Write your name here

Surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number	Candidate Number

Further Mathematics

Advanced Subsidiary
Paper 1: Core Pure Mathematics

Monday 14 May 2018 – Afternoon
Time: 1 hour 40 minutes

Paper Reference
8FM0/01

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 80.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

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Pearson

Answer ALL questions. Write your answers in the spaces provided.

1.

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -2 & 1 \\ 3 & 5 & -2 \end{pmatrix}$$

(a) Find $\mathbf{M}^{-1}$ giving each element in exact form.

(2)

(b) Solve the simultaneous equations

$$\begin{aligned} 2x + y - 3z &= -4 \\ 4x - 2y + z &= 9 \\ 3x + 5y - 2z &= 5 \end{aligned}$$

(2)

(c) Interpret the answer to part (b) geometrically.

(1)

(a) $\underline{\underline{M}} = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -2 & 1 \\ 3 & 5 & -2 \end{pmatrix} \therefore \underline{\underline{M}}^{-1} = \frac{1}{69}\begin{pmatrix} 1 & 13 & 5 \\ -11 & -5 & 14 \\ -26 & 7 & 8 \end{pmatrix}$

(b) $2x + y - 3z = -4$

$4x - 2y + z = 9$

$3x + 5y - 2z = 5$

In matrix form: $\begin{pmatrix} 2 & 1 & -3 \\ 4 & -2 & 1 \\ 3 & 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -2 & 1 \\ 3 & 5 & -2 \end{pmatrix}^{-1}\begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

$\therefore x = 2 \quad y = 1 \quad z = 3$

(c) The point where three planes meet

2. The cubic equation

$$z^3 - 3z^2 + z + 5 = 0$$

has roots α, β and γ. $a=1,\ b=-3,\ c=1,\ d=5$

Without solving the equation, find the cubic equation whose roots are $(2\alpha + 1)$, $(2\beta + 1)$ and $(2\gamma + 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

$\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{-3}{1} = 3$

$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} = \frac{1}{1} = 1$

$\alpha\beta\gamma = -\frac{d}{a} = -\frac{5}{1} = -5$

SUM $= (2\alpha + 1) + (2\beta + 1) + (2\gamma + 1)$

$= 2(\alpha + \beta + \gamma) + 3$

$= 2(3) + 3$

$= 9$

PAIR SUM $= (2\alpha + 1)(2\beta + 1) + (2\alpha + 1)(2\gamma + 1) + (2\beta + 1)(2\gamma + 1)$

$= 4\alpha\beta + 2\alpha + 2\beta + 1 + 4\alpha\gamma + 2\alpha + 2\gamma + 1 + 4\beta\gamma + 2\beta + 2\gamma + 1$

$= 4(\alpha\beta + \beta\gamma + \alpha\gamma) + 4(\alpha + \beta + \gamma) + 3$

$= 4(1) + 4(3) + 3$

$= 19$

PRODUCT $= (2\alpha + 1)(2\beta + 1)(2\gamma + 1)$

$= (2\alpha + 1)(4\beta\gamma + 2\beta + 2\gamma + 1)$

$= 8\alpha\beta\gamma + 4\alpha\beta + 4\alpha\gamma + 2\alpha + 4\beta\gamma + 2\beta + 2\gamma + 1$

Question 2 continued

$= 8(\alpha\beta\gamma) + 4(\alpha\beta + \beta\gamma + \alpha\gamma) + 2(\alpha + \beta + \gamma) + 1$

$= 8(-5) + 4(1) + 2(3) + 1$

$= -29$

$\therefore w^3 - 9w^2 + 19w + 29 = 0$

(Total for Question 2 is 5 marks)

3. (a) Shade on an Argand diagram the set of points

$$\{z \in \mathbb{C} : |z - 1 - \mathrm{i}| \leqslant 3\} \cap \left\{z \in \mathbb{C} : \frac{\pi}{4} \leqslant \arg(z-2) \leqslant \frac{3\pi}{4}\right\}$$

(5)

The complex number w satisfies

$$|w - 1 - \mathrm{i}| = 3 \text{ and } \arg(w-2) = \frac{\pi}{4}$$

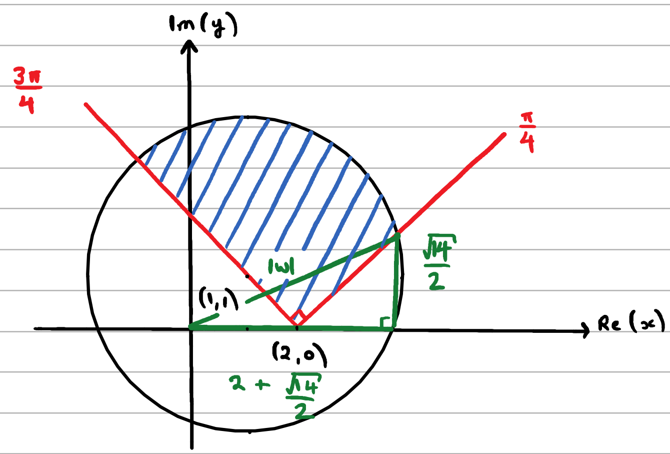
(b) Find, in simplest form, the exact value of $|w|^2$

(4)

(a) $\{z \in \mathbb{C} : |z - (1+i)| \leq 3\} \cap \{z \in \mathbb{C} : \frac{\pi}{4} \leq \arg(z-(2)) \leq \frac{3\pi}{4}\}$

↓ CIRCLE, $c(1,1), r=3$

↓ HALF LINE FROM $(2,0)$



(b) In equivalent cartesian form: $(x-1)^2 + (y-1)^2 = 9$ ① and $y = x - 2$ ②

Solve ① and ② simultaneously:

$(x-1)^2 + (x-2-1)^2 = 9$

$(x-1)^2 + (x-3)^2 = 9$

$x^2 - 2x + 1 + x^2 - 6x + 9 = 9$

Question 3 continued

$2x^2 - 8x + 1 = 0$

$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(2)(1)}}{2(2)} = \frac{4 \pm \sqrt{14}}{2}$

$\therefore x = 2 + \frac{\sqrt{14}}{2}$

$\therefore y = \left(2 + \frac{\sqrt{14}}{2}\right) - 2 = \frac{\sqrt{14}}{2}$

using pythagoras : $|w|^2 = \left(2 + \frac{\sqrt{14}}{2}\right)^2 + \left(\frac{\sqrt{14}}{2}\right)^2$

$\therefore |w|^2 = 11 + 2\sqrt{14}$

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4. Part of the mains water system for a housing estate consists of water pipes buried beneath the ground surface. The water pipes are modelled as straight line segments. One water pipe, W, is buried beneath a particular road. With respect to a fixed origin O, the road surface is modelled as a plane with equation $3x - 5y - 18z = 7$, and W passes through the points $A(-1, -1, -3)$ and $B(1, 2, -3)$. The units are in metres.

(a) Use the model to calculate the acute angle between W and the road surface.

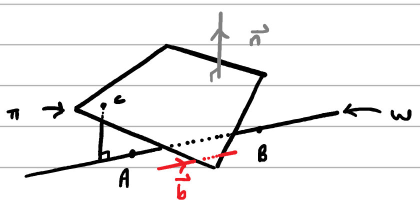
(5)

A point $C(-1, -2, 0)$ lies on the road. A section of water pipe needs to be connected to W from C.

(b) Using the model, find, to the nearest cm, the shortest length of pipe needed to connect C to W.

(6)

(a) $3x - 5y - 18z = 7$



$\vec{r} \cdot \begin{pmatrix} 3 \\ -5 \\ -18 \end{pmatrix} = 7$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

$\sin\theta = \left| \frac{\vec{b} \cdot \vec{n}}{|\vec{b}| \times |\vec{n}|} \right| = \left| \frac{(2\underline{i} + 3\underline{j} + 0\underline{k}) \cdot (3\underline{i} - 5\underline{j} - 18\underline{k})}{\sqrt{(2)^2 + (3)^2 + (0)^2} \times \sqrt{(3)^2 + (-5)^2 + (-18)^2}} \right|$

$= \left| \frac{(2)(3) + (3)(-5) + (0)(-18)}{\sqrt{13} \times \sqrt{358}} \right| = \left| \frac{-9}{\sqrt{13} \times \sqrt{358}} \right| = 0.131\ldots$

$\therefore \theta = 0.13$ radians $= 7.58$ degrees

(b) line W has equation

$\vec{r} = \begin{pmatrix} -1 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

$\overrightarrow{CW}$ is perpendicular to $\overrightarrow{AB}$

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Question 4 continued

$$\overrightarrow{CW} = \overrightarrow{OW} - \overrightarrow{OC} = \left\{ \begin{pmatrix} -1 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\} - \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

$$\therefore \overrightarrow{CW} = \begin{pmatrix} 2\lambda \\ 3\lambda + 1 \\ -3 \end{pmatrix}$$

hence $\begin{pmatrix} 2\lambda \\ 3\lambda + 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = 0$

$(2)(2\lambda) + (3)(3\lambda + 1) + (0)(-3) = 0$

$4\lambda + 9\lambda + 3 = 0$

$13\lambda = -3 \quad \therefore \lambda = -\frac{3}{13}$

<u>So:</u> $\overrightarrow{CW} = \begin{pmatrix} 2 \times -3/13 \\ 3 \times -3/13 + 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -6/13 \\ 4/13 \\ -3 \end{pmatrix}$

$|\overrightarrow{CW}| = \sqrt{(-6/13)^2 + (4/13)^2 + (-3)^2} = 3.05 \text{ m}$

$\therefore$ shortest distance is 305cm

5.

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix **A**.

(3)

The transformation V, represented by the 2×2 matrix **B**, is a reflection in the line $y = -x$

(b) Write down the matrix **B**.

(1)

Given that U followed by V is the transformation T, which is represented by the matrix **C**,

(c) find the matrix **C**.

(2)

(d) Show that there is a real number k for which the point $(1, k)$ is invariant under T.

(4)

(a) $\cos\theta = -\frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$

$\frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}/2}{-1/2}$

$\tan\theta = -\sqrt{3}$

$\theta = -\frac{\pi}{3} + \pi$

$\therefore \theta = \frac{2\pi}{3}^{c}$

$\therefore$ rotation by $\frac{2\pi}{3}$ radians anticlockwise about O

(b) Reflection in $y = -x$: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(c) Reflection, followed by rotation :

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} (0)(-\frac{1}{2}) + (-1)(\frac{\sqrt{3}}{2}) & (0)(-\frac{\sqrt{3}}{2}) + (-1)(-\frac{1}{2}) \\ (-1)(-\frac{1}{2}) + (0)(\frac{\sqrt{3}}{2}) & (-1)(-\frac{\sqrt{3}}{2}) - (0)(-\frac{1}{2}) \end{pmatrix}$$

Question 5 continued

$\therefore \underline{\underline{C}} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

d) Points on the line $y=x$ are invariant points, and the lines $y=-x$ and $y=x+k$ for any value of k are invariant lines. The invariant points do not change under the transformation, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$$

$$\begin{pmatrix} (-\frac{\sqrt{3}}{2})(1) + (\frac{1}{2})(k) \\ (\frac{1}{2})(1) + (\frac{\sqrt{3}}{2})(k) \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} + \frac{1}{2}k \\ \frac{1}{2} + \frac{\sqrt{3}}{2}k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$$

<u>so:</u> $-\frac{\sqrt{3}}{2} + \frac{1}{2}k = 1$ ①

$\frac{1}{2} + \frac{\sqrt{3}}{2}k = k$ ②

solving ①: $\frac{1}{2}k = \frac{2+\sqrt{3}}{2}$

$\therefore k = 2 + \sqrt{3}$

solving ②: $-\frac{2+\sqrt{3}}{2}k = -\frac{1}{2}$

$\therefore k = 2 + \sqrt{3}$

$\therefore k = 2 + \sqrt{3}$, which is real

6. (a) Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that

$$\sum_{r=1}^{n}(3r-2)^2 = \frac{1}{2}n\left[6n^2 - 3n - 1\right]$$

for all positive integers n.

(5)

(b) Hence find any values of n for which

$$\sum_{r=5}^{n}(3r-2)^2 + 103\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 3n^3$$

(5)

(a) $(3r-2)^2 = 9r^2 - 12r + 4$

$\therefore \sum_{r=1}^{n}(9r^2 - 12r + 4) = 9\sum_{r=1}^{n} r^2 - 12\sum_{r=1}^{n} r + 4\sum_{r=1}^{n} 1$

$= 9 \times \frac{1}{6}n(n+1)(2n+1) - 12 \times \frac{1}{2}n(n+1) + 4n$

$= \frac{3}{2}n(n+1)(2n+1) - 6n(n+1) + 4n$

$= \frac{1}{2}n\left[3(n+1)(2n+1) - 12(n+1) + 8\right]$

$= \frac{1}{2}n\left[3(2n^2 + 3n + 1) - 12n - 12 + 8\right]$

$= \frac{1}{2}n\left[6n^2 + 9n + 3 - 12n - 12 + 8\right]$

$= \frac{1}{2}n(6n^2 - 3n - 1)$

(b) $\sum_{r=5}^{n}(3n-2)^2 + 103\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 3n^3$

$\sum_{r=1}^{n}(3n-2)^2 - \sum_{r=1}^{4}(3n-2)^2 + 103\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 3n^3$

$\frac{1}{2}n(6n^2 - 3n - 1) - \frac{1}{2}\times 4 \times (6\times 4^2 - 3\times 4 - 1) + 103\left[0 - 2 + 0 + 4 + 0 - 6 + 0 + 8\right.$
$\left. 0 - 10 + 0 + 12 + \cdots 0 - 26 + 0 + 28\right] = 3n^3$

$\frac{1}{2}n(6n^2 - 3n - 1) - 166 + 103\left[2 + 2 + \cdots + 2\right] = 3n^3$

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Question 6 continued

$\frac{1}{2}n(6n^2 - 3n - 1) - 166 + 103(2 \times 7) = 3n^3$

$\cancel{3n^3} - \frac{3}{2}n^2 - \frac{1}{2}n - 166 + 1442 = \cancel{3n^3}$

$-\frac{3}{2}n^2 + \frac{1}{2}n - 1276 = 0$

$3n^2 + n - 2552 = 0$

$n = \frac{-(1) \pm \sqrt{(1)^2 - 4(3)(-2552)}}{2(3)}$

$n = -\frac{88}{3}, 29$

$\therefore n = 29$

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7.

$$f(z) = z^3 + z^2 + pz + q$$

where p and q are real constants.

The equation $f(z) = 0$ has roots z_1, z_2 and z_3
When plotted on an Argand diagram, the points representing z_1, z_2 and z_3 form the vertices of a triangle of area 35

$z = 3$ is a root of $f(z) = 0$

Given that $z_1 = 3$, find the values of p and q.

(7)

Through factor theorem: $f(3)=0$: $(3)^3 + (3)^2 + p(3) + q = 0$ $\therefore 3p + q = -36$

$\therefore (z-3)$ is a linear factor of $f(z)$

$\therefore$ We can factorise $f(z)$ as the product of a linear factor and a quadratic factor

$(z-3)(z^2 + Az + B) \equiv z^3 + z^2 + pz + q$

compare coefficients:

z^2: $A - 3 = 1$ $\therefore A = 4$

z: $B - 3A = p$

$B - 12 = p$ $\therefore B = p + 12$

No°: $-3B = q$

$\therefore f(z) = (z-3)(z^2 + 4z + p + 12)$

If we solve $f(z)=0$, the quadratic factor has roots which form a complex conjugate, i.e.

$\therefore z_{2,3} = x \pm iy$

so: $z^2 + 4z + p + 12 = 0$ has roots $z = x \pm iy$

$a = 1$, $b = 4$, $c = p + 12$

Sum $= \alpha + \beta = -\frac{b}{a} = -\frac{4}{1} = -4$

Question 7 continued

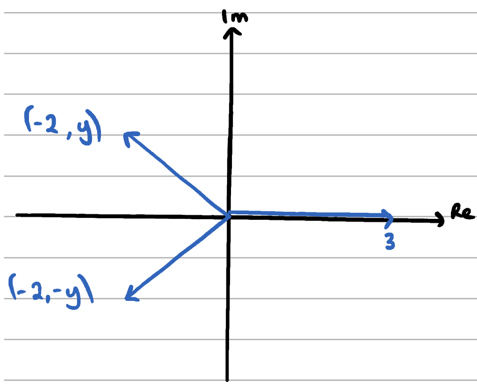
$(x+iy) + (x-iy) = -4$

$2x = -4 \quad \therefore x = -2$

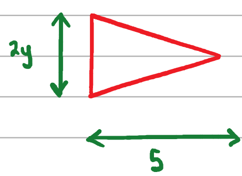
$\therefore z_{2,3} = -2 \pm iy$

PRODUCT : $\alpha\beta = \frac{c}{a} = \frac{p+12}{1} = p + 12$

Now draw roots on argand diagram



We know area of triangle is 35



$\frac{1}{2} \times 2y \times 5 = 35 \quad \therefore y = 7$

$\therefore z_{2,3} = -2 \pm 7i$

hence : $(-2+7i)(-2-7i) = p + 12$

$53 = p + 12 \quad \therefore p = 41$

but $-3B = q$ and $B = p + 12$

Question 7 continued

so: $-3(p+12) = q$

$-3(41+12) = q \quad \therefore q = -159$

$\therefore f(z) = z^3 + z^2 + 41z - 159$

8. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

we want to show :

$$\begin{pmatrix} 4(k+1)+1 & -8(k+1) \\ 2(k+1) & 1-4(k+1) \end{pmatrix}$$

(6)

(ii) Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 4^{n+1} + 5^{2n-1}$$

is divisible by 21

(6)

(i) STEP 1 Basis step

$n=1$

$LHS = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^1 = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$

$RHS = \begin{pmatrix} 4\times1+1 & -8\times1 \\ 2\times1 & 1-4\times1 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$

$\therefore$ statement is true for $n=1$

STEP 2 Assumption step

Assume result is true for $n=k$

$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}$

STEP 3 Inductive step

Try for $n=k+1$

$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1}$

$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^1$

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Question 8 continued

$$\begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} (4k+1)(5)+(-8k)(2) & (4k+1)(-8)+(-8k)(-3) \\ (2k)(5)+(1-4k)(2) & (2k)(-8)+(1-4k)(-3) \end{pmatrix}$$

$$\begin{pmatrix} 20k+5-16k & -32k-8+24k \\ 10k+2-8k & -8k-3+12k \end{pmatrix}$$

$$\begin{pmatrix} 4k+5 & -8k-8 \\ 2k+2 & -4k+3 \end{pmatrix}$$

$$\begin{pmatrix} 4(k+1)+1 & -8(k+1) \\ 2(k+1) & 1-4(k+1) \end{pmatrix}$$

STEP 4 Conclusion step

If true for $n=k$, then true for $n=k+1$, true for $n=1$, so true for all $n \in \mathbb{Z}^+$

(ii) $f(n) = 4^{n+1} + 5^{2n+1}$

STEP 1 Basis step

$f(1) = 4^{(1)+1} + 5^{2(1)-1} = 4^2 + 5^1 = 21$ (divisible by 21)

$\therefore$ statement is true when $n=1$

STEP 2 Assumption step

Assume true when $n=k$, i.e.

$f(k) = 4^{k+1} + 5^{2k-1}$ is divisible by 21

STEP 3 Inductive step

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Question 8 continued

$f(k+1) = 4^{(k+1)+1} + 5^{2(k+1)-1}$

$= 4^{k+1} \cdot 4^1 + 5^{2k+2-1}$

$= 4(4^{k+1}) + (5^2)5^{2k-1}$

$= 4(4^{k+1}) + 25(5^{2k-1})$

so $f(k+1) - f(k) = [4(4^{k+1}) + 25(5^{2k-1})] - [4^{k+1} + 5^{2k-1}]$

$= 4(4^{k+1}) + 25(5^{2k-1}) - 4^{k+1} - 5^{2k-1}$

$= 3(4^{k+1}) + 24(5^{2k-1})$

$= 3(4^{k+1}) + 3(5^{2k-1}) + 21(5^{2k-1})$

$= 3[4^{k+1} + 5^{2k-1}] + 21(5^{2k-1})$

$= 3f(k) + 21(5^{2k-1})$

i.e. $f(k+1) = f(k) + 3f(k) + 21(5^{2k-1})$

$\therefore f(k+1) = 4f(k) + 21(5^{2k-1})$

STEP 4 Conclusion Step

If true for $n=k$, then true for $n=k+1$, true for $n=1$ so true for all $n \in \mathbb{Z}^+$

9.

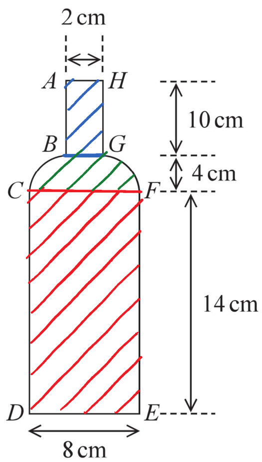

Figure 1

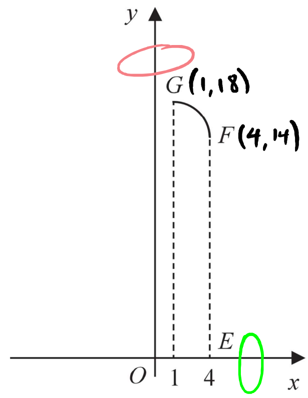

Figure 2

A mathematics student is modelling the profile of a glass bottle of water. Figure 1 shows a sketch of a central vertical cross-section *ABCDEFGHA* of the bottle with the measurements taken by the student.

The horizontal cross-section between *CF* and *DE* is a circle of diameter 8 cm and the horizontal cross-section between *BG* and *AH* is a circle of diameter 2 cm. } cylinder

The student thinks that the curve *GF* could be modelled as a curve with equation

$$y = ax^2 + b \qquad 1 \leqslant x \leqslant 4$$

where *a* and *b* are constants and *O* is the fixed origin, as shown in Figure 2.

(a) Find the value of *a* and the value of *b* according to the model. **(2)**

(b) Use the model to find the volume of water that the bottle can contain. **(7)**

(c) State a limitation of the model. **(1)**

The label on the bottle states that the bottle holds approximately 750 cm^3 of water.

(d) Use this information and your answer to part (b) to evaluate the model, explaining your reasoning. **(1)**

(a) (1,18) : $18 = a(1)^2 + b$ (4,14) : $14 = a(4)^2 + b$

$18 = a + b$ ① $14 = 16a + b$ ②

Solving ① and ② simultaneously

Question 9 continued

$15a = -4 \quad \therefore a = -4/15$

$a = -4/15 : \; -4/15 + b = 18 \quad \therefore b = 274/15$

$\therefore y = -\frac{4}{15}x^2 + \frac{274}{15}$ ③ $\quad \therefore x^2 = \frac{1}{4}(274 - 15y)$

(b) volume of cylinders $= \pi(1)^2(10) + \pi(4)^2(14) = 10\pi + 224\pi = 234\pi$ cm^3

volume of curved bit $= \pi\int_\alpha^\beta x^2\,dy = \pi\int_{14}^{18}\frac{1}{4}(274 - 15y)\,dy = \frac{\pi}{4}\int_{14}^{18}(274 - 15y)\,dy$

$= \frac{\pi}{4}\left[274y - \frac{15y^2}{2}\right]_{14}^{18} = \frac{\pi}{4}\left[\left\{274(18) - \frac{15(18)^2}{2}\right\} - \left\{274(14) - \frac{15(14)^2}{2}\right\}\right]$

$= \frac{\pi}{4}(136) = 34\pi$ cm^3

$\therefore$ total volume $= 234\pi + 34\pi = 268\pi$ cm^3

(c) The measurements may not be accurate

(d) Actual volume of bottle $= 268\pi$ cm$^3 \simeq 841.9\ldots$cm$^3 > 750$cm^3

This value of 750 cm^3 makes sense as 841 cm^3 represents the volume of space in the bottle; it would make sense that the volume of water in the bottle is less so there is space for air in the bottle.

Therefore, the model makes sense and is suitable.