Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Mathematical Analysis - Lecture Notes
Computer Science, Academic Year: 2020/2021

Lecture 6

Local extrema and derivatives

Definition 1. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. We say that f

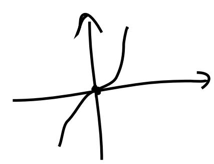
- *attains a local maximum (local minimum)* at $c \in A$: if there exists $V \in \mathcal{V}(c)$ such that c is a maximum point (minimum point) for $f|_{A\cap V}$. In this case c is called a *local maximum point (minimum point)* for f.
- *attains a local extremum* at $c \in A$: if it attains either a local maximum or a local minimum at c. In this case c is called a *local extremum point* for f.

Theorem 1 (Fermat)**.** *Let $a, b \in \overline{\mathbb{R}}$ with $a < b$, $f : (a, b) \to \mathbb{R}$, and $c \in (a, b)$. If f has a derivative at c and f attains a local extremum at c, then $f'(c) = 0$.*

Remark 1. Let $a, b \in \overline{\mathbb{R}}$ with $a < b$, $f : (a, b) \to \mathbb{R}$, $c \in (a, b)$, and suppose that f has a derivative at c.

$$f'(c) = 0 \quad \not\Rightarrow \quad f \text{ attains a local extremum at } c$$

$f:(-1,1)\to\mathbb{R}$, $f(x)=x^3$, $c=0$



Remark 2. The conclusion in Fermat's Theorem may not hold if

- f is not assumed to have a derivative at c: $f:(-1,1)\to\mathbb{R}$, $f(x)=|x|$, $c=0$
- the open interval is replaced by a closed one: $f:[0,1]\to\mathbb{R}$, $f(x)=x$, $c=0$

Theorem 2 (Darboux)**.** *Let $a, b \in \mathbb{R}$, $a < b$ and let $f : [a, b] \to \mathbb{R}$ be differentiable. If $\gamma \in \mathbb{R}$ satisfies $f'(a) < \gamma < f'(b)$ or $f'(b) < \gamma < f'(a)$, then there exists a point $c \in (a, b)$ such that $f'(c) = \gamma$.*

Remark 3. The derivative of a differentiable function is not always continuous. Take $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0,\\ 0, & \text{if } x = 0.\end{cases}$$

$\lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \dfrac{x^2 \sin\frac{1}{x}}{x} = \lim\limits_{x\to 0} x \sin\frac{1}{x} = 0 \Rightarrow f$ is diff. at 0 and $f'(0)=0$

f is diff on $\mathbb{R}$

$x \neq 0$, $f'(x) = 2x\sin\frac{1}{x} + x^2\cdot\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$

$$f'(x) = \begin{cases} 0, & x = 0\\ 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0\end{cases}$$

f' is not cont at 0:

$x_n = \frac{1}{2n\pi}$, $n\in\mathbb{N}$, $x_n \to 0$, $f'(x_n) = -1 \to -1$, but $f'(0)=0$

Definition 2. A function is called *continuously differentiable* if it is differentiable and its derivative is continuous.

Theorem 3 (Rolle)**.** *Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}$. If f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem 4 (Mean Value Theorem, Lagrange)**.** *Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}$. If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Theorem 5 (Generalized Mean Value Theorem, Cauchy)**.** *Let $a, b \in \mathbb{R}$, $a < b$ and $f, g : [a, b] \to \mathbb{R}$. If f, g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Higher order derivatives

Definition 3. Let $A \subseteq \mathbb{R}$, $c \in A \cap A'$ and $f : A \to \mathbb{R}$. We say that f is *twice differentiable at* c if $\exists V \in \mathcal{V}(c)$ such that f is differentiable on $A \cap V$ and f' is differentiable at c. If f is twice differentiable at c, then we write $f^{(2)}(c) = f''(c) = (f')'(c)$.
In general, for $n \geq 2$, we say that f is *n-times differentiable at c* if $\exists V \in \mathcal{V}(c)$ such that f is $(n-1)$-times differentiable on $A \cap V$ and $f^{(n-1)}$ is differentiable at c. If f is n-times differentiable at c, then we write $f^{(n)}(c) = \left(f^{(n-1)}\right)'(c)$.
If B is a subset of A, we say that f is *n-times differentiable on B* if it is n-times differentiable at every point of B. In this case, the function $f^{(n)} : B \to \mathbb{R}$, $x \in B \mapsto f^{(n)}(x)$ is called the *n^{th} derivative of f on B*.
We say that f is *infinitely differentiable at c* if for every $n \in \mathbb{N}$, f is n-times differentiable at c.
Notation: $f^{(0)} = f$, $f^{(1)} = f'$.

Local extrema and derivatives (revisited)

Theorem 6 (Second Derivative Test)**.** *Let $a, b \in \overline{\mathbb{R}}$ with $a < b$, $f : (a, b) \to \mathbb{R}$, and $c \in (a, b)$. If f is twice differentiable at c, $f'(c) = 0$, and $f''(c) \neq 0$, then*

(i) if $f''(c) > 0$, then f attains a local minimum at c.

(ii) if $f''(c) < 0$, then f attains a local maximum at c.

Justification:

$f''(c) = \lim\limits_{x \to c} \dfrac{f'(x) - f'(c)}{x - c} = \lim\limits_{x \to c} \dfrac{f'(x)}{x - c} > 0 \Rightarrow \dfrac{f'(x)}{x - c} > 0$ for x near c

$\Rightarrow$ the slope is negative to the left of c and positive to the right of c

$\Rightarrow$ c is a local minimum point of f.

Remark 4. If $f''(c) = 0$, the Second Derivative Test gives no information.

$f, g : (-1, 1) \to \mathbb{R}$, $f(x) = x^3$, $g(x) = x^4$

$f'(0) = f''(0) = 0$ 0 is not a local extremum point for f

$g'(0) = g''(0) = 0$ 0 is a global minimum point for g

Example 1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 9x^2 + 15x + 2$.

$f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 3(x-1)(x-5)$; $f''(x) = 6x - 18$

$f''(1) = 6 - 18 = -12 < 0 \Rightarrow 1$ is a local max. point for f (but not a global one since $\lim_{x \to \infty} f(x) = \infty$).

$f''(5) = 30 - 18 = 12 > 0 \Rightarrow 5$ is a local min. point for f (but not a global one since $\lim_{x \to -\infty} f(x) = -\infty$).

Taylor polynomials

Let $I \subseteq \mathbb{R}$ be a nonempty interval, $x_0 \in I$, $f : I \to \mathbb{R}$ and $n \in \mathbb{N}$. Suppose that f is n-times differentiable at x_0.
Goal: Approximate f by finding a polynomial function $T_n : \mathbb{R} \to \mathbb{R}$ of degree (at most) n such that

$$T_n(x_0) = f(x_0), \quad T_n'(x_0) = f'(x_0), \quad T_n''(x_0) = f''(x_0), \quad \ldots, \quad T_n^{(n)}(x_0) = f^{(n)}(x_0). \tag{1}$$

We are looking for T_n of the form

$$T_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n.$$

Clearly, from (1), we obtain that

$$a_0 = f(x_0), \quad a_1 = f'(x_0), \quad a_2 = \frac{f''(x_0)}{2!}, \quad \ldots, \quad a_n = \frac{f^{(n)}(x_0)}{n!}.$$

The polynomial function $T_n : \mathbb{R} \to \mathbb{R}$,

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{2}$$

is called the n^{th} *Taylor polynomial of f at the point x_0*.
Notation: The complete notation for the n^{th} Taylor polynomial of f at the point x_0 would be $T_n(f; x_0)(x)$. However, to simplify the writing we keep the notation $T_n(x)$.

Remark 5. There is a unique polynomial function of degree (at most) n that satisfies (1).

If P is another polynomial fct of degree (at most) n s.t.
$P(x_0) = f(x_0)$, $P'(x_0) = f'(x_0)$, $\ldots$, $P^{(n)}(x_0) = f^{(n)}(x_0)$,
then taking $Q : \mathbb{R} \to \mathbb{R}$, $Q(x) = P(x) - T_n(x)$ we get that Q is also a polynomial fct of degree (at most) n and $Q(x_0) = Q'(x_0) = \ldots = Q^{(n)}(x_0) = 0$
$\Rightarrow x_0$ is a zero of order $n+1$ for $Q \Rightarrow Q = 0 \Rightarrow P = T_n$.

We are interested to establish the quality of the approximation of f at points in I near x_0. To this end we consider the function $R_n : I \to \mathbb{R}$, $R_n(x) = f(x) - T_n(x)$ called the *remainder of the approximation of f by T_n around x_0* (in other words, R_n represents the error between f and T_n). If R_n is given explicitly, the formula $f(x) = T_n(x) + R_n(x), \forall x \in I$, is called *Taylor's formula.*

Theorem 7 (Taylor-Lagrange)**.** *Let $I \subseteq \mathbb{R}$ be an interval, $n \in \mathbb{N}_0$ and $f : I \to \mathbb{R}$ be $(n+1)$-times differentiable. Then $\forall x, x_0 \in I$ with $x \neq x_0$, there exists a point c strictly between x and x_0 such that*

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{3}$$

In other words, $f(x) = T_n(x) + R_n(x)$*, where*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{4}$$

Remark 6. (i) The above formula (4) for the remainder term R_n is known as the Lagrange form (there are also other expressions of the remainder).

(ii) If we can bound $|f^{(n+1)}(c)|$, then we can estimate the error of approximation of $f(x)$ by $T_n(x)$.

Local extrema and derivatives (revisited once again)

Corollary 1. *Let* $a, b \in \overline{\mathbb{R}}$ *with* $a < b$*,* $f : (a,b) \to \mathbb{R}$*, and* $c \in (a,b)$*. If* f *is* n*-times differentiable* ($n \in \mathbb{N}$, $n \geq 2$) *at* c*,* $f'(c) = f''(c) = \ldots = f^{(n-1)}(c) = 0$*, and* $f^{(n)}(c) \neq 0$*, then*

(i) if n *is even and* $f^{(n)}(c) > 0$*, then* f *attains a local minimum at* c*.*

(ii) if n *is even* $f^{(n)}(c) < 0$*, then* f *attains a local maximum at* c*.*

(iii) if n *is odd, then* f *does not attain a local extremum at* c*.*

Taylor series

Definition 4. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to R$ be infinitely differentiable. For $x_0 \in I$ and $x \in \mathbb{R}$, the series

$$\sum_{n \geq 0} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the *Taylor series of* f *around* x_0.

Problem: At which points x is the above series convergent? If so, is its sum $f(x)$ (when $x \in I$)?

Note that the partial sums of the above series are $T_n(x)$, so the series is convergent $\Leftrightarrow$ $(T_n(x))_n$ is convergent. In this case,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \lim_{n\to\infty} T_n(x) \in \mathbb{R}$$

For $x \in I$, $f(x) = T_n(x) + R_n(x)$, so $f(x) = \lim_{n\to\infty} T_n(x) \Leftrightarrow \lim_{n\to\infty} R_n(x) = 0$

Definition 5. If $J \subseteq I$ is a nonempty set such that for all $x \in J$, the Taylor series of f around x_0 converges and its sum is $f(x)$, i.e.,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \tag{5}$$

we say that f can be *expanded as a Taylor series around* x_0 *on* J. In this case, the formula (5) is called the *Taylor series expansion of* $f(x)$ *around* x_0.

Remark 7. f can be expanded as a Taylor series around x_0 on J if and only if

$$\lim_{n\to\infty} R_n(x) = 0, \quad \forall x \in J.$$

Example 2 (Taylor series expansion of the exponential function around 0).

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$; $f^{(k)}(x) = e^x$, $\forall x \in \mathbb{R}$, $\forall k \in \mathbb{N}_0$

$f^{(k)}(0) = 1$, $\forall k \in \mathbb{N}_0$

Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then $\exists c$ btw 0 and x s.t.

$$e^x = 1 + \frac{1}{1!} \cdot x + \dots + \frac{1}{n!} \cdot x^n + \underbrace{\frac{e^c}{(n+1)!} x^{n+1}}_{R_n(x)}$$

$$0 \le |c| \le |x| \Rightarrow 0 \le |R_n(x)| \le \frac{e^{|x|}}{(n+1)!} |x|^{n+1}$$

$\lim\limits_{n \to \infty} \frac{|x|^n}{n!} = 0$ $\left(\text{take } N \text{ s.t. } |x| < N : \text{ for } n \geq N, \frac{|x|^n}{n!} = \frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \ldots \cdot \frac{|x|}{N} \cdot \ldots \cdot \frac{|x|}{n} \le \frac{|x|^{N-1}}{(N-1)!} \cdot \underbrace{\left(\frac{|x|}{N}\right)^{n-N+1}}_{\xrightarrow{n \to \infty} 0} \right)$

By the Squeeze Thm, $\lim\limits_{n \to \infty} R_n(x) = 0$

$\Rightarrow f$ can be expanded as a Taylor series around 0 on $\mathbb{R}$

Remark 8. Taylor polynomials and Taylor series play an important role in computer science (e.g. they are used in computer graphics to approximate trigonometric functions used in rendering objects).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x \in \mathbb{R}$$