Mark Scheme

Q1.

Question Number	Scheme	Notes	Marks
(a)	$y=\ln(\tanh 2x)\Rightarrow\frac{dy}{dx}=\frac{1}{\tanh 2x}\times 2\operatorname{sech}^2 2x$ **or** $y=\ln(\tanh 2x)\Rightarrow e^y=\tanh 2x\Rightarrow e^y\frac{dy}{dx}=2\operatorname{sech}^2 2x\Rightarrow\frac{dy}{dx}=\frac{2\operatorname{sech}^2 2x}{\tanh 2x}$ M1: Applies the chain rule or eliminates the "ln" and differentiates implicitly to obtain to obtain $\frac{dy}{dx}=\frac{k\operatorname{sech}^2 2x}{\tanh 2x}$ A1: Correct derivative in any form **Note that some candidates now convert to exponential form to complete this part – see below in the alternative for scoring the final M1A1**		M1A1
	$=\frac{2\cosh 2x}{\sinh 2x}\times\frac{1}{\cosh^2 2x}=\frac{2}{\sinh 2x\cosh 2x}$	Converts to $\sinh 2x$ and $\cosh 2x$ correctly to obtain $\frac{k}{\sinh 2x\cosh 2x}$	M1
	$=\frac{2}{\frac{1}{2}\sinh 4x}=4\operatorname{cosech}4x$	Correct answer. Note that this is not a given answer so you can allow if e.g. a sinh becomes a sin but is then recovered but if there are any obvious errors this mark should be withheld.	A1
			(4)

	Alternative using exponentials:		
	$y=\ln(\tanh 2x)=\ln\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)$ $\frac{dy}{dx}=\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}\left(\frac{(e^{2x}+e^{-2x})(2e^{2x}+2e^{-2x})-(e^{2x}-e^{-2x})(2e^{2x}-2e^{-2x})}{(e^{2x}+e^{-2x})^2}\right)$ **or** $y=\ln(\tanh 2x)=\ln\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)=\ln(e^{2x}-e^{-2x})-\ln(e^{2x}+e^{-2x})$ $\frac{dy}{dx}=\frac{2e^{2x}+2e^{-2x}}{e^{2x}-e^{-2x}}-\frac{2e^{2x}-2e^{-2x}}{e^{2x}+e^{-2x}}$ M1: Writes $\tanh 2x$ correctly in terms of exponentials and applies the chain rule and quotient rule or uses the subtraction law of logs and applies the chain rule A1: Correct derivative in any form		M1A1
	$=\frac{2(e^{2x}+e^{-2x})^2-2(e^{2x}-e^{-2x})^2}{e^{4x}-e^{-4x}}=\frac{8}{e^{4x}-e^{-4x}}$ Obtains $\frac{k}{e^{4x}-e^{-4x}}$		M1
	$=\frac{4}{\sinh 4x}=4\operatorname{cosech}4x$	Correct answer. Note that this is not a given answer so you can allow if e.g. a sinh becomes a sin but is then recovered but if there are any obvious errors this mark should be withheld.	A1

(b) Way 1	$4\text{cosech}4x=1\Rightarrow \sinh 4x=4\Rightarrow 4x=\ln\left(4+\sqrt{4^2+1}\right)$ Changes to sinh $4x$ =... and uses the **correct** logarithmic form of arsinh to reach $4x$ = ...		M1
	$x=\frac{1}{4}\ln\left(4+\sqrt{17}\right)$	This value only. Allow e.g. $x=\ln\left(4+\sqrt{17}\right)^{\frac{1}{4}}$	A1
			(2)
(b) Way 2	$4\text{cosech}4x=1\Rightarrow 4\times\frac{2}{e^{4x}-e^{-4x}}=1\Rightarrow e^{8x}-8e^{4x}-1=0$ Changes to the **correct** exponential form to reach $\frac{k}{e^{4x}-e^{-4x}}$, obtains a 3TQ in e^{4x}, solves and takes ln's to reach $4x$ = ... (usual rules for solving a 3TQ do not apply as long as the above conditions are met)		M1
	$x=\frac{1}{4}\ln\left(4+\sqrt{17}\right)$	This value only. Allow e.g. $x=\ln\left(4+\sqrt{17}\right)^{\frac{1}{4}}$	A1
			(2)
			Total 6

Q2.

Question Number	Scheme	Notes	Marks
(a)	$y=\text{arsech}\left(\frac{x}{2}\right)\Rightarrow \text{sech}\, y=\frac{x}{2}$ $\Rightarrow\frac{dx}{dy}=-2\,\text{sech}\, y\tanh y$	Takes "sech" of both sides and differentiates to obtain $\frac{dx}{dy}=k\,\text{sech}\, y\tanh y$ or equivalent.	M1
	$\Rightarrow\frac{dx}{dy}=-2\left(\frac{x}{2}\right)\sqrt{1-\left(\frac{x}{2}\right)^2}$ M1: Replaces sech y with $\frac{x}{2}$ and tanh y with $\sqrt{1-\left(\frac{x}{2}\right)^2}$ A1: Correct equation involving $\frac{dx}{dy}$ or $\frac{dy}{dx}$ in any form in terms of x only.		M1A1
	$\Rightarrow\frac{dy}{dx}=\frac{-2}{x\sqrt{4-x^2}}$	Correct derivative in the required form or correct values for p and q.	A1
			(4)

<table>
<tr><td rowspan="3">(a)
Way 2</td><td>$y = \text{arsech}\left(\frac{x}{2}\right) \Rightarrow \text{sech}\, y = \frac{x}{2}$
$\Rightarrow \cosh y = \frac{2}{x} \Rightarrow \sinh y \frac{dy}{dx} = -\frac{2}{x^2}$</td><td>Takes "sech" of both sides, changes to "cosh" and differentiates to obtain $\sinh y \frac{dy}{dx} = \frac{k}{x^2}$ or equivalent.</td><td>M1</td></tr>
<tr><td colspan="2">$\Rightarrow \frac{dy}{dx} = -\frac{2}{x^2 \sinh y} = -\frac{2}{x^2\sqrt{\left(\frac{2}{x}\right)^2 - 1}}$
M1: Replaces $\sinh y$ with $\sqrt{\left(\frac{2}{x}\right)^2 - 1}$
A1: Correct equation involving $\frac{dx}{dy}$ or $\frac{dy}{dx}$ in any form in terms of x only.</td><td>M1A1</td></tr>
<tr><td>$\Rightarrow \frac{dy}{dx} = \frac{-2}{x\sqrt{4-x^2}}$</td><td>Correct derivative in the required form or correct values for p and q.</td><td>A1</td></tr>
<tr><td rowspan="3">(a)
Way 3</td><td colspan="2">$y = \text{arsech}\left(\frac{x}{2}\right) \Rightarrow y = \text{arcosh}\left(\frac{2}{x}\right)$
Changes to "arcosh" correctly. Score this as the second M mark on EPEN.</td><td>M1</td></tr>
<tr><td colspan="2">$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\left(\frac{2}{x}\right)^2 - 1}} \times -\frac{2}{x^2}$
M1: Differentiates to the form $\frac{k}{x^2\sqrt{\left(\frac{2}{x}\right)^2 - 1}}$ oe
A1: Correct equation involving $\frac{dx}{dy}$ or $\frac{dy}{dx}$ in any form in terms of x only.
Score this as the first M mark and first A mark on EPEN.</td><td>M1A1</td></tr>
<tr><td>$\Rightarrow \frac{dy}{dx} = \frac{-2}{x\sqrt{4-x^2}}$</td><td>Correct derivative in the required form or correct values for p and q.</td><td>A1</td></tr>
</table>

<table>
<tr><td rowspan="3">(a)
Way 4</td><td colspan="2">$$y=\text{arsech}\left(\frac{x}{2}\right)\Rightarrow \text{sech}\, y=\frac{x}{2}\Rightarrow\left(\frac{x}{2}\right)^2=\text{sech}^2 y\Rightarrow \tanh y=\sqrt{1-\left(\frac{x}{2}\right)^2}$$ $$\Rightarrow \text{sech}^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=-x\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$$ Differentiates to $\text{sech}^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=kx\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$ or equivalent</td><td>M1</td></tr>
<tr><td colspan="2">$$\Rightarrow \text{sech}^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=-x\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}\Rightarrow\frac{x^2}{4}\frac{\mathrm{d}y}{\mathrm{d}x}=-x\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{4}{x}\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$$ M1: Replaces $\text{sech}^2 y$ with $\left(\frac{2}{x}\right)^2$
A1: Correct equation involving $\frac{\mathrm{d}x}{\mathrm{d}y}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$ in any form in terms of x only.</td><td>M1A1</td></tr>
<tr><td>$$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2}{x\sqrt{4-x^2}}$$</td><td>Correct derivative in the required form or correct values for p and q.</td><td>A1</td></tr>
<tr><td rowspan="3">(a)
Way 5</td><td colspan="2">$$y=\text{arsech}\left(\frac{x}{2}\right)\Rightarrow \text{sech}\, y=\frac{x}{2}\Rightarrow y=\text{artanh}\left(\sqrt{1-\left(\frac{x}{2}\right)^2}\right)$$ Changes to "artanh" correctly. Score this as the second M mark on EPEN.</td><td>M1</td></tr>
<tr><td colspan="2">$$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{1}{2}\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}}{1-\left(1-\frac{x^2}{4}\right)}\times-\frac{x}{2}$$ M1: Differentiates to the form $\dfrac{kx\left(1-\frac{x^2}{4}\right)^{\frac{1}{2}}}{1-\left(1-\frac{x^2}{4}\right)}$ oe
A1: Correct equation involving $\frac{\mathrm{d}x}{\mathrm{d}y}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$ in any form in terms of x only.
Score this as the first M mark and first A mark on EPEN.</td><td>M1A1</td></tr>
<tr><td>$$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2}{x\sqrt{4-x^2}}$$</td><td>Correct derivative in the required form or correct values for p and q.</td><td>A1</td></tr>
</table>

There may be other methods used.
If you are in any doubt if the method deserves any marks use Review.

(b)	$$\mathrm{f}(x)=\tanh^{-1}(x)+\mathrm{sech}^{-1}\left(\frac{x}{2}\right)\Rightarrow \mathrm{f}'(x)=\frac{1}{1-x^2}-\frac{2}{x\sqrt{4-x^2}}$$ Correct f '(x) following through their (a) of the form $\frac{p}{x\sqrt{q-x^2}}$ Also allow with "made up" p and q or the letters p and q.		B1ft
	$$\frac{1}{1-x^2}-\frac{2}{x\sqrt{4-x^2}}=0\Rightarrow 2\left(1-x^2\right)=x\sqrt{4-x^2}\Rightarrow 4\left(1-x^2\right)^2=x^2\left(4-x^2\right)$$ Sets $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ with their (a) of the form $\frac{p}{x\sqrt{q-x^2}}$ and squares both sides to reach a quartic equation		M1
	$5x^4-12x^2+4=0$	Correct quartic	A1
	$5x^4-12x^2+4=0\Rightarrow x^2=2,\ 0.4$ $\Rightarrow x=\ldots$	Solves their quartic equation to obtain a value for x^2 and proceeds to a value for x. Apply usual rules for solving and check if necessary. Allow complex roots.	M1
	$x=\sqrt{\frac{2}{5}}$	Correct exact answer (allow equivalents e.g. $\frac{\sqrt{10}}{5}$). If any extra answers given score A0 e.g. $x=\pm\sqrt{\frac{2}{5}}$	A1
			(5)
			Total 9

Special case:

It is possible for a correct solution in (b) following a sign error in (a) e.g.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{x\sqrt{4-x^2}}$$

$$\mathrm{f}(x)=\tanh^{-1}(x)+\mathrm{sech}^{-1}\left(\frac{x}{2}\right)\Rightarrow \mathrm{f}'(x)=\frac{1}{1-x^2}+\frac{2}{x\sqrt{4-x^2}}$$

$$\frac{1}{1-x^2}+\frac{2}{x\sqrt{4-x^2}}=0\Rightarrow 2\left(1-x^2\right)=-x\sqrt{4-x^2}\Rightarrow 4\left(1-x^2\right)^2=x^2\left(4-x^2\right)\text{ etc.}$$

This is likely to score M1M1A0A0 in (a) but allow full recovery in (b) if it leads to the correct answer.

Q3.

Question Number	Scheme	Notes	Marks
(a)	$\left\{\frac{\mathrm{d}y}{\mathrm{d}x}=\right\}\operatorname{arcosh}5x+\frac{ax}{\sqrt{bx^2-1}}$ or $\operatorname{arcosh}5x+\frac{cx}{\sqrt{x^2-d}}$ (M1) $\Rightarrow \operatorname{arcosh}(5x)+\frac{5x}{\sqrt{25x^2-1}}$ (A1) M1: Differentiates to obtain expression of the correct form $a,\ b,\ c,\ d \neq 0$ A1: Correct differentiation. Any equivalent form.		M1 A1
			(2)
(b)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\operatorname{arcosh}(5x)\right)=\operatorname{arcosh}(5x)+\text{"}\frac{5x}{\sqrt{25x^2-1}}\text{"}\Rightarrow\int\operatorname{arcosh}(5x)\,\mathrm{d}x=x\operatorname{arcosh}(5x)-\int\text{"}\frac{5x}{\sqrt{25x^2-1}}\text{"}\mathrm{d}x$ M1: Rearranges **their** answer to (a) correctly and integrates or uses the correct formula to apply parts to $1\times\operatorname{arcosh}5x$ to obtain the above.		M1
	$\int\operatorname{arcosh}(5x)\,\mathrm{d}x=x\operatorname{arcosh}(5x)-\int\frac{5x}{\sqrt{25x^2-1}}\mathrm{d}x$ A1: **Correct expression** – but **see note below on limited ft**		A1 **(limited ft)**
	$=x\operatorname{arcosh}(5x)-\frac{1}{5}\left(25x^2-1\right)^{\frac{1}{2}}\ (+c)$	M1: $\int\frac{Ax}{\sqrt{Bx^2-1}}\mathrm{d}x\to C\left(Bx^2-1\right)^{\frac{1}{2}}$ A1: Fully correct expression with $x\operatorname{arcosh}(5x)$ - see note below for limited ft	M1 A1 **(limited ft)**
	Note: Substitutions : $u=5x\Rightarrow\left(u^2-1\right)^{\frac{1}{2}}\Rightarrow\left[\frac{1}{5}\sqrt{u^2-1}\right]_{\frac{5}{4}}^{3}$ $\quad u=25x^2-1\Rightarrow\left[\frac{1}{5}\sqrt{u}\right]_{\frac{9}{16}}^{8}$ M1: Correct form A1: Fully correct expression with $x\operatorname{arcosh}(5x)$		
	A limited ft for <u>one</u> of the errors in (a) shown below applies for the first two A marks. **However also allow the following if this error occurs in part (b)** which is most likely to come from not rearranging and effectively restarting by using parts. Note that substitutions could be used. $a=1\Rightarrow x\operatorname{arcosh}(5x)-\int\frac{x}{\sqrt{25x^2-1}}\mathrm{d}x\Rightarrow x\operatorname{arcosh}(5x)-\frac{1}{25}\left(25x^2-1\right)^{\frac{1}{2}}\ (+c)$ $b=5\Rightarrow x\operatorname{arcosh}(5x)-\int\frac{5x}{\sqrt{5x^2-1}}\mathrm{d}x\Rightarrow x\operatorname{arcosh}(5x)-\left(5x^2-1\right)^{\frac{1}{2}}\ (+c)$ $a=-5\Rightarrow x\operatorname{arcosh}(5x)+\int\frac{5x}{\sqrt{25x^2-1}}\mathrm{d}x\Rightarrow x\operatorname{arcosh}(5x)+\frac{1}{5}\left(25x^2-1\right)^{\frac{1}{2}}\ (+c)$		

	$\int_{\frac{1}{4}}^{\frac{3}{5}}\operatorname{arcosh}5x\,\mathrm{d}x=\frac{3}{5}\operatorname{arcosh}(3)-\frac{1}{5}\sqrt{25\times\frac{9}{25}-1}-\left(\frac{1}{4}\operatorname{arcosh}\left(\frac{5}{4}\right)-\frac{1}{5}\sqrt{25\times\frac{1}{16}-1}\right)$ Applies appropriate limits (note substitutions above) with subtraction the right way round seen to obtain an expression of the form $x\operatorname{arcosh}(5x)\pm\mathrm{f}(x)$ **where f(x) has come from integration**		M1
	$=\frac{3}{5}\operatorname{arcosh}(3)-\frac{2\sqrt{2}}{5}-\frac{1}{4}\operatorname{arcosh}\left(\frac{5}{4}\right)+\frac{3}{20}$	Correct answer seen in any form. **Must not follow clearly incorrect work.**	A1
	$\operatorname{arcosh}3=\ln\left(3+\sqrt{3^2-1^2}\right)$ or $\operatorname{arcosh}\left(\frac{5}{4}\right)=\ln\left(\frac{5}{4}+\sqrt{\left(\frac{5}{4}\right)^2-1^2}\right)$ $\left\{\Rightarrow\frac{3}{5}\ln\left(3+\sqrt{8}\right)-\frac{2\sqrt{2}}{5}-\frac{1}{4}\ln 2+\frac{3}{20}\right\}$	Converts $\operatorname{arcosh}(3)$ or $\operatorname{arcosh}\left(\frac{5}{4}\right)$ to any correct log form. Independent mark but must have obtained $x\operatorname{arcosh}(5x)\pm\mathrm{f}(x)$ **where f(x) has come from integration**	M1
	$=\frac{3}{20}-\frac{2\sqrt{2}}{5}+\ln\left(3+2\sqrt{2}\right)^{\frac{3}{5}}-\frac{1}{4}\ln 2$ **Must not follow clearly incorrect work.**	Correct answer. Terms in any order but otherwise written as shown. Allow values for p, q, r & k	A1
			(8)
			Total 10

Q4.

Question	Scheme	Marks
(a)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\frac{1}{\sqrt{1-k\sqrt{x}^2}}\times\ldots x^{-\frac{1}{2}}$ or $\cos y=2x^{\frac{1}{2}}\Rightarrow\pm\sin y\frac{\mathrm{d}y}{\mathrm{d}x}=..x^{-\frac{1}{2}}$	M1
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\frac{1}{\sqrt{1-4x}}\times\left(Kx^{-\frac{1}{2}}\right)$ or $\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\frac{Kx^{-\frac{1}{2}}}{\sqrt{1-\left(2\sqrt{x}\right)^2}}$	dM1
	$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\sqrt{x}\sqrt{1-4x}}$ oe e.g. $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\sqrt{x-4x^2}}$	A1
		(3)
(b) Way 1	$\int y\,\mathrm{d}x=\int 1\times\arccos\left(2\sqrt{x}\right)\mathrm{d}x=x\arccos\left(2\sqrt{x}\right)-\int x\frac{-1}{\sqrt{x}\sqrt{1-4x}}\mathrm{d}x$	M1
	$=x\arccos\left(2\sqrt{x}\right)+\int\frac{\sqrt{x}}{\sqrt{1-4x}}\mathrm{d}x$ *	A1*
		(2)
Way 2	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\arccos\left(2\sqrt{x}\right)\right)=1.\arccos\left(2\sqrt{x}\right)+x.\frac{-1}{\sqrt{x}\sqrt{1-4x}}$	M1
	$\Rightarrow\int\arccos\left(2\sqrt{x}\right)\mathrm{d}x=x\arccos\left(2\sqrt{x}\right)+\int\frac{\sqrt{x}}{\sqrt{1-4x}}\mathrm{d}x$ *	A1*
		(2)

(c)	$\frac{1}{2\sqrt{x}}\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\frac{1}{2}\sin\theta$, $\mathrm{d}x=-\sqrt{x}\sin\theta\,\mathrm{d}\theta$, $\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\frac{1}{2}\sin\theta\cos\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\frac{1}{4}\sin2\theta$	B1
	$\int\frac{\sqrt{x}}{\sqrt{1-4x}}\mathrm{d}x=\int\frac{-\left(\frac{1}{2}\cos\theta\right)^2\sin\theta}{\sqrt{1-4\left(\frac{1}{2}\cos\theta\right)^2}}\mathrm{d}\theta$	M1
	$=-\frac{1}{4}\int\frac{\cos^2\theta\sin\theta}{\sqrt{1-\cos^2\theta}}\mathrm{d}\theta=-\frac{1}{4}\int\cos^2\theta\,\mathrm{d}\theta$	A1
	$x=0\Rightarrow\theta=\frac{\pi}{2}$ $x=\frac{1}{8}\Rightarrow\theta=\frac{\pi}{4}$ So $\int_0^{\frac{1}{8}}\frac{\sqrt{x}}{\sqrt{1-4x}}\mathrm{d}x=\frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta$	A1
		(4)

(d)	$\frac{1}{4}\int\frac{1}{2}(1+\cos 2\theta)\,d\theta = K\left(\theta \pm \frac{1}{2}\sin 2\theta\right)$	M1
	$\int_0^{\frac{1}{8}}\frac{\sqrt{x}}{\sqrt{1-4x}}\,dx = \frac{1}{8}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = ...\left(=\frac{\pi}{32}-\frac{1}{16}\right)$ or e.g. $\int_0^{\frac{1}{8}}\frac{\sqrt{x}}{\sqrt{1-4x}}\,dx = \frac{1}{8}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{1}{8}\left[\arccos 2\sqrt{x}+\frac{1}{2}\sin 2\arccos 2\sqrt{x}\right]_0^{\frac{1}{8}}$ $= ...\left(=-\frac{1}{8}\left(\frac{\pi}{4}+\frac{1}{2}-\frac{\pi}{2}\right)\right)$	dM1
	$\Rightarrow \int_0^{\frac{1}{8}}\arccos\left(2\sqrt{x}\right)dx = \left[x\arccos 2\sqrt{x}\right]_0^{\frac{1}{8}}+\frac{\pi}{32}-\frac{1}{16} = \frac{1}{8}\arccos\frac{1}{\sqrt{2}}-0+\frac{\pi}{32}-\frac{1}{16}$	dM1
	$=\frac{\pi}{16}-\frac{1}{16}$ oe	A1
		(4)
		(13 marks)

Notes:

(a)

M1: Attempts to apply the arccos derivative formula together with chain rule. Look for $\frac{dy}{dx} = \pm\frac{1}{\sqrt{1-k\sqrt{x}^2}}\times f(x)$ where f (x) is an attempt at differentiating $2\sqrt{x}$ where $f(x) \neq \alpha\sqrt{x}$

Note that k may be 1 for this mark.

Alternatively, takes cosine of both sides and differentiates to the form shown in the scheme.

dM1: Correct form for the overall derivative achieved, may be errors in sign or constants with $k \neq 1$

Alternatively, divides through by sin y and applies Pythagorean identity to achieve derivative in terms of x.

A1: Correct derivative, but need not be simplified. Award when first seen and isw.

(b) Way 1

M1: Attempts to apply integration by parts to $1\times\arccos\left(2\sqrt{x}\right)$.

Look for $x\arccos\left(2\sqrt{x}\right)-\int x\text{"their (a)"}\,dx$ or $u=\arccos\left(2\sqrt{x}\right)\Rightarrow\frac{du}{dx}=\text{part}(a),\ \frac{dv}{dx}=1\Rightarrow v=x$

A1*: Correct work leading to the printed answer. There must be a clear statement for the integration by parts before the given answer is stated.

So e.g. $u=\arccos\left(2\sqrt{x}\right)\Rightarrow\frac{du}{dx}=\text{part}(a),\ \frac{dv}{dx}=1\Rightarrow v=x$

$\Rightarrow\int\arccos\left(2\sqrt{x}\right)dx = x\arccos\left(2\sqrt{x}\right)+\int\frac{\sqrt{x}}{\sqrt{1-4x}}\,dx$* scores M1A0

You can condone $\int\arccos\left(2\sqrt{x}\right)dx = x\arccos\left(2\sqrt{x}\right)+\int\frac{x^{\frac{1}{2}}}{\sqrt{1-4x}}\,dx$*

Way 2

M1: Applies the product rule to $x\arccos\left(2\sqrt{x}\right)$, look for $1.\arccos\left(2\sqrt{x}\right)+x.$"their (a)".

A1*: Rearranges and integrates to achieve the given result, with no errors seen.

(c)

B1: Any correct expression involving $\mathrm{d}x$ and $\mathrm{d}\theta$, see examples in scheme.

M1: Makes a complete substitution in the integral $\int \frac{\sqrt{x}}{\sqrt{1-4x}}\,\mathrm{d}x$ to achieve an integral in θ only.

Ignore attempts at substitution into the $x\arccos\left(2\sqrt{x}\right)$.

A1: A correct simplified integral aside from limits. May be implied by e.g. $\frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta$

Note that this mark depends on the B mark.

A1: Finds correct limits for θ and applies to the integral by reversing the sign – i.e. correct answer with limits and sign all correct. Accept equivalent limits e.g. $-\frac{\pi}{4}$ to $-\frac{\pi}{4}$ or $\frac{\pi}{2}$ to $\frac{3\pi}{4}$

Note that this mark depends on the B mark.

(d)

M1: Applies double angle identity to get the integral in a suitable form and attempts to integrate.

Accept $\cos^2\theta=\frac{1}{2}(\pm1\pm\cos 2\theta)$ used as identity and look for $1\rightarrow\theta$ and $\cos 2\theta\rightarrow\pm\frac{1}{2}\sin 2\theta$

dM1: Applies their limits (either way round) to their integral in θ or reverse substitution and applies limits 0 and $\frac{1}{8}$.

Depends on the previous method mark.

dM1: Applies limits of 0 and $\frac{1}{8}$ to the $x\arccos\left(2\sqrt{x}\right)$ to obtain a value (or their limits either way round if they applied the substitution to this to obtain a value) and combines with the result of the other integral.

Depends on both previous method marks.

A1: Correct final answer.

Q5.

Question Number	Scheme	Notes	Marks
	$y=\operatorname{artanh}\left(\dfrac{\cos x+a}{\cos x-a}\right)$		
	$\dfrac{dy}{dx}=\dfrac{1}{1-\left(\dfrac{\cos x+a}{\cos x-a}\right)^2}\times\dfrac{(\cos x-a)\times-\sin x-(\cos x+a)\times-\sin x}{(\cos x-a)^2}$ or $\dfrac{dy}{dx}=\dfrac{1}{1-\left(\dfrac{\cos x+a}{\cos x-a}\right)^2}\times\left(-\sin x\times(\cos x-a)^{-1}+(\cos x+a)\times\sin x(\cos x-a)^{-2}\right)$ M1: Correct method for the derivative. This requires $\dfrac{1}{1-\left(\dfrac{\cos x+a}{\cos x-a}\right)^2}\times$ An attempt at the quotient (or product) rule. A1: Correct derivative in any form		M1A1
	$=\dfrac{(\cos x-a)^2}{(\cos x-a)^2-(\cos x+a)^2}\times\dfrac{2a\sin x}{(\cos x-a)^2}=\dfrac{2a\sin x}{-4a\cos x}=\ldots$ Uses correct processing to reach $\lambda\dfrac{\sin x}{\cos x}$ or $\lambda\tan x$ **Depends on the first method mark.**		dM1
	$=-\dfrac{1}{2}\tan x$	cso	A1 **(4)**
Way 2	$y=\operatorname{artanh}\left(\dfrac{\cos x+a}{\cos x-a}\right)\Rightarrow\tanh y=\dfrac{\cos x+a}{\cos x-a}\Rightarrow\operatorname{sech}^2 y\dfrac{dy}{dx}=\dfrac{2a\sin x}{(\cos x-a)^2}$ Takes tanh of both sides, obtains $\operatorname{sech}^2 y\dfrac{dy}{dx}=$ an attempt at the quotient or product rule		M1
	$\dfrac{dy}{dx}=\dfrac{1}{1-\left(\dfrac{\cos x+a}{\cos x-a}\right)^2}\times\dfrac{2a\sin x}{(\cos x-a)^2}$ Correct derivative in any form		A1
	$=\dfrac{(\cos x-a)^2}{(\cos x-a)^2-(\cos x+a)^2}\times\dfrac{2a\sin x}{(\cos x-a)^2}=\dfrac{2a\sin x}{-4a\cos x}=\ldots$ Uses correct processing to reach $\lambda\dfrac{\sin x}{\cos x}$ or $\lambda\tan x$ **Depends on the first method mark.**		dM1
	$=-\dfrac{1}{2}\tan x$	cso	A1 **(4)**

Way 3	Uses substitution $u=\frac{\cos x+a}{\cos x-a}$, obtains $\frac{du}{dx}\left(=\frac{2a\sin x}{(\cos x-a)^2}\right)$ by quotient rule and $\frac{dy}{du}\left(=\frac{1}{1-u^2}\right)$ followed by chain rule to obtain $\frac{dy}{dx}=\frac{1}{1-\left(\frac{\cos x+a}{\cos x-a}\right)^2}\times\frac{2a\sin x}{(\cos x-a)^2}$		M1
	Correct derivative in any form		A1
	Uses correct processing to reach $\lambda\frac{\sin x}{\cos x}$ or $\lambda\tan x$ **Depends on the first method mark.**		dM1
	$=-\frac{1}{2}\tan x$	cso	A1 (4)
			Total 4
Way 4	$y=\frac{1}{2}\ln\left(\frac{1+\frac{\cos x+a}{\cos x-a}}{1-\frac{\cos x+a}{\cos x-a}}\right)=\frac{1}{2}\ln\left(-\frac{\cos x}{a}\right)$ $\frac{dy}{dx}=\frac{1}{2}\times\frac{1}{-\frac{\cos x}{a}}\times\left(\frac{\sin x}{a}\right)$	M1: Converts to correct ln form and uses chain rule to differentiate A1: Correct derivative in any form	M1A1
	Uses correct processing to reach $\lambda\frac{\sin x}{\cos x}$ or $\lambda\tan x$ **Depends on the first method mark.**		dM1
	$=-\frac{1}{2}\tan x$	cso	A1
			(4)

Q6.

Question	Scheme		Marks
	$y=9\cosh x+3\sinh x+7x$		
	$\frac{dy}{dx}=9\sinh x+3\cosh x+7$	Correct derivative	B1
	$9\frac{(e^x-e^{-x})}{2}+3\frac{(e^x+e^{-x})}{2}+7=0$	Replaces sinh$x$ and cosh$x$ by the correct exponential forms	M1
	Note that the first 2 marks can score the other way round: M1: $y=9\frac{(e^x+e^{-x})}{2}+3\frac{(e^x-e^{-x})}{2}+7x$ B1: $\frac{dy}{dx}=9\frac{(e^x-e^{-x})}{2}+3\frac{(e^x+e^{-x})}{2}+7$		
	$12e^{2x}+14e^x-6=0$ oe	M1: Obtains a quadratic in e^x A1: Correct quadratic	M1 A1
	$(3e^x-1)(2e^x+3)=0\Rightarrow e^x=...$	Solves their quadratic as far as $e^x=...$	M1
	$x=\ln\left(\frac{1}{3}\right)$	cso (Allow $-\ln 3$) $e^x=-\frac{3}{2}$ need not be seen. Extra answers, award A0	A1

	Alternative		
	$\frac{dy}{dx}=9\sinh x+3\cosh x+7$	Correct derivative	B1
	$9\sinh x=-3\cosh x-7\Rightarrow 81\sinh^2 x=9\cosh^2 x+42\cosh x+49$		
	$72\cosh^2 x-42\cosh x-130=0$	Squares and attempts quadratic in cosh$x$	M1
	$(3\cosh x-5)(12\cosh x+13)=0\Rightarrow\cosh x=\frac{5}{3}$	M1: Solves quadratic A1: Correct value	M1 A1
	$x=\ln\left(\frac{5}{3}\pm\sqrt{\left(\frac{5}{3}\right)^2-1}\right)$	Use of ln form of arcosh	M1
	$x=\ln\left(\frac{1}{3}\right)$	cso (Allow $-\ln 3$)	A1
	NB: Ignore any attempts to find the y coordinate		
			(6 marks)

Q7.

Question Number	Scheme	Notes	Marks
(a)	$1-\tanh^2 x \equiv \text{sech}^2 x$		
	$1-\tanh^2 x = 1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2$	Replaces the tanh x on the lhs with a **correct** expression in terms of exponentials.	B1
	$=\frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}=\frac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}$ or e.g. $\frac{2e^{2x}\times 2e^{-2x}}{(e^x+e^{-x})^2}$ Attempts to find common denominator and expand numerator		M1
	$=\left(\frac{4}{(e^x+e^{-x})^2}\right)=\text{sech}^2 x$ *	Obtains the rhs with no errors.	A1cso
			(3)
ALT 1	$1-\tanh^2 x = (1-\tanh x)(1+\tanh x)$ $=\left(1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)\right)\left(1+\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)\right)$	Uses the difference of 2 squares on the lhs and replaces the tanh x with a **correct** expression in terms of exponentials.	B1
	$=\left(\frac{2e^{-x}}{e^x+e^{-x}}\right)\left(\frac{2e^x}{e^x+e^{-x}}\right)$	Attempt to find common denominators and simplify numerators.	M1
	$=\left(\frac{4}{(e^x+e^{-x})^2}\right)=\text{sech}^2 x$ *	Obtains the rhs with no errors.	A1cso
ALT 2	$\text{sech}^2 x = \frac{4}{(e^x+e^{-x})^2}$	Replaces the sech x on the rhs with a **correct** expression in terms of exponentials.	B1
	$=\frac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}=\frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$ Attempts to express the "4" in terms of the denominator.		M1
	$=1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2=1-\tanh^2 x$ *	Obtains the lhs with no errors.	A1cso

(b)	$2\,\text{sech}^2 x+3\tanh x=3 \Rightarrow 2(1-\tanh^2 x)+3\tanh x=3$ $\Rightarrow 2\tanh^2 x-3\tanh x+1=0$ Uses $\text{sech}^2 x=1-\tanh^2 x$ and forms a 3 term quadratic in tanh x		M1
	$(2\tanh x-1)(\tanh x-1)=0 \Rightarrow \tanh x=\ldots$	Solves 3TQ by any valid method including calculator.	M1
	$\tanh x=\frac{1}{2} \to x=\ln\sqrt{3}$	$\ln\sqrt{3}$. Accept $\frac{1}{2}\ln 3,\ -\frac{1}{2}\ln\frac{1}{3}$ And no other answers.	A1
			(3)
ALT	$2\,\text{sech}^2 x+3\tanh x=3 \Rightarrow 2\left(\frac{4}{(e^x+e^{-x})^2}\right)+3\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)=3$ $\Rightarrow 8+3(e^{2x}-e^{-2x})=3(e^{2x}+2+e^{-2x})\Rightarrow\ldots$ Substitutes the correct exponential forms, attempts to eliminate fractions and collect terms		M1
	$6e^{-2x}=2 \Rightarrow e^{-2x}=\frac{1}{3}$	Rearranges to reach $e^{-2x}=\ldots$	M1
	$x=\ln\sqrt{3}$	$\ln\sqrt{3}$. Accept $\frac{1}{2}\ln 3,\ -\frac{1}{2}\ln\frac{1}{3}$ And no other answers.	A1
			Total 6

Q8.

Question Number	Scheme	Notes	Marks
(a)	$$8\cosh^4 x = 8\left(\frac{e^x + e^{-x}}{2}\right)^4 = \frac{8}{16}\left(e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}\right)$$ Applies $\cosh x = \frac{e^x + e^{-x}}{2}$ and attempts to expand the bracket to at least 4 different and no more than 5 different terms of the correct form but they may be "uncollected" depending on how they do the expansion. Allow unsimplified terms e.g. $\left(e^x\right)^3 e^{-x}$. May see $8\left(\frac{e^x + e^{-x}}{2}\right)^2\left(\frac{e^x + e^{-x}}{2}\right)^2$ but must attempt to expand as above		M1
	$=\frac{1}{2}\left(e^{4x} + e^{-4x}\right) + 4\left(\frac{e^{2x} + e^{-2x}}{2}\right) + 3 = \ldots$	Collects appropriate terms and reaches the form $\cosh 4x + p\cosh 2x + q$ or obtains values of p and q.	M1
	$=\cosh 4x + 4\cosh 2x + 3$	Correct expression or values e.g. $p = 4$ and $q = 3$	A1
	No marks are available in (a) if exponentials are not used but note that they may appear in combination with the use of hyperbolic identities e.g.: $$8\cosh^4 x = 8\left(\cosh^2 x\right)^2 = 8\left(\frac{\cosh 2x + 1}{2}\right)^2 = 2\left(\frac{e^{2x} + e^{-2x}}{2} + 1\right)^2$$ $$= 2\left(\frac{e^{4x} + 2 + e^{-4x}}{4} + e^{2x} + e^{-2x} + 1\right) = \frac{e^{4x} + e^{-4x}}{2} + 4\left(\frac{e^{2x} + e^{-2x}}{2}\right) + 2$$ $$= \cosh 4x + 4\cosh 2x + 3$$ **Allow to "meet in the middle" e.g. expands as above and compares with** $$\frac{1}{2}\left(e^{4x} + e^{-4x}\right) + p\left(\frac{e^{2x} + e^{-2x}}{2}\right) + q \Rightarrow p = \ldots, q = \ldots$$ **but to score any marks the expansion must be attempted.**		
			(3)

<table>
<tr><td rowspan="6">(b)
Way 1</td><td colspan="2">$\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow 8\cosh^4 x - 4\cosh 2x - 3 - 17\cosh 2x + 9 = 0$
$\Rightarrow 8\cosh^4 x - 21\cosh 2x + 6 = 0 \Rightarrow 8\cosh^4 x - 21(2\cosh^2 x - 1) + 6 = 0$
Uses their result from part (a) and $\cosh 2x = \pm 2\cosh^2 x \pm 1$
to obtain a quadratic equation in $\cosh^2 x$
or
$\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow 2(2\cosh^2 x - 1)^2 - 1 - 17(2\cosh^2 x - 1) + 9 = 0$
Uses $\cosh 4x = \pm 2\cosh^2 2x \pm 1$ and $\cosh 2x = \pm 2\cosh^2 x \pm 1$
to obtain a quadratic equation in $\cosh^2 x$</td><td>M1</td></tr>
<tr><td>$\Rightarrow 8\cosh^4 x - 42\cosh^2 x + 27 = 0$</td><td>Correct 3TQ in $\cosh^2 x$</td><td>A1</td></tr>
<tr><td>$\Rightarrow 8\cosh^4 x - 42\cosh^2 x + 27 = 0$
$\Rightarrow \cosh^2 x = \frac{9}{2}\left(, \frac{3}{4}\right)$</td><td>Solves 3TQ in $\cosh^2 x$ (apply usual rules if necessary) to obtain
$\cosh^2 x = k \ \ (k \in \mathbb{R} \text{ and } > 1)$. May be implied by their values – check if necessary.</td><td>M1</td></tr>
<tr><td colspan="2">$\cosh^2 x = \frac{9}{2} \Rightarrow \cosh x = \frac{3}{\sqrt{2}} \Rightarrow x = \pm\ln\left(\frac{3}{\sqrt{2}} + \sqrt{\frac{9}{2} - 1}\right)$
or
$\cosh x = \frac{3}{\sqrt{2}} \Rightarrow \frac{e^x + e^{-x}}{2} = \frac{3}{\sqrt{2}} \Rightarrow \sqrt{2}e^{2x} - 6e^x + \sqrt{2} = 0 \Rightarrow e^x = ... \Rightarrow x = ...$
or
$\cosh^2 x = \frac{9}{2} \Rightarrow \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{9}{2} \Rightarrow e^{4x} - 16e^{2x} + 1 = 0 \Rightarrow e^{2x} = ... \Rightarrow x = ...$
Takes square root to obtain $\cosh x = k \ \ (k > 1)$ and applies the correct logarithmic form for arcosh or uses the correct exponential form for $\cosh x$ to obtain at least one value for x
The root(s) must be real to score this mark.</td><td>M1</td></tr>
<tr><td colspan="2">$x = \pm\ln\left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{14}}{2}\right)$
Both correct and exact including brackets.
Accept simplified equivalents e.g. $x = \ln\left(\frac{3}{\sqrt{2}} \pm \frac{\sqrt{7}}{\sqrt{2}}\right)$ but withhold this mark if additional answers are given unless they are the same e.g. allow $x = \pm\ln\left(\frac{3\sqrt{2}}{2} \pm \frac{\sqrt{14}}{2}\right)$</td><td>A1</td></tr>
<tr><td></td><td></td><td>(5)</td></tr>
</table>

(b) Way 2	$\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow 2\cosh^2 2x - 1 - 17\cosh 2x + 9 = 0$ Applies $\cosh 4x = \pm 2\cosh^2 2x \pm 1$ to obtain a quadratic equation in $\cosh 2x$		M1
	$2\cosh^2 2x - 17\cosh 2x + 8 = 0$	Correct 3TQ in $\cosh 2x$	A1
	$2\cosh^2 2x - 17\cosh 2x + 8 = 0$ $\Rightarrow \cosh 2x = 8\left(, \frac{1}{2}\right)$	Solves 3TQ in $\cosh 2x$ (apply usual rules if necessary) to obtain $\cosh 2x = k \quad (k \in \mathbb{R} \text{ and } > 1)$	M1
	$\cosh 2x = 8 \Rightarrow 2x = \pm\ln\left(8 + \sqrt{8^2 - 1}\right)$ or $\cosh 2x = 8 \Rightarrow \frac{e^{2x} + e^{-2x}}{2} = 8 \Rightarrow e^{4x} - 16e^{2x} + 1 = 0 \Rightarrow e^{2x} = \ldots \Rightarrow 2x = \ldots$ Applies the correct logarithmic form for arcosh from $\cosh 2x = k \quad (k > 1)$ or uses the correct exponential form for $\cosh 2x$ to obtain at least one value for $2x$ The root(s) must be real to score this mark.		M1
	$x = \pm\frac{1}{2}\ln\left(8 + 3\sqrt{7}\right)$ or e.g. $x = \pm\ln\left(8 + 3\sqrt{7}\right)^{\frac{1}{2}}$	Both correct and exact with brackets. Accept simplified equivalents e.g. $x = \frac{1}{2}\ln\left(8 \pm \sqrt{63}\right)$ but withhold this mark if additional answers are given unless they are the same as above.	A1
(b) Way 3	$\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow \frac{e^{4x} + e^{-4x}}{2} - \frac{17}{2}\left(e^{2x} + e^{-2x}\right) + 9 = 0$ $\Rightarrow e^{8x} - 17e^{6x} + 18e^{4x} - 17e^{2x} + 1 = 0$ M1: Applies the correct exponential forms and attempts a quartic equation in e^{2x} A1: Correct equation		M1A1
	$e^{8x} - 17e^{6x} + 18e^{4x} - 17e^{2x} + 1 = 0$ $\Rightarrow e^{2x} = 8 \pm 3\sqrt{7}, \ldots$	Solves and proceeds to a value for e^{2x} where $e^{2x} > 1$ and real.	M1
	$\Rightarrow e^{2x} = 8 \pm 3\sqrt{7} \Rightarrow 2x = \ln\left(8 \pm 3\sqrt{7}\right)$	Takes ln's to obtain at least one value for $2x$ The root(s) must be real to score this mark.	M1
	$x = \frac{1}{2}\ln\left(8 \pm 3\sqrt{7}\right)$ or e.g. $x = \ln\left(8 \pm 3\sqrt{7}\right)^{\frac{1}{2}}$	Both correct and exact with brackets. Accept simplified equivalents e.g. $x = \pm\frac{1}{2}\ln\left(8 + 3\sqrt{7}\right)$ but withhold this mark if additional answers are given unless they are the same as above.	A1
			Total 8

Q9.

Question Number	Scheme	Notes	Marks
(a)	$(\cosh A\cosh B+\sinh A\sinh B=)\left(\frac{e^{A}+e^{-A}}{2}\right)\left(\frac{e^{B}+e^{-B}}{2}\right)+\left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{B}-e^{-B}}{2}\right)$ $=\frac{e^{A+B}+e^{A-B}+e^{B-A}+e^{-A-B}+e^{A+B}-e^{A-B}-e^{B-A}+e^{-A-B}}{4}$ Expresses the lhs in terms of exponentials correctly, combines terms and combines fractions with common denominator (Brackets not needed due to fraction lines)		M1
	$=\frac{2e^{A+B}+2e^{-(A+B)}}{4}=\frac{e^{A+B}+e^{-(A+B)}}{2}=\cosh(A+B)$* Fully correct proof with no errors		A1*
			(2)
(b)	$\cosh(x+\ln 2)=\cosh x\cosh(\ln 2)+\sinh x\sinh(\ln 2)$ $=\left(\frac{2+\frac{1}{2}}{2}\right)\cosh x+\left(\frac{2-\frac{1}{2}}{2}\right)\sinh x$ Applies the result from part (a) and evaluates both cosh(ln2) and sinh(ln2) **Use of (a) must be seen**		M1
	$\frac{5}{4}\cosh x+\frac{3}{4}\sinh x=5\sinh x$ $\Rightarrow\frac{5}{4}\cosh x=\frac{17}{4}\sinh x$	Collects terms and reaches $a\cosh x=b\sinh x$ oe Depends on the first M mark	dM1
	$5\cosh x=17\sinh x$ oe	Correct equation	A1
	$x=\frac{1}{2}\ln\left(\frac{1+\frac{5}{17}}{1-\frac{5}{17}}\right)$ Or $\frac{e^{2x}-1}{e^{2x}+1}=\frac{5}{17}\Rightarrow x=\ldots$	Moves to tanh x and uses the correct logarithmic form for artanh$x$ or reverts to exponential forms and solves for x Depends on both M marks	ddM1
	$x=\frac{1}{2}\ln\left(\frac{11}{6}\right)$	Cao $\left(\text{Accept integer multiples of }\frac{11}{6}\right)$	A1
			(5)
			Total 7

Way 2			
(b)	$\cosh(x+\ln 2)=\cosh x\cosh(\ln 2)+\sinh x\sinh(\ln 2)$ $=\left(\frac{2+\frac{1}{2}}{2}\right)\cosh x+\left(\frac{2-\frac{1}{2}}{2}\right)\sinh x$ Applies the result from part (a) and evaluates both cosh(ln2) and sinh(ln2) **Use of (a) must be seen**		M1
	$\Rightarrow 5\cosh x=17\sinh x$ dM1: Collects terms and reaches an equation of form $A\cosh x=B\sinh x$ A1: Correct equation		dM1A1
	$5\left(\frac{e^x+e^{-x}}{2}\right)=17\left(\frac{e^x-e^{-x}}{2}\right)$		
	$12e^x=22e^{-x}\Rightarrow e^{2x}=\frac{22}{6}\Rightarrow x=\ldots$	Changes to exponentials (correct forms) And solves for x	ddM1
	$x=\frac{1}{2}\ln\left(\frac{11}{6}\right)$	Cao $\left(\text{Accept integer multiples of } \frac{11}{6}\right)$	A1
Way 3			
	$\cosh(x+\ln 2)=\cosh x\cosh(\ln 2)+\sinh x\sinh(\ln 2)$ $\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^{\ln 2}+e^{-\ln 2}}{2}\right)+\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^{\ln 2}-e^{-\ln 2}}{2}\right)=5\left(\frac{e^x-e^{-x}}{2}\right)$ Applies the result from part (a) and uses the exponential forms of the hyperbolic functions. **Use of (a) must be seen**		M1
	eg $5e^x+5e^{-x}=17e^x-17e^{-x}$ oe	Evaluates $e^{\ln 2}$ and $e^{-\ln 2}$ and starts to collect terms	dM1
	$12e^{2x}=22\Rightarrow e^{2x}=\frac{11}{6}$	Correct value for e^{2x}	A1
	$x=\ldots$	Solves for x	ddM1
	$x=\frac{1}{2}\ln\left(\frac{11}{6}\right)$	Cao $\left(\text{Accept integer multiples of } \frac{11}{6}\right)$	A1

NB: Squaring and obtaining a value for sinh*x* or cosh*x* introduces extra answers. If these extra answers are then eliminated M1A1 is available but if no attempt at elimination is made award M0A0

Q10.

Question Number	Scheme	Notes	Marks
	$7\cosh x + 3\sinh x = 2e^x + 7 \Rightarrow$ $7\left(\frac{e^x+e^{-x}}{2}\right)+3\left(\frac{e^x-e^{-x}}{2}\right)=2e^x+7$ $\left\{\frac{7}{2}e^x+\frac{7}{2}e^{-x}+\frac{3}{2}e^x-\frac{3}{2}e^{-x}=2e^x+7\right\}$	Substitutes at least one correct exponential form for either of the hyperbolic terms and achieves an **equation** in exponentials and constants alone	M1
	$\Rightarrow 7\left(e^{2x}+1\right)+3\left(e^{2x}-1\right)=4e^{2x}+14e^x$ $\left\{\Rightarrow 5e^{2x}+2=2e^{2x}+7e^x\right\}$	Multiplies through by e^x to obtain any equation that would form a 3TQ in e^x if like terms were collected	M1
	$\Rightarrow 6e^{2x}-14e^x+4=0 \quad \left\{3e^{2x}-7e^x+2=0\right\}$	A correct three term quadratic in e^x. Could be implied by a correct root even if terms have not been collected.	A1
	$\Rightarrow \left(3e^x-1\right)\left(e^x-2\right)=0 \Rightarrow e^x=\ldots$	Solves their 3TQ - usual rules. One correct root for their quadratic if no working. Ignore labelling of the roots even if e.g.,"x" is used.	M1
	$x=\ln 2,\ \ln\frac{1}{3}$	Both correct and simplified but do not isw if there are **other answers**. Allow $-\ln\frac{1}{2}$ for ln 2 and –ln 3 or ln 3^{-1} for $\ln\frac{1}{3}$	A1
	Answer only is 0/5		**Total 5**

	Note that it is possible to multiply through by e^{-x} to form an equation in e^{-2x}, e^{-x} and constants. Score as main scheme, e.g., $\frac{7}{2}e^x+\frac{7}{2}e^{-x}+\frac{3}{2}e^x-\frac{3}{2}e^{-x}=2e^x+7$ $\Rightarrow \frac{7}{2}+\frac{7}{2}e^{-2x}+\frac{3}{2}-\frac{3}{2}e^{-2x}=2+7e^{-x}$ (M1) $\Rightarrow 2e^{-2x}-7e^{-x}+3=0$ (A1) $\left(2e^{-x}-1\right)\left(e^{-x}-3\right)=0 \Rightarrow e^{-x}=\frac{1}{2},\ 3$ (M1) $\Rightarrow e^x=2,\ \frac{1}{3} \Rightarrow x=\ln 2,\ \ln\frac{1}{3}$ (A1)	

Q11.

Question	Scheme		Marks
(a)	$y = \text{artanh}(\cos x)$		
	$\frac{dy}{dx} = \frac{1}{1-\cos^2 x} \times -\sin x$	Correct use of the chain rule	M1
	$= \frac{-\sin x}{\sin^2 x} = \frac{-1}{\sin x} = -\text{cosec}\, x$ *	A1: Correct completion with no errors	A1
			(2)
	Alternative 1		
	$\tanh y = \cos x \Rightarrow \text{sech}^2 y \frac{dy}{dx} = -\sin x$		
	$\frac{dy}{dx} = \frac{-\sin x}{\text{sech}^2 y} = \frac{-\sin x}{1-\cos^2 x}$	Correct differentiation to obtain a function of x	M1
	$= \frac{-\sin x}{\sin^2 x} = \frac{-1}{\sin x} = -\text{cosec}\, x$ *	A1: Correct completion with no errors	A1
			(2)
	Alternative 2		
	$\text{artanh}(\cos x) = \frac{1}{2}\ln\left(\frac{1+\cos x}{1-\cos x}\right)$		
	$\frac{dy}{dx} = \frac{1}{2} \times \frac{1-\cos x}{1+\cos x} \times \frac{-\sin x(1-\cos x) - \sin x(1+\cos x)}{(1-\cos x)^2}$	Correct differentiation to obtain a function of x	M1
	$= \frac{-2\sin x}{2(1-\cos^2 x)} = -\text{cosec}\, x$ *	A1: Correct completion with no errors	A1
			(2)

(b)	$\int \cos x\, \text{artanh}(\cos x)\, dx = \sin x\, \text{artanh}(\cos x) - \int \sin x \times -\text{cosec} x\, dx$ M1: Parts in the correct direction A1: Correct expression		M1 A1
	$\left[\sin x\, \text{artanh}(\cos x) + x\right]_0^{\frac{\pi}{6}} = \frac{1}{2}\text{artanh}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}(-(0))$ M1: Correct use of limits on either part (provided both parts are integrated). Lower limit need not be shown		M1
	$= \frac{1}{4}\ln\left(\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right) + \frac{\pi}{6}$	Use of the logarithmic form of artanh	M1
	$= \frac{1}{4}\ln(7+4\sqrt{3}) + \frac{\pi}{6}$ or $\frac{1}{2}\ln(2+\sqrt{3}) + \frac{\pi}{6}$	Cao (oe)	A1
	The last 2 M marks may be gained in reverse order.		(5)
			(7 marks)

Q12.

Question Number	Scheme	Notes	Marks
	$x=4\cosh\theta \Rightarrow \dfrac{dx}{d\theta}=4\sinh\theta$ $\Rightarrow \displaystyle\int \frac{1}{(x^2-16)^{\frac{3}{2}}}\,dx=\int \frac{4\sinh\theta}{(16\cosh^2\theta-16)^{\frac{3}{2}}}\,d\theta$ Full attempt to use the given substitution. Award for $\displaystyle\int \frac{1}{(x^2-16)^{\frac{3}{2}}}\,dx=k\int \frac{\sinh\theta}{((4\cosh\theta)^2-16)^{\frac{3}{2}}}\,d\theta$ Condone $4\cosh^2\theta$ for $(4\cosh\theta)^2$		M1
	$=\displaystyle\int \frac{4\sinh\theta}{(16\sinh^2\theta)^{\frac{3}{2}}}\,d\theta=\int \frac{4\sinh\theta}{64\sinh^3\theta}\,d\theta$ Simplifies $(16\cosh^2\theta-16)^{\frac{3}{2}}$ to the form $k\sinh^3\theta$ which may be implied by: $\displaystyle\int \frac{1}{(x^2-16)^{\frac{3}{2}}}\,dx=k\int \frac{1}{\sinh^2\theta}\,d\theta$ **Note that this is not dependent on the first M**		M1
	$=\displaystyle\int \frac{1}{16\sinh^2\theta}\,d\theta$ Fully correct simplified integral. Allow equivalents e.g. $\dfrac{1}{16}\displaystyle\int \operatorname{cosech}^2\theta\,d\theta,\ \int \frac{1}{(4\sinh\theta)^2}\,d\theta,\ \int (4\sinh\theta)^{-2}\,d\theta$ etc. May be implied by subsequent work.		A1
	$=\displaystyle\int \frac{1}{16\sinh^2\theta}\,d\theta=\frac{1}{16}\int \operatorname{cosech}^2\theta\,d\theta=-\frac{1}{16}\coth\theta(+c)$ Integrates to obtain $k\coth\theta$. **Depends on both previous method marks.**		dM1
	$=-\dfrac{1}{16}\dfrac{\cosh\theta}{\sinh\theta}+c=-\dfrac{1}{16}\dfrac{\frac{x}{4}}{\sqrt{\frac{x^2}{16}-1}}+c$ or e.g. $-\dfrac{1}{4}\dfrac{\frac{x}{4}}{\sqrt{x^2-16}}+c$ Substitutes back **correctly** for x by replacing $\cosh\theta$ with $\dfrac{x}{4}$ or equivalent e.g. $4\cosh\theta$ with x and $\sinh\theta$ with $\sqrt{\left(\frac{x}{4}\right)^2-1}$ or equivalent e.g. $4\sinh\theta$ with $\sqrt{x^2-16}$ **Depends on all previous method marks and must be fully correct work for their** "$-\dfrac{1}{16}$"		dM1
	$\dfrac{-x}{16\sqrt{x^2-16}}(+c)$ oe e.g. $\dfrac{-\frac{1}{16}x}{\sqrt{x^2-16}}(+c)$	Correct answer. Award once the correct answer is seen and apply isw if necessary. Condone the omission of "$+c$"	A1
	Note that you can condone the omission of the "$d\theta$" throughout		
			(6)
			Total 6

Q13.

Question Number	Scheme	Notes	Marks
(i)	$x^2-4x+5=(x-2)^2+1$	Attempts to complete the square. Allow for $(x-2)^2+c,\ c>0$	M1
	$\int\frac{1}{(x-2)^2+1}\mathrm{d}x=\arctan(x-2)$	Allow for $k\arctan\mathrm{f}(x)$.	M1
	$[\arctan(x-2)]_1^2=0-\left(-\frac{\pi}{4}\right)=\frac{\pi}{4}$	$\frac{\pi}{4}$ cao	A1
			(3)
(ii)	$\int\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=-\frac{\sqrt{x^2-3}}{x}+\int\frac{1}{\sqrt{x^2-3}}\mathrm{d}x$ Uses integration by parts and obtains $A\frac{\sqrt{x^2-3}}{x}+B\int\frac{1}{\sqrt{x^2-3}}\mathrm{d}x$		M1
	$=-\frac{\sqrt{x^2-3}}{x}+\mathrm{arcosh}\frac{x}{\sqrt{3}}$	$B\int\frac{1}{\sqrt{x^2-3}}\mathrm{d}x=k\mathrm{arcosh}\,\mathrm{f}(x)$	M1
		All correct	A1
	$\int_{\sqrt{3}}^{3}\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=\left[-\frac{\sqrt{x^2-3}}{x}+\mathrm{arcosh}\frac{x}{\sqrt{3}}\right]_{\sqrt{3}}^{3}=\left(-\frac{\sqrt{6}}{3}+\mathrm{arcosh}\sqrt{3}\right)-(0+\mathrm{arcosh}1)$ Applies the limits 3 and √3 **Depends on both previous M marks**		dM1
	$\mathrm{arcosh}\sqrt{3}-\frac{1}{3}\sqrt{6}=\ln(\sqrt{2}+\sqrt{3})-\frac{1}{3}\sqrt{6}$	Accept either of these forms.	A1
			(5)
(ii) ALT 1	$\int\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=\int\frac{\sqrt{3\cosh^2u-3}}{3\cosh^2u}\sqrt{3}\sinh u\mathrm{d}u$	A complete substitution using $x=\sqrt{3}\cosh u$	M1
	$=\int\tanh^2u\mathrm{d}u$	Obtains $k\int\tanh^2u\mathrm{d}u$	M1
	$=\int(1-\mathrm{sech}^2u)\mathrm{d}u=u-\tanh u$	Correct integration	A1
	$\int_{\sqrt{3}}^{3}\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=[u-\tanh u]_0^{\mathrm{arcosh}\sqrt{3}}=\mathrm{arcosh}\sqrt{3}-\tanh\left(\mathrm{arcosh}\sqrt{3}\right)-0$ Applies the limits 0 and arcosh√3 **Depends on both previous M marks**		dM1
	$\mathrm{arcosh}\sqrt{3}-\frac{1}{3}\sqrt{6}=\ln(\sqrt{2}+\sqrt{3})-\frac{1}{3}\sqrt{6}$	Accept either of these forms.	A1

(ii) ALT 2	$\int \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = \int \frac{\sqrt{3\sec^2 u - 3}}{3\sec^2 u}\sqrt{3}\sec u \tan u \mathrm{d}u$	A complete substitution using $x = \sqrt{3}\sec u$	M1
	$= \int \frac{\tan^2 u}{\sec u}\mathrm{d}u$	Obtains $k\int \frac{\tan^2 u}{\sec u}\mathrm{d}u$	M1
	$= \ln(\sec u + \tan u) - \sin u$	Correct integration	A1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = [\ln(\sec u + \tan u) - \sin u]_0^{\operatorname{arcsec}\sqrt{3}}$ $= \ln\left(\sec\left(\operatorname{arcsec}\sqrt{3}\right) + \tan\left(\operatorname{arcsec}\sqrt{3}\right)\right) - \ln\left(\sec(0) + \tan(0)\right) - \sin\left(\operatorname{arcsec}\sqrt{3}\right)$ Applies the limits 0 and arcsec√3 **Depends on both previous M marks**		dM1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = \ln(\sqrt{2}+\sqrt{3}) - \frac{1}{3}\sqrt{6}$	Correct answer.	A1
			Total 8

Note that there may be other ways to perform the integration in part (ii) e.g. subsequent substitutions. Marks can be awarded if the method leads to something that is integrable and should be awarded as in the main scheme e.g. M1 for a complete method, M2 for simplifying and reaching an expression that itself can be integrated or can be integrated after rearrangement, A1 for correct integration, dM3 for using appropriate limits and A2 as above.

Alternative approach:

$$\int \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = \int \frac{x^2-3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \int \frac{1}{\sqrt{x^2-3}}\mathrm{d}x - \int \frac{3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \operatorname{arcosh}\frac{x}{\sqrt{3}} - \ldots$$

Can score **M0M1A0dM0A0** if there is no creditable attempt at the second integral.

If the second integral is attempted, it must be using a suitable method
e.g. with either $x = \sqrt{3}\cosh u$ or $x = \sqrt{3}\sec u$:

$$\int \frac{3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \int \frac{3}{3\cosh^2 u\sqrt{3\cosh^2 u - 3}}\sqrt{3}\sinh u \,\mathrm{d}u = \int \operatorname{sech}^2 u \,\mathrm{d}u = \tanh u + c$$

or

$$\int \frac{3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \int \frac{3}{3\sec^2 u\sqrt{3\sec^2 u - 3}}\sqrt{3}\sec u \tan u \mathrm{d}u = \int \cos u \,\mathrm{d}u = \sin u + c$$

In these cases the first M can then be awarded and the other marks as defined with the appropriate limits used.

Q14.

Question Number	Scheme	Notes	Marks
(i)	$f(x) = x\arccos x,\ -1 \le x \le 1,$		
	$f'(x) = \arccos x - \dfrac{x}{\sqrt{1-x^2}}$ M1: Differentiates using the product rule to obtain an expression of the form: $\arccos x \pm \dfrac{x}{\sqrt{1-x^2}}$ A1: Correct derivative		M1A1
	$f'(0.5) = \arccos 0.5 - \dfrac{0.5}{\sqrt{1-0.5^2}} = \dfrac{\pi - \sqrt{3}}{3}$	$\dfrac{\pi-\sqrt{3}}{3}$ oe e.g. $\dfrac{\pi}{3} - \dfrac{1}{\sqrt{3}}$	A1
			(3)
(ii)	$g(x) = \arctan(e^{2x})$		
	$g'(x) = \dfrac{2e^{2x}}{e^{4x}+1}$ M1: Differentiates using the chain rule to obtain an expression of the form: $\dfrac{ke^{2x}}{\left(e^{2x}\right)^2+1}$ A1: Correct derivative in any form		M1A1
	$g'(x) = \dfrac{2}{e^{2x}+e^{-2x}} = \text{sech}(2x)$	Introduces $\text{sech}(2x)$. Depends on previous M.	dM1
	$g''(x) = -2\,\text{sech}(2x)\tanh(2x)$	Differentiates $\text{sech}(u) \to \pm\text{sech}\,u\tanh u$ Depends on both previous M's.	dM1
		Correct expression.	A1
			(5)

(ii) ALT 1	$g'(x) = \dfrac{2e^{2x}}{e^{4x}+1}$ M1: Differentiates using the chain rule to obtain an expression of the form: $\dfrac{ke^{2x}}{\left(e^{2x}\right)^2+1}$ A1: Correct derivative in any form		M1A1
	$g''(x) = \dfrac{4e^{2x}\left(1+e^{4x}\right) - 4e^{4x}\times 2e^{2x}}{(e^{4x}+1)^2}$	Differentiates using quotient or product rule. Depends on first M.	dM1
	$= \dfrac{4e^{2x}-4e^{6x}}{(e^{4x}+1)^2} = \dfrac{-4(e^{2x}-e^{-2x})}{(e^{2x}+e^{-2x})^2}$	Multiply through by e^{-4x}. Depends on both previous M's.	dM1
	$= -2\dfrac{2}{e^{2x}+e^{-2x}}\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$ $= -2\,\text{sech}\,2x\tanh 2x$	Correct expression.	A1
	Note that the first derivative may be obtained implicitly in either method e.g. $y = \arctan\left(e^{2x}\right) \Rightarrow \tan y = e^{2x} \Rightarrow \sec^2 y\dfrac{dy}{dx} = 2e^{2x} \Rightarrow \dfrac{dy}{dx} = \dfrac{2e^{2x}}{1+\left(e^{2x}\right)^2}$		
			Total 8

Q15.

Question Number	Scheme	Notes	Marks
(i)	$x^2-3x+5=\left(x-\frac{3}{2}\right)^2+\frac{11}{4}$	Correct completion of the square	B1
	$\int \frac{1}{\sqrt{x^2-3x+5}}\,dx=\int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2+\frac{11}{4}}}\,dx=\sinh^{-1}\frac{2x-3}{\sqrt{11}}(+c)$ M1: Use of $\sinh^{-1}$ A1: Fully correct expression (condone omission of $+c$) Allow equivalent correct expressions e.g. $\sinh^{-1}\frac{x-\frac{3}{2}}{\sqrt{\frac{11}{4}}}(+c)$, $\sinh^{-1}\frac{x-\frac{3}{2}}{\frac{\sqrt{11}}{2}}(+c)$ Allow equivalents for $\sinh^{-1}$ e.g. arsinh, arcsinh but **not** arsin or arcsin		M1A1
	You may see logarithmic forms for the answer: e.g. $\ln\left(\frac{2x-3}{\sqrt{11}}+\sqrt{\left(\frac{2x-3}{\sqrt{11}}\right)^2+1}\right)$, $\ln\left(x-\frac{3}{2}+\sqrt{\left(x-\frac{3}{2}\right)^2+\frac{11}{4}}\right)$ but apply isw once a correct answer is seen.		
			(3)

(ii)	$63+4x-4x^2=-4\left(x^2-x-\frac{63}{4}\right)$ $=-4\left(\left(x-\frac{1}{2}\right)^2-\frac{64}{4}\right)$	Obtains $-4\left(\left(x-\frac{1}{2}\right)^2\pm\ldots\right)$ or $-4\left(x-\frac{1}{2}\right)^2\pm\ldots$ or $\ldots-(2x-1)^2$	M1
	$-4\left(\left(x-\frac{1}{2}\right)^2-16\right)$ or $64-4\left(x-\frac{1}{2}\right)^2$ or $64-(2x-1)^2$	Correct completion of the square	A1
	$\int \frac{1}{\sqrt{63+4x-4x^2}}\,dx=\frac{1}{2}\sin^{-1}\left(\frac{2x-1}{8}\right)(+c)$ M1: Use of $\sin^{-1}$ A1: Fully correct expression (condone omission of $+c$) Allow equivalent correct expressions e.g. $\frac{1}{2}\sin^{-1}\frac{x-\frac{1}{2}}{4}(+c)$, $-\frac{1}{2}\sin^{-1}\frac{\frac{1}{2}-x}{4}(+c)$ Allow equivalents for $\sin^{-1}$ e.g. arsin, arcsin but not arsinh or arcsinh		M1A1
			(4)
	In (ii) there are no marks for using $\int \frac{1}{\sqrt{63+4x-4x^2}}\,dx=-\int \frac{1}{\sqrt{4x^2-63-4x}}\,dx$ But if completion of square attempted first allow M1A1 e.g. for $\int \frac{1}{\sqrt{63+4x-4x^2}}\,dx=\int \frac{1}{\sqrt{64-(2x-1)^2}}\,dx$ but then M0 for $=\int \frac{-1}{\sqrt{(2x-1)^2-64}}\,dx$		
			Total 7

Q16.

Question Number	Scheme	Notes	Marks
(i)	Throughout both parts of this question do not penalise omission of dx or $d\theta$		
	$5+4x-x^2=9-(x-2)^2$ oe	Correct completion of the square Any correct result	B1
	$\int \frac{1}{\sqrt{5+4x-x^2}}\,dx=\int \frac{1}{\sqrt{9-(x-2)^2}}\,dx=\sin^{-1}\left(\frac{x-2}{3}\right)(+c)$ M1: Obtains $k\sin^{-1}f(x)$ A1: Correct integration ($+c$ not needed)		M1A1
			(3)

(ii)	$x=6\Rightarrow\theta=\frac{\pi}{3}$ $x=2\sqrt{3}\Rightarrow\theta=\frac{\pi}{6}$	Correct θ limits in radians	B1
	$\int \frac{18}{(x^2-9)^{\frac{3}{2}}}\,dx=\int \frac{18\times 3\sec\theta\tan\theta}{(9\sec^2\theta-9)^{\frac{3}{2}}}\,d\theta$ M1: For $\int \frac{18}{((3\sec\theta)^2-9)^{\frac{3}{2}}}\times\left(\text{their}\frac{dx}{d\theta}\right)d\theta$		M1
	$\int \frac{54\sec\theta\tan\theta}{(9\sec^2\theta-9)^{\frac{3}{2}}}\,d\theta=54\int \frac{\sec\theta\tan\theta}{27\tan^3\theta}\,d\theta=2\int \frac{\sin\theta\cos^3\theta}{\cos^2\theta\sin^3\theta}\,d\theta$ $2\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$ oe eg $2\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$ Correct simplified integral		A1
	$2\int \frac{\cos\theta}{\sin^2\theta}\,d\theta=2\int \operatorname{cosec}\theta\cot\theta\,d\theta=-2\operatorname{cosec}\theta(+c)$ Obtains $k\operatorname{cosec}\theta(+c)$		M1
	$[-2\operatorname{cosec}\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=-2\operatorname{cosec}\frac{\pi}{3}+2\operatorname{cosec}\frac{\pi}{6}$	Uses changed limits correctly. **Depends on all previous method marks.**	dM1
	$=4-\frac{4}{3}\sqrt{3}$	Cao Allow these 2 marks if limits have been given in degrees	A1
			(6)
			Total 9

ALT	For B1 and final dM1A1 of (ii)	
	dM1: Reverse the substitution A1: Correct reversed result A1: **enter as B1 on e-PEN** Correct final answer	

Q17.

Question	Scheme		Marks
(a)	$x^2+4x+13\equiv(x+2)^2+9$		B1
	$\int\frac{1}{(x+2)^2+9}\mathrm{d}x=\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)$	M1: $k\arctan f(x)$. A1: Correct expression	M1 A1
	$\left[\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)\right]_{-2}^{1}=\frac{1}{3}(\arctan 1-\arctan 0)$	Correct use of limits arctan0 need not be shown	M1
	$\frac{\pi}{12}$	cao	A1
			(5)
	Alternative		
	Sub $x+2=3\tan t$		
	$x^2+4x+13\equiv(x+2)^2+9$		B1
	$\frac{\mathrm{d}x}{\mathrm{d}t}=3\sec^2 t \qquad x=-2, \tan t=0,\ t=0;\ x=1, \tan t=1,\ t=\frac{\pi}{4}$		
	$\int\frac{3\sec^2 t}{9\tan^2 t+9}\mathrm{d}t=\frac{1}{3}\int \mathrm{d}t=\frac{1}{3}t$	M1 sub and integrate inc use of $\tan^2+1=\sec^2$ A1 Correct expression Ignore limits	M1 A1
	$\left[\frac{\pi}{12}\right]_0^{\frac{\pi}{4}}$.	Either change limits and substitute Or reverse substitution and substitute original imits	M1
	$\frac{\pi}{12}$	cao	A1
			(5)

(b)	$4x^2-12x+34=4\left(x-\frac{3}{2}\right)^2+25$ or $(2x-3)^2+25$	M1: $4(x\pm p)^2\pm q,\ (p,q\neq 0)$ A1: $4\left(x-\frac{3}{2}\right)^2+25$	M1 A1
	$\int\frac{1}{\sqrt{4\left(x-\frac{3}{2}\right)^2+25}}\mathrm{d}x=\frac{1}{2}\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2+\frac{25}{4}}}\mathrm{d}x=\frac{1}{2}\operatorname{arsinh}\left(\frac{x-\frac{3}{2}}{\frac{5}{2}}\right)$ M1: $k\operatorname{arsinh}\mathrm{f}(x)$. A1: Correct expression		M1 A1
	$\left[\frac{1}{2}\operatorname{arsinh}\left(\frac{x-\frac{3}{2}}{\frac{5}{2}}\right)\right]_{-1}^{4}=\frac{1}{2}(\operatorname{arsinh}(1)-\operatorname{arsinh}(-1))$	Correct use of limits	M1
	$=\frac{1}{2}\left(\ln\left(1+\sqrt{2}\right)-\ln\left(-1+\sqrt{2}\right)\right)$	Uses the logarithmic form of arsinh	M1
	$=\frac{1}{2}\ln\left(3+2\sqrt{2}\right)$ or $\ln\left(1+\sqrt{2}\right)$	cao	A1
			(7)
	Alternative: Second M1 A1		
	Sub $2x-3=u$ or $2x-3=5\sinh u$		
	$\int_{\operatorname{arsinh}-1}^{\operatorname{arsinh}1}\frac{1}{\sqrt{25\sinh^2 u+25}}5\cosh u\,\mathrm{d}u=\left[\frac{1}{2}\operatorname{arsinh}\left(\frac{u}{5}\right)\right]_{-5}^{5}$		M1 A1
	$\int_{-5}^{5}\frac{1}{2\sqrt{u^2+25}}\mathrm{d}u=\left[\frac{1}{2}\operatorname{arsinh}\left(\frac{u}{5}\right)\right]_{-5}^{5}$		
			(12 marks)

Q18.

Question Number	Scheme	Notes	Marks
(a)	$5\mathbf{i}+3\mathbf{j}-8\mathbf{k}$ and $2\mathbf{i}-3\mathbf{j}-6\mathbf{k}$ lie in Π_1	Identifies 2 correct vectors lying in Π_1	B1
	$\mathbf{n}=\begin{pmatrix}5\\3\\-8\end{pmatrix}\times\begin{pmatrix}2\\-3\\-6\end{pmatrix}=\begin{pmatrix}-18-24\\-(-30+16)\\-15-6\end{pmatrix}$ Attempts the vector product between 2 **correct** vectors in Π_1 If no working is shown, look for at least 2 correct elements. Or e.g. Let $\mathbf{n}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ then $(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot(5\mathbf{i}+3\mathbf{j}-8\mathbf{k})=0,\ (a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot(2\mathbf{i}-3\mathbf{j}-6\mathbf{k})=0$ $\Rightarrow 5a+3b-8c=0,\ 2a-3b-6c=0 \Rightarrow a=2c,\ 3b=-2c \Rightarrow \mathbf{n}=\ldots$		M1
	$=\begin{pmatrix}-42\\14\\-21\end{pmatrix}$ or e.g. $\begin{pmatrix}6\\-2\\3\end{pmatrix}$	Correct normal vector	A1
	$(6\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}+\mathbf{k})=\ldots$ Attempts scalar product between their normal vector and position vector of a point in Π_1. Do not allow this mark if the "5" (or equivalent) just 'appears'. There must be some evidence for its origin e.g. **a**.**n** = … where **a** and **n** have been defined earlier. **Depends on the first method mark.**		dM1
	$6x-2y+3z=5$ *	Correct proof	A1*
			(5)

	Alternative 1 for (a):		
	E.g. Let equation of Π_1 be $ax+by+z=c$ 3 points on Π_1 are (1, 2, 1), (3, −1, −5) and e.g. (8, 2, −13)		B1
	$a+2b+1=c,\ 3a-b-5=c,\ 8a+2b-13=c \Rightarrow a=\ldots, b=\ldots, c=\ldots$ Solves simultaneously for a, b and c using **correct** points		M1
	$\Rightarrow a=2, b=-\frac{2}{3}, c=\frac{5}{3}$	Correct values	A1
	$2x-\frac{2}{3}y+z=\frac{5}{3}$	Forms Cartesian equation	dM1
	$6x-2y+3z=5$ *	Correct proof	A1*
	Alternative 2 for (a):		
	$(1,2,1)\rightarrow 6x-2y+3z=6-4+3=5$ Shows (1, 2, 1) lies on Π_1		B1
	$\frac{x-3}{5}=\frac{y+1}{3}=\frac{z+5}{-8}\rightarrow \mathbf{r}=\begin{pmatrix}3\\-1\\-5\end{pmatrix}+\lambda\begin{pmatrix}5\\3\\-8\end{pmatrix}$ or equivalent M1: Converts l to **correct** parametric form <u>seen as part of an attempt at this alternative</u> allow 1 slip with one of the elements A1: Correct form		M1A1
	$6(3+5\lambda)-2(-1+3\lambda)+3(-5-8\lambda)=5$ Shows l lies in Π_1		dM1
	P lies in Π_1 and l lies in Π_1 so $6x-2y+3z=5$ * All correct with conclusion		A1*

(b) Way 1	$d=\dfrac{\lvert 6(2)-2k+3(-7)-5\rvert}{\sqrt{6^2+2^2+3^2}}$	Correct method for the shortest distance	M1
	$=\dfrac{1}{7}\lvert -2k-14\rvert=\dfrac{2}{7}\lvert k+7\rvert$*	Correct completion	A1*
			(2)
(b) Way 2	Distance O to Π_1 is $\dfrac{5}{\sqrt{6^2+2^2+3^2}}$. Distance O to parallel plane containing Q is $\dfrac{(6\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(2\mathbf{i}+k\mathbf{j}-7\mathbf{k})}{\sqrt{6^2+2^2+3^2}}=\dfrac{-9-2k}{7}$ $d=\left\lvert\dfrac{5}{7}-\dfrac{-9-2k}{7}\right\rvert$ Correct method for the shortest distance		M1
	$=\dfrac{1}{7}\lvert 2k+14\rvert=\dfrac{2}{7}\lvert k+7\rvert$*	Correct completion	A1*
(b) Way 3	$d=\left\lvert\dfrac{\overrightarrow{PQ}\cdot\mathbf{n}}{\lvert\mathbf{n}\rvert}\right\rvert=\left\lvert\dfrac{(\mathbf{i}+(k-2)\mathbf{j}-8\mathbf{k})\cdot(-42\mathbf{i}+14\mathbf{j}-21\mathbf{k})}{\sqrt{42^2+14^2+21^2}}\right\rvert$ Correct method for the shortest distance		M1
	$=\left\lvert\dfrac{-42+14k-28+168}{49}\right\rvert=\left\lvert\dfrac{14k+98}{49}\right\rvert=\dfrac{2}{7}\lvert k+7\rvert$*	Correct completion	A1*

(c)	$\dfrac{2}{7}\lvert k+7\rvert=\dfrac{\lvert 8(2)-4k-7+3\rvert}{\sqrt{8^2+4^2+1^2}}$ Correctly attempts the distance between $(2, k, -7)$ and Π_2 and sets equal to the result from (a). May see alternative methods here for the distance between $(2, k, -7)$ and Π_2 e.g. finds the coordinates of a point on Π_2 e.g. $R(1, 1, -7)$ and then finds $d=\left\lvert\dfrac{\overrightarrow{RQ}\cdot(8\mathbf{i}-4\mathbf{j}+\mathbf{k})}{\lvert 8\mathbf{i}-4\mathbf{j}+\mathbf{k}\rvert}\right\rvert=\left\lvert\dfrac{(\mathbf{i}+(k-1)\mathbf{j})\cdot(8\mathbf{i}-4\mathbf{j}+\mathbf{k})}{\sqrt{8^2+4^2+1^2}}\right\rvert=\left\lvert\dfrac{8-4k+4}{9}\right\rvert=\left\lvert\dfrac{12-4k}{9}\right\rvert$		M1
	$\dfrac{2}{7}(k+7)=\text{"}\dfrac{1}{9}(12-4k)\text{"}\Rightarrow k=\ldots$ or $\dfrac{2}{7}(k+7)=\text{"}\dfrac{1}{9}(4k-12)\text{"}\Rightarrow k=\ldots$ Attempts to solve one of these equations where their distance from Q to Π_2 is of the form $ak+b$ where a and b are non-zero. **or** $\dfrac{2}{7}(k+7)=\text{"}\dfrac{1}{9}(12-4k)\text{"}\Rightarrow\dfrac{4}{49}(k+7)^2=\text{"}\dfrac{1}{81}(12-4k)^2\text{"}$ $\Rightarrow 23k^2-462k-441=0\Rightarrow k=\ldots$ Squares both sides and attempts to solve resulting quadratic. Condone poor attempts at squaring the brackets and there is no requirement to follow the usual guidance for solving the quadratic		dM1
	$k=-\dfrac{21}{23}$ or $k=21$	One correct value. Must be 21 but allow equivalent exact fractions for $-\dfrac{21}{23}$	A1
	$k=-\dfrac{21}{23}$ **and** $k=21$	Both correct values. Must be 21 but allow equivalent exact fractions for $-\dfrac{21}{23}$ and no other values.	A1
			(4)
			Total 11

Q19.

Question Number	Scheme	Notes	Marks
(a)	$\frac{x-3}{4}=\frac{y-5}{-2}=\frac{z-4}{7}\Rightarrow \mathbf{r}=\begin{pmatrix}3\\5\\4\end{pmatrix}\pm\lambda\begin{pmatrix}4\\-2\\7\end{pmatrix}$	Converts to parametric form. "r =" is not required	M1
	$2x+4y-z=1$ $\Rightarrow 2(3+4\lambda)+4(5-2\lambda)-4-7\lambda=1$ $\Rightarrow \lambda=...(3)\Rightarrow P$ is ...	Correct strategy for finding *P*. Condone the use of $2x+4y-z=0$ for the plane equation.	M1
	$(15,\ -1,\ 25)$	Correct coordinates. Condone if given as a vector.	A1
			(3)
(a) Way 2	$\frac{x-3}{4}=\frac{y-5}{-2}\Rightarrow x=13-2y$	Uses the Cartesian equation to find *x* in terms of *y*	M1
	$2x+4y-z=1\Rightarrow 26-4y+4y-z=1$ $\Rightarrow z=...,\ x=...,\ y=...$	Correct strategy for finding *P*. Condone the use of $2x+4y-z=0$ for the plane equation.	M1
	$(15,\ -1,\ 25)$	Correct coordinates. Condone if given as a vector.	A1

(b)	$\begin{pmatrix}4\\-2\\7\end{pmatrix}\bullet\begin{pmatrix}2\\4\\-1\end{pmatrix}=8-8-7=-7$	Applies the scalar product between the direction of l_1 and the normal to the plane	M1
	Examples: $\phi=\cos^{-1}\frac{\pm7}{\sqrt{69}\sqrt{21}}=...\quad \phi=\sin^{-1}\frac{\pm7}{\sqrt{69}\sqrt{21}}=...$ Attempts to find a relevant angle in degrees or radians. **Depends on the first method mark.**		dM1
	$\theta=10.6°$	Allow awrt 10.6 but do **not** isw and mark the final answer. For reference $\theta=10.5965654°$	A1
			(3)
(b) Way 2	$\begin{pmatrix}4\\-2\\7\end{pmatrix}\times\begin{pmatrix}2\\4\\-1\end{pmatrix}=\begin{pmatrix}26\\-18\\-20\end{pmatrix}$	Attempts vector product of normal to Π and direction of l_1	M1
	$\sqrt{26^2+18^2+20^2}=\sqrt{21}\sqrt{69}\sin\alpha$ $\sin\alpha=\frac{10\sqrt{46}}{69}\Rightarrow\alpha=...$	Attempts to find a relevant angle. **Depends on the first method mark.**	dM1
	$\theta=10.6°$	Allow awrt 10.6 but do **not** isw and mark the final answer. For reference $\theta=10.5965654°$	A1

(c)	$\mathbf{a}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&4&-1\\4&-2&7\end{vmatrix}=\begin{pmatrix}26\\-18\\-20\end{pmatrix}$	Attempts vector product of normal to Π and direction of l_1. If no method is seen expect at least 2 correct components.	M1
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\13&-9&-10\\2&4&-1\end{vmatrix}=\begin{pmatrix}49\\-7\\70\end{pmatrix}$	Attempts vector product of "a" with normal to Π to find direction of l_2	M1
		Correct direction for l_2	A1
	$\mathbf{r}=\begin{pmatrix}15\\-1\\25\end{pmatrix}+\mu\begin{pmatrix}7\\-1\\10\end{pmatrix}$	Depends on both previous M marks Attempts vector equation using their direction vector and their *P*	ddM1
		Correct equation or any equivalent correct vector equation	A1
			(5)

(c) Way 2	$\lambda = 1 \Rightarrow (7,\ 3,\ 11)$ lies on l_1 $$\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ 11 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$ $$\Rightarrow 2(7+2t)+4(3+4t)-11+t=1$$ $$t = -\frac{2}{3} \Rightarrow \left(\frac{17}{3}, \frac{1}{3}, \frac{35}{3}\right)$$ is on l_2	Complete method to find a point on l_2	M1
	Direction of l_2 is $\begin{pmatrix} 15 \\ -1 \\ 25 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 17 \\ 1 \\ 35 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 28 \\ -4 \\ 40 \end{pmatrix}$	Uses their point and their P to find direction of l_2	M1
		Correct direction for l_2	A1
	$\mathbf{r} = \begin{pmatrix} 15 \\ -1 \\ 25 \end{pmatrix} + \mu\begin{pmatrix} 7 \\ -1 \\ 10 \end{pmatrix}$	Attempts vector equation using their direction vector and their point on l_2	ddM1
		Correct equation or any equivalent correct vector equation. Must have $\mathbf{r} =$ and not e.g. $l_2 = \ldots$	A1
(c) Way 3	Normal to plane from l_1 $$\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$ $$\Rightarrow 2(3+2t)+4(5+4t)-(4-t)=1$$ $t = -1 \Rightarrow (1, 1, 5)$ is on l_2	Complete method to find a point on l_2	M1
	Direction of l_2 is $\begin{pmatrix} 15 \\ -1 \\ 25 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 14 \\ -2 \\ 20 \end{pmatrix}$	Uses their point and their P to find direction of l_2	M1
		Correct direction for l_2	A1
	$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} 7 \\ -1 \\ 10 \end{pmatrix}$	Attempts vector equation using their direction vector and their point on l_2	ddM1
		Correct equation or any equivalent correct vector equation. Must have $\mathbf{r} =$ and not e.g. $l_2 = \ldots$	A1
			Total 11

Q20.

Question Number	Scheme	Notes	Marks
(a)	$(\mathbf{r}=)\begin{pmatrix}-4\\-5\\3\end{pmatrix}+t\begin{pmatrix}3\\4\\-1\end{pmatrix}$	Forms the parametric form of the line	M1
	$3(3t-4)+4(4t-5)-(3-t)=17$ $\Rightarrow t=(2)$	Substitutes the parametric form for the line into the plane equation and solves for "*t*". **Depends on the first mark.**	dM1
	$\begin{pmatrix}-4\\-5\\3\end{pmatrix}+"2"\begin{pmatrix}3\\4\\-1\end{pmatrix}$	Uses their value of *t* correctly to find *Q*. **Depends on the previous mark.**	dM1
	$(2, 3, 1)$	Correct coordinates Accept if written as a column vector but not with **i**, **j**, **k**	A1 **(4)**
Way 2	$\frac{x+4}{3}=\frac{y+5}{4}=\frac{z-3}{-1}$ eg $x=\mathrm{f}(y)$ $z=\mathrm{g}(y)$	Forms the Cartesian equation of the line, rearranges twice to get 2 of *x*, *y*, *z* as functions of the third	M1
		Substitutes these into the plane equation and solves for one coordinate	dM1
		Obtains the other 2 coordinates	dM1
	$(2, 3, 1)$	Correct coordinates Accept if written as a column vector but not with **i**, **j**, **k**	A1
			(4)
(b)	$\mathbf{PQ}=\begin{pmatrix}2+4\\3+5\\1-3\end{pmatrix}, \mathbf{PR}=\begin{pmatrix}-1+4\\6+5\\4-3\end{pmatrix}, \mathbf{RQ}=\begin{pmatrix}2+1\\3-6\\1-4\end{pmatrix}$	Attempts 2 vectors in plane *PQR* (Must use the given coordinates of *P*, *R* and their coordinates of *Q*	M1
	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\6 & 8 & -2\\3 & 11 & 1\end{vmatrix}=\begin{pmatrix}30\\-12\\42\end{pmatrix}$	Attempt vector product between 2 vectors in *PQR*. **Depends on the first mark.**	dM1
	$\begin{pmatrix}5\\-2\\7\end{pmatrix}\bullet\begin{pmatrix}2\\3\\1\end{pmatrix}=11$	Uses any of *P*, *Q* or *R* to find constant. **Depends on the previous mark.**	dM1
	$5x-2y+7z=11$	Any correct Cartesian equation	A1
			(4)

Way 2	$-4a-5b-3c=1$ $2a+3b+c=1$ $-a+6b+4c=1$	Uses the Cartesian form of the equation of a plane, $ax+by+cz=1$, and substitutes the coordinates of each of the 3 points	M1
	Solves to get a value for any of a, b or c		dM1
	Obtains values for the other 2		dM1
	$\frac{5}{11}x-\frac{2}{11}y+\frac{7}{11}z=1$	Any correct Cartesian equation	A1
			(4)
(c)	Reflection of P in Π is $\begin{pmatrix}-4\\-5\\3\end{pmatrix}+2\times"2"\left(\begin{pmatrix}3\\4\\-1\end{pmatrix}\right)=\left(\begin{pmatrix}8\\11\\-1\end{pmatrix}\right)$	Correct strategy for another point on l_3	M1
	$\left(\begin{pmatrix}8\\11\\-1\end{pmatrix}-\begin{pmatrix}-1\\6\\4\end{pmatrix}\right)=\left(\begin{pmatrix}9\\5\\-5\end{pmatrix}\right)$	Attempts direction of l_3. **Depends on the first mark.**	dM1
	$\mathbf{r}=\begin{pmatrix}-1\\6\\4\end{pmatrix}+\lambda\begin{pmatrix}9\\5\\-5\end{pmatrix}$	Forms the equation of l_3 using R (or their reflected P) and their direction. **Depends on the previous mark.**	ddM1
		Any correct equation in vector form	A1 **(4)**
			Total 12

Q21.

Question Number	Scheme	Notes	Marks
(a)	Normal to plane given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \ldots$	Attempt cross product of direction vectors. If the method is unclear, look for at least 2 correct components.	M1
	$= 6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$	Or any multiple of this vector.	A1
	Substitute appropriate point into $6x + 2y - 2z = d$ e.g. (1, 1, 1) or (2, 1, 4) to find "*d*"	Use a valid point and use scalar product with normal or substitute into Cartesian equation.	M1
	$6x + 2y - 2z = 6$ $3x + y - z = 3$ *	Given answer. No errors seen	A1* cso
			(4)
(a) ALT	$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ $\Rightarrow x = 1 + \lambda + \mu,\ y = 1 - 2\mu,\ z = 1 + 3\lambda + \mu$ M1: Forms equation of plane using (1, 1, 1) and direction vectors and extracts 3 equations for *x*, *y* and *z* in terms of λ and μ A1: Correct equations		M1A1
	$x = 1 + \frac{1}{2} - \frac{1}{2}y + \frac{1}{3}z - \frac{1}{2} + \frac{1}{6}y$	Eliminates λ and μ and achieves an equation in x, y and z only.	M1
	$3x + y - z = 3$ *	Given answer. No errors seen.	A1
(b)	$s = -3$	cao	B1
			(1)
(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & 1 & -1 \end{vmatrix} = \mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$	Attempts cross product of normal vectors. If the method is unclear, look for at least 2 correct components.	M1
	e.g. $x = 0, 2y - 2z = 6, y - 2z = 3$ $\Rightarrow y = 3, z = 0$	Any valid attempt to find a point on the line.	M1
	e.g. (0,3,0)	Any valid point on the line	A1
	$\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$	Correct equation including "$\mathbf{r} =$" or equivalent e.g. $x = \frac{y-3}{-5} = \frac{z}{-2}$	A1
			(4)

(c) ALT 1	$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k}),\ \mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 3$ $\Rightarrow 1 + \lambda + \mu + 1 - 2\mu - 2 - 6\lambda - 2\mu = 3$ Forms equation of first plane using (1, 1, 1) and direction vectors and substitutes into the second plane to form an equation in λ and μ		M1
	$\Rightarrow \mu = \frac{1}{3}(-5\lambda - 3)$	Solves to obtain μ in terms of λ or λ in terms of μ	M1
		Correct equation	A1
	E.g. $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \frac{1}{3}(-5\lambda - 3)(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ Correct equation including "$\mathbf{r} =$"		A1
(c) ALT 2	$3x + y - z = 3,\ x + y - 2z = 3 \Rightarrow 2x + z = 0$	Uses the Cartesian equations of both planes and eliminates one variable	M1
	$z = \lambda \Rightarrow x = -\frac{1}{2}\lambda,\ y = 3 + 2z - x = 3 + \frac{5}{2}\lambda$	Introduces parameter and expresses other 2 variables in terms of the parameter	M1
		Correct equations	A1
	$\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$	Correct equation including "$\mathbf{r} =$" or equivalent e.g. $x = \frac{y-3}{-5} = \frac{z}{-2}$	A1

(c) ALT 3	$3x+y-z=3,\ x+y-2z=3 \Rightarrow 2x+z=0$	Uses the Cartesian equations of both planes and eliminates one variable	M1
	$3x+y-z=3,\ x+y-2z=3 \Rightarrow 5x+y=3$	Uses the Cartesian equations of both planes and eliminates another variable	M1
	$\Rightarrow x=-\frac{z}{2},\quad x=\frac{3-y}{5}$	Correct equations for one variable in terms of the other 2	A1
	$x=\frac{y-3}{-5}=\frac{z}{-2}$	Correct equation or equivalent e.g. $x=\frac{3-y}{5}=\frac{z}{-2}$	A1
(d)	$(3\mathbf{i}+\mathbf{j}-\mathbf{k})\square(\mathbf{i}+\mathbf{j}-2\mathbf{k})=6$	Correct value for scalar product	B1
	$\cos\theta=\frac{(3\mathbf{i}+\mathbf{j}-\mathbf{k}).(\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{9+1+1}\sqrt{1+1+4}}=\sqrt{\frac{6}{11}}$	Full scalar product attempt to reach a value for $\cos\theta$	M1
		For $\cos\theta=\sqrt{\frac{6}{11}}$	A1
	$\theta=42.4°$	Correct value. Mark their final answer.	A1
			(4)
(d) ALT	$\lvert(3\mathbf{i}+\mathbf{j}-\mathbf{k})\times(\mathbf{i}+\mathbf{j}-2\mathbf{k})\rvert=\sqrt{30}$	Correct value for magnitude of cross product	B1
	$\sin\theta=\frac{\lvert(3\mathbf{i}+\mathbf{j}-\mathbf{k}).(\mathbf{i}+\mathbf{j}-2\mathbf{k})\rvert}{\sqrt{9+1+1}\sqrt{1+1+4}}=\frac{\sqrt{55}}{11}$	Full attempt to reach a value for $\sin\theta$	M1
		For $\sin\theta=\frac{\sqrt{55}}{11}$	A1
	$\theta=42.4°$	Correct value. Mark their final answer.	A1
			Total 13

Q22.

Question	Scheme	Marks
	$\cosh y=x, y<0 \Rightarrow y=\ln\left[x-\sqrt{x^2-1}\right]$	
	$\cosh y=x \Rightarrow x=\frac{e^y+e^{-y}}{2}$	B1
	$\Rightarrow 2xe^y=e^{2y}+1$	M1
	$\Rightarrow e^{2y}-2xe^y+1=0 \Rightarrow e^y=\frac{2x\pm\sqrt{(2x)^2-4\times1\times1}}{2}$ or $\Rightarrow e^{2y}-2xe^y+1=0 \Rightarrow (e^y-x)^2+1-x^2=0 \Rightarrow e^y=...$	M1
	$=x\pm\sqrt{x^2-1}$	A1
	So $y=\ln\left[x-\sqrt{x^2-1}\right]$ *	A1*
	since $y<0 \Rightarrow e^y<1$ so need $x-\sqrt{x^2-1}$ (as $x>1$ so must subtract)	B1
		(6)
		(6 marks)

Notes:
B1: Correct statement for x in terms of exponentials. $\cosh y = \dfrac{e^x + e^{-x}}{2}$ scores B0.   **M1:** Multiplies through by e^y to achieve a quadratic in e^y. (Terms need not be gathered.)   **M1:** Uses the quadratic formula or other valid method (e.g. completing the square) to solve for e^y.   **A1:** Correct solution(s) for e^y. Accept if only the negative one is given. Accept $\dfrac{2x \pm \sqrt{4x^2 - 4}}{2}$ **A1*:** Completely correct work leading to the given answer regardless of the justification why the negative root is taken (correct or incorrect). Must be no errors seen.   **B1:** Suitable justification for taking the negative root given.   E.g. $y < 0$ so $y = \ln\left[x - \sqrt{x^2 - 1}\right]$. Condone $x \pm \sqrt{x^2 - 1} < 1$ so $y = \ln\left[x - \sqrt{x^2 - 1}\right]$.   **Note that the B1 can only be awarded if all previous marks have been awarded.**   But the reason may be given before or after ln has been taken.   E.g. $\left(e^y - x\right)^2 + 1 - x^2 = 0 \Rightarrow e^y - x = \pm\sqrt{x^2 - 1}$ but $y < 0$ so $e^y - x = -\sqrt{x^2 - 1}$ **<u>Working backwards:</u>** $y = \ln\left[x - \sqrt{x^2 - 1}\right] \Rightarrow e^y = x - \sqrt{x^2 - 1}\,(\text{B1}) \Rightarrow e^y + e^{-y} = x - \sqrt{x^2 - 1} + \dfrac{1}{x - \sqrt{x^2 - 1}}(\text{M1})$ $x - \sqrt{x^2 - 1} + \dfrac{1}{x - \sqrt{x^2 - 1}} = \dfrac{2x\left(x - \sqrt{x^2 - 1}\right)}{x - \sqrt{x^2 - 1}}(\text{M1}) = 2x(\text{A1}) \Rightarrow x = \dfrac{e^y + e^{-y}}{2} = \cosh y\,(\text{A1})$ **Final B1 unlikely to be available.**

Q23.

Question	Scheme	Marks
(a)	$(5\mathbf{i} + \mathbf{j}) \times (8\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 0 \\ 8 & -2 & 3 \end{vmatrix} = \ldots$ Or $\left.\begin{array}{r} (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}).(5\mathbf{i} + \mathbf{j}) = 0 \\ (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}).(8\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 0 \end{array}\right\} \Rightarrow \begin{array}{c} 5u + v = 0 \\ 8u - 2v + 3w = 0 \end{array} \Rightarrow u, v, w = \ldots$	M1
	$\mathbf{n} = 3\mathbf{i} - 15\mathbf{j} - 18\mathbf{k}$ or $\alpha\left(\mathbf{i} - 5\mathbf{j} - 6\mathbf{k}\right)$ for any $\alpha \neq 0$	A1
		(2)
(b)	(i) $\mathbf{r} = \left(2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}\right) + s\left(8\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}\right) + t\left(5\mathbf{i} + \mathbf{j}\right)$	B1
		(1)
	(ii) $\left(2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}\right).\left(3\mathbf{i} - 15\mathbf{j} - 18\mathbf{k}\right) = \ldots (= -6)$	M1
	So $\mathbf{r}.\left(3\mathbf{i} - 15\mathbf{j} - 18\mathbf{k}\right) = -6$ oe such as $\mathbf{r}.\left(-\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}\right) = 2$	A1
		(2)

(c) **Way 1**	Distance from plane in (b) to origin is $\frac{\pm 6}{\sqrt{3^2+15^2+18^2}}$ oe e.g. $\frac{2}{\sqrt{1^2+5^2+6^2}}$ Or attempts similar for parallel plane containing l_1, e.g. $\frac{(\mathbf{i}+2\mathbf{j}-5\mathbf{k}).(3\mathbf{i}-15\mathbf{j}-18\mathbf{k})}{\sqrt{3^2+15^2+18^2}}=\ldots$	**M1**
	$=\pm\frac{2}{\sqrt{62}}$ (oe evaluated) or $\mp\frac{21}{\sqrt{62}}$ if considering other plane.	**A1**
	Both $\frac{\pm 6}{\sqrt{3^2+15^2+18^2}}$ oe and $\frac{(\mathbf{i}+2\mathbf{j}-5\mathbf{k}).(3\mathbf{i}-15\mathbf{j}-18\mathbf{k})}{\sqrt{3^2+15^2+18^2}}=\ldots$ attempted	**M1**
	Hence shortest distance between lines is $\frac{2}{\sqrt{62}}+\frac{21}{\sqrt{62}}=\ldots$	**M1**
	$=\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$	**A1**
		(5)
Way 2	$\overrightarrow{AB}=\pm((\mathbf{i}+2\mathbf{j}-5\mathbf{k})-(2\mathbf{i}-4\mathbf{j}+4\mathbf{k}))=\pm(-\mathbf{i}+6\mathbf{j}-9\mathbf{k})$	**M1 A1**
	$d=AB\cos\theta=\frac{\overrightarrow{AB}.\mathbf{n}}{\lvert\mathbf{n}\rvert}=\frac{\pm(-\mathbf{i}+6\mathbf{j}-9\mathbf{k}).(3\mathbf{i}-15\mathbf{j}-18\mathbf{k})}{\sqrt{3^2+15^2+18^2}}$ oe	**M1**
	$=\frac{\pm(-3-90+162)}{\sqrt{558}}=\frac{\pm 69}{\sqrt{558}}=\ldots$	**M1**
	$=\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$	**A1**
		(5)

Way 3	$(2\mathbf{i}-4\mathbf{j}+4\mathbf{k})+\mu(8\mathbf{i}-2\mathbf{j}+3\mathbf{k})-((\mathbf{i}+2\mathbf{j}-5\mathbf{k})+\lambda(5\mathbf{i}+\mathbf{j}))$ $=(1+8\mu-5\lambda)\mathbf{i}+(-6-2\mu-\lambda)\mathbf{j}+(9+3\mu)\mathbf{k}$	**M1 A1**
	$((1+8\mu-5\lambda)\mathbf{i}+(-6-2\mu-\lambda)\mathbf{j}+(9+3\mu)\mathbf{k}).(5\mathbf{i}+\mathbf{j})=0$ $\Rightarrow 38\mu-26\lambda=1$ $((1+8\mu-5\lambda)\mathbf{i}+(-6-2\mu-\lambda)\mathbf{j}+(9+3\mu)\mathbf{k}).(8\mathbf{i}-2\mathbf{j}+3\mathbf{k})=0$ $\Rightarrow 77\mu-38\lambda=-47$ $\Rightarrow \lambda=-\frac{207}{62},\ \mu=-\frac{70}{31}$	**M1**
	$(2\mathbf{i}-4\mathbf{j}+4\mathbf{k})+\mu(8\mathbf{i}-2\mathbf{j}+3\mathbf{k})-((\mathbf{i}+2\mathbf{j}-5\mathbf{k})+\lambda(5\mathbf{i}+\mathbf{j}))$ $=-\frac{23}{62}\mathbf{i}+\frac{115}{62}\mathbf{j}+\frac{69}{31}\mathbf{k}$ $d=\sqrt{\left(\frac{23}{62}\right)^2+\left(\frac{115}{62}\right)^2+\left(\frac{69}{31}\right)^2}$	**M1**
	$=\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$	**A1**
		(5)
		(10 marks)

Notes:
Accept equivalent vector notation, e.g. column vectors, throughout. **(a)** **M1:** Any correct method to find a vector perpendicular to the two direction vectors of the lines. Look for the cross product between the two direction vectors, but may use dot products and solving equations. In the latter case the method should lead to values for u, v and w. For the vector product, if no method is shown look for at least 2 correct components. **A1:** Any correct vector, a scalar multiple of $-\mathbf{i}+5\mathbf{j}+6\mathbf{k}$ **(b)** **B1:** Any correct equation. Must have **r** = ... or e.g. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ...$ **M1:** Uses their normal vector from (a) with any point on the plane (probably $(2\mathbf{i}-4\mathbf{j}+4\mathbf{k})$ to find p Condone slips with the calculation so $(2\mathbf{i}-4\mathbf{j}+4\mathbf{k}).(3\mathbf{i}-15\mathbf{j}-18\mathbf{k})$ evaluated as a scalar is sufficient for M1. May also be implied by $p=-6$ **A1:** Any correct equation of the correct form. **(c)** **Way 1** **M1:** Uses the plane equation from (b) (or otherwise) OR the parallel plane containing l_1 to find the distance of one of these planes to the origin. **A1:** Correct distance between one of the planes and the origin, accept ± here. **M1:** Attempts distance of both the parallel planes containing l_1 and l_2 from the origin.

M1: Correct method for finding the distance between lines – i.e. subtracts their distances either way round. **A1:** Correct answer. Accept $\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$ **Way 2** **M1:** Subtracts position vectors of points on the lines (either way around). Implied by two correct coordinates if method not shown. (Forms suitable hypotenuse.) **A1:** Correct vector or as coordinates, either direction. **M1:** Correct formula for the distance using their vectors, $d = AB\cos\theta = \frac{\overline{AB}.\mathbf{n}}{\lvert\mathbf{n}\rvert}$ with their $\overline{AB}$ and **n**. **M1:** Complete evaluation of the formula. **A1:** Correct answer. Accept $\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$ but must be positive. **Way 3** **M1:** Subtracts position vectors of general points on each line (either way around). Implied by two correct coordinates if method not shown. **A1:** Correct vector or as coordinates, either direction. **M1:** Forms scalar product of the general vector with both direction vectors, sets = 0 and solves simultaneously **M1:** Substitutes the values of their parameters back into the general vector and attempts its magnitude **A1:** Correct answer. Accept $\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$ but must be positive.