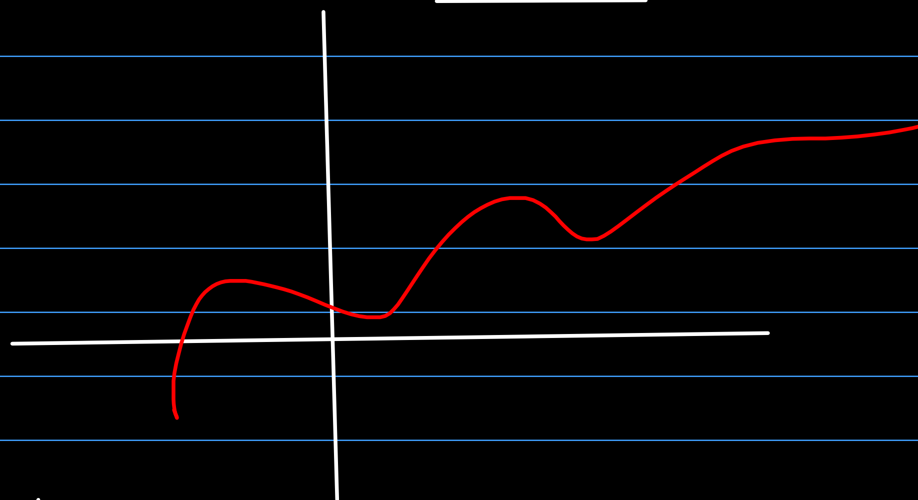
Seminar W8 - 915

Curves



$\mathcal{C}$ curve

• given parametrically: $\mathcal{C}: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$

$$\mathcal{C}: \ \gamma: \mathbb{R} \longrightarrow \mathbb{R}^2 \ (\text{resp. } \mathbb{R}^3)$$
$$t \longmapsto (x(t), y(t), z(t))$$

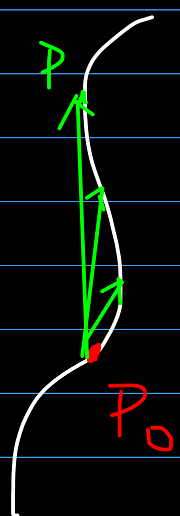
$\mathcal{C} = \mathcal{C}(O, r)$: $\mathcal{C}: \begin{cases} x = r\cos t \\ y = r\sin t \end{cases}, t \in [0, 2\pi)$

• given implicitly

→ in 2D : $f(x, y) = 0$

→ in 3D : $\begin{cases} f_1(x, y, z) = 0 \\ f_2(x, y, z) = 0 \end{cases}$

• $\mathcal{C}$ curve, the tangent to $\mathcal{C}$ in the point P_0 is a line that contains the point P_0 whose direction is given by:



$$\vec{t} = \lim_{\substack{P \to P_0 \\ P \in \mathcal{C}}} \frac{\overrightarrow{P_0 P}}{\|\overrightarrow{P_0 P}\|}$$

• If $\mathcal{C}$ is given parametrically by:

$$\mathcal{C} : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \iff \gamma = \gamma(t)$$

$$\lim_{P \to P_0} \frac{\overrightarrow{P_0P}}{\|\overrightarrow{P_0P}\|} = \frac{\left(x(t) - x(t_0),\ y(t) - y(t_0),\ z(t) - z(t_0)\right)}{\|\overrightarrow{P_0P}\|} =$$

$$= \lim_{t \to t_0} \left(\frac{x(t) - x(t_0)}{t - t_0},\ \frac{y(t) - y(t_0)}{t - t_0},\ \frac{z(t) - z(t_0)}{t - t_0} \right)$$

$$\Rightarrow T_{\mathcal{C}}(t = t_0) = \gamma'(t_0)$$

$$T_{\mathcal{C}}(t = t_0) : \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}$$

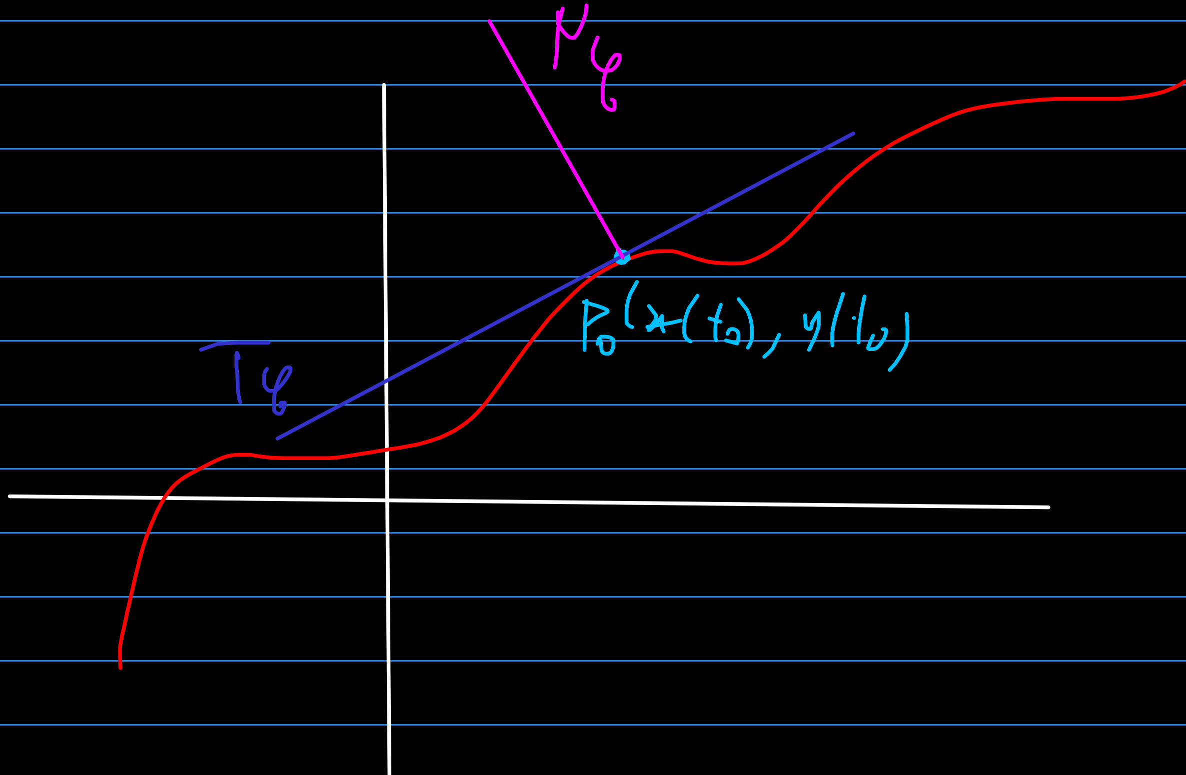
$$(\text{in 2D})\ m_{T_{\mathcal{C}}}(t = t_0) = \frac{y'(t_0)}{x'(t_0)}$$

• if $\mathscr{C}$ is planar (in 2D):

normal line in P_0 = line that contains P_0 and is perpendicular to the tangent

$$N_{\mathscr{C}}(t=t_0): -\frac{x'(t_0)}{y'(t_0)} \cdot (x - x(t_0)) = y - y(t_0) \iff$$

$$\iff y'(t_0)\cdot(y - y(t_0)) + x'(t_0)\cdot(x - x(t_0)) = 0$$



· If $\mathcal{C}$ is spatial (in 3D)

normal plane in P_0 = the plane perpendicular to the tangent line in P_0 and contains P_0

$$N_{\mathcal{C}}(t=t_0): \ x'(t_0)\cdot(x-x(t_0))+y'(t_0)\cdot(y-y(t_0))+$$
$$+z'(t_0)\cdot(z-z(t_0))=0$$

· If $\mathcal{C}$ is given implicitly (and is planar), then:

$\mathcal{C}: f(x,y)=0$

$$T_{\mathcal{C}}(x_0,y_0): \ \underbrace{f'_x(x_0,y_0)}_{\frac{\partial f}{\partial x}(x_0,y_0)}\cdot(x-x_0)+\underbrace{f'_y(x_0,y_0)}_{\frac{\partial f}{\partial y}(x_0,y_0)}\cdot(y-y_0)=0$$

$$N_{\mathscr{C}}(x_0, y_0): \quad \frac{x - x_0}{f'_x(x_0, y_0)} = \frac{y - y_0}{f'_y(x_0, y_0)}$$

8.1. Show that the angle between the tangent of the circular helix:

$$\mathscr{C}: \begin{cases} x = a \cdot \cos t \\ y = a \cdot \sin t \\ z = bt \end{cases}, t \in \mathbb{R}$$

and the z-axis is constant.

$$x'(t) = -a \sin t$$

$$y'(t) = a \cos t$$

$$z'(t) = b$$

$$T_{\mathscr{C}}(t = t_0): \frac{x - a\cos t_0}{-a \sin t_0} = \frac{y - a \sin t_0}{a \cos t_0} = \frac{z - bt_0}{b}$$

$$\vec{u}_{T_{\mathcal{C}}}\left(-a\sin t_0,\ a\cos t_0,\ b\right)$$

$$\vec{u}_{Oz}\left(0,0,1\right)$$

$$m\left(\widehat{T_{\mathcal{C}}(t=t_0),\ Oz}\right) = \arccos \frac{\vec{u}_{T_{\mathcal{C}}} \cdot \vec{u}_{Oz}}{\|\vec{u}_{T_{\mathcal{C}}}\| \cdot \|\vec{u}_{Oz}\|} =$$

$$= \arccos \frac{b}{\sqrt{a^2\sin^2 t_0 + a^2\cos^2 t_0 + b^2}} =$$

$$= \arccos \frac{b}{\sqrt{a^2(\sin^2 t_0 + \cos^2 t_0) + b^2}} =$$

$$= \arccos \frac{b}{\sqrt{a^2+b^2}}$$ does not depend on t_0 $\Rightarrow$ the angle between the tangent and the z-axis is constant.

7-8. Write the equations of the tangent line and the normal plane for the following curve.

$$\mathcal{C}: \begin{cases} x = e^t \cos 3t \\ y = e^t \sin 3t \\ z = e^{-2t} \end{cases}$$ at the

points corresponding to the values $t = 0$ and $t = \frac{\pi}{4}$ of the parameter.

$x'(t) = e^t \cdot \cos(3t) - 3 \cdot e^t \sin(3t) \Rightarrow x'(0) = 1$

$y'(t) = e^t \cdot \sin(3t) + 3 e^t \cos(3t) \Rightarrow y'(0) = 3$

$z'(t) = -2 \cdot e^{-2t} \Rightarrow z'(0) = -2$

$$T_{\mathcal{C}}(t=0): \quad \frac{x - x(0)}{x'(0)} = \frac{y - y(0)}{y'(0)} = \frac{z - z(0)}{z'(0)}$$

$x(0) = 1, \quad y(0) = 0, \quad z(0) = 1$

$$\Rightarrow T_{\mathcal{C}}(t=0): \frac{x-1}{1} = \frac{y}{3} = \frac{z-1}{-2}$$

$$\Rightarrow N_{\mathcal{C}}(t=0): x'(0)\cdot(x - x(0)) + y'(0)\cdot(y-y(0)) + z'(0)\cdot(z - z(0)) = 0$$

$$\Rightarrow N_{\mathcal{C}}(t=0): 1\cdot(x-1) + 3\cdot y + (-2)\cdot(z-1) = 0$$

8.2 Find the equation of the tangent line and the normal line to the curve

$$\mathcal{C}: x^3 - x^2y^4 + y - x + 2 = 0$$

in the point $P_0(0, -2)$.

$$f(x,y) := x^3 - x^2y^4 + y - x + 2$$

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 2xy^4 - 1$$

$$\frac{\partial f}{\partial y}(x,y) = -4x^2y^3 + 1$$

$$\frac{\partial f}{\partial x}(0,2) = -1$$

$$\frac{\partial f}{\partial y}(0,2) = 1$$

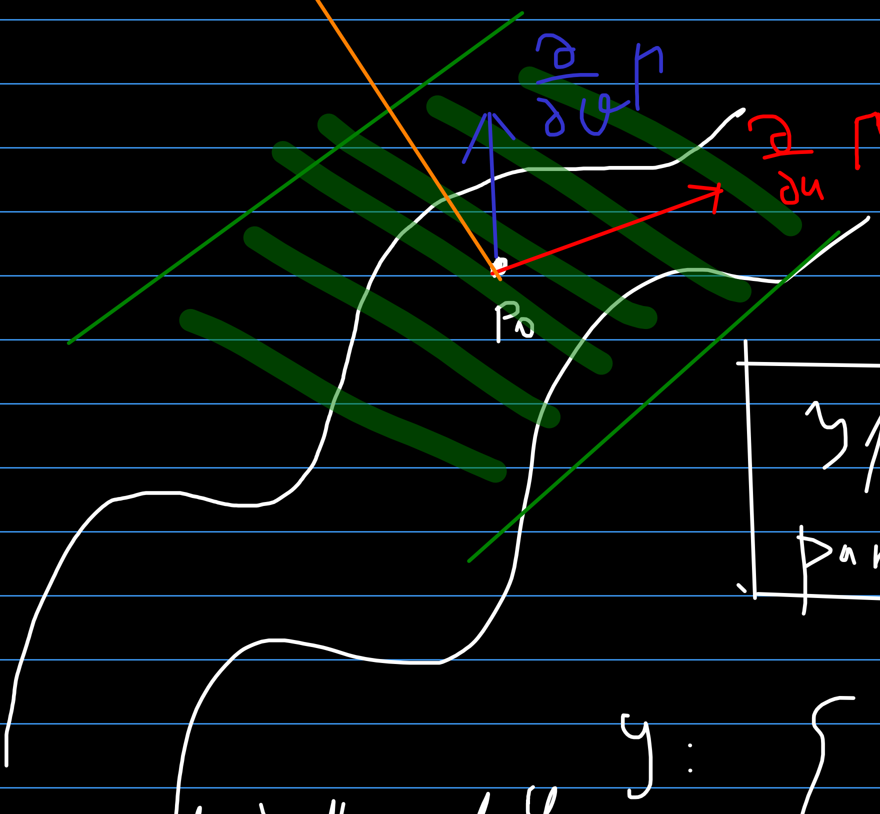
$$\Rightarrow \quad T_{\mathcal{C}}(0,2): \; -x + y - 2 = 0$$

$$N_{\mathcal{C}}(0,2): \; \frac{x}{-1} = \frac{y-2}{1}$$

$N_\varphi(u=u_0, v=v_0) = N_\varphi(P_0)$.

Surfaces

tangent plane.



$P_0\big(x(u_0,v_0),\ y(u_0,v_0),\ z(u_0,v_0)\big)$

$\boxed{\varphi \,/\, \varphi \text{ is given parametrically}}$

$$\varphi : \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

parameterization of φ

$\Gamma : \mathbb{R}^2 \to \mathbb{R}^3$

$$(u,v) \mapsto \big(x(u,v),\ y(u,v),\ z(u,v)\big)$$

__Ex.__: φ sphere centered in O with radius 1

$$\varphi : \begin{cases} x = \cos u \cdot \cos v \\ y = \cos u \cdot \sin v \\ z = \sin u \end{cases}$$

$T_y\,(u=u_0,\ v=v_0)$:

$$\begin{vmatrix} x-x(u_0,v_0) & y-y(u_0,v_0) & z-z(u_0,v_0) \\ \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \\ \frac{\partial x}{\partial v}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0) \end{vmatrix} = 0$$

$\Rightarrow$ $T_y\,(u=u_0,\ v=v_0)$:

$$\underbrace{\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}}_{\frac{\partial(y,z)}{\partial(u,v)}} \cdot \big(x-x(u_0,v_0)\big) +$$

$$+ \underbrace{\begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix}}_{\frac{\partial(z,x)}{\partial(u,v)}} \cdot \big(y-y(u_0,v_0)\big) +$$

$$+\underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}}_{\frac{\partial(x,y)}{\partial(u,v)}} \cdot \left(z - z(u_0, v_0)\right) = 0$$

$$N_y(P_0): \frac{x - x(u_0,v_0)}{\frac{\partial(y,z)}{\partial(u,v)}} = \frac{y - y(u_0,v_0)}{\frac{\partial(z,x)}{\partial(u,v)}} = \frac{z - z(u_0,v_0)}{\frac{\partial(x,y)}{\partial(u,v)}}$$

If y is given implicitly

$$y : f(x,y,z) = 0$$

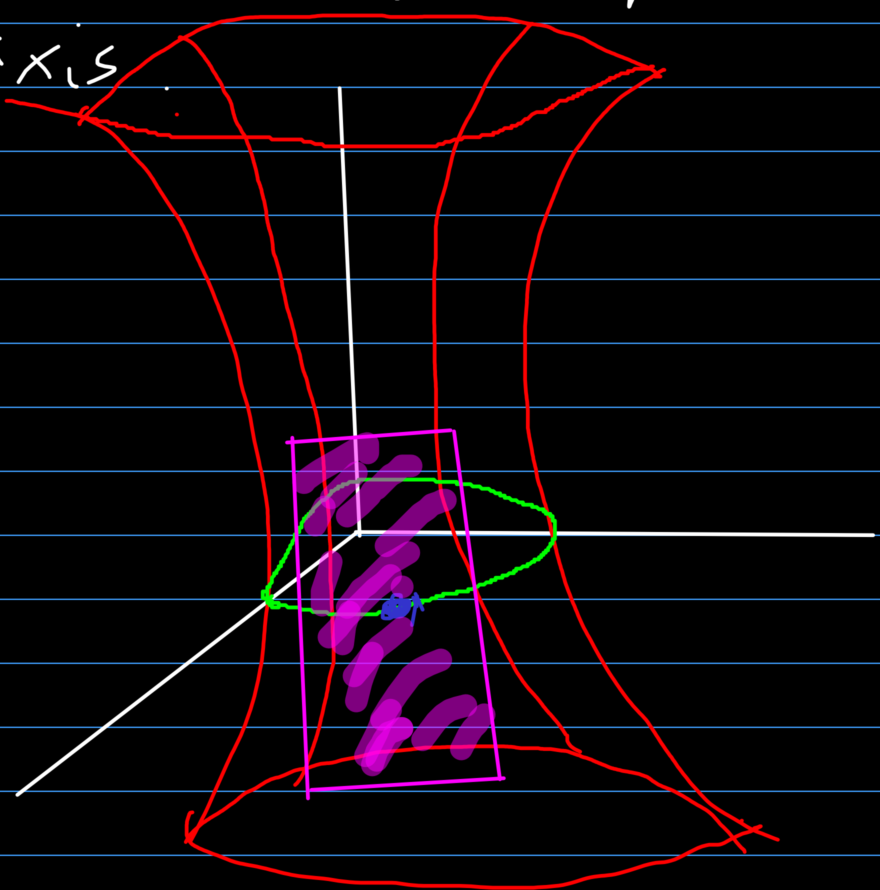
$T_y(x_0, y_0, z_0)$:

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)\cdot(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)\cdot(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)\cdot(z - z_0) = 0$$

$N_y(x_0, y_0, z_0)$:

$$\frac{x - x_0}{\frac{\partial f}{\partial x}(x_0, y_0, z_0)} = \frac{y - y_0}{\frac{\partial f}{\partial y}(x_0, y_0, z_0)} = \frac{z - z_0}{\frac{\partial f}{\partial z}(x_0, y_0, z_0)}$$

8.9. Write the equations of the tangent planes of the hyperboloid of one sheet
$$\mathcal{H}: x^2+y^2-z^2=1$$
at the points of the form $(x_0, y_0, 0)$ and show that they are parallel to the z-axis.



$$f(x,y,z) = x^2+y^2-z^2-1=0$$

$$\frac{\partial f}{\partial x} = 2x \quad , \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = -2z$$

$$\Rightarrow T_{\mathcal{H}}(x_0, y_0, 0) :$$

$$2x_0 \cdot (x - x_0) + 2y_0(y - y_0) -$$

$$- 2z_0(z - z_0) = 0$$

$$\Rightarrow T_{\mathcal{H}} : x_0(x - x_0) + y_0(y - y_0) = 0$$

$$\vec{n}_{T_{\mathcal{H}}(x_0, y_0, 0)} (x_0, y_0, 0).$$

$$\vec{u}_{Oz} = (0, 0, 1)$$

Because $\vec{n}_{T_{\mathcal{H}}(x_0, y_0, z_0)} \cdot \vec{u}_{Oz} = 0$

$$\Rightarrow Oz \parallel T_{\mathcal{H}}(x_0, y_0, 0).$$