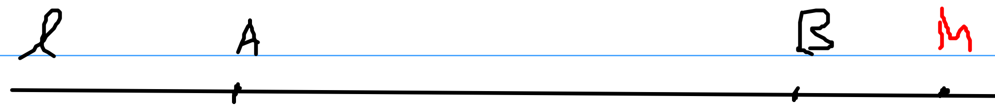
Seminar W2 — 9.7



l line in the Euclidean plane

$A, B \in l$

Say we fix a reference system.

Then $\forall M \in l \quad \exists! \lambda \in \mathbb{R}$ s.t.:

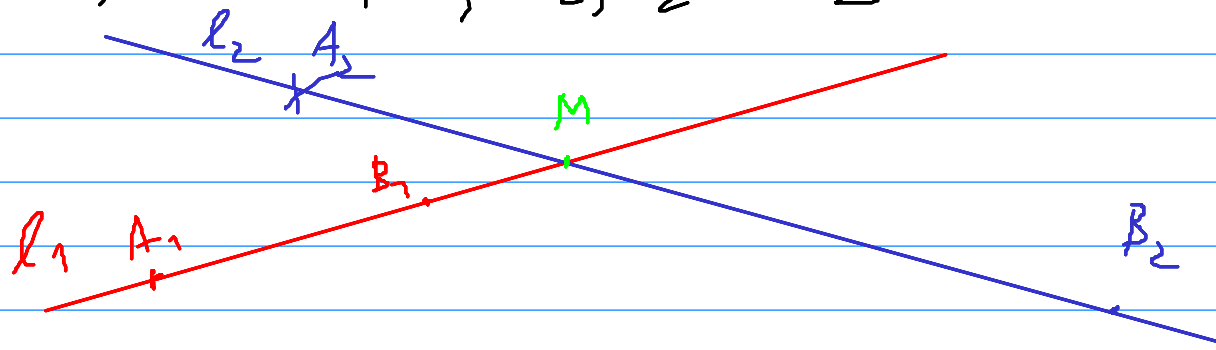
$$\vec{r}_M = \lambda \vec{r}_A + (1-\lambda)\vec{r}_B$$

If $M \in [AB]$ and $\frac{AM}{MB} = \alpha \in \mathbb{R}$

$$\Rightarrow \quad \vec{r}_M = \frac{\alpha}{\alpha+1}\vec{r}_B + \frac{1}{\alpha+1}\vec{r}_A$$

Let l_1, l_2 be lines, $l_1 \cap l_2 = \{M\}$

Let $A_1, B_1 \in l_1$, $A_2, B_2 \in l_2$



Objective : Find $\vec{r_M}$

Template for proofs involving concurrence of lines

Step 1 : Write M as a point on both lines:

(1) $M \in \ell_1 \Rightarrow \exists \lambda \in \mathbb{R} : \vec{r_M} = \lambda \vec{r_{A_1}} + (1-\lambda)\vec{r_{B_1}}$

(2) $M \in \ell_2 \Rightarrow \exists \mu \in \mathbb{R} : \vec{r_M} = \mu \vec{r_{A_2}} + (1-\mu)\vec{r_{B_2}}$

$$\Rightarrow \lambda \vec{r_{A_1}} + (1-\lambda)\vec{r_{B_1}} = \mu \vec{r_{A_2}} + (1-\mu)\vec{r_{B_2}} \quad (*)$$

Step 2 : We choose two linearly independent vectors $\vec{v}$ and $\vec{w}$

Step 3 : We write $(*)$ in the basis $(\vec{v}, \vec{w})$

(we write $\vec{r_{A_1}}, \vec{r_{A_2}}, \vec{r_{B_1}}, \vec{r_{B_2}}$ as linear combinations of $\vec{v}$ and $\vec{w}$)

Step 4: We have obtained:

$$\alpha(\lambda,\mu)\cdot\vec{u}+\beta(\lambda,\mu)\cdot\vec{w}=\vec{0}$$

$\vec{u},\vec{w}$ lin. indep. $\Rightarrow$ $(S)\begin{cases}\alpha(\lambda,\mu)=0\\ \beta(\lambda,\mu)=0\end{cases}$

Step 5: Solve the system (S) and obtain λ (or μ)

Step 6: Replace λ (or μ) in (1) (or (2))

Step 7: We have obtained a formula:

$$\boxed{\vec{r_M}=t_1\cdot\vec{u}+t_2\cdot\vec{w}}$$

2.1. $\triangle ABC$, G centroid, H orthocenter

I incenter, O circumcenter

$\downarrow$

point where all the

perpendicular bisectors

meet

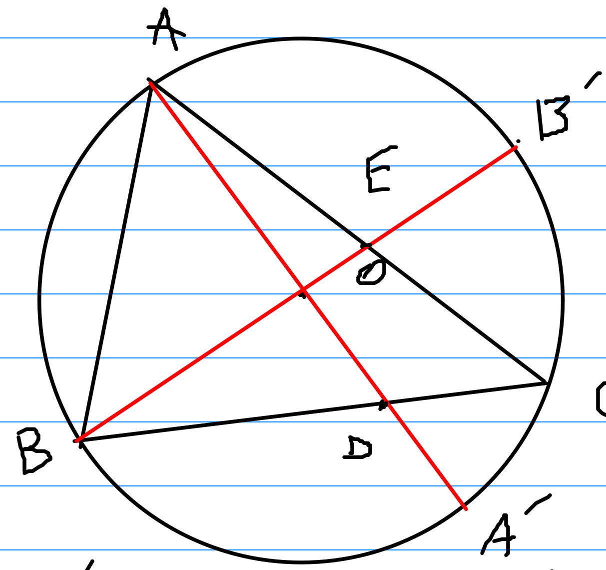
We fix a reference system.

(a) $$\vec{r}_G = \frac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}$$

(b) $$\vec{r}_I = \frac{a\vec{r}_A + b\vec{r}_B + c\vec{r}_C}{a+b+c}, \quad a = BC,\ b = CA,\ c = AB$$

(c) $$\vec{r}_H = \frac{\tan A \cdot \vec{r}_A + \tan B \cdot \vec{r}_B + \tan C \cdot \vec{r}_C}{\tan A + \tan B + \tan C}$$

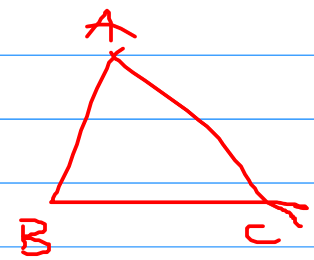
(d) $$\vec{r}_O = \frac{\sin 2A \cdot \vec{r}_A + \sin 2B \cdot \vec{r}_B + \sin 2C \cdot \vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}$$



We draw the diameters AA' and BB'

$AA' \cap BC = \{D\}$, $BB' \cap CA = \{E\}$

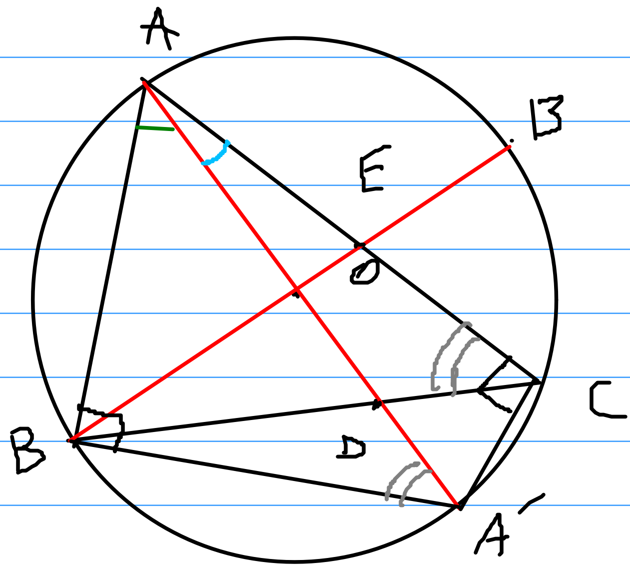
Prove that $$\frac{BD}{DC} = \frac{\sin(2C)}{\sin(2B)}$$



The sine theorem: in ΔABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

R = radius of the circumscribed circle



$$\frac{BD}{DC} = \frac{A_{ABD}}{A_{ADC}} = \frac{AB \cdot AD \cdot \sin(\widehat{BAD})}{AC \cdot AD \cdot \sin(\widehat{CAD})} = \frac{c}{b} \cdot \frac{\sin(\widehat{BAD})}{\sin(\widehat{CAD})}$$

AA' diameter $\Rightarrow m(\widehat{ABA'}) = m(\widehat{ACA'}) = \frac{\pi}{2}$

$\widehat{ACB} \equiv \widehat{AA'B} \equiv \widehat{AB} \Rightarrow m(\widehat{A'AB}) = \frac{\pi}{2} - m(\widehat{AA'B}) =$

$= \frac{\pi}{2} - \hat{C}$

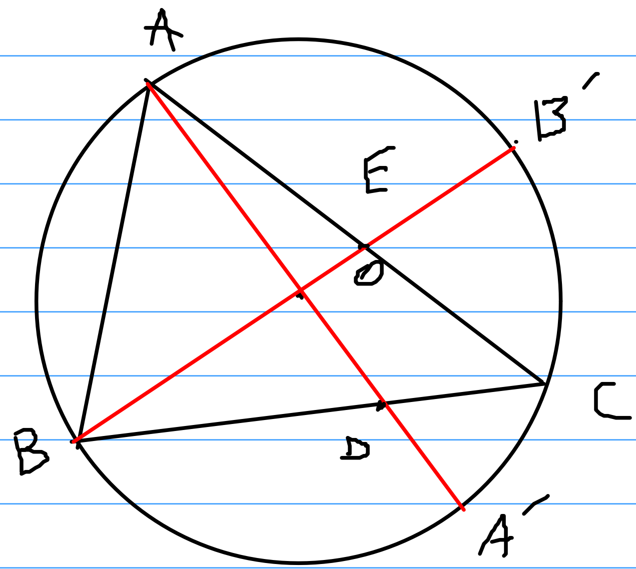
$m(\widehat{A'AC}) = \frac{\pi}{2} - \hat{B} \Rightarrow \frac{\sin(\widehat{BAD})}{\sin(\widehat{CAD})} = \frac{\sin(\frac{\pi}{2} - C)}{\sin(\frac{\pi}{2} - B)} =$

$$= \frac{\cos C}{\cos B}$$

From the sine theorem $\frac{c}{b} = \frac{\sin C}{\sin B}$

$$\Rightarrow \frac{BD}{DC} = \frac{\sin C}{\sin B} \cdot \frac{\cos C}{\cos B} = \frac{\sin 2C}{\sin 2B}$$

Now we can come back to 1.1 d



We need to show

$$\vec{r_O} = \frac{\sin 2A \cdot \vec{r_A} + \sin 2B \vec{r_B} + \sin 2C \cdot \vec{r_C}}{\sin 2A + \sin 2B + \sin 2C}$$

We will denote

$\sin 2A =: \alpha$

$\sin 2B =: \beta$

$\sin 2C =: \gamma$

$$\Rightarrow \frac{BD}{DC} = \frac{\gamma}{\beta}, \quad \frac{AE}{EC} = \frac{\gamma}{\alpha}$$

$O \in AD \Rightarrow \exists \lambda \in \mathbb{R}: \vec{r_O} = \lambda \vec{r_A} + (1-\lambda)\vec{r_D}$

$O \in BE \Rightarrow \exists \mu \in \mathbb{R}: \vec{r_O} = \mu \vec{r_B} + (1-\mu)\vec{r_E}$

$$\vec{r_D} = \frac{1}{\frac{\gamma}{\beta}+1} \cdot \vec{r_B} + \frac{\frac{\gamma}{\beta}}{\frac{\gamma}{\beta}+1} \vec{r_C} = \frac{\beta}{\beta+\gamma} \vec{r_B} + \frac{\gamma}{\beta+\gamma} \vec{r_C}$$

$$\vec{r_E} = \frac{1}{\frac{\gamma}{\alpha}+1}\vec{r_A} + \frac{\frac{\gamma}{\alpha}}{\frac{\gamma}{\alpha}+1}\vec{r_C} = \frac{\alpha}{\alpha+\gamma}\vec{r_A} + \frac{\gamma}{\alpha+\gamma}\vec{r_C}$$

Because $\lambda\vec{r_A} + (1-\lambda)\vec{r_D} = \mu\vec{r_B} + (1-\mu)\vec{r_E}$

we have:

$$\lambda\vec{r_A} + \frac{\beta(1-\lambda)}{\beta+\gamma}\vec{r_B} + \frac{\gamma(1-\lambda)}{\beta+\gamma}\vec{r_C} =$$

$$= \mu\vec{r_B} + \frac{\alpha(1-\mu)}{\alpha+\gamma}\vec{r_A} + \frac{\gamma(1-\mu)}{\alpha+\gamma}\vec{r_C}$$

$$\Rightarrow \left(\lambda - \frac{\alpha(1-\mu)}{\alpha+\gamma}\right)\vec{r_A} + \left(\frac{\beta(1-\lambda)}{\beta+\gamma} - \mu\right)\vec{r_B} +$$

$$+ \left(\frac{\gamma(1-\lambda)}{\beta+\gamma} - \frac{\gamma(1-\mu)}{\alpha+\gamma}\right)\cdot\vec{r_C} = \vec{0}$$

$\triangle ABC$ is non-degenerate $\Rightarrow$ $\vec{AB}$ and $\vec{AC}$ are linearly independent, $\vec{v} := \vec{AB}$, $\vec{w} = \vec{AC}$

$\Rightarrow \vec{r_B} = \vec{r_A} + \vec{v}$, $\vec{r_C} = \vec{r_A} + \vec{w}$

We perform the substitutions:

$$\left(\lambda - \frac{\alpha(1-\mu)}{\alpha+\gamma}\right)\vec{r_A} + \left(\frac{\beta(1-\lambda)}{\beta+\gamma} - \mu\right)\cdot\left(\vec{r_A}+\vec{u}\right) +$$

$$+\left(\frac{\gamma(1-\lambda)}{\beta+\gamma} - \frac{\gamma(1-\mu)}{\alpha+\gamma}\right)\cdot\left(\vec{r_A}+\vec{w}\right) = \vec{0}$$

$$\Rightarrow \left(\lambda - \frac{\alpha(1-\mu)}{\alpha+\gamma} + \frac{\beta(1-\lambda)}{\beta+\gamma} - \mu + \frac{\gamma(1-\lambda)}{\beta+\gamma} - \frac{\gamma(1-\mu)}{\alpha+\gamma}\right)\cdot\vec{r_A} + \left(\frac{\beta(1-\lambda)}{\beta+\gamma} - \mu\right)\cdot\vec{u} +$$

$$+\left(\frac{\gamma(1-\lambda)}{\beta+\gamma} - \frac{\gamma(1-\mu)}{\alpha+\gamma}\right)\cdot\vec{w} = \vec{0}$$

$$\Rightarrow \left(\frac{\beta(1-\lambda)}{\beta+\gamma} - \mu\right)\cdot\vec{u} + \left(\frac{\gamma(1-\lambda)}{\beta+\gamma} - \frac{\gamma(1-\mu)}{\alpha+\gamma}\right)\cdot\vec{w} = \vec{0}$$

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$$\Rightarrow \begin{cases} \dfrac{\beta(1-\lambda)}{\beta+\gamma} = \mu \\[2ex] \dfrac{\gamma(1-\lambda)}{\beta+\gamma} - \dfrac{\gamma}{\alpha+\gamma}\cdot(1-\mu) = 0 \end{cases}$$

$$\Rightarrow \frac{\gamma(1-\lambda)}{\beta+\gamma} - \frac{\gamma}{\alpha+\gamma}\cdot\left(1 - \frac{\beta(1-\lambda)}{\beta+\gamma}\right) = 0$$

$$\Rightarrow \frac{\gamma}{\beta+\gamma} - \frac{\lambda\gamma}{\beta+\gamma} - \frac{\gamma}{\alpha+\gamma} +$$

$$+ \frac{\beta\gamma}{(\alpha+\gamma)(\beta+\gamma)} - \frac{\beta\gamma\lambda}{(\alpha+\gamma)(\beta+\gamma)} = 0$$

$(\alpha+\gamma)(\beta+\gamma)\,/$

$$\Rightarrow \lambda = \frac{\dfrac{\gamma}{\beta+\gamma} - \dfrac{\gamma}{\alpha+\gamma} + \dfrac{\beta\gamma}{(\alpha+\gamma)(\beta+\gamma)}}{\dfrac{\gamma}{\beta+\gamma} + \dfrac{\beta\gamma}{(\alpha+\gamma)(\beta+\gamma)}} \quad \backslash\,\gamma$$

$$\Rightarrow \lambda = \frac{\alpha+\gamma-\beta-\gamma+\beta}{(\alpha+\gamma)+\beta} = \frac{\alpha}{\alpha+\beta+\gamma}$$

$$\vec{r}_O = \lambda\vec{r}_A + \frac{\beta(1-\lambda)}{\beta+\gamma}\vec{r}_B + \frac{\gamma(1-\lambda)}{\beta+\gamma}\vec{r}_C$$

$$\Rightarrow \vec{r}_O = \frac{\alpha}{\alpha+\beta+\gamma}\vec{r}_A + \frac{\beta}{\alpha+\beta+\gamma}\vec{r}_B + \frac{\gamma}{\alpha+\beta+\gamma}\vec{r}_C$$

$$1-\lambda = 1-\frac{\alpha}{\alpha+\beta+\gamma} = \frac{\beta+\gamma}{\alpha+\beta+\gamma}$$

$$\frac{\beta}{\beta+\gamma}(1-\lambda) = \frac{\beta}{\beta+\gamma}\cdot\frac{\beta+\gamma}{\alpha+\beta+\gamma} = \frac{\beta}{\alpha+\beta+\gamma}$$

2.1. $\widehat{BOB'}$ nonzero angle

$A \in [OB]$, $A' \in [OB']$

$\vec{OA} =: \vec{v}$ $\quad \vec{OA'} = \vec{w}$

$\vec{OB} = m \cdot \vec{v}$ $\quad \vec{OB'} = n \cdot \vec{w}$

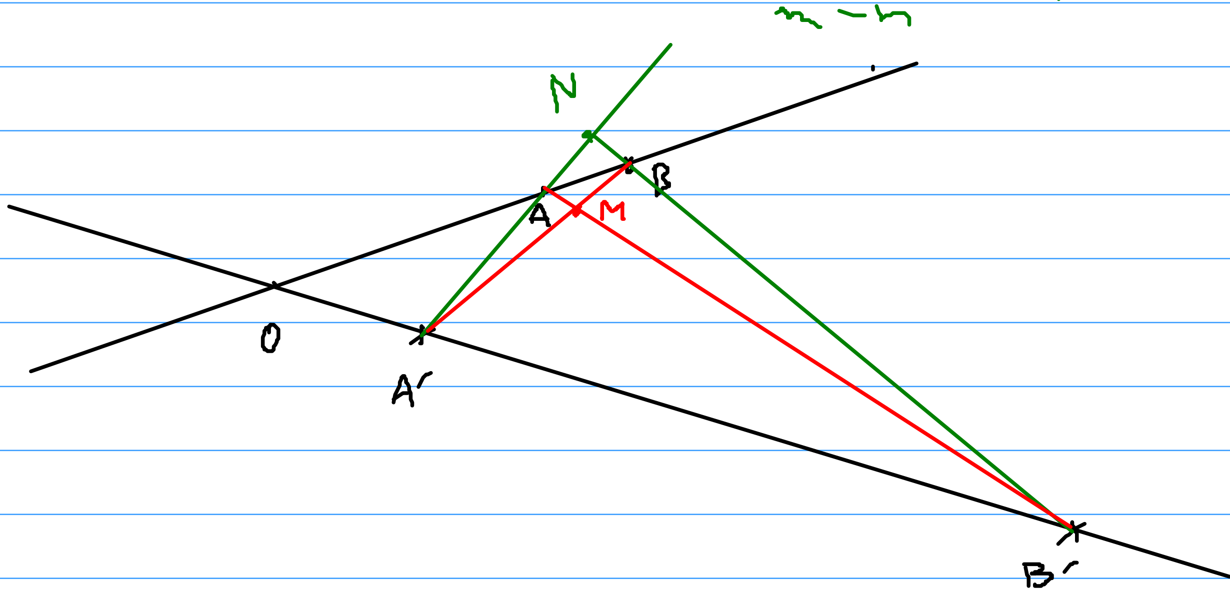
$AA' \cap BB' = \{N\}$

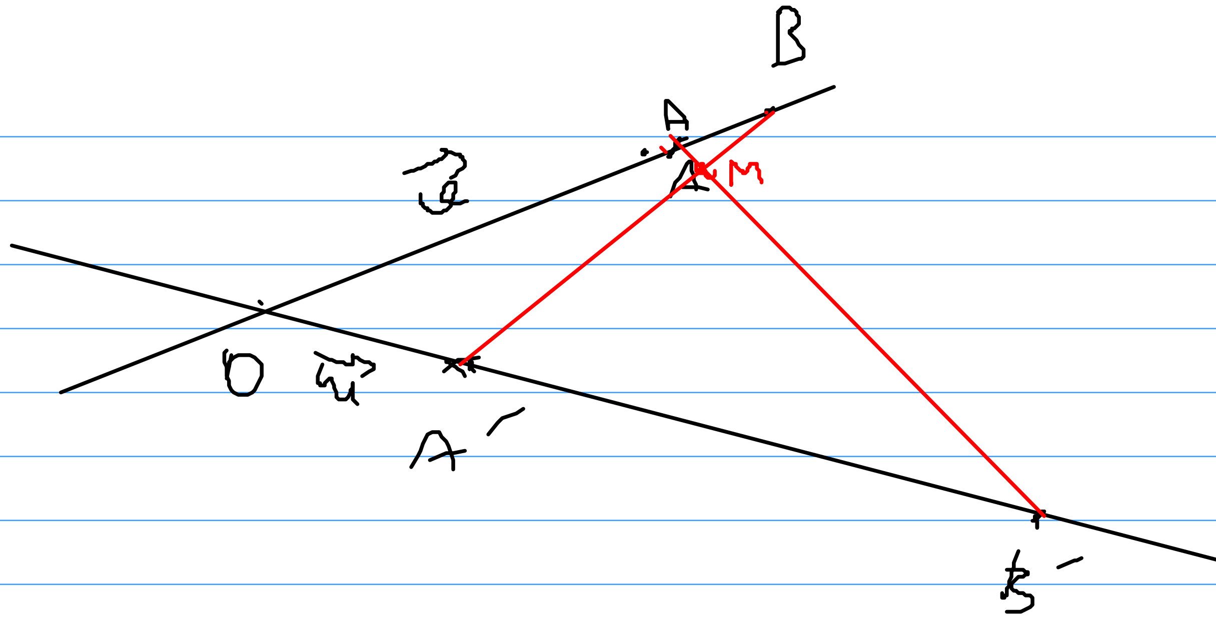
$AB' \cap BA' = \{M\}$

Show that

$$\vec{OM} = m \frac{1-n}{1-mn} \vec{OA} + n \cdot \frac{1-m}{1-mn} \vec{OA'}$$

$$\vec{ON} = m \frac{n-1}{n-m} \vec{OA} + n \cdot \frac{m-1}{m-n} \vec{OA'}$$





$$\vec{OM} = \lambda \vec{OA} + (1-\lambda)\vec{OB'} =$$
$$= \mu \cdot \vec{OA'} + (1-\mu)\vec{OB}$$

$$\Rightarrow \vec{OM} = \lambda \vec{w} + (1-\lambda)\cdot n \cdot \vec{w}$$
$$= \mu \cdot m \cdot \vec{u} + (1-\mu)\cdot \vec{w}$$
$$\Rightarrow (\lambda - \mu m)\cdot \vec{u} + ((1-\lambda)n - 1 + \mu)\cdot \vec{w} = \vec{0}$$

$\widehat{BOB'}$ nonzero $\Rightarrow$ $\vec{u}, \vec{w}$ lin. indep.

$$\Rightarrow \begin{cases} \lambda - \mu m = 0 \\ (1-\lambda)n - 1 + \mu = 0 \end{cases}$$

$$\Rightarrow \quad \lambda = \mu m$$

$$\Rightarrow (1-\mu m)n - 1 + \mu = 0$$

$$\Rightarrow \quad \mu(-mn+1) + n - 1 = 0$$

$$\Rightarrow \quad \mu = \frac{1-n}{1-mn} \Rightarrow 1-\mu = \frac{n-mn}{1-mn}$$

$$\vec{OM} = \mu \cdot m \cdot \vec{u} + (1-\mu) \cdot \vec{w} =$$

$$= \frac{m(1-n)}{1-mn} \cdot \vec{u} + \frac{n(1-m)}{1-mn} \cdot \vec{w}$$

Same thing for $\vec{ON}$