Seminar W6 - 913

Cross product (vector product)

$\vec{u}, \vec{w}$ vectors

if $\vec{u}$ and $\vec{w}$ are linearly dependent, then

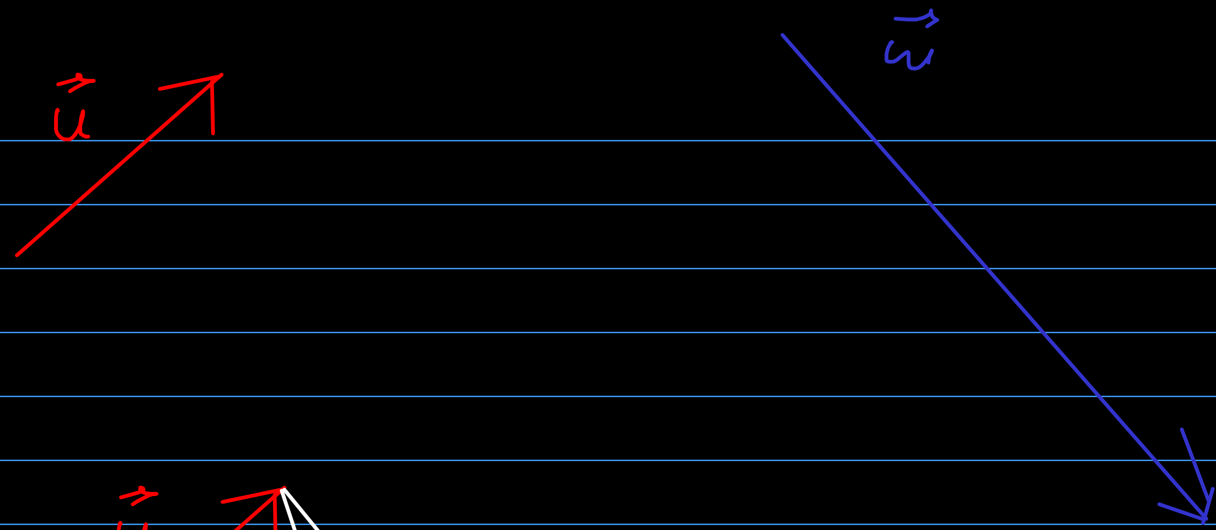
$$\vec{u} \times \vec{w} = \vec{0}$$

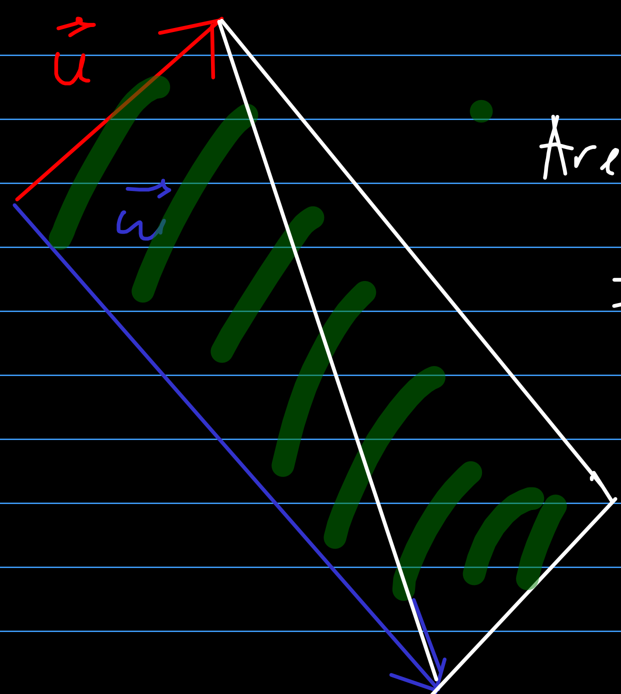
if $\vec{u}$ and $\vec{w}$ are linearly independent

$\longrightarrow$ <u>direction</u>: perpendicular, to both $\vec{u}$ and $\vec{w}$ (therefore perpendicular to $\langle \vec{u}, \vec{w} \rangle$)

$\longrightarrow$ <u>norm</u>: $\|\vec{u} \times \vec{w}\| = \|\vec{u}\| \cdot \|\vec{w}\| \cdot \sin(\widehat{\vec{u}, \vec{w}})$

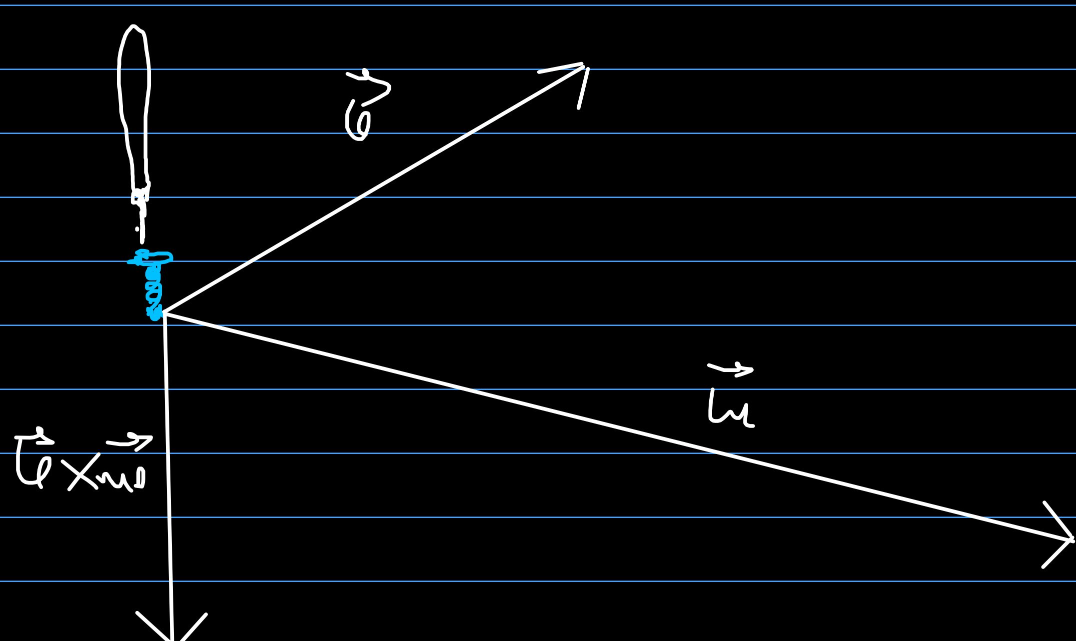
= area of the parallelogram constructed on the two vectors



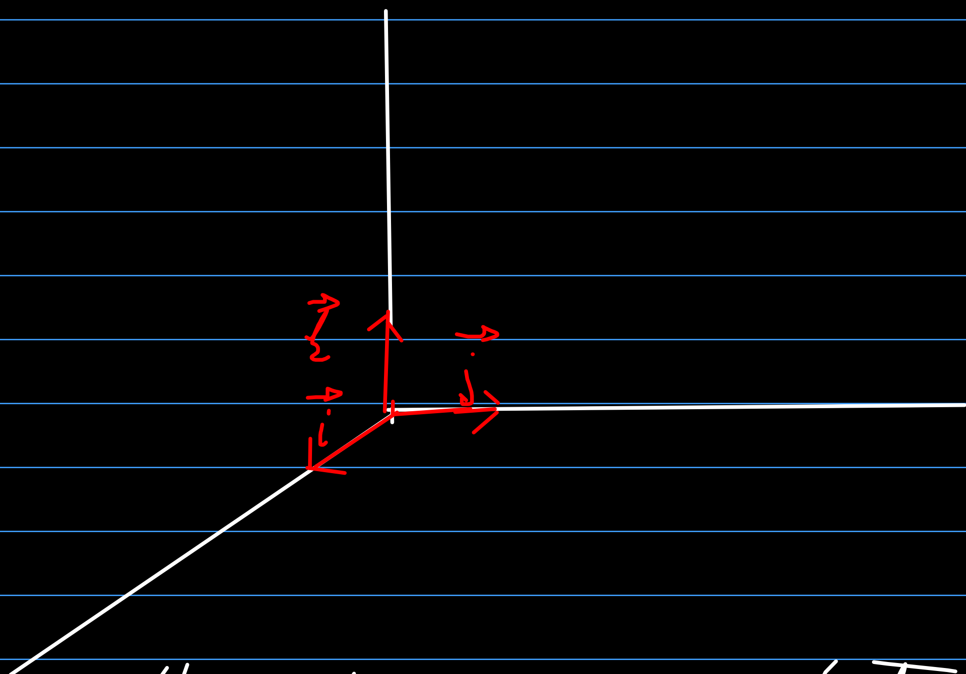


$\text{Area}(\text{parallelogram}(\vec{u}, \vec{w})) =$
$= \|\vec{u} \times \vec{w}\|$

- orientation



If the reference system $(O, [\vec{i}, \vec{j}, \vec{k}])$ is orthonormal and direct $\to \vec{i} \times \vec{j} = \vec{k}$



then the cross product is computed as follows: $\vec{u}(a_1, b_1, c_1)$, $\vec{w}(a_2, b_2, c_2)$

$$\vec{u} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} =$$

$$= (b_1 c_2 - b_2 c_1,\ a_2 c_1 - a_1 c_2,\ a_1 b_2 - a_2 b_1)$$

The cross product is anti-commutative:

$$\vec{v} \times \vec{w} = - \vec{w} \times \vec{v}$$

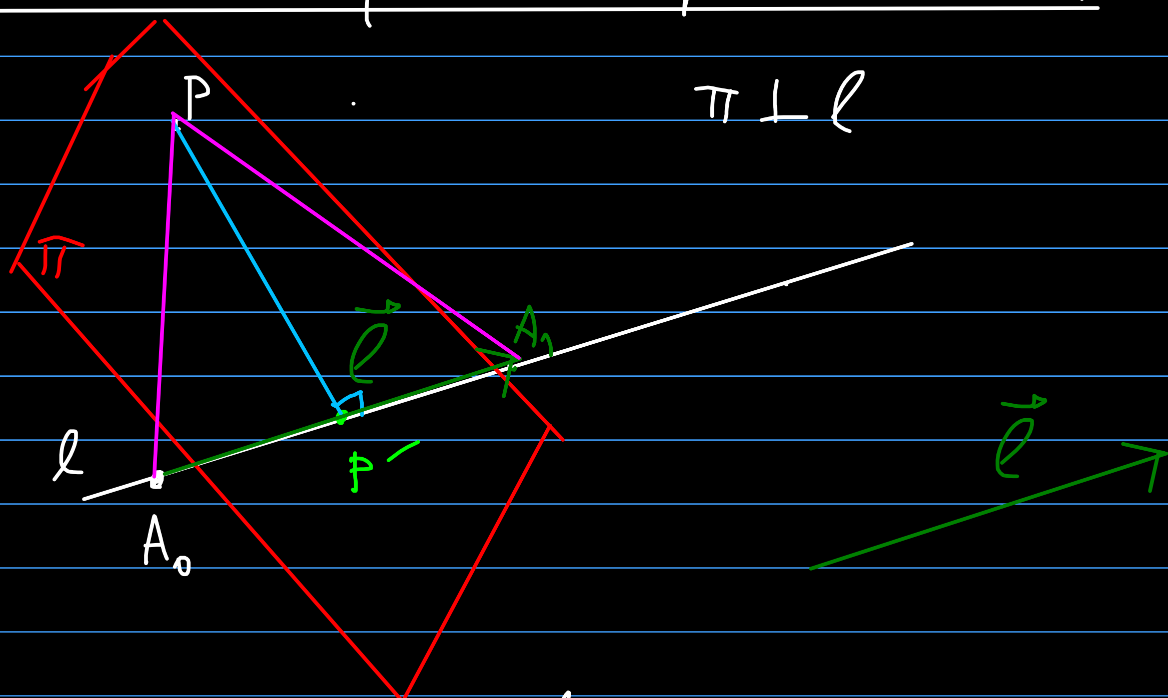
For $\forall \alpha, \beta \in \mathbb{R}$, $\vec{v_1}, \vec{v_2}, \vec{w} \in V$

$$(\alpha \vec{v_1} + \beta \vec{v_2}) \times \vec{w} = \alpha \cdot (\vec{v_1} \times \vec{w}) + \beta (\vec{v_2} \times \vec{w})$$

Computing the area of a triangle

$$A_{ABC} = \frac{1}{2} \| \vec{AB} \times \vec{AC} \|$$

The distance from a point to a line



$A_0 \in \ell$, $\vec{\ell}$ vector of ℓ

$\exists A_1 \in l: \quad \overrightarrow{A_0A_1} = \vec{l}$

In ΔPA_0A_1, PP' is a height

$$\Rightarrow dist(P, l) = PP' = \frac{S_{PA_0A_1}}{A_0A_1} =$$

$$= \frac{\|\overrightarrow{PA_0} \times \overrightarrow{A_0A_1}\|}{\|\overrightarrow{A_0A_1}\|} = \frac{\|\overrightarrow{PA_0} \times \vec{l}\|}{\|\vec{l}\|}.$$

6.4. Find the distance from the point $P(1, 2, -1)$ to the straight line

$$l: x = y = z$$

We choose $A_0(1,1,1) \in l$ and $\vec{l}(1,1,1)$

$\overrightarrow{PA_0}(0, -1, 2)$.

$$\overrightarrow{PA_0} \times \vec{l} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} =$$

$$= -\vec{i} + 2\vec{j} + \vec{k} - 2\vec{i} = -3\vec{i} + 2\vec{j} + \vec{k}$$

$$\Rightarrow \overrightarrow{PA_0} \times \vec{l} = (-3, 2, 1)$$

$$\Rightarrow \|\overrightarrow{PA_0} \times \vec{l}\| = \sqrt{9+4+1} = \sqrt{14}$$

$$\text{dist}(P, l) = \frac{\|\overrightarrow{PA_0} \times \vec{l}\|}{\|\vec{l}\|} = \frac{\sqrt{14}}{\sqrt{3}}$$

6.5. Find the area of the triangle ABC and the lengths of its heights, where $A(-1, 1, 2)$, $B(2, -1, 1)$ and $C(2, -3, -2)$.

$$S_{ABC} = \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\|$$

$$\overrightarrow{AB}(3, -2, -1) \qquad \overrightarrow{AC}(3, -4, -4)$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -1 \\ 3 & -4 & -4 \end{vmatrix} =$$

$$= \vec{i} \cdot \begin{vmatrix} -2 & -1 \\ -4 & -4 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 3 & -1 \\ 3 & -4 \end{vmatrix} +$$

$$+ \vec{k} \cdot \begin{vmatrix} 3 & -2 \\ 3 & -4 \end{vmatrix} = 4\vec{i} + 9\vec{j} + (-6)\vec{k}$$

$$\Rightarrow \|\vec{AB} \times \vec{AC}\| = \sqrt{16+81+36} = \sqrt{133}$$

$$\Rightarrow S_{ABC} = \frac{\sqrt{133}}{2}$$

$$h_A = \frac{2 \cdot S_{ABC}}{\|\vec{BC}\|} = \frac{\sqrt{133}}{\sqrt{4+9}} = \sqrt{\frac{133}{13}}$$

$$h_B = \frac{2 S_{ABC}}{\|\vec{AC}\|} = \frac{\sqrt{133}}{\sqrt{41}} = \sqrt{\frac{133}{41}}$$

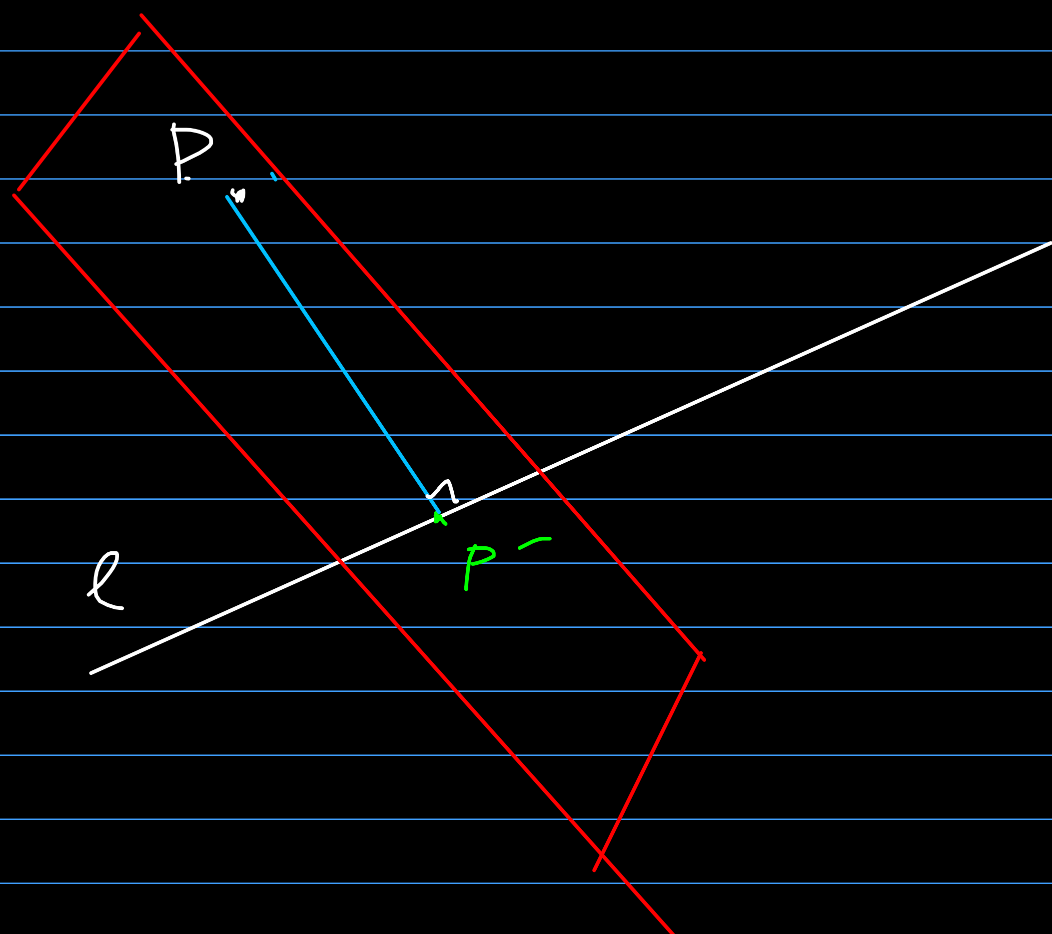
$$h_C = \frac{2 S_{ABC}}{\|\vec{AB}\|} = \frac{\sqrt{133}}{\sqrt{14}} = \sqrt{\frac{133}{14}}$$

Ex. 6.11. Consider the line:

$$\ell: \begin{cases} x - 2y + z + 6 = 0 \\ 3x + y + z - 1 = 0 \end{cases}$$

and a point $P(1, 2, 3)$

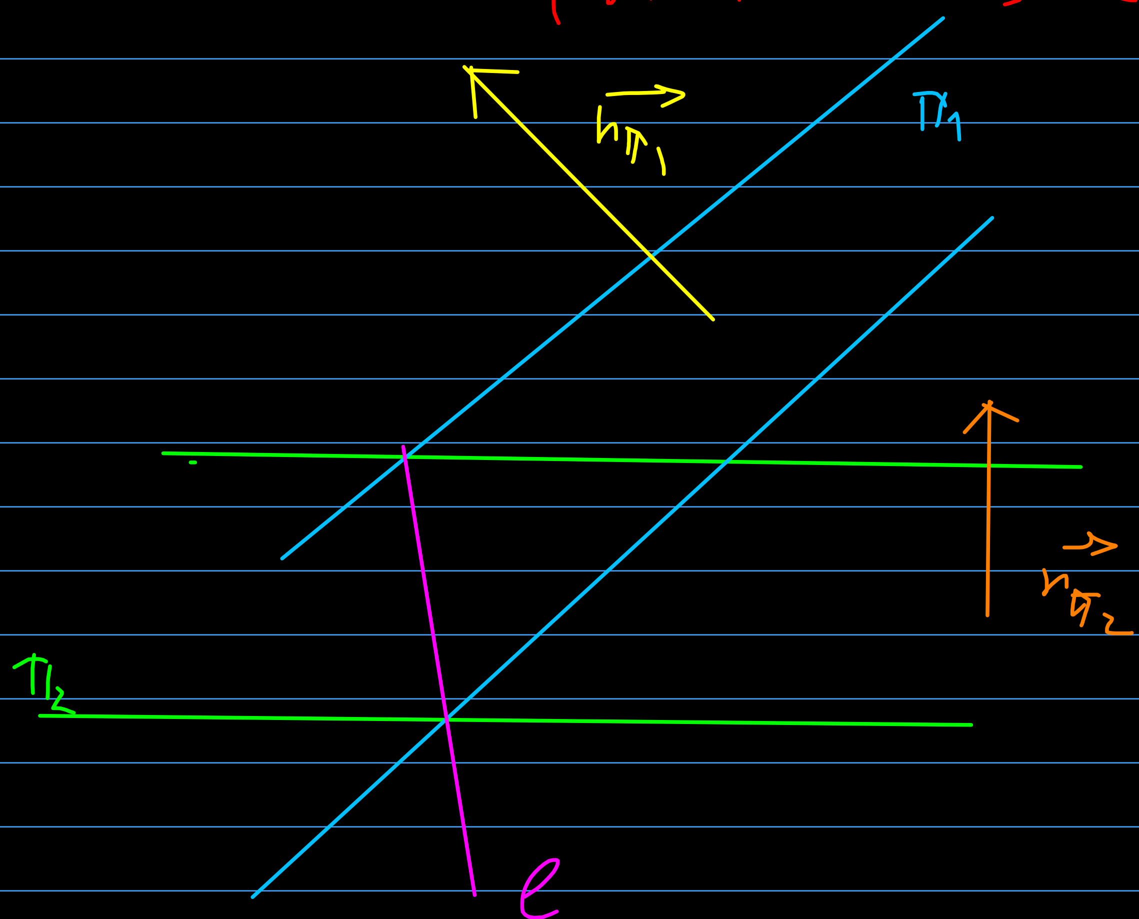
Find the equations of the perpendicular from P onto ℓ.



∃/ $\ell: \begin{cases} \Pi_1: A_1x + B_1y + C_1z + D_1 = 0 \\ \Pi_2: A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$

$\Pi_1 \nparallel \Pi_2$

∃/ all we want is a director vector of the line ℓ, then we can choose $\vec{n}_{\Pi_1} \times \vec{n}_{\Pi_2}$ as our director vector (in other words $\ell \parallel \vec{n}_{\Pi_1} \times \vec{n}_{\Pi_2}$)



$$\left.\begin{array}{l} \vec{n}_{\Pi_1} \perp \Pi_1 \Rightarrow \vec{n}_{\Pi_1} \perp \ell \\ \vec{n}_{\Pi_2} \perp \Pi_2 \Rightarrow \vec{n}_{\Pi_2} \perp \ell \end{array}\right\} \Rightarrow \ell \parallel (\vec{n}_{\Pi_1} \times \vec{n}_{\Pi_2})$$

$$\ell: \begin{cases} \pi_1: x - 2y + z + 6 = 0 \\ \pi_2: 3x + y + z - 1 = 0 \end{cases}$$

$$\vec{n_{\pi_1}} = (1, -2, 1)$$

$$\vec{n_{\pi_2}} = (3, 1, 1)$$

$$\vec{n_{\pi_1}} \times \vec{n_{\pi_2}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 3 & 1 & 1 \end{vmatrix} =$$

$$= -3\vec{i} + 2\vec{j} + 7k$$

$$\Rightarrow \vec{\ell}(-3, 2, 7)$$

We will find the equation of the plane π, that contains P and is perpendicular to the line ℓ.

$$\pi: \quad -3x + 2y + 7z + D = 0$$

$$P \in \pi \Rightarrow -3 \cdot 1 + 2 \cdot 2 + 7 \cdot 3 + D = 0$$

$\Rightarrow D = -22$

$\Rightarrow \Pi : -3x+2y+7z-22=0$

$$D' = \ell \cap \Pi : \begin{cases} \begin{cases} x-2y+z+6=0 \\ 3x+y+z-1=0 \end{cases} \\ -3x+2y+7z-22=0 \end{cases}$$

$$\begin{pmatrix} 1 & -2 & 1 & -6 \\ 3 & 1 & 1 & 1 \\ -3 & 2 & 7 & 22 \end{pmatrix} \begin{array}{l} L_2 \leftarrow L_2 - 3L_1 \\ \sim \\ L_3 \leftarrow L_3 + 3L_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & -2 & 1 & -6 \\ 0 & 7 & -2 & 19 \\ 0 & -4 & 10 & 4 \end{pmatrix} \sim$$

$$\begin{array}{l} L_3 \leftarrow \frac{1}{7}L_2 + \frac{1}{4}L_3 \\ \sim \end{array} \begin{pmatrix} 1 & -2 & 1 & -6 \\ 0 & 7 & -2 & 19 \\ 0 & 0 & \frac{10}{4}-\frac{2}{7} & 1+\frac{19}{7} \end{pmatrix}$$

$$\Rightarrow \begin{cases} x - 2y + z = 6 \\ 7y - 2z = -19 \\ \frac{62}{28} \cdot z = -\frac{26}{7} \end{cases}$$

$$\Rightarrow z_{p'} = -\frac{52}{31}$$

$$\Rightarrow y_{p'} = \frac{1}{7}\left(2 \cdot \frac{-52}{31} - 19\right) =$$

$$x_{p'} = \frac{2}{7}\left(2 \cdot \frac{-52}{31} - 19\right) + 6 + \frac{52}{31}$$

$\Rightarrow PP'$: the eqn. of the line through P and P'.

The double cross product (double vector product)

$\vec{a}, \vec{b}, \vec{c} \in V$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{b} & \vec{a}\vec{c} \end{vmatrix} = (\vec{a}\cdot\vec{c})\cdot\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{a} \end{vmatrix}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b}) =$$

$$= \vec{c} \times (\vec{b} \times \vec{a}) = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c}\vec{b} & \vec{c}\cdot\vec{a} \end{vmatrix}$$

$\Rightarrow$ the cross product is not associative!!!