Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel Level 3 GCE

Thursday 25 May 2023

Afternoon
(Time: 1 hour 30 minutes)

Paper reference **9FM0/01**

Further Mathematics

Advanced
PAPER 1: Core Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

Pearson

1. The cubic equation

$$x^3 - 7x^2 - 12x + 6 = 0$$

has roots α, β and γ.

Without solving the equation, determine a cubic equation whose roots are $(\alpha + 2)$, $(\beta + 2)$ and $(\gamma + 2)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

$x^3 - 7x^2 - 12x + 6 = 0$

$w = x + 2 \quad \therefore x = w - 2$

$\therefore (w-2)^3 - 7(w-2)^2 - 12(w-2) + 6 = 0$

$\therefore w^3 - 6w^2 + 12w - 8 - 7(w^2 - 4w + 4) - 12w + 24$

$+ 6 = 0$

$\therefore w^3 - 6w^2 + 12w - 8 - 7w + 28w - 28 - 12w + 24$

$+ 6 = 0$

$\therefore w^3 - 13w^2 + 28w - 6 = 0$

$\therefore p = -13, \; q = 28, \; r = -6$

Question 1 continued

(Total for Question 1 is 5 marks)

2. (a) Write $x^2 + 4x - 5$ in the form $(x+p)^2 + q$ where p and q are integers.

(1)

(b) Hence use a standard integral from the formula book to find

$$\int \frac{1}{\sqrt{x^2 + 4x - 5}}\,dx$$

(2)

(c) Determine the mean value of the function

$$f(x) = \frac{1}{\sqrt{x^2 + 4x - 5}} \qquad 3 \leqslant x \leqslant 13$$

giving your answer in the form $A \ln B$ where A and B are constants in simplest form.

(3)

(a) $x^2 + 4x - 5 \equiv (x+2)^2 - 2^2 - 5$

$\equiv (x+2)^2 - 4 - 5$

$\equiv (x+2)^2 - 9$

(b) $\int \frac{1}{\sqrt{x^2 + 4x - 5}}\,dx \equiv \int \frac{1}{\sqrt{(x+2)^2 - 9}}\,dx$

$\equiv \int \frac{1}{\sqrt{(x+2)^2 - 3^2}}\,dx$

$= \text{arcosh}\left(\frac{x+2}{3}\right) + c$

$= \ln\left\{(x+2) + \sqrt{(x+2)^2 - 3^2}\right\} + c$

(c) MEAN VALUE $= \frac{1}{13-3}\left[\ln\left\{(x+2) + \sqrt{(x+2)^2 - 3^2}\right\}\right]_3^{13}$

$= \frac{1}{10}\left[\ln\left\{(13+2) + \sqrt{(13+2)^2 - 3^2}\right\} - \right.$

$\left. \ln\left\{(3+2) + \sqrt{(3+2)^2 - 3^2}\right\}\right]$

Question 2 continued

$$= \frac{1}{10}\left[\ln\{15+\sqrt{216}\}-\ln\{5+\sqrt{16}\}\right]$$

$$= \frac{1}{10}\ln\left[\frac{5+2\sqrt{6}}{3}\right]$$

(Total for Question 2 is 6 marks)

3. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

$$z_1 = -4 + 4\text{i}$$

(a) Express z_1 in the form $r(\cos\theta + \text{i}\sin\theta)$, where $r \in \mathbb{R}$, $r > 0$ and $0 \leqslant \theta < 2\pi$

(2)

$$z_2 = 3\left(\cos\frac{17\pi}{12} + \text{i}\sin\frac{17\pi}{12}\right)$$

(b) Determine in the form $a + \text{i}b$, where a and b are exact real numbers,

(i) $\frac{z_1}{z_2}$

(2)

(ii) $(z_2)^4$

(2)

(c) Show on a single Argand diagram

(i) the complex numbers z_1, z_2 and $\frac{z_1}{z_2}$

(ii) the region defined by $\{z \in \mathbb{C} : |z - z_1| < |z - z_2|\}$

(4)

(a) $r = |z_1| = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$

$\arg(z_1) = \tan^{-1}\left(\frac{4}{-4}\right) + \pi = \frac{3\pi}{4}$

$\therefore z_1 = 4\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

(b)(i) $z_2 = 3\left[\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right]$

$\left|\frac{z_1}{z_2}\right| = \frac{4\sqrt{2}}{3}$

$\arg\left(\frac{z_1}{z_2}\right) = \frac{3\pi}{4} - \frac{17\pi}{12} = -\frac{2\pi}{3}$

Question 3 continued

$$\therefore \frac{z_1}{z_2} = \frac{4\sqrt{2}}{3}\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$$

$$= \frac{4\sqrt{2}}{3}\left[-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$$

$$= -\frac{2\sqrt{2}}{3} - i\,\frac{2\sqrt{6}}{3}$$

(ii) $$[z_2]^4 = \left\{3\left[\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right]\right\}^4$$

$$= 3^4\left[\cos\left(4\times\frac{17\pi}{12}\right) + i\sin\left(4\times\frac{17\pi}{12}\right)\right]$$

$$= 81\left[\cos\frac{17\pi}{3} + i\sin\frac{17\pi}{3}\right]$$

$$= 81\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$$

$$= \frac{81}{2} - i\,\frac{81\sqrt{3}}{2}$$

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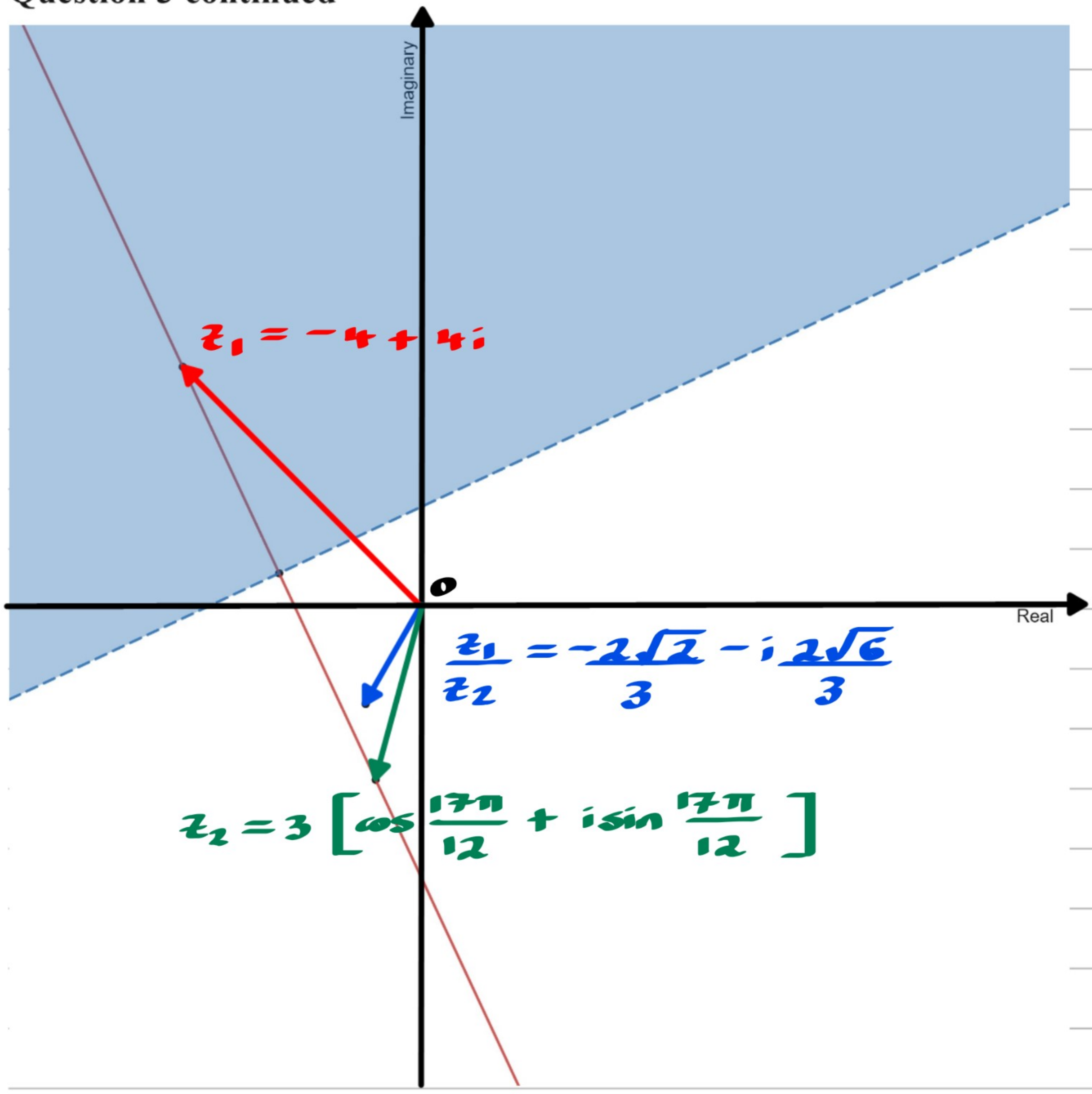
Question 3 continued

(c)



Question 3 continued

(Total for Question 3 is 10 marks)

4. Prove by induction that for $n \in \mathbb{N}$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix}$$

(5)

WE WANT TO SHOW

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & -2(k+1) \\ 0 & 1 \end{bmatrix}$$

$$\text{LHS} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\text{RHS} = \begin{bmatrix} 1 & -2(1) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ TRUE FOR $n = 1$

ASSUME TRUE FOR $n = k$:

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{k} = \begin{bmatrix} 1 & -2k \\ 0 & 1 \end{bmatrix}$$

TRY FOR $n = k+1$:

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{k} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{1}$$

$$= \begin{bmatrix} 1 & -2k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2-2k \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2(k+1) \\ 0 & 1 \end{bmatrix}$$

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Question 4 continued

STATEMENT IS TRUE FOR $n=1$, AND IF TRUE FOR $n=k$, THEN TRUE FOR $n=k+1$

$\therefore$ TRUE FOR $n \in \mathbb{N}$

(Total for Question 4 is 5 marks)

5. The line l_1 has equation $\dfrac{x+5}{1} = \dfrac{y+4}{-3} = \dfrac{z-3}{5}$

The plane Π_1 has equation $2x + 3y - 2z = 6$

(a) Find the point of intersection of l_1 and Π_1

(2)

The line l_2 is the reflection of the line l_1 in the plane Π_1

(b) Show that a vector equation for the line l_2 is

$$\mathbf{r} = \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$$

where μ is a scalar parameter.

(5)

The plane Π_2 contains the line l_1 and the line l_2

(c) Determine a vector equation for the line of intersection of Π_1 and Π_2

(2)

The plane Π_3 has equation $\mathbf{r}.\begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix} = b$ where a and b are constants.

Given that the planes Π_1, Π_2 and Π_3 form a sheaf,

(d) determine the value of a and the value of b.

(3)

(a) $\dfrac{x+5}{1} = \dfrac{y+4}{-3} = \dfrac{z-3}{5} = \lambda$

$\underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$

$\pi_1 : 2x + 3y - 2z = 6$

$\therefore 2(-5+\lambda) + 3(-4-3\lambda) - 2(3+5\lambda) = 6$

$\therefore -10 + 2\lambda - 12 - 9\lambda - 6 - 10\lambda = 6$

$\therefore -17\lambda = 34$

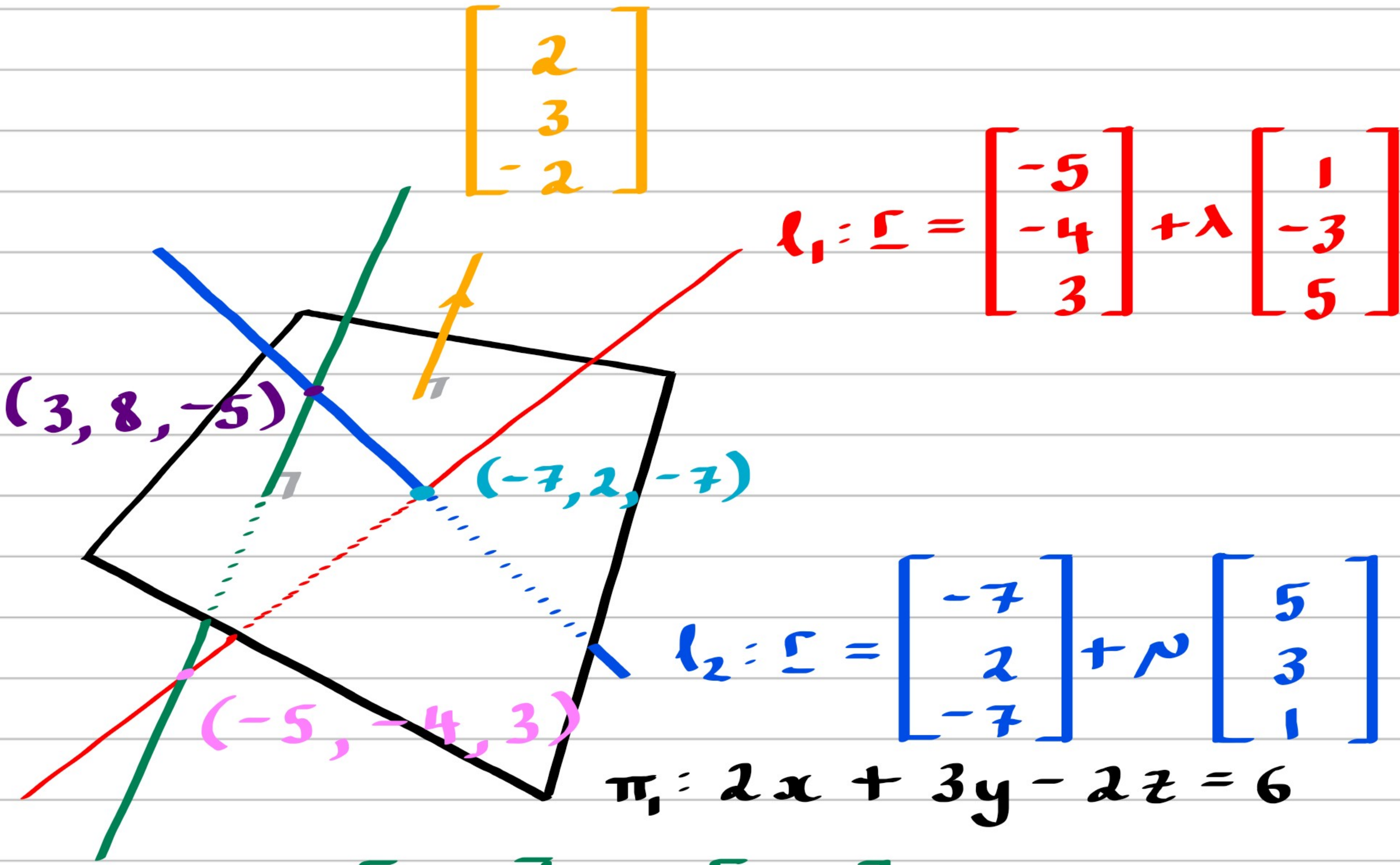
Question 5 continued

$\therefore \lambda = -2$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \\ -7 \end{bmatrix}$$

$\therefore$ POINT IS $(-7, 2, -7)$

(b)



$$l_3 : \underline{r} = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

$(-5, -4, 3)$ LIES ON l_1

$\therefore$ LINE l_3 NORMAL TO Π_1 IS $\underline{r} = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$

THIS MEETS Π_1 WHEN:

$$2(-5+2t) + 3(-4+3t) - 2(3-2t) = 6$$

Question 5 continued

$-10 + 4t - 12 + 9t - 6 + 4t = 6$

$17t = 34$

$\therefore t = 2$

$\therefore l_3$ MEETS l_2 WHEN $t = 4$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ -5 \end{bmatrix}$$

$\therefore (3, 8, -5)$ LIES ON l_2

$$\therefore \text{DIRECTION VECTOR OF } l_2 = \begin{bmatrix} 3 \\ 8 \\ -5 \end{bmatrix} - \begin{bmatrix} -7 \\ 2 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 6 \\ 2 \end{bmatrix}$$

$$= 2\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

$$\therefore l_2 : \underline{r} = \begin{bmatrix} -7 \\ 2 \\ -7 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

(c) WE REQUIRE NORMAL VECTOR TO DETERMINE Π_2. THIS CAN BE FOUND BY FINDING THE CROSS PRODUCT OF THE DIRECTION VECTORS OF l_1 AND l_2

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P 7 2 7 9 4 A 0 1 4 2 8

Question 5 continued

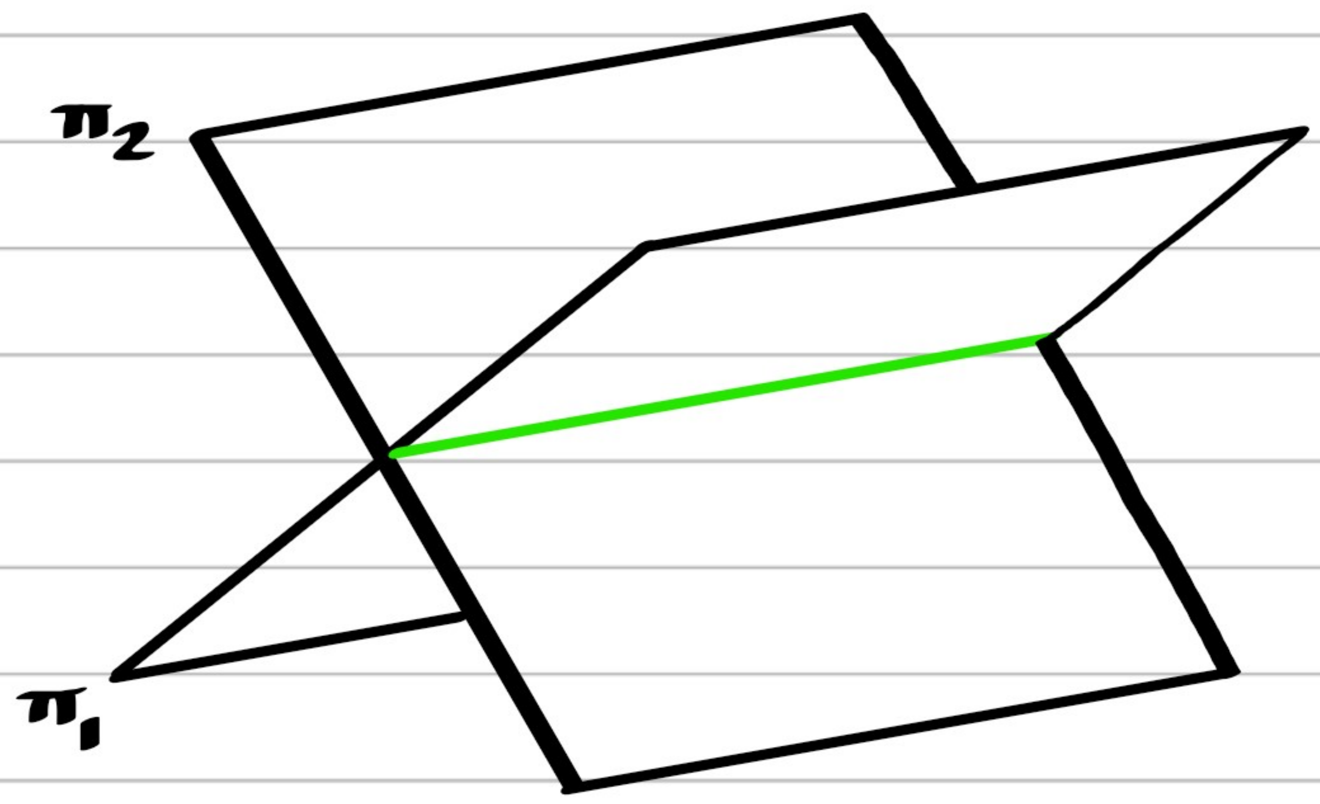
$$\therefore \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 3 & 1 \\ 1 & -3 & 5 \end{vmatrix} = \begin{bmatrix} 18 \\ -24 \\ -18 \end{bmatrix}$$

$\therefore$ NORMAL VECTOR IS $\begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix}$

$\therefore \Pi_2 : 3x - 4y - 3z = d$

AS $(-7, 2, -7)$ LIES ON Π_2 : $d = 3(-7) - 4(2) - 3(-7) = -8$

$\therefore \Pi_2 : 3x - 4y - 3z = -8$



DIRECTION VECTOR OF LINE $= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -4 & -3 \\ 2 & 3 & -2 \end{vmatrix} = \begin{bmatrix} 17 \\ 0 \\ 17 \end{bmatrix}$

$z = 0 : \quad 2x + 3y = 6$

$3x - 4y = -8$

$\therefore x = 0, \; y = 2$

(Total for Question 5 is 12 marks)

$\therefore$ LINE OF INTERSECTION HAS EQUATION

$$\underline{r} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(d) $\Pi_1: \ 2x + 3y - 2z = 6 \qquad ①$

$\Pi_2: \ 3x - 4y - 3z = -8 \qquad ②$

$\Pi_3: \ x + y + az = b \qquad ③$

WHEN $y = 2$: ① BECOMES $2x + 6 - 2z = 6 \quad \therefore x = z$

② BECOMES $3x - 8 - 3z = -8 \quad \therefore x = z$

$\therefore$ ③ BECOMES $x + 2 + az = b$

AS $x = z$: $z + 2 + az = b$

$(1 + a)z + 2 = b$

$\therefore b = 2$

$\therefore 1 + a = 0$

$\therefore a = -1$

6. Water is flowing into and out of a large tank.

Initially the tank contains 10 litres of water.

The rate of flow of the water is modelled so that

- there are V litres of water in the tank at time t minutes after the water begins to flow
- water enters the tank at a rate of $\left(3 - \dfrac{4}{1 + e^{0.8t}}\right)$ litres per minute
- water leaves the tank at a rate proportional to the volume of water remaining in the tank

Given that when $t = 0$ the volume of water in the tank is decreasing at a rate of 3 litres per minute, use the model to

(a) show that the volume of water in the tank at time t satisfies

$$\frac{dV}{dt} = 3 - \frac{4}{1 + e^{0.8t}} - 0.4V$$

(3)

(b) Determine $\dfrac{d}{dt}\left(\arctan e^{0.4t}\right)$

(2)

Hence, by solving the differential equation from part (a),

(c) determine an equation for the volume of water in the tank at time t.

Give your answer in simplest form as $V = f(t)$

(6)

After 10 minutes, the volume of water in the tank was 8 litres.

(d) Evaluate the model in light of this information.

(1)

(a) WATER ENTERS TANK AT RATE $= 3 - \dfrac{4}{1+e^{0.8t}}$ $\dfrac{\ell}{\text{MIN}}$

WATER LEAVES TANK AT RATE $= KV$ $\dfrac{\ell}{\text{MIN}}$

$$\therefore \frac{dV}{dt} = \left[3 - \frac{4}{1+e^{0.8t}}\right] - KV$$

$$t = 0, \frac{dV}{dt} = -3, V = 10: \left[3 - \frac{4}{1+e^{0}}\right] - 10K = -3$$

$$1 - 10K = -3$$

Question 6 continued

$\therefore k = 0.4$

$\therefore \dfrac{dV}{dt} = 3 - \dfrac{4}{1+e^{0.8t}} - 0.4V$

(b) $\dfrac{d}{dt}\left[\arctan(e^{0.4t})\right] = \dfrac{1}{1+(e^{0.4t})^2} \times 0.4e^{0.4t}$

$= \dfrac{0.4e^{0.4t}}{1+e^{0.8t}}$

(c) $\dfrac{dV}{dt} = 3 - \dfrac{4}{1+e^{0.8t}} - 0.4V$

$\dfrac{dV}{dt} + 0.4V = 3 - \dfrac{4}{1+e^{0.8t}}$

$IF = e^{0.4dt} = e^{0.4t}$

$\therefore e^{0.4t}V = \int e^{0.4t}\left[3 - \dfrac{4}{1+e^{0.8t}}\right]dt + c$

$\therefore e^{0.4t}V = \int \left[3e^{0.4t} - \dfrac{4e^{0.4t}}{1+e^{0.8t}}\right]dt + c$

$\therefore e^{0.4t}V = \int \left[3e^{0.4t} - 10 \times \dfrac{0.4e^{0.4t}}{1+e^{0.8t}}\right]dt + c$

$\therefore Ve^{0.4t} = \dfrac{15e^{0.4t}}{2} - 10\arctan(e^{0.4t}) + c$

$t=0, V=10: \; 10e^{0} = \dfrac{15e^{0}}{2} - 10\arctan(e^{0}) + c$

$10 = \dfrac{15}{2} - \dfrac{5\pi}{2} + c$

$$\therefore c = \frac{5(1+\pi)}{2}$$

$$\therefore Ve^{0.4t} = \frac{15e^{0.4t}}{2} - 10\arctan(e^{0.4t}) + \frac{5(1+\pi)}{2}$$

$$\therefore V = \frac{15e^{0.4t}}{2e^{0.4t}} - \frac{20}{2e^{0.4t}}\arctan(e^{0.4t}) + \frac{5(1+\pi)}{2e^{0.4t}}$$

$$\therefore V = \frac{5\{3e^{0.4t} - 4\arctan(e^{0.4t}) + 1 + \pi\}}{2e^{0.4t}}$$

(d) $t = 10$: $V = \dfrac{5\{3e^{4} - 4\arctan(e^{4}) + 1 + \pi\}}{2e^{4}}$

$$\therefore V = 7.4052\ldots$$

$$\therefore \%\ \text{ERROR} = \frac{8 - 7.4052\ldots}{8} \times 100$$

$$= 7.43\ldots\ \%$$

$\therefore$ MODEL IS GOOD

Question 6 continued

(Total for Question 6 is 12 marks)

7. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(a) Explain why, for $n \in \mathbb{N}$

$$\sum_{r=1}^{2n} (-1)^r \mathrm{f}(r) = \sum_{r=1}^{n} \left(\mathrm{f}(2r) - \mathrm{f}(2r-1)\right)$$

for any function $\mathrm{f}(r)$.

(2)

(b) Use the standard summation formulae to show that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{2n} r\left((-1)^r + 2r\right)^2 = n(2n+1)(8n^2 + 4n + 5)$$

(6)

(c) Hence evaluate

$$\sum_{r=14}^{50} r\left((-1)^r + 2r\right)^2$$

(4)

(a) $(-1)^r = 1$ WHEN r IS EVEN

$(-1)^r = -1$ WHEN r IS ODD

$\therefore \sum_{r=1}^{2n} (-1)^r f(r)$ ALTERNATES BETWEEN +VE AND -VE TERMS

THE INTEGERS FROM 1 TO $2n$ CAN BE SPLIT INTO:

* EVEN INTEGERS OF THE FORM $2r$ FROM 1 TO n
* ODD INTEGERS OF THE FORM $2r-1$ FROM 1 TO n

$\therefore \sum_{r=1}^{2n} (-1)^r f(r) = \sum_{r=1}^{n} [f(2r) - f(2r-1)]$

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Question 7 continued

(b) $\sum_{r=1}^{2n} r\left[(-1)^r + 2r\right]^2 = \sum_{r=1}^{2n} r\left[(-1)^{2r} + 4r(-1)^r + 4r^2\right]$

$= \sum_{r=1}^{2n} r\left[1 + 4r(-1)^r + 4r^2\right]$

$= \sum_{r=1}^{2n}\left[4r^3 + r\right] + \sum_{r=1}^{2n}\left[4r^2(-1)^r\right]$

$= \sum_{r=1}^{2n}\left[4r^3 + r\right] + 4\sum_{r=1}^{2n}\left[(-1)^r r^2\right]$

$= \sum_{r=1}^{2n}\left[4r^3 + r\right] +$

$4\sum_{r=1}^{n}\left[(2r)^2 - (2r-1)^2\right]$

$= \sum_{r=1}^{2n}\left[4r^3 + r\right] +$

$4\sum_{r=1}^{n}\left[4r^2 - (4r^2 - 4r + 1)\right]$

$= \sum_{r=1}^{2n}\left[4r^3 + r\right] + 4\sum_{r=1}^{n}\left[4r - 1\right]$

$= \sum_{r=1}^{2n}\left[4r^3 + r\right] + \sum_{r=1}^{n}\left[16r - 4\right]$

$= \frac{1}{2}\cdot(2n)\cdot(2n+1) + 4\cdot\frac{1}{4}\cdot(2n)^2\cdot(2n+1)^2$

$+ 16\cdot\frac{1}{2}\cdot n(n+1) - 4n$

Question 7 continued

$= n(2n+1) + (2n)^2(2n+1)^2 + 8n(n+1)$

$- 4n$

$= n(2n+1) + (2n)^2(2n+1)^2 +$

$4n[2(n+1)-1]$

$= n(2n+1) + 4n^2(2n+1)^2 +$

$4n(2n+1)$

$= n(2n+1)[1 + 4n(2n+1) + 4]$

$= n(2n+1)(8n^2 + 4n + 5)$

(c) $\sum_{r=14}^{50} r\left[(-1)^r + 2r\right]^2 = \sum_{r=1}^{50} r\left[(-1)^r + 2r\right]^2$

$- 13\left[(-1)^{13} + 2\times 13\right]$

$- \sum_{r=1}^{12} r\left[(-1)^r + 2r\right]^2$

$= 25(2\times 25 + 1)\left[8\times 25^2 + 4\times 25 + 5\right]$

$- 13\left[(-1)^{13} + 2\times 13\right] -$

$6(2\times 6 + 1)\left[8\times 6^2 + 4\times 6 + 5\right]$

$= 6508875 - 8125 - 24726$

$= 6476024$

Question 7 continued

(Total for Question 7 is 12 marks)

8. A colony of small mammals is being studied.
In the study, the mammals are divided into 3 categories

N (newborns)	0 to less than 1 month old
J (juveniles)	1 to 3 months old
B (breeders)	over 3 months old

(a) State one limitation of the model regarding the division into these categories.

(1)

A model for the population of the colony is given by the matrix equation

$$\begin{pmatrix} N_{n+1} \\ J_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix} \begin{pmatrix} N_n \\ J_n \\ B_n \end{pmatrix}$$

where a and b are constants, and N_n, J_n and B_n are the respective numbers of the mammals in each category n months after the start of the study.

At the start of the study the colony has breeders only, with no newborns or juveniles.

According to the model, after 2 months the number of newborns is 48 and the number of juveniles is 40

(b) (i) Determine the number of mammals in the colony at the start of the study.

(ii) Show that $a = 0.8$

(4)

(c) Determine, in terms of b,

$$\begin{pmatrix} 0 & 0 & 2 \\ 0.8 & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}^{-1}$$

(3)

Given that the model predicts approximately 1015 mammals **in total** at the start of a particular month, and approximately 596 **newborns**, 464 **juveniles** and 437 **breeders** at the start of the next month,

(d) determine the value of b, giving your answer to 2 decimal places.

(3)

It is decided to monitor the number of **newborn** males and females as a part of the study. Assuming that 42% of newborns are male,

(e) refine the matrix equation for the model to reflect this information, giving a reason for your answer.
(*There is no need to estimate any unknown values for the refined model, but any known values should be made clear.*)

(2)

Question 8 continued

(a) THE MAMMALS WILL NOT GO STRAIGHT FROM NOT BREEDING AT THE SAME RATE AS ADULTS AT 3 MONTHS; THERE WILL BE A GRADUAL TRANSITION

(b) LET NUMBER OF MAMMALS AT START $= x$

$$\therefore \begin{bmatrix} N_0 \\ J_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

$$\therefore \text{AFTER 1 MONTH} = \begin{bmatrix} N_1 \\ J_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{bmatrix} \begin{bmatrix} N_0 \\ J_0 \\ B_0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

$$= \begin{bmatrix} 2x \\ 0 \\ 0.96x \end{bmatrix}$$

$$\therefore \text{AFTER 2 MONTHS} = \begin{bmatrix} N_2 \\ J_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{bmatrix} \begin{bmatrix} N_1 \\ J_1 \\ B_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{bmatrix} \begin{bmatrix} 2x \\ 0 \\ 0.96x \end{bmatrix}$$

$$= \begin{bmatrix} 1.92x \\ 2ax \\ 0.9216x \end{bmatrix}$$

$$\therefore 1.92x = 48$$

Question 8 continued

$\therefore x = 25$

$\therefore$ 25 MAMMALS AT START OF STUDY

$\therefore 2ax = 40$

$\therefore 2 \times a \times 25 = 40$

$\therefore a = 0.8$

LET $M = \begin{bmatrix} 0 & 0 & 2 \\ 0.8 & b & 0 \\ 0 & 0.48 & 0.96 \end{bmatrix}$

$\therefore \det(M) = \begin{vmatrix} 0 & 0 & 2 \\ 0.8 & b & 0 \\ 0 & 0.48 & 0.96 \end{vmatrix}$

$= 0 \begin{vmatrix} b & 0 \\ 0.48 & 0.96 \end{vmatrix} - 0 \begin{vmatrix} 0.8 & 0 \\ 0 & 0.96 \end{vmatrix} + 2 \begin{vmatrix} 0.8 & b \\ 0 & 0.48 \end{vmatrix}$

$= 2 [0.8 \times 0.48 - 0 \times b]$

$= 0.768$

MATRIX OF MINORS

$= \begin{bmatrix} \begin{vmatrix} b & 0 \\ 0.48 & 0.96 \end{vmatrix} & \begin{vmatrix} 0.8 & 0 \\ 0 & 0.96 \end{vmatrix} & \begin{vmatrix} 0.8 & b \\ 0 & 0.48 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 0.48 & 0.96 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 0.96 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 0 & 0.48 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ b & 0 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0.8 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 0.8 & b \end{vmatrix} \end{bmatrix}$

Question 8 continued

$$= \begin{bmatrix} 0.96b & 0.768 & 0.384 \\ -0.96 & 0 & 0 \\ -2b & -1.6 & 0 \end{bmatrix}$$

MATRIX OF COFACTORS $= \begin{bmatrix} 0.96b & -0.768 & 0.384 \\ 0.96 & 0 & 0 \\ -2b & 1.6 & 0 \end{bmatrix}$

TRANSPOSE $= \begin{bmatrix} 0.96b & 0.96 & -2b \\ -0.768 & 0 & 1.6 \\ 0.384 & 0 & 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} 0 & 0 & 2 \\ 0.8 & b & 0 \\ 0 & 0.48 & 0.96 \end{bmatrix}^{-1} = \frac{1}{0.768} \begin{bmatrix} 0.96b & 0.96 & -2b \\ -0.768 & 0 & 1.6 \\ 0.384 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.25b & 1.25 & -2.6042b \\ -1 & 0 & 2.80333 \\ 0.5 & 0 & 0 \end{bmatrix}$$

(d) $\begin{bmatrix} 596 \\ 464 \\ 437 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{bmatrix} \begin{bmatrix} N\lambda \\ J\lambda \\ B\lambda \end{bmatrix}$

$$\begin{bmatrix} N\lambda \\ J\lambda \\ B\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{bmatrix}^{-1} \begin{bmatrix} 596 \\ 464 \\ 437 \end{bmatrix}$$

$$= \begin{bmatrix} 1.25b & 1.25 & -2.6042b \\ -1 & 0 & 2.80333 \\ 0.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 596 \\ 464 \\ 437 \end{bmatrix}$$

$$= \begin{bmatrix} 580 - 393.035b \\ 314.415 \\ 298 \end{bmatrix}$$

Question 8 continued

$\therefore 580 - 393.0356 + 314.415 + 298 = 1015$

$\therefore b = 0.45$ (2dp)

(e) WE CAN SPLIT NEWBORNS INTO MALES AND FEMALES, SO, USING A 4×4 MATRIX

$$\therefore \begin{bmatrix} N,M_{n+1} \\ N,F_{n+1} \\ J_{n+1} \\ B_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0.84 \\ 0 & 0 & 0 & 1.16 \\ ? & ? & 0.45 & 0 \\ 0 & 0 & 0.48 & 0.96 \end{bmatrix} \begin{bmatrix} N,M_{n} \\ N,F_{n} \\ J_{n} \\ B_{n} \end{bmatrix}$$

(Total for Question 8 is 13 marks)

TOTAL FOR PAPER IS 75 MARKS