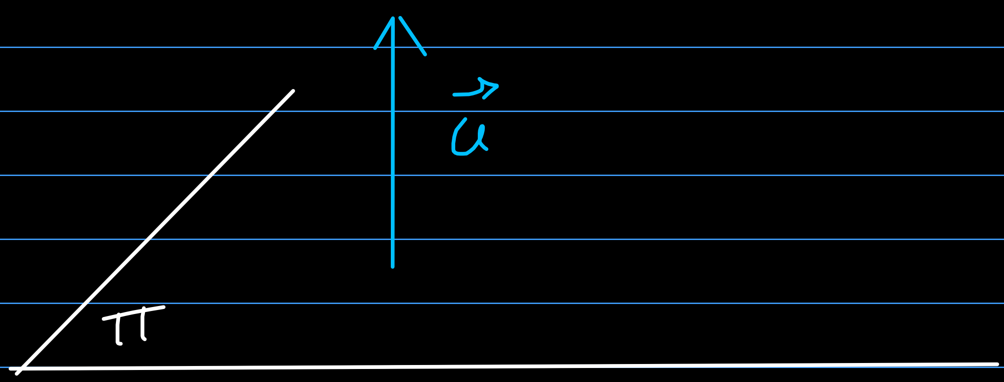
Seminar W4 – 513

$\Pi: \quad Ax + By + Cz + D = 0$

$\vec{n_\Pi}(A, B, C)$ normal vector for the plane:



(i.e. $\vec{u} \perp \vec{w}$, $\forall \vec{w}$ s.t. $\vec{w} \parallel \Pi$)
($\vec{w} \in \vec{\Pi}$)

If we have Π s.t. $\vec{n_\Pi} = (\alpha, \beta, \gamma)$ and we know that $A \in \Pi$, then

$$\Pi: \quad \alpha(x - x_A) + \beta(y - y_A) + \gamma(z - z_A) = 0$$

- A vector $\vec{w}(x_{\vec{w}}, y_{\vec{w}}, z_{\vec{w}})$ is parallel to Π $\iff$

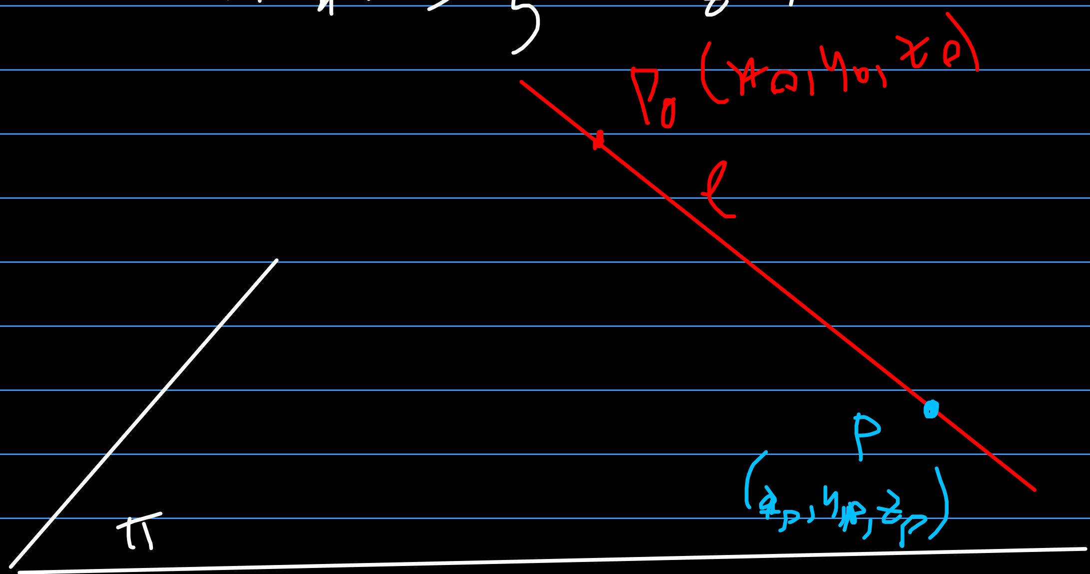
$$A x_{\vec{w}} + B y_{\vec{w}} + C z_{\vec{w}} = 0$$

- $\pi : Ax + By + Cz + D = 0$

$$\ell : \begin{cases} x = x_0 + \lambda\, x_{\vec{u}} \\ y = y_0 + \lambda \cdot y_{\vec{u}} \\ z = z_0 + \lambda \cdot z_{\vec{u}} \end{cases}$$

We assume that $\ell \nparallel \pi$, i.e. we assume that

$$Au_x + Bu_y + Cu_z \neq 0$$



Then $\{P\} = \ell \cap \pi$ has the coordinates

$$\begin{cases} x_P = x_0 - x_{\vec{u}} \cdot \dfrac{Ax_0 + By_0 + Cz_0 + D}{Ax_u + By_u + Cz_u} \\ y_P = y_0 - y_{\vec{u}} \cdot \dfrac{Ax_0 + By_0 + Cz_0 + D}{Ax_u + By_u + Cz_u} \\ z_P = z_0 - z_{\vec{u}} \cdot \dfrac{Ax_0 + By_0 + Cz_0 + D}{Ax_u + By_u + Cz_u} \end{cases}$$

Ex. I: $\pi: x+2y-3z+5=0$

$$\ell: \frac{x-1}{3} = \frac{y+2}{2} = \frac{z+1}{5}$$

Find the coordinates of the intersection point $\{P\} = \ell \cap \pi$ (without using the formulas above)

$$P: \begin{cases} x+2y-3z+5=0 \\ \frac{x-1}{3} = \frac{y+2}{2} = \frac{z+1}{5} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x-1 = \frac{3y+6}{2} \\ z+1 = \frac{5y+10}{2} \\ x+2y-3z+5=0 \end{cases} \Leftrightarrow \begin{cases} x = \frac{3y+8}{2} \\ z = \frac{5y+8}{2} \\ x+2y-3z+5=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = \frac{3y+8}{2} \\ z = \frac{5y+8}{2} \\ \frac{3y+8}{2} + \frac{4y}{2} - \frac{15y+24}{2} + \frac{10}{2} = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} -8y - 6 = 0 \\ x = \dfrac{3y+8}{2} \\ z = \dfrac{5y+8}{2} \end{cases} \Leftrightarrow \begin{cases} y = -\dfrac{3}{4} \\ x = \dfrac{3y+8}{2} \\ z = \dfrac{5y+8}{2} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = -\dfrac{3}{4} \\ x = \dfrac{-9}{8} + 4 = \dfrac{23}{8} \\ z = \dfrac{-15}{8} + 4 = \dfrac{17}{8} \end{cases}$$

If we were to use the formulas:

$$\begin{cases} x_p = x_0 - x_{\vec{u}} \cdot \dfrac{A x_0 + B y_0 + C z_0 + D}{A x_u + B y_u + C z_u} \\ y_p = y_0 - y_{\vec{u}} \cdot \dfrac{A x_0 + B y_0 + C z_0 + D}{A x_u + B y_u + C z_u} \\ z_p = z_0 - z_{\vec{u}} \cdot \dfrac{A x_0 + B y_0 + C z_0 + D}{A x_u + B y_u + C z_u} \end{cases}$$

$$(x_{\vec{u}}, y_{\vec{u}}, z_{\vec{u}}) = (3, 2, 5)$$

$$(x_0, y_0, z_0) = (1, -2, -1)$$

$$(A, B, C, D) = (1, 2, -3, 5)$$

$$\frac{Ax_0 + By_0 + Cz_0 + D}{Ax_{\vec{u}} + By_{\vec{u}} + Cz_{\vec{u}}} = \frac{5}{-8} = -\frac{5}{8}$$

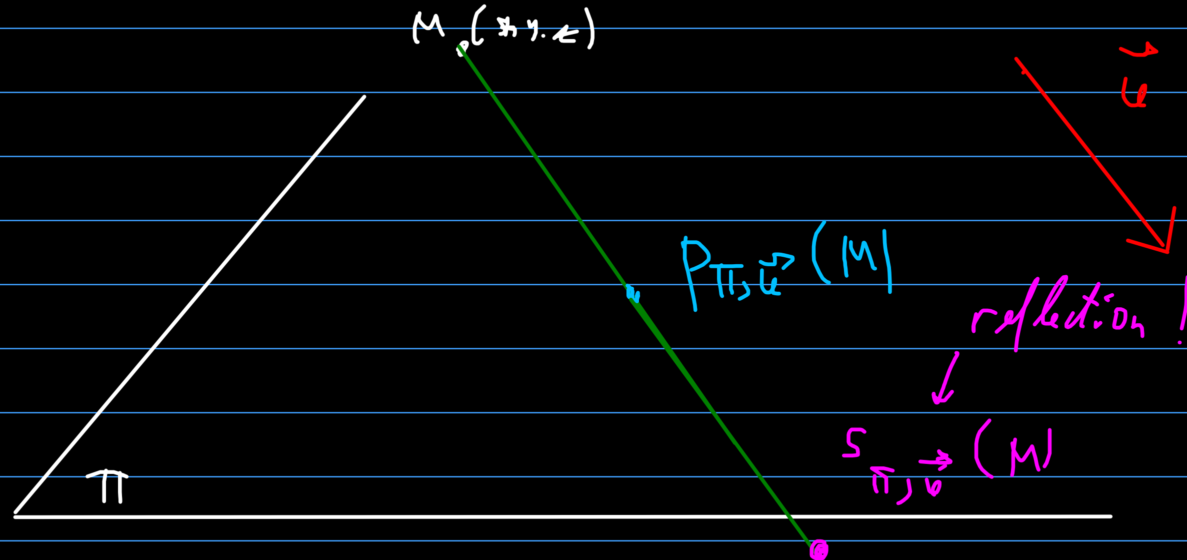
$$\Rightarrow \begin{cases} x_p = 1 - 3 \cdot \left(-\frac{5}{8}\right) = \frac{23}{8} \\ y_p = -2 - 2 \cdot \left(-\frac{5}{8}\right) = -\frac{3}{4} \\ z_p = -1 - 5 \cdot \left(-\frac{5}{8}\right) = \frac{17}{8} \end{cases}$$

Projections

$\Pi : Ax+By+Cz+D=0$

$\vec{u}(p,q,r)$

Assume that $\vec{u} \nparallel \Pi$, i.e. $Ap+Bq+Cr \neq 0$



We define the projection onto the plane Π with direction $\vec{u}$ as the function:

$$P_{\Pi,\vec{u}} : \mathbb{R}^3 \longrightarrow \Pi$$

$$(x,y,z) \longmapsto (x',y',z')$$

$$\begin{cases} x' = x - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot \vec{u} \\ y' = y - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot \vec{u} \\ z' = z - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot \vec{u} \end{cases}$$

We have that $\vec{r}_M (x, y, z)$

$$\vec{r}_{P_{\Pi,\vec{u}}(M)} = (x', y', z')$$

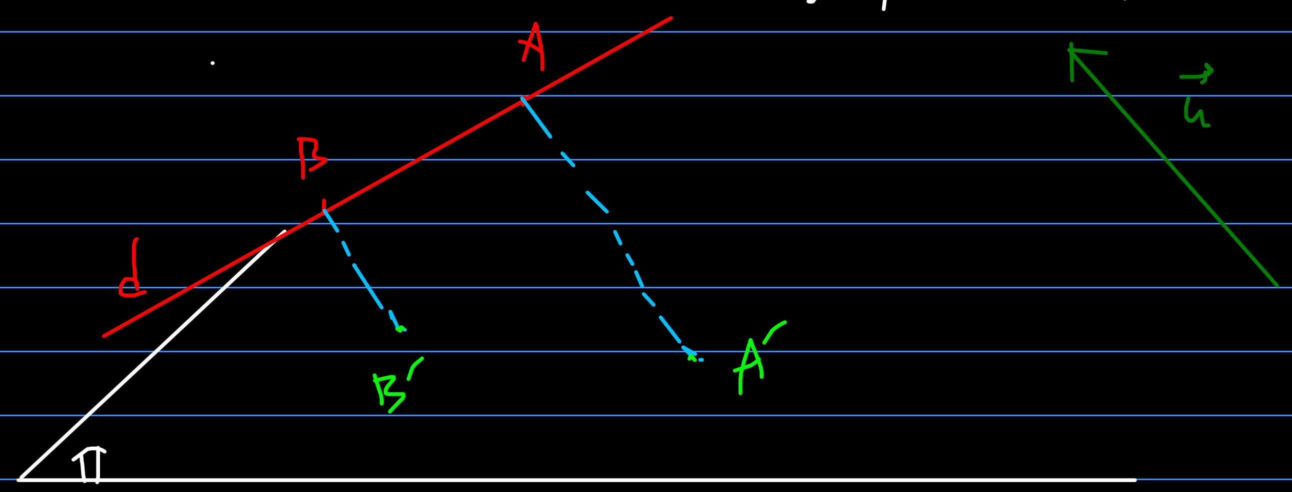
$$\vec{r}_{S_{\Pi,\vec{u}}(M)} = 2\,\vec{r}_{P_{\Pi,\vec{u}}(M)} - \vec{r}_M$$

4.3. Write the equations of the projection of the line

$$(d)\ \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

on the plane $\pi : x + 2y - z = 0$ parallel to the direction $\vec{u}(1, 1, -2)$

(Homework: do the same thing for the reflection)



$$\vec{r}_{P_{\pi,\vec{u}}(M)} = \vec{r}_M - \frac{F(x_M, y_M, z_M)}{\vec{n}_\pi \cdot \vec{u}} \cdot \vec{u}$$

$$F(x, y, z) = Ax + By + Cz + D$$

$$(d): \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -y + z - 1 \\ -2y + 2z - 2 - y + z - 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = -y + z - 1 \\ -3y + 3z - 3 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -y + z - 1 \\ y = z - 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 1 - z + z - 1 \\ y = z - 1 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = z - 1 \\ z = z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = t - 1 \\ z = t \end{cases}$$

$$\frac{F(x,y,z)}{\vec{n_\pi} \cdot \vec{u}} = \frac{x + 2y - z}{(1,2,-1)\cdot(1,1,-2)} = \frac{2(t-1) - t}{1 + 2 + 2} =$$

$$= \frac{t-2}{5}$$

$$\begin{cases} x' = x - \dfrac{F(x,y,z)}{\vec{n_\pi} \cdot \vec{u}} \cdot x_{\vec{u}} \\ y' = y - \dfrac{F(x,y,z)}{\vec{n_\pi} \cdot \vec{u}} \cdot y_{\vec{u}} \\ z' = z - \dfrac{F(x,y,z)}{\vec{n_\pi} \cdot \vec{u}} \cdot z_{\vec{u}} \end{cases}$$

$$\Rightarrow \begin{cases} x' = 0 - \dfrac{t-2}{5} \cdot 1 \\ y' = t-1 - \dfrac{t-2}{5} \cdot 1 \\ z' = t - \dfrac{t-2}{5} \cdot (-2) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x' = -\dfrac{t}{5} + \dfrac{2}{5} \\ y' = \dfrac{4t}{5} - \dfrac{3}{5} \\ z' = \dfrac{7t}{5} + \dfrac{4}{5} \end{cases}$$

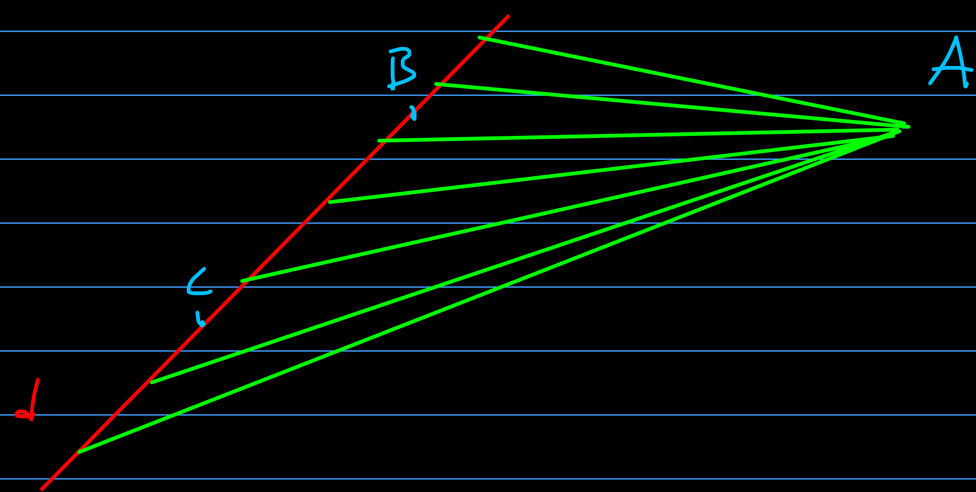
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = t \cdot \begin{pmatrix} -\frac{1}{5} \\ \frac{4}{5} \\ \frac{7}{5} \end{pmatrix} + \begin{pmatrix} \frac{2}{5} \\ -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

This is the equation of a line, hence the projection is a line.

4.1. Write the equation of the plane determined by the line

$$(d)\ \begin{cases} x-2y+3z=0 \\ 2x+z-3=0 \end{cases}$$

and the point $A(-1,2,6)$



$$(d):\ \begin{cases} x-2y+3z=0 \\ 2x+z-3=0 \end{cases} \Leftrightarrow \begin{cases} x=\frac{3-z}{2} \\ x-2y+3z=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=\frac{3-z}{2} \\ \frac{3-z}{2}-2y+3z=0 \end{cases} \Leftrightarrow \begin{cases} x=\frac{3-z}{2} \\ 3-z-4y+6z=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=\frac{3-z}{2} \\ 3+5z-4y=0 \end{cases} \Leftrightarrow \begin{cases} x=\frac{3-z}{2} \\ y=\frac{5z+3}{4} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = -\frac{z}{2} + \frac{3}{2} \\ y = \frac{5}{4} z + \frac{3}{4} \\ z = z \end{cases}$$

For $z = 1$ we get $B\left(1, 2, 1\right) \in d$

For $z = 0$ we get $C\left(\frac{3}{2}, \frac{3}{4}, 0\right) \in d$

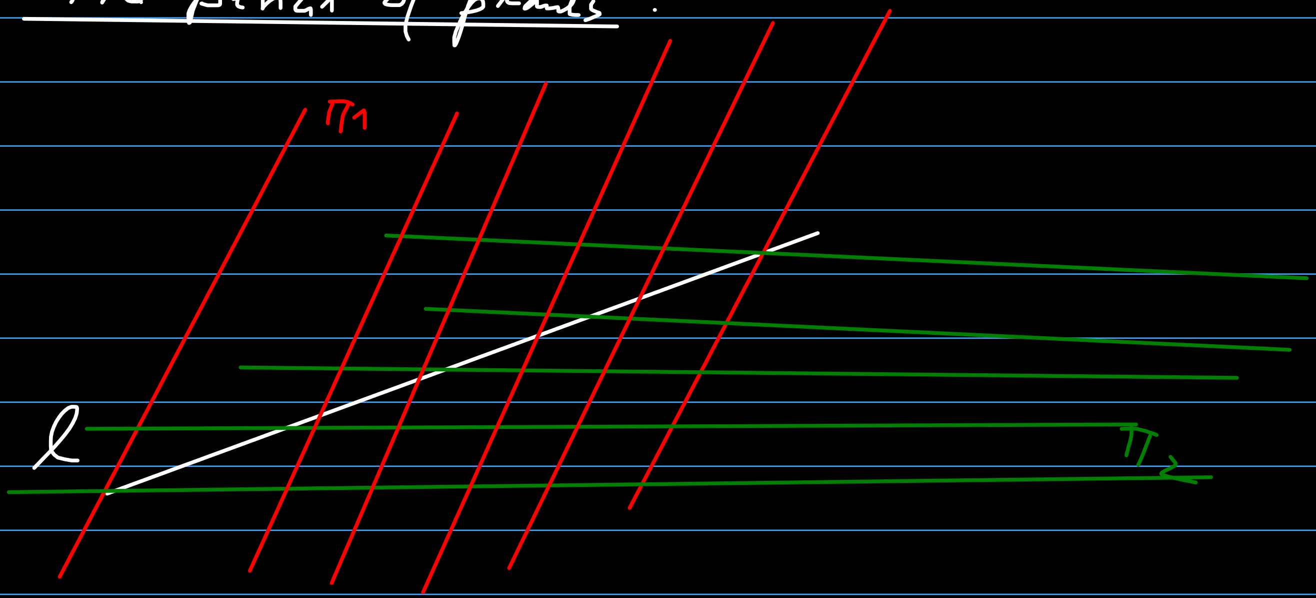
$$\pi : \begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0$$

$$\pi : \begin{vmatrix} x+1 & y-2 & z-6 \\ 2 & 0 & -5 \\ \frac{5}{2} & -\frac{5}{4} & -6 \end{vmatrix} = 0$$

$$\Rightarrow -\frac{5}{2}(z-6) - \frac{25}{2}(y-2) - \frac{25}{4}(x+1) +$$

$$+ 12(y-2) = 0 \Rightarrow -\frac{25}{4}x - \frac{y}{2} - \frac{5}{2}z + \frac{39}{4} = 0$$

The pencil of planes:



$$\ell : \begin{cases} \pi_1 : A_1 x + B_1 y + C_1 z + D_1 = 0 \\ \pi_2 : A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$

$$\pi_{\alpha,\beta} : \alpha \cdot (A_1 x + B_1 y + C_1 z + D_1) + \\ + \beta (A_2 x + B_2 y + C_2 z + D_2) = 0$$

$\hookrightarrow$ every plane that contains ℓ has this form.

Ex. II: We consider the line:

$$l: \begin{cases} x - 2y + z - 6 = 0 \\ 2x + y + z + 3 = 0 \end{cases}$$

Find the plane that contains the line l and is parallel to the vector $\vec{u}(1,1,1)$.

We start by writing the pencil of planes:

$$\Pi_{\alpha,\beta}: \alpha(x - 2y + z - 6) + \beta \cdot (2x + y + z + 3) = 0$$

$$\Rightarrow \Pi_{\alpha,\beta}: (\alpha + 2\beta)x + (-2\alpha + \beta) \cdot y + (\alpha + \beta)z + (-6\alpha + 3\beta) = 0$$

$$\Pi_{\alpha,\beta} \parallel \vec{u} \Leftrightarrow \vec{n}_{\Pi_{\alpha,\beta}} \cdot \vec{u} = 0 \Leftrightarrow$$

$$\Leftrightarrow (\alpha + 2\beta, -2\alpha + \beta, \alpha + \beta) \cdot (1,1,1) = 0 \Leftrightarrow$$

$$\Leftrightarrow \alpha + 2\beta - 2\alpha + \beta + \alpha + \beta = 0 \Leftrightarrow$$

$$\Leftrightarrow \beta = 0$$

$\Rightarrow$ the plane that we want is $\Pi_{\alpha,0}$

$\pi_{\alpha,0}$: $\alpha(x-2y+z-6)=0$

$\Rightarrow \pi : x-2y+z-6=0$

This is the plane that we want.