $x = 1$, crosses the x-axis at the point A and crosses the y-axis at the point B.

Given that O is the origin,

(b) show that the area of the triangle OAB is $\frac{1}{54}\left(p\sqrt{3} + q\pi + r\sqrt{3}\pi^2\right)$ where p, q and r are integers to be determined.

(5)

(a) $y = \cos^{-1}\left(\frac{x}{2}\right)$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(x/2)^2}} \times \frac{1}{2}$$

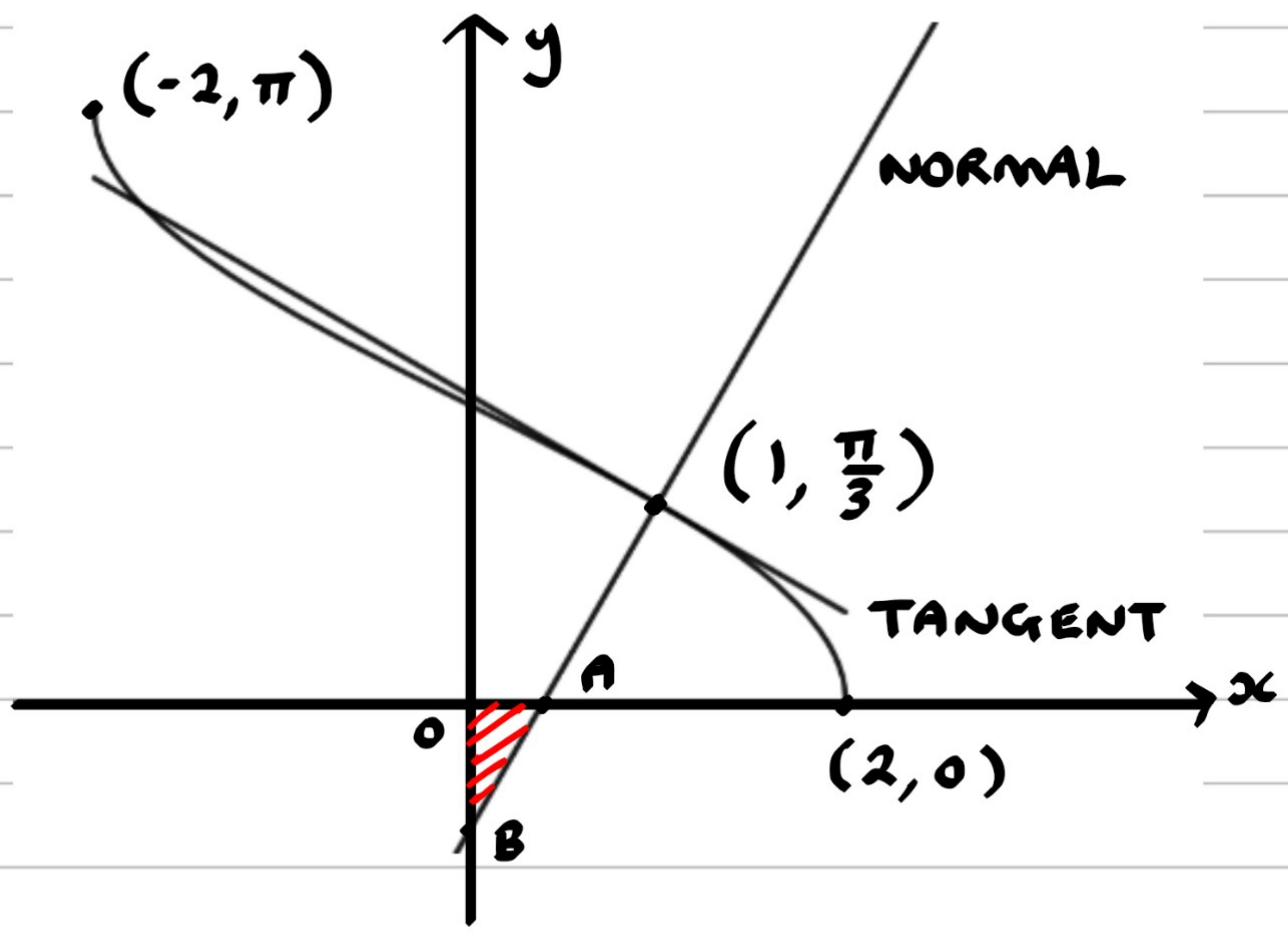
$$dy/dx = 0: \quad -\frac{1}{\sqrt{1-(x/2)^2}} \times \frac{1}{2} = 0$$

$$\frac{1}{\sqrt{1-x^2/4}} = 0$$

$1 = 0$, WHICH IS A CONTRADICTION

$\therefore$ C HAS NO SP

(b)



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Question 5 continued

$x = 1:\quad y = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - 1^2/4}} \times \frac{1}{2} = -\frac{1}{\sqrt{3}}$ (GRADIENT OF TANGENT)

$\therefore$ GRADIENT OF NORMAL $= \frac{-1}{-1/\sqrt{3}} = \sqrt{3}$

$\therefore y - \frac{\pi}{3} = \sqrt{3}(x - 1)$

$y = 0:\quad 0 - \frac{\pi}{3} = \sqrt{3}(x - 1)$

$-\frac{\pi}{3\sqrt{3}} = x - 1$

$x = 1 - \frac{\pi}{3\sqrt{3}}$

$\therefore A\left(1 - \frac{\pi}{3\sqrt{3}}, 0\right)$

$x = 0:\quad y - \frac{\pi}{3} = \sqrt{3}(0 - 1)$

$y = \frac{\pi}{3} - \sqrt{3}$

$\therefore B\left(0, \frac{\pi}{3} - \sqrt{3}\right)$

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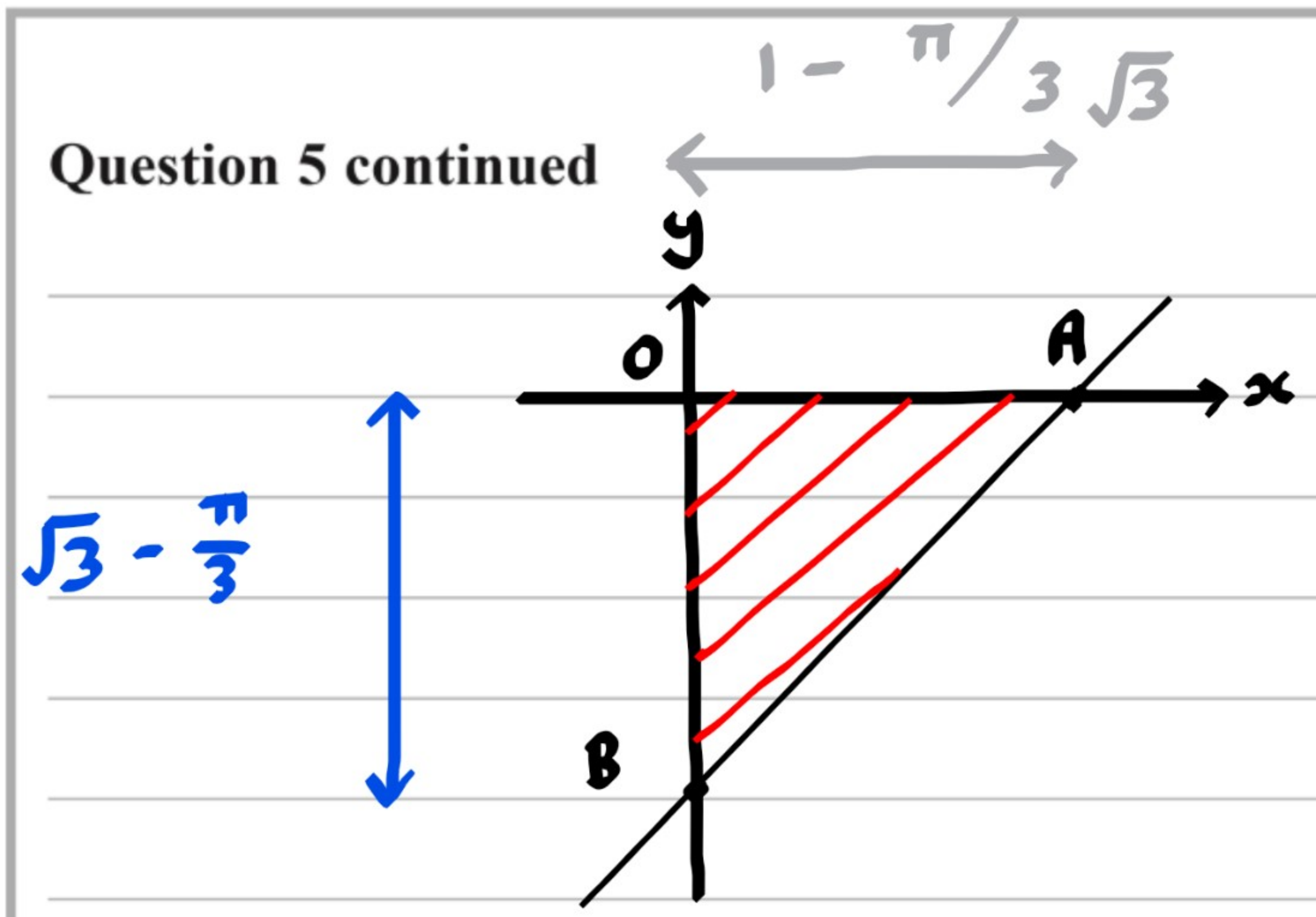
Question 5 continued



$$\text{AREA} = \frac{1}{2}\left[1-\frac{\pi}{3\sqrt{3}}\right]\left[\sqrt{3}-\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[\sqrt{3}-\frac{\pi}{3}-\frac{\pi}{3}+\frac{\pi^2}{9\sqrt{3}}\right]$$

$$= \frac{\sqrt{3}}{2}-\frac{2\pi}{6}+\frac{\sqrt{3}\pi^2}{54}$$

$$= \frac{1}{54}\left[27\sqrt{3}-18\pi+\sqrt{3}\pi^2\right]$$

$\therefore p = 27,\ q = -18,\ r = 1$

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6. The curve C has equation

$$r = a(p + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

where a and p are positive constants and $p > 2$

There are exactly four points on C where the tangent is perpendicular to the initial line.

(a) Show that the range of possible values for p is

$$2 < p < 4$$

(5)

(b) Sketch the curve with equation

$$r = a(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi \qquad \text{where } a > 0$$

(1)

John digs a hole in his garden in order to make a pond.

The pond has a uniform horizontal cross section that is modelled by the curve with equation

$$r = 20(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

where r is measured in centimetres.

The depth of the pond is 90 centimetres.

Water flows through a hosepipe into the pond at a rate of 50 litres per minute.

Given that the pond is initially empty,

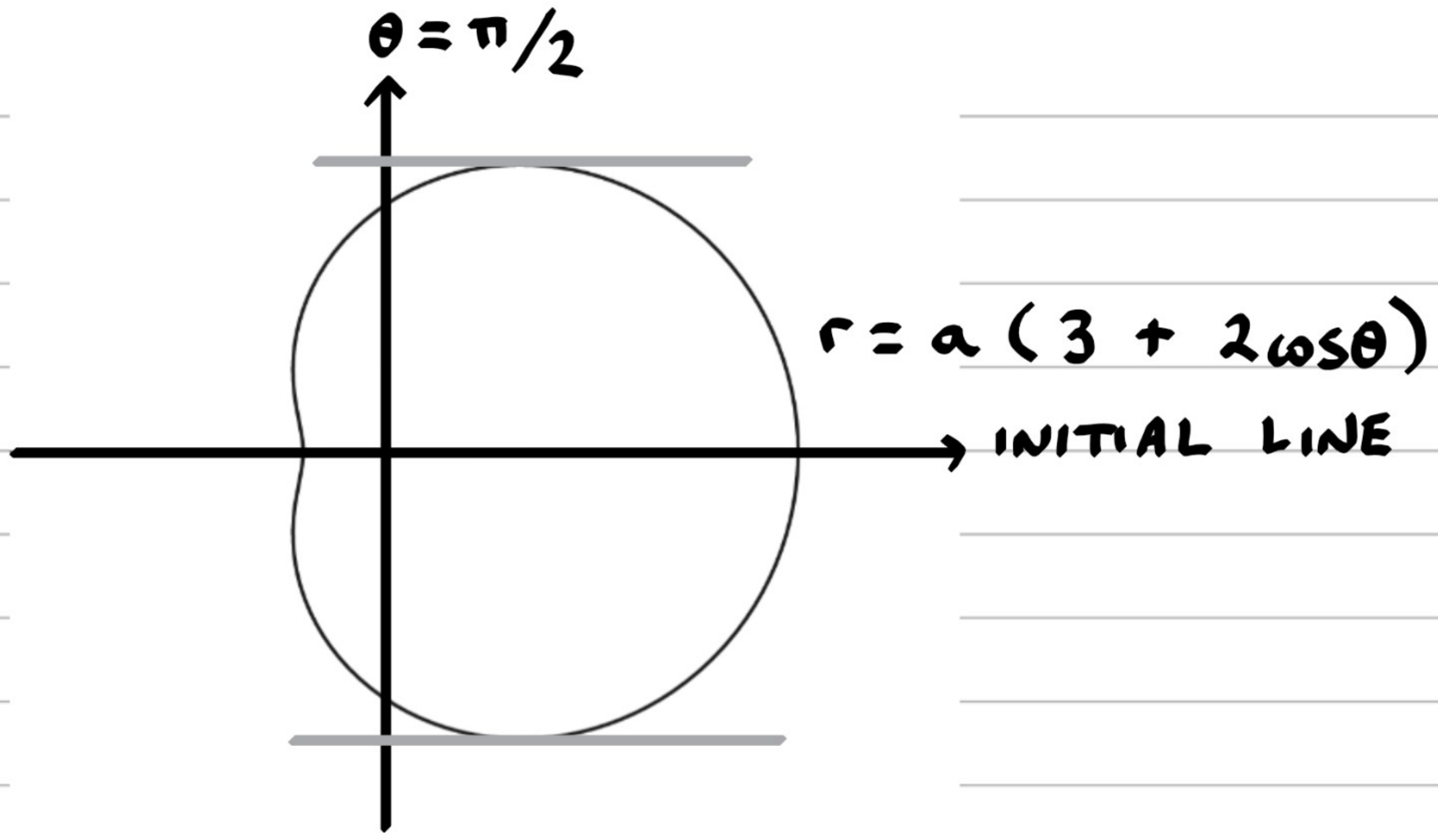
(c) determine how long it will take to completely fill the pond with water using the hosepipe, according to the model. Give your answer to the nearest minute.

(7)

(d) State a limitation of the model.

(1)

(a)



$x = r\cos\theta = a(p + 2\cos\theta)\cos\theta$

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Question 6 continued

$\frac{dx}{d\theta} = a[\cos\theta(-2\sin\theta) + (p + 2\cos\theta)(-\sin\theta)]$

$\frac{dx}{d\theta} = a[-2\sin\theta\cos\theta - \sin\theta(p + 2\cos\theta)]$

$dx/d\theta = 0 : a[-2\sin\theta\cos\theta - \sin\theta(p + 2\cos\theta)] = 0$

$-2\sin\theta\cos\theta - p\sin\theta - 2\sin\theta\cos\theta = 0$

$-4\sin\theta\cos\theta - p\sin\theta = 0$

$\sin\theta(-4\cos\theta - p) = 0$

$\sin\theta = 0, \ -4\cos\theta - p = 0$

$\sin\theta = 0, \ \cos\theta = -\frac{p}{4}$

$\theta = 0, \pi \qquad -\frac{p}{4} > -1 \quad \because -1 \leq \cos\theta \leq 1$

$p < 4$

AS $p > 2$

$\therefore 2 < p < 4$

(b) SEE ABOVE

(c) AREA $= \frac{1}{2}\int_0^{2\pi} [20(3 + 2\cos\theta)]^2 \, d\theta$

$= 200\int_0^{2\pi} (9 + 12\cos\theta + 4\cos^2\theta) \, d\theta$

$= 200\int_0^{2\pi} (9 + 12\cos\theta + 4 \times \frac{1}{2}(1 + \cos 2\theta)) \, d\theta$

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$= \int_0^{2\pi} (2200 + 2400\cos\theta + 400\cos 2\theta)\, d\theta$

$= \left[2200\theta + 2400\sin\theta + 200\sin 2\theta\right]_0^{2\pi}$

$= \{2200 \times 2\pi + 0 + 0\} - \{0 + 0 + 0\}$

$= 4400\pi \text{ cm}^2$

VOL $= 4400\pi \times 90$

$= 396000\pi \text{ cm}^3$

$= 396\pi \ \ell$

TIME $= \dfrac{396\pi \ell}{50 \ \ell/\text{min}}$

$= 24.88\ldots$ min

$= 25$ min

(d) POLAR EQUATION NOT LIKELY TO BE ACCURATE

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7. **Solutions based entirely on graphical or numerical methods are not acceptable.**

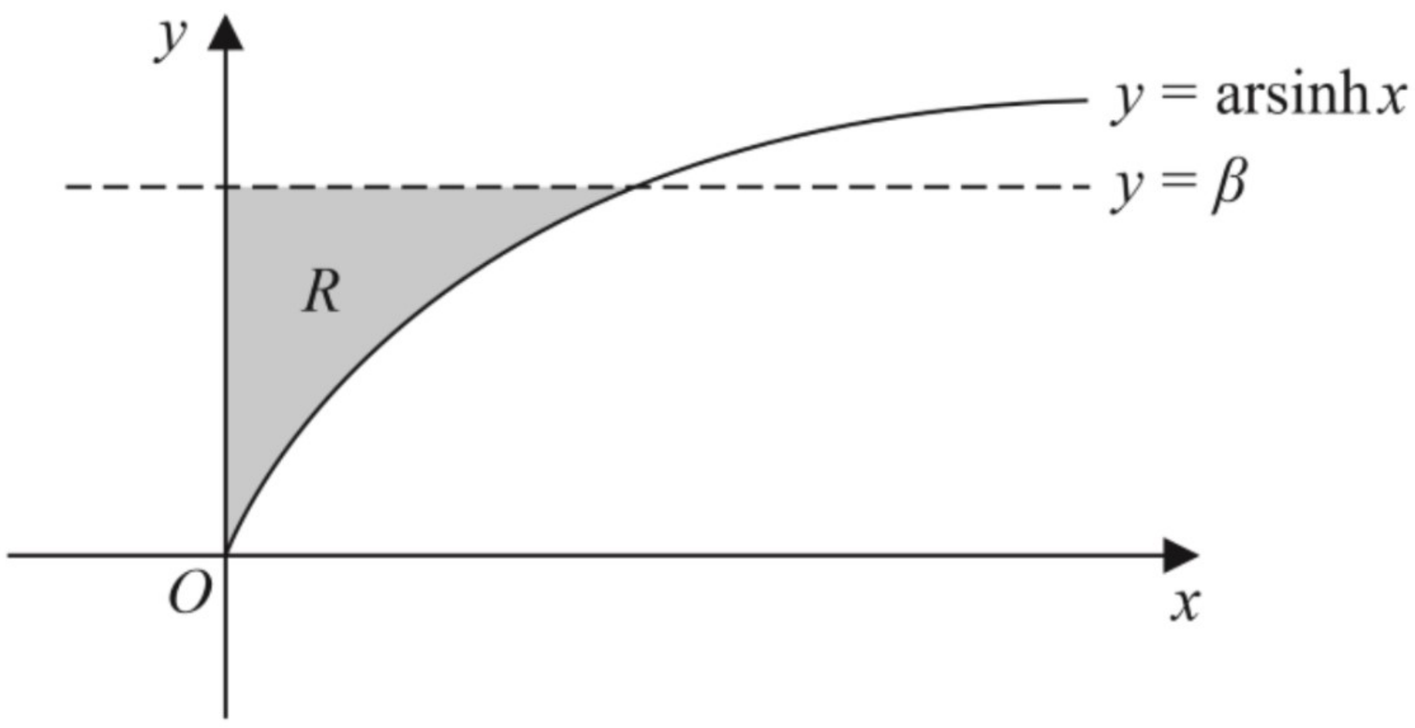

Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = \text{arsinh}\,x \qquad x \geqslant 0$$

and the straight line with equation $y = \beta$

The line and the curve intersect at the point with coordinates (α, β)

Given that $\beta = \frac{1}{2}\ln 3$

(a) show that $\alpha = \frac{1}{\sqrt{3}}$

(3)

The finite region R, shown shaded in Figure 1, is bounded by the curve with equation $y = \text{arsinh}\,x$, the y-axis and the line with equation $y = \beta$

The region R is rotated through 2π radians about the y-axis.

(b) Use calculus to find the exact value of the volume of the solid generated.

(6)

(a) (α, β) : $\beta = \text{arsinh}\,\alpha$

$\ln(\alpha + \sqrt{\alpha^2+1}) = \frac{1}{2}\ln 3$

$\ln(\alpha + \sqrt{\alpha^2+1}) = \ln 3^{1/2}$

$\alpha + \sqrt{\alpha^2+1} = 3^{1/2}$

$\sqrt{\alpha^2+1} = \sqrt{3} - \alpha$

$\alpha^2 + 1 = (\sqrt{3} - \alpha)^2$

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Question 7 continued

$\alpha^2 + 1 = 3 - 2\sqrt{3}\alpha + \alpha^2$

$2\sqrt{3}\alpha = 2$

$\therefore \alpha = \dfrac{1}{\sqrt{3}}$

(b) $\text{VOL} = \pi \int_{\mu}^{\lambda} x^2 \, dy$

IF $y = \text{arsinh}\,x$, THEN $x = \sinh y$

$\therefore x^2 = \sinh^2 y$

$\therefore \text{VOL OF } R = \pi \int_0^{\frac{1}{2}\ln 3} \sinh^2 y \, dy$

$= \pi \int_0^{\frac{1}{2}\ln 3} \left(\dfrac{e^y - e^{-y}}{2}\right)^2 dy$

$= \dfrac{\pi}{4} \int_0^{\frac{1}{2}\ln 3} (e^{2y} - 2e^y e^{-y} + e^{-2y}) dy$

$= \dfrac{\pi}{4} \int_0^{\frac{1}{2}\ln 3} (e^{2y} - 2 + e^{-2y}) dy$

$= \dfrac{\pi}{4} \left[\tfrac{1}{2}e^{2y} - 2y - \tfrac{1}{2}e^{-2y}\right]_0^{\frac{1}{2}\ln 3}$

$= \dfrac{\pi}{4} \left[\left\{\tfrac{1}{2}e^{2\times\frac{1}{2}\ln 3} - 2\times\tfrac{1}{2}\ln 3 - \tfrac{1}{2}e^{-2\times\frac{1}{2}\ln 3}\right\}\right.$

$\left. - \left\{\tfrac{1}{2}e^0 - 2\times 0 - \tfrac{1}{2}e^0\right\}\right]$

$= \dfrac{\pi}{4}\left[\tfrac{1}{2}e^{\ln 3} - \ln 3 - \tfrac{1}{2}e^{-\ln 3}\right]$

$= \dfrac{\pi}{4}\left[\tfrac{1}{2}\times 3 - \ln 3 - \tfrac{1}{2}\times 3^{-1}\right]$

$= \dfrac{\pi}{4}\left[\tfrac{4}{3} - \ln 3\right]$

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8. (i) The point P is one vertex of a regular pentagon in an Argand diagram.
The centre of the pentagon is at the origin.

Given that P represents the complex number $6 + 6\text{i}$, determine the complex numbers that represent the other vertices of the pentagon, giving your answers in the form $r\text{e}^{\text{i}\theta}$

(5)

(ii) (a) On a single Argand diagram, shade the region, R, that satisfies both

$$|z - 2\text{i}| \leqslant 2 \qquad \text{and} \qquad \frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{3}\pi$$

(2)

(b) Determine the exact area of R, giving your answer in simplest form.

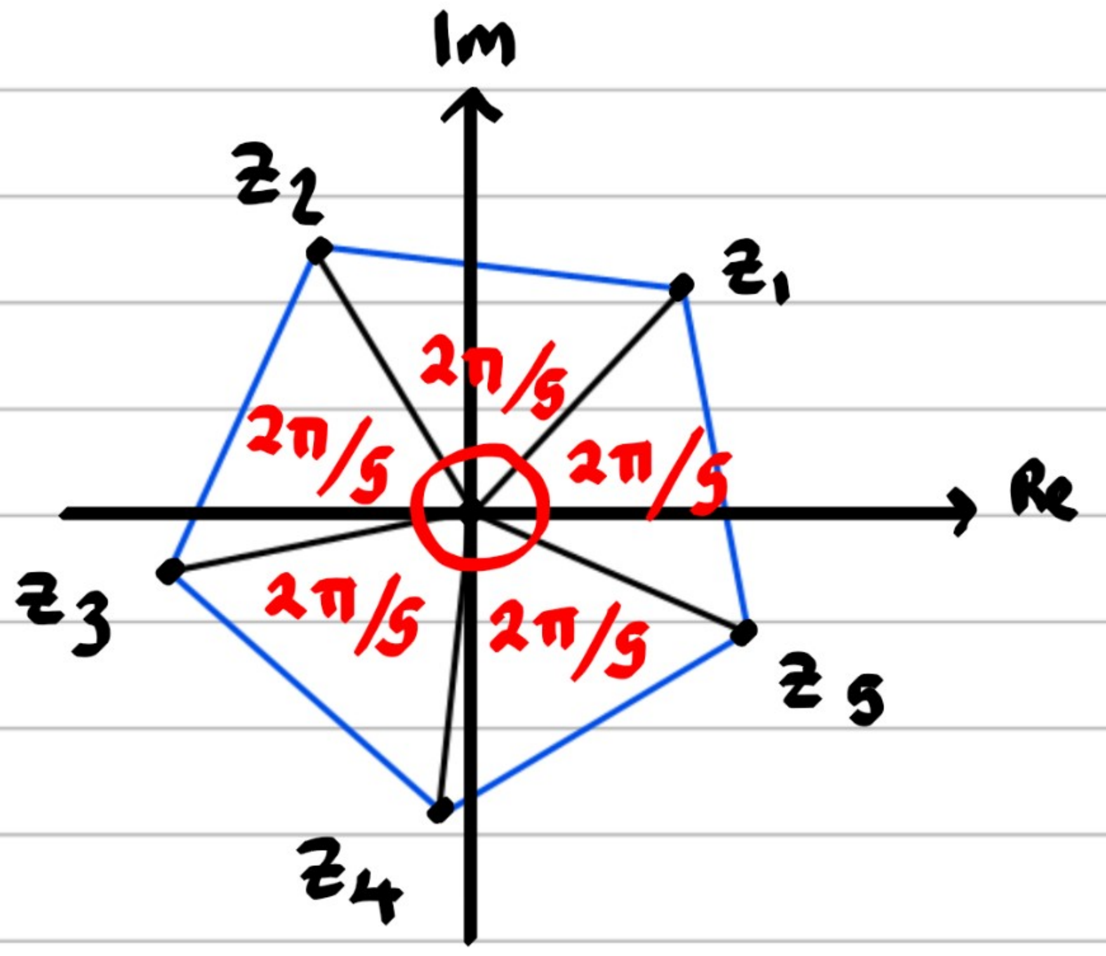
(4)

(i) LET $z_1 = 6 + 6i$

$|z_1| = \sqrt{6^2 + 6^2} = 6\sqrt{2}$

$\arg(z_1) = \tan^{-1}\left(\frac{6}{6}\right) = \frac{\pi}{4}$

$\therefore z_1 = 6\sqrt{2}\, e^{i\pi/4}$



i.e. $z = 6\sqrt{2}\, e^{i(\pi/4 + 2\pi K/5)}$

$K = 0: \ z_1 = 6\sqrt{2}\, e^{i\pi/4}$

$K = 1: \ z_2 = 6\sqrt{2}\, e^{i13\pi/20}$

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Question 8 continued

$K=2: z_3 = 6\sqrt{2}\, e^{-i19\pi/20}$

$K=3: z_4 = 6\sqrt{2}\, e^{-i11\pi/20}$

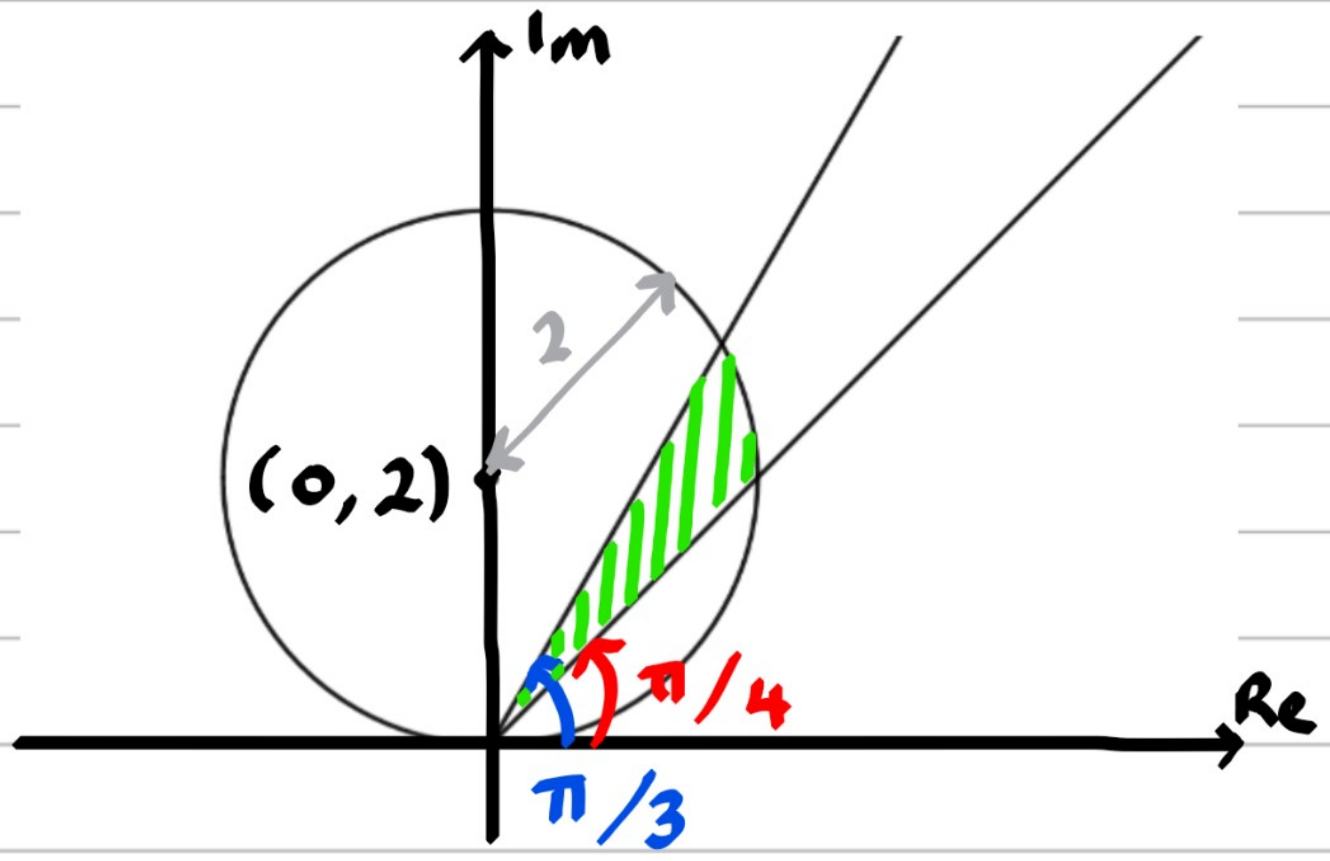
$K=4: z_5 = 6\sqrt{2}\, e^{-i3\pi/20}$

(ii) $|z-(2i)| \leq 2$ AND $\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{3}$

$\downarrow$

CIRCLE, $c\,(0,2)$, $r=2$

(a)



(b) IN CARTESIAN FORM: $x^2 + (y-2)^2 = 4$

IN POLAR FORM: $(r\cos\theta)^2 + (r\sin\theta - 2)^2 = 4$

$r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta + 4 = 4$

$r^2 = 4r\sin\theta$

$r = 4\sin\theta$

$\therefore$ AREA $= \frac{1}{2}\int_{\pi/4}^{\pi/3} (4\sin\theta)^2\, d\theta$

$= 8\int_{\pi/4}^{\pi/3} \sin^2\theta\, d\theta$

$= 8\int_{\pi/4}^{\pi/3} \frac{1}{2}(1-\cos 2\theta)\, d\theta$

Question 8 continued

$$= 4\int_{\pi/4}^{\pi/3} (1 - \cos 2\theta)\, d\theta$$

$$= \left[4\theta - 2\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

$$= \left\{4\left(\tfrac{\pi}{3}\right) - 2\sin\left(2\times\tfrac{\pi}{3}\right)\right\} - \left\{4\left(\tfrac{\pi}{4}\right) - 2\sin\left(2\times\tfrac{\pi}{4}\right)\right\}$$

$$= \frac{4\pi}{3} - \sqrt{3} - \pi + 2$$

$$= \left(\frac{\pi}{3} - \sqrt{3} + 2\right) \text{ UNITS}^2$$

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9. (a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \ldots$$

(1)

(b) Given that $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots \quad = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(5)

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ cannot be purely imaginary, giving a reason for your answer.

(2)

(a) $1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$

(b) $z = \frac{1}{2}(\cos\theta + i\sin\theta)$:

$\Rightarrow 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \left[\frac{1}{2}(\cos\theta + i\sin\theta)\right]^2 + \left[\frac{1}{2}(\cos\theta + i\sin\theta)\right]^3$

$+ \cdots = \frac{1}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)}$

$\Rightarrow 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \frac{1}{8}(\cos 3\theta + i\sin 3\theta)$

$+ \cdots = \frac{2}{2 - (\cos\theta + i\sin\theta)}$

$\Rightarrow 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \frac{1}{8}(\cos 3\theta + i\sin 3\theta)$

$+ \cdots = \frac{2}{(2 - \cos\theta) - i\sin\theta} \quad \frac{(2-\cos\theta) + i\sin\theta}{(2-\cos\theta) + i\sin\theta}$

$\Rightarrow 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \frac{1}{8}(\cos 3\theta + i\sin 3\theta)$

$+ \cdots = \frac{4 - 2\cos\theta + 2i\sin\theta}{(2 - \cos\theta)^2 - i^2\sin^2\theta}$

P 6 6 7 9 7 A 0 2 8 3 2

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Question 9 continued

$$\Rightarrow 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \frac{1}{8}(\cos 3\theta + i\sin 3\theta) + \cdots = \frac{4 - 2\cos\theta + 2i\sin\theta}{4 - 4\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$\Rightarrow 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \frac{1}{8}(\cos 3\theta + i\sin 3\theta) + \cdots = \frac{4 - 2\cos\theta + 2i\sin\theta}{4 - 4\cos\theta + 1}$$

$$\Rightarrow 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \frac{1}{8}(\cos 3\theta + i\sin 3\theta) + \cdots = \frac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cos\theta}$$

COMPARING IMAGINARY PARTS:

$$\therefore \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \cdots = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(ii) IF PURELY IMAGINARY, THEN REAL PART OF SUM IS 0:

$$\frac{4 - 2\cos\theta}{5 - 4\cos\theta} = 0$$

$$4 - 2\cos\theta = 0$$

$$\cos\theta = 2$$

AS $-1 \leq \cos\theta \leq 1$, THEN $\frac{4 - 2\cos\theta}{5 - 4\cos\theta} \neq 0$

$\therefore$ THERE IS ALSO A REAL PART

$\therefore$ SUM CANNOT BE PURELY IMAGINARY