Seminar W13 - 9.14

The projective plane:

$$\mathbb{RP}^2 = \mathbb{P}^2(\mathbb{R}) = \mathbb{P}_2(\mathbb{R}) =$$

$$= \left\{ [x : y : z] \;\middle|\; \begin{array}{l} x, y, z \in \mathbb{R} \\ (x,y,z) \neq (0,0,0) \end{array} \right\}$$

$$[x : y : z] = [\lambda x : \lambda y : \lambda z], \; \forall \lambda \in \mathbb{R} \setminus \{0\}$$

homogeneous coordinates

$$\mathbb{RP}^2 = \frac{\mathbb{R}^3 \setminus \{0\}}{\sim}$$

where $(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \Leftrightarrow \exists \lambda \in \mathbb{R} \setminus \{0\}$

$$x_1 = \lambda x_2$$
$$y_1 = \lambda y_2$$
$$z_1 = \lambda z_2$$

$\mathbb{RP}^2 \simeq$ all the lines in $\mathbb{R}^3$ that contain the origin

$$\mathbb{RP}^2 = \{ [x:y:z] \in \mathbb{RP}^2 \mid z \neq 0 \} \cup$$

$$\cup \{ [x:y:z] \in \mathbb{RP}^2 \mid z = 0 \}$$

$$\mathbb{RA}^2 \simeq \{ [x:y:z] \in \mathbb{RP}^2 \mid z \neq 0 \}$$

$\wr\wr$

$\mathbb{R}^2$ if $z \neq 0 \Rightarrow [x:y:z] = [\underbrace{\tfrac{x}{z}}_{= X} : \underbrace{\tfrac{y}{z}}_{= Y} : 1] =$

$= [X:Y:1]$

$$\mathbb{RA}^2 \simeq \mathbb{R}^2$$

$$[x:y:z] \longmapsto \left(\frac{x}{y}, \frac{y}{z} \right)$$

$$\mathbb{R}_\infty \simeq \left\{ [x:y:0] \;\middle|\; \begin{array}{l} x, y \in \mathbb{R} \\ x^2+y^2 > 0 \end{array} \right\}$$

↳ line at infinity

$[x:y:0]$ is the intersection of all the parallel lines that have the direction vector (x, y)

Why we care!

φ_1, φ_2 affine transformations

$$\varphi_1 \begin{pmatrix} x \\ y \end{pmatrix} = M_1 \begin{pmatrix} x \\ y \end{pmatrix} + v_1$$

$$\varphi_2 \begin{pmatrix} x \\ y \end{pmatrix} = M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_2$$

In order to compose these transformations:

$$\varphi_2 \circ \varphi_1 \begin{pmatrix} x \\ y \end{pmatrix} = \varphi_2 \left(M_1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_1 \right) =$$

$$= M_2 \cdot M_1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + M_2 v_1 + v_2$$

Instead of defining:

$$\varphi \begin{pmatrix} x \\ y \end{pmatrix} = M \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_0$$

we define

$$\varphi \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M} & v_0 \\ \hline 0 & 0 & 1 \end{array} \right)$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad v_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$\Rightarrow$ instead of

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

we define:

$$\varphi\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \left(\begin{array}{cc|c} a & b & x_0 \\ c & d & y_0 \\ \hline 0 & 0 & 1 \end{array}\right) \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} ax + by + x_0 \\ cx + dy + y_0 \\ 1 \end{bmatrix} \simeq \begin{pmatrix} ax + by + x_0 \\ cx + dy + y_0 \end{pmatrix}$$

$$[\varphi] = \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

This leads us to more general transformations the projective transformation:

$$\varphi\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

→ out of these, the affine transformations are the ones for which $a_{31} = a_{32} = 0$ and $a_{33} \neq 0$ (to simplify, they will look like this:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

13-1. Find the concatenation of an anticlockwise rotation about the origin through an angle of $\frac{3\pi}{2}$, followed by a scaling by a factor of 3 units in the x direction and 2 units in the y direction

$$[S(3,2)] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$[R_{\frac{3\pi}{2}}] = \begin{pmatrix} \cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2} & 0 \\ \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[S(3,2) \circ R_{\frac{3\pi}{2}}] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

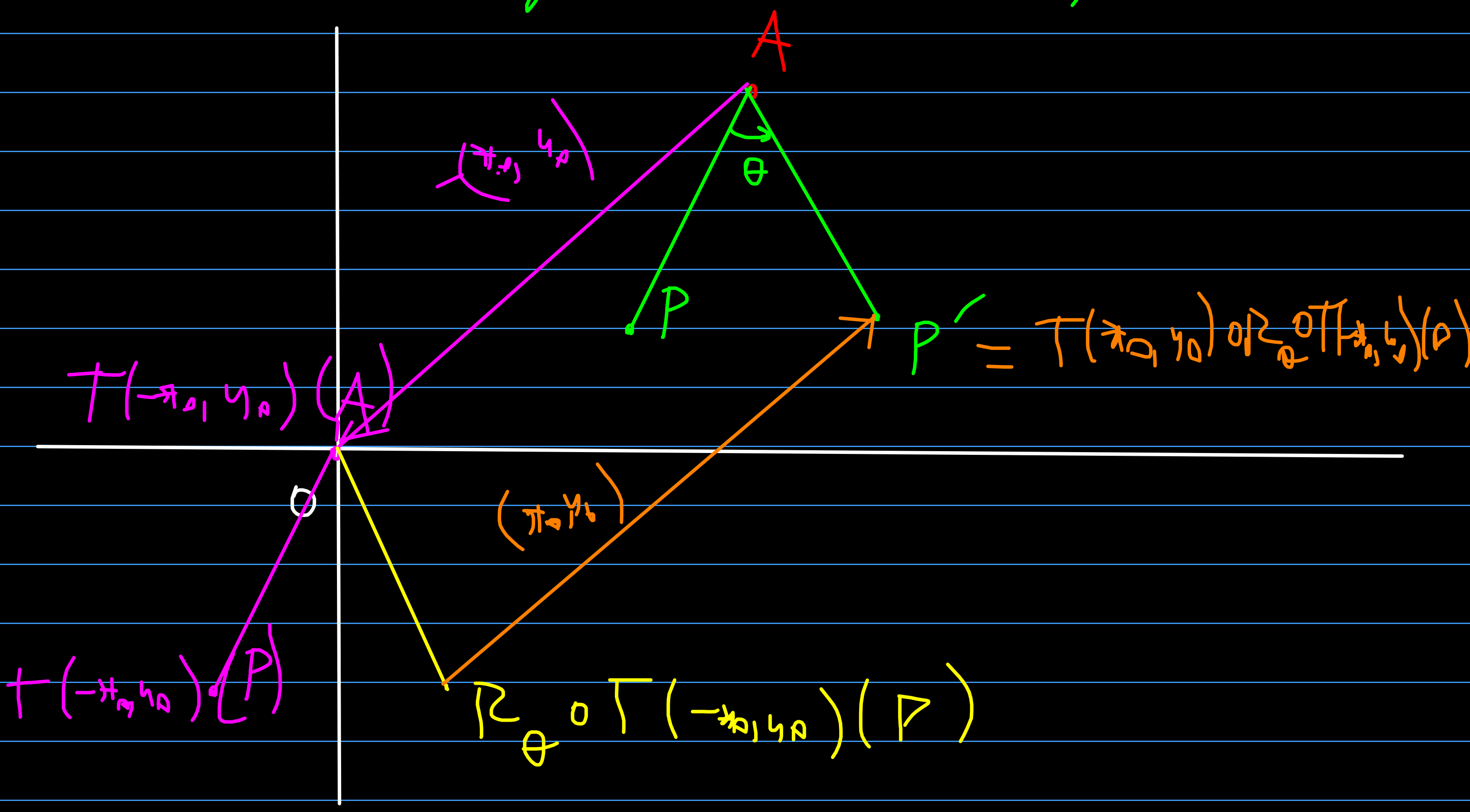
$$\Rightarrow \varphi \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} 3y \\ -2x \\ 1 \end{bmatrix}$$

$$\varphi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ -2x \end{pmatrix}$$

13.3.

$$R_\theta(x_0, y_0) = T(x_0, y_0) \circ R_\theta \circ T(-x_0, -y_0)$$

the counterclockwise rotation by an angle θ around a point $A(x_0, y_0)$



$$[R_\theta(x_0, y_0)] = \begin{bmatrix} \cos\theta & -\sin\theta & -x_0\cos\theta + y_0\sin\theta + x_0 \\ \sin\theta & \cos\theta & -x_0\sin\theta - y_0\cos\theta + y_0 \\ 0 & 0 & 1 \end{bmatrix}.$$

13 *. Show that the composition of two reflections with regards to two lines l_1 and l_2 is a translation, if $l_1 \parallel l_2$

l_1, l_2 lines, $l_1 \parallel l_2$

Show that $r_{l_1} \circ r_{l_2}$ is a translation.

$$l_1 : ax + by + c_1 = 0$$
$$l_2 : ax + by + c_2 = 0$$

$$[r_{l_1}] = \begin{pmatrix} \frac{b^2 - a^2}{b^2 + a^2} & \frac{-2ab}{a^2+b^2} & -\frac{2ac_1}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2bc_1}{a^2+b^2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$[r_{\ell_2}] = \begin{pmatrix} \frac{b^2-a^2}{b^2+a^2} & \frac{-2ab}{a^2+b^2} & -\frac{2ac_2}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2bc_2}{a^2+b^2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$[r_{\ell_1}] = \begin{pmatrix} b^2-a^2 & -2ab & -2ac_1 \\ -2ab & a^2-b^2 & -2bc_1 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

$$[r_{\ell_2}] = \begin{pmatrix} b^2-a^2 & -2ab & -2ac_2 \\ -2ab & a^2-b^2 & -2bc_2 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

$$[r_{\ell_1} \circ r_{\ell_2}] = \begin{pmatrix} b^2-a^2 & -2ab & -2ac_1 \\ -2ab & a^2-b^2 & -2bc_1 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} b^2-a^2 & -2ab & -2ac_2 \\ -2ab & a^2-b^2 & -2bc_2 \\ 0 & 0 & a^2+b^2 \end{pmatrix} = (C)$$

$$c_{11} = (b^2-a^2)^2 + (-2ab)^2 = b^4 - 2a^2b^2 + a^4 + 4a^2b^2 =$$
$$= (a^2+b^2)^2$$

$$c_{12} = -2ab(b^2-a^2) - 2ab(a^2-b^2) = 0$$

$$c_{13} = -2ac_2(b^2-a^2) - 4b^2ac_2 - 2ac_1(a^2+b^2) = \alpha$$

$$c_{21} = -2ab(b^2-a^2) - 2ab(a^2-b^2) = 0$$

$$c_{22} = (b^2-a^2)^2 + 4a^2b^2 = (a^2+b^2)^2$$

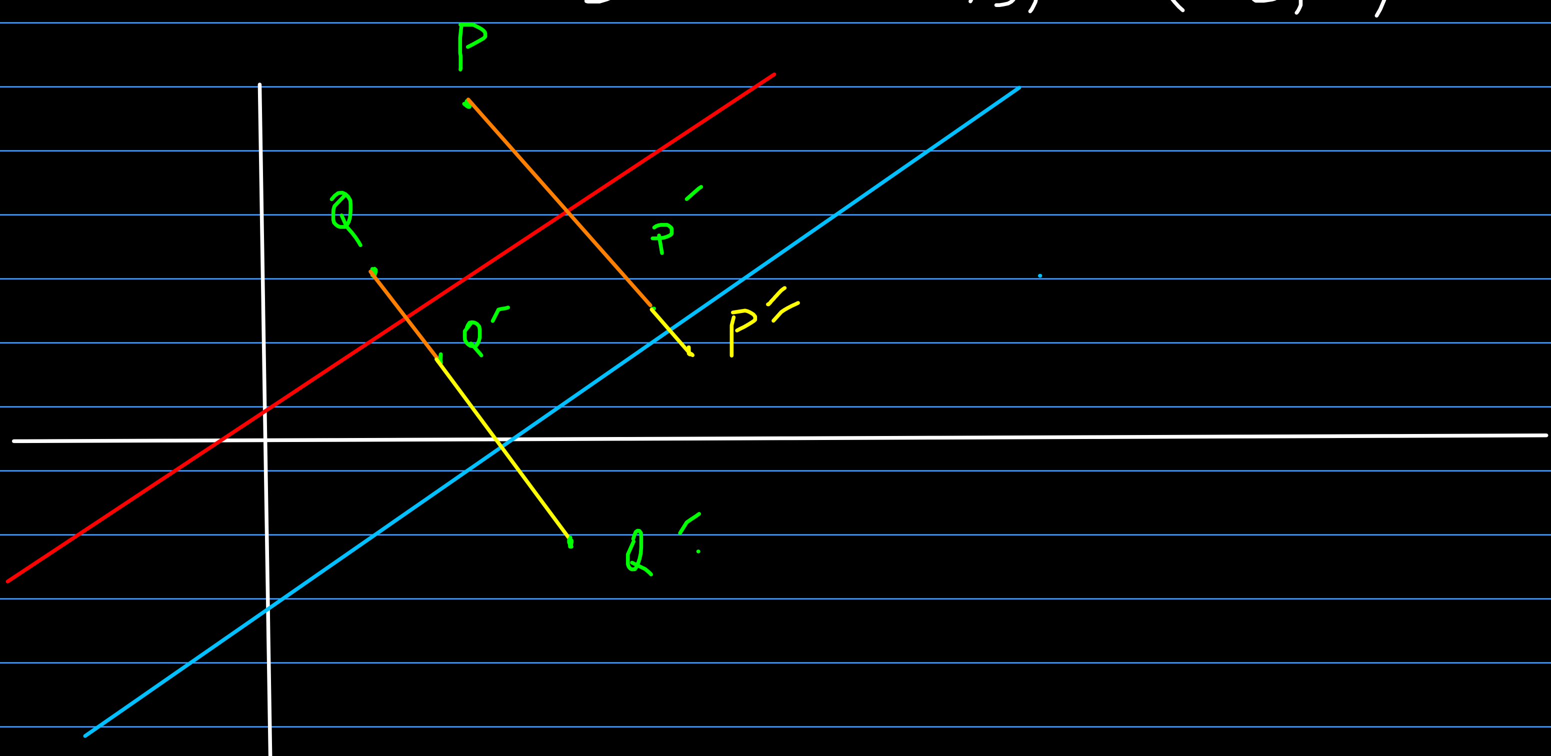
$$c_{23} = 4a^2bc_2 - 2(a^2-b^2) \cdot bc_2 -$$
$$- 2bc_1(a^2+b^2) = \beta$$

$c_{31} = 0 \qquad c_{32} = 0 \qquad c_{33} = (a^2+b^2)^2$

$$\Rightarrow [r_{l_1} \circ r_{l_2}] = \begin{pmatrix} (a^2+b^2)^2 & 0 & \alpha \\ 0 & (a^2+b^2)^2 & \beta \\ 0 & 0 & (a^2+b^2)^2 \end{pmatrix} \simeq$$

$$\simeq \begin{pmatrix} 1 & 0 & \frac{\alpha}{(a^2+b^2)^2} \\ 0 & 1 & \frac{\beta}{(a^2+b^2)^2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow r_{l_1} \circ r_{l_2} = T\left(\frac{\alpha}{(a^2+b^2)^2}, \frac{\beta}{(a^2+b^2)^2} \right)$$



13.4. $P(x_0, y_0)$, $Q(x_1, y_1)$

$Q \neq P$

$\theta \in \mathbb{R}$.

show that $R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0)$ is a translation.

Keep in mind that

$$[R_\theta(x_0, y_0)] = \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_0 = -x_0 \cos\theta + y_0 \sin\theta + x_0$$

$$\beta_0 = -x_0 \sin\theta - y_0 \cos\theta + y_0$$

$$[R_{-\theta}(x_1, y_1)] = \begin{pmatrix} \cos\theta & \sin\theta & \alpha_1 \\ -\sin\theta & \cos\theta & \beta_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_1 = -x_1 \cos\theta - y_1 \sin\theta + x_1$$

$$\beta_1 = x_1 \sin\theta - y_1 \cos\theta + y_1$$

$$[R_{-\theta}(x_1, y_1) \cdot R_{\theta}(x_0, y_0)] =$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & \alpha_1 \\ -\sin\theta & \cos\theta & \beta_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= C$$

$$c_{11} = \cos^2\theta + \sin^2\theta = 1$$

$$c_{12} = -\cos\theta \sin\theta + \sin\theta \cos\theta = 0$$

$$c_{21} = -\sin\theta \cos\theta + \sin\theta \cos\theta = 0$$

$$c_{22} = \sin^2\theta + \cos^2\theta = 1$$

$$c_{31} = 0, \quad c_{32} = 0, \quad c_{33} = 1$$

$$\Rightarrow \left(R_{-\theta}(x_1, y_1) \circ R_{\theta}(x_0, y_0)\right) = \begin{pmatrix} 1 & 0 & c_{13} \\ 0 & 1 & c_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ this transformation is a translation