Seminar W8 — 916

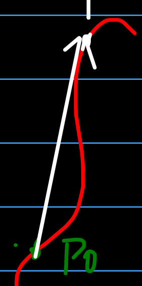
Curves: (2D/3D)

→ given parametrically: $\mathcal{C}: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$

→ given implicitly

- (planar) $f(x,y) = 0$
- (spatial) $\begin{cases} f_1(x,y,z) = 0 \\ f_2(x,y,z) = 0 \end{cases}$

The tangent to the curve $\mathcal{C}$ at the point P_0 is a line that contains P_0 and has a direction specified by the vector:



$$\vec{v} = \lim_{\substack{P \to P_0 \\ P \in \mathcal{C}}} \frac{\overrightarrow{P_0P}}{\|\overrightarrow{P_0P}\|}$$

. If $\mathcal{C}$ is given parametrically by

$$\mathcal{C}: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

then the tangent line to $\mathcal{C}$ in the point P_0 $(t = t_0)$ is:

$$T_{\mathcal{C}}(t=t_0): \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}$$

→ if $\mathcal{C}$ is in 2D

normal line = line ⊥ the tangent and contains the point P_0

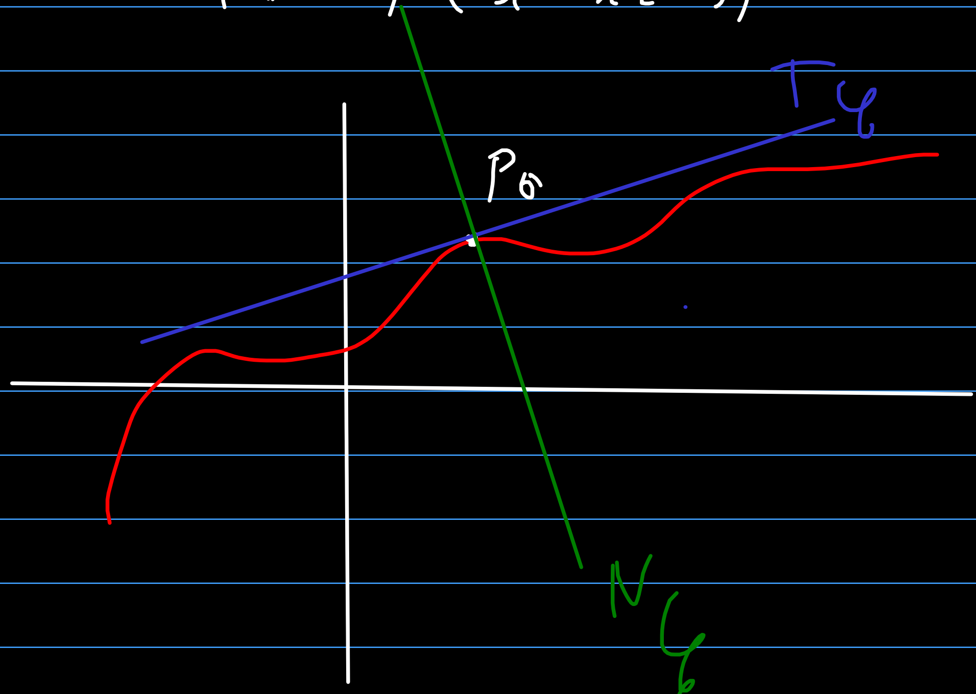
$$m_{T_{\mathcal{C}}} = \frac{y'(t_0)}{x'(t_0)} \Rightarrow m_{N_{\mathcal{C}}} = \frac{-x'(t_0)}{y'(t_0)}$$

$$N_{\mathcal{C}}(t = t_0): \frac{-x'(t_0)}{y'(t_0)} \cdot (x - x(t_0)) =$$

$$= y - y(t_0)$$

$N_{\mathcal{C}}(t=t_0): y'(t_0)\cdot(y-y(t_0)) +$

$+ x'(t_0)\cdot(x-x(t_0)) = 0$



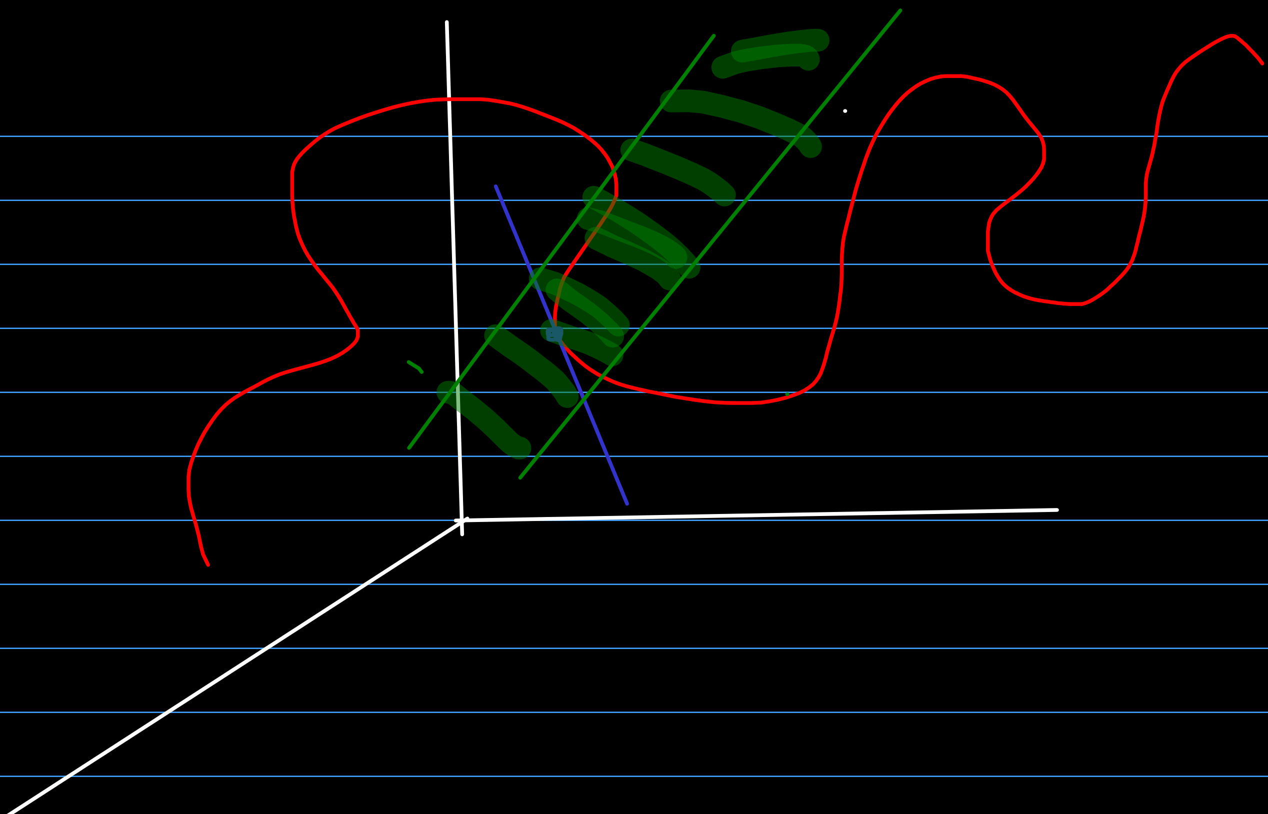
- if $\mathcal{C}$ is in 3D

normal plane = plane that is perp to the tangent and contains P_0

$N_{\mathcal{C}}(t=t_0)$:

$x'(t_0)\cdot(x-x(t_0)) + y'(t_0)\cdot(y-y(t_0)) +$

$+ z'(t_0)\cdot(z-z(t_0)) = 0$



· If $\mathcal{C}$ is given implicitly (and is planar) then
i.e. it is in 2D

$$\mathcal{C}: \quad f(x,y) = 0$$

$$T_{\mathcal{C}}(x_0,y_0): \quad f'_x(x_0,y_0)\cdot(x-x_0) + f'_y(x_0,y_0)\cdot(y-y_0) = 0$$

$$N_{\mathcal{C}}(x_0,y_0): \quad \frac{x-x_0}{f'_x(x_0,y_0)} = \frac{y-y_0}{f'_y(x_0,y_0)}$$

8.1. Show that the angle between the tangent of the circular helix

$$\mathcal{C}: \begin{cases} x = a\cos t \\ y = a\sin t \\ z = b \cdot t \end{cases}$$

and the z-axis is constant.

$$T_{\mathcal{C}}(t=t_0): \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}$$

$$\Leftrightarrow \frac{x - a\cos(t_0)}{-a\sin t_0} = \frac{y - a\sin(t_0)}{a\cos t_0} = \frac{z - bt_0}{b}$$

$$\Rightarrow \vec{u}_{T_{\mathcal{C}}}(t=t_0)\ (-a\sin t_0, a\cos t_0, b)$$

$$\vec{u}_{Oz}(0,0,1)$$

$$m(\widehat{T_{\mathcal{C}}, Oz}) = \arccos \frac{\vec{u}_{T_{\mathcal{C}}} \cdot \vec{u}_{Oz}}{\|\vec{u}_{T_{\mathcal{C}}}\| \cdot \|\vec{u}_{Oz}\|} =$$

$$= \arccos \frac{b}{\sqrt{a^2 \sin^2 t_0 + a^2 \cos^2 t_0 + b^2}} =$$

$$= \arccos \frac{b}{\sqrt{a^2+b^2}}$$ does not depend on t_0 $\Rightarrow$ q. e. d.

8.8. Write the equation of the tangent line and the normal plane for the curve:

$$\mathcal{C}: \begin{cases} x = e^t \cos 3t \\ y = e^t \sin 3t \\ z = e^{-2t} \end{cases}$$

at the point $t = 0$.

$$x'(t) = e^t \cos 3t - 3 e^t \sin 3t$$

$$y'(t) = e^t \cdot \sin 3t + 3 e^t \cos 3t$$

$$z'(t) = -2 e^{-2t}$$

$\Rightarrow x'(0) = 1 \quad , y'(0) = 3 \quad , z'(0) = -2$

$x(0) = 1 \quad y(0) = 0 \quad , z(0) = 1$

$T_{\mathcal{C}}(t = t_0) : \dfrac{x - x(t_0)}{x'(t_0)} = \dfrac{y - y(t_0)}{y'(t_0)} = \dfrac{z - z(t_0)}{z'(t_0)}$

$T_{\mathcal{C}}(t = 0) : \dfrac{x - 1}{1} = \dfrac{y}{3} = \dfrac{z - 1}{-2}$

$N_{\mathcal{C}}(t = 0) : 1 \cdot (x - 1) + 3 \cdot y + (-2) \cdot (z - 1) = 0$

8.? Find the equation of the tangent line and normal line to the curve $\mathcal{C}$ at the point $P_0(1, 0)$

$\mathcal{C} : x^3 - x^2 y + y^4 - 1 = 0$

$f(x, y) = x^3 - x^2 y + y^4 - 1$

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 2xy$$

$$\frac{\partial f}{\partial y}(x,y) = -x^2 + 4y^3$$

$$T_{\mathcal{C}}(x_0, y_0): \frac{\partial f}{\partial x}(x_0,y_0) \cdot (x - x_0) +$$

$$+ \frac{\partial f}{\partial y}(x_0,y_0) \cdot (y - y_0) = 0$$

$$\Rightarrow T_{\mathcal{C}}(1,0): 3 \cdot (x-1) - y = 0$$

$$N_{\mathcal{C}}(1,0): \frac{x-1}{3} = \frac{y}{-1}$$

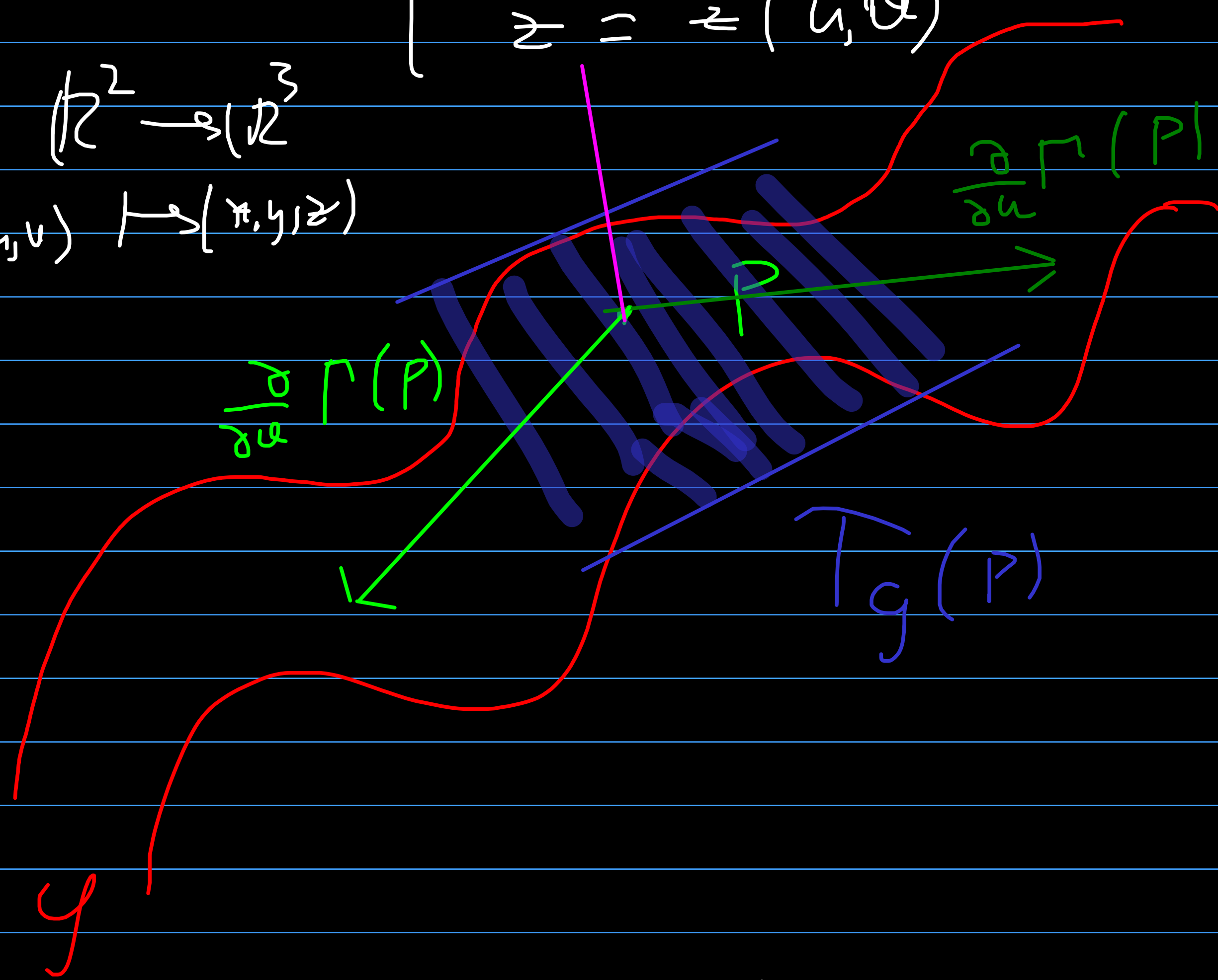
Surfaces

→ given parametrically:

$$\mathcal{S}: \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

$\Gamma : \mathbb{R}^2 \to \mathbb{R}^3$

$(u,v) \mapsto (x,y,z)$



The tangent plane to $\mathcal{S}$ in $P(u_0,v_0)$

$$T_{\mathcal{Y}}(u=u_0,\ v=v_0): \begin{vmatrix} x-x(u_0,v_0) & y-y(u_0,v_0) & z-z(u_0,v_0) \\ \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \\ \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0) \end{vmatrix}$$

$$= 0$$

$$N_{\mathcal{Y}}(u=u_0,\ v=v_0):$$

$$\frac{x-x(u_0,v_0)}{\boxed{\frac{\partial(y,z)}{\partial(u,v)}}} = \frac{y-y(u_0,v_0)}{\frac{\partial(z,x)}{\partial(u,v)}} = \frac{z-z(u_0,v_0)}{\frac{\partial(x,y)}{\partial(u,v)}}$$

→ if the surface is given implicitly

$$\mathcal{Y}: f(x,y,z)=0$$

$T_y(x_0, y_0, z_0): f'_x(x_0, y_0, z_0)\cdot(x - x_0) + f'_y(x_0, y_0, z_0)\cdot$
$\cdot(y - y_0) + f'_z(x_0, y_0, z_0)\cdot(z - z_0) = 0$

$N_y(x_0, y_0, z_0): \dfrac{x - x_0}{f'_x(x_0, y_0, z_0)} = \dfrac{y - y_0}{f'_y(x_0, y_0, z_0)} = \dfrac{z - z_0}{f'_z(x_0, y_0, z_0)}$

8.9. Write the equations of the tangent planes of the hyperboloid of one sheet

$$x^2 + y^2 - z^2 = 1$$

at the points of the form $(x_0, y_0, 0)$ and show that they are parallel to the z-axis.

$f(x, y, z) = x^2 + y^2 - z^2 - 1$

$\dfrac{\partial f}{\partial x}(x, y, z) = 2x \quad \dfrac{\partial f}{\partial y}(x, y, z) = 2y$

$$\frac{\partial f}{\partial z}(x,y,z) = -2z$$

$$T_{\mathcal{Y}(x_0,y_0,0)} : 2x_0 \cdot (x - x_0) + 2y_0 \cdot (y - y_0) =$$
$$= 0$$

$$\vec{n}_{T_{\mathcal{Y}(x_0,y_0,0)}} (2x_0, 2y_0, 0)$$

$$\vec{u}_{Oz} (0,0,1)$$

$$\vec{n}_{T_{\mathcal{Y}(x_0,y_0,0)}} \cdot \vec{n}_{Oz} = 0 \Rightarrow$$

$$\Rightarrow Oz \perp \vec{n}_{T_{\mathcal{Y}(x_0,y_0,0)}} \Rightarrow T_{\mathcal{Y}(x_0,y_0,0)} \parallel Oz$$

8.?? $\mathcal{Y}$ sphere

$$\mathcal{Y}: \begin{cases} x = \cos u \cdot \cos v \\ y = \cos u \cdot \sin v \\ z = \sin u \end{cases} \Leftrightarrow r = r(t)$$

Write the equation of the tangent plane to: $\mathcal{Y}$ at the point $(0,0,1)$

(i.e. the point corresponding to $u = \frac{\pi}{2}, v = 0$)

$$\frac{\partial}{\partial u} r = \left(-\sin u \cos v, -\sin u \sin v, \cos u\right)$$

$$\frac{\partial}{\partial v} r = \left(-\cos u \sin v, \cos u \cos v, 0\right)$$

$$T_{\mathcal{Y}}: \begin{vmatrix} x & y & z-1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\Rightarrow$ the parametrization is wrong!