Mark Scheme

Q1.

Question	Scheme	Marks
(a)	$u_n = ar^{n-1} \Rightarrow ar + ar^2 = 6$ and $ar^3 = 8$	M1
	$\Rightarrow \dfrac{ar+ar^2}{ar^3} = \dfrac{6}{8} \Rightarrow 1 + r = \dfrac{3}{4}r^2$	M1
	$\Rightarrow 3r^2 - 4r - 4 = 0$*	A1*
		(3)
(a) Way 2	$u_4 = 8 \Rightarrow u_3 = \dfrac{8}{r},\ u_2 = \dfrac{8}{r^2}$	M1
	$\dfrac{8}{r} + \dfrac{8}{r^2} = 6$	M1
	$\Rightarrow 3r^2 - 4r - 4 = 0$*	A1*
		(3)
(b)	$r = -\dfrac{2}{3}$	B1
	$ar^3 = 8 \Rightarrow a = \dfrac{8}{\left(-\frac{2}{3}\right)^3} = ...$	M1
	$u_1 = -27$	A1ft
		(3)
(c)	$S_\infty = \dfrac{-27}{1-\left(-\frac{2}{3}\right)} = ...$	M1
	$= -\dfrac{81}{5}$	A1
		(2)
		(8 marks)

Notes:

(a) **Ignore labelling and mark (a), (b) and (c) together**

M1: Uses the correct n-th term formula for a GP to set up two equations in a and r

May also be in terms of u_1 and r or u_2 and r or u_3 and r e.g.

$u_1r + u_1r^2 = 6,\ u_1r^3 = 8$ or $u_2 + u_2r = 6,\ u_2r^2 = 8$ or $\dfrac{u_3}{r} + u_3 = 6,\ ru_3 = 8$

Must be using the correct term formula so e.g. $ar^2 + ar^3 = 6$ and $ar^4 = 8$ is M0

Alternatively may use the correct sum formula for the first equation, $a\dfrac{1-r^3}{1-r} - a = 6$ oe.

M1: Attempts to solve their two equations to get an equation in r. Look for an attempt to divide the two equations, or an attempt to find a in terms of r from one and substitute into the other.

Allow slips but the algebra should essentially be correct so do not allow use of e.g. $ar^3 = (ar)^3$

Alternatively, attempts to eliminate r from the equation, $r = \frac{2}{\sqrt[3]{a}} \Rightarrow \frac{2a}{\sqrt[3]{a}} + \frac{4a}{\sqrt[3]{a^2}} = 6$

$\Rightarrow 2a^{\frac{2}{3}} + 4a^{\frac{1}{3}} = 6$. Award when they reach a quadratic in $a^{\frac{1}{3}}$ in this case.

A1*: CSO Note that as we are marking (a), (b) and (c) together, allow the printed answer to appear anywhere as long as it follows correct work but the "=0" must be seen not implied.

Way 2:

M1: Uses $u_4 = 8$ to write u_2 and u_3 in terms of r

M1: Uses $u_2 + u_3 = 6$ to get an equation in r

A1*: CSO

(b)

B1: Correct value of r seen or used in their working even if subsequently rejected and ignore any other value offered e.g. $r = 2$

M1: Uses a value for r from solving the equation given in (a) where $|r|<1$ in one of their equations from (a) to find a value for u_1 or a. Allow slips but the algebra should essentially be correct

A1ft: Correct value and no other values. Follow through on $\frac{8}{(\text{their } r)^3}$ for a value of r with $|r|<1$

(c)

M1: Uses the correct sum formula with their a (u_1) obtained from $|r|<1$ and r where $|r|<1$, to find the sum to infinity.

A1: Correct answer and no other values. Allow equivalents e.g. -16.2

Q2.

Question	Scheme		Marks
(a)	$S_\infty = \frac{20}{1-\frac{7}{8}}$; $= 160$	Use of a correct S_∞ formula	M1
		160	A1
			(2)
(b)	$S_{12} = \frac{20\left(1-\left(\frac{7}{8}\right)^{12}\right)}{1-\frac{7}{8}}$; $= 127.77324...$	M1: Use of a correct S_n formula with $n = 12$ (condone missing brackets around $\frac{7}{8}$)	M1 A1
	$= 127.8$ (1 dp)	A1: awrt 127.8	
			(2)

(c)	$160 - \dfrac{20(1-(\frac{7}{8})^N)}{1-\frac{7}{8}} < 0.5$	Applies S_N (**GP only**) with $a = 20$, $r = \frac{7}{8}$ and "uses" 0.5 and their S_∞ at any point in their working.	M1
	$160\left(\dfrac{7}{8}\right)^N < (0.5)$ or $\left(\dfrac{7}{8}\right)^N < \left(\dfrac{0.5}{160}\right)$	Attempt to isolate $+160(\frac{7}{8})^N$ or $(\frac{7}{8})^N$	dM1
	$N\log\left(\dfrac{7}{8}\right) < \log\left(\dfrac{0.5}{160}\right)$	Uses the law of logarithms to obtain an equation or an inequality of the form $N\log\left(\dfrac{7}{8}\right) < \log\left(\dfrac{0.5}{\text{their } S_\infty}\right)$ or $N > \log_{0.875}\left(\dfrac{0.5}{\text{their } S_\infty}\right)$	M1
	$N > \dfrac{\log\left(\frac{0.5}{160}\right)}{\log\left(\frac{7}{8}\right)} = 43.19823...$ cso $\Rightarrow N = 44$	$N = 44$ (Allow $N \geq 44$ but no $N > 44$	A1 cso
	An incorrect **inequality** statement at any stage in a candidate's working loses the final mark. Some candidates do not realise that the direction of the inequality is reversed in the final line of their solution. **BUT** it is possible to gain full marks for using =, as long as no incorrect working seen.		
			(4)
	Alternative: Trial & Improvement Method in (c):		
	Attempts $160 - S_N$ or S_N with at least one value for $N > 40$		M1
	Attempts $160 - S_N$ or S_N with $N = 43$ or $N = 44$		dM1
	For evidence of examining $160 - S_N$ or S_N for **both** $N = 43$ **and** $N = 44$ with **both** values correct to 2 DP Eg: $160 - S_{43} =$ awrt 0.51 and $160 - S_{44} =$ awrt 0.45 or $S_{43} =$ awrt 159.49 and $S_{44} =$ awrt 159.55		M1
	$N = 44$		A1 cso
	Answer of $N = 44$ only with no working scores no marks		
			(4)
			(8 marks)

Q3.

Question Number	Scheme	Marks
(i)	States or implies that $4\times 6^{n-1}>10^{100}$ Takes logs correctly to produce an equation without powers $4\times 6^{n-1}>10^{100}\Rightarrow \log 4+(n-1)\log 6>100\log 10$ $n=129$	B1 M1 A1 (3)
(ii) (a)	States $ar=-6$ and $\dfrac{a}{1-r}=25$ Combines to form an equation in $r\Rightarrow \dfrac{-6}{r(1-r)}=25$ $\Rightarrow -6=25r(1-r)\Rightarrow 25r^2-25r-6=0$ *	B1 M1 A1* (3)
(b)	$r=\dfrac{6}{5},-\dfrac{1}{5}$	B1 (1)
(c)	$r=-\dfrac{1}{5}$ as $\lvert r\rvert<1$ (for S_∞ to exist)	B1 (1)
(d)	Attempts $S_4=\dfrac{a(1-r^n)}{1-r}$ with $n=4, r=\text{their}(c)$ and $a=\dfrac{-6}{\text{their}(c)}$ $S_4=\dfrac{30\left(1-\left(-\frac{1}{5}\right)^4\right)}{1-\left(-\frac{1}{5}\right)}=24.96$ o.e.	M1 A1 (2)
		(10 marks)

(i)

B1: States or implies the solution can be found by solving $4\times 6^{n-1}\ldots 10^{100}$ where ... can be = or any inequality

M1: Shows a correct method of solving an equation of the form $4\times 6^{N}\ldots 10^{100}$ by correctly taking logs to produce an equation without powers. The log and index work must be correct, but allow slips in rearranging terms.

A1: $n=129$

(ii)(a)

B1: States or implies the correct two equations in a and r. $ar=-6$ and $\dfrac{a}{1-r}=25$

M1: Combines $ar=-6$ and $\dfrac{a}{1-r}=25$ to form a single equation in r

A1*: Proceeds to $25r^2-25r-6=0$ showing at least one correct simplified intermediate line and no errors (ii)(b)

B1: $r = \frac{6}{5}, -\frac{1}{5}$ or exact equivalent – award when seen even if not in part (b).

(ii)(c)

B1: $r = -\frac{1}{5}$ as $|r| < 1$ (for S_∞ to exist). Requires a minimal reason, but accept reasons that reject $\frac{6}{5}$ since it would mean all the terms are negative so cannot give a positive sum. Do not accept just "as the GS is convergent".

(ii)(d)

M1: Attempts $S_4 = \frac{a\left(1-r^n\right)}{1-r}$ with $n = 4, r = \text{their}(c)$ and $a = \frac{-6}{\text{their}(c)}$. If there was no attempt to answer (c) accept with either value from (b) for r.

If the correct formula is quoted then you can allow slips in substitution, but if the correct formula is not quoted then the equation should be correct for their r and a.

Alternatively, they may find and add the first 4 terms.

A1: 24.96 or equivalent such as $\frac{624}{25}$

Q4.

Question Number	Scheme	Marks
(a)	$\frac{dy}{dx} - 3y\tan x = e^{4x}\sec^3 x$	
	$e^{-3\int \tan x\,dx} = e^{-3\ln\sec x} = \sec^{-3} x$ or $\cos^3 x$	M1A1
	$\cos^3 x\frac{dy}{dx} - 3y\sin x\cos^2 x = e^{4x}\cos^3 x\sec^3 x$	
	$\frac{d}{dx}\left(y\cos^3 x\right) = e^{4x} \Rightarrow y\cos^3 x = \int e^{4x}\,dx$	M1
	$y\cos^3 x = \frac{1}{4}e^{4x}\quad(+c)$	M1
	$y = \left(\frac{1}{4}e^{4x} + c\right)\sec^3 x$ or $y = \left(\frac{1}{4}e^{4x} + c\right)\cos^{-3} x$ oe	A1
		(5)
(b)	$y = 4,\ x = 0\quad 4 = \left(\frac{1}{4} + c\right)$	
	$c = \frac{15}{4}$	M1
	$y = \frac{1}{4}\left(e^{4x} + 15\right)\sec^3 x$ or $\frac{1}{4}\left(e^{4x} + 15\right)\cos^{-3} x$ oe	A1
		(2)
		[7]

	Notes
(a)	
M1	Attempt the integrating factor, including integration of (-3)tan x; $\ln\cos$ or $\ln\sec$ seen
A1	Correct simplified integrating factor $\sec^{-3} x$ or $\cos^3 x$
M1	Multiply the equation by the integrating factor and integrate the LHS. Look for $y \times \text{their IF} = \int\left(e^{4x}\sec^3 x \times \text{their IF}\right)dx$ (condone missing dx)
M1	Integrate RHS, constant not needed. Must be a function they can integrate and a valid attempt (e.g. allowing coefficient slips).
A1	Correct result in the demanded form, including $y = ..$, constant included
(b)	
M1	Use the given initial conditions to obtain a value for c
A1	Fully correct final answer. Must include $y = ..$ but allow A1 if missing and penalised in (a). May be in the form $y\cos^3 x = ...$ or $4y\cos^3 x = ...$

Q5.

Question	Scheme		Marks
(a)	$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}-3y=2\sin x$$		
	AE: $m^2-2m-3=0$		
	$m^2-2m-3=0\Rightarrow m=...(-1,3)$	Forms Auxiliary Equation and attempts to solve (usual rules)	M1
	$(y=)Ae^{3x}+Be^{-x}$	Cao	A1
	PI: $(y=)p\sin x+q\cos x$	Correct form for PI	B1
	$(y'=)p\cos x-q\sin x$ $(y''=)-p\sin x-q\cos x$		
	$-p\sin x-q\cos x-2(p\cos x-q\sin x)-3p\sin x-3q\cos x=2\sin x$ Differentiates twice and substitutes		M1
	$2q-4p=2,\ 4q+2p=0$	Correct equations	A1
	$p=-\frac{2}{5},\ q=\frac{1}{5}$	A1A1 both correct A1A0 one correct	A1 A1
	$y=\frac{1}{5}\cos x-\frac{2}{5}\sin x$		
	$y=Ae^{3x}+Be^{-x}+\frac{1}{5}\cos x-\frac{2}{5}\sin x$	Follow through their p and q and their CF	B1ft
			(8)

(b)	$y'=3Ae^{3x}-Be^{-x}-\frac{1}{5}\sin x-\frac{2}{5}\cos x$	Differentiates their GS	M1
	$0=A+B+\frac{1}{5},\ 1=3A-B-\frac{2}{5}$	M1: Uses the given conditions to give two equations in A and B A1: Correct equations	M1 A1
	$A=\frac{3}{10},\ B=-\frac{1}{2}$	Solves for A and B Both correct	A1
	$y=\frac{3}{10}e^{3x}-\frac{1}{2}e^{-x}+\frac{1}{5}\cos x-\frac{2}{5}\sin x$	Sub their values of A and B in their GS	A1ft
			(5)
			(13 marks)

Q6.

Question Number	Scheme	Marks
(a)	$m^2+2m+5=0 \Rightarrow m=-1\pm 2i$	M1
	C F: $y=\mathrm{e}^{-x}\left(A\cos 2x+B\sin 2x\right)$ OR $y=\mathrm{e}^{-x}\left(P\mathrm{e}^{\mathrm{i}2x}+Q\mathrm{e}^{-\mathrm{i}2x}\right)$ or $y=P\mathrm{e}^{(-1+2\mathrm{i})x}+Q\mathrm{e}^{(-1-2\mathrm{i})x}$	A1
	PI: $y=a\cos x+b\sin x$	B1
	$y'=-a\sin x+b\cos x \quad y''=-a\cos x-b\sin x$	
	$-a\cos x-b\sin x-2a\sin x+2b\cos x+5a\cos x+5b\sin x=6\cos x$	M1
	$-b-2a+5b=0 \quad -a+2b+5a=6$	M1
	$a=\frac{6}{5} \quad b=\frac{3}{5}$	A1
	GS: $y=$ their CF $+\frac{6}{5}\cos x+\frac{3}{5}\sin x$	A1ft (7)

(b)	$x=0, y=0 \;\; 0=A+\frac{6}{5} \Rightarrow A=-\frac{6}{5}$	M1
	$y'=-\mathrm{e}^{-x}\left(A\cos 2x+B\sin 2x\right)+\mathrm{e}^{-x}\left(-2A\sin 2x+2B\cos 2x\right)$ $-\frac{6}{5}\sin x+\frac{3}{5}\cos x$	M1A1ft
	$x=0 \;\; \frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow 0=+\frac{6}{5}+2B+\frac{3}{5} \Rightarrow B=-\frac{9}{10}$	dM1
	PS: $y=\mathrm{e}^{-x}\left(-\frac{6}{5}\cos 2x-\frac{9}{10}\sin 2x\right)+\frac{6}{5}\cos x+\frac{3}{5}\sin x$	A1 (5)
		[12]
ALT	$y=\mathrm{e}^{-x}\left(P\mathrm{e}^{\mathrm{i}2x}+Q\mathrm{e}^{-\mathrm{i}2x}\right)+\frac{6}{5}\cos x+\frac{3}{5}\sin x$	
	$x=0 \;\; y=0 \;\; 0=P+Q+\frac{6}{5}$	M1
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{-x}\left(2\mathrm{i}P\mathrm{e}^{\mathrm{i}2x}-2\mathrm{i}Q\mathrm{e}^{-\mathrm{i}2x}\right)-\mathrm{e}^{-x}\left(P\mathrm{e}^{\mathrm{i}2x}+Q\mathrm{e}^{-\mathrm{i}2x}\right)-\frac{6}{5}\sin x+\frac{3}{5}\cos x$	M1A1ft
	$0=2\mathrm{i}P-2\mathrm{i}Q+\frac{9}{5}$	
	$P+Q=-\frac{6}{5} \quad P-Q=\frac{9}{10}\mathrm{i}$	
	$P=\frac{1}{2}\left(-\frac{6}{5}+\frac{9}{10}\mathrm{i}\right) \quad Q=\frac{1}{2}\left(-\frac{6}{5}-\frac{9}{10}\mathrm{i}\right)$	dM1
	PS: $y=\frac{1}{2}\mathrm{e}^{-x}\left(-\frac{6}{5}+\frac{9}{10}\mathrm{i}\right)\mathrm{e}^{2\mathrm{i}x}+\frac{1}{2}\mathrm{e}^{-x}\left(-\frac{6}{5}-\frac{9}{10}\mathrm{i}\right)\mathrm{e}^{-2\mathrm{i}x}+\frac{6}{5}\cos x+\frac{3}{5}\sin x$	A1 (5)

Question Number	Scheme	Marks
(a)		
M1	Form and solve the auxiliary equation	
A1	Correct CF, either form (Often not seen until GS stated)	
B1	Correct form for the PI	
M1	Differentiate twice and sub in the original equation	
M1	Obtain a pair of simultaneous equations and attempt to solve	
A1	Correct values for both unknowns	
A1ft	Form the GS. Must start $y = ...$ Follow through their CF (writing CF scores A0) Must have scored a minimum of 2 of the M marks	
(b)		
	For CF $y = e^{-x}(A\cos 2x + B\sin 2x)$	
M1	Sub $x = 0,\ y = 0$ in their GS and obtain a value for A	
M1	Differentiate their GS Product rule must be used	
A1ft	Correct differentiation of their GS provided this has 4 terms	
dM1	Sub $x = 0,\ \frac{dy}{dx} = 0$ and their A and obtain a value for B Depends on both previous M marks	
A1	Fully correct PS. Must start $y = ...$	
ALT(b)		
	For CF $y = e^{-x}\left(Pe^{i2x} + Qe^{-i2x}\right)$ or $y = Pe^{(-1+2i)x} + Qe^{(-1-2i)x}$	
M1	Sub $x = 0,\ y = 0$ in their GS and obtain an equation in P and Q	
M1	Differentiate their GS Product rule must be used if $y = e^{-x}\left(Pe^{i2x} + Qe^{-i2x}\right)$ used	
A1ft	Correct differentiation of their GS	
dM1	Sub $x = 0,\ \frac{dy}{dx} = 0$ to obtain a second equation and solve the pair of equations The solution must allow for P and Q to be complex	
A1	Fully correct PS. Must start $y = ...$	

Q7.

Question Number	Scheme	Marks
(a)	$m^2-6m+8=0$	
	$(m-2)(m-4)=0,\ m=2,4$	M1
	$(\text{CF}=)\quad Ae^{2x}+Be^{4x}$	A1
	$\text{PI}:\ y=\lambda x^2+\mu x+\nu$	B1
	$y'=2\lambda x+\mu \quad y''=2\lambda$	
	$2\lambda-6(2\lambda x+\mu)+8(\lambda x^2+\mu x+\nu)=2x^2+x$	M1
	$\lambda=\frac{1}{4},\ -12\lambda+8\mu=1,\ 2\lambda-6\mu+8\nu=0$	M1
	$\lambda=\frac{1}{4},\ \mu=\frac{1}{2},\ \nu=\frac{5}{16}$	A1A1
	$y=Ae^{2x}+Be^{4x}+\frac{1}{4}x^2+\frac{1}{2}x+\frac{5}{16}$	A1ft (8)
(a)M1	Form aux equation and attempt to solve (any valid method). Equation need not be shown if CF is correct or complete solution $(m=2,4)$ is shown	
A1	Correct CF $y=$.. not needed.	
B1	Correct form for PI	
M1	Their PI (minimum 2 terms) differentiated twice and substituted in the equation	
M1	Coefficients equated	
A1	Any 2 values correct	
A1	All 3 values correct	
A1ft	A complete solution, follow through their CF and PI. All 3 M marks must have been earned. Must start $y=$...	

(b)	$y=Ae^{2x}+Be^{4x}+\frac{1}{4}x^2+\frac{1}{2}x+\frac{5}{16}$	
	$1=A+B+\frac{5}{16}$	M1
	$\frac{dy}{dx}=2Ae^{2x}+4Be^{4x}+\frac{1}{2}x+\frac{1}{2} \qquad 0=2A+4B+\frac{1}{2}$	M1
	$A=\frac{13}{8}\quad B=-\frac{15}{16}$ oe	dM1A1
	$y=\frac{13}{8}e^{2x}-\frac{15}{16}e^{4x}+\frac{1}{4}x^2+\frac{1}{2}x+\frac{5}{16}$ oe	A1ft (5)
		[13]
(b) M1	Substitute $y=1$ and $x=0$ in their complete solution from (a)	
M1	Differentiate and substitute $\frac{dy}{dx}=0$, $x=0$	
dM1	Solve the 2 equations to $A=$... or $B=$ .... Depends on the two previous M marks	
A1	Both values correct	
A1ft	Particular solution, follow through their general solution and A and B. Must start $y=$ ...	

Q8.

Question	Scheme		Marks
(a)	$\theta=\frac{\pi}{3}\Rightarrow r=\sqrt{3}\sin\left(\frac{\pi}{3}\right)=\frac{3}{2}$	Attempt to verify coordinates in at least one of the polar equations	M1
	$\theta=\frac{\pi}{3}\Rightarrow r=1+\cos\left(\frac{\pi}{3}\right)=\frac{3}{2}$	Coordinates verified in both curves (Coordinate brackets not needed)	A1
			(2)
	Alternative		
	Equate rs: $\sqrt{3}\sin\theta=1+\cos\theta$ and verify (by substitution) that $\theta=\frac{\pi}{3}$ is a solution or solve by using $t=\tan\frac{\theta}{2}$ or writing $\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta=\frac{1}{2}$ $\quad \sin\left(\theta-\frac{\pi}{6}\right)=\frac{1}{2}$ $\quad \theta=\frac{\pi}{3}$ Squaring the original equation allowed as θ is known to be between 0 and π		M1
	Use $\theta=\frac{\pi}{3}$ in either equation to obtain $r=\frac{3}{2}$		A1
			(2)

(b)	$\frac{1}{2}\int(\sqrt{3}\sin\theta)^2\,d\theta,\quad \frac{1}{2}\int(1+\cos\theta)^2\,d\theta$	Correct formula used on at least one curve (1/2 may appear later) Integrals may be separate or added or subtracted.	M1
	$=\frac{1}{2}\int 3\sin^2\theta\,d\theta,\quad \frac{1}{2}\int(1+2\cos\theta+\cos^2\theta)\,d\theta$		
	$=\left(\frac{1}{2}\right)\int\frac{3}{2}(1-\cos 2\theta)\,d\theta,\quad \left(\frac{1}{2}\right)\int(1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta))\,d\theta$ Attempt to use $\sin^2\theta$ or $\cos^2\theta=\pm\frac{1}{2}\pm\frac{1}{2}\cos 2\theta$ on either integral Not dependent 1/2 may be missing		M1
	$=\frac{3}{4}\left[\theta-\frac{1}{2}\sin 2\theta\right]_{(0)}^{\left(\frac{\pi}{3}\right)},\quad \frac{1}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_{\left(\frac{\pi}{3}\right)}^{(\pi)}$ Correct integration (ignore limits) A1A1 or A1A0		A1, A1
	$R=\frac{3}{4}\left[\frac{\pi}{3}-\frac{\sqrt{3}}{4}(-0)\right]+\frac{1}{2}\left[\frac{3\pi}{2}-\left(\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right)\right]$	Correct use of limits for both integrals Integrals must be added. Dep on both previous M marks	ddM1
	$=\frac{3}{4}\left(\pi-\sqrt{3}\right)$	Cao No equivalents allowed	A1
			(6)
			(8 marks)

Q9.

Question Number	Scheme	Marks
(a)	$y = r\sin\theta = \sin\theta + \sin\theta\cos\theta$ OR $r\sin\theta = \sin\theta + \frac{1}{2}\sin 2\theta$	B1
	$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \cos\theta - \sin^2\theta + \cos^2\theta$ OR $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \cos\theta + \cos 2\theta$ $0 = \cos\theta + 2\cos^2\theta - 1 \quad = (2\cos\theta - 1)(\cos\theta + 1)$	M1
	$\cos\theta = \frac{1}{2}$ $(\cos\theta = -1$ outside range for $\theta)$ $\theta = \frac{\pi}{3}$	M1
	A is $\left(1\frac{1}{2}, \frac{\pi}{3}\right)$	A1 (4)
(b)	Area $= \frac{1}{2}\int_0^{\frac{\pi}{3}} (1+\cos\theta)^2 \mathrm{d}\theta$	B1
	$= \frac{1}{2}\int \left(1 + 2\cos\theta + \frac{1}{2}(\cos 2\theta + 1)\right)\mathrm{d}\theta$	M1A1
	$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$	dM1A1
	$= \frac{\pi}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{16} = \frac{\pi}{4} + \frac{9\sqrt{3}}{16}$	A1 (6)
		[10]

(a)	
B1	Use of $r\sin\theta$ Award if not seen explicitly but a correct result following use of double angle formula is seen.
M1	Differentiate $r\sin\theta$ or $r\cos\theta$
M1	Set $\frac{\mathrm{d}(r\sin\theta)}{\mathrm{d}\theta} = 0$ and solve the resulting equation. Only the solution used need be shown.
A1	Correct coordinates of A
(b)B1	Use of Area $= \frac{1}{2}\int r^2 \mathrm{d}\theta$ with $r = 1 + \cos\theta$, limits not needed.
M1	Attempt $(1+\cos\theta)^2$ (minimum accepted is $(1 + k\cos\theta + \cos^2\theta)$) and change $\cos^2\theta$ to an expression in $\cos 2\theta$ using $\cos^2\theta = \frac{1}{2}(\pm\cos 2\theta \pm 1)$
A1	Correct integrand; limits not needed. $\frac{1}{2}$ may be missing.
dM1	Attempt to integrate all terms. $\cos 2\theta \to \pm\frac{1}{k}\sin 2\theta$ $k = \pm 1$ or ± 2 Limits not needed. Depends on the previous M mark
A1	Correct integration and correct limits seen
A1	Substitute correct limits and obtain the correct answer in the required form.

Question Number	Scheme	Marks
Cont.	*Alternative for* (b) *using integration by parts* (Very rare but may be seen)	
	Area $=\frac{1}{2}\int_0^{\frac{\pi}{3}}(1+\cos\theta)^2\mathrm{d}\theta$	B1
	$=\frac{1}{2}\left[\int(1+2\cos\theta)\mathrm{d}\theta+\int\cos^2\theta\,\mathrm{d}\theta\right]$	
	$=\frac{1}{2}\left[\int(1+2\cos\theta)\mathrm{d}\theta+\cos\theta\sin\theta+\int\sin^2\theta\,\mathrm{d}\theta\right]$	M1A1
	$=\frac{1}{2}\left[\theta+2\sin\theta+\sin\theta\cos\theta+\int(1-\cos^2\theta)\mathrm{d}\theta\right]_0^{\frac{\pi}{3}}$	
	$=\frac{1}{2}\left[\theta+2\sin\theta+\frac{1}{2}(\sin\theta\cos\theta+\theta)\right]_0^{\frac{\pi}{3}}$	dM1A1
	$=\frac{\pi}{4}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{16}=\frac{\pi}{4}+\frac{9\sqrt{3}}{16}$	A1

B1	Use of Area $=\frac{1}{2}\int r^2\mathrm{d}\theta$ with $r=1+\cos\theta$, limits not needed.
M1	Attempt $(1+\cos\theta)^2$ (minimum accepted is $(1+k\cos\theta+\cos^2\theta)$) and attempt first stage of $\int\cos^2\theta\,\mathrm{d}\theta$ by parts. Reach $\int\cos^2\theta\,\mathrm{d}\theta=\cos\theta\sin\theta\pm\int\sin^2\theta\,\mathrm{d}\theta$ Limits not needed
A1	Correct so far. Limits not needed.
dM1	Attempt to integrate all terms. $\int(1+2\cos\theta)\mathrm{d}\theta$ **and** attempt to complete $\int\cos^2\theta\,\mathrm{d}\theta$ using Pythagoras identity. Limits not needed. Depends on the previous M mark
A1A1	As main scheme

Q10.

Question Number	Scheme	Marks
	$r = 6(1+\cos\theta) \qquad 0 \leqslant \theta \leqslant \pi$	
(a)	$\theta = 0,\ r = 6(1+\cos 0) \Rightarrow 12$ or $(12, 0)$	B1
		(1)
(b)	$\dfrac{\mathrm{d}}{\mathrm{d}\theta}(r\sin\theta)$: $= 6\sin\theta(1+\cos\theta) \Rightarrow$ $6\sin\theta(-\sin\theta)+6\cos\theta(1+\cos\theta)$ $= 6\sin\theta + 6\sin\theta\cos\theta \Rightarrow$ $6\cos\theta + 6(\cos^2\theta - \sin^2\theta)$ $= 6\sin\theta + 3\sin 2\theta \Rightarrow$ $6\cos\theta + 6\cos 2\theta$	M1
	$\Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0$ $[(2\cos\theta - 1)(\cos\theta+1) = 0 \Rightarrow] \quad \cos\theta = \dfrac{1}{2} \quad [\text{or } -1]$	dM1
	$\theta = \dfrac{\pi}{3} \qquad r = 6\left(1+\cos\dfrac{\pi}{3}\right) = 9 \qquad$ or $\left(9, \dfrac{\pi}{3}\right)$	A1 A1
		(4)
(c)	$\left[\tfrac{1}{2}\right]\int r^2\,\mathrm{d}\theta = \left[\tfrac{1}{2}\right]\int 36(1+\cos\theta)^2\,[\mathrm{d}\theta]$	M1
	$\int r^2\,\mathrm{d}\theta = 36\int(1+2\cos\theta+\cos^2\theta)[\mathrm{d}\theta] = 36\int\left(\tfrac{3}{2}+2\cos\theta+\tfrac{1}{2}\cos 2\theta\right)[\mathrm{d}\theta]$	M1
	$\int(a+b\cos\theta+c\cos 2\theta)[\mathrm{d}\theta] = a\theta \pm b\sin\theta \pm \tfrac{c}{2}\sin 2\theta$	M1
	$18\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]^{\dots}_{\dots}$	A1
	$18\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}} = 18\left(\dfrac{\pi}{2}+\sqrt{3}+\dfrac{\sqrt{3}}{8}\right)\ [-0] \qquad \left[= 9\pi + \dfrac{81}{4}\sqrt{3}\right]$	dM1
	E.g. $OB = 6\left(1+\cos\dfrac{'\pi'}{3}\right) \Rightarrow BQ = 6\left(1+\cos\dfrac{'\pi'}{3}\right)\sin\dfrac{'\pi'}{3} \quad \left[=\dfrac{9\sqrt{3}}{2}\right]$	M1
	$BP = 12 - 9\cos\dfrac{\pi}{3} = 12 - \dfrac{9}{2} = \dfrac{15}{2}$ $\Rightarrow$ area $OBPA = \dfrac{1}{2}\left(12+\dfrac{15}{2}\right)\left(\dfrac{9\sqrt{3}}{2}\right)$ or $\dfrac{1}{2}\times\dfrac{9}{2}\times\dfrac{9\sqrt{3}}{2}+\dfrac{9\sqrt{3}}{2}\times\dfrac{15}{2} \quad \left[=\dfrac{351}{8}\sqrt{3}\right]$	A1
	area of region $R = \dfrac{351}{8}\sqrt{3} - \dfrac{81}{4}\sqrt{3} - 9\pi = \dfrac{189}{8}\sqrt{3} - 9\pi$	A1
		(8)
		Total 13

Notes
(a)   B0: Correct values for θ and r or correct coordinates. Condone (0, 12).   (b)   M1: Differentiates $r\sin\theta$. Allow sign errors only. The "6" may be missing   dM1: **Dependent on previous M mark.** Uses correct identity/identities to reach a 3TQ in $\cos\theta$ and solves.   Apply usual rules or by calculator must obtain at least one real consistent solution where $-1 \leqslant \cos\theta \leqslant 1$.   Accept $\theta = \dfrac{\pi}{3}$ following a correct derivative set equal to zero to imply the M if no incorrect working shown (by calculator or by inspection).   A1: Either r or θ correct.   A1: Both coordinates correct. Only accept $\dfrac{\pi}{3}$ and 9. Withhold last mark if additional answers offered and not rejected, but isw after correct answers seen if only a miscopy is made.

(c)

M1: Attempts $\int r^2 \, d\theta$ which may include the $\frac{1}{2}$ for the area formula. The multiple 36 may be missing or wrong. Condone poor squaring - allow this mark if there are only two terms.

M1: Uses $\cos^2\theta = \pm\frac{1}{2}\cos 2\theta \pm \frac{1}{2}$ and obtains an integrand of the form $a + b\cos\theta + c\cos 2\theta$ (constants may be uncollected)

M1: Integrates and obtains a form $a\theta \pm b\sin\theta \pm \frac{c}{2}\sin 2\theta$ (sign errors on trig terms only). May be two terms in θ

A1: Fully correct expression for the area of the sector after integration. Ignore limits but $\frac{1}{2}$ must have been used or appear later.

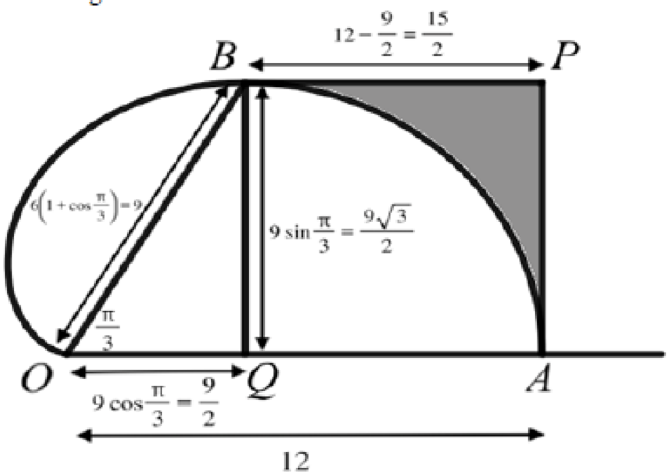
dM1: **Dependent on previous M mark.** Evidence of substitution of "correct" limits into the integral, so their $\frac{\pi}{3}\left(\text{provided } 0 < \theta < \frac{\pi}{2}\right)$ and the lower limit must be 0 and lead to zero but this can be implied by omission.

M1: Correct method for the perpendicular distance between l_1 and the initial line with their r and θ $\left(\text{provided } 0 < \theta < \frac{\pi}{2}\right)$ for point B. This may only be seen embedded in their attempt at an area, so may be implied if not explicit. E.g. look for "their r sin their $\frac{\pi}{3}$" or may find r again from the equation of curve.

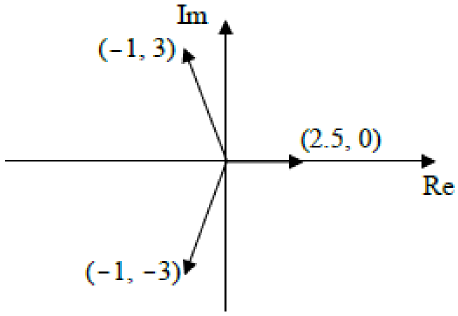
A1: Correct expression for area of trapezium $OBPA$ - may be given as as a sum of separate areas, e.g. triangle OBQ + rectangle $QBPA$. Other formulations are possible.

A1: Correct answer in the correct form or $p = \frac{189}{8}$ or exact equivalent, $q = -9$

Useful diagram:

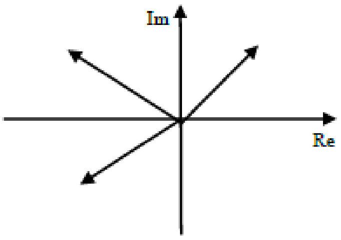


Q11.

Question Number	Scheme	Notes	Marks
	$\mathrm{f}(z)=2z^3-z^2+az+b$		
(a)	$(z=)-1+3\mathrm{i}$	Correct complex number	B1
			(1)
(b)	$z=-1\pm 3\mathrm{i}\Rightarrow\left(z-(-1+3\mathrm{i})\right)\left(z-(-1-3\mathrm{i})\right)\rightarrow z^2+...z+...$ Or e.g. Sum $=-2$, Product $=(-1)^2-(3\mathrm{i})^2=10\rightarrow z^2+...z+...$ Correct strategy to find the quadratic factor		M1
	$z^2+2z+10$	Correct expression	A1
	$\mathrm{f}(z)=\left(z^2+2z+10\right)\left(2z-5\right)$	Uses an appropriate method to find the linear factor	M1
	$\Rightarrow \mathrm{f}(z)=2z^3-z^2+10z-50$ or $a=10,\quad b=-50$	Correct cubic or correct constants	A1
			(4)
(c)	 	$-1\pm 3\mathrm{i}$ correctly plotted with vectors or dots or crosses etc. May be labelled by coordinates or on axes. Do not be concerned about scale but should look like reflections in the real line.	B1
		(2.5, 0) plotted correctly or follow through their non-zero real root correctly plotted. May be labelled by coordinates or on axes. Do not be too concerned about scale but e.g (2.5,0) should be further from O than (−1,0) is.	B1ft
			(2)

Alt (b)	$\mathrm{f}(-1+3\mathrm{i})=0\Rightarrow 2(-1+3\mathrm{i})^3-(-1+3\mathrm{i})^2+a(-1+3\mathrm{i})+b=0$ $\mathrm{Im}\left(\mathrm{f}(-1+3\mathrm{i})\right)=0\Rightarrow 2(9-27)-(-6)+3a=0\Rightarrow a=...$ Or e.g. $\mathrm{f}(-1+3\mathrm{i})-\mathrm{f}(-1-3\mathrm{i})=0\Rightarrow 2\left(2(9\mathrm{i}-27\mathrm{i})-(-6\mathrm{i})+3a\mathrm{i}\right)=0\Rightarrow a=...$ Correct full strategy to find one constant.		M1
	$a=10$ or $b=-50$	One correct value.	A1
	E.g. $\mathrm{Re}(\mathrm{f}(-1+3\mathrm{i}))=0\Rightarrow 2(-1+27)-(1-9)-a+b=0\Rightarrow b=...$ Correct method to find the second constant.		M1
	$a=10$ and $b=-50$ or $\mathrm{f}(z)=2z^3-z^2+10z-50$	Correct constants or correct cubic	A1
			(4)

Q12.

Question Number	Scheme	Notes	Marks
(a)	$z_1 = 3+3\text{i} \qquad z_2 = p+q\text{i} \qquad p,q \in \mathbb{R}$		
	$\lvert z_1\rvert = \sqrt{3^2+3^2}$ $\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert \Rightarrow \lvert z_2\rvert\sqrt{18} = 15\sqrt{2} \Rightarrow \lvert z_2\rvert = \ldots$	Attempts $\lvert z_1\rvert$ using Pythagoras and uses $\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$ to find $\lvert z_2\rvert$	M1
	$\lvert z_2\rvert = 5$	Cao	A1
			(2)
ALT	$\lvert z_1 z_2\rvert = 15\sqrt{2}$ $\lvert(3p-3q)+\text{i}(3p+3q)\rvert = 15\sqrt{2}$ $\sqrt{18p^2+18q^2} = 15\sqrt{2}$ $p^2+q^2 = 25$ $\lvert z_2\rvert = \sqrt{p^2+q^2} = 5$	Uses $\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$ to reach $p^2+q^2 = \ldots$	M1 A1 **(2)**
(b)	$\lvert z_2\rvert = 5 \Rightarrow p^2+q^2 = 25$ $\Rightarrow (-4)^2+q^2 = 25 \Rightarrow q = \ldots$	Uses $p^2+q^2 = "5"^2$ with $p = \pm 4$ leading to a value for q.	M1
	$q = \pm 3$	Both values. Must be clear $p = 4$ has not been used	A1
			(2)
(c)	 Points to be in the correct quadrants and either with correct numbers on the axes or labelled correctly	3 + 3i plotted correctly and labelled Vectors/ lines not needed; point(s) alone are sufficient	B1
		A conjugate pair plotted correctly following through their q.	B1ft
			(2)
			Total 6

Q13.

Question Number	Scheme	Notes	Marks
	$\mathrm{f}(z)=2z^4-19z^3+Az^2+Bz-156$		
(a)	$(z=)5+\mathrm{i}$	Correct complex number	B1
			(1)
	Mark (b) and (c) together – ignore any labelling seen. Award marks in the order given for their choice of method		
(b)/(c) With (b) first	$z=5\pm\mathrm{i}\Rightarrow(z-(5+\mathrm{i}))(z-(5-\mathrm{i}))=...$ Or e.g. Sum of roots = 10 Product of roots = 26	Correct strategy to find the quadratic factor using the conjugate pair	M1
	$z^2-10z+26$	Correct quadratic	A1
	$\mathrm{f}(z)=(z^2-10z+26)(2z^2+...z+k)$	Attempts to find the other quadratic. May use inspection (apply rules for quadratic factorisation ie "26"$\lvert k\rvert=156$) or e.g. long division with quotient $2z^2+...z+...$	M1
	NB long division gives quotient $2z^2+z+(A-42)$ and remainder $(10A+B-446)z+936-26A$		
	$2z^2+z-6$	Correct quadratic	A1
	$\Rightarrow z=\frac{3}{2},\ -2\ \ (,5\pm\mathrm{i})$	Correct real roots. The complex roots do not have to be stated.	A1
			(5)
	$\mathrm{f}(z)=(z^2-10z+26)(2z^2+z-6)$ $=...$	Multiplies out both quadratics or extracts the terms needed	M1
	$A=36, B=86$	Correct values (can be seen in the quartic equation)	A1
			(2)
			Total 8

(b)/(c) With (c) first	952+960i-2090-24A+10Ai+5Bi-156=0	Substitute $(5+\mathrm{i})$ into the quartic (by calculator) and equate real and imag parts (can be done with $(5-\mathrm{i})$)	M1
	-1294+24A+5B=0 -446+10A+B=0	Correct equations	A1
	A=36 B=86	M1 Solve simultaneously A1 One correct A1 Both correct	M1 A1A1
			(5)
	$2z^4-19z^3+36z^2+86z-156=0$ $z=...$	Solve the equation by long division, inspection or by calculator	M1
	$\Rightarrow z=\frac{3}{2},\ -2\ \ (,5\pm\mathrm{i})$	Correct real roots. The complex roots do not have to be stated.	A1
			(2)
			Total 8

Q14.

Question Number	Scheme	Notes	Marks
(a)	Condone use of e.g. $C+\mathrm{i}S$ for $\cos x+\mathrm{i}\sin x$ if the intention is clear.		
	$(\cos 5x \equiv)\mathrm{Re}(\cos x+\mathrm{i}\sin x)^5 \equiv \cos^5 x+\binom{5}{2}\cos^3 x(\mathrm{i}\sin x)^2+\binom{5}{4}\cos x(\mathrm{i}\sin x)^4$ Identifies the correct terms of the binomial expansion of $(\cos x+\mathrm{i}\sin x)^5$ They may expand $(\cos x+\mathrm{i}\sin x)^5$ completely but there must be an attempt to extract the real terms which must have the correct binomial coefficients combined with the correct powers of sin x and cos x. Condone use of a different variable e.g. θ.		M1
	$(\cos 5x \equiv)\cos^5 x-10\cos^3 x\sin^2 x+5\cos x\sin^4 x$ Correct simplified expression. Condone use of a different variable e.g. θ.		A1
	$\equiv \cos x\left(\cos^4 x-10\cos^2 x\sin^2 x+5\sin^4 x\right)$ $\equiv \cos x\left(\left(1-\sin^2 x\right)^2-10\left(1-\sin^2 x\right)\sin^2 x+5\sin^4 x\right)$ Applies $\cos^2 x=1-\sin^2 x$ to obtain an expression in terms of sin x inside the bracket. Condone use of a different variable e.g. θ.		M1
	$\equiv \cos x\left(16\sin^4 x-12\sin^2 x+1\right)$	Correct expression. Must be in terms of x now. The "cos5x =" is not required.	A1
			(4)

(b)	Allow use of a different variable in (b) e.g. x for <u>all</u> marks.		
	$\cos 5\theta = \sin 2\theta \sin\theta - \cos\theta$ $\Rightarrow \cos\theta\left(16\sin^4\theta-12\sin^2\theta+1\right)=2\sin^2\theta\cos\theta-\cos\theta$ $\Rightarrow \cos\theta\left(16\sin^4\theta-14\sin^2\theta+2\right)=0$ Uses the result from part (a) with $\sin 2\theta = 2\sin\theta\cos\theta$ and collects terms		M1
	$16\sin^4\theta-14\sin^2\theta+2=0$ $\Rightarrow \sin^2\theta=\frac{7\pm\sqrt{17}}{16}\Rightarrow \sin\theta=...$ Solves for $\sin^2\theta$ by any method including calculator and takes square root to obtain at least one value for $\sin\theta$. Depends on the first mark. May be implied by their values of $\sin\theta$ or θ. NB $\frac{7\pm\sqrt{17}}{16}=0.69519...,\ 0.17980...$		dM1
	$\sin\theta=\sqrt{\frac{7\pm\sqrt{17}}{16}}\Rightarrow\theta=...$ NB $\sqrt{\frac{7\pm\sqrt{17}}{16}}=0.833783...,\ \ 0.424035...$ A full method to reach at least one value for θ. Depends on the previous mark. May be implied by their values of θ		ddM1
	$(\theta=)0.986,\ 0.438$	Correct values and no others in range. Allow awrt these values.	A1
			(4)
			Total 8

<u>Note that it is possible to do (b) by changing to cos θ e.g.</u>

$$\cos\theta\left(16\sin^4\theta-12\sin^2\theta+1\right)=\cos\theta\left(16\left(1-\cos^2\theta\right)^2-12\left(1-\cos^2\theta\right)+1\right)$$

$$\cos\theta\left(16\left(1-\cos^2\theta\right)^2-12\left(1-\cos^2\theta\right)+1\right)=2\sin^2\theta\cos\theta-\cos\theta$$

$$16\cos^4\theta-18\cos^2\theta+4=0$$

$$\cos^2\theta=\frac{9\pm\sqrt{17}}{16}\Rightarrow\cos\theta=\sqrt{\frac{9\pm\sqrt{17}}{16}}$$

$$(\theta=)0.986,\ 0.438$$

This is acceptable as they used part (a) and can be scored as:

M1: Uses part (a) with $\sin^2\theta=1-\cos^2\theta$ and $\sin 2\theta=2\sin\theta\cos\theta$ and collects terms.

dM1: Solves for $\cos^2\theta$ by any method including calculator and takes square root to obtain at least one value for $\cos\theta$. Depends on the first mark. May be implied by their values of $\cos\theta$ or θ.

NB $\frac{9\pm\sqrt{17}}{16}=0.82019...,\ 0.30480...$

dM1: A full method to reach at least one value for θ.
Depends on the previous mark. May be implied by their values of θ

NB $\sqrt{\frac{9\pm\sqrt{17}}{16}}=0.0.743029...,\ 0.552092...$

A1: $(\theta=)0.986,\ 0.438$

Q15.

Question Number	Scheme	Marks
(a)	$z^n = e^{in\theta} = \cos n\theta + i\sin n\theta$	
	$\frac{1}{z^n} = e^{-in\theta} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$	
	$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta$ *	M1A1cso (2)
(b)	$\left(z+\frac{1}{z}\right)^6 = z^6 + 6z^5 \times \frac{1}{z} + \frac{6\times5}{2!}z^4 \times \frac{1}{z^2} + \frac{6\times5\times4}{3!}z^3 \times \frac{1}{z^3}$ $+\frac{6\times5\times4\times3}{4!}z^2 \times \frac{1}{z^4} + \frac{6\times5\times4\times3\times2}{5!}z \times \frac{1}{z^5} + \frac{1}{z^6}$	M1A1
	$(2\cos\theta)^6 = z^6 + 6z^4 + 15z^2 + 20 + 15\times\frac{1}{z^2} + 6\times\frac{1}{z^4} + \frac{1}{z^6}$	
	$64\cos^6\theta = z^6 + \frac{1}{z^6} + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$	M1
	$64\cos^6\theta = 2\cos 6\theta + 6\times 2\cos 4\theta + 15\times 2\cos 2\theta + 20$	M1
	$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$ *	A1* (5)

(c)	$\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 = 10$	
	$32\cos^6\theta = 10$	M1A1
	$\cos\theta = \pm\sqrt[6]{\frac{5}{16}}$	
	$\theta = 0.6027...,\ 2.5388...\qquad \theta = 0.603,\ 2.54$	M1A1 (4)
(d)	$\int_0^{\frac{\pi}{3}} \left(32\cos^6\theta - 4\cos^2\theta\right) d\theta$	
	$= \int_0^{\frac{\pi}{3}} \left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 - 4\cos^2\theta\right) d\theta$	
	$= \int_0^{\frac{\pi}{3}} \left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 - 2 - 2\cos 2\theta\right) d\theta$	M1
	$= \left[\frac{1}{6}\sin 6\theta + \frac{3}{2}\sin 4\theta + \frac{13}{2}\sin 2\theta + 8\theta\right]_0^{\frac{\pi}{3}}$	M1A1
	$= (0) + \frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{13}{2}\times\frac{\sqrt{3}}{2} + \frac{8\pi}{3}\quad (-0)$	dM1
	$= \frac{5\sqrt{3}}{2} + \frac{8\pi}{3}$ oe	A1 (5)
		[16]

Question Number	Scheme	Marks
(a)		
M1	Attempt to obtain $z^n + \frac{1}{z^n}$	
A1cso	Reach the given result with clear working and no errors Must see $\cos(-n\theta)+\mathrm{i}\sin(-n\theta)$ changed to $\cos n\theta - \mathrm{i}\sin n\theta$ (ie both included)	
(b)	*The first 3 marks apply to the binomial expansion only*	
M1	Apply the binomial expansion to $\left(z+\frac{1}{z}\right)^6$ Coefficients must be numerical (ie ${}^nC_r$ is not acceptable). The expansion must have 7 terms with at least 4 correct	
A1	Correct expansion, terms need not be simplified	
M1	Simplify the coefficients and pair the appropriate terms on RHS (At least 2 pairs must be correct)	
M1	Use the result from (a) throughout. Must include 2^6 or 64 now	
A1*	Obtain the given result with no errors in the working	
(c)		
M1	Use the result from (b) to simplify the given equation	
A1	Reach $32\cos^6\theta = 10$ oe	
M1	Solve to obtain at least one correct value for θ, in radians and in the given range, 3 sf or better	
A1	2 correct values, and no extras, in radians and in the given range. Must be 3 sf here Ignore extras outside the range	
(d)		
M1	Use the result in (b) to change $\cos^6\theta$ to a sum of multiple angles ready for integration and use $\cos^2\theta = \pm\frac{1}{2}(\cos 2\theta \pm 1)$ on $\cos^2\theta$ Limits not needed, ignore any shown	
M1	Integrate their expression to obtain an expression containing terms in $\sin 6\theta, \sin 4\theta, \sin 2\theta$ and θ Limits not needed	
A1	Correct integration Limits not needed	
dM1	Substitute limit pi/3. Depends second M mark	
A1	Correct, exact, answer (any equivalent to that shown). AwardM1A1 for a correct final answer following fully correct working.	
	There are other ways to integrate the function in (d), eg parts on one or both of the powers of $\cos\theta$, using $\cos^6\theta = \left(\cos^2\theta\right)^3 = \frac{1}{8}(1+\cos 2\theta)^3 = \ldots$ If in doubt about the marking of alternative methods which are not completely correct, send to review	

Q16.

Question Number	Scheme	Notes	Marks
	$z^5 - 32\mathrm{i} = 0 \Rightarrow r^5 = 32 \Rightarrow r = 2$	Correct value for r. May be shown explicitly or used correctly.	B1
	$5\theta = \frac{\pi}{2} + 2n\pi \Rightarrow \theta = \frac{\pi}{10} + \frac{2n\pi}{5}$	Applies a correct strategy for establishing at least 2 values of θ. This can be awarded if if the initial angle $\left(\frac{\pi}{2} \text{ or } \frac{\pi}{10}\right)$ is incorrect but otherwise their strategy is correct.	M1
	$z = 2e^{\frac{i\pi}{10}}, 2e^{\frac{i\pi}{2}}, 2e^{\frac{i9\pi}{10}}, 2e^{\frac{i13\pi}{10}}, 2e^{\frac{i17\pi}{10}}$ or $z = 2e^{\left(\frac{\pi}{10}+\frac{2n\pi}{5}\right)i}, \quad n = 0,1,2,3,4$	At least 2 correct, follow through their r	A1ft
		All correct. Must have $r = 2$	A1
			(4)
			Total 4

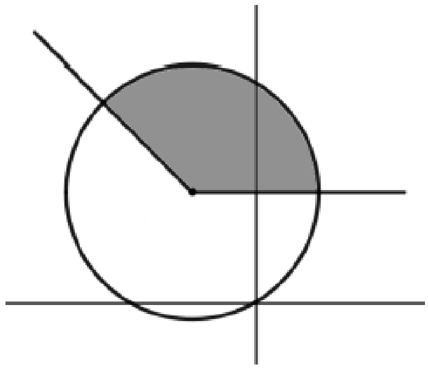
Q17.

Question Number	Scheme	Marks
(a)	$(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$	B1
	$= \cos^5\theta + 5\cos^4(\mathrm{i}\sin\theta) + \frac{5\times4}{2!}\cos^3\theta(\mathrm{i}\sin\theta)^2$ $+\frac{5\times4\times3}{3!}\cos^2\theta(\mathrm{i}\sin\theta)^3 + \frac{5\times4\times3\times2}{4!}\cos\theta(\mathrm{i}\sin\theta)^4 + (\mathrm{i}\sin\theta)^5$	M1
	$= \cos^5\theta + \underline{5\mathrm{i}\cos^4\theta\sin\theta} - 10\cos^3\theta\sin^2\theta \underline{-10\mathrm{i}\cos^2\theta\sin^3\theta} + 5\cos\theta\sin^4\theta \underline{+\mathrm{i}\sin^5\theta}$	A1
	$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ $= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta\frac{dy}{dx}$ $= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$	M1
	$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ *	A1* (5)
	Alternative: Using "$z - \frac{1}{z}$" $\quad z^5 - \frac{1}{z^5} = 2\mathrm{i}\sin 5\theta$ oe	B1
	Binomial expansion of $\left(z - \frac{1}{z}\right)^5$	M1
	$32\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta$	A1
	Uses double angle formulae etc to obtain $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ and then use it in their expansion	M1
	$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ *	A1* (5)
(b)	Let $x = \sin\theta \quad 16x^5 - 20x^3 + 5x = -\frac{1}{5} \Rightarrow \sin 5\theta = ...$	M1A1
	$\Rightarrow \theta = \frac{1}{5}\sin^{-1}\left(\pm\frac{1}{5}\right) = 38.306$ (or $-2.307, 69.692. 110.306, 141.693, 182.306$)	dM1
	(or in radians $-0.0402\ldots 0.6685\ldots, 1.216\ldots, 1.925\ldots, 2.473\ldots$)	
	Two of (awrt) $x = \sin\theta = -0.963, -0.555, -0.040, 0.620, 0.938$	A1
	All of (awrt) $x = \sin\theta = -0.963, -0.555, -0.040, 0.620, 0.938$	A1 (5)

(c)	$\int_0^{\frac{\pi}{4}}\left(4\sin^5\theta-5\sin^3\theta-6\sin\theta\right)\mathrm{d}\theta=\left(\int_0^{\frac{\pi}{4}}\frac{1}{4}(\sin 5\theta-5\sin\theta)-6\sin\theta\right)\mathrm{d}\theta$	M1
	$=\left[\frac{1}{4}\left(-\frac{1}{5}\cos 5\theta+5\cos\theta\right)+6\cos\theta\right]_0^{\frac{\pi}{4}}\left(=\left[-\frac{1}{20}\cos 5\theta+\frac{29}{4}\cos\theta\right]_0^{\frac{\pi}{4}}\right)$	A1
	$\frac{1}{4}\left[-\frac{1}{5}\cos\frac{5\pi}{4}+5\cos\frac{\pi}{4}-\left(-\frac{1}{5}+5\right)\right]+6\cos\frac{\pi}{4}-6$	
	$=\frac{1}{4}\left[\frac{1}{5}\times\frac{1}{\text{Ö}2}+\frac{5}{\text{Ö}2}-4\frac{4}{5}\right]+\frac{6}{\sqrt{2}}-6$	dM1
	$=\frac{73\sqrt{2}}{20}-\frac{36}{5}$ oe	A1 (4)
		[14]

	Notes
(a)	
B1	Applies de Moivre correctly. Need not see full statement, but must be correctly applied.
M1	Use binomial theorem to expand $(\cos\theta+\mathrm{i}\sin\theta)^5$ May only show imaginary parts - ignore errors in real parts. Binomial coefficients must be evaluated.
A1	Simplify coefficients to obtain a simplified result with all imaginary terms correct
M1	Equate imaginary parts and obtain an expression for $\sin 5\theta$ in terms of powers of $\sin\theta$ No $\cos\theta$ now
A1*	Correct **given** result obtained from fully correct working with at least one intermediate line wit the $(1-\sin^2\theta)^2$ expanded. Must see both sides of answer (may be split across lines). A0 if equating of imaginary terms is not clearly implied.
(b)	
	Note Answers only with no working score no marks as the "hence" has not been used. But if the first M1A1 gained then dM1 may be implied by a correct answer.
M1	Use substitution $x=\sin\theta$ and attempts to use the result from (a) to obtain a value for $\sin 5\theta$
A1	Correct value for $\sin 5\theta$
dM1	Proceeds to apply arcsin and divide by 5 to obtain at least one value for θ. Note for $\sin 5\theta=\frac{1}{5}$ the values you may see are the negatives of the true answers. FYI: $(5\theta=-11.53...,191.53..., 348.46..., 551.53..., 708.46..., 911.53...)$ (Or in radians $-0.201...$ $3.3428..., 6.0819..., 9.6260..., 12.365..., 15.909...$)
A1	Proceeds to take sin and achieve at least 2 different correct values for x or $\sin\theta$
A1	For all 5 values of x or $\sin\theta$ awrt 3 d.p. (allow 0.62 and -0.04)
(c)	
M1	Use previous work to change the integrand into a function that can be integrated
A1	Correct result after integrating. Any limits shown can be ignored
dM1	Substitute given limits, subtracts and uses exact numerical values for trig functions
A1	Final answer correct (oe provided in the given form)

Q18.

Question Number	Scheme		Marks
(a)	$\left(\cos\frac{5\pi}{12}+\mathrm{i}\sin\frac{5\pi}{12}\right)^4=\cos\frac{20\pi}{12}+\mathrm{i}\sin\frac{20\pi}{12}$ or/and $\left(\cos\frac{\pi}{3}-\mathrm{i}\sin\frac{\pi}{3}\right)^3=\cos(-\pi)+\mathrm{i}\sin(-\pi)$		M1 / A1
	$(z_1=)\dfrac{\cos\frac{5\pi}{3}+\mathrm{i}\sin\frac{5\pi}{3}}{\cos(-\pi)+\mathrm{i}\sin(-\pi)}=\cos\left(\frac{5\pi}{3}-(-\pi)\right)+\mathrm{i}\sin\left(\frac{5\pi}{3}-(-\pi)\right)$ Alt : $(z_1=)\dfrac{\cos\frac{5\pi}{3}+\mathrm{i}\sin\frac{5\pi}{3}}{-1}=-\cos\frac{5\pi}{3}-\mathrm{i}\sin\frac{5\pi}{3}$		M1
	$=\cos\frac{8\pi}{3}+\mathrm{i}\sin\frac{8\pi}{3}\quad=\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}$ * Alt: if denominator -1 used via e.g. $\cos\left(\frac{5\pi}{3}-\pi\right)+\mathrm{i}\sin\left(-\frac{5\pi}{3}\right)$		A1*
			(4)
(b)	$\lvert z-z_1\rvert\leqslant 1 \qquad 0\leqslant\arg(z-z_1)\leqslant\frac{3\pi}{4}$		
		A circle in any position (may just see the minor arc)	M1
		A pair of **half**–lines in correct directions from their centre, one with negative gradient and one parallel to (but not) the x–axis. *If full lines are used the M marks can be implied by their shading*	M1
		Area shaded inside their circle between the two half lines from from the parallel one anticlockwise to the negative gradient line.	M1
	Fully correct shaded sector. See notes.		A1
			(4)
(c)	$\arctan\left(\frac{\text{'}\frac{\sqrt{3}}{2}\text{'}}{\text{'}\frac{1}{2}\text{'}}\right)=\ldots\qquad\frac{\pi}{3}\quad(\text{or }60^\circ)$		M1 A1
			(2)
			Total 10

Notes

(a)

M1: One correct use of de Moivre in polar (or exponential) form. Allow use of $e^{i\theta}$ for $\cos\theta+\mathrm{i}\sin\theta$ throughout until the final A.

A1: Both correct (unsimplified) in polar form. Accept for both marks use of

$\left(\cos\frac{\pi}{3}-\mathrm{i}\sin\frac{\pi}{3}\right)^3\rightarrow\cos\pi-\mathrm{i}\sin\pi$ but denominator directly to $\cos\pi+\mathrm{i}\sin\pi$ with no evidence of dealing with the negative between terms is A0.

M1: A correct method shown for the division of the two complex numbers. No need to simplify for this mark, a difference of argument in the trig terms is fine. Look for the subtraction of the arguments. Sight of the $\frac{8\pi}{3}$ can imply the mark if no incorrect work is seen. Note it is M0 if the arguments are added (so

$\cos\left(\frac{5\pi}{3}-\pi\right)+\mathrm{i}\sin\left(\frac{5\pi}{3}-\pi\right)$ is M0 unless the denominator has clearly been written as $\cos\pi+\mathrm{i}\sin\pi$ first.)

May write the denominator as -1 first, which is correct, score for $-\cos\frac{5\pi}{3}-\mathrm{i}\sin\frac{5\pi}{3}$.

Accept methods that convert both numbers into exact Cartesian form, apply a correct process to realise the denominator and convert back to polar form.

Do not allow going straight to $\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}$ for this mark as it is a given answer. Justification is required and an incorrect method is M0.

A1cso: Obtains $\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}$ with suitable intermediate step shown and no errors in the work. Must have scored the preceding 3 marks. This will usually be via $\cos\frac{8\pi}{3}+\mathrm{i}\sin\frac{8\pi}{3}$ unless equivalent suitable working has been shown to justify the correct modulus (e.g. proceeding via -1 in the denominator).

(b)

Do not be concerned about lines being dashed or dotted in this part. Accept either.

M1: A circle in any position. You may see just the minor arc of a circle, which is acceptable as long as it is clearly an arc of a circle (e.g. implied by their shading) and not just a angle demarcation.

M1: Draws or indicates a pair of half–lines from their centre (which need not be in quadrant 2) with one with negative gradient proceeding up and left and one parallel to the x–axis proceeding right but not the x-axis itself. If full lines are used this can be implied by the shading of the correct region between lines.

M1: Shades the area between their pair of rays (the second ray may have positive gradient for this mark) and inside their circle, anticlockwise from the horizontal line. The half line need not stem from the centre of the circle for this mark, and accept the x-axis as the horizontal line for this mark.

A1: Fully correct shaded sector. Must

- be in quadrants 1 and 2
- have approximately correct gradients for the half-lines (-1 and 0)
- have circle with centre in quadrant 2 and (if whole circle shown) passing roughly through the origin. If only an arc is shown apply bod as long as the position is reasonable.

Ignore any centre coordinates and axes intersections of any major sector. If extra regions are shaded they must make clear which their region R is in order to access the mark.

(c)

M1: Identifies the correct point for their sector, which must be from a circle with centre in quadrant 1 or 2 (above the real axis) and ray parallel to the real axis, and attempts the relevant angle. Look for selecting the point (c, d) at the "3 o'clock" position having identified a suitable sector and proceeding to find a relevant positive angle (e.g., allow if $\pi-\theta$ is found) (accept $\arctan\pm\frac{c}{d}$ or $\arctan\pm\frac{d}{c}$ as an attempt at the angle).

Their point must be in quadrants 1 or 2. If no shading was shown in (b) allow for attempts at the relevant point of horizontal ray and circle intersection. Could use other trig.

A1 : Either correct value. Mark final answer.

Q19.

Question	Scheme	Marks
	$\det \mathbf{M} = 3x\times(2-x)-(4x+1)\times 7 = \ldots$	M1
	$= -3x^2-22x-7$ or $3x^2+22x+7$	A1
	$-3x^2-22x-7=0 \Rightarrow (-3x-1)(x+7)=0 \Rightarrow x=\ldots$	M1
	$-3x^2-22x-7>0 \Rightarrow "-7"<x<"-\frac{1}{3}"$	M1
	So range is $-7<x<-\frac{1}{3}$ or $(x\in)\left(-7,-\frac{1}{3}\right)$	A1
		(5)
		(5 marks)

Notes:

M1: Attempts to expand the determinant of **M**. Allow with + between the 2 products.

A1: Correct simplified quadratic with = or an inequality sign or neither

M1: Attempts to solve their three term quadratic, any valid means (usual rules – see front pages). Correct answers seen implies correct method. Can be awarded even if the roots are complex.

M1: Chooses the inside region for their roots, accept with strict or loose inequalities.

A1: Correct answer. Accept $x>-7 \cap x<-\frac{1}{3}$

Q20.

Question Number	Scheme	Marks
(a)	$k(k+5)-6=0$	M1
	$k^2+5k-6=0$	
	$\left((k-1)(k+6)=0 \Rightarrow\right) \quad k=1,\ -6$	A1 (2)
(b)	$\dfrac{1}{"k^2+5k-6"}\begin{pmatrix} k & 2 \\ 3 & k+5 \end{pmatrix}$	M1A1 (2)
		[4]
(a)		
M1	Attempts determinant and sets equal to zero (or equivalent method) to obtain an unsimplified quadratic equation	
A1	Correct values for k (may solve the quadratic by any valid means)	
(b)		
M1	Forms the matrix of signed minors (must have at least three correct elements) divided or multiplied by an attempt at the determinant	
A1	Fully correct inverse	

Q21.

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td>(a)</td><td>$y=\ln(5+3x)\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{5+3x}$</td><td>Correct first derivative</td><td>B1</td></tr>
<tr><td></td><td colspan="2">$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{5+3x}\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{9}{(5+3x)^2}\Rightarrow\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{54}{(5+3x)^3}$
M1: Continues the process of differentiating and reaches $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{k}{(5+3x)^3}$ oe
Note this may be achieved via the quotient rule e.g. $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{-9\times-2\times3(5+3x)}{(5+3x)^4}$
A1: Correct **simplified** third derivative. Allow e.g. $\frac{54}{(5+3x)^3}$ or $54(5+3x)^{-3}$.</td><td>M1 A1</td></tr>
<tr><td></td><td></td><td></td><td>**(3)**</td></tr>
<tr><td>(b)</td><td colspan="2">$y_0=\ln5,\ y_0'=\frac{3}{5},\ y_0''=-\frac{9}{25},\ y_0'''=\frac{54}{125}$
$\Rightarrow\ln(5+3x)\approx\ln5+\frac{3}{5}x-\frac{9}{25}\frac{x^2}{2!}+\frac{54}{125}\frac{x^3}{3!}+\ldots$
Attempts all values at $x=0$ and applies Maclaurin's theorem.
Evidence for attempting the values can be taken from at least 2 terms.
The form of the expansion must be correct including the factorials or their values.
Note that this is "Hence" and so do not allow other methods e.g. Formula Book.</td><td>M1</td></tr>
<tr><td></td><td colspan="2">$\ln(5+3x)\approx\ln5+\frac{3}{5}x-\frac{9}{50}x^2+\frac{9}{125}x^3+\ldots$
Correct expansion. The "ln(5 + 3x) =" is not required.</td><td>A1</td></tr>
<tr><td></td><td colspan="2"></td><td>**(2)**</td></tr>
</table>

<table>
<tr><td>(c)</td><td colspan="2">$\ln(5-3x)\approx\ln5-\frac{3}{5}x-\frac{9}{50}x^2-\frac{9}{125}x^3+\ldots$
Correct expansion even if obtained "from scratch"
OR for a correct follow through with signs changed on the coefficients of the odd powers of x only in an expansion of the correct form e.g. a polynomial in ascending powers of x.</td><td>B1ft</td></tr>
<tr><td></td><td></td><td></td><td>**(1)**</td></tr>
<tr><td>(d)</td><td colspan="2">$\ln\frac{(5+3x)}{(5-3x)}=\ln(5+3x)-\ln(5-3x)$
$\ln5+\frac{3}{5}x-\frac{9}{50}x^2+\frac{9}{125}x^3+\ldots-\left(\ln5-\frac{3}{5}x-\frac{9}{50}x^2-\frac{9}{125}x^3+\ldots\right)$
Subtracts **their** 2 **different** series to obtain at least 2 non-zero terms in ascending powers of x.</td><td>M1</td></tr>
<tr><td></td><td colspan="2">$=\frac{6}{5}x+\frac{18}{125}x^3+\ldots$
Correct terms. Allow e.g. $0+\frac{6}{5}x+0x^2+\frac{18}{125}x^3+\ldots$</td><td>A1</td></tr>
<tr><td></td><td colspan="2">Allow both marks to score in (d) provided the **correct** series have been obtained in (b) and (c) by **any** means.</td><td></td></tr>
<tr><td></td><td></td><td></td><td>**(2)**</td></tr>
<tr><td></td><td colspan="2"></td><td>**Total 8**</td></tr>
</table>

Q22.

Question	Scheme	Marks
(a)	$\begin{pmatrix} 2 & -1 & 3 \\ -2 & 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & -3 \\ 2k & 2 \end{pmatrix} = \begin{pmatrix} 2+0+6k & 2k+3+6 \\ -2+0+0 & -2k-9+0 \end{pmatrix}$	M1
	$= \begin{pmatrix} 2+6k & 2k+9 \\ -2 & -2k-9 \end{pmatrix}$	A1cao
		(2)
(b)	$\det \mathbf{AB} = (2+6k)(-2k-9)-(-2)(2k+9)$	M1
	$\det \mathbf{AB} = 0 \Rightarrow -12k^2 - 54k = 0 \Rightarrow k = ...$	dM1
	$k = -\dfrac{9}{2}$	A1
		(3)
		(5 marks)

Notes:
(a) **M1:** Obtains a 2×2 matrix with at least two entries correct, unsimplified. **A1cao:** Correct matrix with terms simplified. (b) **M1:** Attempts the determinant, be tolerant of minor slips, such as sign slips with the negatives, if the correct "$ad-bc$" form is apparent. They may give the $-(-2)(\ldots)$ as just $+\,2(\ldots)$. Accept if seen as part of the attempt at the inverse matrix. **dM1:** Expands their determinant to a **quadratic**, sets equal to zero (may be implied) and achieves a value for k via correct method (allow if a factor k is cancelled, use of formula or calculator (a correct value for their quadratic)). **A1:** cso for $-\dfrac{9}{2}$. Accept as decimal or equivalent fractions, such as $-\dfrac{54}{12}$. Ignore any reference to the 0 solution.

Q23.

Question	Scheme		Marks
(a)	$\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3=\left(z^2-\frac{1}{z^2}\right)^3$		
	$=z^6-3z^2+\frac{3}{z^2}-z^{-6}$	M1: Attempt to expand A1: Correct expansion	M1A1
	$=z^6-\frac{1}{z^6}-3\left(z^2-\frac{1}{z^2}\right)$	Correct answer with no errors seen	A1
			(3)
	Alternative		
	$\left(z+\frac{1}{z}\right)^3=z^3+3z+\frac{3}{z}+\frac{1}{z^3}$, $\left(z-\frac{1}{z}\right)^3=z^3-3z+\frac{3}{z}-\frac{1}{z^3}$		M1A1
	M1: Attempt to expand both cubic brackets A1: Correct expansions		
	$=z^6-\frac{1}{z^6}-3\left(z^2-\frac{1}{z^2}\right)$	Correct answer with no errors	A1
			(3)
(b)(i)(ii)	$z^n=\cos n\theta+i\sin n\theta$	Correct application of de Moivre	B1
	$z^{-n}=\cos(-n\theta)+i\sin(-n\theta)=\pm\cos n\theta\pm\sin n\theta$ but must be different from their z^n	Attempt z^{-n}	M1
	$z^n+\frac{1}{z^n}=2\cos n\theta*$, $z^n-\frac{1}{z^n}=2i\sin n\theta*$	$z^{-n}=\cos n\theta-i\sin n\theta$ must be seen	A1*
			(3)

(c)	$\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3=(2\cos\theta)^3(2i\sin\theta)^3$		B1
	$z^6-\frac{1}{z^6}-3\left(z^2-\frac{1}{z^2}\right)=2i\sin 6\theta-6i\sin 2\theta$	Follow through their k in place of 3	B1ft
	$-64i\sin^3\theta\cos^3\theta=2i\sin 6\theta-6i\sin 2\theta$	Equating right hand sides and simplifying $2^3\times(2i)^3$ (B mark needed for each side to gain M mark)	M1
	$\cos^3\theta\sin^3\theta=\frac{1}{32}(3\sin 2\theta-\sin 6\theta)$ *		A1cso
			(4)

(d)	$\int_0^{\frac{\pi}{8}} \cos^3\theta \sin^3\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{8}} \frac{1}{32}(3\sin 2\theta - \sin 6\theta)\, \mathrm{d}\theta$		
	$= \frac{1}{32}\left[-\frac{3}{2}\cos 2\theta + \frac{1}{6}\cos 6\theta\right]_0^{\frac{\pi}{8}}$	M1: $p\cos 2\theta + q\cos 6\theta$ A1: Correct integration Differentiation scores M0A0	M1 A1
	$= \frac{1}{32}\left[\left(-\frac{3}{2\sqrt{2}} - \frac{1}{6\sqrt{2}}\right) - \left(-\frac{3}{2} + \frac{1}{6}\right)\right] = \frac{1}{32}\left(\frac{4}{3} - \frac{5\sqrt{2}}{6}\right)$	dM1: Correct use of limits – lower limit to have non-zero result. Dep on previous M mark A1: Cao (oe) but must be exact	dM1 A1
			(4)
			(14 marks)

Q24.

Question Number	Scheme	Marks
(a)	$(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$	
	$\cos^4\theta + 4\cos^3\theta(\mathrm{i}\sin\theta) + \frac{4\times 3}{2!}\cos^2\theta(\mathrm{i}\sin\theta)^2 + \frac{4\times 3\times 2}{3!}\cos\theta(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4$	M1
	$= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta + \mathrm{i}^2 6\cos^2\theta\sin^2\theta + 4\mathrm{i}^3\cos\theta\sin^3\theta + \mathrm{i}^4\sin^4\theta$	A1
	$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$	M1
	$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$	A1
	$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$	
	$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ *	M1A1* (6)
(b)	$x = \tan\theta \quad \frac{2\tan\theta - 2\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = \frac{1}{2}\tan 4\theta = 1$	
	$\tan 4\theta = 2$	M1
	$x = \tan\theta = 0.284,\ 1.79$	A1A1 (3) [9]

(a)	
M1	Correct use of de Moivre and attempt the complete expansion
A1	Correct expansion. Coefficients to be single numbers but powers of i may still be present.
M1	Equate the real and imaginary parts
A1	Correct expressions for $\cos 4\theta$ and $\sin 4\theta$
M1	Use $\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$ and divide numerator and denominator by $\cos^4\theta$ Only tangents now.
A1*	Correct **given** answer, no errors seen.
(b)	
M1	Substitute $x = \tan\theta$ and re-arrange to $\tan 4\theta = \pm 2$ or $\pm\frac{1}{2}$
A1A1	A1 for either solution; A2 for both. Deduct one mark only for failing to round either or both to 3 sf (One correct answer but not rounded scores A0A0; two correct answers neither rounded scores A1A0; two correct answers, only one rounded, scores A1A0)

Q25.

Question Number	Scheme	Notes	Marks
(a)	$$\frac{1}{(2n-1)(2n+1)(2n+3)} \equiv \frac{A}{2n-1}+\frac{B}{2n+1}+\frac{C}{2n+3}$$ $$\Rightarrow A=...,B=...,C=...$$ Correct partial fraction attempt to obtain values for A, B and C		M1
	$$\frac{1}{8(2n-1)}-\frac{1}{4(2n+1)}+\frac{1}{8(2n+3)} \text{ or e.g } \frac{1}{16n-8}-\frac{1}{8n+4}+\frac{1}{16n+24}$$ $$\text{or e.g. } \frac{\frac{1}{8}}{(2n-1)}-\frac{\frac{1}{4}}{(2n+1)}+\frac{\frac{1}{8}}{(2n+3)}$$ Correct **partial fractions**. (May be seen in (b)) This mark is **not** for the correct values of A, B and C, it is for the correct fractions.		A1
			(2)

(b)	$$\frac{1}{8}\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{2}{2r+1}+\frac{1}{2r+3}\right)=$$ $$\frac{1}{8}\left(\frac{1}{1}-\frac{2}{3}+\cancel{\frac{1}{5}}\right.$$ $$+\frac{1}{3}-\cancel{\frac{2}{5}}+\cancel{\frac{1}{7}}$$ $$+\cancel{\frac{1}{5}}-\cancel{\frac{2}{7}}+\cancel{\frac{1}{9}}$$ $$.$$ $$.$$ $$+\cancel{\frac{1}{2n-3}}-\cancel{\frac{2}{2n-1}}+\frac{1}{2n+1}$$ $$\left.+\cancel{\frac{1}{2n-1}}-\frac{2}{2n+1}+\frac{1}{2n+3}\right)$$ Uses the method of differences to find sufficient terms to establish cancelling. E.g. 3 rows at the start and 2 rows at the end or vice versa This may be implied if they extract the correct non-cancelling terms.		M1
	$$=\frac{1}{8}\left(1-\frac{1}{3}-\frac{1}{2n+1}+\frac{1}{2n+3}\right) \text{or e.g. } =\frac{1}{8}\left(1-\frac{2}{3}+\frac{1}{3}+\frac{1}{2n+1}-\frac{2}{2n+1}+\frac{1}{2n+3}\right)$$ Identifies the correct non-cancelling terms. May be unsimplified.		A1
	$$=\frac{1}{8}\left(\frac{2(2n+1)(2n+3)-3(2n+3)+3(2n+1)}{3(2n+1)(2n+3)}\right)=...$$ Attempts to combine terms into one fraction. There must have been at least one constant term and at least 2 different algebraic terms with at least 3 terms in the numerator when combining the fractions.		dM1
	$$=\frac{n(n+2)}{3(2n+1)(2n+3)}$$	cao	A1
			(4)
			Total 6

Q26.

Question Number	Scheme	Notes	Marks
(a)	$(x+1)\frac{dy}{dx}-xy=e^{3x} \qquad x>-1$		
	$\frac{dy}{dx}-\frac{xy}{(x+1)}=\frac{e^{3x}}{(x+1)}$	Correctly rearranged equation	B1
	$I=e^{\int\frac{-x}{x+1}dx}=e^{\int\left(-1+\frac{1}{x+1}\right)dx}$	Correct strategy for the integrating factor including an attempt at the integration	M1
	$=e^{-x+\ln(x+1)}$	For $-x+\ln(x+1)$	A1
	$=(x+1)e^{-x}$	Correct integrating factor	A1
	$y(x+1)e^{-x}=\int\frac{e^{3x}}{x+1}\times(x+1)e^{-x}\,dx$	Uses their integrating factor to reach the form $yI=\int QI\,dx$	M1
	$y(x+1)e^{-x}=\frac{1}{2}e^{2x}+c$	Correct equation (with or without $+c$)	A1
	$y=\frac{e^{3x}}{2(x+1)}+\frac{ce^{x}}{(x+1)}$	Correct answer (allow equivalent forms). Must have $y=$...	A1
			(7)
(b)	$x=0, y=5\Rightarrow 5=\frac{1}{2}+c\Rightarrow c=\frac{9}{2}$	Substitutes $x=0$ and $y=5$ and attempts to find a value for c.	M1
	$y=\frac{e^{3x}}{2(x+1)}+\frac{9e^{x}}{2(x+1)}$	Cao (oe) Must have $y=$...	A1
			(2)
			Total 9

Q27.

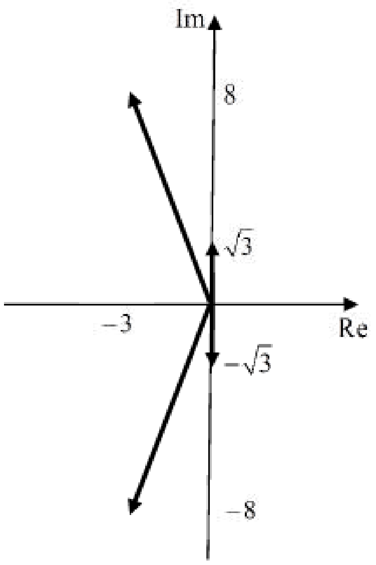
Question	Scheme		Marks
	$2z + z^* = \dfrac{3+4\text{i}}{7+\text{i}}$		
	Way 1		
	$\{2z+z^* =\}\ 2(a+\text{i}b) + (a-\text{i}b)$	Left hand side $= 2(a+\text{i}b) + (a-\text{i}b)$ Can be implied by eg. $3a + \text{i}b$ **Note:** This can be seen anywhere in their solution	B1
	$\ldots\ldots\ldots = \dfrac{(3+4\text{i})}{(7+\text{i})}\dfrac{(7-\text{i})}{(7-\text{i})}$	Multiplies numerator **and** denominator of the right hand side by $7-\text{i}$ or $-7+\text{i}$	M1
	$\ldots\ldots\ldots = \dfrac{25+25\text{i}}{50}$	Applies $\text{i}^2 = -1$ to and collects like terms to give right hand side $= \dfrac{25+25\text{i}}{50}$ or equivalent	A1
	So, $3a + \text{i}b = \dfrac{1}{2} + \dfrac{1}{2}\text{i}$ $\Rightarrow a = \dfrac{1}{6}, b = \dfrac{1}{2}$ **or** $z = \dfrac{1}{6} + \dfrac{1}{2}\text{i}$	**dependent on the previous B and M marks** Equates either real parts or imaginary parts to give at least one of $a = \ldots$ or $b = \ldots$	ddM1
		Either $a = \dfrac{1}{6}$ **and** $b = \dfrac{1}{2}$ **or** $z = \dfrac{1}{6} + \dfrac{1}{2}\text{i}$	A1
			(5)

	Way 2		
	$\{2z+z^* =\}\ 2(a+\text{i}b) + (a-\text{i}b)$	Left hand side $= 2(a+\text{i}b) + (a-\text{i}b)$ Can be implied by eg. $3a + \text{i}b$	B1
	$(3a + \text{i}b)(7+\text{i}) = \ldots\ldots\ldots$	Multiplies their $(3a + \text{i}b)$ by $(7+\text{i})$	M1
	$21a + 3a\text{i} + 7b\text{i} - b = \ldots\ldots\ldots$	Applies $\text{i}^2 = -1$ to give left hand side $= 21a + 3a\text{i} + 7b\text{i} - b$	A1
	So, $(21a - b) + (3a + 7b) = 3 + 4\text{i}$ gives $21a - b = 3,\ 3a + 7b = 4$ $\Rightarrow a = \dfrac{1}{6}, b = \dfrac{1}{2}$ **or** $z = \dfrac{1}{6} + \dfrac{1}{2}\text{i}$	**dependent on the previous B and M marks** Equates **both** real parts and imaginary parts to give at least one of $a = \ldots$ or $b = \ldots$	ddM1
		Either $a = \dfrac{1}{6}$ **and** $b = \dfrac{1}{2}$ **or** $z = \dfrac{1}{6} + \dfrac{1}{2}\text{i}$	A1
			(5)
			(5 marks)

Notes:
Some candidates may let $z = x + \text{i}y$ and $z^* = x - \text{i}y$. So apply the mark scheme with $x \equiv a$ and $y \equiv b$. For the final A1 mark, you can accept exact equivalents for a, b.

Q28.

Question	Scheme		Marks
(a)	$\text{f}(z) = z^4 + 6z^3 + 76z^2 + az + b,\quad a, b$ are real constants. $z_1 = -3 + 8\text{i}$ is given.		
	$-3 - 8\text{i}$	$-3 - 8\text{i}$	B1
			(1)
(b)	$z^2 + 6z + 73$	Attempt to expand $(z - (-3+8\text{i}))(z - (-3-8\text{i}))$ or any valid method ***to establish a quadratic factor*** eg $z = -3 \pm 8\text{i} \Rightarrow z + 3 = \pm 8\text{i} \Rightarrow z^2 + 6z + 9 = -64$ or sum of roots -6, product of roots 73 to give $z^2 \pm (\text{sum})z + \text{product}$	M1
		$z^2 + 6z + 73$	A1
	$\text{f}(z) = (z^2 + 6z + 73)(z^2 + 3)$	Attempts to find the other quadratic factor. eg. using long division to get as far as $z^2 + \ldots$ or eg. $\text{f}(z) = (z^2 + 6z + 73)(z^2 + \ldots)$	M1
		$z^2 + 3$	A1
	$\{z^2 + 3 = 0 \Rightarrow z =\}\ \pm\sqrt{3}\text{i}$	**dependent on only the previous M mark** Correct method of solving the 2nd quadratic factor	dM1
		$\sqrt{3}\text{i}$ and $-\sqrt{3}\text{i}$	A1
			(6)

(c)		**<u>Criteria</u>** • $-3 \pm 8\text{i}$ plotted correctly in quadrants 2 and 3 with some evidence of symmetry • Their other two *complex roots* (which are found from solving their 2nd quadratic in (b)) are plotted correctly with some evidence of symmetry about the x-axis	
		Satisfies at least one of the two criteria	B1 ft
		Satisfies both criteria with some indication of scale or coordinates stated. All points (arrows) must be in the correct positions relative to each other.	B1 ft
			(2)
			(9 marks)

Notes:
(b) Give 3rd M1 for $z^2 + k = 0,\ k > 0 \Rightarrow$ **at least one of either** $z = \sqrt{k}\,\text{i}$ **or** $z = -\sqrt{k}\,\text{i}$ Give 3rd M0 for $z^2 + k = 0,\ k > 0 \Rightarrow z = \pm k\text{i}$ Give 3rd M0 for $z^2 + k = 0,\ k > 0 \Rightarrow z = \pm k$ or $z = \pm\sqrt{k}$ Candidates do not need to find $a = 18, b = 219$

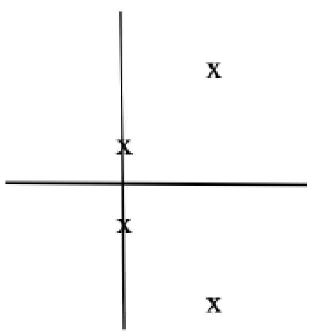
Q29.

Qn No	Scheme	Notes	Marks
(a)	$\dfrac{z_2z_3}{z_1}=\dfrac{(p-\mathrm{i})(p+\mathrm{i})(2+\mathrm{i})}{(2-\mathrm{i})(2+\mathrm{i})}$	Multiply top and bottom by complex conjugate of their denominator. (The two M's may be scored if the given numbers are wrongly placed.)	M1
	$=\dfrac{(p^2+1)(2+\mathrm{i})}{5}$	Simplifies numerator with evidence that $\mathrm{i}^2=-1$ and denominator real. Accept any equivalent form in the numerator as long as there are not i^2 terms if expanded.	M1
	$=\dfrac{2(p^2+1)}{5}+\dfrac{(p^2+1)}{5}\mathrm{i}$	Correct real +imaginary form with i factored out. Accept as single fraction with numerator in correct form. Accept $'a='$ and $'b='$.	A1
			(3)
ALT	$\dfrac{z_2z_3}{z_1}=\dfrac{(p-\mathrm{i})(p+\mathrm{i})}{(2-\mathrm{i})}=a+b\mathrm{i}$ $p^2+1=(a+b\mathrm{i})(2-\mathrm{i})$ $\left.\begin{array}{l}2a+b=p^2+1\\2b-a=0\end{array}\right\}$	Cross multiplies by $2-\mathrm{i}$ (or their denominator), expands and equates real and imaginary parts. (The two M's may be scored if the given numbers are wrongly placed.)	M1
	$\left.\begin{array}{l}2a+b=p^2+1\\2b-a=0\end{array}\right\}\Rightarrow a=....,b=...$	Attempt to solve their equations.	M1
	$a+b\mathrm{i}=\dfrac{2(p^2+1)}{5}+\dfrac{(p^2+1)}{5}\mathrm{i}$	Correct real +imaginary form with i factored out. Accept as single fraction with numerator in correct form. Accept $'a='$ and $'b='$.	A1
			(3)

(b)	$\left\|\dfrac{z_2z_3}{z_1}\right\|^2=\dfrac{4(p^2+1)^2}{25}+\dfrac{(p^2+1)^2}{25}$	Correct attempt at the modulus or modulus squared. Accept with their answers to part (a). Any erroneous i or i^2 is M0.	M1
	$\dfrac{4(p^2+1)^2}{25}+\dfrac{(p^2+1)^2}{25}=\left(2\sqrt{5}\right)^2$	Their $\left\|\dfrac{z_2z_3}{z_1}\right\|^2=\left(2\sqrt{5}\right)^2$	dM1
	$(p^2+1)^2=100\Rightarrow p=\pm3$	Attempt to solve and achieves $p=...$ (may be scored from use of $\|..\|^2=2\sqrt{5}$) $p=\pm3$	M1 A1
			(4)
ALT 1	$\left\|\dfrac{z_2z_3}{z_1}\right\|=2\sqrt{5}\Rightarrow\|z_2z_3\|=\sqrt{4+1}\times2\sqrt{5}$	Cross multiplies and attempts $\|z_1\|$	M1
	$\Rightarrow\|z_2\|^2=\sqrt{4+1}\times2\sqrt{5}\Rightarrow p^2+1=...$	Attempts $\|z_2z_3\|$ either directly or using $\|z_2z_2^*\|=\|z_2\|^2$ to get an equation in p.	dM1
	$(p^2+1)=10\Rightarrow p=\pm3$	Attempt to solve and achieves $p=...$ $p=\pm3$	M1 A1
			(4)
			Total 7

Q30.

Question Number	Scheme	Notes	Marks
	$2x^4-8x^3+29x^2-12x+39=0,\quad x=2+3\text{i}$ Condone work in e.g., z throughout		
(a)	2–3i	Correct conjugate	B1
			(1)
(b)	$(x-(2-3\text{i}))(x-(2+3\text{i}))$ or $(x-2+3\text{i})(x-2-3\text{i})$ $=\ldots\ \{x^2-4x+13\}$ or sum $=4$, product $=13$ $\Rightarrow x^2\pm4x\pm13$ or $x^2\pm13x\pm4$ or $x^2-(2+3\text{i}+2-3\text{i})x+(2+3\text{i})(2-3\text{i})$ $\Rightarrow\ldots\ \{x^2-4x+13\}$	Attempts to multiply the two correct factors to obtain a 3 term quadratic with real coefficients. Could use $(x-2)^2=(\pm3\text{i})^2$ or $x^2-2ax+a^2+b^2$ with $a=2,\ b=\pm3$ Or uses the correct sum and product of the roots to obtain an expression of the form shown (must be some minimal working – but if just a quadratic is given the next 2 marks are available) or $x^2-(\alpha+\beta)x+\alpha\beta$ to obtain a 3 term quadratic with real coefficients.	M1
	$2x^4-8x^3+29x^2-12x+39\Rightarrow(x^2-4x+13)(2x^2+3)$	Uses their 2 or 3 term quadratic factor with real coefficients to obtain a second 2 or 3 term quadratic of the form $2x^2+\ldots$ by long division, equating coefficients or inspection. Ignore any remainder from long division. Can follow M0	M1
	$2x^2+3(=0)\Rightarrow$ $x=\pm\dfrac{\sqrt{6}}{2}\text{i}$ or $\pm\text{i}\sqrt{\dfrac{3}{2}}$ or $\pm\dfrac{\sqrt{3}}{\sqrt{2}}\text{i}$ or $\sqrt{1.5}\,\text{i}$ $\sqrt{1.5\text{i}}$ is M0 1.2247...i is M1 A0	dM1: Solves their second quadratic factor = 0. If 2 term must get one correct non-zero root. (Usual rules if 3TQ and one correct root if no working) Could be inexact. **Requires previous method mark.** A1: Both correct exact roots with "i" **Requires all previous marks.**	dM1 A1

	Solving by calculator, sometimes followed by attempts to reconstruct factors. e.g., $\text{f}(x)=(x^2-4x+13)\left(x^2+\dfrac{3}{2}\right)$ is first M1 only and working for the 3TQ must be seen		**(4)**
(c)		Allow ft on their answers to (b) if they are of the form $\pm k\text{i}$ or $\pm k\sqrt{-1},\ k\neq0$ regardless of how they were obtained 1st B1: One of the two pairs of roots in correct positions 2nd B1: Both pairs of roots in correct positions and correct relative to each other for their k Allow any suitable indication of the roots such as vectors. Ignore all labelling and scaling but each pair should be reasonably symmetric in x-axis for any marks (for each pair -distance of one to x-axis not less than $\frac{1}{2}$ of the other)	B1 B1 (ft on (b))
			(2)
			Total 7

Q31.

Question Number	Scheme	Notes	Marks
(a)	$\lvert z_1+z_2\rvert\{=\lvert 3+2\text{i}+2+3\text{i}\rvert=\lvert 5+5\text{i}\rvert\}=\sqrt{5^2+5^2}$	Attempts the sum (allow one slip) and uses Pythagoras correctly	M1
	$\sqrt{50}$ or $5\sqrt{2}$	Either correct exact answer	A1
	Answer only is no marks but working can be minimal e.g., $\lvert 5+5\text{i}\rvert=5\sqrt{2}$		(2)
(b)	$\frac{z_2z_3}{z_1}=\frac{(2+3\text{i})(a+b\text{i})}{(3+2\text{i})}=\frac{(2+3\text{i})(a+b\text{i})}{(3+2\text{i})}\times\frac{(3-2\text{i})}{(3-2\text{i})}$ or $\frac{z_2}{z_1}=\frac{2+3\text{i}}{3+2\text{i}}\times\frac{3-2\text{i}}{3-2\text{i}}$ or $\frac{z_3}{z_1}=\frac{a+b\text{i}}{3+2\text{i}}\times\frac{3-2\text{i}}{3-2\text{i}}$	Substitutes complex numbers and correct multiplier to rationalise the denominator seen or implied. See note below Could use $\times\frac{-3+2\text{i}}{-3+2\text{i}}$	M1
	$(3+2\text{i})(3-2\text{i})=13$	13 <u>obtained from</u> $(3+2\text{i})(3-2\text{i})$ Could be implied.	B1
	$\frac{z_2z_3}{z_1}=\frac{12a-5b}{13}+\frac{5a+12b}{13}\text{i}$ or $\frac{1}{13}(12a-5b)+\frac{\text{i}}{13}(5a+12b)$ or $\frac{12}{13}a-\frac{5}{13}b+\text{i}\left(\frac{5}{13}a+\frac{12}{13}b\right)$ etc. Condone $\frac{(12a-5b)+(5a+12b)\text{i}}{13}$	dM1: Attempts to simplify the numerator and collects terms to obtain $pa+qb+ra\text{i}+sb\text{i}$ with at least three of p, q, r and s non-zero. **Requires previous M mark.** A1: Correct answer in any form with a single "i". Correct bracketing where needed. Allow $x=\ldots$, $y=\ldots$	dM1 A1
	Note: The following marks are accessible if complex numbers are substituted in the wrong places: z_2 as denominator max 1010, z_3 as denominator max 1000		(4)
(c)	$\frac{12a-5b}{13}=\frac{4}{13},\ \frac{5a+12b}{13}=\frac{58}{13}\Rightarrow a=\ldots,\ b=\ldots$	Equates their x to $\frac{4}{13}$ and their y to $\frac{58}{13}$ to obtain 2 linear equations in both a and b and solves to obtain values for both a and b.	M1
	No need to check values but must be some working between equations and values. "$\frac{12a-5b}{13}=\frac{4}{13},\ \frac{5a+12b}{13}=\frac{58}{13}\quad 12a-5b=4,\ 5a+12b=58\quad a=2,\ b=4$" is M0A0 Values can immediately follow if equations are produced with coefficients of a or b of the same magnitude		
	$a=2$ and $b=4$	Correct values for a and b from correct equations with working.	A1
	SC: Allow access to both marks for the exact $a=-\frac{242}{169}$ and $b=\frac{716}{169}$ from using $w=\frac{z_1z_3}{z_2}=\frac{12a+5b}{13}+\frac{12b-5a}{13}\text{i}$ There are no marks in (c) if z_3 was used as the denominator in (b) [leads to a = b = 0]		(2)

(d)	$\arctan\left(\frac{\frac{58}{13}}{\frac{4}{13}}\right)\{=1.5019\ldots \text{ or } 86.05\ldots^\circ\}$ or $\arctan\left(\frac{\frac{4}{13}}{\frac{58}{13}}\right)\{=0.068856\ldots \text{ or } 3.945\ldots^\circ\}$	Either correct arctan or $\tan^{-1}$ seen or implied by a correct 2sf value (awrt 1.5, 86, 0.069/0.068, 3.9) Could use equivalent trig. Note: $\tan\frac{58}{4}=-2.634$ or 0.258	M1
	1.502	1.502 only (**not** awrt) Mark final answer if 1.502 is followed by e.g., $\frac{\pi}{2}-1.502=0.06880$	A1
			(2)
			Total 10

Q32.

Question	Scheme		Marks
	$\sum_{r=1}^{n} r(r^2-3) = \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$		
	$= \frac{1}{4}n^2(n+1)^2 - 3\left(\frac{1}{2}n(n+1)\right)$	Attempts to expand $r(r^2-3)$ and attempts to substitute at least one correct standard formula into their resulting expression.	M1
		Correct expression (or equivalent)	A1
	$= \frac{1}{4}n(n+1)\left[n(n+1)-6\right]$	**dependent on the previous M mark** Attempt to factorise at least $n(n+1)$ having attempted to substitute both the standard formulae	dM1
	$= \frac{1}{4}n(n+1)\left[n^2+n-6\right]$	{this step does not have to be written]	
	$= \frac{1}{4}n(n+1)(n+3)(n-2)$	Correct completion with no errors	A1 cso
			(4)
			(4 marks)

Notes:

Applying eg. $n=1,\ n=2,\ n=3$ to the printed equation without applying the standard formulae to give $a=1, b=3, c=-2$ or another combination of these numbers is M0A0M0A0.

Alternative Method:

Obtains $\sum_{r=1}^{n} r(r^2-3) \equiv \frac{1}{4}n(n+1)\left[n(n+1)-6\right] \equiv \frac{1}{4}n(n+a)(n+b)(n+c)$

So $a=1$. $n=1 \Rightarrow -2 = \frac{1}{4}(1)(2)(1+b)(1+c)$ and $n=2 \Rightarrow 0 = \frac{1}{4}(2)(3)(2+b)(2+c)$

leading to either $b=-2, c=3$ or $b=3, c=-2$

dM1: **dependent on the previous M mark.**
Substitutes in values of n and solves to find $b = \ldots$ and $c = \ldots$

A1: Finds $a=1, b=3, c=-2$ or another combination of these numbers.

Using **only** a method of "proof by induction" scores 0 marks unless there is use of the standard formulae when the first M1 may be scored.

Allow final dM1A1 for $\frac{1}{4}n^4 + \frac{1}{2}n^3 - \frac{5}{4}n^2 - \frac{3}{2}n$ or $\frac{1}{4}n(n^3+2n^2-5n-6)$

or $\frac{1}{4}(n^4+2n^3-5n^2-6n) \rightarrow \frac{1}{4}n(n+1)(n+3)(n-2)$, from no incorrect working.

Give final A0 for eg. $\frac{1}{4}n(n+1)\left[n^2+n-6\right] \rightarrow = \frac{1}{4}n(n+1)(x+3)(x-2)$ unless recovered.

Q33.

Question Number	Scheme	Notes	Marks
	$\sum_{r=1}^{n} r^2(r+2)=\sum_{r=1}^{n} r^3+2\sum_{r=1}^{n} r^2$ or $\sum_{r=1}^{n} r^3+\sum_{r=1}^{n} 2r^2$	Correct split with 2 summations. Could be implied by correct work. Condone missing or incorrect summation limits.	B1
	$=\frac{1}{4}n^2(n+1)^2+2\times\frac{1}{6}n(n+1)(2n+1)$	Attempts to use both standard results and obtains an expression of the form $pn^2(n+1)^2+qn(n+1)(2n+1)$ $p,\ q\neq 0$ Could be implied by immediate expansion	M1
	$=\frac{1}{12}n(n+1)[3n(n+1)+4(2n+1)]$ $=\frac{1}{12}n(n+1)(3n^2+11n+4)$	dM1: Attempts factorisation to obtain $\frac{1}{12}n(n+1)(an^2+bn+c)$ $a,b,c\neq 0$. Condone poor algebra. Could follow cubic or quartic. Allow a consistent $a=$ ..., $b=$ ..., $c=$ ... if quadratic never seen simplified **Requires previous M mark.** A1: Correct expression or $a=3,\ b=11,\ c=4$ Allow e.g., $\frac{1}{12}n(n+1)$ written as $\frac{n}{12}(n+1)$	dM1 A1
	Note: $n(n+1)(3n^2+11n+4)=3n^4+14n^3+15n^2+4n$		**Total 4**

Q34.

Question	Scheme		Marks
(a)	$\frac{1}{(r+6)(r+8)}$		
	$\frac{1}{2(r+6)}-\frac{1}{2(r+8)}$ oe	Correct partial fractions, any equivalent form	B1
			(1)
(b)	$=\left(2\times\frac{1}{2}\right)\left(\frac{1}{7}-\frac{1}{9}+\frac{1}{8}-\frac{1}{10}+\frac{1}{9}-\frac{1}{11}...+\frac{1}{n+5}-\frac{1}{n+7}+\frac{1}{n+6}-\frac{1}{n+8}\right)$ Expands at least 3 terms at start and 2 at end (may be implied) The partial fractions obtained in (a) can be used without multiplying by 2. Fractions may be $\frac{1}{2}\times\frac{1}{7}-\frac{1}{2}\times\frac{1}{9}$ etc These comments apply to both M1 and A1		M1
	$=\frac{1}{7}+\frac{1}{8}-\frac{1}{n+7}-\frac{1}{n+8}$	Identifies the terms that do not cancel	A1
	$=\frac{15(n+7)(n+8)-56(2n+15)}{56(n+7)(n+8)}$	Attempt common denominator Must have multiplied the fractions from (a) by 2 now	M1
	$=\frac{n(15n+113)}{56(n+7)(n+8)}$		A1 cso
			(4)
			(5 marks)

Q35.

Question Number	Scheme	Marks
(a)	$\dfrac{r+2}{r(r+1)}-\dfrac{r+3}{(r+1)(r+2)}=\dfrac{(r+2)^2-r(r+3)}{r(r+1)(r+2)}$	M1
	$=\dfrac{r^2+4r+4-r^2-3r}{r(r+1)(r+2)}=\dfrac{r+4}{r(r+1)(r+2)}$ *	A1* (2)
(b)	$r=1 \quad \dfrac{3}{1\times 2}-\dfrac{4}{2\times 3}$ $\qquad r=n-1 \quad \dfrac{n+1}{(n-1)n}-\dfrac{n+2}{n(n+1)}$	
	$r=2 \quad \dfrac{4}{2\times 3}-\dfrac{5}{3\times 4}$ $\qquad r=n \quad \dfrac{n+2}{n(n+1)}-\dfrac{n+3}{(n+1)(n+2)}$	M1
	$r=3 \quad \dfrac{5}{3\times 4}-\dfrac{6}{4\times 5}$	
	$\sum_{r=1}^{n}\dfrac{r+4}{r(r+1)(r+2)}=\dfrac{3}{2}-\dfrac{n+3}{(n+1)(n+2)}$	A1
	$\sum_{r=1}^{n}\dfrac{r+4}{r(r+1)(r+2)}=\dfrac{3(n+1)(n+2)-2n-6}{2(n+1)(n+2)}=\dfrac{n(3n+7)}{2(n+1)(n+2)}$	dM1 A1cao (4) [6]
(a) M1	Attempt a single fraction with the correct denominator (or 2 separate fractions with the correct common denominator)	
A1*	Correct result obtained with no errors in the working. Must include LHS as shown in question or LHS = ...	
(b) M1	Show sufficient terms to demonstrate the cancelling, min 3 at start and 1 at end or 2 at start and 2 at end. Award by implication if the correct 2 remaining terms are seen	
A1	Extract the correct 2 remaining terms	
dM1	Attempt common denominator of the form $k(n+1)(n+2)$	
A1cao	Correct result obtained. No need to show a, b and c explicitly.	

Q36.

Question Number	Scheme	Marks
(a)	$\dfrac{2}{r(r+1)(r-1)}=\dfrac{1}{r-1}-\dfrac{2}{r}+\dfrac{1}{r+1}$	M1A1A1 (3)
(b)	$r=2 \quad 1-\dfrac{2}{2}+\dfrac{1}{3}$	
	$r=3 \quad \dfrac{1}{2}-\dfrac{2}{3}+\dfrac{1}{4}$	
	$r=4 \quad \dfrac{1}{3}-\dfrac{2}{4}+\dfrac{1}{5}$	M1
	$r=n-1 \quad \dfrac{1}{n-2}-\dfrac{2}{n-1}+\dfrac{1}{n}$	
	$r=n \quad \dfrac{1}{n-1}-\dfrac{2}{n}+\dfrac{1}{n+1}$	M1
	$\sum_{r=2}^{n}\left(\dfrac{1}{r-1}-\dfrac{2}{r}+\dfrac{1}{r+1}\right)=\left(1-\dfrac{2}{2}+\dfrac{1}{2}+\dfrac{1}{n}-\dfrac{2}{n}+\dfrac{1}{n+1}\right)$	A1
	$\dfrac{1}{2}\sum_{r=1}^{n}\dfrac{2}{r(r+1)(r-1)}=\dfrac{1}{2}\times\left(\dfrac{1}{2}-\dfrac{1}{n}+\dfrac{1}{n+1}\right)=\dfrac{n^2+n-2}{4n(n+1)}$	dM1A1 (4)
		[7]

(a)	
M1	Attempt PFs by any valid method (by implication if 3 correct fractions seen)
A1A1	A1 any 2 fractions correct; A1 third fraction correct
(b)	Method of differences with at least 3 terms at start and 2 at end OR 2 at start and 3 at end.
M1	Must start at 2 and end at n One M mark for the initial terms and a second for the final.
M1	Last lines may be missing $k/(n-1)$ and $c/(n-2)$ These 2 M marks may be implied by a correct extraction of terms. If starting from 1, M0M1 can be awarded.
A1	Extract the remaining terms. $1-2/2$ may be missing and $1/n-2/n$ may be combined
dM1	Include the 1/2 and attempt a common denominator of the required form. Depends on both previous M marks
A1	$\dfrac{n^2+n-2}{4n(n+1)}$

Question Number	Scheme	Marks
(a)	**Special Case:** $\dfrac{2}{r(r^2-1)}=\dfrac{2r}{r^2-1}-\dfrac{2}{r}$ seen, award M1A1A0 Award M1A0A0 provided of the form $\dfrac{2}{r(r^2-1)}=\dfrac{Ar}{r^2-1}-\dfrac{B}{r}$	
(b)	Terms listed as described above – award M1M1. Further progress unlikely as too many terms needed to establish the cancellation.	

Q37.

Question Number	Scheme	Marks
(a)	$\dfrac{2}{\sqrt{r}+\sqrt{r-2}}\times\dfrac{\sqrt{r}-\sqrt{r-2}}{\sqrt{r}-\sqrt{r-2}}$	M1
	$\dfrac{2\left(\sqrt{r}-\sqrt{r-2}\right)}{r-(r-2)}=\sqrt{r}-\sqrt{r-2}$ *	A1*
		(2)
(b)	$r=2:\ \sqrt{2}-\sqrt{2-2}\ \left(=\sqrt{2}-0\right)$ $\quad\ldots r=n-2:\ \sqrt{n-2}-\sqrt{n-4}$ $r=3:\ \sqrt{3}-\sqrt{3-2}\ \left(=\sqrt{3}-1\right)$ $\quad r=n-1:\ \sqrt{n-1}-\sqrt{n-3}$ $r=4:\ \sqrt{4}-\sqrt{4-2}\ \left(=2-\sqrt{2}\right)$ $\quad r=n:\ \sqrt{n}-\sqrt{n-2}$	M1
	$\left[\sum_{r=2}^{n}\dfrac{2}{\sqrt{r}+\sqrt{r-2}}=\right]\sqrt{n}+\sqrt{n-1}-1$	A1 A1
		(3)
(c)	$\left[\sum_{r=4}^{50}\dfrac{2}{\sqrt{r}+\sqrt{r-2}}=\right]\mathrm{f}(50)-\mathrm{f}(3)=\sqrt{50}+\sqrt{49}-1-\left(\sqrt{3}+\sqrt{2}-1\right)$	M1
	$\left(=5\sqrt{2}+7-1-\sqrt{3}-\sqrt{2}+1\right)=7+4\sqrt{2}-\sqrt{3}$	A1
		(2)
		Total 7

Notes
(a) M1: Indicates intention to multiply either side by a correct fraction, may use $\dfrac{\sqrt{r-2}-\sqrt{r}}{\sqrt{r-2}-\sqrt{r}}$. The "2" may be missing. May work in reverse from right hand side to left, and this is fine. Alternatively, may multiply the initial expression through by $\sqrt{r}+\sqrt{r-2}$ and use a sequence of equivalences (though accept with $\Rightarrow$ or nothing between lines). A1: Fully correct proof. A result from rationalisation that is not the given answer must be seen. If using a sequences of equivalences there must be a minimal conclusion. (b) M1: Correct process of differences evidenced in their work, e.g. attempts any three of the 6 expression shown. There should be enough evidence of at least one pair of cancelling terms. Ignore any attempts at any of $r = 0$ or 1 if they start earlier than $r = 2$. A1: Correct algebraic terms **or** correct constant term(s) extracted. Accept unsimplified expressions such as "$-\sqrt{1}-\sqrt{0}$" A1: Fully correct simplified expression. Must have simplified the $\sqrt{1}$ to 1 (c) M1: Attempts f(50) – f(3) using **their answer to (b)**. Must be indication of subtraction. The '-1's may be omitted, and allow if a slip is made. They may subtract the sum of their results from $r = 2$ and $r = 3$ from (b) or by using (a) again to obtain f(3) but their answer to (b) must be used for f(50). Allow from attempts starting at $r = 1$ in their summations. f(50) −f(4) is M0. A1: Correct expression or $A = 7$, $B = 4$, $C = -1$ **following a correct answer to (b)**. The decimal answer 10.92480344… without evidence of the M mark is 0/2

Q38.

Question	Scheme	Marks
(i)(a)	Reflection or in the line $y=-x$	M1
	Reflection in the line $y=-x$	A1
		(2)
(b)	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ or $6\times\begin{pmatrix} \pm\cos 240° & \pm\sin 240° \\ \pm\sin 240° & \pm\cos 240° \end{pmatrix}$	M1
	$\begin{pmatrix} -3 & 3\sqrt{3} \\ -3\sqrt{3} & -3 \end{pmatrix}$	A1
		(2)
(c)	$\mathbf{R}=\mathbf{QP}=\begin{pmatrix} -3 & 3\sqrt{3} \\ -3\sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}=\dots$	M1
	$=\begin{pmatrix} -3\sqrt{3} & 3 \\ 3 & 3\sqrt{3} \end{pmatrix}$ **QP** correctly found	A1
		(2)
(ii)	$\begin{pmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix}\begin{pmatrix} \lambda \\ 1 \end{pmatrix}=\begin{pmatrix} 4\lambda \\ 4 \end{pmatrix}\Rightarrow\begin{pmatrix} -2\lambda+2\sqrt{3} \\ 2\lambda\sqrt{3}+2 \end{pmatrix}=\begin{pmatrix} 4\lambda \\ 4 \end{pmatrix}$	M1
	$-2\lambda+2\sqrt{3}=4\lambda$ or $2\lambda\sqrt{3}+2=4$	A1
	$\Rightarrow 6\lambda=2\sqrt{3}\Rightarrow\lambda=\dots$ or $2\sqrt{3}\lambda=2\Rightarrow\lambda=\dots$	dM1
	$\lambda=\frac{\sqrt{3}}{3}$ or $\frac{1}{\sqrt{3}}$ oe	A1
	Both $-2\lambda+2\sqrt{3}=4\lambda$ and $2\lambda\sqrt{3}+2=4$ solved leading to $\lambda=\frac{\sqrt{3}}{3}$ or $\frac{1}{\sqrt{3}}$	A1
		(5)
		(11 marks)

(a)

M1: Identifies the transformation as a reflection or identifies the correct line of reflection.

A1: Fully correct description, with the equation of the line of reflection or suitable description (e.g. in the line through angle 135° with the positive x-axis). Ignore any references to a centre of reflection.

(b)

M1: Either the correct matrix for the rotation (with trig ratios evaluated) or an attempt at scaling a matrix of form shown by a factor 6 (need not evaluate ratio) – if no trig ratios seen this may be implied by the exact values in the right places. The correct answer implies the M.

A1: Correct matrix.

(c)

M1: Attempts to multiply **Q** and **P** in the correct order.

A1: QP correct

(ii)

M1: Attempts the product $\begin{pmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix}\begin{pmatrix} \lambda \\ 1 \end{pmatrix}$ and sets equal to $\begin{pmatrix} 4\lambda \\ 4 \end{pmatrix}$. Allow for poor notation as long as the intention is clear, and it may be implied by one correct equation or follow through equation.

A1: Extracts at least one correct equation (not part of the matrix equation). May be implied by correct value for λ following correct matrix equation.

dM1: Attempts to solve the equation. May be implied by the correct value following a correct matrix equation with no extraction of separate equations.

A1: Correct value for λ from at least one equation and isw if incorrectly simplified (allow if their second equation does not concur).

A1: Correct value for λ coming from both equations, solved explicitly, or checks the value of λ from the first equation works in the second equation.

Alt (ii)	$\begin{pmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix}\begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} 4\lambda \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \frac{1}{-4-12}\begin{pmatrix} 2 & -2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix}\begin{pmatrix} 4\lambda \\ 4 \end{pmatrix}$ $\Rightarrow \begin{pmatrix} \lambda \\ 1 \end{pmatrix} = -\frac{1}{16}\begin{pmatrix} 8\lambda - 8\sqrt{3} \\ -8\lambda\sqrt{3} - 8 \end{pmatrix}$	**M1**
	$2\lambda = \sqrt{3} - \lambda$ or $2 = \lambda\sqrt{3} + 1$	**A1**
	$\Rightarrow 3\lambda = \sqrt{3} \Rightarrow \lambda = ...$ or $\sqrt{3}\lambda = 1 \Rightarrow \lambda = ...$	**dM1**
	$\lambda = \frac{\sqrt{3}}{3}$ or $\frac{1}{\sqrt{3}}$	**A1**
	Both $2\lambda = \sqrt{3} - \lambda$ and $2 = \lambda\sqrt{3} + 1$ solved leading to $\lambda = \frac{\sqrt{3}}{3}$ or $\frac{1}{\sqrt{3}}$	**A1**
		(5)

Notes:

M1: Correct attempt at inverse, attempts the product $\begin{pmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix}^{-1} \begin{pmatrix} 4\lambda \\ 4 \end{pmatrix}$ and sets equal to $\begin{pmatrix} \lambda \\ 1 \end{pmatrix}$.

Allow for poor notation as long as the intention is clear, and it may be implied by one correct equation or follow through equation.

A1: Extracts at least one correct equation (not part of the matrix equation). May be implied by correct value for λ following correct matrix equation.

dM1: Attempts to solve the equation. May be implied by the correct value following a correct matrix equation with no extraction of separate equations.

A1: Correct value for λ from at least one equation (allow if their second equation does not concur).

A1: Correct value for λ coming from both equations, solved explicitly, or checks the value of λ from the first equation works in the second equation.

Q39.

Question	Scheme	Marks
(a)	Two of: Rotation; about O; through $60°$ $\left(\frac{\pi}{3}\right)$ (anticlockwise)	M1
	All of: Rotation about O through $60°$ $\left(\frac{\pi}{3}\right)$ (anticlockwise)	A1
		(2)
(b)	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	B1
		(1)
(c)	$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	M1
	$= \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ **QP** correctly found	A1

(d)	$3\mathbf{R}=\begin{pmatrix}-\frac{3\sqrt{3}}{2} & -\frac{3}{2}\\ -\frac{3}{2} & \frac{3\sqrt{3}}{2}\end{pmatrix}$ or correctly deals with 3 as a multiple.	B1ft
	Required matrix is $(3\mathbf{R})^{-1}=\frac{1}{\left(-\frac{3\sqrt{3}}{2}\right)\left(\frac{3\sqrt{3}}{2}\right)-\left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)}\begin{pmatrix}\frac{3\sqrt{3}}{2} & \frac{3}{2}\\ \frac{3}{2} & -\frac{3\sqrt{3}}{2}\end{pmatrix}=...$ Or $(\mathbf{R})^{-1}=\frac{1}{\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)}\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1\sqrt{3}}{2}\end{pmatrix}=...$	M1
	$(3\mathbf{R})^{-1}=\frac{1}{-9}\begin{pmatrix}\frac{3\sqrt{3}}{2} & \frac{3}{2}\\ \frac{3}{2} & -\frac{3\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}-\frac{\sqrt{3}}{6} & -\frac{1}{6}\\ -\frac{1}{6} & \frac{\sqrt{3}}{6}\end{pmatrix}$	A1
		(3)
		(8 marks)

Notes:
(a) **M1:** Two aspects of the type, centre of rotation and angle correct. Accept equivalent angles or angle in radians. (E.g. 300° clockwise is fine). Assume anticlockwise unless otherwise stated. **A1:** Fully correct description. Accept just 60° for the angle, but 60° clockwise is incorrect (b) **B1:** Correct matrix. (c) **M1:** Attempts to multiply **Q** and **P** in the correct order. **A1:** **QP** correct (d) **B1ft:** Multiplies all elements of their matrix by 3, or multiplies all elements of their $\mathbf{R}^{-1}$ by $\frac{1}{3}$ **M1:** Attempts the inverse of their 3**R** or **R**. This must be a complete method – ie must transpose and evaluate the determinant and use it. Alternatively, they may attempt an inverse from first principles. Award this mark if a slip is made in solving their simultaneous equations. **A1:** Correct answer. Accept alternative forms

Q40.

Question Number	Scheme	Notes	Marks
(a)	$\lvert z+1-13\text{i}\rvert = 3\lvert z-7-5\text{i}\rvert \Rightarrow (x+1)^2+(y-13)^2 = 9\{(x-7)^2+(y-5)^2\}$ Correct application of Pythagoras Accept 3 or 9 on RHS		M1
	$\Rightarrow x^2+y^2-16x-8y+62=0$	Correct equation in any form with terms collected	A1
	Centre (8, 4)	Correct centre. i included scores A0	A1
	$r^2 = 64+16-62=...$	Correct method for r or r^2	M1
	$r=\sqrt{18}$ or $3\sqrt{2}$	Correct radius. Must be exact.	A1
			(5)
(b)	$\arg(z-8-6\text{i}) = -\frac{3\pi}{4} \Rightarrow y-6=x-8$	Converts the given locus to the correct Cartesian form	B1
	$\Rightarrow x^2+y^2-16x-8y+62=0$ $\Rightarrow x^2+(x-2)^2-16x-8(x-2)+62=0 \Rightarrow x=...$ or $\Rightarrow (y+2)^2+y^2-16x-8(y+2)+62=0 \Rightarrow y=...$	Uses both Cartesian equations to obtain an equation in one variable and attempts to solve	M1
	$x=7-2\sqrt{2}$ or $y=5-2\sqrt{2}$	One correct "coordinate"	A1
	R is $7-2\sqrt{2}+(5-2\sqrt{2})\text{i}$ **or** $x=7-2\sqrt{2}$ and $y=5-2\sqrt{2}$	Correct complex number or coordinates and no others. Must be exact	A1
			(4)
			Total 9

Q41.

Question	Scheme		Marks
(a)	$\mathbf{A} = \begin{pmatrix} k & 3 \\ -1 & k+2 \end{pmatrix}$ where k is a constant and let $g(k) = k^2 + 2k + 3$		
	$\{\det(\mathbf{A}) =\}\ k(k+2)+3$ or k^2+2k+3	Correct $\det(\mathbf{A})$, un-simplified or simplified	B1
	Way 1		
	$= (k+1)^2 - 1 + 3$	Attempts to complete the square [usual rules apply]	M1
	$= (k+1)^2 + 2 > 0$	$(k+1)^2 + 2$ and > 0	A1 cso
			(3)
	Way 2		
	$\{\det(\mathbf{A}) =\}\ k(k+2)+3$ or k^2+2k+3	Correct $\det(\mathbf{A})$, un-simplified or simplified	B1
	$\{b^2 - 4ac =\}\ 2^2 - 4(1)(3)$	Applies "$b^2 - 4ac$" to their $\det(\mathbf{A})$	M1
	All of • $b^2 - 4ac = -8 < 0$ • some reference to k^2+2k+3 being above the x-axis • so $\det(\mathbf{A}) > 0$	Complete solution	A1 cso
			(3)
	Way 3		
	$\{g(k) = \det(\mathbf{A}) =\}\ k(k+2)+3$ or k^2+2k+3	Correct $\det(\mathbf{A})$, un-simplified or simplified	B1
	$g'(k) = 2k+2 = 0 \Rightarrow k = -1$ $g_{\min} = (-1)^2 + 2(-1) + 3$	Finds the value of k for which $g'(k) = 0$ and substitutes this value of k into $g(k)$	M1
	$g_{\min} = 2$, so $\det(\mathbf{A}) > 0$	$g_{\min} = 2$ and states $\det(\mathbf{A}) > 0$	A1 cso
			(3)

(b)	$\mathbf{A}^{-1} = \dfrac{1}{k^2+2k+3}\begin{pmatrix} k+2 & -3 \\ 1 & k \end{pmatrix}$	$\dfrac{1}{\text{their } \det(\mathbf{A})}\begin{pmatrix} k+2 & -3 \\ 1 & k \end{pmatrix}$	M1
		Correct answer in terms of k	A1
			(2)
			(5 marks)

Notes:
(a) **B1:** Also allow $k(k+2)--3$ **Way 2:** Proving $b^2-4ac=-8<0$ by itself could mean that $\det(\mathbf{A})>0$ or $\det(\mathbf{A})<0$. To gain the final A1 mark for Way 2, candidates need to show $b^2-4ac=-8<0$ **and** make some reference to k^2+2k+3 being above the x-axis (eg. states that coefficient of k^2 is positive **or** evaluates $\det(\mathbf{A})$ for any value of k to give a positive result **or** sketches a quadratic curve that is above the x-axis) before then stating that $\det(\mathbf{A})>0$. Attempting to solve $\det(\mathbf{A})=0$ by applying the quadratic formula or finding $-1\pm\sqrt{2}\mathrm{i}$ is enough to score the M1 mark for Way 2. To gain A1 these candidates need to make some reference to k^2+2k+3 being above the x-axis (eg. states that coefficient of k^2 is positive **or** evaluates $\det(\mathbf{A})$ for any value of k to give a positive result **or** sketches a quadratic curve that is above the x-axis) before then stating that $\det(\mathbf{A})>0$.
(b) **A1:** Allow either $\dfrac{1}{(k+1)^2+2}\begin{pmatrix} k+2 & -3 \\ 1 & k \end{pmatrix}$ or $\begin{pmatrix} \dfrac{k+2}{k^2+2k+3} & \dfrac{-3}{k^2+2k+3} \\ \dfrac{1}{k^2+2k+3} & \dfrac{k}{k^2+2k+3} \end{pmatrix}$ or equivalent.

Q42.

Question	Scheme		Marks
(a)	Rotation	Rotation	B1
	67 degrees (anticlockwise)	Either $\arctan\left(\frac{12}{5}\right)$, $\tan^{-1}\left(\frac{12}{5}\right)$, $\sin^{-1}\left(\frac{12}{13}\right)$, $\cos^{-1}\left(\frac{5}{13}\right)$, awrt 67 degrees, awrt 1.2, truncated 1.1 (anticlockwise), awrt 293 degrees clockwise or awrt 5.1 clockwise	B1 o.e.
	about $(0, 0)$	**The mark is dependent on at least one of the previous B marks being awarded.** About $(0, 0)$ or about O or about the origin	dB1
	Note: Give 2nd B0 for 67 degrees clockwise o.e.		**(3)**
(b)	$\{\mathbf{Q} =\}\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Correct matrix	B1
			(1)
(c)	$\{\mathbf{R} = \mathbf{PQ}\}\ \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; = \begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$	Multiplies **P** by their **Q** in the correct order and finds at least one element	M1
		Correct matrix	A1
			(2)

(d)	Way 1		
	$\begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}\begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} x \\ kx \end{pmatrix}$	Forming the equation "their matrix **R**" $\begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} x \\ kx \end{pmatrix}$ **Allow x being replaced by any non-zero number eg. 1.** Can be implied by at least one correct ft equations below.	M1
	$-\dfrac{12}{13}x + \dfrac{5kx}{13} = x$ or $\dfrac{5}{13}x + \dfrac{12kx}{13} = kx \Rightarrow k = ...$	Uses their matrix equation to form an equation in k and progresses to give $k =$ numerical value	M1
	So $k = 5$	**dependent on only the previous M mark** $k = 5$	A1 cao
	Dependent on all previous marks being scored in this part. Either • Solves both $-\dfrac{12}{13}x + \dfrac{5kx}{13} = x$ and $\dfrac{5}{13}x + \dfrac{12kx}{13} = kx$ to give $k = 5$ • Finds $k = 5$ and checks that it is true for the other component • Confirms that $\begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}\begin{pmatrix} x \\ 5x \end{pmatrix} = \begin{pmatrix} x \\ 5x \end{pmatrix}$		A1 cso
			(4)

(d) *continued*	Way 2		
	Either $\cos 2\theta = -\dfrac{12}{13}$, $\sin 2\theta = \dfrac{5}{13}$ or $\tan 2\theta = -\dfrac{5}{12}$	Correct follow through equation in 2θ based on their matrix **R**	M1
	$\{k =\}\ \tan\left(\dfrac{1}{2}\arccos\left(-\dfrac{12}{13}\right)\right)$	Full method of finding 2θ, then θ and applying $\tan\theta$	M1
		$\tan\left(\dfrac{1}{2}\arccos\left(-\dfrac{12}{13}\right)\right)$ or $\tan(\text{awrt } 78.7^\circ)$ or $\tan(\text{awrt } 1.37)$. **Can be implied.**	A1
	So $k = 5$	**$k = 5$ by a correct solution only**	A1
			(4)
			(10 marks)

Notes:

(a)

Condone "Turn" for the 1st B1 mark.

Penalise the first B1 mark for candidates giving a combination of transformations.

(c)

Allow 1st M1 for eg. "their matrix **R**" $\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$ or "their matrix **R**" $\begin{pmatrix} k \\ k^2 \end{pmatrix} = \begin{pmatrix} k \\ k^2 \end{pmatrix}$

or "their matrix **R**" $\begin{pmatrix} \frac{1}{k} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{k} \\ 1 \end{pmatrix}$ or equivalent

$y = (\tan\theta)x : \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = \begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$

Q43.

Question Number	Scheme	Notes	Marks
(a)	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 3 & 1 \\ -2 & -2 & 0 \end{pmatrix}$	Attempt to multiply in the correct order with at least four correct elements. May be done as three separate calculations, so look for at least 4 correct values.	M1
	$(1,-2)$, $(3,-2)$ and $(1,0)$	Accept as individual column vectors but not as a single 2x3 matrix.	A1
			(2)
(b)	Rotation	Accept rotate or turn oe.	B1
	270° (anticlockwise) about the origin	Accept $-90°$ (anticlockwise) or 90° clockwise (must be stated) and (0,0) or O. *Assume anticlockwise unless otherwise stated.*	B1
			(2)
(c)	$\mathbf{Q}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{R}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	One correct, both correct	B1,B1
			(2)
(d)	$\mathbf{RQ}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Multiplication in correct order for their matrices and at least 1 row or 1 column correct.	M1
	$=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Correct matrix	A1
			(2)
(e)	Reflection	Correct type identified.	B1
	in (the line) $y = x$	Correct line of reflection specified, accepting equivalent forms (e.g. line at angle 45° (anticlockwise) to the (positive) x-axis).	B1
			(2)
			Total 10

Q44.

Question Number	Scheme	Notes	Marks
(a)	$\mathbf{A}^{-1} = \frac{1}{3\times -2 - a\times -2}\begin{pmatrix} -2 & -a \\ 2 & 3 \end{pmatrix}$	Complete method for the inverse. Allow slips in the determinant and at most one error in the adjoint matrix.	M1
	$= \frac{1}{2a-6}\begin{pmatrix} -2 & -a \\ 2 & 3 \end{pmatrix}$	Correct inverse	A1
			(2)
(b)	$\mathbf{A}+\mathbf{A}^{-1}=\mathbf{I} \Rightarrow \begin{pmatrix} 3 & a \\ -2 & -2 \end{pmatrix} + \frac{1}{2a-6}\begin{pmatrix} -2 & -a \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Sets up the correct matrix equation with their $\mathbf{A}^{-1}$. The identity may be left as **I** as long at least one equation is processed correctly and no equation implies an incorrect identity matrix.	M1
	$3+\frac{2}{6-2a}=1,\ a+\frac{a}{6-2a}=0,\ -2+\frac{2}{2a-6}=0,\ -2+\frac{3}{2a-6}=1 \Rightarrow a=...$ Uses one of the elements to set up a suitable equation and solves for a. Allow a sign slip in the $6-2a$ but have correct coefficient of **I**		dM1
	$a=\frac{7}{2}$ oe	Correct value and no others	A1
			(3)
			Total 5

Q45.

Question Number	Scheme	Notes	Marks
	$\mathbf{A}=\begin{pmatrix}-3 & 8\\ -3 & k\end{pmatrix}$ $\quad\mathbf{B}=\begin{pmatrix}a & -4\\ 2 & 3\end{pmatrix}$ $\quad\mathbf{BC}=\begin{pmatrix}2 & 5 & 1\\ 1 & 4 & 2\end{pmatrix}$		
(i)	$\det\mathbf{A}=-3k-8(-3)\ \{=-3k+24\}$ Could be implied	Attempts det **A** and obtains $\pm3k\pm8(\pm3)$ or $\pm3k\pm24$	M1
	$-3k+24=3$ or $-3k+24=-3$ $\Rightarrow k=\dots$ May see $(-3k+24)^2=+9\Rightarrow 9k^2-144k+567=0\Rightarrow\dots$	Equates their det **A** of form $ak+b$ $a,b\neq0$ to 3 or -3 or equivalent work and solves for k (usual rules if quadratic and must use +9)	M1
	$k=7,\ k=9$ 1st A1: Either correct value of k from correct work. Allow e.g., $\frac{-21}{-3}$ or $\frac{-27}{-3}$ 2nd A1: Both correct values of k from correct work. 7 and 9 only. No extra		A1 A1
			(4)
(ii)	$\det\mathbf{B}=1\times3a-(-4)\times2\ \{=3a+8\}$	Correct unsimplified expression for det **B**. Could be implied	B1
	$\mathbf{B}^{-1}=\frac{1}{"3a+8"}\begin{pmatrix}3 & 4\\ -2 & a\end{pmatrix}$	Correct $\mathbf{B}^{-1}$ with their det **B**. Adj(**B**) to be correct but allow elements to have their det **B** as denominators if incorporated.	M1
	$\mathbf{C}=\mathbf{B}^{-1}\mathbf{BC}=\frac{1}{3a+8}\begin{pmatrix}3 & 4\\ -2 & a\end{pmatrix}\begin{pmatrix}2 & 5 & 1\\ 1 & 4 & 2\end{pmatrix}=\dots$ Access to this mark is allowed if there is no determinant or if $\mathbf{B}^{-1}=$ det **B** × Adj(**B**) used	Multiplies **BC** by their $\mathbf{B}^{-1}$ (changed – and not just by incorporation of their determinant) **the correct way round.** Expect four correct elements for their matrices if the method is unclear. The incorrect order scores M0 even if the correct result is obtained.	M1
	$\mathbf{C}=\frac{1}{3a+8}\begin{pmatrix}10 & 31 & 11\\ a-4 & 4a-10 & 2a-2\end{pmatrix}$ Ignore any reference to inapplicable values of a $\left(a\neq-\frac{8}{3}\right)$	Correct **C** or equivalent with like terms collected and single fractions if necessary. e.g., $\begin{pmatrix}\frac{10}{3a+8} & \frac{31}{3a+8} & \frac{11}{3a+8}\\ \frac{a-4}{3a+8} & \frac{2(2a-5)}{3a+8} & \frac{2(a-1)}{3a+8}\end{pmatrix}$	A1
			(4)

Alt **Sim. equations**	$\begin{pmatrix}a & -4\\ 2 & 3\end{pmatrix}\begin{pmatrix}p & q & r\\ s & t & u\end{pmatrix}=\begin{pmatrix}2 & 5 & 1\\ 1 & 4 & 2\end{pmatrix}\Rightarrow \begin{matrix}ap-4s=2 & aq-4t=5 & ar-4u=1\\ 2p+3s=1 & 2q+3t=4 & 2r+3u=2\end{matrix}$ Multiplies in the correct order to obtain at least **three** correct equations	B1
	$(3a+8)p=10 \quad (3a+8)q=31 \quad (3a+8)r=11$ $p=\frac{10}{3a+8} \quad q=\frac{31}{3a+8} \quad r=\frac{11}{3a+8}$ $s=\frac{1}{3}\left(1-\frac{20}{3a+8}\right) \quad t=\frac{1}{3}\left(4-\frac{62}{3a+8}\right) \quad u=\frac{1}{3}\left(2-\frac{22}{3a+8}\right)$ $s=\frac{a-4}{3a+8} \quad t=\frac{4a-10}{3a+8} \quad u=\frac{2a-2}{3a+8}$ $\Rightarrow\begin{pmatrix}\frac{10}{3a+8} & \frac{31}{3a+8} & \frac{11}{3a+8}\\ \frac{a-4}{3a+8} & \frac{4a-10}{3a+8} & \frac{2a-2}{3a+8}\end{pmatrix}$ M1: Solves their equations to find expressions in terms of a for three elements M1: Finds expressions in terms of a for all six elements A1: Correct matrix – like terms collected and single fractions	M1 M1 A1
		Total 8

Q46.

Question Number	Scheme	Notes	Marks
(i)(a)	$\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$	Correct matrix	B1
			(1)
(b)	$\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$	Correct matrix	B1
			(1)
(c)	$\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$	Attempt to multiply the right way round. Implied by a correct answer (for their (a) and (b)) if no working is shown, but M0 if incorrect with no working.	M1
	$\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{5}{2} & -\frac{5\sqrt{3}}{2} \end{pmatrix}$	Correct matrix	A1
			(2)
(ii)(a)	$\begin{vmatrix} k & k+3 \\ -5 & 1-k \end{vmatrix} = k(1-k)-(-5)(k+3)$	Correct method for the determinant. (Allow miscopy slips only. So $k(1-k)-5(k+3)$ is M0 without further evidence.)	M1
	$= -k^2+6k+15$	Correct simplified expression	A1
			(2)
(b)	$-k^2+6k+15=\frac{16k}{2} \Rightarrow k=...$ or $-k^2+6k+15=-\frac{16k}{2} \Rightarrow k=...$	Correct strategy for establishing at least one value for k	M1
	One of $k=-5,\ 3,\ -1,\ 15$	Any one correct value. Note that the negative values may be rejected here.	A1
	$k=3$ **and** $k=15$ or $k=-5,\ 3$ **and** $k=-1,\ 15$	Both correct positive values and no others. Condone the inclusion of the negative values if given.	A1
			(3)
			Total 9

Q47.

Question Number	Scheme	Notes	Marks
(a)	$\mathbf{A}^2 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	Correct matrix	B1
			(1)
(b)	Rotation $-60°$ (anticlockwise) about the origin	Rotation	M1
		$-60°$ (anticlockwise) (Or $60°$ clockwise or $300°$ (anticlockwise)) about (0, 0)	A1
			(2)
(c)	$n = 12$	Cao but can be embedded ie $A^{12} = I$	B1
			(1)
(d)	$\mathbf{B} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$	Correct matrix	B1
			(1)
(e)	$\mathbf{C} = \mathbf{BA} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$	Multiplies the right way round.	M1
	$\mathbf{C} = \begin{pmatrix} -2\sqrt{3} & -2 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$	Correct matrix Accept unsimplified	A1
			(2)
(f)	$\det \mathbf{C} = -2\sqrt{3} \times -\frac{\sqrt{3}}{2} - \frac{1}{2}(-2) = 4$ So area of P is $\frac{20}{\det \mathbf{C}} = \ldots$	Attempts determinant of **C** (or deduces area scale factor is 4) and divides into 20	M1
	$= 5$	Cao Must follow a correct matrix in (e)	A1
			(2)
			Total 9

Q48.

Question Number	Scheme	Notes	Marks
(a)	$\mathbf{M}^{-1}=\frac{1}{5k-3k}\begin{pmatrix}5 & -k\\ -3 & k\end{pmatrix}$	Attempts $\mathbf{M}^{-1}=\frac{1}{\det\mathbf{M}}\times\text{adj}(\mathbf{M})$ Either part correct but $\text{adj}(\mathbf{M})=\mathbf{M}$ scores M0	M1
	$=\frac{1}{2k}\begin{pmatrix}5 & -k\\ -3 & k\end{pmatrix}$ or $\begin{pmatrix}\frac{5}{2k} & -\frac{1}{2}\\ \frac{-3}{2k} & \frac{1}{2}\end{pmatrix}$	Correct matrix $2k$ must be seen for this mark	A1
			(2)
(b)	$(\mathbf{MN})^{-1}=\mathbf{N}^{-1}\mathbf{M}^{-1}=\frac{1}{2k}\begin{pmatrix}k & k\\ 4 & -1\end{pmatrix}\begin{pmatrix}5 & -k\\ -3 & k\end{pmatrix}$	Applies $(\mathbf{MN})^{-1}=\mathbf{N}^{-1}\mathbf{M}^{-1}$	M1
	$=\frac{1}{2k}\begin{pmatrix}2k & 0\\ 23 & -5k\end{pmatrix}$ or e.g. $\begin{pmatrix}1 & 0\\ \frac{23}{2k} & \frac{-5}{2}\end{pmatrix}$	Correct matrix	A1
			(2)
ALT (b)	Find **N** (ie inverse of $\mathbf{N}^{-1}$) Find **MN** $=-\frac{1}{5k}\begin{pmatrix}-5k & 0\\ -23 & 2k\end{pmatrix}$ Find $(\mathbf{MN})^{-1}$	Complete method needed	M1
	$=\frac{1}{2k}\begin{pmatrix}2k & 0\\ 23 & -5k\end{pmatrix}$ or e.g. $\begin{pmatrix}1 & 0\\ \frac{23}{2k} & \frac{-5}{2}\end{pmatrix}$	Correct matrix	A1
			(2)
			Total 4

Q49.

Question Number	Scheme	Notes	Marks
(a)	$n^5-(n-1)^5=n^5-\left(n^5-5n^4+10n^3-10n^2+5n-1\right)=...$ Starts the proof by expanding the bracket		M1
	$5n^4-10n^3+10n^2-5n+1$*	Correct proof with no errors. Full expansion of $(n-1)^5$ must be shown.	A1*
			(2)
(b)	$1^5-0^5=5(1)^4-10(1)^3+10(1)^2-5(1)+1$ $2^5-1^5=5(2)^4-10(2)^3+10(2)^2-5(2)+1$ .... $(n-1)^5-(n-2)^5=5(n-1)^4-10(n-1)^3+10(n-1)^2-5(n-1)+1$ $(n)^5-(n-1)^5=5(n)^4-10(n)^3+10(n)^2-5(n)+1$ $n^5=5\sum_{r=1}^{n}r^4-10\sum_{r=1}^{n}r^3+10\sum_{r=1}^{n}r^2-5\sum_{r=1}^{n}r+n$ M1: Applies the result from part (a) between 1 and n and sums **both** sides Min 3 lines shown A1: Correct equation If only the last line is seen, award M1A1 These marks can be implied by a correct following stage.		M1A1
	$n^5=5\sum_{r=1}^{n}r^4-10\times\frac{1}{4}n^2(n+1)^2+10\times\frac{1}{6}n(n+1)(2n+1)-5\times\frac{1}{2}n(n+1)+n$ M1: Introduces at least 2 correct summation formulae A1: Correct equation		M1A1
	$5\sum_{r=1}^{n}r^4=\frac{5}{2}n^2(n+1)^2-\frac{5}{3}n(n+1)(2n+1)+\frac{5}{2}n(n+1)+n^5-n=...$ $5\sum_{r=1}^{n}r^4=n(n+1)\left[\frac{5}{2}n(n+1)-\frac{5}{3}(2n+1)+\frac{5}{2}+n^3-n^2+n-1\right]$ Makes $5\sum_{r=1}^{n}r^4$ or $\sum_{r=1}^{n}r^4$ the subject and takes out a factor of $n(n+1)$		M1
	$\sum_{r=1}^{n}r^4=\frac{1}{30}n(n+1)\left[15n(n+1)-10(2n+1)+15+6\left(n^3-n^2+n-1\right)\right]$ $=\frac{1}{30}n(n+1)\left[6n^3+9n^2+n-1\right]=\frac{1}{30}n(n+1)(2n+1)(...)$ Takes out a factor of $n(n+1)(2n+1)$ Depends on all previous method marks		dM1
	$=\frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right)$	cao	A1
			(7)
			Total 9

Q50.

Question	Scheme		Marks
(a)	$\mathrm{f}(x)=x^2+\dfrac{3}{x}-1,\quad x<0$		
	$\mathrm{f}'(x)=2x-3x^{-2}$	At one of either $x^2 \to \pm Ax$ or $\dfrac{3}{x} \to \pm Bx^{-2}$ where A and B are non-zero constants.	M1
		Correct differentiation	A1
	$\mathrm{f}(-1.5)=-0.75$, $\mathrm{f}'(-1.5)=-\dfrac{13}{3}$	Either $\mathrm{f}(-1.5)=-0.75$ or $\mathrm{f}'(-1.5)=-\dfrac{13}{3}$ or awrt -4.33 **or a correct numerical expression for** either $\mathrm{f}(-1.5)$ or $\mathrm{f}'(-1.5)$ **Can be implied by later working**	B1
	$\left\{\alpha \simeq -1.5-\dfrac{\mathrm{f}(-1.5)}{\mathrm{f}'(-1.5)}\right\} \Rightarrow \alpha \simeq -1.5-\dfrac{-0.75}{-4.333333...}$	**dependent on the previous M mark** Valid attempt at Newton-Raphson using their values of $\mathrm{f}(-1.5)$ and $\mathrm{f}'(-1.5)$	dM1
	$\left\{\alpha=-1.67307692...\text{ or }-\dfrac{87}{52}\right\} \Rightarrow \alpha=-1.67$	**dependent on all 4 previous marks** -1.67 on their first iteration (Ignore any subsequent iterations)	A1 cso cao
	Correct differentiation followed by a correct answer scores full marks in (a) **Correct answer with <u>no</u> working scores no marks in (a)**		
			(5)

(b)	**Way 1**		
	$\mathrm{f}(-1.675)=\ \ 0.01458022...$ $\mathrm{f}(-1.665)=-0.0295768...$	Chooses a suitable interval for x, which is within ± 0.005 of their answer to (a) and at least one attempt to evaluate $\mathrm{f}(x)$.	M1
	Sign change (positive, negative) (and $\mathrm{f}(x)$ is continuous) therefore (a root) $\alpha=-1.67$ (2 dp)	Both values correct awrt (or truncated) 1 sf, sign change and conclusion.	A1 cso
			(2)

(b) *continued*	**Way 2**		
	Alt 1: Applying Newton-Raphson again Eg. Using $\alpha=-1.67, -1.673$ or $-\dfrac{87}{52}$		
	• $\alpha \simeq -1.67-\dfrac{-0.007507185629...}{-4.415692926...}\ \{=-1.671700115...\}$ • $\alpha \simeq -1.673-\dfrac{0.005743106396...}{-4.41783855...}\ \{=-1.671700019...\}$ • $\alpha \simeq -\dfrac{87}{52}-\dfrac{0.006082942257...}{-4.417893838...}\ \{=-1.67170036...\}$	Evidence of applying Newton-Raphson for a second time on their answer to part (a)	M1
	So $\alpha=-1.67$ (2 dp)	$\alpha=-1.67$	A1
			(2)
			(7 marks)

Notes:

(a)

Incorrect differentiation followed by their estimate of α with no evidence of applying the NR formula is final dM0A0.

B1: B1 can be given for a correct numerical expression for either $\mathrm{f}(-1.5)$ or $\mathrm{f}'(-1.5)$

Eg. either $(-1.5)^2+\dfrac{3}{(-1.5)}-1$ or $2(-1.5)-\dfrac{3}{(-1.5)^2}$ are fine for B1.

Final -This mark can be implied by applying at least one correct value of either $\mathrm{f}(-1.5)$ or $\mathrm{f}'(-1.5)$

dM1: in $-1.5-\dfrac{\mathrm{f}(-1.5)}{\mathrm{f}'(-1.5)}$. So just $-1.5-\dfrac{\mathrm{f}(-1.5)}{\mathrm{f}'(-1.5)}$ with an incorrect answer and no other evidence scores final dM0A0.

Give final dM0 for applying $1.5-\dfrac{\mathrm{f}(-1.5)}{\mathrm{f}'(-1.5)}$ without first quoting the correct N-R formula.

(b)

A1: **Way 1:** correct solution only

Candidate needs to state both of their values for $\mathrm{f}(x)$ to awrt (or truncated) 1 sf along with a reason and conclusion. Reference to change of sign or eg. $\mathrm{f}(-1.675)\times\mathrm{f}(-1.665)<0$ or a diagram or <0 and >0 or one positive, one negative are sufficient reasons. There must be a (minimal, not incorrect) conclusion, eg. $\alpha=-1.67$, root (or α or part (a)) is correct, QED and a square are all acceptable. Ignore the presence or absence of any reference to continuity.

A minimal acceptable reason and conclusion is "change of sign, hence root".

No explicit reference to 2 decimal places is required.

Stating "root is in between -1.675 and -1.665" without some reference to is not sufficient for A1

Accept 0.015 as a correct evaluation of $\mathrm{f}(-1.675)$

(b)

A1: **Way 2:** correct solution only

Their conclusion in Way 2 needs to convey that they understand that $\alpha=-1.67$ to 2 decimal places. Eg. "therefore my answer to part (a) [which must be -1.67] is correct" is fine for A1.

$-1.67-\dfrac{\mathrm{f}(-1.67)}{\mathrm{f}'(1.67)}=-1.67$ (2 dp) is sufficient for M1A1 in part (b).

The root of $\mathrm{f}(x)=0$ is $-1.67169988...$, so candidates can also choose x_1 which is less than $-1.67169988...$ and choose x_2 which is greater than $-1.67169988...$ with both x_1 and x_2 lying in the interval $[-1.675, -1.665]$ and evaluate $\mathrm{f}(x_1)$ and $\mathrm{f}(x_2)$.

Helpful Table

x	$\mathrm{f}(x)$
-1.675	0.014580224
-1.674	0.010161305
-1.673	0.005743106
-1.672	0.001325627
-1.671	-0.003091136
-1.670	-0.007507186
-1.669	-0.011922523
-1.668	-0.016337151
-1.667	-0.020751072
-1.666	-0.025164288
-1.665	-0.029576802

Q51.

Question Number	Scheme	Notes	Marks
	$f(x)=x^3+4x-6$		
(i)(a)	$f(1)=-1$ $f(1.5)=3.375\left(=\frac{27}{8}\right)$	Attempts to evaluate at both end points. If substitution not seen accept $f(1)=-1$ **or** $f(1.5)=3.375$ as evidence, with any value for the other end.	M1
	Sign change and $f(x)$ continuous therefore α is between $x=1$ and $x=1.5$	$f(1)=-1$ **and** $f(1.5)=3.375$ both correct and mentions/indicates sign change, continuous and conclusion.	A1
			(2)

(i)(b)	$f'(x)=3x^2+4$	Correct derivative. Can be implied by a correct expression seen later such as $3(1.5)^2+4$	B1
	$x_2=1.5-\frac{f(1.5)}{f'(1.5)}=1.5-\frac{3.375}{10.75}$	Attempt Newton-Raphson using the **correct** formula.	M1
	$x_2=1.186...$	Accurate first application either awrt 1.186, $\frac{51}{43}$ or a correct numerical expression.	A1
	$x_3=\left(1.186...-\frac{0.412...}{8.220...}\right)=1.1358...$		
	$\alpha\approx 1.136$	cao	A1
			(4)
(ii)	$g(1.4)=3.442116...$ **or** $g(1.5)=-3.601419...$	Evidence of at least one value correct to 3 d.p. or better. May be implied by correct answer if never seen.	B1
	$\frac{1.5-\beta}{\beta-1.4}=\frac{0-g(1.5)}{g(1.4)-0}$ or $\frac{\beta-1.4}{0-g(1.4)}=\frac{1.5-1.4}{g(1.5)-g(1.4)}$ oe	A correct linear interpolation statement as shown oe (with correct signs). May omit the zeroes or use evaluated values (e.g 0.1 instead of 1.5 – 1.4). Other forms are possible.	B1
	$\Rightarrow \beta\approx 1.448869...$	Evaluates β from an attempt at a linear interpolation statement, allow if signs are incorrect. $\beta\approx$ awrt 1.449	M1 A1
			(4)
			Total 10

Q52.

Question Number	Scheme	Notes	Marks
(a)	$x = r\cos\theta = (1+\sin\theta)\cos\theta$ $\Rightarrow \dfrac{dx}{d\theta} = \cos^2\theta - (1+\sin\theta)\sin\theta$ or $\Rightarrow \dfrac{dx}{d\theta} = -\sin\theta + \cos 2\theta$	Differentiates $r\cos\theta$ using product rule or double angle formula	M1
		Correct derivative in any form	A1
	$\cos^2\theta - (1+\sin\theta)\sin\theta = 0 \Rightarrow 1-\sin^2\theta - \sin\theta - \sin^2\theta = 0 \Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$ or $-\sin\theta + \cos 2\theta = 0 \Rightarrow -\sin\theta + 1 - 2\sin^2\theta = 0 \Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$ Sets $\dfrac{dx}{d\theta} = 0$ and proceeds to a 3TQ in $\sin\theta$ Depends on the first M mark		dM1
	$\Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$ $\Rightarrow \sin\theta = \dfrac{1}{2}, (-1) \Rightarrow \theta = ...$	Solves for θ. Depends o both M marks above.	ddM1
	$\left(\dfrac{3}{2}, \dfrac{\pi}{6}\right)$	Correct coordinates and no others. Need not be in coordinate brackets.	A1
			(5)

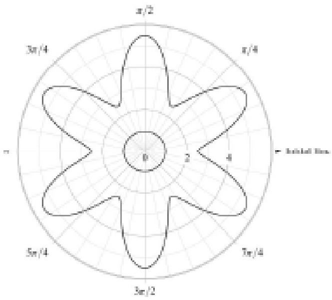
(b)	$\int (1+\sin\theta)^2\, d\theta = \int (1+2\sin\theta+\sin^2\theta)\, d\theta$ $= \int \left(1+2\sin\theta+\dfrac{1}{2}-\dfrac{1}{2}\cos 2\theta\right) d\theta$	Attempts $\left(\dfrac{1}{2}\right)\int r^2\, d\theta$ and applies $\sin^2\theta = \pm\dfrac{1}{2} \pm \dfrac{1}{2}\cos 2\theta$ Ignore any limits shown	M1
	$\int (1+\sin\theta)^2\, d\theta = \dfrac{3}{2}\theta - 2\cos\theta - \dfrac{1}{4}\sin 2\theta (+c)$	Correct integration (Ignore limits)	A1
	$\dfrac{1}{2}\left[\dfrac{3}{2}\theta - 2\cos\theta - \dfrac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ $= \dfrac{1}{2}\left[\dfrac{3\pi}{4} - \left(\dfrac{\pi}{4} - \sqrt{3} - \dfrac{\sqrt{3}}{8}\right)\right] \left(= \dfrac{\pi}{4} + \dfrac{9\sqrt{3}}{16}\right)$	Applies the limits of $\dfrac{\pi}{2}$ and their $\dfrac{\pi}{6}$ Substitution must be shown but no simplification needed	M1
	Trapezium: $\dfrac{1}{2}\left(2 + \left(2 - \dfrac{3}{2}\sin\dfrac{\pi}{6}\right)\right) \times \dfrac{3}{2}\cos\dfrac{\pi}{6}$ $\left(= \dfrac{39\sqrt{3}}{32}\right)$	Uses a correct strategy for the area of trapezium *OQSP*	M1
	Area of $R = \dfrac{39\sqrt{3}}{32} - \dfrac{\pi}{4} - \dfrac{9\sqrt{3}}{16}$	Fully correct method for the required area. Depends on all previous method marks.	dM1
	$\dfrac{1}{32}\left(21\sqrt{3} - 8\pi\right)$	Cao	A1
			(6)
			Total 11

Q53.

Question Number	Scheme	Notes	Mark
(a)	$y = r\sin\theta = (1-\sin\theta)\sin\theta = \sin\theta - \sin^2\theta$ $\Rightarrow \frac{dy}{d\theta} = \cos\theta - 2\sin\theta\cos\theta$ or e.g. $\Rightarrow \frac{dy}{d\theta} = \cos\theta - \sin 2\theta$	Differentiates $(1-\sin\theta)\sin\theta$ to achieve $\pm\cos\theta \pm k\sin\theta\cos\theta$ or equivalent. Use of $y = r\cos\theta$ or $x = r\cos\theta$ scores M0	M1
		Correct derivative in any form.	A1
	$\cos\theta - 2\sin\theta\cos\theta = 0 \Rightarrow \cos\theta(1-2\sin\theta) = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = ...$ Solves to find a value for θ. Depends on the first M.		dM1
	$\left(\frac{1}{2}, \frac{\pi}{6}\right)$ Correct coordinates and no others. Isw if necessary e.g. if written as $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ after correct values seen or implied award A1. Allow e.g. $\theta = \frac{\pi}{6}, r = \frac{1}{2}$. **The value of r must be seen in (a) – i.e. do not allow recovery in (b).**		A1
			(4)
(b) **Way 1**	Note that the $\frac{1}{2}$ in $\frac{1}{2}\int r^2\, d\theta$ is not required for the first 4 marks		
	$\int (1-\sin\theta)^2\, d\theta = \int (1-2\sin\theta+\sin^2\theta)\, d\theta$ $= \int \left(1-2\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right) d\theta$	Attempts $\left(\frac{1}{2}\right)\int r^2\, d\theta$ and applies $\sin^2\theta = \pm\frac{1}{2}\pm\frac{1}{2}\cos 2\theta$	M1
	$\int (1-\sin\theta)^2\, d\theta = \frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta (+c)$ Correct integration. Condone mixed variables e.g. $\int (1-\sin\theta)^2\, d\theta = \frac{3}{2}x + 2\cos\theta - \frac{1}{4}\sin 2\theta (+c)$		A1
	$\left(\frac{1}{2}\right)\left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{6}} = \left(\frac{1}{2}\right)\left[\left(\frac{\pi}{4}+\sqrt{3}-\frac{\sqrt{3}}{8}\right)-(2)\right]\left(=\frac{\pi}{8}+\frac{7\sqrt{3}}{16}-1\right)$ Applies the limits of 0 and their $\frac{\pi}{6}$ to their integration. The $\frac{1}{2}$ is not required. For the integration look for at least $\pm\int \sin\theta\, d\theta \to \pm\cos\theta$		M1
	Triangle: $\frac{1}{2}\times\frac{1}{2}\sin\frac{\pi}{6}\times\frac{1}{2}\cos\frac{\pi}{6}\left(=\frac{\sqrt{3}}{32}\right)$ Uses a correct strategy for the area of the triangle		M1
	Area of $R = \frac{\pi}{8}+\frac{7\sqrt{3}}{16}-1+\frac{\sqrt{3}}{32}$	Fully correct method for the required area. **Depends on all previous method marks.**	dM1
	$\frac{1}{32}\left(4\pi+15\sqrt{3}-32\right)$	cao	A1
			(6)
			Total 10
	Note that the $\frac{1}{2}$ in $\frac{1}{2}\int r^2\, d\theta$ is not required for the first 3 marks		

<table>
<tr><td rowspan="7">(b)
Way 2</td><td>$\int(1-\sin\theta)^2\,d\theta=\int(1-2\sin\theta+\sin^2\theta)d\theta$
$=\int\left(1-2\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta$</td><td>Attempts $\left(\frac{1}{2}\right)\int r^2\,d\theta$ and applies
$\sin^2\theta=\pm\frac{1}{2}\pm\frac{1}{2}\cos 2\theta$.</td><td>M1</td></tr>
<tr><td colspan="2">$\int(1-\sin\theta)^2\,d\theta=\frac{3}{2}\theta+2\cos\theta-\frac{1}{4}\sin 2\theta(+c)$
Correct integration. Condone mixed variables e.g.
$\int(1-\sin\theta)^2\,d\theta=\frac{3}{2}x+2\cos\theta-\frac{1}{4}\sin 2\theta(+c)$</td><td>A1</td></tr>
<tr><td colspan="2">$\left(\frac{1}{2}\right)\left[\frac{3}{2}\theta+2\cos\theta-\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}=\left(\frac{1}{2}\right)\left[\left(\frac{3\pi}{4}+0-0\right)-(2)\right]\left(=\frac{3\pi}{8}-1\right)$
Evidence of use of both limits 0 and $\frac{\pi}{2}$ to their integration. The $\frac{1}{2}$ is not required.
For the integration look for at least $\pm\int\sin\theta\,d\theta\to\pm\cos\theta$</td><td>M1</td></tr>
<tr><td colspan="2">Triangle – "Segment":
$\frac{1}{2}\times\frac{1}{2}\sin\frac{\pi}{6}\times\frac{1}{2}\cos\frac{\pi}{6}-\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1-\sin\theta)^2\,d\theta$
$\frac{\sqrt{3}}{32}-\frac{1}{2}\left[\frac{3}{2}\theta+2\cos\theta-\frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(=\frac{15\sqrt{3}}{32}-\frac{\pi}{4}\right)$
Uses a fully correct strategy for the area above the curve between O and P.
Requires a correct method for the triangle as in Way 1 and a correct method for the "segment" using both their $\frac{\pi}{6}$ and $\frac{\pi}{2}$.</td><td>M1</td></tr>
<tr><td>Area of $R=\frac{3\pi}{8}-1+\frac{15\sqrt{3}}{32}-\frac{\pi}{4}$</td><td>Fully correct method for the required area. Depends on all previous method marks.</td><td>dM1</td></tr>
<tr><td>$\frac{1}{32}\left(4\pi+15\sqrt{3}-32\right)$</td><td>cao</td><td>A1</td></tr>
<tr><td colspan="2"></td><td>(6)</td></tr>
</table>

Q54.

Question	Scheme	Marks
(a)	 Completes to a closed loop with "petals" containing circle of radius 1 (whether the circle is drawn or not)	**M1**
	Fully correct – 6 petals in roughly the right places, but allow if curvature is not quite smooth.	**A1**
	Circle centre O radius 1.	**B1**
		(3)
(b)	$\left(\frac{1}{2}\right)\int r^2\,\mathrm{d}\theta = \left(\frac{1}{2}\right)\int\left(\underline{16-12\cos 6\theta+\frac{9}{4}\cos^2 6\theta}\right)\mathrm{d}\theta$	**M1**
	$=\frac{1}{2}\int_0^{2\pi}\left(16-12\cos 6\theta+\underline{\frac{9}{8}(1+\cos 12\theta)}\right)\mathrm{d}\theta$	**M1**
	$=\frac{1}{2}\left[16\theta-2\sin 6\theta+\frac{9}{8}\left(\theta+\frac{1}{12}\sin 12\theta\right)\right]_0^{2\pi}$	**M1** **A1**
	$A_{outer}=\frac{1}{2}\int_0^{2\pi} r^2\,\mathrm{d}\theta=\frac{1}{2}\int_0^{2\pi}\left(16-12\cos 6\theta+\frac{9}{4}\cos^2 6\theta\right)\mathrm{d}\theta$ $=\frac{1}{2}\left(32\pi-0+\frac{9}{8}(2\pi+0)-(0)\right)$	**dM1**
	So Area required is $\frac{1}{2}\left(32\pi+\frac{9\pi}{4}\right)-\pi(1^2)=\ldots$	**B1**
	$=\frac{129}{8}\pi$	**A1**
		(7)
		(10 marks)

Notes:
(a) **M1:** Allow for any closed loop that oscillates, though may not have the correct number of "petals" but require at least 4 . Need not have correct places of maximum radius. **A1:** Fully correct sketch, 6 "petals" in the right places, with maximum radius between the 5 and 6 radius lines, minimum between the 2 and 3 radius lines. **B1:** For a circle of radius 1 and centre O drawn. **(b)** **M1:** Attempts to square r as part of an integral for the outer curve, achieving a 3 term quadratic in $\cos 6\theta$ **M1:** Applies the double angle formula to the $\cos^2$ term from their expansion (not dependent on the first M, but must have a $\cos^2$ term). Accept $\cos^2 6\theta \to \frac{1}{2}(\pm 1 \pm \cos 12\theta)$ **M1:** Attempts to integrate, achieving the form $\alpha\theta + \beta\sin 6\theta + \gamma\sin 12\theta$ where $\alpha, \beta, \gamma \neq 0$ **A1:** Correct integration – limits and the $\frac{1}{2}$ not needed. Look for $16\theta - 2\sin 6\theta + \frac{9}{8}\left(\theta + \frac{1}{12}\sin 12\theta\right)$ oe. **dM1:** Depends on at least two of the previous M's being scored. For a correct overall strategy for the area contained in the outer loop, with an attempt at the r^2 (should be 3 term expansion). Correct appropriate limits and the $\frac{1}{2}$ should be present or implied by working, but note variations on the scheme are possible, e.g. $2 \times \frac{1}{2}\int_0^{\pi} r^2\, d\theta$, in which the $2 \times \frac{1}{2}$ may be implied rather than seen. **B1:** Subtracts correct area of π for inner circle **A1:** cso. Check carefully the integration was correct as the sin terms disappear with the limits.

Q55.

Question	Scheme		Marks
	$u_1 = 5,\ u_{n+1} = 3u_n + 2,\ n \geq 1$. Required to prove the result, $u_n = 2 \times (3)^n - 1,\ n \in \mathbb{Z}^+$		
(i)	$n = 1:\ u_1 = 2(3) - 1 = 5$	$u_1 = 2(3) - 1 = 5$ or $u_1 = 6 - 1 = 5$	B1
	(Assume the result is true for $n = k$)		
	$u_{k+1} = 3\left(2(3)^k - 1\right) + 2$	Substitutes $u_k = 2(3)^k - 1$ into $u_{k+1} = 3u_k + 2$	M1
	$= 2(3)^{k+1} - 1$	**dependent on the previous M mark** Expresses u_{k+1} in term of 3^{k+1}	dM1
		$u_{k+1} = 2(3)^{k+1} - 1$ **by correct solution only**	A1
	If the result is true for $n = k$, then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n		A1 cso
			(5)
	Required to prove the result $\sum_{r=1}^{n} \frac{4r}{3^r} = 3 - \frac{(3+2n)}{3^n},\ n \in \mathbb{Z}^+$		

(ii)	$n=1: \text{LHS} = \frac{4}{3}, \text{ RHS} = 3 - \frac{5}{3} = \frac{4}{3}$	Shows or states **both** $\text{LHS} = \frac{4}{3}$ and $\text{RHS} = \frac{4}{3}$ or states $\text{LHS} = \text{RHS} = \frac{4}{3}$	B1
	(Assume the result is true for $n = k$)		
	$\sum_{r=1}^{k+1} \frac{4r}{3^r} = 3 - \frac{(3+2k)}{3^k} + \frac{4(k+1)}{3^{k+1}}$	Adds the $(k+1)^{\text{th}}$ term to the sum of k terms	M1
	$= 3 - \frac{3(3+2k)}{3^{k+1}} + \frac{4(k+1)}{3^{k+1}}$	**dependent on the previous M mark** Makes 3^{k+1} or $(3)3^k$ a common denominator for their fractions.	dM1
		Correct expression with common denominator 3^{k+1} or $(3)3^k$ for their fractions.	A1
	$= 3 - \left(\frac{3(3+2k) - 4(k+1)}{3^{k+1}}\right)$ $= 3 - \left(\frac{5+2k}{3^{k+1}}\right)$		
	$= 3 - \frac{(3+2(k+1))}{3^{k+1}}$	$3 - \frac{(3+2(k+1))}{3^{k+1}}$ **by correct solution only**	A1
	If the result is true for $n=k$, then it is true for $n=k+1$. As the result has been shown to be true for $n=1$, then the result is true for all n		A1 cso
			(6)
			(11 marks)

Notes:
(i) & (ii) **Final A1 for parts (i) and (ii)** is dependent on all previous marks being scored in that part. It is gained by candidates conveying the ideas of **all** four underlined points **either** at the end of their solution or as a narrative in their solution.
(i) $u_1 = 5$ by itself is not sufficient for the 1st B1 mark in part (i). $u_1 = 3+2$ without stating $u_1 = 2(3) - 1 = 5$ or $u_1 = 6-1 = 5$ is B0
(ii) $\text{LHS} = \text{RHS}$ by itself is not sufficient for the 1st B1 mark in part (ii).

Q56.

Question Number	Scheme	Marks
(i)	$n=1 \quad u_1 = 3\times\frac{2}{3}-1=1$ (so true for $n=1$ (†))	B1
	Assume true for $n=k$ ie $u_k = 3\left(\frac{2}{3}\right)^k - 1$ (†)	
	$u_{k+1} = \frac{1}{3}(2u_k - 1) = \frac{1}{3}\left(2\left(3\left(\frac{2}{3}\right)^k - 1\right) - 1\right) = \frac{1}{3}\left(6\left(\frac{2}{3}\right)^k - 2 - 1\right)$	M1A1
	$= \frac{1}{3}\left(2\times 3\left(\frac{2}{3}\right)^{k+1}\times\left(\frac{3}{2}\right) - 2 - 1\right)$	dM1
	$= \frac{1}{3}\times 2\times 3\left(\frac{2}{3}\right)^{k+1}\times\left(\frac{3}{2}\right) + \frac{1}{3}(-2-1)$	
	$= 3\left(\frac{2}{3}\right)^{k+1} - 1$	A1
	$\therefore$ if true for $n=k$, also true for $n=k+1$ (†)	
	(True for $n=1$) so $u_n = 3\left(\frac{2}{3}\right)^n - 1$ is true for all $n\in\mathbb{Z}^+$	A1cso (6)
(ii)	$\mathrm{f}(1) = 2^3 + 3^3 = 8 + 27 = 35$ (Multiple of 7) (so true for $n=1$ (†))	B1
	Assume $\mathrm{f}(k)$ is a multiple of 7 $\mathrm{f}(k) = 2^{k+2} + 3^{2k+1}$ is a multiple of 7 (†)	
	$\mathrm{f}(k+1) - M\mathrm{f}(k) = 2^{k+3} + 3^{2k+3} - M\left(2^{k+2} + 3^{2k+1}\right)$	M1
	$= 2^{k+2}(2-M) + 3^{2k+1}(3^2 - M)$	A1
	$= (2-M)\left(2^{k+2} + 3^{2k+1}\right) + 3^{2k+1}\times 7$ or $(9-M)\left(2^{k+2} + 3^{2k+1}\right) - 7\times 2^{k+2}$ oe	dM1
	$\therefore \mathrm{f}(k+1) = 2\mathrm{f}(k) + 7\times 3^{2k+1}$ oe e.g. $9\mathrm{f}(k) - 7\times 2^{k+2}$	A1
	Or e.g. $7\times 3^{2k+1}$ is a multiple of 7, so if f(k) is a multiple of 7 <u>then f(k+1) is also a multiple of 7</u>	
	If the result is true for $n=k$ it is also true for $n=k+1$ (†)	
	As the result has been shown to be true for $n=1$, it is true for all $n\in\mathbb{Z}^+$	A1 cso (6) [12]

(i)	
B1	Check that the formula gives 1 when $n = 1$ Working must be shown. (Need not state true for $n = 1$ for this mark – but see final A)
M1	(Assume true for $n = k$ and) attempts to substitute the formula for u_k into $u_{k+1} = \frac{1}{3}(2u_k - 1)$ or equivalent with suffixes increased. Allow slips.
A1	Correct substitution.
dM1	Obtain an expression with $\left(\frac{2}{3}\right)^{k+1}$ and no other k. Alternatively, expands u_{k+1} to a matching expression (ie work from both directions).
A1	Correct expression when $n = k + 1$ At least one intermediate stage of working must be shown and no errors (though notational slips may be condoned). If working from both directions, it is for correct work to reach matching expressions.
A1cso	Correct concluding statements following correct solution which has included each of the points (†) at some stage during the working. Depends on all except the first B mark (e.g. if they think they have checked $n = 1$ but have really checked $n = 2$). Note: Allow the M's and first two A's for students who go from k+1 to k+2 but treat it as k to $k + 1$.
(ii)	
B1	Checks the case $n = 1$. Minimum statement of f(1) = 35
M1	Attempts an expression for $f(k+1) - Mf(k)$ with any value of M. Need not be simplified. Most likely with $M = 1$ but may be seen with other values of M. With $M = 0$, $f(k+1) = 2^{k+3} + 3^{2k+3}$ is all that is required.
A1	A correct expression with terms 2^{k+2} and 3^{2k+1} clearly identified.
dM1	Attempts to extract/identify f(k) within a correct expression to give terms divisible by 7. With $M = 0$ look for $f(k+1) = 2\times\left(2^{k+2} + 3^{3k+1}\right) + 7\times 3^{2k+1}$ or $9\times\left(2^{k+2} + 3^{3k+1}\right) - 7\times 2^{k+2}$ oe and similar for other value of M.
A1	One of the correct expressions for f(k+1) shown (or with powers of 2 and 3) or full reason why f(k+1) is divisible by 7, following a suitable expression.
A1cso	Correct concluding statements following correct solution which has included each of the points (†) at some stage during the working. Depends on all previous marks.

Q57.

Question Number	Scheme	Notes	Marks
	$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$		
(a)	$n=1$, LHS$=1^2=1$, RHS$=\frac{1}{6}.2.3=1$	Shows both LHS=1 and RHS=1 Accept LHS = 1 but must see at least $\frac{1}{6}.2.3=1$ for RHS.	B1
	Assume true for $n=k$		
	When $n=k+1$ $\sum_{r=1}^{k+1} r^2 = \frac{k}{6}(k+1)(2k+1)+(k+1)^2$	Adds $(k+1)^2$ to result for $n=k$	M1
	$=\frac{(k+1)}{6}(k(2k+1)+6(k+1))$	Attempt to factorise by $\frac{(k+1)}{6}$	dM1
	$=\frac{(k+1)}{6}(k+2)(2k+3)$ $=\frac{(k+1)}{6}((k+1)+1))((2(k+1)+1)$	Either factorised form. SC allow dM1A0 for fully factorising to a cubic expression and going direct to the fully factorised expression with no intermediate quadratic seen.	A1
	True for $n=1$. If true for $n=k$ then true for $n=k+1$ therefore true for all n.	Complete proof with no errors and these 4 statements seen anywhere. Depends on both M's and the A, but may be scored if the B is lost as long as some indication of true for $n=1$ is given.	A1cso
			(5)

(b)	$\sum_{r=1}^{n}(r^2+2)=\sum_{r=1}^{n} r^2+\sum_{r=1}^{n} 2$ $=\frac{n}{6}(n+1)(2n+1)+\ldots$	Split into the addition of 2 sums and applies the result of (a).	M1
	$=\frac{n}{6}(n+1)(2n+1)+2n$	Correct expression.	A1
	$=\frac{n}{6}(2n^2+3n+13)$	Factorises out the $\frac{n}{6}$ - must have a common factor n to achieve this mark; Simplifies to correct answer.	M1; A1
	$(a=2, b=3, c=13)$		
			(4)
(c)	$\sum_{r=10}^{25}(r^2+2)=S_{25}-S_9$ $=\frac{25}{6}.(2\times25^2+3\times25+13)-\frac{9}{6}(2\times9^2+3\times9+13$	Attempts $S_{25}-S_9$ or $S_{25}-S_{10}$ with some substitution.	M1
	$=\frac{25}{6}\times1338-\frac{9}{6}\times202=5575-303=5272$	For 5272	A1
	Note: Answer only (from calculator) is M0A0 as question requires use of part (b).		
			(2)
			Total 11

Q58.

Question Number	Scheme	Notes	Marks
	$f(n) = 4^{n+2} + 5^{2n+1}$ divisible by 21		
	$n = 1, 4^3 + 5^3 = 189 = 9 \times 21$ (Or $n = 0,\ 4^2 + 5^1 = 21$)	$f(1) = 21 \times 9$ Accept f(0) = 21 as an alternative starting point.	B1
	Assume that for $n = k$, $f(k) = (4^{k+2} + 5^{2k+1})$ is divisible by 21 for $k \in \mathbb{Z}^+$.		
	$f(k+1) - f(k) = 4^{k+3} + 5^{2k+3} - (4^{k+2} + 5^{2k+1})$	Applies $f(k+1)$ with at least 1 power correct. May be just as $f(k+1)$, or as part of an expression in $f(k+1)$ and $f(k)$.	M1
	$= 4.4^{k+2} + 25.5^{2k+1} - 4^{k+2} - 5^{2k+1}$	For a correct expression in $f(k+1)$, and possibly $f(k)$, with powers reduced to those of $f(k)$.	A1
	$= 3.4^{k+2} + 24.5^{2k+1}$		
	$= 3f(k) + 21.5^{2k+1}$ or $= 24f(k) - 21.4^{k+2}$	For one of these expression or equivalent with obvious factor of 21 in each.	A1
	$f(k+1) = 4f(k) + 21.5^{2k+1}$	Makes $f(k+1)$ the subject or gives clear reasoning of each term other than $f(k+1)$ being divisible by 21. Dependent on at least one of the previous accuracy marks being awarded.	dM1

	{ $f(k+1)$ is divisible by 21 as both $f(k)$ and 21 are both divisible by 21}		
	If the result is **true for $n = k$**, then it is now **true for $n = k + 1$**. As the result has shown to be **true for $n = 1$ (or 0)**, then the result is **true for all n** ($\in \mathbb{Z}^+$).	Correct conclusion seen at the end. Condone true for $n = 1$ stated earlier. Depends on both M's andA's, but may be scored if the B is lost as long as at least f(1) = 189 was reached (so e.g. if the 21×9 was not shown)	A1 cso
			(6)
ALT for first 4 marks	$n = 1, 4^3 + 5^3 = 189 = 9 \times 21$ (Or $n = 0,\ 4^2 + 5^1 = 21$)	As main scheme.	B1
	$f(k+1) - \alpha f(k) = 4^{k+3} + 5^{2k+3} - \alpha(4^{k+2} + 5^{2k+1})$	Attempts $f(k+1)$ in any equation (as main scheme).	M1
	$f(k+1) - \alpha f(k) = (4-\alpha)4^{k+2} + (25-\alpha)5^{2k+1}$	For a correct expression with any α, with powers reduced to match $f(k)$.	A1
	$f(k+1) - \alpha f(k) = (4-\alpha)(4^{k+2} + 5^{2k+1}) + 21.5^{2k+1}$ $f(k+1) - \alpha f(k) = (25-\alpha)(4^{k+2} + 5^{2k+1}) - 21.4^{k+2}$	Any suitable equation with powers sorted appropriately to match $f(k)$	A1
	NB: $\alpha = 0, \alpha = 4, \alpha = 25$ will make relevant terms disappear, but marks should be awarded accordingly.		
			Total 6

Q59.

Question Number	Scheme	Notes	Marks
(i)	$n=1 \Rightarrow u_1 = 3\times 2 - 2\times 3 = 0$ $n=2 \Rightarrow u_2 = 3\times 2^2 - 2\times 3^2 = -6$	Shows the result is true for n = 1 **and** n = 2. Ignore references to n = 3.	B1
	Substitutes $u_k = 3\times 2^k - 2\times 3^k$ and $u_{k+1} = 3\times 2^{k+1} - 2\times 3^{k+1}$ into $(u_{k+2} =)5u_{k+1} - 6u_k = 5\left(3\times 2^{k+1} - 2\times 3^{k+1}\right) - 6\left(3\times 2^k - 2\times 3^k\right)$ (The inductive assumption may be tacit for this mark.)		M1
	$(u_{k+2}) = 15\times 2^{k+1} - 10\times 3^{k+1} - 18\times 2^k + 12\times 3^k$ $= 15\times 2^{k+1} - 9\times 2^{k+1} - 10\times 3^{k+1} + 4\times 3^{k+1}$ $= 6\times 2^{k+1} - 6\times 3^{k+1}$	Gathers to a correct two term expression. Accept alternative forms such as $12\times 2^k - 18\times 3^k$	A1
	$u_{k+2} = 3\times 2^{k+2} - 2\times 3^{k+2}$	Achieves this result with no errors – must be clear it is u_{k+2} but this may have been seen at the start.	A1
	If the result is true for $n = k$ **and** $n = k + 1$ **then** it is **true for** $n = k + 2$. As the result has been shown to be **true for** $n = 1$ **and** $n = 2$ then the result is **true for all** n. Correct conclusion including all the bold points in some form. Depends on all previous marks.		A1cso
			(5)

(ii)	$\mathrm{f}(n) = 3^{3n-2} + 2^{4n-1}$		
	$\mathrm{f}(1) = 3^1 + 2^3 = 11$	Shows the result is true for n = 1	B1
	$\mathrm{f}(k+1) - \mathrm{f}(k) = 3^{3(k+1)-2} + 2^{4(k+1)-1} - 3^{3k-2} - 2^{4k-1}$ $= 27\times 3^{3k-2} + 16\times 2^{4k-1} - 3^{3k-2} - 2^{4k-1} = 26\times 3^{3k-2} + 15\times 2^{4k-1}\left(= \frac{26}{9}3^{3k} + \frac{15}{2}2^{4k}\right)$ Attempts f (k + 1) – f (k) and reaches $\alpha\times 3^{3k-2} + \beta\times 2^{4k-1}$ or $\alpha\times 3^{3k} + \beta\times 2^{4k}$ (Mark variations on the theme as appropriate here and in the following marks.)		M1
	$= 15\times\left(3^{3k-2} + 2^{4k-1}\right) + 11\times 3^{3k-2}$ **or** $= 26\times\left(3^{3k-2} + 2^{4k-1}\right) - 11\times 2^{4k-1}$	Correct expression with f(k) evident.	A1
	$\mathrm{f}(k+1) = 16\mathrm{f}(k) + 11\times 3^{3k-2}$ **or** $\mathrm{f}(k+1) = 27\mathrm{f}(k) - 11\times 2^{4k-1}$	Makes f(k + 1) the subject and states divisible by 11 (oe – may be implied by conclusion), or gives full reason why f(k + 1) is divisible by 11. **Dependent on first M**	dM1
	If the result is true for $n = k$ **then** it is **true for** $n = k + 1$. As the result has been shown to be **true for** $n = 1$, then the result is **true for all** n. Correct conclusion including all the bold points in some form. Depends on all previous marks.		A1cso
			(5)
			Total 10

Q60.

Question Number	Scheme	Notes	Marks
	$f(n)=4^n+6n-10 \quad n\in\mathbb{Z} \quad n\geqslant 2$		

General guidance:
Apply the way that best fits the overall approach.
Condone work in e.g., n instead of k.
Attempts with no induction e.g., not using f(k) in an equation with f(k +1) score a max of 11000.
Using e.g., f(k + 2) – f(k + 1) requires a clear indication of assuming f(k + 1) is true to access the last three marks.
Alternative explanations are unlikely to access the last three marks unless there is a fully convincing justification of divisibility, e.g., $f(k+1)-f(k)=3\times4^k+6$ followed by "Since 3×4^k is a multiple of both 3 and 4 and hence 12, $3\times4^k+6$ is divisible by 18" is not a sound argument. Attempts that involve further induction on different expressions must be complete methods to access the last 3 marks.
Allow use of –18 but if any different multiples of 18 are involved e.g., 36, the first A1 requires "36 is a multiple of/divisible by (but not "factor of") 18" oe for each case
B1: Any correct numerical expression that is not just "18" is sufficient for this mark e.g., 16 + 12 –10, 28 – 10, 4^2 + 2. Starting with e.g., f(3) scores a max of 01110.
Ignore an extra evaluation of f(1) but a comment on f(1)'s divisibility is final A0 since $n\geqslant2$
Final A1: There must be evidence that true for $n=k \Rightarrow$ true for $n=k+1$ but it could be minimal and be scored in a conclusion or a narrative or via both. So if e.g., "Assume true for $n=k$ ..." is seen in the work followed by "true for $n=k+1$" in a conclusion this is sufficient.
Condone "for all $n\in\mathbb{Z}$", "all $n\in\mathbb{Z}$ $n>2$", "all $\mathbb{Z}>$ (or $\geqslant$) 2" but **not** $n\in\mathbb{R}$

Way 1 $f(k+1)-f(k)$	$f(2)=4^2+6\times2-10=18$	Obtains f(2) = 18 with substitution	B1
	$f(k+1)=4^{k+1}+6(k+1)-10$	Attempts $f(k+1)$	M1
	$f(k+1)-f(k)=4^{k+1}+6(k+1)-10-(4^k+6k-10)$ $=4^{k+1}-4^k+6=3\times4^k+6$ $=3(4^k+6k-10)-18k+36$	Attempts $f(k+1)-f(k)$, uses $4^{k+1}=4\times4^k$ & obtains $pf(k)+g(k)$ with g(k) linear (allow constant $\neq 0$)	M1
	$f(k+1)=4f(k)+18(2-k)$ f(k) may be written in full	Correct factorised expression Allow $4f(k)+18\times2-18\times k$ If f(k + 1) is not made the subject then e.g., "true for f(k + 1) – f(k)" is also required	A1
	True for n = 2, if true for n = k then true for n = $k+1$ so **true for all** $n\in\mathbb{Z}$ $(n\geqslant2)$ Minimum in bold.	Full conclusion/narrative and no errors. All marks needed but allow if B0 provided this mark was only withheld for insufficient working.	A1
			(5)

Way 2 $f(k+1)=...$	$f(2)=4^2+6\times2-10=18$	Obtains f(2) = 18 with substitution	B1
	$f(k+1)=4^{k+1}+6(k+1)-10$	Attempts $f(k+1)$	M1
	$=4\times4^k+6k-4$ $=4(4^k+6k-10)-18k+36$	Uses $4^{k+1}=4\times4^k$ & obtains $pf(k)+g(k)$ with g(k) linear (allow constant $\neq 0$)	M1
	$=4f(k)+18(2-k)$ f(k) may be written in full	Correct factorised expression Allow $4f(k)+18\times2-18\times k$	A1
	True for n = 2, if true for n = k then true for n = $k+1$ so **true for all** $n\in\mathbb{Z}$ $(n\geqslant2)$ Minimum in bold.	Full conclusion/narrative and no errors. All marks needed but allow if B0 provided this mark was only withheld for insufficient working.	A1
			(5)

Question Number	Scheme	Notes	Marks
cont.	$f(n)=4^n+6n-10 \quad n\in\mathbb{Z} \quad n\geqslant 2$		
Way 3 $f(k+1)-mf(k)$	$f(2)=4^2+6\times 2-10=18$	Obtains f(2) = 18 with substitution	B1
	$f(k+1)=4^{k+1}+6(k+1)-10$	Attempts $f(k+1)$	M1
	$f(k+1)-mf(k)=4^{k+1}+6(k+1)-10-m(4^k+6k-10)$ $=(4-m)4^k+(6-6m)k-4+10m$ e.g. $m=-14\Rightarrow 18\times 4^k+90k-144$ e.g. $m=4\Rightarrow -18k+36$	Attempts $f(k+1)-mf(k)$ and uses a value of m to obtain $c\times 4^k+\ldots$ where c is a multiple of their 18 or uses $m=4$	M1
	e.g., $f(k+1)=-14f(k)+18(4^k+5k-8)$ $f(k+1)=4f(k)+18(2-k)$ f(k) may be written in full	A correct factorised expression Allow $-14f(k)+18\times 4^k+18\times 5k-18\times 8$ If f(k + 1) is not made the subject then e.g., "true for f(k + 1) – mf(k)" is also required	A1
	True for n = 2, if true for n = k then true for n = $k+1$ so **true for all** $n\in\mathbb{Z}$ $(n\geqslant 2)$ Minimum in bold.	Full conclusion/narrative and no errors. All marks needed but allow if B0 provided this mark was only withheld for insufficient working.	A1
			(5)
Way 4 $f(k)=18M$	$f(2)=4^2+6\times 2-10=18$	Obtains f(2) = 18 with substitution	B1
	$f(k+1)=4^{k+1}+6(k+1)-10$	Attempts $f(k+1)$	M1
	$f(k)=18M,\quad f(k+1)=4\times 4^k+6k-4$ $=4\times 18M-18k+36$	Sets $f(k)=18M$, uses $4^{k+1}=4\times 4^k$ & obtains $pf(k)+g(k)$ with g(k) linear (allow constant $\neq 0$)	M1
	$f(k+1)=18(4M+2-k)$	A correct factorised expression Allow $18\times 4M+18\times 2-18\times k$	A1
	True for n = 2, if true for n = k then true for n = $k+1$ so **true for all** $n\in\mathbb{Z}$ $(n\geqslant 2)$ Minimum in bold.	Full conclusion/narrative and no errors. All marks needed but allow if B0 provided this mark was only withheld for insufficient working.	A1
			(5)

Q61.

Question	Scheme	Marks
(a)	For $n = 1$, $\sum_{r=1}^{1} r^3 = 1$ and $\frac{1}{4}(1^2)(1+1)^2 = \frac{1}{4}\times 1 \times 4 = 1$ So true for $n = 1$	B1
	(Assume the result is true for $n = k$, so $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$) Then $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$	M1
	$= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right] = \frac{1}{4}(k+1)^2\left[k^2+4k+4\right]$	M1
	$= \frac{1}{4}(k+1)^2(k+2)^2$	A1
	$\left[= \frac{1}{4}(k+1)^2((k+1)+1)^2\right]$ Hence result is true for $n = k+1$. As true for $n = 1$ and have shown if true for $n = k$ then it is true for $n = k+1$, so it is true for all $n \in \mathbb{N}$ by induction.	A1
		(5)

(b)	$\sum_{r=1}^{n} r(r+1)(r-1) = \sum_{r=1}^{n} r^3 - r$	B1
	$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1)$	M1
	(Please note the mark above is incorrectly labelled as A1 on e-PEN)	
	$= \frac{1}{4}n(n+1)\left[n^2+n-2\right] = \frac{1}{4}n(n+1)(n+...)(n+...)$	M1
	$= \frac{1}{4}n(n+1)(n-1)(n+2)$	A1
		(4)
(c)	$\sum_{r=n}^{2n} r^2 = \frac{1}{6}(2n)(2n+1)(2(2n)+1) - \frac{1}{6}(n-1)(n)(2(n-1)+1)$	M1 A1
	$3\sum_{r=1}^{n} r(r+1)(r-1) = 17\sum_{r=n}^{2n} r^2$ $\Rightarrow \frac{3}{4}n(n+1)(n-1)(n+2) = \frac{17}{6}n\left(2(8n^2+6n+1) - (2n^2-3n+1)\right)$ $\Rightarrow 18(n+1)(n-1)(n+2) = 68\left(14n^2+15n+1\right) = 68(14n+1)(n+1)$	dM1
	$\Rightarrow 18(n-1)(n+2) = 68(14n+1)$ $\Rightarrow 18n^2 - 934n - 104 = 0 \Rightarrow n = ...$	ddM1
	$n = 52$	A1
		(5)
		(14 marks)

Notes:

(a)

B1: Checks the result for $n = 1$. Should see a clear substitution into both sides, accept minimum of seeing $\frac{1}{4}\times1\times4$, or $\frac{1}{4}\times1\times2^2$, or $\frac{1}{4}\times1\times(1+1)^2=1$ for right hand side.

M1:(Makes or assumes the inductive assumption, and) adds $(k + 1)^3$ to the result for $n = k$

M1: Attempts to take at least $(k+1)^2$ as a factor out of the expression. Allow if an expansion to a quartic is followed by the factorised expression.

A1: Reaches the correct expression for $n = k + 1$from correct working with sufficient working seen, so expect at least seeing the quadratic before a factorised form.. Need not see the "k+1" explicitly for this mark.

A1: Completes the induction by demonstrating the result clearly, with suitable conclusion conveying "true for $n = 1$", "assumed true for $n = k$" and "shown true for $n = k + 1$", and "hence true for all n". All these statements (or equivalents) must be seen in their conclusion (not simply scattered through the work). Depends on all except the B mark, though a check for $n = 1$ must have been attempted.

(b)

B1: Correct expansion.

M1: Applies the standard formula for $\sum r$ and the result from (a) to their sum.

If the expansion is given as $\sum r^3 - r^2$, allow the use of $\sum r^2$ instead of $\sum r$

M1: Takes out the common factors n and $(n+1)$ and attempts to simplify to required form OR factorises their quartic.

A1: Correct answer. (Ignore A, B and C listed explicitly.) Correct answer can be obtained from a cubic or a quartic. Award M1A1 in either of these cases.

(c)

M1:Attempts to apply $\sum_{r=n}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2$ with the standard result for $\sum r^2$ Accept with n instead of $n-1$ in second expression.

A1: Correct expression for the RHS seen, no need to be simplified.

dM1:Applies the summations to the equation in the question and cancels/factorises out the factor n. Depends on the first M mark of (c)

ddM1: Simplifies the quadratic factor of the right hand side and cancels/factors out the n+1 and solves the resulting quadratic. Note M1M1 is implied by sights of the correct roots $-\frac{1}{9}, 52, 0, -1$ of the quartic. Depends on both previous M marks in (c)

A1: Correct answer. Must reject other roots. The correct answer obtained from a quartic or cubic equation solved by calculator gains all relevant marks.

Q62.

Question Number	Scheme	Marks
(a)	$\alpha+\beta=-\frac{5}{2}\quad \alpha\beta=\frac{7}{2}$	B1
	$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=\left(-\frac{5}{2}\right)^3-3\left(\frac{7}{2}\right)\left(-\frac{5}{2}\right)$	M1
	$=\frac{85}{8}$	A1 (3)
(b)	$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=...$	M1
	$=\left(\frac{85}{8}\right)\times\left(\frac{2}{7}\right)=\frac{85}{28}$	A1
	$\frac{\alpha^2}{\beta}\times\frac{\beta^2}{\alpha}=\alpha\beta=\frac{7}{2}$	B1ft
	$x^2-\text{"}\frac{85}{28}\text{"}x+\text{"}\frac{7}{2}\text{"}\ (=0)$	M1
	$28x^2-85x+98=0$	A1 (5) [8]
	Note: if a candidate solves the equation and uses the roots to answer the question, then send to review.	
(a) B1	Both correct. (Seen anywhere in the working)	
M1	Uses their sum and product of roots in a correct expression for $\alpha^3+\beta^3$.	
A1	Correct value. Must be exact. Accept 10.625	
(b) M1	substitutes their values for $\alpha^3+\beta^3$ and $\alpha\beta$ into $\frac{\alpha^3+\beta^3}{\alpha\beta}=...$ (allow slips in substitution).	
A1	Correct sum as a single fraction (may be seen or implied in their equation)	
B1ft	Correct product or follow through their product	
M1	Use $x^2-\text{sum of roots}\times x+\text{product of roots}$ with their values for sum and product. "= 0" may be missing.	
A1	A correct final equation as shown or any integer multiple of this. " = 0" must be included.	

Q63.

Question	Scheme	Marks
(a)	$\alpha+\beta=-\frac{3}{4}$	B1
	$\alpha\beta=\frac{k}{4}$	B1
		(2)
(b)	$\frac{\alpha}{\beta^2}+\frac{\beta}{\alpha^2}=\frac{\alpha^3+\beta^3}{\alpha^2\beta^2}$	B1
	$=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^2};=\frac{\left(-\frac{3}{4}\right)^3-3\left(\frac{k}{4}\right)\left(-\frac{3}{4}\right)}{\left(\frac{k}{4}\right)^2}=\ldots$	M1; M1
	$=\frac{36k-27}{4k^2}=\frac{9}{k}-\frac{27}{4k^2}$	A1
		(4)
(c)	Product of roots is $\frac{\alpha\beta}{\alpha^2\beta^2}=\frac{1}{\alpha\beta}=\frac{4}{k}$	B1ft
	Equation is $x^2-\left(\frac{36k-27}{4k^2}\right)x+\frac{4}{k}=0$	M1
	$4k^2x^2-(36k-27)x+16k=0$	A1
		(3)
		(9 marks)

Notes:

(a)

B1: Correct expression for $\alpha+\beta$

B1: Correct expression for $\alpha\beta$

(b)

B1: Combines the fractions correctly.

M1: For a correct identity for the sum of cubes.

M1: Substitutes their values for $\alpha+\beta$ and $\alpha\beta$ into their equation for sum of $\frac{\alpha}{\beta^2}+\frac{\beta}{\alpha^2}$ (not dependent, so there may be a slip in the identity used for $\alpha^3+\beta^3$).

A1: Correct expression in terms of k in a simplified form - e.g. either form as shown in scheme.

(c)

B1ft: Correct product of roots in terms of k, or follow through $\frac{1}{\text{their }\alpha\beta}$ from part (a).

M1: Applies $x^2-(\text{their sum of roots})x+\text{their product of roots}(=0)$. Allow without the "=0" for this mark.

A1: Correct equation, as shown or an integer multiple thereof. Accept equivalents for the x term (e.g. $4k^2x^2+(27-36k)x+16k=0$. Must include the "=0".

Q64.

Question	Scheme		Marks
(a)	$2x^2+4x-3=0$ has roots α, β		
	$\alpha+\beta=-\frac{4}{2}$ or -2, $\alpha\beta=-\frac{3}{2}$	**Both** $\alpha+\beta=-\frac{4}{2}$ **and** $\alpha\beta=-\frac{3}{2}$. This may be seen or implied anywhere in this question.	B1
(i)	$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\ldots\ldots$	Use of a **correct** identity for $\alpha^2+\beta^2$ (May be implied by their work)	M1
	$=(-2)^2-2\left(-\frac{3}{2}\right)=7$	7 **from correct working**	A1 cso
(ii)	$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=\ldots\ldots$ or $=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)=\ldots\ldots$	Use of an **appropriate** and **correct** identity for $\alpha^3+\beta^3$ (May be implied by their work)	M1
	$=(-2)^3-3\left(-\frac{3}{2}\right)(-2)=-17$ or $=(-2)\left(7--\frac{3}{2}\right)=-17$	-17 **from correct working**	A1 cso
			(5)

(b)	Sum $=\alpha^2+\beta+\beta^2+\alpha$ $=\alpha^2+\beta^2+\alpha+\beta$ $=7+(-2)=5$	Uses at least one of their $\alpha^2+\beta^2$ or $\alpha+\beta$ in an attempt to find a **numerical value** for the sum of $(\alpha^2+\beta)$ and $(\beta^2+\alpha)$	M1
	Product $=(\alpha^2+\beta)(\beta^2+\alpha)$ $=(\alpha\beta)^2+\alpha^3+\beta^3+\alpha\beta$ $=\left(-\frac{3}{2}\right)^2-17-\frac{3}{2}=-\frac{65}{4}$	Expands $(\alpha^2+\beta)(\beta^2+\alpha)$ and uses at least one of their $\alpha\beta$ or $\alpha^3+\beta^3$ in an attempt to find a **numerical value** for the product of $(\alpha^2+\beta)$ and $(\beta^2+\alpha)$	M1
	$x^2-5x-\frac{65}{4}=0$	Applies $x^2-(\text{sum})x+\text{product}$ (Can be implied) ("$=0$" not required)	M1
	$4x^2-20x-65=0$	Any integer multiple of $4x^2-20x-65=0$, including the "$=0$"	A1
			(4)

(b) *continued*	**Alternative:** Finding $\alpha^2+\beta$ and $\beta^2+\alpha$ explicitly			
	Eg. Let $\alpha = \dfrac{-4+\sqrt{40}}{4}$, $\beta = \dfrac{-4+\sqrt{40}}{4}$ and so $\alpha^2+\beta = \dfrac{5-3\sqrt{10}}{2}$, $\beta^2+\alpha = \dfrac{5+3\sqrt{10}}{2}$			
	$\left(x-\left(\dfrac{5-3\sqrt{10}}{2}\right)\right)\left(x-\left(\dfrac{5+3\sqrt{10}}{2}\right)\right)$	Uses $\left(x-(\alpha^2+\beta)\right)\left(x-(\beta^2+\alpha)\right)$ with exact numerical values. (May expand first)		M1
	$= x^2-\left(\dfrac{5+3\sqrt{10}}{2}\right)x-\left(\dfrac{5-3\sqrt{10}}{2}\right)x+\left(\dfrac{5-3\sqrt{10}}{2}\right)\left(\dfrac{5+3\sqrt{10}}{2}\right)$		Attempts to expand using exact numerical values for $\alpha^2+\beta$ and $\beta^2+\alpha$	M1
	$\Rightarrow x^2-5x-\frac{65}{4}=0$	Collect terms to give a 3TQ. ("$=0$" not required)		M1
	$4x^2-20x-65=0$	Any integer multiple of $4x^2-20x-65=0$, including the "$=0$"		A1
				(4)
				(9 marks)

Notes:

(a)

1st A1: $\alpha+\beta=2,\ \alpha\beta=-\dfrac{3}{2} \Rightarrow \alpha^2+\beta^2=4-2\left(-\frac{3}{2}\right)=7$ is M1A0 cso

Finding $\alpha+\beta=-2,\ \alpha\beta=-\dfrac{3}{2}$ by writing down or applying $\dfrac{-4+\sqrt{40}}{4}, \dfrac{-4+\sqrt{40}}{4}$ but then writing $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=4+3=7$ and $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=-8-9=-17$ scores B0M1A0M1A0 in part (a).

Applying $\dfrac{-4+\sqrt{40}}{4}, \dfrac{-4+\sqrt{40}}{4}$ explicitly in part (a) will score B0M0A0M0A0

Eg: Give no credit for $\left(\dfrac{-4+\sqrt{40}}{4}\right)^2+\left(\dfrac{-4+\sqrt{40}}{4}\right)^2=7$

or for $\left(\dfrac{-4+\sqrt{40}}{4}\right)^3+\left(\dfrac{-4+\sqrt{40}}{4}\right)^3=-17$

(b)

Candidates are allowed to apply $\dfrac{-4+\sqrt{40}}{4}, \dfrac{-4+\sqrt{40}}{4}$ explicitly in part (b).

A correct method leading to a candidate stating $a=4,\ b=-20,\ c=-65$ without writing a final answer of $4x^2-20x-65=0$ is final M1A0

Q65.

Question Number	Scheme	Notes	Marks
	Solutions that rely entirely on solving the equation are generally unlikely to score but there may be attempts which include some of the work below which can receive appropriate credit.		
(a)	$\alpha+\beta=6 \quad \alpha\beta=3$	Correct sum and product. Could be implied. Allow $\frac{6}{1}$ and $\frac{3}{1}$	B1
	$(\alpha^2+1)(\beta^2+1)=\alpha^2\beta^2+\alpha^2+\beta^2+1$	Multiplies $(\alpha^2+1)(\beta^2+1)$ to obtain 3 or 4 terms with 3 correct. Do not condone $\alpha\beta^2$ for $(\alpha\beta)^2$ unless implied later	M1
	$=\alpha^2\beta^2+(\alpha+\beta)^2-2\alpha\beta+1$	Uses $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$	M1
	$\{=3^2+6^2-2\times3+1\}$ $=40$	Correct answer from correct work. Use of e.g., $\alpha+\beta=-6$ is A0	A1
			(4)

(b)	Allow use of their $(\alpha^2+1)(\beta^2+1)$ which could be from (a) or a first or reattempt in (b). Numerator must be correct		
	$\frac{\alpha}{(\alpha^2+1)}+\frac{\beta}{(\beta^2+1)}=\frac{\alpha(\beta^2+1)+\beta(\alpha^2+1)}{"(\alpha^2+1)(\beta^2+1)"}$	Any correct expression with their $(\alpha^2+1)(\beta^2+1)$ for the new sum as a single fraction (or two fractions both with the common denominator)	B1
	$=\frac{\alpha\beta(\beta+\alpha)+(\alpha+\beta)}{"(\alpha^2+1)(\beta^2+1)"}=\frac{"3"\times"6"+"6"}{"40"}=\ldots$	Uses a correct expression with their $(\alpha^2+1)(\beta^2+1)$ for the new sum to obtain a correct numerical expression with their denominator, $\alpha+\beta$ & $\alpha\beta$ and achieves a value.	M1
	$\frac{\alpha\beta}{"(\alpha^2+1)(\beta^2+1)"}=\frac{"3"}{"40"}$	Uses a correct expression with their $(\alpha^2+1)(\beta^2+1)$ for the new product to obtain a correct value with their denominator and $\alpha\beta$	M1
	new sum $=\frac{24}{40}\left\{=\frac{3}{5}\right\}$ or new product $=\frac{3}{40}$	One value for new sum or new product correct. Any equivalent fractions. Not ft. **Requires appropriate previous M mark.**	A1
	$x^2-\frac{24}{40}x+\frac{3}{40}\quad\{=0\}$	Correctly uses $x^2-(\text{sum of roots})x+(\text{product of roots})$ or equivalent work with their new sum and product. Condone use of a different variable. Allow appropriate values for *p*, *q* and *r*	M1
	$40x^2-24x+3=0$	Any correct equation with integer coefficients and "= 0". Condone use of a different variable. Allow e.g., $p=40$, $q=-24$, $r=3$. Requires all marks.	A1
			(6)
	Note that although $(\alpha^2+1)(\beta^2+1)$ may be attempted or reattempted in (b) there is no credit for work in (a) that is only seen in (b)		**Total 10**

Q66.

Question	Scheme	Marks
(a)	$4-3\text{i}$	B1
		(1)
(b)	$(x-(4+3\text{i}))(x-(4-3\text{i}))=\ldots$	M1
	$x^2-8x+25$	A1
		(2)
(c)	E.g. Product of roots is 225, so product of real roots is $\frac{225}{25}=9$ Or $x^4+Ax^3+Bx^2+Cx+225=(x^2-8x+25)(x^2+\ldots+9)$	M1
	Hence (as root is positive) repeated real root is 3	A1
		(2)
(d)	$(x^2-8x+25)(x^2-6x+9)$ $=x^4-6x^3+9x^2-8x^3+48x^2-72x+25x^2-150x+225$	M1
	$=x^4-14x^3+82x^2-222x+225$ Two correct middle term coefficients	A1
	So $A=-14$, $B=82$ and $C=-222$ (or accept in the quartic)	A1
		(3)
		(8 marks)

Notes:

(a)

B1: For $4-3\text{i}$

(b)

M1: Correct strategy to find a quadratic factor. May expand as shown in scheme, or may look for sum of roots and product of roots first and then write down the factor.

A1: Correct quadratic factor. Can be written down – give M1A1 if correct, M0A0 if incorrect. Ignore "= 0" with their quadratic factor.

Alt for (b):

M1: Product of complex roots is 25, so product of real roots is $\frac{225}{25}=9$, so the (positive) real root is "3", hence quadratic factor is $(x-"3")^2$

A1: x^2-6x+9 or $(x-3)^2$

(c)

M1: A complete strategy to deduce the real root or its square. May consider product of roots, as in scheme, or may first attempt to factorise/long division to find the other quadratic factor – award at the point the quadratic factor with real roots is found. May have been seen in (b)

A1: Real root is 3. (No need to see rejection of the negative possibility.)

Not a "show that" so award M1A1 if correct root is written down with no working.

(d)

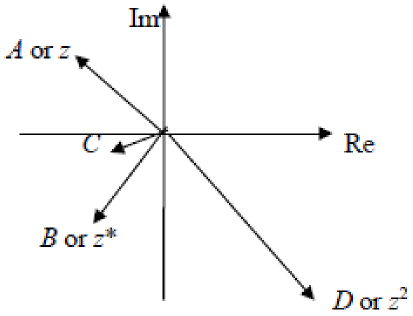
M1: Attempts to expand the two quadratic factors – one of which must have a repeated root, so $(x^2\pm9)$ scores M0. (Alternative, may apply –(sum of roots) to find A, pair sum to find B etc – accept method for at least two constants.)

A1: Two correct values of the three. Accept as embedded in a quartic equation.

A1: All three correct. Accept as embedded in their quartic equation.

If their answers are wrong a correct method would get M1A0A0 but w/o some working score M0

Q67.

Question Number	Scheme	Marks
(a)	$\lambda = 4$	B1 (1)
(b)	$\arctan\frac{3}{"4"}$ or $\arctan\frac{-3}{"4"}$	M1
	(Second quadrant so arg z = 2.498...) = 2.5 (rad)	A1 (2)
(c)(i)	$\frac{z+3i}{2-4i}=\frac{-4+6i}{2-4i}\times\frac{2+4i}{2+4i}$ or $\frac{z+3i}{2-4i}=a+ib \Rightarrow -4+6i=(a+ib)(2-4i)$	M1
	$=\frac{-8+12i-16i+24i^2}{4+16}=-\frac{8}{5}-\frac{1}{5}i$ Accept e.g. $\frac{-32-4i}{20}$ Or $2a+4b=-4, 2b-4a=6 \Rightarrow a=..,b=..$	dM1A1
(ii)	$z^2=(-4+3i)^2=16-24i+9i^2=16-24i-9$	M1
	$=7-24i$	A1ft (5)
(d)		B1 B1ft B1ft (3) [11]

(a) B1	Correct answer. No working needed.
(b) M1	For $\arctan\left(\pm\frac{3}{"4"}\right)$ with their "4". Can be awarded from $\tan\theta=\pm\frac{3}{"4"}\Rightarrow\theta=...$ or by implication if correct value for either arctan or correct final answer (rounded or not rounded, may be degrees) is seen.
A1	Cao 2.5
(c)(i) M1	Multiplies numerator and denominator by complex conjugate of denominator. Award if denominator of 4+16 or 20 is seen instead of product. May still have z at this stage, or even allow with $\lambda+3i$ as numerator. Alternatively, sets equal to $a+ib$ and cross multiplies.
dM1	Using their λ or 4 substitutes **correctly** for z, fully expands the numerator and uses $i^2=-1$ Alt, uses $i^2=-1$, equates real and imaginary terms and solves their equations for a and b
A1	Correct answer only, as shown or single fraction accepting equivalent fractions or with exact decimals $(-1.6-0.2i)$.
(ii) M1	Squaring an expression of form $k+3i$ (with a real value for k) to get 3 terms (may be implied) and uses $i^2=-1$
A1ft	Correct answer, follow through their $\lambda>0$ (ie for $"\lambda^2-9"-"6\lambda"i$ must be negative i term)
(d)	NB: Penalise once only (in the first mark due) for mislabelling or failing to label points as long as they look to be placed correctly. Award if lines/arrows not included. Points may be labelled by letter, name or their Cartesian coordinates (which may be given on the axes).
B1	Plots z in second quadrant and $z*$ as mirror image in the Real axis. Both must be labelled.
B1ft	Plot and label C for their solution to (c)(ii) It must be the correct side of B (for their

B1ft	answers) and a correct relative scale (so noticeably closer to O than their B if correct values). Plot and label their D (- 24 need not be to scale, but should be further from O than their B).

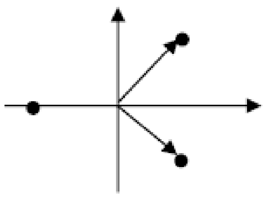
Q68.

Question Number	Scheme	Marks
(a)	$\frac{3}{8}-\frac{\sqrt{71}}{8}\mathrm{i}$	B1 (1)
(b)	$\left(x-\frac{3}{8}-\frac{\sqrt{71}}{8}\mathrm{i}\right)\left(x-\frac{3}{8}+\frac{\sqrt{71}}{8}\mathrm{i}\right)\ \left((x-4)=0\right)$	
	$\left(x^2-\frac{3}{4}x+\frac{5}{4}\right)\ \left((x-4)=0\right)$	M1A1
	$x^3-\frac{19}{4}x^2+\frac{17}{4}x-5\ (=0)$	dM1
	$4x^3-19x^2+17x-20\ (=0)\quad p=17,\ q=-20$	A1 (4) [5]
(a) B1 (b)	Correct answer only	
M1	Attempt the multiplication of the 2 brackets with the complex terms. Allow $(x \pm \text{root})$ for the brackets. Allow "invisible" brackets.	
A1	Correct quadratic obtained - may have multiplied by the 4 (or other constant factor) and this is fine. (Need not be fully simplified but must have real terms)	
dM1	Attempt to multiply their quadratic by $(x-4)$ or may divide their quadratic into the cubic or other full method leading to at least one of p or q.	
A1	Correct values. Values of p and q need not be shown explicitly but may been seen in a cubic, provided the cubic starts $4x^3-19x^2$ (isw after a correct cubic)	
	Note if a candidate uses a hybrid method, mark under main scheme unless an Alt scores more marks.	

Alt 1 (b)	$-\frac{q}{4}=4\times\left(\frac{3}{8}+\frac{\sqrt{71}}{8}\mathrm{i}\right)\times\left(\frac{3}{8}-\frac{\sqrt{71}}{8}\mathrm{i}\right)=...\to q=..$ **or** $\frac{p}{4}=4\left(\frac{3}{8}+\frac{\sqrt{71}}{8}\mathrm{i}\right)+4\left(\frac{3}{8}-\frac{\sqrt{71}}{8}\mathrm{i}\right)+\left(\frac{3}{8}+\frac{\sqrt{71}}{8}\mathrm{i}\right)\left(\frac{3}{8}-\frac{\sqrt{71}}{8}\mathrm{i}\right)\to p=...$	M1
	$\Rightarrow q=-16\times\left(\frac{9}{64}+\frac{71}{64}\right)=-20$ **or** $p=17$	A1
	E.g. $f(4)=0\Rightarrow 4(4)^3-19(4)^2+4p-"20"=0\Rightarrow p=...$ or $\frac{p}{4}=4\left(\frac{3}{8}+\frac{\sqrt{71}}{8}\mathrm{i}\right)+4\left(\frac{3}{8}-\frac{\sqrt{71}}{8}\mathrm{i}\right)+\left(\frac{3}{8}+\frac{\sqrt{71}}{8}\mathrm{i}\right)\left(\frac{3}{8}-\frac{\sqrt{71}}{8}\mathrm{i}\right)\to p=...$	dM1
	$p=17,\ q=-20$	A1 (4)

Question Number	Scheme	Marks
M1	A correct attempt to use product of roots is $-\frac{q}{4}$ to find a value for q or pair sum is $\frac{p}{4}$ to find a value of p.	
A1	Correct value for p or q	
dM1	Correct full method to find both p and q.	
A1	Correct values for both	
Alt 2	Attempts at using the factor theorem are possible but unlikely to succeed. Score as follows: M1: Uses the factor theorem to generate two equations in the two unknowns (note they will need to use a complex root to achieve this and equate real and imaginary parts.). A1: Correct equations. dM1: Solves their two equations to find values for p and q. A1: Correct values Send to review if unsure.	
Alt 3 (b)	$x-4\,\overline{)\,4x^3-19x^2+px+q}$ with quotient $4x^2-3x+p-12$ $4x^3-16x^2$ $-3x^2+px+q$ $-3x^2+12x$ $(p-12)x+q$ $(p-12)x-4(p-12)$	M1 A1
	$\Rightarrow q+4(p-12)=0\ \ \&\ \ \frac{p-12}{4}=\left(\frac{3}{8}+\frac{\sqrt{71}}{8}\mathrm{i}\right)\left(\frac{3}{8}-\frac{\sqrt{71}}{8}\mathrm{i}\right)=\frac{5}{4}$	dM1
	$P=17,\ q=-20$	A1 (4)
(b) **M1**	Divides $x-4$ into the cubic to achieve a 3TQ quotient and a remainder	
A1	Correct quotient and remainder	
dM1	Correct full method to find p or q	
A1	Correct values	

Q69.

Question	Scheme	Marks
	$f(z) = 4z^3 + pz^2 - 24z + 108$, -3 a root.	
(a)	$f(-3) = 0 \Rightarrow 4(-3)^3 + p(-3)^2 - 24(-3) + 108 = 0 \Rightarrow p = ...$	M1
	$p = -8$	A1
		(2)
(b)	$4z^3 - 8z^2 - 24z + 108 = (z+3)(4z^2 + ...z + 36)$	M1
	$= (z+3)(4z^2 - 20z + 36)$	A1
	$4z^2 - 20z + 36 = 0 \Rightarrow z = \dfrac{20 \pm \sqrt{400 - 4\times4\times36}}{8} = ...$	dM1
	Roots are $-3, \dfrac{5 \pm i\sqrt{11}}{2}$	A1
		(4)
(c)	e.g. Product of complex roots is $\dfrac{36}{4} = 9$, so modulus is $\sqrt{"9"}$ or Modulus is $\sqrt{\left(\dfrac{5}{2}\right)^2 + \left(\dfrac{\sqrt{11}}{2}\right)^2}$	M1
	Hence modulus is 3	A1
		(2)
(d)	 Complex conjugate pair in correct quadrant for their roots	M1
	All three roots correctly positioned.	A1
		(2)
		(10 marks)

Notes:
Mark the question as a whole - do not be concerned part labelling. **(a)** **M1:** A complete method to find the value of p. Use of the factor theorem is most direct, look for setting $f(-3) = 0$ and solving for p. May attempt to factor out $(z+3)$ and compare coefficients, e.g. $f(z) = 4z^3 + pz^2 - 24z + 108 = (z+3)(4z^2 + bz + 36) \Rightarrow 3b + 36 = -24, 12 + b = p \Rightarrow b = ..., p = ...$ or may attempt long division and set remainder equal to zero to find p or variations on these. **A1:** For $p = -8$ **(b)** **Note:** Allow marks in (b) for work seen in (a) e.g. via attempts in (a) by long division. **M1:** Correct strategy to find a quadratic factor. If factorising, look for correct first and last terms. May use long division, in which case look for the correct first term and attempt to use it - may have been seen in (a). Question instructs use of algebra so an algebraic method must be seen. **A1:** Correct quadratic factor - may have been seen in (a). **dM1:** Uses the quadratic formula or completing the square or calculator to find the roots of their quadratic factor (allow for attempts at a quadratic factor via long division which had non-zero remainder). If a calculator is used (no method shown), there must be at least one correct **complex** root for their equation. Factorisation is M0. **A1:** Correct roots in simplest form. All three should be included at some point in the solution in (b). **(c)** **M1:** Any correct method to find the modulus of the complex roots. Most likely to see Pythagoras, but some may deduce from product of roots. They must have complex roots to score the marks in (c). **A1:** Modulus 3 only. If -3 is also given as a modulus then score A0. **(d)** **Note:** Allow the marks in (d) if the i's were missing in their roots in (b) but they clearly mean the correct complex roots on their diagram. **M1:** Plots the complex roots as a conjugate pair in the correct quadrants for their roots. **A1:** Fully correct diagram with one root on the negative real axis, and the other as a complex pair of roughly the same length in quadrants 1 and 4.

Q70.

Question	Scheme	Marks
(a)	(i) $\alpha+\beta=-\frac{5}{A}$	B1
	(ii) $\alpha\beta=-\frac{12}{A}$	B1
		(2)
(b)	$\left(\alpha-\frac{3}{\beta}\right)+\left(\beta-\frac{3}{\alpha}\right)=(\alpha+\beta)-3\left(\frac{\alpha+\beta}{\alpha\beta}\right)=-\frac{5}{A}-3\left(\frac{-5}{A}\right)\times\frac{-A}{12}$	M1
	$-\frac{5}{A}-\frac{15}{12}=\frac{5}{4}\Rightarrow A=\ldots$	dM1
	$A=-2$	A1
		(3)
(c)	$\left(\alpha-\frac{3}{\beta}\right)\left(\beta-\frac{3}{\alpha}\right)=\alpha\beta-6+\frac{9}{\alpha\beta}=-\frac{12}{A}-6+\frac{9}{-12/A}$	M1
	$-\frac{12}{\text{"}-2\text{"}}-6-\frac{9\text{"}-2\text{"}}{12}=\frac{B}{4}\Rightarrow B=\ldots$	dM1
	$B=6$	A1
		(3)
		(8 marks)

Notes:

(a)
(i) B1: Correct expression for $\alpha+\beta$
(ii) B1: Correct expression for $\alpha\beta$
(b)
M1: Attempts the sum of roots for second equation in terms of A using results from (a). Allow slips in signs.
dM1: Equates the sum of roots to $\frac{5}{4}$ and solves for A. Depends on the previous M mark.
A1: $A=-2$
(c)
M1: Attempts the product of roots for second equation in terms of A using results from (a). Allow slips in signs. May be using their value of A or A itself
dM1: Equates the product of roots to $\frac{B}{4}$ and solves for B using their value of A. Depends on first M mark of (c).
A1: $B=6$

Q71.

Question Number	Scheme	Notes	Marks
	$2x^2-5x+7=0$		
(a)	$\alpha+\beta=\frac{5}{2},\quad \alpha\beta=\frac{7}{2}$	Both	B1
			(1)
(b)(i)	$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$	Attempts to use a correct identity	M1
	$=\left(\frac{5}{2}\right)^2-2\left(\frac{7}{2}\right)=-\frac{3}{4}$	cso – must have scored the B1	A1
(ii)	$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$	Attempts to use a correct identity	M1
	$=\left(\frac{5}{2}\right)^3-3\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)=-\frac{85}{8}$	cso – must have scored the B1	A1
			(4)
(c)	$\text{Sum}=\frac{1}{\alpha^2+\beta}+\frac{1}{\beta^2+\alpha}=\frac{\alpha^2+\beta+\beta^2+\alpha}{(\alpha^2+\beta)(\beta^2+\alpha)}$ $=\frac{\alpha^2+\beta^2+\alpha+\beta}{\alpha^2\beta^2+\alpha^3+\beta^3+\alpha\beta}=\frac{-\frac{3}{4}+\frac{5}{2}}{\frac{49}{4}-\frac{85}{8}+\frac{7}{2}}\left(=\frac{14}{41}\right)$ Attempts sum – substitutes their into a correct numerator must but allow slips in the denominator as long as 4 terms are produced from the expansion.		M1
	$\text{Product}=\frac{1}{\alpha^2+\beta}\times\frac{1}{\beta^2+\alpha}=\frac{1}{\alpha^2\beta^2+\alpha^3+\beta^3+\alpha\beta}=\frac{1}{\frac{49}{4}-\frac{85}{8}+\frac{7}{2}}\left(=\frac{8}{41}\right)$ Attempts product – must be correct expansion of denominator with their values.		M1
	$x^2-\frac{14}{41}x+\frac{8}{41}(=0)$	Applies $x^2-(\text{their sum})x+\text{their product }(=0)$ Depends on at least one previous M awarded.	dM1
	$41x^2-14x+8=0$	Allow any integer multiple. Must include "=0"	A1
			(4)
			Total 9

Q72.

Question Number	Scheme	Notes	Marks
	$2x^2-3x+5=0$		
(a)	$\alpha+\beta=\frac{3}{2},\quad \alpha\beta=\frac{5}{2}$	Both	B1
			(1)
(b)(i)	$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$	Uses a correct identity	M1
	$=\left(\frac{3}{2}\right)^2-2\left(\frac{5}{2}\right)=-\frac{11}{4}\ (=-2.75)$	Correct value Allow to come from $\alpha+\beta=-\frac{3}{2}$	A1
(ii)	$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$	Reaches an identity ready for substitution	M1
	$=\left(\frac{3}{2}\right)^3-3\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)=-\frac{63}{8}\ (=-7.875)$	Correct value	A1
			(4)
(c)	$\text{Sum}=\alpha^3+\beta^3-(\alpha+\beta)=-\frac{63}{8}-\frac{3}{2}\left(=-\frac{75}{8}\right)$	Attempts sum Allow eg $(\alpha^3-\beta)+(\beta^3-\alpha)$ followed by $(\alpha^3+\beta^3)+(\alpha+\beta)=\ldots$	M1
	$\text{Prod}=(\alpha\beta)^3-\alpha^4-\beta^4+\alpha\beta$ and $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2(\alpha\beta)^2$	Expands $(\alpha^3-\beta)(\beta^3-\alpha)$ and uses a correct identity for $\alpha^4+\beta^4$	M1
	Alt identities: $\alpha^4+\beta^4=$ $(\alpha+\beta)^4-4\alpha\beta(\alpha^2+\beta^2)-6\alpha^2\beta^2;\ \alpha^4+\beta^4=(\alpha^3+\beta^3)(\alpha+\beta)-\alpha\beta(\alpha^2+\beta^2)$		
	$(\alpha\beta)^3-\alpha^4-\beta^4+\alpha\beta=\left(\frac{5}{2}\right)^3+\frac{5}{2}-\left(\left(-\frac{11}{4}\right)^2-2\left(\frac{5}{2}\right)^2\right)=\frac{369}{16}$		A1
	$x^2+\frac{75}{8}x+\frac{369}{16}(=0)$	Applies x^2-(their sum)$x+$ their prod $(=0)$	M1
	$16x^2+150x+369=0$	Allow any integer multiple	A1
			(5)
			Total 10

Q73.

Question	Scheme	Marks
	$2m^2-5m-3=0 \Rightarrow (2m+1)(m-3)=0 \Rightarrow m=...$	M1
	So C.F. is $(y_{CF}=)\,Ae^{-\frac{1}{2}x}+Be^{3x}$	A1
	P.I. is $y_{PI}=axe^{3x}$	B1
	$\frac{dy_{PI}}{dx}=3axe^{3x}+ae^{3x},\ \frac{d^2y_{PI}}{dx^2}=9axe^{3x}+3ae^{3x}+3ae^{3x}$ $\Rightarrow 2(9ax+6a)e^{3x}-5(3ax+a)e^{3x}-3axe^{3x}=2e^{3x} \Rightarrow a=...$	M1
	$a=\frac{2}{7}$	A1
	General solution is $y=Ae^{-\frac{1}{2}x}+Be^{3x}+\frac{2}{7}xe^{3x}$	B1ft
		(6)
		(6 marks)

Notes:

M1: Forms and solves the auxiliary equation.
A1: Correct complementary function (no need for $y = \ldots$)

B1: Correct form for the particular integral. Accept any PI that includes axe^{3x}, so e.g. $(ax+b)e^{3x}$ is fine.

M1: Attempts to differentiate their PI twice and substitutes into the left hand side of the equation. The derivatives must be changed functions. There is no need to reach a value for the unknown(s) but their PI must contain an unknown constant.

A1: Correct value of a (and any other coefficients as zero). Must have had a suitable PI

B1ft: For y = their CF + their PI. Must include the $y =$. The PI must be a function of x that matches their initial choice of PI, with their constants substituted.

Q74.

Question Number	Scheme	Marks
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-8\frac{\mathrm{d}y}{\mathrm{d}x}+16y=48x^2-34$	
(a)	(AE:) $m^2-8m+16=0$ $\left[\Rightarrow(m-4)^2=0\right]$ $\Rightarrow m=4$	M1
	(CF: $y=$) $(A+Bx)\mathrm{e}^{4x}$	A1
	(PI: $y=$) $\lambda x^2+\mu x+\nu$	B1
	$y'=2\lambda x+\mu \quad y''=2\lambda$ $2\lambda-8(2\lambda x+\mu)+16(\lambda x^2+\mu x+\nu)=48x^2-34$ $16\lambda x^2=48x^2 \; (-16\lambda+16\mu)x=0 \quad 2\lambda-8\mu+16\nu=-34$ $\lambda=3 \quad \mu=3 \quad \nu=-1$	M1
	$y=\text{"}(A+Bx)\mathrm{e}^{4x}\text{"}+3x^2+3x-1$	A1ft
		(5)
(b)	$(0,4)\Rightarrow 4=A-1 \quad (A=5)$	M1
	$\frac{\mathrm{d}y}{\mathrm{d}x}=4(A+Bx)\mathrm{e}^{4x}+B\mathrm{e}^{4x}+6x+3$	M1
	$21=4A+B+3\Rightarrow B=-2 \quad A=5$ $[y]=(5-2x)\mathrm{e}^{4x}+3x^2+3x-1$	M1A1
		(4)
(c)	$(x=-2\Rightarrow y=)\; 9\mathrm{e}^{-8}+5$	M1 A1
		(2)
		Total 11

Notes
(a) M1: Forms the auxiliary equation (condone one slip/copying error) and solves 3TQ. Usual rules. One consistent solution if no working (could be complex). Implied by a correct CF if no incorrect working shown. A1: Correct complementary function $y=$ ... not required. May only be seen in final answer. B1: Correct form for particular integral $y=$ ... not required. M1: Correct method to obtain value for constants (or constant - but PI must be a quadratic - but could have 1 or 2 terms) - so differentiates twice (powers reduced) and substitutes, equates terms and solves equations. Allow if there are minor slips in the process if the holistic approach is correct. A1ft: A correct general solution following through on their CF only - the PI must be correct. Must have " $y=$ " e.g., not " GS = " (b) M1: Uses (0, 4) in their answer to (a) and forms an equation in one or both of their constants. M1: Differentiates their GS, which must contain a term "$Bx\mathrm{e}^{kx}$", to obtain an expression of the correct form for their GS - product rule must be used, powers reduced, but may have errors in coefficients. M1: Substitutes $x=0,\ \frac{\mathrm{d}y}{\mathrm{d}x}=21$ into their equation where their derivative is a changed function and finds values for their B (and their A if not found already). A1: Any correct equation Condone e.g., " PS =... " (c) M1: Substitutes $x=-2$ into their particular solution and obtains an expression of the right form or non-zero values for p, q and r, which need not be integers for the M but should be gathered terms. Implied by a correct answer as long as it fits their answer to (b) if no method shown. A1: Correct expression or values

Q75.

Question Number	Scheme	Notes	Marks
(a)	$r(r-1)(r-3)=r^3-4r^2+3r$	Correct expansion	B1
	$\sum_{r=1}^{n}(r^3-4r^2+3r)=\frac{1}{4}n^2(n+1)^2-4\frac{1}{6}n(n+1)(2n+1)+3\times\frac{1}{2}n(n+1)$ M1: Attempt to use at least two of the standard formulae correctly A1: Correct expression		M1A1
	$=\frac{1}{12}n(n+1)[3n(n+1)-8(2n+1)+18]$	Attempt to factorise $\frac{1}{12}n(n+1)$ from an expression with these factors. Depends on previous M.	dM1
	Note: for attempts that first expand to a quartic this mark may be awarded at the point the relevant factors are taken out provided a suitable quadratic factor is seen before the final answer.		
	$=\frac{1}{12}n(n+1)[3n^2-13n+10]$ $=\frac{1}{12}n(n+1)(n-1)(3n-10)$*	Cso with $3n^2-13n+10$ (or another appropriate correct quadratic) seen before the final printed answer.	A1*
			(5)
(b)	$\sum_{r=n+1}^{2n+1}r(r-1)(r-3)=\frac{1}{12}(2n+1)(2n+2)2n(6n-7)-\frac{1}{12}n(n+1)(n-1)(3n-10)$ Attempts f (2*n* + 1) – f (*n*)		M1
	$=\frac{1}{12}n(n+1)[4(2n+1)(6n-7)-(n-1)(3n-10)]=\frac{1}{12}n(n+1)(\ldots n^2+\ldots n+\ldots)$ Attempt to factor out $\frac{1}{12}n(n+1)$ and simplify the rest to 3 term quadratic expression. For attempts expanding to a quartic first, score for reaching an expression of the correct form.		dM1
	$=\frac{1}{12}n(n+1)(45n^2-19n-38)$	Cao	A1
			(3)
			Total 8

Q76.

Question Number	Scheme	Notes	Marks
	$\sum_{r=0}^{n}(r+1)(r+2)$		
(a)	$\sum_{r=0}^{n} r^2+3r+2 = 2+\frac{1}{6}n(n+1)(2n+1)+\frac{3}{2}n(n+1)+2n$ M1: Attempt to use at least one of the standard formulae correctly A1: For $\frac{1}{6}n(n+1)(2n+1)+\frac{3}{2}n(n+1)+(2n \text{ or } 2n+2)$ A1:Fully correct expression		M1A1A1
	$\frac{1}{6}n(n+1)(2n+1)+\frac{3}{2}n(n+1)+2n+2=(n+1)\left[\frac{1}{6}n(2n+1)+\frac{3}{2}n+2\right]$ Attempt to factorise $(n+1)$ It is a "show" question so this must be seen (in any equivalent form). If their expression does not allow for factorising $(n+1)$ score M0		M1
	$\frac{1}{3}(n+1)\left[n^2+5n+6\right]$	May obtain a cubic and extract a different factor ie $n+2$ or $n+3$	
	$\frac{1}{3}(n+1)(n+2)(n+3)$*	Cso At least one intermediate step in the working must be seen.	A1*
			(5)
(a) **Way 2**	$\sum_{r=0}^{n}(r+1)(r+2)=\sum_{r=1}^{n+1}r(r+1)$ $=\sum_{r=1}^{n+1}r^2+r=\frac{1}{6}(n+1)(n+2)(2(n+1)+1)+\frac{1}{2}(n+1)(n+2)$ M1: Attempt to use at least one of the standard formulae correctly with $n=n+1$ A1: For $\frac{1}{6}(n+1)(n+2)(2(n+1)+1)$ or $\frac{1}{2}(n+1)(n+2)$ A1:Fully correct expression		M1A1A1
	$\frac{1}{6}(n+1)(n+2)(2(n+1)+1)+\frac{1}{2}(n+1)(n+2)=(n+1)\left[\frac{1}{6}(n+1)(2n+3)+\frac{1}{2}(n+2)\right]$ Attempt to factorise $(n+1)$ (see additional comments above)		M1
	$\frac{1}{3}(n+1)\left[n^2+5n+6\right]$	May obtain a cubic and extract a different factor ie $n+2$ or $n+3$	
	$\frac{1}{3}(n+1)(n+2)(n+3)$*	Cso At least one intermediate step in the working must be seen.	A1*

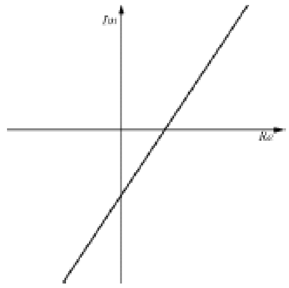
(b)	Upper limit = 99	Correct upper limit	B1
	$10\times11+11\times12+12\times13+...+100\times101=\sum_{r=0}^{99}(r+1)(r+2)-\sum_{r=0}^{8}(r+1)(r+2)$ Fully correct strategy for the sum using their upper limit for the first sum and upper limit 8 for the second in the result from (a). Lower limits 0 or 1		M1
	$=\frac{1}{3}(100)(101)(102)-\frac{1}{3}(9)(10)(11)$ $=343\,070$	Correct value	A1
			(3)
			Total 8

Q77.

Question Number	Scheme	Marks
(a)	$2n+1=A(n+1)^2+Bn^2 \Rightarrow 2n+1=An^2+2An+1+Bn^2$	
	$A=1\ \ B=-1$ or $\frac{1}{n^2}-\frac{1}{(n+1)^2}$	B1 (1)
(b)	$\sum_{r=5}^{n}\frac{2r+1}{r^2(r+1)^2}=\sum_{r=5}^{n}\left(\frac{1}{r^2}-\frac{1}{(r+1)^2}\right)$ $=\left(\frac{1}{5^2}-\cancel{\frac{1}{6^2}}\right)+\left(\cancel{\frac{1}{6^2}}-\cancel{\frac{1}{7^2}}\right)+\ldots\left(\cancel{\frac{1}{n^2}}-\frac{1}{(n+1)^2}\right)$	M1
	$\sum_{r=5}^{n}\frac{2r+1}{r^2(r+1)^2}=\frac{1}{5^2}-\frac{1}{(n+1)^2}$	A1
	$=\frac{n^2+2n+1-25}{25(n+1)^2}=\frac{n^2+2n-24}{25(n+1)^2}$	M1A1 (4)
		[5]

	Notes	
(a) B1	Both values correct with or without working seen, may be in the expression. Ignore incorrect working.	
(b) M1	Show sufficient terms to demonstrate the cancelling. Require at least one cancelling term seen. Must start at $r=5$ - M0 if starting at e.g $r=1$ unless there is a full process to complete the difference method (same condition) **and** apply f(n) — f(4)	
A1	Extract the two correct terms, or in the Alt obtains a correct overall expression.	
M1	Write the terms with a (non-zero) common denominator with at least numerator correct for their terms. Not dependent - may be scored following M0 if no cancelling terms were shown, but must have had exactly two terms to combine from differences.	
A1	Correct answer in the required form or accept correct values stated following an unsimplified form. (Allow as long as correct terms were extracted, even if no cancelling terms were shown.) Note: this means M0A0M1A1 can be scored for answers which show only the last two lines of the scheme with no cancelling process shown. Note: if e.g. r is used in place of n allow full marks if recovered, but A0 if left in terms of r.	
Alt (b) for first two marks	$\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}=\sum_{r=1}^{n}\left(\frac{1}{r^2}-\frac{1}{(r+1)^2}\right)$ $=\left(\frac{1}{1^2}-\cancel{\frac{1}{2^2}}\right)+\left(\cancel{\frac{1}{2^2}}-\cancel{\frac{1}{3^2}}\right)+\ldots\left(\cancel{\frac{1}{n^2}}-\frac{1}{(n+1)^2}\right)=1-\frac{1}{(n+1)^2}$ $\rightarrow\sum_{r=5}^{n}\frac{2r+1}{r^2(r+1)^2}=1-\frac{1}{(n+1)^2}-\left(1-\frac{1}{25}\right)\left(=\frac{n^2+2n}{(n+1)^2}-\frac{24}{25}\right)$	M1A1

Q78.

Question Number	Scheme	Notes	Marks
(a)		A straight line anywhere that is not vertical or horizontal which does not pass through the origin. It may be solid or dotted. Clear "V" shapes score M0.	M1
		A straight line in the correct position. Must have a positive gradient and lie in quadrants 1, 3 and 4. Ignore any intercepts correct or incorrect. If there are other lines that are clearly "construction" lines e.g. a line from 2i to 3 they can be ignored. The line may be solid or dotted. However, if there are clearly several lines then score A0.	A1
			(2)

Part (b)

The approaches below are the ones that have been seen most often.
Apply the mark scheme to the overall method the candidate has chosen.
There may be several attempts:

- **If none are crossed out, mark all attempts and score the best single complete attempt**
- **If some attempts are crossed out, mark the uncrossed out work**
- **If everything is crossed out, mark all the work and score the best single complete attempt**

Note that the question does not specify the variables the candidates should work in so they may use: e.g. $z = x + iy$ and $w = u + iv$ or $w = x + iy$ and $z = u + iv$ or any other letters so please check the work carefully.

Note that the M marks are all dependent on each other.

(b) Way 1	$w=\frac{iz}{z-2i}\Rightarrow z=\frac{2wi}{w-i}$	Attempts to make z the subject. Must obtain the form $\frac{awi}{bw+ci}$, a, b, c real and non-zero.	M1
	$z=\frac{2(u+iv)i}{u+iv-i}$ or e.g. $z=\frac{2(x+iy)i}{x+iy-i}$ $z=\frac{2(u+iv)i}{u+(v-1)i}\times\frac{u-(v-1)i}{u-(v-1)i}$ or equivalent Introduces $w=u+iv$ or e.g. $w=x+iy$ and attempts to multiply numerator and denominator by the complex conjugate of the denominator. The above statement would be sufficient e.g. no expansion is needed for this mark.		dM1
	$z=\frac{-2u}{u^2+(v-1)^2}+\frac{2u^2+2v(v-1)}{u^2+(v-1)^2}i$ or e.g. $z=\frac{-2x}{x^2+(y-1)^2}+\frac{2x^2+2y(y-1)}{x^2+(y-1)^2}i$ or $z=\frac{-2uv+2u(v-1)+(2u^2+2v(v-1))i}{u^2+(v-1)^2}$ or e.g. $z=\frac{-2xy+2x(y-1)+(2x^2+2y(y-1))i}{x^2+(y-1)^2}$ Correct expression for z in terms of their variables with real and imaginary parts identified. May be embedded as above or stated explicitly.		A1
	$\lvert z-2i\rvert=\lvert z-3\rvert\Rightarrow y-1=\frac{3}{2}\left(x-\frac{3}{2}\right)\left(y=\frac{3}{2}x-\frac{5}{4},\ 6x-4y=5\right)$ $\Rightarrow\frac{2u^2+2v(v-1)}{u^2+(v-1)^2}-1=\frac{3}{2}\left(\frac{-2u}{u^2+(v-1)^2}-\frac{3}{2}\right)$ Attempts the Cartesian equation of the locus of z and substitutes for x and y or equivalent using their variables to obtain an equation in u and v (or their variables). Condone slips with the locus of z but must be a linear equation in any form but with a non-zero constant term. **or** $\lvert z-2i\rvert=\lvert z-3\rvert\Rightarrow\left\lvert\frac{-2u}{u^2+(v-1)^2}+\frac{2u^2+2v(v-1)}{u^2+(v-1)^2}i-2i\right\rvert=\left\lvert\frac{-2u}{u^2+(v-1)^2}+\frac{2u^2+2v(v-1)}{u^2+(v-1)^2}i-3\right\rvert$ $\Rightarrow\left(\frac{-2u}{u^2+(v-1)^2}\right)^2+\left(\frac{2u^2+2v(v-1)}{u^2+(v-1)^2}-2\right)^2=\left(\frac{-2u}{u^2+(v-1)^2}-3\right)^2+\left(\frac{2u^2+2v(v-1)}{u^2+(v-1)^2}\right)^2$ Substitutes their z into the locus of z and applies Pythagoras correctly to obtain an equation in u and v (or their variables). Note that here, further progress is unlikely.		ddM1

	$13u^2+13v^2+12u-18v+5=0\Rightarrow u^2+v^2+\frac{12}{13}u-\frac{18}{13}v+\frac{9}{13}=\frac{4}{13}$ $\Rightarrow\left(u+\frac{6}{13}\right)^2+\left(v-\frac{9}{13}\right)^2=\frac{4}{13}$ Attempts to complete the square on their equation in u and v where u^2 and v^2 have the same coefficient. Award for e.g. $u^2+v^2+\alpha u+\beta v+...=\left(u+\frac{\alpha}{2}\right)^2+\left(v+\frac{\beta}{2}\right)^2+...=...$		dddM1
	Attempts using the form $u^2+v^2+2gu+2fv+c=0$ send to review.		
	$\left\lvert w-\left(-\frac{6}{13}+\frac{9}{13}i\right)\right\rvert=\frac{2}{\sqrt{13}}$	Correct equation in the required form	A1
			Total 8

(b) **Way 2**	$w=\frac{\mathrm{i}z}{z-2\mathrm{i}}\Rightarrow z=\frac{2w\mathrm{i}}{w-\mathrm{i}}$	Attempts to make z the subject. Must obtain the form $\frac{aw\mathrm{i}}{bw+c\mathrm{i}}$, a, b, c real and non-zero.	M1
	$\lvert z-2\mathrm{i}\rvert=\lvert z-3\rvert\Rightarrow\left\lvert\frac{2w\mathrm{i}}{w-\mathrm{i}}-2\mathrm{i}\right\rvert=\left\lvert\frac{2w\mathrm{i}}{w-\mathrm{i}}-3\right\rvert$ $\Rightarrow\left\lvert\frac{2w\mathrm{i}-2w\mathrm{i}-2}{w-\mathrm{i}}\right\rvert=\left\lvert\frac{2w\mathrm{i}-3w+3\mathrm{i}}{w-\mathrm{i}}\right\rvert$ Introduces z in terms of w into the given locus and attempts to combine terms		dM1
	$\left\lvert\frac{-2}{w-\mathrm{i}}\right\rvert=\left\lvert\frac{2w\mathrm{i}-3w+3\mathrm{i}}{w-\mathrm{i}}\right\rvert\Rightarrow\lvert-2\rvert=\lvert 2w\mathrm{i}-3w+3\mathrm{i}\rvert$ Correct equation with fractions removed		A1
	$\lvert 2(u+\mathrm{i}v)\mathrm{i}-3(u+\mathrm{i}v)+3\mathrm{i}\rvert=2\Rightarrow(3u+2v)^2+(3v-2u-3)^2=4$ Introduces e.g. $w=u+\mathrm{i}v$ and applies Pythagoras correctly		ddM1
	$13u^2+13v^2+12u-18v+9=4\Rightarrow u^2+v^2+\frac{12}{13}u-\frac{18}{13}v+\frac{9}{13}=\frac{4}{13}$ $\Rightarrow\left(u+\frac{6}{13}\right)^2+\left(v-\frac{9}{13}\right)^2=\frac{4}{13}$ Attempts to complete the square on their equation in u and v where u^2 and v^2 have the same coefficient. Award for e.g. $u^2+v^2+\alpha u+\beta v+...=\left(u+\frac{\alpha}{2}\right)^2+\left(v+\frac{\beta}{2}\right)^2+...=...$ Attempts using the form $u^2+v^2+2gu+2fv+c=0$ send to review.		dddM1
	$\left\lvert w-\left(-\frac{6}{13}+\frac{9}{13}\mathrm{i}\right)\right\rvert=\frac{2}{\sqrt{13}}$	Correct equation in the required form	A1
			Total 8

(b) **Way 3**	$w=\frac{\mathrm{i}z}{z-2\mathrm{i}}\Rightarrow z=\frac{2w\mathrm{i}}{w-\mathrm{i}}$	Attempts to make z the subject. Must obtain the form $\frac{aw\mathrm{i}}{bw+c\mathrm{i}}$, a, b, c real and non-zero.	M1
	$\lvert z-2\mathrm{i}\rvert=\lvert z-3\rvert\Rightarrow\left\lvert\frac{2w\mathrm{i}}{w-\mathrm{i}}-2\mathrm{i}\right\rvert=\left\lvert\frac{2w\mathrm{i}}{w-\mathrm{i}}-3\right\rvert$ $\Rightarrow\left\lvert\frac{2w\mathrm{i}-2w\mathrm{i}-2}{w-\mathrm{i}}\right\rvert=\left\lvert\frac{2w\mathrm{i}-3w+3\mathrm{i}}{w-\mathrm{i}}\right\rvert$ Introduces z and attempts to combine terms		dM1
	$\left\lvert\frac{-2}{w-\mathrm{i}}\right\rvert=\left\lvert\frac{2w\mathrm{i}-3w+3\mathrm{i}}{w-\mathrm{i}}\right\rvert\Rightarrow\lvert-2\rvert=\lvert 2w\mathrm{i}-3w+3\mathrm{i}\rvert$ Correct equation with fractions removed		A1
	$\lvert w(2\mathrm{i}-3)+3\mathrm{i}\rvert=\left\lvert(2\mathrm{i}-3)\left(w+\frac{3\mathrm{i}}{2\mathrm{i}-3}\right)\right\rvert=\lvert 2\mathrm{i}-3\rvert\left\lvert w+\frac{6-9\mathrm{i}}{13}\right\rvert=2$ Attempts to isolate w and rationalise denominator of other term		ddM1
	$\sqrt{13}\left\lvert w-\left(-\frac{6}{13}+\frac{9}{13}\mathrm{i}\right)\right\rvert=2\Rightarrow\left\lvert w-\left(-\frac{6}{13}+\frac{9}{13}\mathrm{i}\right)\right\rvert=\frac{2}{\sqrt{13}}$ M1: Completes the process by dividing by their $\lvert 2\mathrm{i}-3\rvert$ A1: Correct equation in the required form		dddM1A1
			(6)

Q79.

Question	Scheme	Marks
	$(7r-5)^2 = 49r^2 - 70r + 25$	B1
	$\sum_{r=1}^{n}(7r-5)^2 = 49\sum_{r=1}^{n} r^2 - 70\sum_{r=1}^{n} r + \sum_{r=1}^{n} 25$ $= 49\times\underline{\frac{n}{6}(n+1)(2n+1)} - 70\times\underline{\frac{n}{2}(n+1)} + 25\times\underline{n}$	M1 A1ft
	$= \frac{n}{6}\left(49(2n^2+3n+1) - 210(n+1) + 150\right)$	M1
	$= \frac{n}{6}\left(98n^2 - 63n - 11\right)$	A1
	$= \frac{n(7n+1)(14n-11)}{6}$	A1
		(6)
		(6 marks)

Notes:

B1: Correct expansion.

M1: Attempts the summations with at least two of the underlined formulae correct.

A1ft: Fully correct application of all three summations. Follow through on their expansion as long as there are 3 terms.

M1: Attempts to factor out at least the factor of n from their **three term** expansion – must have a common factor of n throughout to be able to score this mark which must be extracted from each term. (If the last term is +25, it is M0.) Allow if there are minor slips but the process must be correct.

Alternatively allow this mark for an attempt to expand $\frac{n}{6}(7n+1)(An+B)$ and compare coefficients with their expanded equation.

A1: Gathers terms appropriately and achieves the correct quadratic. In the alternative approach allow for $A = 14$ and $B = -11$ stated from their comparison.

A1cso: Correct answer from correct work. Any values found from the comparison approach must be substituted back in to achieve the result. Note from a correct unsimplified quadratic to correct answer, A0A1 can be awarded.

Q80.

Question Number	Scheme	Marks
(a)	$\sum_{r=1}^{n}(r+1)(r+5)=\sum_{r=1}^{n}(r^2+6r+5)$	B1
	$=\sum_{r=1}^{n}r^2+6\sum_{r=1}^{n}r+5n$	
	$=\frac{n}{6}(n+1)(2n+1)+6\frac{n}{2}(n+1)+5n$	M1A1
	$=\frac{n}{6}(2n^2+3n+1+18n+18+30)$	dM1
	$=\frac{n}{6}(2n^2+21n+49)=\frac{n}{6}(n+7)(2n+7)$ *	A1 * (5)
(b)	$\sum_{r=n+1}^{2n}=\sum_{r=1}^{2n}-\sum_{r=1}^{n}=\frac{2n}{6}(2n+7)(4n+7)-\frac{n}{6}(n+7)(2n+7)$	M1
	$=\frac{n}{6}(2n+7)\{8n+14-(n+7)\}$	
	$=\frac{7n}{6}(2n+7)(n+1)$	A1 (2)
		[7]

(a)	
B1	Brackets multiplied out correctly. Summation signs not needed.
M1	Use at least two correct formulae from $\sum_{r=1}^{n}r$, $\sum_{r=1}^{n}r^2$ and $\sum_{r=1}^{n}1=n$.
A1	Fully correct expression.
dM1	Attempt to remove factor $\frac{n}{6}$ from an expression with common factor n present. (if "$5n$" is just 5 then this mark will not be scored). Must be seen before the given answer is quoted. No need to simplify the remaining quadratic factor.
A1*	Obtain the correct 3 term quadratic and factorise. This is a "show that" question, so the 3 TQ must be seen. No errors seen.
(b)	
M1	Use $\sum_{r=n+1}^{2n}=\sum_{r=1}^{2n}-\sum_{r=1}^{n}$
A1	Simplify to the correct answer.

Q81.

Question Number	Scheme	Marks
(a)	$\frac{\mathrm{d}(r\sin\theta)}{\mathrm{d}\theta}=4a\cos\theta+4a\cos^2\theta-4a\sin^2\theta$ or $4a\cos\theta+4a\cos 2\theta$ oe (Or allow $\frac{\mathrm{d}(r\cos\theta)}{\mathrm{d}\theta}=-4a\sin\theta-8a\cos\theta\sin\theta$ or $-4a\sin\theta-4a\sin 2\theta$)	M1
	E.g. $4a\cos\theta+4a\cos^2\theta-4a\sin^2\theta=0\Rightarrow\cos\theta+\cos^2\theta-(1-\cos^2\theta)=0$	M1
	$2\cos^2\theta+\cos\theta-1=0$ terms in any order	A1
	$(2\cos\theta-1)(\cos\theta+1)=0\Rightarrow\cos\theta=...$	ddM1
	$\left(\cos\theta=\frac{1}{2}\Rightarrow\right)\theta=\frac{\pi}{3}$ $(\theta=\pi$ need not be seen)	A1
	$r=4a\times\frac{3}{2}=6a$	A1 (6)
(b)	Area $=\frac{1}{2}\int r^2\mathrm{d}\theta=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}16a^2(1+\cos\theta)^2\,\mathrm{d}\theta$	
	$=\frac{16a^2}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1+2\cos\theta+\cos^2\theta)\mathrm{d}\theta$	M1
	$=8a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(1+2\cos\theta+\frac{1}{2}(\cos 2\theta+1)\right)\mathrm{d}\theta$	M1
	$=8a^2\left[\theta+2\sin\theta+\frac{1}{2}\left(\frac{1}{2}\sin 2\theta+\theta\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$	dM1A1
	$8a^2\left[\frac{\pi}{3}+\sqrt{3}+\frac{1}{4}\times\frac{\sqrt{3}}{2}+\frac{\pi}{6}-\left(\frac{\pi}{6}+1+\frac{1}{4}\times\frac{\sqrt{3}}{2}+\frac{\pi}{12}\right)\right]$	A1
	$8a^2\left[\frac{\pi}{4}+\sqrt{3}-1\right]$	
	Area $R=8a^2\left[\frac{\pi}{4}+\sqrt{3}-1\right]-6a^2\left(1+\frac{\sqrt{3}}{2}\right)=a^2\left(2\pi+5\sqrt{3}-14\right)$	M1A1 (7)
		[13]

	Notes
(a) M1	Attempt the differentiation of $r\sin\theta$ using product rule or $\sin 2\theta=2\sin\theta\cos\theta$ **OR** for this mark only allow differentiation of $r\cos\theta$, inc use of product rule, chain rule or $\cos^2\theta=\frac{1}{2}(1\pm\cos 2\theta)$ Allow errors in coefficients as long as the form is correct.
M1	Sets their derivative of $r\sin\theta$ equal to zero and achieves a quadratic expression in $\cos\theta$
A1	Correct 3 term quadratic in $\cos\theta$ (any multiple, including a)
ddM1	Dep on both M marks. Solve their quadratic (usual rules) giving one or two roots
A1	Correct quadratic solved to give $\theta=\frac{\pi}{3}$

	Notes
A1*	Correct r obtained from an intermediate step. Accept as shown in scheme, or $r = 4a\left(1+\cos\frac{\pi}{3}\right) = 6a$ or equivalent in stages. No need to see coordinates together in brackets
(b) **M1**	Note : first 4 marks of (b) do not require limits. Use of correct area formula, $\frac{1}{2}$ may be seen later, inc squaring the bracket to obtain 3 terms - limits need not be shown.
M1	Use double angle formula (formula to be of form $\cos^2\theta = \pm\frac{1}{2}(\cos 2\theta \pm 1)$) to obtain an integrable function - limits need not be shown, $\frac{1}{2}$ from area formula may be missing,
dM1	Attempt the integration $\cos\theta \to \pm k\sin\theta$ and $\cos 2\theta \to \pm m\sin 2\theta$ - limits not needed – dep on 2nd M mark but not the first. Note if only two terms arise from squaring allow for $\cos 2\theta \to \pm m\sin 2\theta$
A1	Correct integration – substitution of limits not required (NB Not follow through)
A1	Include the $\frac{1}{2}$ and substitute the correct limits in a correct integral. Note may be attempted via integral from 0 to $\frac{\pi}{3}$ minus integral from 0 to $\frac{\pi}{6}$ - but attempts at sector formula for the latter is A0.
M1	Attempt the area of the triangle - accept valid attempt even if not subtracted from area. E.g. attempts $\frac{1}{2}OA.OB\sin\frac{\pi}{6}$
A1	Correct final answer in the demanded the form.

Q82.

Question Number	Scheme	Marks
(a)	$x = r\cos\theta = 3\sin 2\theta\cos\theta$	B1
	$\dfrac{dx}{d\theta} = 6\cos 2\theta\cos\theta - 3\sin 2\theta\sin\theta = 0$	M1
	$2\cos\theta\left(\cos^2\theta - 2\sin^2\theta\right) = 0$	M1
ALT	For the 2 M marks:	
	$x = 6\sin\theta\cos^2\theta \Rightarrow \dfrac{dx}{d\theta} = 6\cos^3\theta - 12\sin^2\theta\cos\theta = 0$	
	$\tan\phi = \dfrac{1}{\sqrt{2}}$ *	A1* (4)
(b)	$\tan\phi = \dfrac{1}{\sqrt{2}} \Rightarrow \sin\phi = \dfrac{1}{\sqrt{3}},\ \cos\phi = \dfrac{\sqrt{2}}{\sqrt{3}}$	M1
	$R = 3\times 2\times\dfrac{1}{\sqrt{3}}\times\dfrac{\sqrt{2}}{\sqrt{3}} = 2\sqrt{2}$	A1 (2)
(c)	Area of sector $= \dfrac{1}{2}\int r^2 d\theta = \dfrac{9}{2}\int \sin^2 2\theta\, d\theta$	M1
	$= \dfrac{9}{2}\int_0^{\arctan\left(\frac{1}{\sqrt{2}}\right)} \dfrac{1}{2}(1-\cos 4\theta)\, d\theta$	M1
	$= \dfrac{9}{2}\left[\dfrac{1}{2}\left(\theta - \dfrac{1}{4}\sin 4\theta\right)\right]_0^{\arctan\frac{1}{\sqrt{2}}}$	M1A1
	$= \dfrac{9}{4}\left[\arctan\dfrac{1}{\sqrt{2}} - \dfrac{1}{4}\sin 4\left(\arctan\dfrac{1}{\sqrt{2}}\right) - 0\right]$	dM1
	$\sin 4\phi = 2\sin 2\phi\cos 2\phi = 4\sin\phi\cos\phi\left(2\cos^2\phi - 1\right)$ $= 4\times\dfrac{1}{\sqrt{3}}\times\dfrac{\sqrt{2}}{\sqrt{3}}\left(2\times\dfrac{2}{3} - 1\right) = \dfrac{4\sqrt{2}}{9}$	M1
	Area of sector $= \dfrac{9}{4}\left(\arctan\dfrac{1}{\sqrt{2}} - \dfrac{1}{4}\times\dfrac{4\sqrt{2}}{9}\right) = \dfrac{9}{4}\arctan\dfrac{1}{\sqrt{2}} - \dfrac{\sqrt{2}}{4}$	A1 (7)
		[13]

Question Number	Scheme	Marks
(a)		
B1	State $x=(r\cos\theta)=3\sin 2\theta\cos 2\theta$ May be given by implication	
M1	Attempt to differentiate $x=r\cos\theta$ or $x=r\sin\theta$ Product rule must be used	
M1	Use a correct double angle formula **and** equate the derivative of $r\cos\theta$ to 0	
ALT	**M1** Attempt the differentiation of $x=r\cos\theta$ or $x=r\sin\theta$ using the product rule (after using a double angle formula) **M1** Use a correct double angle formula **and** equate the derivative of $r\cos\theta$ to 0	
A1*	Complete to the given answer and no extras with no errors in the working. Accept θ or ϕ All values seen must be exact	
(b)		
M1	Attempt exact values for $\sin\theta$ and $\cos\theta$ and use these to obtain a value for R. Values for $\sin\theta$ and/or $\cos\theta$ may have been seen in (a)	
A1	A correct, exact value for R, as shown or any equivalent. Award M1A1 for a correct exact answer	
(c)		
M1	Use of Area $=\frac{1}{2}\int r^2\mathrm{d}\theta$ Limits not needed (ignore any shown)	
M1	Use the double angle formula to obtain $k\int \frac{1}{2}(1\pm\cos 4\theta)\mathrm{d}\theta$ Ignore any limits given This is NOT dependent NB: There are other, lengthy, methods of reaching this point	
M1	Attempt the integration $\cos 4\theta\to\pm\frac{1}{4}\sin 4\theta$ (Not dependent)	
A1	Correct integration of $1-\cos 4\theta$	
dM1	Correct use of correct limits. Depends on second and third M marks 0 at lower limit need not be shown	
M1	Attempt an exact numerical value for $\sin 4\left(\arctan\frac{1}{\sqrt{2}}\right)$	
A1	Correct final answer. Award M1A1 for a correct exact final answer	

Q83.

Question Number	Scheme		Notes	Marks
(a)(i)	$x = t^{\frac{1}{2}} \Rightarrow \frac{dx}{dy} = \frac{1}{2}t^{-\frac{1}{2}}\frac{dt}{dy} \Rightarrow \frac{dy}{dx} = \ldots$ or $t = x^2 \Rightarrow \frac{dt}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \ldots$ Applies the chain rule and proceeds to an expression for $\frac{dy}{dx}$			M1
	$\Rightarrow \frac{dy}{dx} = 2t^{\frac{1}{2}}\frac{dy}{dt}$		Any correct expression for $\frac{dy}{dx}$ in terms of y and t	A1
(a)(ii)	$\frac{dy}{dx} = 2t^{\frac{1}{2}}\frac{dy}{dt} \Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dt}t^{-\frac{1}{2}}\frac{dt}{dx} + 2t^{\frac{1}{2}}\frac{d^2y}{dt^2}\frac{dt}{dx}$ **dM1**: Uses the product rule to differentiate an equation of the form $\frac{dy}{dx} = kt^{\frac{1}{2}}\frac{dy}{dt}$ or equivalent e.g. $\frac{dy}{dx} = kx\frac{dy}{dt}$ to obtain $\frac{d^2y}{dx^2} = \alpha t^{-\frac{1}{2}}\frac{dy}{dt}\frac{dt}{dx} + \ldots$ or $\frac{d^2y}{dx^2} = \ldots + \beta t^{\frac{1}{2}}\frac{d^2y}{dt^2}\frac{dt}{dx}$ **<u>or equivalent expressions</u>** where … is non-zero **A1**: <u>Any</u> correct expression for $\frac{d^2y}{dx^2}$			dM1A1
	$\frac{dy}{dt}t^{-\frac{1}{2}}\frac{dt}{dx} + 2t^{\frac{1}{2}}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{dy}{dt}t^{-\frac{1}{2}} \times 2t^{\frac{1}{2}} + 2t^{\frac{1}{2}}\frac{d^2y}{dt^2}\frac{dt}{dx}$ $\frac{d^2y}{dx^2} = 2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2}$ Correct expression in terms of y and t			A1
				(5)
(b)	$x\frac{d^2y}{dx^2} - (6x^2+1)\frac{dy}{dx} + 9x^3y = x^5 \Rightarrow t^{\frac{1}{2}}\left(2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2}\right) - (6t+1)2t^{\frac{1}{2}}\frac{dy}{dt} + 9t^{\frac{3}{2}}y = t^{\frac{5}{2}}$ Substitutes their expressions from part (a) and replaces x with $t^{\frac{1}{2}}$			M1
	$2t^{\frac{1}{2}}\frac{dy}{dt} + 4t^{\frac{3}{2}}\frac{d^2y}{dt^2} - 12t^{\frac{3}{2}}\frac{dy}{dt} - 2t^{\frac{1}{2}}\frac{dy}{dt} + 9t^{\frac{3}{2}}y = t^{\frac{5}{2}}$ $\Rightarrow 4\frac{d^2y}{dt^2} - 12\frac{dy}{dt} + 9y = t$ * Obtains the given answer with no errors and sufficient working shown – at least one intermediate line after substitution but check working. **Must follow full marks in (a) apart from SC below.**			A1*
				(2)

<u>Special case in (a) and (b) for those who do not have (a) in terms of y and t only:</u>

$t = x^2 \Rightarrow \frac{dt}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \ldots$ **Scores M1.** $\ldots = 2x\frac{dy}{dt}$ **scores A0 in (a)(i)**

$\frac{dy}{dx} = 2x\frac{dy}{dt} \Rightarrow \frac{d^2y}{dx^2} = 2\frac{dy}{dt} + 2x\frac{d^2y}{dt^2}\frac{dt}{dx} = 2\frac{dy}{dt} + 4x^2\frac{d^2y}{dt^2}$ **Scores dM1A1A0 in (a)(ii)**

$x\frac{d^2y}{dx^2} - (6x^2+1)\frac{dy}{dx} + 9x^3y = x^5 \Rightarrow t^{\frac{1}{2}}\left(2\frac{dy}{dt} + 4x^2\frac{d^2y}{dt^2}\right) - (6t+1)2x\frac{dy}{dt} + 9t^{\frac{3}{2}}y = t^{\frac{5}{2}}$

$\Rightarrow t^{\frac{1}{2}}\left(2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2}\right) - (6t+1)2t^{\frac{1}{2}}\frac{dy}{dt} + 9t^{\frac{3}{2}}y = t^{\frac{5}{2}} \Rightarrow 4\frac{d^2y}{dt^2} - 12\frac{dy}{dt} + 9y = t$ * **Scores M1A1 in (b)**

Mark (c) and (d) together

(c)	$4m^2-12m+9=0 \Rightarrow m=\frac{3}{2}$	Attempts to solve $4m^2-12m+9=0$ Apply general guidance for solving a 3TQ if necessary.	M1
	$(y=)e^{\frac{3}{2}t}(At+B)$	Correct CF. No need for "$y=\ldots$" Condone $(y=)e^{\frac{3}{2}x}(Ax+B)$ here but must be in terms of t in the GS. Allow equivalents for the $\frac{3}{2}$.	A1
	$(y=)at+b \Rightarrow \frac{dy}{dt}=a \Rightarrow \frac{d^2y}{dt^2}=0$ $\Rightarrow -12a+9(at+b)=t$ Starts with the **correct** PI form and differentiates to obtain $\frac{dy}{dt}=a$ and $\frac{d^2y}{dt^2}=0$ and substitutes. NB starting with a PI of $y=at$ is M0		M1
	$9a=1 \Rightarrow a=\ldots$ $9b-12a=0 \Rightarrow b=\ldots$	Complete method to find a and b by comparing coefficients. **Depends on the previous method mark.**	dM1
	$y=e^{\frac{3}{2}t}(At+B)+\frac{1}{9}t+\frac{4}{27}$	Correct GS including "$y=\ldots$" and must be in terms of t (no x's). Allow equivalent exact fractions for the constants.	A1
			(5)
(d)	$y=e^{\frac{3}{2}x^2}(Ax^2+B)+\frac{1}{9}x^2+\frac{4}{27}$ Correct equation including "$y=\ldots$" (follow through their answer to (c)). Allow equivalent exact fractions for the constants. For the ft, the answer to (c) must be in terms of t and the answer to (d) should be the same as (c) with t replaced with x^2. If there is no final answer to (c) you can award B1ft if the equation is correct in terms of x if it follows the previous work.		B1ft
			(1)
			Total 13

Q84.

Question Number	Scheme	Marks
(a)	$\begin{vmatrix} 4 & -5 \\ -3 & 2 \end{vmatrix} = 8-15=-7 \Rightarrow \text{Area } T' = \text{"}\pm 7\text{"} \times 23 = \ldots$	M1
	$\text{Area } T' = 161$	A1 (2)
(b)	$\begin{pmatrix} 4 & -5 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 3p+2 \\ 2p-1 \end{pmatrix} = \begin{pmatrix} 17 \\ -18 \end{pmatrix}$ or $\begin{pmatrix} 3p+2 \\ 2p-1 \end{pmatrix} = \frac{1}{8-15}\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 17 \\ -18 \end{pmatrix}$	
	$4(3p+2)-5(2p-1)=17$ or $-3(3p+2)+2(2p-1)=-18$ or $3p+2=-\frac{1}{7}(34-90)$ or $2p-1=-\frac{1}{7}(51-72)$ $(\text{e.g. } 2p+13=17 \Rightarrow \ldots\ldots)$	M1
	$p=2$	A1 (2)
(c)	Rotation; through 90° clockwise (or 270° anticlockwise) about origin	B1;B1 (2)
(d)	$\mathbf{CA}=\mathbf{B}$	
	$\mathbf{A}^{-1} = -\frac{1}{7}\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$ or $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 4 & -5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 4a-3b & -5a+2b \\ 4c-3d & -5c+2d \end{pmatrix}$	B1
	$\mathbf{C} = -\frac{1}{7}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} = \ldots$ or $\begin{cases} 4a-3b=0 & -5a+2b=1 \\ 4c-3d=-1 & -5c+2d=0 \end{cases} \Rightarrow \ldots$	M1
	$\mathbf{C} = -\frac{1}{7}\begin{pmatrix} 3 & 4 \\ -2 & -5 \end{pmatrix}$ or $\begin{pmatrix} -\frac{3}{7} & -\frac{4}{7} \\ \frac{2}{7} & \frac{5}{7} \end{pmatrix}$ oe	A1 (3)
		[9]

(a)	
M1	Attempts to find the determinant of **M** and use as a scale factor. Accept if a slip in calculation is made and accept if negative is used for this mark. Dividing by the determinant is M0.
A1	Correct answer only. No working needed (correct answer implies the method).
(b)	
M1	Form a matrix equation using either **A** or an attempt at $\mathbf{A}^{-1}$, obtain a linear equation and solve for *p*.
A1	Correct value for *p*, obtained from a correct equation. (No need to check in other equation.)
(c)	
B1	Rotation, rotates, rotate or rotating (oe) Accept "turn"
B1	Correct angle (degrees or radians) with direction specified and about origin or (0, 0)
(d)	
B1	Correct matrix for $\mathbf{A}^{-1}$ May have been found in (b) but must be used in (d). Alternatively, correct **CA** with unknowns for entries of **C**.
M1	Multiply **B** by $\mathbf{A}^{-1}$ **on the right.** Alternatively, sets **CA** equal to **B** and solves equations.
A1	Correct matrix **C**(isw after a correct answer).

Q85.

Question Number	Scheme	Notes	Mark
(a)	$x^2\frac{dy}{dx}+xy=2y^2 \qquad y=\frac{1}{z}$		
	$\frac{dy}{dx}=-\frac{1}{z^2}\frac{dz}{dx}$	Correct differentiation	B1
	$-\frac{x^2}{z^2}\frac{dz}{dx}+\frac{x}{z}=\frac{2}{z^2}$	Substitutes into the given differential equation	M1
	$\frac{dz}{dx}-\frac{z}{x}=-\frac{2}{x^2}$ *	Achieves the printed answer with no errors. Allow this to be written down following a correct substitution i.e. with no intermediate step.	A1*
			(3)
(a) Way 2	$y=\frac{1}{z}\Rightarrow zy=1\Rightarrow y\frac{dz}{dx}+z\frac{dy}{dx}=0$	Correct differentiation	B1
	$-\frac{y}{z}x^2\frac{dz}{dx}+\frac{x}{z}=-\frac{2}{z^2}$	Substitutes into the given differential equation	M1
	$\frac{dz}{dx}-\frac{z}{x}=-\frac{2}{x^2}$ *	Achieves the printed answer with no errors. Allow this to be written down following a correct substitution i.e. with no intermediate step.	A1*
(a) Way 3	$y=\frac{1}{z}\Rightarrow z=\frac{1}{y}\Rightarrow\frac{dz}{dx}=-\frac{1}{y^2}\frac{dy}{dx}$	Correct differentiation	B1
	$-\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{xy}=-\frac{2}{x^2}$	Substitutes into differential equation (II)	M1
	$x^2\frac{dy}{dx}+xy=2y^2$	Obtains differential equation (I) with no errors. Allow this to be written down following a correct substitution i.e. with no intermediate step.	A1*

(b)	$I=e^{-\int\frac{1}{x}dx}=e^{-\ln x}=\frac{1}{x}$	Correct integrating factor of $\frac{1}{x}$	B1
	$\frac{z}{x}=-\int\frac{2}{x^3}dx$	For $Iz=-\int\frac{2I}{x^2}dx$. Condone the "dx" missing.	M1
	$\frac{z}{x}=\frac{1}{x^2}+c$	Correct equation including constant	A1
	$z=\frac{1}{x}+cx$	Correct equation in the required form	A1
			(4)
(c)	$\frac{1}{y}=\frac{1}{x}+cx\Rightarrow-\frac{8}{3}=\frac{1}{3}+3c\Rightarrow c=-1$	Reverses the substitution and uses the given conditions to find their constant	M1
	$\frac{1}{y}=\frac{1}{x}-x\Rightarrow y=\frac{x}{1-x^2}$	Correct equation for y in terms of x. Allow any correct equivalents e.g. $y=\frac{1}{x^{-1}-x}$, $y=\frac{1}{\frac{1}{x}-x}$	A1
			(2)
			Total 9