Seminar 1

We use the following notation: $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Principle of Mathematical Induction: Let $n_0 \in \mathbb{N}$ and let $P(n)$ be a property defined for any $n \in \mathbb{N}$, $n \geq n_0$. Suppose that:

i) $P(n_0)$ is true;

ii) if $P(k)$ is true for some $k \in \mathbb{N}$, $k \geq n_0$, then $P(k+1)$ is also true.

Then $P(n)$ is true, $\forall n \in \mathbb{N}$, $n \geq n_0$.

Exercise 1. Prove that for every $n \in \mathbb{N}$, $n \geq 4$, we have $n! \geq 2^n$.

Exercise 2. Prove that for every $n \in \mathbb{N}$ we have $4 \displaystyle\sum_{m=1}^{n} m^3 = n^2(n+1)^2$.

Exercise 3. Prove that for every $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that $m_n^2 \leq n < (m_n + 1)^2$.

Exercise 4. Prove that for every $n \in \mathbb{N}$ with $n \geq 2$ and for any real numbers $a_1, a_2, \ldots, a_n > 0$ satisfying $a_1 \cdot a_2 \cdot \ldots \cdot a_n = 1$, we have $a_1 + a_2 + \ldots + a_n \geq n$.

Exercise 5. Given $n \in \mathbb{N}$ with $n \geq 2$ and the real numbers $x_1, x_2, \ldots, x_n > 0$, denote

$$\begin{aligned} H(x_1, \ldots, x_n) &= \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} && \text{(the harmonic mean)}, \\ G(x_1, \ldots, x_n) &= \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} && \text{(the geometric mean)}, \\ A(x_1, \ldots, x_n) &= \frac{x_1 + x_2 + \ldots + x_n}{n} && \text{(the arithmetic mean)}. \end{aligned}$$

Prove that $H(x_1, \ldots, x_n) \leq G(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_n)$.

Exercise 6. Prove that:

a) for every $x \in [-1, \infty)$ and every $n \in \mathbb{N}$ we have

$$(1+x)^n \geq 1 + nx \quad \text{(Bernoulli's inequality)};$$

b) for every $y \in \mathbb{R}$ and every even $m \in \mathbb{N}$ we have $(1+y)^m \geq 1 + my$.

Recall that the *absolute value* of $x \in \mathbb{R}$, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Exercise 7. Let $x, y \in \mathbb{R}$. Prove that:

a) $|x + y| \leq |x| + |y|$ (the triangle inequality);

b) $||x| - |y|| \leq |x - y|$.

Seminar 2

Exercise 1. Suppose that B is a nonempty, bounded subset of $\mathbb{R}$ and let A be a nonempty subset of B. Prove that

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

If, in addition, $\inf A = \inf B$ and $\sup A = \sup B$, does it follow that $A = B$?

Exercise 2. For each set A_i from below find $\operatorname{lb}(A_i)$ and $\operatorname{ub}(A_i)$ (as subsets of $\mathbb{R}$), $\min A_i$ and $\max A_i$ (if they exist), and $\inf A_i$ and $\sup A_i$ (in $\overline{\mathbb{R}}$).

$$\begin{aligned} A_1 &= (-2, 1) \cup \{7\}, & A_4 &= \left\{ \frac{n}{n+1} \mid n \in \mathbb{N} \right\}, \\ A_2 &= \left\{ \frac{1}{n} \mid n \in \mathbb{Z} \setminus \{0\} \right\}, & A_5 &= \left\{ \frac{n}{n+m} \mid m, n \in \mathbb{N} \right\}, \\ A_3 &= \left\{ x^2 \mid x \in \mathbb{Z} \right\}, & A_6 &= \left\{ x \in \mathbb{Q} \mid x^2 \leq 2 \right\}. \end{aligned}$$

Exercise 3. Is $\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset$? What can be said about the Nested Interval Property when dropping the assumption that the intervals are closed?

Exercise 4. Decide which of the following sets are neighborhoods of 0. Justify.

$$\begin{aligned} A_1 &= [-1, 1] \cup \{2\}, & A_4 &= \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z} \setminus \{0\} \right\}, \\ A_2 &= (-1, 1) \cap \mathbb{Q}, & & \\ A_3 &= (-1, 0) \cup (0, 1), & A_5 &= \bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n} \right]. \end{aligned}$$

Exercise 5. Prove that if $x, y \in \mathbb{R}$, $x \neq y$, there exist $U \in \mathcal{V}(x)$ and $V \in \mathcal{V}(y)$ such that $U \cap V = \emptyset$.

Exercise 6. Let $A \subseteq \mathbb{R}$ be a nonempty set which is bounded below (resp. above) by $\alpha \in \mathbb{R}$. Prove that $\inf A = \alpha$ (resp. $\sup A = \alpha$) if and only if $V \cap A \neq \emptyset$ for all $V \in \mathcal{V}(\alpha)$.

Seminar 3

Exercise 1. Let $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ nonempty. Prove that:

a) if A is bounded above, then $x = \sup A \iff \begin{cases} x \in \operatorname{ub}(A); \\ \exists (a_n) \subseteq A \text{ such that } \lim\limits_{n\to\infty} a_n = x. \end{cases}$

b) if A is bounded below, then $x = \inf A \iff \begin{cases} x \in \operatorname{lb}(A); \\ \exists (a_n) \subseteq A \text{ such that } \lim\limits_{n\to\infty} a_n = x. \end{cases}$

Exercise 2. Find the limit (as $n \to \infty$) of the sequence whose general term x_n, $n \in \mathbb{N}$, is given below:

a) $\left(\sin \frac{\pi}{7}\right)^n$, b) $\frac{\alpha n^3 + \beta n^2 + \gamma n + 1}{n^2 - n + 1}$, where $\alpha, \beta, \gamma \in \mathbb{R}$, c) $\frac{e^n - 2^n}{\pi^n - 3^n}$, d) $\left(1 + \frac{1}{n}\right)^{\frac{3n}{n+1}}$,

e) $\left(\frac{n^2 + n + 1}{n^2 + 1}\right)^{\frac{2n^2+n+1}{n+1}}$, f) $\sqrt{n}\left(\sqrt{n} - \sqrt{n+3}\right)$, g) $\frac{2^n}{n!}$, h) $\frac{n^\alpha}{(1+\beta)^n}$, where $\alpha \in \mathbb{N}$, $\beta > 0$,

i) $\frac{1^p + 2^p + \ldots + n^p}{n^{p+1}}$, where $p \in \mathbb{N}$, j) $\sqrt[n]{n}$, k) $\sqrt[n]{n!}$, l) $\frac{\sqrt[n]{n!}}{n}$, m) $\sqrt[n]{\sin^2(n) + 2\cos^2(n)}$,

n) $\sin(\pi\sqrt{n^2 + 1})$, o) $n\left(\sqrt[n]{e} - 1\right)$.

Exercise 3. Decide whether for an arbitrary sequence (x_n) in $\mathbb{R}$ the next statements hold true:

a) if (x_n) converges, then $(|x_n|)$ converges.

b) if $(|x_n|)$ converges, then (x_n) converges.

Seminar 4

Exercise 1. Prove that the sequence $(\sin n)$ has no limit.

Exercise 2. In each of the following cases, study if the sequence (x_n) is bounded, monotone and convergent (if possible, find also its limit).

a) $x_n = \dfrac{(-1)^n}{n}$, $n \in \mathbb{N}$. Does the sequence $(1/x_n)$ have a limit?

b) $x_n = (-1)^n + \dfrac{n+1}{n}$, $n \in \mathbb{N}$, c) $x_n = \dfrac{n!}{n^n}$, $n \in \mathbb{N}$,

d) $x_1 \in (0,1)$, $x_{n+1} = \dfrac{2x_n + 1}{3}$, $n \in \mathbb{N}$, e) $a > 0$, $x_1 > 0$, $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$, $n \in \mathbb{N}$.

Exercise 3. Find the sum of the following series:

a) $\displaystyle\sum_{n\geq 2}\left(-\frac{5}{9}\right)^n$, b) $\displaystyle\sum_{n\geq 1} 2^{1-2n}$, c) $\displaystyle\sum_{n\geq 2}\ln\left(1-\frac{1}{n^2}\right)$, d) $\displaystyle\sum_{n\geq 1}\frac{1}{n(n+1)(n+2)}$,

e) $\displaystyle\sum_{n\geq 1}\frac{1}{(3n-2)(3n+1)}$, f) $\displaystyle\sum_{n\geq 1}\left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right)$, g) $\displaystyle\sum_{n\geq 1}\frac{n+1}{2^n}$.

Seminar 5

Exercise 1. Study if the following series are convergent or divergent:

a) $\displaystyle\sum_{n\geq 1}\sin n$, b) $\displaystyle\sum_{n\geq 1}\arctan n$, c) $\displaystyle\sum_{n\geq 1}\frac{5^{n/2}}{n2^n}$, d) $\displaystyle\sum_{n\geq 1}\frac{e^n}{n+3^n}$, e) $\displaystyle\sum_{n\geq 1}\frac{\sqrt{n+1}}{1+2+\ldots+n}$,

f) $\displaystyle\sum_{n\geq 1}\frac{(n+1)^n}{n^{n+2}}$, g) $\displaystyle\sum_{n\geq 1}\left(\frac{1}{2}\right)^{\ln n}$, h) $\displaystyle\sum_{n\geq 1}\frac{2^n n!}{n^n}$, i) $\displaystyle\sum_{n\geq 1}\frac{n^2}{2^{n^2}}$, j) $\displaystyle\sum_{n\geq 1}\left(1+\frac{1}{n}\right)^{-n^2}$,

k) $\displaystyle\sum_{n\geq 2}(2-\sqrt{e})\cdot(2-\sqrt[3]{e})\cdot\ldots\cdot(2-\sqrt[n]{e})$.

Exercise 2. Define the sequence (x_n) by

$$x_n = \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}, \quad n \in \mathbb{N}.$$

Study if the following series are convergent or divergent:

a) $\displaystyle\sum_{n\geq 1} x_n$, b) $\displaystyle\sum_{n\geq 1}\frac{x_n}{n}$.

Exercise 3. Let $\sum_{n\geq 1} x_n$ be a convergent series with nonnegative terms. Study which of the following series are convergent:

a) $\displaystyle\sum_{n\geq 1}\frac{x_n}{1+x_n}$, b) $\displaystyle\sum_{n\geq 1} x_n^2$, c) $\displaystyle\sum_{n\geq 1}\sqrt{x_n}$, d) $\displaystyle\sum_{n\geq 1}\frac{\sqrt{x_n}}{n}$.

Seminar 6

Exercise 1. Study if the following series are absolutely convergent, semi-convergent, or divergent:

a) $\displaystyle\sum_{n\geq 1}\frac{\sin n}{n^2}$, b) $\displaystyle\sum_{n\geq 1}\frac{\sqrt[3]{n}}{n+1}\cos(n\pi)$, c) $\displaystyle\sum_{n\geq 1}\frac{a^n}{1+a^{2n}}$, where $a \in \mathbb{R}$,

d) $\displaystyle\sum_{n\geq 1}(-1)^{n+1}\frac{x_1+x_2+\ldots+x_n}{n}$, where (x_n) is a decreasing sequence in $\mathbb{R}$ such that $x_1 > 0$, $x_n \geq 0$, $\forall n \in \mathbb{N}$ with $n \geq 2$, and $\displaystyle\lim_{n\to\infty} x_n = 0$.

Exercise 2. For each of the sets $A \subseteq \mathbb{R}$ from below, find A':
a) $A = [1,2) \cup \{3\}$, b) $A = \mathbb{Q}$, c) $A = (-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$.

Exercise 3. Study the existence of the limit of Dirichlet's function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

at every accumulation point of its domain. Then study its continuity and determine the type of its discontinuities.

Exercise 4. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is nowhere continuous, but $|f|$ is continuous on $\mathbb{R}$.

Exercise 5. Study the continuity of the following functions and determine the type of their discontinuities:
a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \lim\limits_{n\to\infty} \dfrac{e^{nx}}{1+e^{nx}}$, b) $g \colon \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$

Exercise 6. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to [a,b]$ be a continuous function. Prove that f has at least one fixed point (that is, there exists $x_0 \in [a,b]$ such that $f(x_0) = x_0$).

Exercise 7. Compute the following limits:
a) $\lim\limits_{x\to-\infty} (-x^3 + 2x)$, b) $\lim\limits_{x\to1} \dfrac{x^2-1}{3x-3}$, c) $\lim\limits_{x\to0} \dfrac{\sqrt{1+2x} - \sqrt{1+x}}{x^2+2x}$, d) $\lim\limits_{x\to0} \dfrac{\sqrt[3]{1+x}-1}{x}$,
e) $\lim\limits_{\substack{x\to1 \\ x>1}} \dfrac{x}{\sqrt[3]{x^2-4x+3}}$, f) $\lim\limits_{x\to-\infty} \left(\dfrac{x^2+x+1}{x^2-x+1}\right)^{\sqrt{-x}}$, g) $\lim\limits_{\substack{x\to1 \\ x<1}} e^{\frac{x^2-2}{x^2-1}}$, h) $\lim\limits_{\substack{x\to1 \\ x>1}} e^{\frac{x^2-2}{x^2-1}}$.

Seminar 7

Exercise 1. Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt[3]{x}$ is not differentiable at 0 although its derivative at 0 exists.

Exercise 2. How many times is the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$ differentiable?

Exercise 3. Find the n^{th} derivative ($n \in \mathbb{N}$) of the following functions:
a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = (\sin x - \cos x)^2 + \sin 2x$, b) $f : (-1, +\infty) \to \mathbb{R}$, $f(x) = \ln(x+1)$,
c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$, d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$, e) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{2x} x^3$.

Exercise 4. Compute $\lim\limits_{x\to0} \dfrac{e - (1+x)^{\frac{1}{x}}}{x}$. Then determine $\lim\limits_{n\to\infty} n \left(e - \left(1 + \dfrac{1}{n}\right)^n\right)$.

Exercise 5. Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a,b] \to \mathbb{R}$. Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Prove that there exists $c \in (a,b)$ such that $(c-a)(c-b)f'(c) = a + b - 2c$.
Hint: Consider the function $g : [a,b] \to \mathbb{R}$, $g(x) = e^{f(x)}(x-a)(x-b)$.

Exercise 6. Let $f : (0,\infty) \to \mathbb{R}$, $f(x) = \sqrt[3]{x}$. Find the second Taylor polynomial $T_2(x)$ of f at 1 and the remainder term $R_2(x)$ of the corresponding Taylor formula in the Lagrange form. If $x \in [0.9, 1.1]$, find an upper bound for $|R_2(x)|$.

Exercise 7. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$. Find the second Taylor polynomial $T_2(x)$ of f at 0 and the remainder term $R_2(x)$ of the corresponding Taylor formula in the Lagrange form. Then show that $\forall x \in \mathbb{R}$, $1 - \dfrac{x^2}{2} \leq \cos x$.

Seminar 8

Exercise 1. Prove that the following functions can be expanded as a Taylor series around 0 on J and find the corresponding Taylor series expansion:

a) $f : (-1, \infty) \to \mathbb{R},\ f(x) = \ln(x+1),\ J = [0,1]$.

b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \cos x,\ J = \mathbb{R}$.

c) $f : \mathbb{R} \to \mathbb{R} = \mathbb{R},\ f(x) = \sin x,\ J = \mathbb{R}$.

Exercise 2. Let $x, y \in \mathbb{R}^n$. Denote by $\alpha = \langle x, y\rangle$, $\beta = ||x||$, and $\gamma = ||y||$.

a) Determine the numbers $\langle x+y, y\rangle$, $\langle x, 2x-3y\rangle$, and $||x-y||$ in terms of α, β and γ.

b) Let $n = 3$, $x = (-1, 2, 3)$ and $y = (-2, 1, -3)$.

  i) Compute α, β and γ.

  ii) Find all reals $r > 0$ such that the open ball $B(x, r)$ does not contain the vector y.

  iii) Find all reals t such that the closed ball $\overline{B}(x, 5)$ contains the vector $(1, -1, t)$.

Exercise 3. Prove that if $x, y \in \mathbb{R}^n$, $x \neq y$, there exist $U \in \mathcal{V}(x)$ and $V \in \mathcal{V}(y)$ such that $U \cap V = \emptyset$.

Exercise 4. Let $x, y \in \mathbb{R}^n$. Prove that:

a) $\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$.

b) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (the parallelogram identity).

Exercise 5. Two vectors $x, y \in \mathbb{R}^n$ are said to be *orthogonal* if $\langle x, y\rangle = 0$. Prove that given $x, y \in \mathbb{R}^n$, the following statements are equivalent:

a) x and y are orthogonal.

b) $\|x+y\| = \|x-y\|$.

c) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Exercise 6. A set $A \subseteq \mathbb{R}^n$ is said to be *convex* if $\forall x, y \in A$, $\forall t \in [0,1]$, $(1-t)x + ty \in A$. Prove that $\forall z \in \mathbb{R}^n$, $\forall r > 0$, the open ball $B(z, r)$ and the closed ball $\overline{B}(z, r)$ are convex.

Seminar 9

Exercise 1. In each of the following cases, determine if the sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ is convergent or not. If the sequence is convergent, find also its limit:

a) $n = 2,\ x^k = \left(\frac{1}{k}, \frac{k^2+4k}{2k^2+1}\right)$, b) $n = 2,\ x^k = \left((-1/2)^k, (-1)^k\right)$,

c) $n = 2,\ x^k = \left(\sin k, \frac{1}{k^2}\right)$, d) $n = 2,\ x^k = \left(\left(\frac{\sqrt{k}}{1+\sqrt{k}}\right)^k, \frac{1^1 + 2^2 + \ldots + k^k}{k^k}\right)$,

e) $n = 3,\ x^k = \left(\frac{2^k}{k!}, \frac{1-4k^7}{k^7+12k}, \frac{\sqrt{k}}{e^{3k}}\right)$, f) $n = 3,\ x^k = \left(e^{-k}\cos k, e^{-k}\sin k, k\right)$,

g) $n = 4,\ x^k = \left(\frac{2^{2k}}{\left(2+\frac{1}{k}\right)^{2k}}, \frac{1}{\sqrt[k]{k!}}, (e^k + k)^{\frac{1}{k}}, \frac{\alpha^k}{k}\right)$, where $\alpha \geq 0$ is fixed.

Exercise 2. In each the following cases, study if the function $f : \mathbb{R}^n \setminus \{0_n\} \to \mathbb{R}$ has a limit at 0_n:

a) $n = 2,\ f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$, b) $n = 2,\ f(x,y) = \dfrac{\sin(x^2)}{x^2+y^2}$, c) $n = 2,\ f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$

d) $f(x_1, x_2, \ldots, x_n) = \dfrac{x_1 \cdot x_2 \cdot \ldots \cdot x_n}{(x_1)^2 + (x_2)^2 + \ldots + (x_n)^2}$.

Exercise 3. In each the following cases, study if the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at 0_2:

a) $f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2, \end{cases}$

b) $p, q \in \mathbb{N},\ f(x,y) = \begin{cases} \dfrac{x^p y^q}{x^2 - xy + y^2} & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2. \end{cases}$

Exercise 4. Find the second order partial derivatives of the following functions:

a) $f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = \sin(xy)$, b) $f : (0,\infty) \times (0,\infty) \to \mathbb{R},\ f(x,y) = x^y$,
c) $f : (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^2 \to \mathbb{R},\ f(x,y,z) = z^2 e^y / x$.