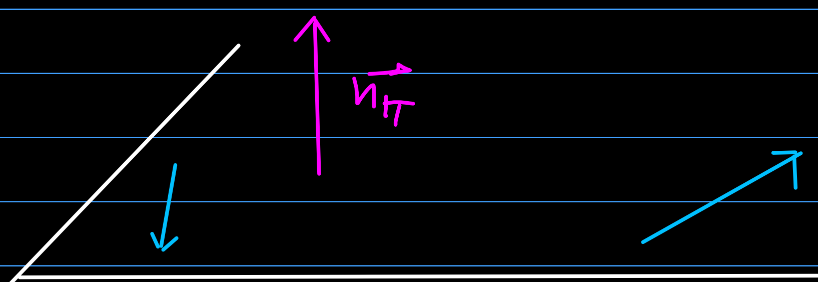
Seminar W4 − 914

$\pi: \ Ax + By + Cz + D = 0$

$\vec{n_\pi}(A, B, C)$ normal vector for π

$\downarrow$

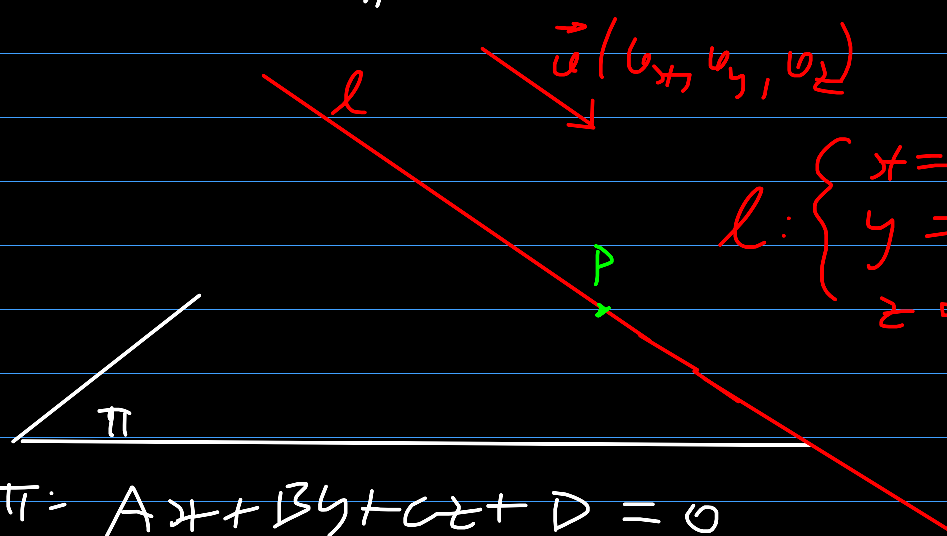
$\forall \vec{u}, \ \vec{u} \parallel \pi : \ \vec{u} \perp \vec{n_\pi}$



· $\vec{w}(p, q, r)$. We have:

$\vec{w} \parallel \pi \iff Ap + Bq + Cr = 0$

$\iff \vec{n_\pi} \cdot \vec{w} = 0$



$$l : \begin{cases} x = x_0 + \lambda u_x \\ y = y_0 + \lambda u_y \\ z = z_0 + \lambda u_z \end{cases}$$

$\pi: \ Ax + By + Cz + D = 0$

If $\ell \nparallel \pi$ (i.e. $Au_x + Bu_y + Cu_z \neq 0$) then the coordinates of the intersection point $\{P\} = \ell \cap \pi$ are:

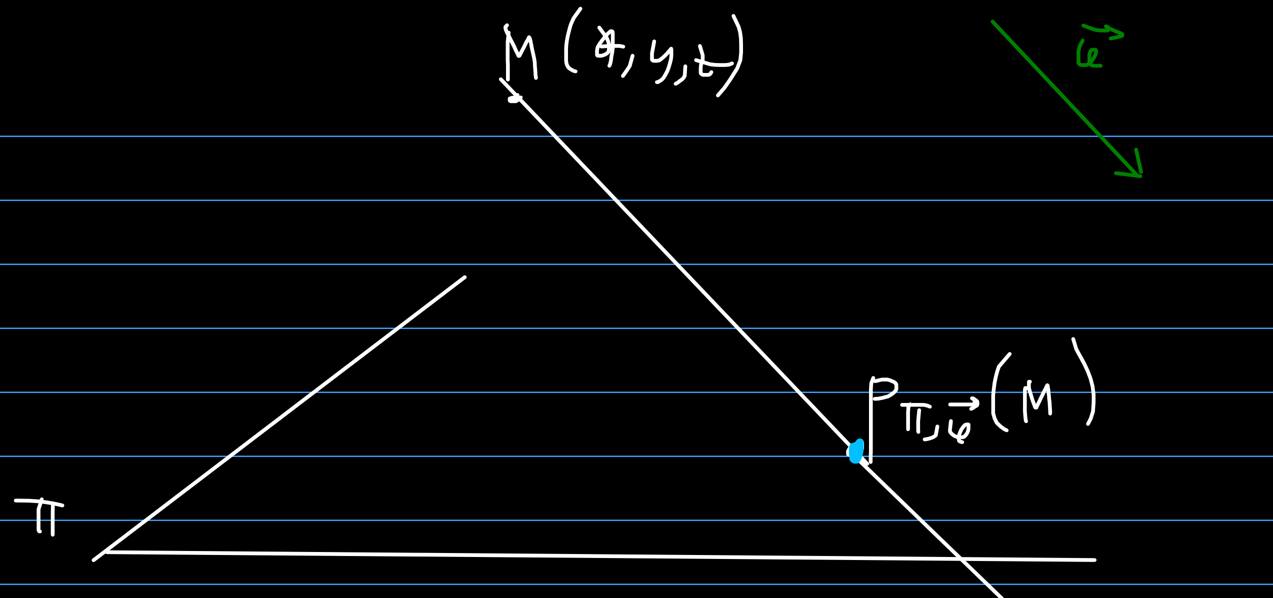
$$\begin{cases} x_P = x_0 - u_x \cdot \dfrac{Ax_0 + By_0 + Cz_0 + D}{Au_x + Bu_y + Cu_z} \\[2ex] y_P = y_0 - u_y \cdot \dfrac{Ax_0 + By_0 + Cz_0 + D}{Au_x + Bu_y + Cu_z} \\[2ex] z_P = z_0 - u_z \cdot \dfrac{Ax_0 + By_0 + Cz_0 + D}{Au_x + Bu_y + Cu_z} \end{cases}$$

$\hookrightarrow$ the coordinates of the point with

$$\lambda = -\frac{Ax_0 + By_0 + Cz_0}{Au_x + Bu_y + Cu_z}$$

Fix $\pi : Ax + By + Cz + D = 0$ and $\vec{u}(u_x, u_y, u_z)$ s.t. $\vec{u} \nparallel \pi$ ($Au_x + Bu_y + Cu_z \neq 0$)

We define the projection onto π, parallel with $\vec{u}$ by:



$$P_{\pi,\vec{\omega}} : \mathbb{R}^3 \longrightarrow \pi$$
$$(x, y, z) \longmapsto (x', y', z')$$

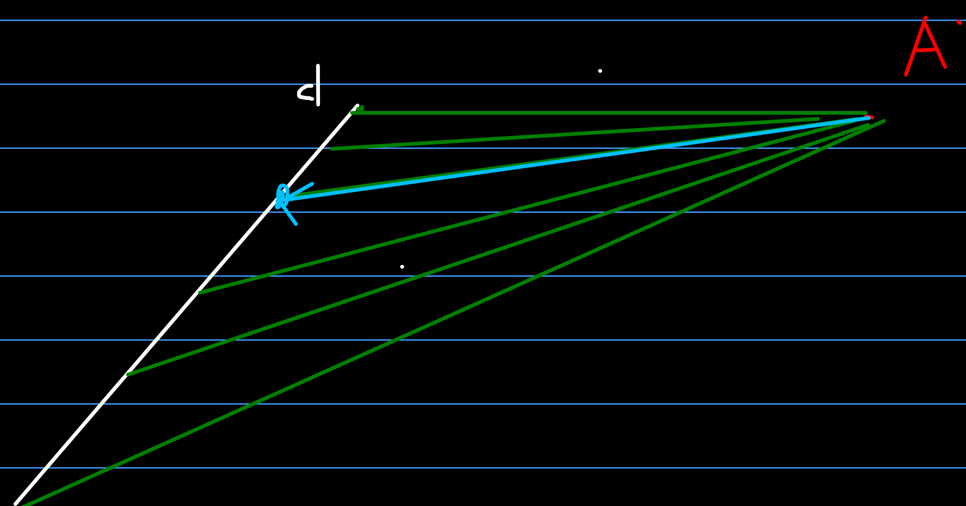
$S_{\pi,\omega}(M)$

$$\begin{cases} x' = x - \omega_x \cdot \dfrac{Ax + By + Cz + D}{A\omega_x + B\omega_y + C\omega_z} \\[2ex] y' = y - \omega_y \cdot \dfrac{Ax + By + Cz + D}{A\omega_x + B\omega_y + C\omega_z} \\[2ex] z' = z - \omega_z \cdot \dfrac{Ax + By + Cz + D}{A\omega_x + B\omega_y + C\omega_z} \end{cases}$$

4.1. Write the equation of the plane determined by the line:

$$(d)\begin{cases} x-2y+3z=0 \\ 2x+z-3=0 \end{cases}$$

and the point $A(-1,2,6)$



$$(d):\begin{cases} x=2y-3z \\ 4y-6z+z-3=0 \end{cases} \Rightarrow \begin{cases} x=2y-3z \\ 4y-5z-3=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x=2y-3z \\ y=\frac{5z+3}{4} \end{cases} \Rightarrow \begin{cases} x=\frac{10z+6}{4}-3z \\ y=\frac{5z+3}{4} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x=-\frac{2z}{4}+\frac{6}{4} \\ y=\frac{5z}{4}+\frac{3}{4} \end{cases} \Rightarrow \begin{cases} x=-\frac{z}{2}+\frac{3}{2} \\ y=\frac{5}{4}z+\frac{3}{4} \end{cases}$$

For $z=1$ we get the point $B(1,2,1)$

For $z=5$ we get the point $C(-1,7,5)$

The plane:

$$\begin{vmatrix} x-x_A & y-y_A & z-z_A \\ x_B-x_A & y_B-y_A & z_B-z_A \\ x_C-x_A & y_C-y_A & z_C-z_A \end{vmatrix} = 0$$

$$\begin{vmatrix} x+1 & y-2 & z-6 \\ 2 & 0 & -5 \\ 0 & 5 & -1 \end{vmatrix} = 0$$

$\Leftrightarrow$ $10z-60+25x+25+2y-4=0$

$\Leftrightarrow$ $25x+2y+10z-39=0$

Ex.: Consider the plane

$$\Pi: x + 2y - z + 5 = 0$$

and a line:

$$\ell: \frac{x-2}{1} = \frac{y}{6} = \frac{z+1}{2}$$

Find the coordinates of $\{P\} = \Pi \cap \ell$
(without using the formulas above)

R:

$$P = \begin{cases} \frac{x-2}{1} = \frac{y}{6} = \frac{z+1}{2} \\ x + 2y - z + 5 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = 6x - 12, \quad z = 2x - 4 - 1 = 2x - 5 \\ x + 2 \cdot y - z + 5 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = 6x - 12 \\ z = 2x - 5 \\ x + 12x - 24 - 2x + 5 + 5 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 6x - 12 \\ z = 2x - 5 \\ 11x = 14 \end{cases} \Leftrightarrow \begin{cases} x = \frac{14}{11} \\ y = 6 \cdot \frac{14}{11} - 12 = \frac{-48}{11} \\ z = 2 \cdot \frac{14}{11} - 5 = \frac{-27}{11} \end{cases}$$

Let's now use the formulas:

$$\begin{cases} x_p = x_0 - u_x \cdot \dfrac{Ax_0 + By_0 + Cz_0 + D}{Au_x + Bu_y + Cu_z} \\ y_p = y_0 - u_y \cdot \dfrac{Ax_0 + By_0 + Cz_0 + D}{Au_x + Bu_y + Cu_z} \\ z_p = z_0 - u_z \cdot \dfrac{Ax_0 + By_0 + Cz_0 + D}{Au_x + Bu_y + Cu_z} \end{cases}$$

$$(x_0, y_0, z_0) = (2, 0, -1)$$

$$(u_x, u_y, u_z) = (1, 6, 2)$$

$$(A, B, C, D) = (1, 2, -1, 5)$$

$$Au_x + Bu_y + Cu_z = 11$$

$$Ax_0 + By_0 + Cz_0 + D = 8$$

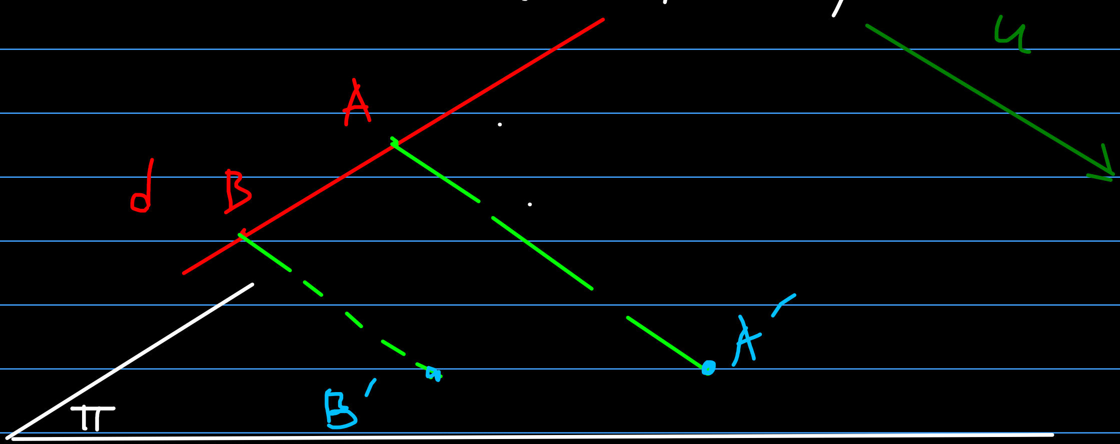
$$\Rightarrow \begin{cases} x = 2 - 1 \cdot \frac{8}{11} = \frac{14}{11} \checkmark \\ y = 0 - 6 \cdot \frac{8}{11} = -\frac{48}{11} \\ z = -1 - 2 \cdot \frac{8}{11} = -\frac{27}{11} \end{cases}$$

4.3. Write the equations of the projection of the line

$$(d)\ \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

on the plane $\pi : x + 2y - z = 0$, parallel to the direction $\vec{u}(1, 1, -2)$.

(Homework: do the same for reflection) → "symmetry" in the notes



$$\overrightarrow{r_{p_{\pi,\vec{u}}(M)}} = \vec{r}_M - \frac{F(M)}{\vec{n}_\pi \cdot \vec{u}} \cdot \vec{u}$$

$$(d): \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x + z - 1 \\ x + 2x + z - 1 - z - 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = 2x + z - 1 \\ x = \frac{2}{3} \end{cases} \Leftrightarrow \begin{cases} x = \frac{2}{3} \\ y = z + \frac{1}{3} \end{cases}$$

$$\Rightarrow \quad d: \begin{cases} x = \dfrac{2}{3} \\ y = t + \dfrac{1}{3} \\ z = t \end{cases}$$

$$\pi: \ x + 2y - z = 0$$

$$\begin{cases} x' = x - u_x \cdot \dfrac{Ax + By + Cz + D}{Au_x + Bu_y + Cu_z} \\ y' = y - u_y \cdot \dfrac{Ax + By + Cz + D}{Au_x + Bu_y + Cu_z} \\ z' = z - u_z \cdot \dfrac{Ax + By + Cz + D}{Au_x + Bu_y + Cu_z} \end{cases}$$

$$\Rightarrow \frac{Ax + By + Cz + D}{Au_x + Bu_y + Cu_z} = \frac{x + 2y - z}{1 \cdot 1 + 2 \cdot 1 + (-2) \cdot (-1)} =$$

$$= \frac{\frac{2}{3} + 2t + \frac{2}{3} - t}{5} = \frac{t + \frac{4}{3}}{5}$$

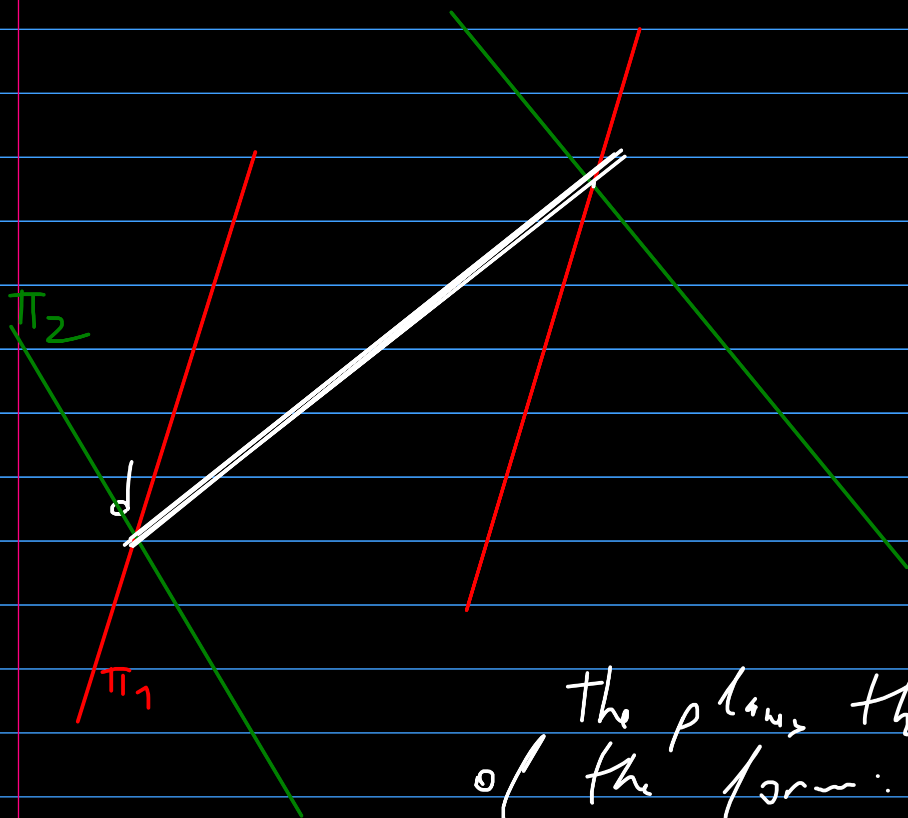
$$\begin{cases} x' = \dfrac{2}{3} - 1 \cdot \dfrac{t + \frac{4}{3}}{5} = -\dfrac{t}{5} + \dfrac{6}{15} = -\dfrac{t}{5} + \dfrac{2}{5} \\ y' = t + \dfrac{1}{3} - 1 \cdot \dfrac{t + \frac{4}{3}}{5} = \dfrac{4}{5} t + \dfrac{1}{15} \\ z' = t + 2 \cdot \dfrac{t + \frac{4}{3}}{5} = \dfrac{7}{5} t + \dfrac{8}{15} \end{cases}$$

$\Rightarrow$ The projection of the line (d) is a line (d'), whose parametric equations are:

$$\begin{cases} x = -\frac{t}{5} + \frac{2}{5} \\ y = \frac{4}{5} t + \frac{1}{15} \\ z = \frac{7}{5} t + \frac{8}{15} \end{cases}$$

$$\forall M : \quad \vec{r}_{s_{\pi,d}(M)} = 2\,\vec{r}_{P_{\pi,d}(M)} - \vec{r}_M$$

The pencil of planes



$$d : \begin{cases} \pi_1 : A_1x + B_1y + C_1z + D_1 = 0 \\ \pi_2 : A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

the planes that contain d are of the form:

$$\pi_{\alpha,\beta} : \alpha \cdot (A_1x + B_1y + C_1z + D_1) + \beta \cdot (A_2x + B_2y + C_2z + D_2) = 0$$

Ex II: Consider the line:

$$d : \begin{cases} 2x - y + z - 1 = 0 \\ -x + 2y + z + 3 = 0 \end{cases}$$

Find the plane π that contains d. and the point $A(1, 1, 1)$

$\Pi_{\alpha,\beta} : \alpha \cdot (2x - y - z - 1) + \beta \cdot (-x + 2y + z + 3) = 0$

$(1,1,1) \in \Pi_{\alpha,\beta} \Rightarrow \alpha + 5\beta = 0 \Rightarrow \alpha = -5\beta$

$\Rightarrow$ the plane that we need is $\Pi_{-5\beta,\beta}$

$\Pi_{-5\beta,\beta} : -5\beta\,(2x - y - z - 1) + \beta \cdot (-x + 2y + z + 3) = 0$

$\Rightarrow \Pi = \Pi_{-5,1} : -5\,(2x - y - z - 1) + (-x + 2y + z + 3) = 0$

$\Rightarrow \Pi : -11x + 7y - 4z + 8 = 0$