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1st Semester, 2020-2021

Algebra (Computer Science)

Bonus Exercises : Week 3

Exercise 1. A **semiring** is an algebraic structure having the same properties as a unital ring, but not requiring the additive inverse property. In other words, $(R, +, \cdot)$ is a semiring if $(R, +)$ is a commutative monoid, $(R, \cdot)$ is a monoid and $\cdot$ is distributive over $+$. We define for every $x, y \in \mathbb{R} \cup \{\infty\}$:

$$x \oplus y := \min(x, y)$$
$$x \otimes y := x + y \text{ (the addition in } \mathbb{R})$$

Prove that $(\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$ is a semiring.

*<u>Trivia:</u> This is the **tropical semiring**, a structure with quite a few applications. To name one that I am aware of, a 2018 paper ("Tropical Geometry of Deep Neural Networks" by Zhang, Naitzat and Kim) has found a connection between Tropical Geometry (essentially geometry over this tropical semiring) and a wide class of neural networks.*

Exercise 2. Let $w \in \mathbb{C}$ be a complex number with trigonometric form

$$w = r(\cos(\theta) + i\sin(\theta))$$

and $n \in \mathbb{N}$, $n \geq 2$. We define the **set of n-th roots of w** by:

$$\sqrt[n]{w} := \{z \in \mathbb{C} | \ z^n = w\}$$

Prove that

$$\sqrt[n]{w} = \{ \sqrt[n]{r} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right) \mid k = 0, \ldots, n-1\}$$

Trivia: *As you might have noticed, this is a generalization of the formula for n-th roots of unity. Geometrically, the points in the complex plane that correspond to the n-th roots of unity are the vertices of a regular n-gon (polygon with n sides) inscribed in the unit circle. In the same way,* $\sqrt[n]{w}$ *consists of points on a circle centered in* 0 *with radius* $\sqrt[n]{|w|}$*, who are also the vertices of a regular n-gon.*

Exercise 3. Let $\mathbb{C}[X]$ be the ring of polynomials with coefficients in $\mathbb{C}$. For any $\alpha \in \mathbb{C}$ we define the map:

$$\begin{aligned} f_\alpha : \mathbb{C}[X] &\to \mathbb{C} \\ F &\mapsto F(\alpha) \end{aligned}$$

Show that F_α is a ring homomorphism. Find the elements in the set:

$$\mathrm{Ker}(f_\alpha) = \{F \in \mathbb{C}[X] |\ f_\alpha(F) = 0\}$$

Trivia: *The* ***kernel*** *can be defined for any homomorphism of algebraic structures (groups, rings, vector spaces, etc.) and it is the set of elements in the domain that are mapped by the homomorphism to the neutral element of the codomain.*

Exercise 4. Let R and S be rings and $f : R \to S$ a ring isomorphism between them. Show that for any $n \in \mathbb{N}$ the rings of matrices $\mathcal{M}_n(R)$ and $\mathcal{M}_n(S)$ are also isomorphic.

Definition. In a commutative unital ring R, an element $a \in R \backslash \{0\}$ is called a **zero divisor** if there exists an element $b \in R \setminus \{0\}$ so that $ab = 0$. An **integral domain** is a commutative unital ring that does not contain any zero divisors.

Trivia: *Zero divisors also make sense if the ring R is not commutative, but in that case a distinction must be made between left and right zero divisors, let us not worry about that.*

Exercise 5. (a) Show that if $\alpha \neq 0$ is a zero divisor in ring R, then it cannot be invertible (with regards to multiplication, of course).

(b) Show that the invertible elements in the ring $\mathbb{Z}_n$ of residue classes modulo n are given by:

$$\mathbb{Z}_n^{\times} = \{\hat{k} |\ (k,n) = 1\}$$

Hint: *Bézout*

(c) Show that $\mathbb{Z}_n$ is a field if and only if n is prime.

Exercise 6. Show that the ring

$$\mathbb{Z}[i] = \{a + ib \in \mathbb{C} |\ a, b \in \mathbb{Z}\}$$

is an integral domain (you do not need to prove that it is a commutative unital ring, you can assume this).

Exercise 7. If R is an integral domain, then the ring of polynomials $R[X]$ is also an integral domain.

Hint: *One way to show that a ring R is an integral domain is to consider $a, b \in R$ so that $ab = 0$ and show that $a = 0$ or $b = 0$.*