Bonus projects for the Algebra course

- Any programming language may be used.
- The solutions will consist of the source code with comments and at least 5 relevant input and output files, and will be sent to the e-mail address: scrivei@gmail.com.
- If necessary, you will be asked to explain your solution.
- The first 10 solutions for each project will be rewarded with at most 0.2 points each.
- You may submit improvements of your solutions, but they will be considered only if they are still in the first 10 solutions for each project.
- Each student may get points for up to 5 projects.
- The final deadline is January 10, 2021.

Project 1 (0.2 points)

- *Input*: non-zero natural number n
- *Output*:
  1. the number of partitions on a set $A = \{a_1, \ldots, a_n\}$
  2. the partitions on a set $A = \{a_1, \ldots, a_n\}$ and their corresponding equivalence relations (for $n \leq 8$)

Example:

- *Input*: $n = 3$
- *Output*:
  1. the number of partitions on a set $A = \{a_1, a_2, a_3\}$ is 5
  2. using the notation $\Delta_A = \{(a_1, a_1), (a_2, a_2), (a_3, a_3)\}$, the partitions on a set $A = \{a_1, a_2, a_3\}$ and their corresponding equivalence relations are:

$\{a_1\}, \{a_2\}, \{a_3\}\} \rightsquigarrow \Delta_A$
$\{a_1, a_2\}, \{a_3\}\} \rightsquigarrow \Delta_A \cup \{(a_1, a_2), (a_2, a_1)\}$
$\{a_1, a_3\}, \{a_2\}\} \rightsquigarrow \Delta_A \cup \{(a_1, a_3), (a_3, a_1)\}$
$\{a_2, a_3\}, \{a_1\}\} \rightsquigarrow \Delta_A \cup \{(a_2, a_3), (a_3, a_2)\}$
$\{\{a_1, a_2, a_3\}\} \rightsquigarrow A \times A$

Project 2 (0.2 points)

- *Input*: non-zero natural number n
- *Output*:
  1. the number of transitive relations on a set $A = \{a_1, \ldots, a_n\}$
  2. the transitive relations on a set $A = \{a_1, \ldots, a_n\}$ (for $n \leq 4$)

Example:

- *Input*: $n = 2$
- *Output*:
  1. the number of transitive relations on a set $A = \{a_1, a_2\}$ is 13
  2. the transitive relations on a set $A = \{a_1, a_2\}$ are:

$R_1 = \emptyset$
$R_2 = \{(a_1, a_1)\}$
$R_3 = \{(a_1, a_2)\}$
$R_4 = \{(a_2, a_1)\}$
$R_5 = \{(a_2, a_2)\}$
$R_6 = \{(a_1, a_1), (a_1, a_2)\}$
$R_7 = \{(a_1, a_1), (a_2, a_1)\}$
$R_8 = \{(a_1, a_1), (a_2, a_2)\}$
$R_9 = \{(a_1, a_2), (a_2, a_2)\}$
$R_{10} = \{(a_2, a_1), (a_2, a_2)\}$
$R_{11} = \{(a_1, a_1), (a_2, a_2), (a_1, a_2)\}$
$R_{12} = \{(a_1, a_1), (a_2, a_2), (a_2, a_1)\}$
$R_{13} = \{(a_1, a_1), (a_1, a_2), (a_2, a_1), (a_2, a_2)\}$

Project 3 (0.2 points)

- *Input*: non-zero natural number n
- *Output*:
  1. the number of associative operations on a set $A = \{a_1, \ldots, a_n\}$
  2. the operation table of each associative operation (for $n \leq 4$)

Example:

- *Input*: $n = 2$
- *Output*:
  1. the number of associative operations on a set $A = \{a_1, a_2\}$ is 8
  2. identifying an operation table $\begin{array}{c|cc} & a_1 & a_2 \\ \hline a_1 & x & y \\ a_2 & z & t \end{array}$ by the matrix $\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(A)$, the operation tables of the associative operations on $A = \{a_1, a_2\}$ are given by the matrices:

$$\begin{pmatrix} a_1 & a_1 \\ a_1 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & a_1 \\ a_1 & a_2 \end{pmatrix}, \begin{pmatrix} a_1 & a_1 \\ a_2 & a_2 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_1 & a_2 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_2 & a_2 \end{pmatrix}, \begin{pmatrix} a_2 & a_1 \\ a_1 & a_2 \end{pmatrix}, \begin{pmatrix} a_2 & a_2 \\ a_2 & a_2 \end{pmatrix}.$$

Project 4 (0.2 points)

- *Input*: non-zero natural number n
- *Output*:
  1. the number of abelian group structures which can be defined on a set $A = \{a_1, \ldots, a_n\}$
  2. the operation table of each such abelian group (for $n \leq 7$)

Example: The operation table of a group G has the property that each element of G appears exactly once on each row and on each column. The operation table of an abelian group is symmetric with respect to the main diagonal. We may identify an operation table by a matrix. Make sure that the operations are associative and have identity element.

- *Input*: $n = 4$
- *Output*:
  1. the number of abelian group structures on a set $G = \{a_1, a_2, a_3, a_4\}$ is 16
  2. the abelian group structures on G with identity element a_1 are given by the matrices:

$$\begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} \\ \mathbf{a_2} & \boxed{a_1} & a_4 & a_3 \\ \mathbf{a_3} & a_4 & a_1 & a_2 \\ \mathbf{a_4} & a_3 & a_2 & a_1 \end{pmatrix}, \begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} \\ \mathbf{a_2} & \boxed{a_1} & a_4 & a_3 \\ \mathbf{a_3} & a_4 & a_2 & a_1 \\ \mathbf{a_4} & a_3 & a_1 & a_2 \end{pmatrix}, \begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} \\ \mathbf{a_2} & \boxed{a_3} & a_4 & a_1 \\ \mathbf{a_3} & a_4 & a_1 & a_2 \\ \mathbf{a_4} & a_1 & a_2 & a_3 \end{pmatrix}, \begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} \\ \mathbf{a_2} & \boxed{a_4} & a_1 & a_3 \\ \mathbf{a_3} & a_1 & a_4 & a_2 \\ \mathbf{a_4} & a_3 & a_2 & a_1 \end{pmatrix}.$$

There are 4 similar abelian group structures for each possible identity element.