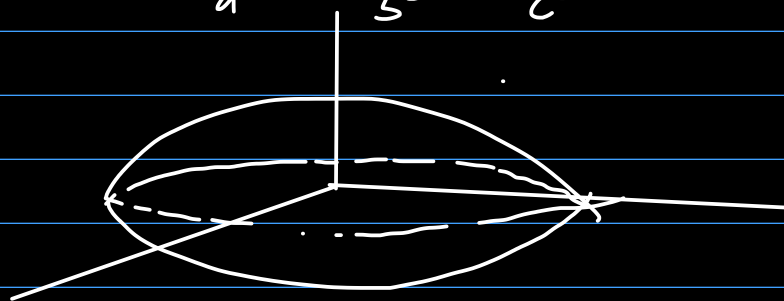
Seminar W10 - 915

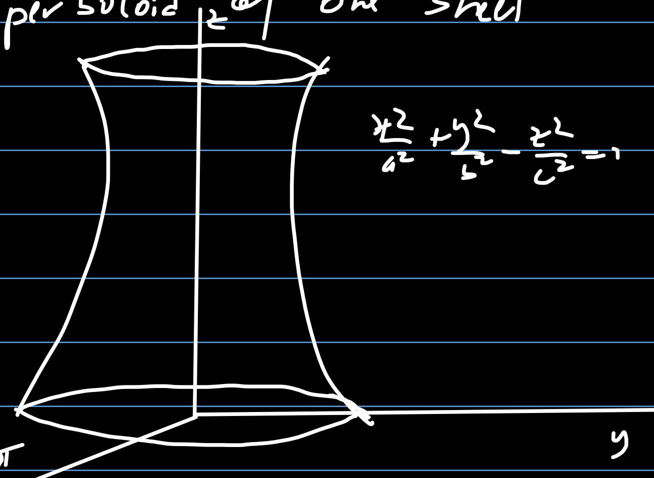
Quadrics:

- ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

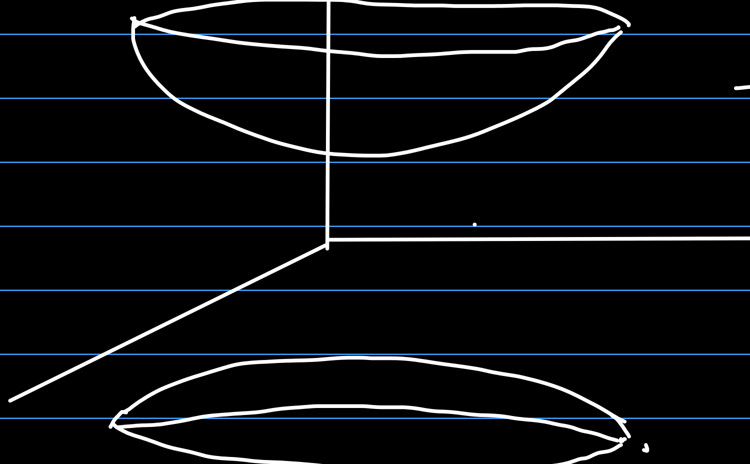


- Hyperboloid of one sheet



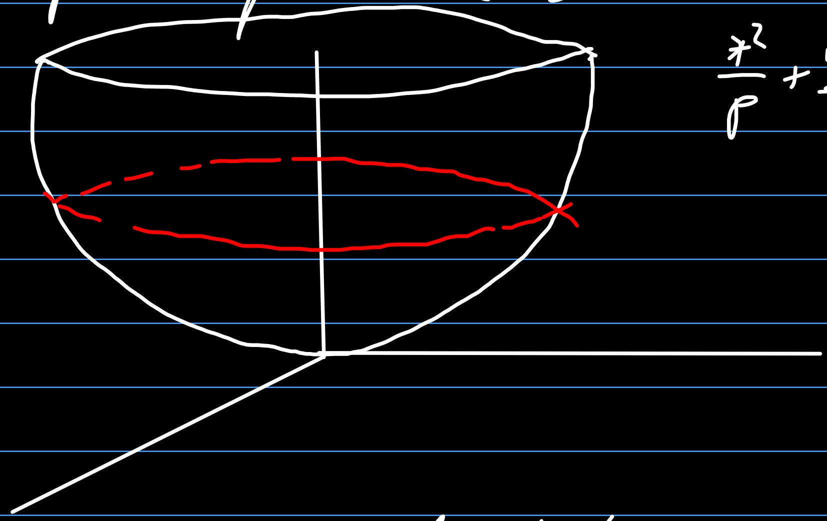
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

- Hyperboloid of two sheets



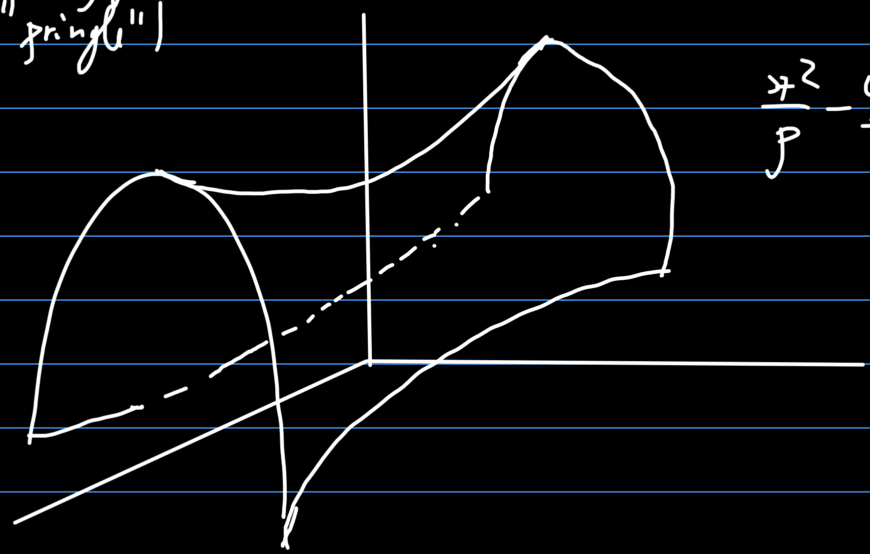
$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

- Elliptic paraboloid



$$\frac{x^2}{p}+\frac{y^2}{q}=2z$$

- Hyperbolic paraboloid (saddle surface) ("pringle")



$$\frac{x^2}{p}-\frac{y^2}{q}=2z$$

- The hyperboloid of one sheet and the hyperbolic paraboloid have rectilinear generatrices (i.e. lines on the surface so that any point on the surface belongs to one of these lines).

If $\mathcal{S}$: $f(x,y,z)=0$ surface

$T_{\mathcal{S}}(x_0,y_0,z_0)$: $f'_x(x_0,y_0,z_0)\cdot(x-x_0)+f'_y(x_0,y_0,z_0)\cdot(y-y_0)+$

tangent plane $\quad + f'_z(x_0,y_0,z_0)\cdot(z-z_0)=0$

$N_{\mathcal{S}}(x_0,y_0,z_0)$: $\dfrac{x-x_0}{f'_x(x_0,y_0,z_0)}=\dfrac{y-y_0}{f'_y(x_0,y_0,z_0)}=\dfrac{z-z_0}{f'_z(x_0,y_0,z_0)}$

normal line

10.1. Find the intersection points of the ellipsoid

$$\mathcal{E}\ \frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$$

with the line:

$$\ell: \frac{x-4}{2}=\frac{y+6}{-3}=\frac{z+2}{-2}$$

and write the equations of the tangent planes as well as the equations of the normal lines to the ellipsoid at the intersection points.

$$\ell: \begin{cases} x=4+2t \\ y=-6-3t \\ z=-2-2t \end{cases}$$

$$\ell \cap \mathcal{E}: \quad \begin{cases} x = 4 + 2t \\ y = -6 - 3t \\ z = -2 - 2t \\ \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} = 1 \end{cases} \iff$$

$$\iff \begin{cases} x = 4 + 2t \\ y = -6 - 3t \\ z = -2 - 2t \\ \frac{(4+2t)^2}{16} + \frac{(6+3t)^2}{12} + \frac{(2+2t)^2}{4} = 1 \end{cases} \iff$$

$$\iff \frac{16+16t+4t^2}{16} + \frac{36+36t+9t^2}{12} + \frac{4+8t+4t^2}{4} = 1$$

$$\iff \frac{4+4t+t^2}{4} + \frac{12+12t+3t^2}{4} + \frac{4+8t+4t^2}{4} = 1$$

$$\iff 8t^2 + 24t + 16 = 0 \iff t^2 + 3t + 2 = 0$$

$$\Delta = 9 - 8 = 1 \Rightarrow t_{1,2} = \frac{-3 \pm 1}{2}$$

$$\Rightarrow t \in \{-2, -1\}$$

$$\ell: \begin{cases} x = 4 + 2t \\ y = -6 - 3t \\ z = -2 - 2t \end{cases}$$

For $t = -2 \Rightarrow P(0, 0, 2)$

For $t = -1 \Rightarrow Q(2, -3, 0)$

$$f(x,y,z) = \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} - 1$$

$$f'_x(x_0, y_0, z_0) = \frac{x_0}{8}$$

$$f'_y(x_0, y_0, z_0) = \frac{y_0}{6}$$

$$f'_z(x_0, y_0, z_0) = \frac{z_0}{2}$$

$$T_\varphi(x_0, y_0, z_0): \quad \frac{x_0}{8}(x - x_0) + \frac{y_0}{6}(y - y_0) + \frac{z_0}{2}(z - z_0) = 0$$

$$T_\varphi(0,0,2): \quad z - 2 = 0$$

$$T_\varphi(2,-3,0): \quad \frac{1}{4}\cdot(x-2) - \frac{1}{2}(y+3) = 0$$

$$N_\varphi(0,0,2): \quad \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$N_\varphi(2,-3,0): \quad \begin{cases} \frac{x-2}{1/4} = \frac{y+3}{-1/2} \\ z = 0 \end{cases}$$

Rectilinear generatrices

1. Hyperboloid of one sheet

$$\mathcal{H}: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}$$

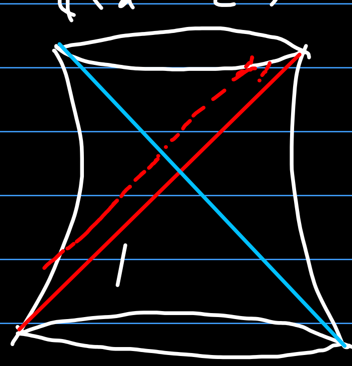
$$\left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right)\cdot\left(1 + \frac{y}{b}\right)$$

$$d_\lambda : \begin{cases} \frac{x}{a} - \frac{z}{c} = \lambda \cdot \left(1 - \frac{y}{b}\right) \\ \lambda \cdot \left(\frac{x}{a} + \frac{z}{c}\right) = 1 + \frac{y}{b} \end{cases}$$

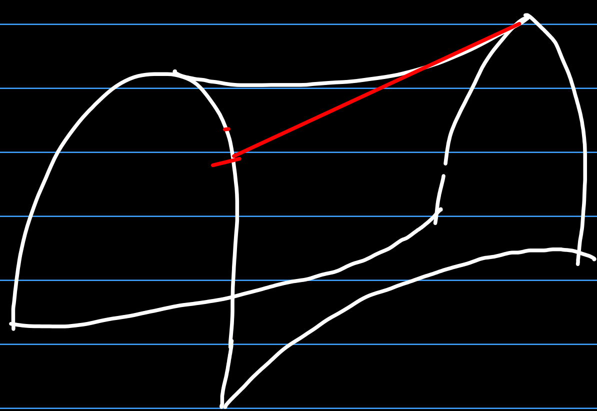
$\hookrightarrow$ parametrized family of lines

($\forall \lambda \in \mathbb{R}$ we get a different line)

$$d'_\mu : \begin{cases} \frac{x}{a} - \frac{z}{c} = \mu \cdot \left(1 + \frac{y}{b}\right) \\ \mu\left(\frac{x}{a} + \frac{z}{c}\right) = 1 - \frac{y}{b} \end{cases}$$



2. Hyperbolic paraboloid



$$\frac{x^2}{p} - \frac{y^2}{q} = 2z$$

$$\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right)\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = 2z$$

$$d_\lambda : \begin{cases} \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = 2\lambda \\ \lambda \cdot \left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = z \end{cases}$$

$$d'_\mu : \begin{cases} \lambda \cdot \left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right) = z \\ \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} = 2\lambda \end{cases}$$

10.2. Find the rectilinear generatrices of the quadric

$$4x^2 - 9y^2 = 36z$$

which pass through the point $P(3\sqrt{2}, 2, 1)$.

$$(2x-3y)(2x+3y)=36z$$

$$d_\lambda : \begin{cases} \lambda\cdot(2x-3y) = 36z \\ 2x+3y = \lambda \end{cases}$$

$$P \in d_\lambda \Rightarrow \begin{cases} \lambda\cdot(6\sqrt{2}-6) = 36 \\ 6\sqrt{2}+6 = \lambda \end{cases} \Leftrightarrow$$

$P(3\sqrt{2}, 2, 1)$

$$\Leftrightarrow \begin{cases} \lambda = \dfrac{36}{6\sqrt{2}-6} = \dfrac{6}{\sqrt{2}-1} \Rightarrow \lambda = 6(\sqrt{2}+1) \\ \lambda = 6(\sqrt{2}+1) \end{cases}$$

$\Rightarrow$ $d_{6(\sqrt{2}+1)}$ is the line that we want

$$d'_\mu : \begin{cases} 2x-3y = \lambda \\ \lambda\cdot(2x+3y) = 36z \end{cases}$$

$$P \in d'_\mu \Rightarrow \begin{cases} 6\sqrt{2}-6 = \mu \\ \mu\cdot(6\sqrt{2}+6) = 36 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu = 6\sqrt{2}-6 = 6(\sqrt{2}-1) \\ \mu = \dfrac{36}{6\sqrt{2}+6} = \dfrac{6}{\sqrt{2}+1} \end{cases}$$

$$\Rightarrow \mu = 6(\sqrt{2}-1)$$

$\Rightarrow$ $d'_{6(\sqrt{2}-1)}$ is the line that we want.

10.3. Find the rectilinear generatrices of the hyperboloid:

$$(\mathcal{H}_1): \frac{x^2}{36}+\frac{y^2}{9}-\frac{z^2}{4}=1$$

which are parallel to the plane

$$\Pi: x+y+z=0$$

$$\frac{x^2}{36}-\frac{z^2}{4}=1-\frac{y^2}{9}$$

$$\left(\frac{x}{6}-\frac{z}{2}\right)=\left(1-\frac{y}{3}\right)\left(1+\frac{y}{3}\right)$$

$$\Rightarrow d_\lambda: \begin{cases} \frac{x}{6}-\frac{z}{2}=\lambda\cdot\left(1-\frac{y}{3}\right) \\ \lambda\cdot\left(\frac{x}{6}+\frac{z}{2}\right)=1+\frac{y}{3} \end{cases}$$

$$\vec{d_\lambda}=\left(\frac{1}{6},\frac{\lambda}{3},-\frac{1}{2}\right)\times\left(\frac{\lambda}{6},-\frac{1}{3},\frac{\lambda}{2}\right)=$$

$$=\begin{vmatrix} i & j & k \\ \frac{1}{6} & \frac{\lambda}{3} & -\frac{1}{2} \\ \frac{\lambda}{6} & -\frac{1}{3} & \frac{\lambda}{2} \end{vmatrix}=\vec{i}\cdot\frac{\lambda^2-1}{6}+\vec{j}\cdot\frac{-\lambda}{6}+ + \vec{k}\cdot\frac{-1-\lambda^2}{18}$$

$$d_\lambda \parallel \Pi \Leftrightarrow d_\lambda \perp \vec{n_\Pi} \Leftrightarrow \vec{d_\lambda}\cdot\vec{n_\Pi}=0 \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{\lambda^2-1}{6},\frac{-\lambda}{6},\frac{-1-\lambda^2}{18}\right)\cdot(1,1,1)=0 \Leftrightarrow$$

$$\Leftrightarrow \frac{\lambda^2-1}{6}-\frac{\lambda}{6}-\frac{1+\lambda^2}{18}=0 \Leftrightarrow$$

$\Leftrightarrow 3\lambda^2 - 3 - 3\lambda - 1 - \lambda^2 = 0 \Leftrightarrow$

$\Leftrightarrow 2\lambda^2 - 3\lambda - 4 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda_{1,2} = \dfrac{3 \pm \sqrt{9+32}}{4} = \dfrac{3 \pm \sqrt{41}}{4}$

$\Rightarrow d_{\frac{3-\sqrt{41}}{2}}$ and $d_{\frac{3+\sqrt{41}}{2}}$ are parallel to π.

Do the same thing for the family

$$d'_\mu : \begin{cases} \dfrac{x}{6} - \dfrac{z}{2} = \mu\left(1 + \dfrac{y}{3}\right) \\ \mu\left(\dfrac{x}{6} + \dfrac{z}{2}\right) = 1 - \dfrac{y}{3} \end{cases}$$