Please check the examination details below before entering your candidate information

| Candidate surname | Other names |
|---|---|
| | |

**Pearson Edexcel**
**Level 3 GCE**

Centre Number | Candidate Number

**Time** 1 hour 30 minutes

**Paper reference** **9FM0/01**

# Further Mathematics

## Advanced

## PAPER 1: Core Pure Mathematics 1

**You must have:**
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

**Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.**

## Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

## Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

## Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- Good luck with your examination.

*Turn over*

P66796A

1/1/1/1/1/

P66796A0136

Pearson

1. The transformation $P$ is an enlargement, centre the origin, with scale factor $k$, where $k > 0$

The transformation $Q$ is a rotation through angle $\theta$ degrees anticlockwise about the origin.

The transformation $P$ followed by the transformation $Q$ is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$$

(a) Determine

(i) the value of $k$,

(ii) the smallest value of $\theta$

**(4)**

A square $S$ has vertices at the points with coordinates $(0, 0)$, $(a, -a)$, $(2a, 0)$ and $(a, a)$ where $a$ is a constant.

The square $S$ is transformed to the square $S'$ by the transformation represented by $\mathbf{M}$.

(b) Determine, in terms of $a$, the area of $S'$

**(2)**

(a) $M = QP$ : $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$

$\begin{pmatrix} K\cos\theta & -K\sin\theta \\ K\sin\theta & K\cos\theta \end{pmatrix} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$

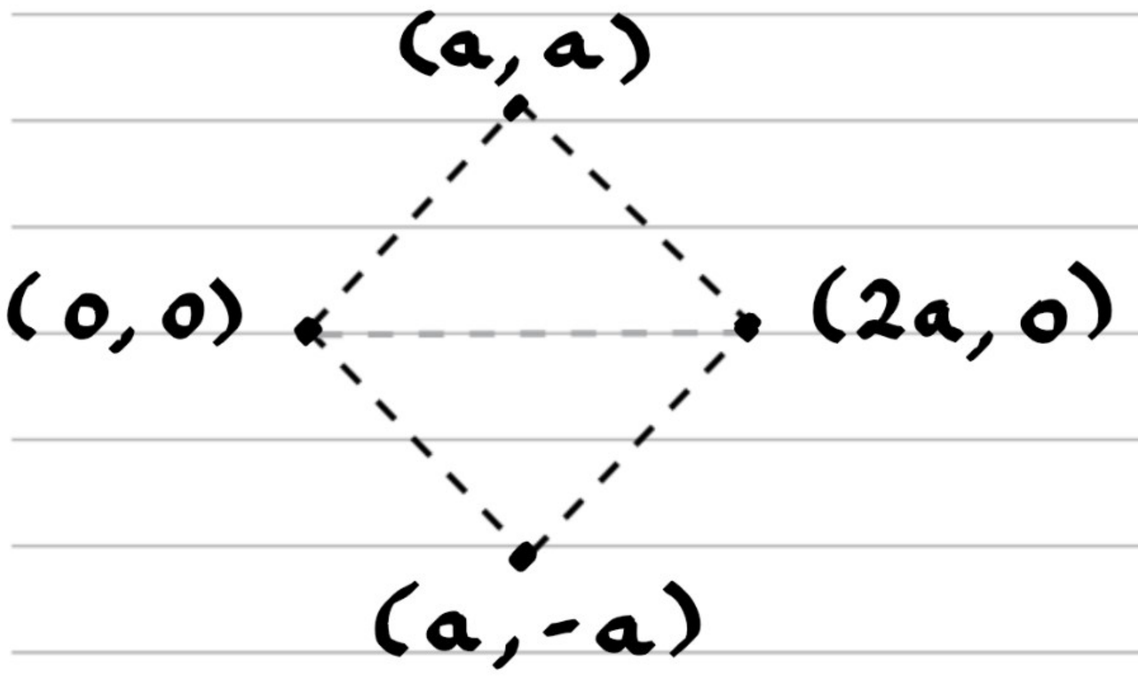
i.e. $K\cos\theta = -4$ AND $\cancel{-}K\sin\theta = \cancel{-}4\sqrt{3}$

i.e. $\dfrac{K\sin\theta}{K\cos\theta} = \dfrac{4\sqrt{3}}{-4}$

$\therefore \tan\theta = -\sqrt{3}$ $\therefore \theta = 120°$

i.e. $K\cos 120 = -4$ $\therefore k = 8$

(b)




$\therefore$ AREA OF $S = 2\left\{\frac{1}{2} \times 2a \times a\right\} = 2a^2$





P 6 6 7 9 6 A 0 2 3 6

**Question 1 continued**

$\therefore$ AREA OF $S' = 2a^2 \times 8^2 = 128a^2$

AREA SCALE FACTOR

2. (a) Use the Maclaurin series expansion for $\cos x$ to determine the series expansion of $\cos^2\left(\frac{x}{3}\right)$ in ascending powers of $x$, up to and including the term in $x^4$

Give each term in simplest form.

**(2)**

(b) Use the answer to part (a) and calculus to find an approximation, to 5 decimal places, for

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{x}\cos^2\left(\frac{x}{3}\right)\right) \mathrm{d}x$$

**(3)**

(c) Use the integration function on your calculator to evaluate

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{x}\cos^2\left(\frac{x}{3}\right)\right) \mathrm{d}x$$

Give your answer to 5 decimal places.

**(1)**

(d) Assuming that the calculator answer in part (c) is accurate to 5 decimal places, comment on the accuracy of the approximation found in part (b).

**(1)**

(a) $\cos^2 \frac{x}{3} = \left(\cos \frac{x}{3}\right)^2$

$= \left[1 - \frac{(x/3)^2}{2!} + \frac{(x/3)^4}{4!} - \dots\right]^2$

$= \left[1 - \frac{x^2}{18} + \frac{x^4}{1944} - \dots\right]\left[1 - \frac{x^2}{18} + \frac{x^4}{1944} - \dots\right]$

$= 1 - \frac{x^2}{18} + \frac{x^4}{1944} - \frac{x^2}{18} + \frac{x^4}{324} + \frac{x^4}{1944} + \dots$

$= 1 - \frac{x^2}{9} + \frac{x^4}{243}$ (UP TO AND INCLUDING $x^4$ TERM)

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA



P 6 6 7 9 6 A 0 6 3 6

**Question 2 continued**

(b) $\int_{\pi/6}^{\pi/2} \frac{1}{x}\cos^2\left(\frac{x}{3}\right)dx = \int_{\pi/6}^{\pi/2} \frac{1}{x}\left(1 - \frac{x^2}{9} + \frac{x^4}{243}\right)dx$

$= \int_{\pi/6}^{\pi/2} \left(\frac{1}{x} - \frac{x}{9} + \frac{x^3}{243}\right)dx$

$= \left[\ln|x| - \frac{x^2}{18} + \frac{x^4}{972}\right]_{\pi/6}^{\pi/2}$

$= \left\{\ln(\pi/2) - \frac{(\pi/2)^2}{18} + \frac{(\pi/2)^4}{972}\right\} -$

$\left\{\ln(\pi/6) - \frac{(\pi/6)^2}{18} + \frac{(\pi/6)^4}{972}\right\}$

$= 0.9829514389$

$= 0.98295$ (5dp)

(c) CALCULATOR VALUE OF $\int_{\pi/6}^{\pi/2} \frac{1}{x}\cos^2\left(\frac{x}{3}\right)dx = 0.9828012317$

$= 0.98280$ (5dp)

(d) THE APPROXIMATION IS CORRECT TO 3dp

3. The cubic equation

$$ax^3 + bx^2 - 19x - b = 0$$

where $a$ and $b$ are constants, has roots $\alpha$, $\beta$ and $\gamma$

The cubic equation

$$w^3 - 9w^2 - 97w + c = 0$$

where $c$ is a constant, has roots $(4\alpha - 1)$, $(4\beta - 1)$ and $(4\gamma - 1)$

Without solving either cubic equation, determine the value of $a$, the value of $b$ and the value of $c$.

**(6)**

LET $w = 4x - 1 \quad \therefore \quad x = \frac{w+1}{4}$

i.e. $a\left(\frac{w+1}{4}\right)^3 + b\left(\frac{w+1}{4}\right)^2 - 19\left(\frac{w+1}{4}\right) - b = 0$

$\Rightarrow \frac{a}{64}(w^3 + 3w^2 + 3w + 1) + \frac{b}{16}(w^2 + 2w + 1) - \frac{19}{4}(w+1) - b = 0$

$\Rightarrow \frac{a}{64}w^3 + \frac{3a}{64}w^2 + \frac{3a}{64}w + \frac{a}{64} + \frac{b}{16}w^2 + \frac{2b}{16}w + \frac{b}{16} - \frac{19}{4}w$

$- \frac{19}{4} - b = 0$

$\Rightarrow \frac{a}{64}w^3 + \left(\frac{3a}{64} + \frac{b}{16}\right)w^2 + \left(\frac{3a}{64} + \frac{2b}{16} - \frac{19}{4}\right)w + \left(\frac{a}{64} + \frac{b}{16} - \frac{19}{4}\right.$

$\left. - b\right) = 0$

$\Rightarrow \frac{a}{64}w^3 + \left(\frac{3a + 4b}{64}\right)w^2 + \left(\frac{3a + 8b - 304}{64}\right)w +$

$\left(\frac{a + 4b - 304 - 64b}{64}\right) = 0$

Question 3 continued

$$\Rightarrow \frac{a}{64}w^3 + \left(\frac{3a+4b}{64}\right)w^2 + \left(\frac{3a+8b-304}{64}\right)w + \left(\frac{a-60b-304}{64}\right) = 0$$

$$\Rightarrow w^3 + \left(\frac{3a+4b}{a}\right)w^2 + \left(\frac{3a+8b-304}{a}\right)w + \left(\frac{a-60b-304}{a}\right) = 0$$

NOW COMPARE COEFFICIENTS:

$w^2$: $\frac{3a+4b}{a} = -9$

$3a + 4b = -9a$

$12a + 4b = 0$ ①

$w$: $\frac{3a+8b-304}{a} = -97$

$3a + 8b - 304 = -97a$

$100a + 8b = 304$ ②

$N0^\circ$: $\frac{a-60b-304}{a} = c$ ③

$\therefore a = 4,\ b = -12$

SUB INTO ③: $c = \frac{(4) - 60(-12) - 304}{(4)}$

$\therefore c = 105$

4. (i) **A** is a 2 by 2 matrix and **B** is a 2 by 3 matrix.

Giving a reason for your answer, explain whether it is possible to evaluate

(a) **AB**

(b) **A** + **B**

**(2)**

(ii) Given that

$$\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix} = \lambda\mathbf{I}$$

where $a$, $b$ and $\lambda$ are constants,

(a) determine

- the value of $\lambda$
- the value of $a$
- the value of $b$

(b) Hence deduce the inverse of the matrix $\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}$

**(3)**

(iii) Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & \cos\theta \\ 0 & \cos 2\theta & \sin 2\theta \end{pmatrix} \qquad \text{where } 0 \leqslant \theta < \pi$$

determine the values of $\theta$ for which the matrix **M** is singular. det m = 0

**(4)**

(i)(a) IT IS POSSIBLE ∵ NO° OF COLUMNS IN A = NO° OF ROWS IN B

(b) IT IS NOT POSSIBLE ∵ A AND B HAVE DIFFERENT DIMENTIONS

(ii)
$$\begin{pmatrix} 5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

(a) $(5)(0) + (3)(2) + (1)(-1) = \lambda$

**Question 4 continued**

$\therefore \lambda = 5$

$(b)(0) + (a)(2) + (b)(-1) = 0$

$\therefore 2a - b = 0$

$(a)(5) + (0)(12) + (0)(-11) = \lambda$

$5a = \lambda$

$5a = 5 \quad \therefore a = 1$

i.e. $2(1) - b = 0 \quad \therefore b = 2$

(b) LET $A = \begin{pmatrix} -5 & 3 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix} \therefore A^{-1} = \frac{1}{5}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix}$

(iii) $M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & \cos\theta \\ 0 & \cos 2\theta & \sin 2\theta \end{pmatrix} \quad ; \quad 0 \leq \theta < \pi$

$\det M = 1(\sin\theta \sin 2\theta - \cos\theta \cos 2\theta) - 0(\sin 2\theta - \cos 2\theta) +$

$0(\cos\theta - \sin\theta)$

$= \sin\theta \sin 2\theta - \cos\theta \cos 2\theta$

$= -(\cos\theta \cos 2\theta - \sin\theta \sin 2\theta)$

$= -(\cos(\theta + 2\theta))$

$= -\cos 3\theta$

i.e. $-\cos 3\theta = 0$

**Question 4 continued**

$\cos 3\theta = 0$

$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$\therefore \quad \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA



P 6 6 7 9 6 A 0 1 6 3 6

5. (i) Evaluate the improper integral

$$\int_1^{\infty} 2e^{-\frac{1}{2}x}\,dx$$

**(3)**

(ii) The air temperature, $\theta$ °C, on a particular day in London is modelled by the equation

$$\theta = 8 - 5\sin\left(\frac{\pi}{12}t\right) - \cos\left(\frac{\pi}{6}t\right) \qquad 0 \leqslant t \leqslant 24$$

where $t$ is the number of hours after midnight.

(a) Use calculus to show that the mean air temperature on this day is 8 °C, according to the model.

**(3)**

Given that the actual mean air temperature recorded on this day was higher than 8 °C,

(b) explain how the model could be refined.

**(1)**

(i) $\int_1^{\infty} 2e^{-x/2}\,dx = \lim_{t\to\infty} \int_1^t 2e^{-x/2}\,dx$

$= \lim_{t\to\infty} \left[-4e^{-x/2}\right]_1^t$

$= \lim_{t\to\infty} \left\{-4e^{-x/2} - (-4e^{-1/2})\right\}$

$= 0 + 4e^{-1/2}$

$= 4e^{-1/2}$

(ii) $\theta = 8 - 5\sin\left(\frac{\pi t}{12}\right) - \cos\left(\frac{\pi t}{6}\right)$ ; $0 \leq t \leq 24$

(a) MEAN TEMPERATURE $= \frac{1}{24-0}\int_0^{24}\left\{8 - 5\sin\left(\frac{\pi t}{12}\right) - \cos\left(\frac{\pi t}{6}\right)\right\}dt$

$= \frac{1}{24}\left[8t + \frac{60}{\pi}\cos\left(\frac{\pi t}{12}\right) - \frac{6}{\pi}\sin\left(\frac{\pi t}{6}\right)\right]_0^{24}$

$= \frac{1}{24}\left[\left\{8(24) + \frac{60}{\pi}\cos\left(\frac{24\pi}{12}\right) - \frac{6}{\pi}\sin\left(\frac{24\pi}{6}\right)\right\} - \left\{8(0)\right.\right.$

**Question 5 continued**

$$+ \frac{60}{\pi}\cos(0) - \frac{6}{\pi}\sin(0)\Big\}\Big]$$

$$= \frac{1}{\cancel{24}}\Big[\Big\{8(\cancel{24}) + \frac{60}{\pi}(1) - \cancel{\frac{6}{\pi}(0)}\Big\} - \Big\{0 + \frac{60}{\pi}(1) - \cancel{\frac{6}{\pi}(0)}\Big\}\Big]$$

$$= 8$$

(b) INCREASE THE VALUE OF THE CONSTANT 8

**(Total for Question 5 is 7 marks)**

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

**6.** A tourist decides to do a bungee jump from a bridge over a river.
One end of an elastic rope is attached to the bridge and the other end of the elastic rope is attached to the tourist.
The tourist jumps off the bridge.

At time $t$ seconds after the tourist reaches their lowest point, their vertical displacement is $x$ metres above a fixed point 30 metres vertically above the river.

When $t = 0$

- $x = -20$
- the velocity of the tourist is $0\,\text{m s}^{-1}$
- the acceleration of the tourist is $13.6\,\text{m s}^{-2}$

In the subsequent motion, the elastic rope is assumed to remain taut so that the vertical displacement of the tourist can be modelled by the differential equation

$$5k\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 17x = 0 \qquad t \geqslant 0$$

where $k$ is a positive constant.

(a) Determine the value of $k$

**(2)**

(b) Determine the particular solution to the differential equation.

**(7)**

(c) Hence find, according to the model, the vertical height of the tourist above the river 15 seconds after they have reached their lowest point.

**(2)**

(d) Give a limitation of the model.

**(1)**

(a) $5k(13.6) + 2k(0) + 17(-20) = 0$

$68k = 340$

$\therefore k = 5$

$\therefore 25\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 17x = 0 \quad ; \quad t \geqslant 0$

(b) AE : $25m^2 + 10m + 17 = 0$

$m = -0.2 \pm 0.8i$

$\therefore x = e^{-0.2t}(A\cos 0.8t + B\sin 0.8t)$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA



P 6 6 7 9 6 A 0 2 0 3 6

**Question 6 continued**

$\frac{dx}{dt} = -0.2e^{-0.2t}(A\cos 0.8t + B\sin 0.8t) +$

$e^{-0.2t}(-0.8A\sin 0.8t + 0.8B\cos 0.8t)$

$t=0, x=-20 : -20 = e^{0}(A\cos 0 + B\sin 0)$

$\therefore -20 = A$

$t=0, dx/dt = 0 : 0 = -0.2e^{0}(A\cos 0 + B\sin 0) + e^{0}(-0.8A\sin 0$

$+ 0.8B\cos 0)$

$0 = -0.2A + 0.8B$

$0 = -0.2\times 20 + 0.8B$

$\therefore B = -5$

$\therefore x_{PS} = e^{-0.2t}(-20\cos 0.8t - 5\sin 0.8t)$

(c) VERTICAL HEIGHT $= 30 + e^{-0.2\times 15}(-20\cos(0.8\times 15) - 5\sin(0.8\times 15))$

$= 29.29\ldots$

$= 29.3\text{ m (3sf)}$

(d) UNLIKELY THAT ROPE WILL REMAIN TAUT

7. The plane $\Pi$ has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

where $\lambda$ and $\mu$ are scalar parameters.

(a) Show that vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ is perpendicular to $\Pi$.

**(2)**

(b) Hence find a Cartesian equation of $\Pi$.

**(2)**

The line $l$ has equation

$$\mathbf{r} = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$$

where $t$ is a scalar parameter.

The point $A$ lies on $l$.

Given that the shortest distance between $A$ and $\Pi$ is $2\sqrt{29}$

(c) determine the possible coordinates of $A$.

**(4)**

(a) $(-\underline{i} + 2\underline{j} + \underline{k}) \cdot (2\underline{i} + 3\underline{j} - 4\underline{k}) = -1 \times 2 + 2 \times 3 + 1 \times -4 = 0$

$(2\underline{i} + 0\underline{j} + \underline{k}) \cdot (2\underline{i} + 3\underline{j} - 4\underline{k}) = 2 \times 2 + 0 \times 3 + 1 \times -4 = 0$

$\therefore$ AS $(2\underline{i} + 3\underline{j} - 4\underline{k})$ IS PERPENDICULAR TO BOTH DIRECTION VECTORS OF $\Pi$, THEN IT MUST BE PERPENDICULAR TO $\Pi$

$\therefore \underline{n} =$

(b) $d = (3\underline{i} + 3\underline{j} + 2\underline{k}) \cdot (2\underline{i} + 3\underline{j} - 4\underline{k})$

$= 3 \times 2 + 3 \times 3 + 2 \times -4$

$= 7$

$\therefore \Pi : \underline{r} \cdot (2\underline{i} + 3\underline{j} - 4\underline{k}) = 7$

**Question 7 continued**

$\therefore \Pi : 2x + 3y - 4z = 7$

(d) LET $A(4+t, -5+6t, 2-3t)$

i.e. $\dfrac{|2(4+t)+3(-5+6t)-4(2-3t)-7|}{\sqrt{(2)^2+(3)^2+(-4)^2}} = 2\sqrt{29}$

$|8+2t-15+18t-8+12t-7| = 58$

$|32t-22| = 58$

$\pm(32t-22) = 58$

$\oplus$ $32t-22=58$ $\quad\ominus$ $22-32t=58$

$t=\frac{5}{2}$ $\quad t=-\frac{9}{8}$

$A(4+\frac{5}{2}, -5+6\times\frac{5}{2}, 2-3\times\frac{5}{2}) \therefore A(\frac{13}{2}, 10, -\frac{11}{2})$

$A(4+-\frac{9}{8}, -5+6\times-\frac{9}{8}, 2-3\times-\frac{9}{8}) \therefore A(\frac{23}{8}, -\frac{47}{4}, \frac{43}{8})$



DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

**8.** Two different colours of paint are being mixed together in a container.

The paint is stirred continuously so that each colour is instantly dispersed evenly throughout the container.

Initially the container holds a mixture of 10 litres of red paint and 20 litres of blue paint.

The colour of the paint mixture is now altered by

- adding red paint to the container at a rate of 2 litres per second
- adding blue paint to the container at a rate of 1 litre per second
- pumping fully mixed paint from the container at a rate of 3 litres per second.

Let $r$ litres be the amount of red paint in the container at time $t$ seconds after the colour of the paint mixture starts to be altered.

(a) Show that the amount of red paint in the container can be modelled by the differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 2 - \frac{r}{\alpha}$$

where $\alpha$ is a positive constant to be determined.

**(2)**

(b) By solving the differential equation, determine how long it will take for the mixture of paint in the container to consist of equal amounts of red paint and blue paint, according to the model. Give your answer to the nearest second.

**(6)**

It actually takes 9 seconds for the mixture of paint in the container to consist of equal amounts of red paint and blue paint.

(c) Use this information to evaluate the model, giving a reason for your answer.

**(1)**

(a) VOLUME OF PAINT $= \underset{\text{RED}}{\underset{\uparrow}{10\ell}} + \underset{\text{BLUE}}{\underset{\uparrow}{20\ell}} = 30\ell$

RATE OF RED IN $= 2\ \ell/\text{SEC}$

RATE OF RED OUT $= 3\ \ell/\text{SEC} \times \dfrac{r\ell}{30\ell} = \dfrac{r}{10}\ \ell/\text{SEC}$

$\therefore \dfrac{dr}{dt} = 2 - \dfrac{r}{10}$ $\ell/\text{SEC}$ $\therefore \alpha = 10$

(b) $\dfrac{dr}{dt} + \dfrac{1}{10}r = 2$

Question 8 continued

$IF = e^{\int \frac{1}{10} dt} = e^{t/10}$

i.e. $re^{t/10} = \int 2e^{t/10}\, dt + c$

$\therefore \quad re^{t/10} = 20e^{t/10} + c$

$t = 0,\ r = 10: \quad 10e^0 = 20e^0 + c$

$\therefore c = -10$

$\therefore re^{t/10} = 20e^{t/10} - 10$

$r = 15$ $\because$ EQUAL AMOUNT OF BLUE AND RED, i.e. $30 \div 2 = 15$

i.e. $15e^{t/10} = 20e^{t/10} - 10$

$5e^{t/10} = 10$

$e^{t/10} = 2$

$\frac{1}{10}t = \ln 2$

$t = 10\ln 2 = 6.931\ldots$

$\therefore t = 7$ SECONDS

(c) THE MODEL PREDICTS 7 SECONDS BUT IT ACTUALLY TAKES 9 SECONDS SO 2 SECONDS OUT

$\therefore$ NOT A GOOD MODEL

**9.** (a) Use a hyperbolic substitution and calculus to show that

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\left[x\sqrt{x^2-1} + \operatorname{arcosh} x\right] + k$$

where $k$ is an arbitrary constant.

**(6)**

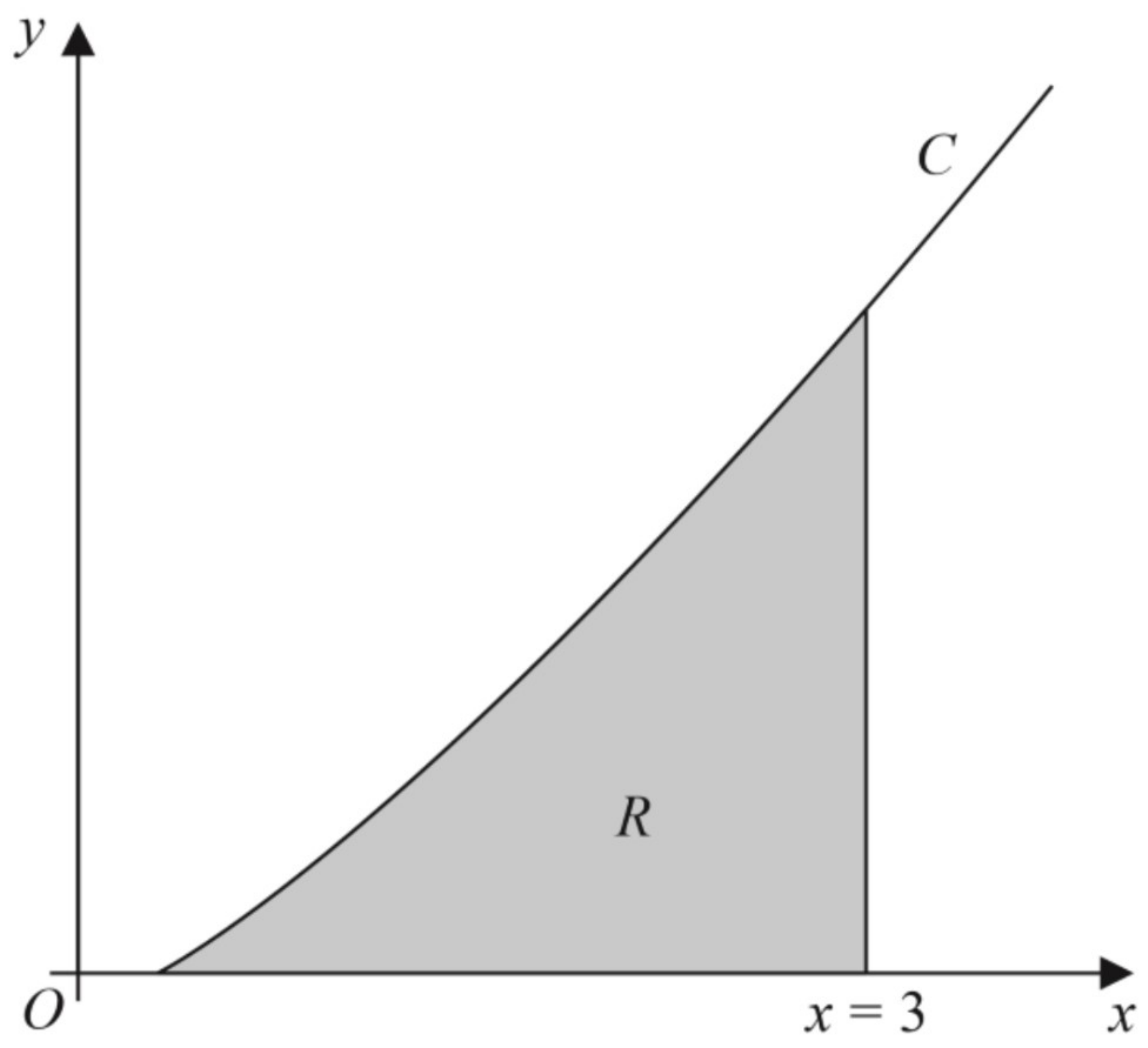



**Figure 1**

Figure 1 shows a sketch of part of the curve $C$ with equation

$$y = \frac{4}{15}x \operatorname{arcosh} x \qquad x \geqslant 1$$

The finite region $R$, shown shaded in Figure 1, is bounded by the curve $C$, the $x$-axis and the line with equation $x = 3$

(b) Using algebraic integration and the result from part (a), show that the area of $R$ is given by

$$\frac{1}{15}\left[17\ln\left(3+2\sqrt{2}\right) - 6\sqrt{2}\right]$$

**(5)**

(a) $\cosh^2 A - \sinh^2 A = 1$

$\sinh^2 A = \cosh^2 A - 1$

$\sinh A = \sqrt{\cosh^2 A - 1}$

LET $x = \cosh u \quad \therefore \quad u = \operatorname{arcosh} x$

$\frac{dx}{du} = \sinh u \quad \therefore \quad dx = \sinh u \, du$

DO NOT WRITE IN THIS AREA



P 6 6 7 9 6 A 0 3 2 3 6

DO NOT WRITE IN THIS AREA

**Question 9 continued**

$\int \frac{x^2}{\sqrt{x^2-1}}\,dx$ BECOMES $\int \frac{\cosh^2 u}{\sqrt{\cosh^2 u - 1}} \sinh u\, du$

$$\equiv \int \frac{\cosh^2 u}{\cancel{\sinh u}} \cancel{\sinh u}\, du$$

$$\equiv \int \cosh^2 u\, du$$

BUT $\cosh 2u \equiv 2\cosh^2 u - 1 \therefore \cosh^2 u \equiv \frac{1}{2}(\cosh 2u + 1)$

$$\equiv \frac{1}{2}\int (\cosh 2u + 1)\, du$$

$$= \frac{1}{2}\left[\frac{1}{2}\sinh 2u + u\right] + K$$

$$= \frac{1}{2}\left[\frac{1}{2} \times 2\sinh u \cosh u + u\right] + K$$

$$= \frac{1}{2}\left[\sqrt{\cosh^2 u - 1}\,\cosh u + u\right] + K$$

$$= \frac{1}{2}\left[x\sqrt{x^2-1} + \operatorname{arcosh} x\right] + K$$

(b) AREA OF $R = \int_1^3 \frac{4}{15} x \operatorname{arcosh} x\, dx$

CONSIDER: $\frac{4}{15}\int x \operatorname{arcosh} x\, dx$

$u = \operatorname{arcosh} x$ $\qquad v = \frac{1}{2}x^2$

$u' = \frac{1}{\sqrt{x^2-1}}$ $\qquad v' = x$

$\therefore \frac{4}{15}\int x \operatorname{arcosh} x\, dx = \frac{4}{15}\left[\frac{1}{2}x^2 \operatorname{arcosh} x - \frac{1}{2}\int \frac{x^2}{\sqrt{x^2-1}}\,dx\right] + A$

$$= \frac{4}{15}\left[\frac{1}{2}x^2 \operatorname{arcosh} x - \frac{1}{2}\left\{\frac{1}{2}\left(x\sqrt{x^2-1} + \operatorname{arcosh} x\right)\right\}\right] + A$$

$$= \frac{2}{15}x^2 \operatorname{arcosh} x - \frac{1}{15}x\sqrt{x^2-1} - \frac{1}{15}\operatorname{arcosh} x + A$$

**Question 9 continued**

$$\therefore \int_1^3 \frac{4}{15} x \operatorname{arcosh} x \, dx = \left[ \frac{2}{15} x^2 \operatorname{arcosh} x - \frac{1}{15} x\sqrt{x^2 - 1} - \frac{1}{15} \operatorname{arcosh} x \right]_1^3$$

$$= \left\{ \frac{2}{15} \times 3^2 \times \operatorname{arcosh}(3) - \frac{1}{15} \times 3 \times \sqrt{3^2 - 1} - \frac{1}{15} \operatorname{arcosh}(3) \right\} - \left\{ \frac{2}{15} \times 1^2 \times \operatorname{arcosh}(1) - \frac{1}{15} \times 1 \times \sqrt{1^2 - 1} - \frac{1}{15} \operatorname{arcosh}(1) \right\}$$

$$= \frac{17}{15} \operatorname{arcosh}(3) - \frac{2\sqrt{2}}{5}$$

$$= \frac{1}{15} \left[ 17 \operatorname{arcosh}(3) - 6\sqrt{2} \right]$$

$$= \frac{1}{15} \left[ 17 \ln(3 + \sqrt{3^2 - 1}) - 6\sqrt{2} \right]$$

$$\therefore \text{AREA OF } R = \frac{1}{15} \left[ 17 \ln(3 + 2\sqrt{2}) - 6\sqrt{2} \right]$$