Seminar W13 - 9.5

The projective plane:

$$\mathbb{RP}^2 = \mathbb{P}\mathbb{R}^2 = \mathbb{P}(\mathbb{R}^2) = \mathbb{P}^2(\mathbb{R}^2)$$

$$\mathbb{RP}^2 = \{ [x : y : z] \mid x, y, z \in \mathbb{R},\ (x,y,z) \neq (0,0,0) \}$$

homogeneous vector (projective)

$[x : y : z] = [\lambda x : \lambda y : \lambda z], \forall \lambda \in \mathbb{R} \setminus \{0\}$

Ex: $[7 : 8 : 9] = [1 : \frac{8}{7} : \frac{9}{7}] = [\frac{7}{9} : \frac{8}{9} : 1]$

$$\mathbb{RP}^2 = \frac{\mathbb{R}^3 \setminus \{0\}}{\sim}$$ (something like this can be defined for a vector space)

where $(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \iff$

$\iff \exists \lambda \in \mathbb{R} \setminus \{0\} : (x_2, y_2, z_2) = \lambda (x_1, y_1, z_1)$

$\mathbb{RP}^2$ = the lines in $\mathbb{R}^3$ that contain the origin $O\ (0,0,0)$

$$\underbrace{\mathbb{RP}^2}_{\text{the projective plane}} = \underbrace{\mathbb{RA}^2}_{\text{the affine plane}} \cup \underbrace{\mathbb{R}\infty}_{\text{the line at infinity}}$$

$$\mathbb{RA}^2 = \{ [x:y:z] \in \mathbb{RP}^2 \mid z \neq 0 \} =$$

$$= \{ [\underbrace{\tfrac{x}{z}}_{X} : \underbrace{\tfrac{y}{z}}_{Y} : 1] \in \mathbb{RP}^2 \mid z \neq 0 \} =$$

$$= \{ [X : Y : 1] \in \mathbb{RP}^2 \mid X, Y \in \mathbb{R} \}$$

$$\mathbb{RA}^2 \longrightarrow \mathbb{R}^2 \quad \text{bijective}$$

$$[x:y:z] \longmapsto \left(\frac{x}{z}, \frac{y}{z} \right)$$

$$\mathbb{R}\infty = \{ [x:y:z] \in \mathbb{RP}^2 \mid z = 0 \} =$$

$$= \{ [x:y:0] \mid x, y \in \mathbb{R},\ (x,y) \neq (0,0) \}$$

→ the projective point $[x : y : 0]$ corresponds to all the parallel lines in $\mathbb{R}^2$ that have the director vector (x, y)

Why we care:

φ_1, φ_2 affine transformations

$$\varphi_1\begin{pmatrix} x \\ y \end{pmatrix} = M_1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_1$$

$$\varphi_2\begin{pmatrix} x \\ y \end{pmatrix} = M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_2$$

$$(\varphi_1 \circ \varphi_2)\begin{pmatrix} x \\ y \end{pmatrix} = \varphi_1\left(M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_2 \right) =$$

$$= M_1 \left(M_2 \begin{pmatrix} x \\ y \end{pmatrix} + v_2 \right) + v_1 =$$

$$= M_1 M_2 \begin{pmatrix} x \\ y \end{pmatrix} + M_1 v_2 + v_1$$

Instead of defining an affine transformation φ as:

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

we pretend that φ is a projective transformation:

$$\varphi\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M} & \begin{array}{c} x_0 \\ y_0 \end{array} \\ 0 & 0 & 1 \end{array}\right) \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} ax + by + x_0 \\ cx + dy + y_0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \varphi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by + x_0 \\ cx + dy + y_0 \end{pmatrix}$$

$$[\varphi \circ \psi] = [\varphi] \cdot [\psi]$$

13.7. Find the concatenation (product, composition) of an anticlockwise rotation about the origin through an angle of $\frac{3\pi}{2}$, followed by a scaling by a factor of 3 in the x-direction and 2 in the y-direction.

$$[S(3,2)] = \left(\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$\left[R_{\frac{3\pi}{2}}\right] = \begin{pmatrix} \cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2} & 0 \\ \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left[S(3,2) \circ R_{\frac{3\pi}{2}}\right] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(S(3,2)\circ R_{\frac{3\pi}{2}}\right)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0&3\\-2&0\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}=$$

$$=\begin{pmatrix}3y\\-2x\end{pmatrix}$$

$$\left(S(3,2)\circ R_{\frac{3\pi}{2}}\right)\begin{bmatrix}x\\y\\1\end{bmatrix}=\begin{pmatrix}0&3&0\\-2&0&0\\0&0&1\end{pmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix}=$$

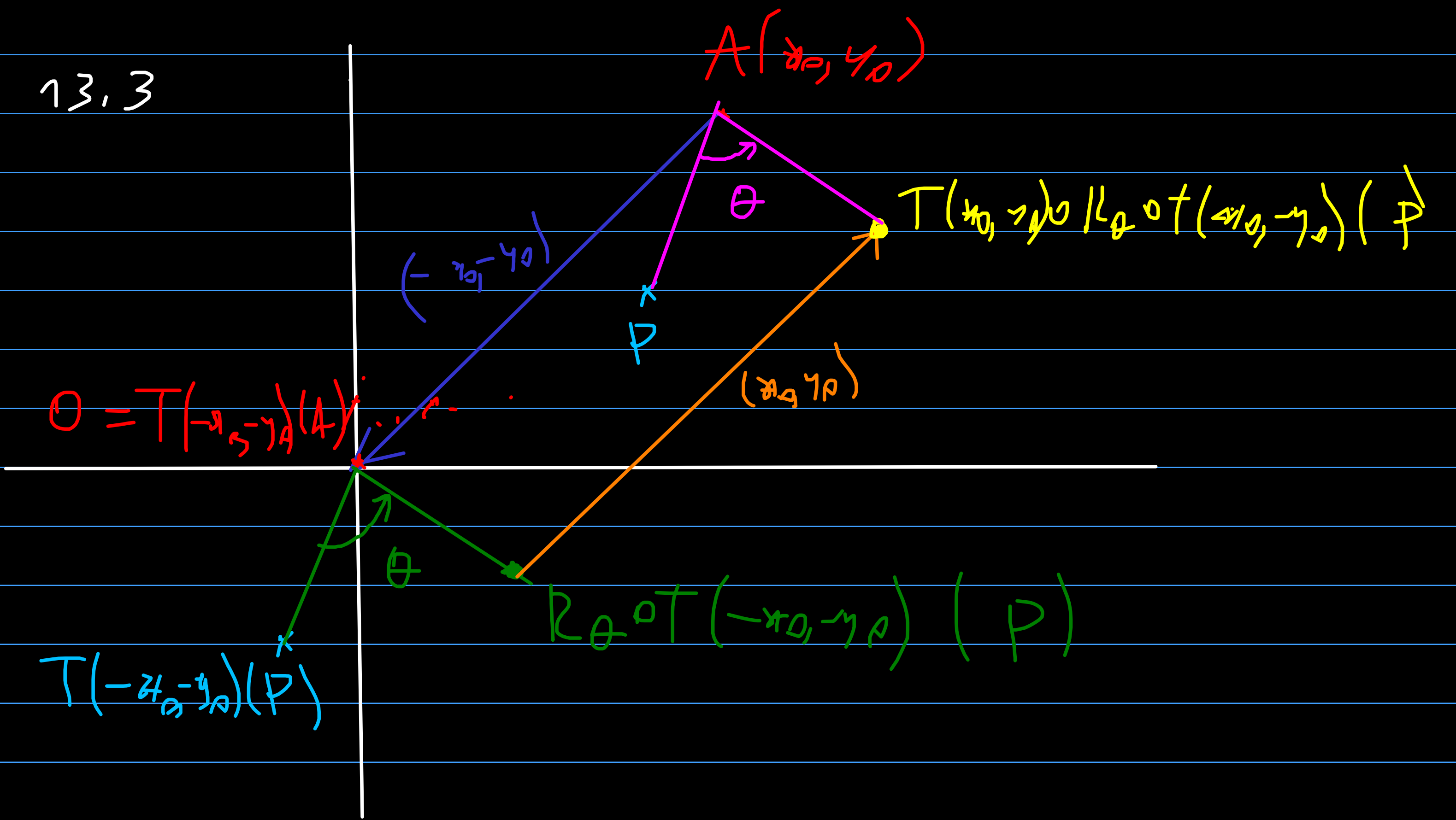
$$=\begin{bmatrix}3y\\-2x\\1\end{bmatrix}$$

$\psi : \mathbb{RP}^2 \to \mathbb{RP}^2$ <u>projective transformation</u>

$$\psi\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}$$

Such a projective transformation is called **affine** if $a_{31}=a_{32}=0$ and $a_{33}\neq 0$

13.3



$$R_\theta(A) = R_\theta(x_0,y_0) = T(x_0,y_0)\circ R_\theta \circ T(-x_0,-y_0)$$

$$[R_\theta(x_0,y_0)] = \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_0 = -x_0\cos\theta + y_0\sin\theta + x_0$$

$$\beta_0 = -x_0\sin\theta - y_0\cos\theta + y_0$$

13. * Let ℓ_1, ℓ_2 be parallel lines ($\ell_1 \parallel \ell_2$)

Show that $r_{\ell_1} \circ r_{\ell_2}$ is a translation.

$\ell_1: \; ax + by + c_1 = 0$

$\ell_2: \; ax + by + c_2 = 0$

$$[r_{\ell_1}] = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} & \frac{-2ac_1}{a^2+b^2} \\ \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & \frac{-2bc_1}{a^2+b^2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$[r_{\ell_1}] = \begin{pmatrix} b^2-a^2 & -2ab & -2ac_1 \\ -2ab & a^2-b^2 & -2bc_1 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

$$[r_{\ell_2}] = \begin{pmatrix} b^2-a^2 & -2ab & -2ac_2 \\ -2ab & a^2-b^2 & -2bc_2 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

$$[r_{l_1} \circ r_{l_2}] = \begin{pmatrix} b^2-a^2 & -2ab & -2ac_1 \\ -2ab & a^2-b^2 & -2bc_1 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \cdot$$

$$\begin{pmatrix} b^2-a^2 & -2ab & -2ac_2 \\ -2ab & a^2-b^2 & -2bc_2 \\ 0 & 0 & a^2+b^2 \end{pmatrix} =: M$$

$$m_{11} = (b^2-a^2)^2 + (-2ab)^2 = (a^2+b^2)^2$$

$$m_{12} = -2ab(b^2-a^2) - 2ab(a^2-b^2) = 0$$

$$m_{13} = -2ac_2\,(b^2-a^2) + 4\,ab^2c_2 -$$

$$-2ac_1\,(a^2+b^2) =$$

$$= 2ac_2\,(2b^2-b^2+a^2) - 2ac_1\,(a^2+b^2) =$$

$$= 2ac_2\,(a^2+b^2) - 2ac_1\,(a^2+b^2) =$$

$$= 2a\,(a^2+b^2)\cdot(c_2-c_1)$$

$$m_{21} = -2ab\,(b^2-a^2) - 2ab\,(a^2-b^2) = 0$$

$$m_{22} = 4a^2b^2 + (a^2-b^2)^2 = (a^2+b^2)^2$$

$$m_{23} = 4a^2bc_2 - 2bc_2\,(a^2-b^2) - 2bc_1\,(a^2+b^2)$$

$$m_{23} = 2bc_2\left(2a^2 - a^2 + b^2\right) - 2bc_1\left(a^2+b^2\right)$$

$$= 2b\left(a^2+b^2\right)\cdot\left(c_2 - c_1\right)$$

$$m_{31} = 0 \qquad m_{32} = 0 \qquad m_{33} = \left(a^2+b^2\right)^2$$

$$\Rightarrow \left[r_{l_1} \circ r_{l_2}\right] = \begin{pmatrix} \left(a^2+b^2\right)^2 & 0 & 2a\left(a^2+b^2\right)\left(c_2-c_1\right) \\ 0 & \left(a^2+b^2\right)^2 & 2b\left(a^2+b^2\right)\left(c_2-c_1\right) \\ 0 & 0 & \left(a^2+b^2\right)^2 \end{pmatrix}$$

$$\left[r_{l_1} \circ r_{l_2}\right] = \begin{pmatrix} 1 & 0 & \frac{2a}{a^2+b^2}\left(c_2-c_1\right) \\ 0 & 1 & \frac{2b}{a^2+b^2}\left(c_2-c_1\right) \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow r_{l_1} \circ r_{l_2} = T\left(\frac{2a}{a^2+b^2}\left(c_2-c_1\right), \frac{2b}{a^2+b^2}\left(c_2-c_1\right)\right)$$

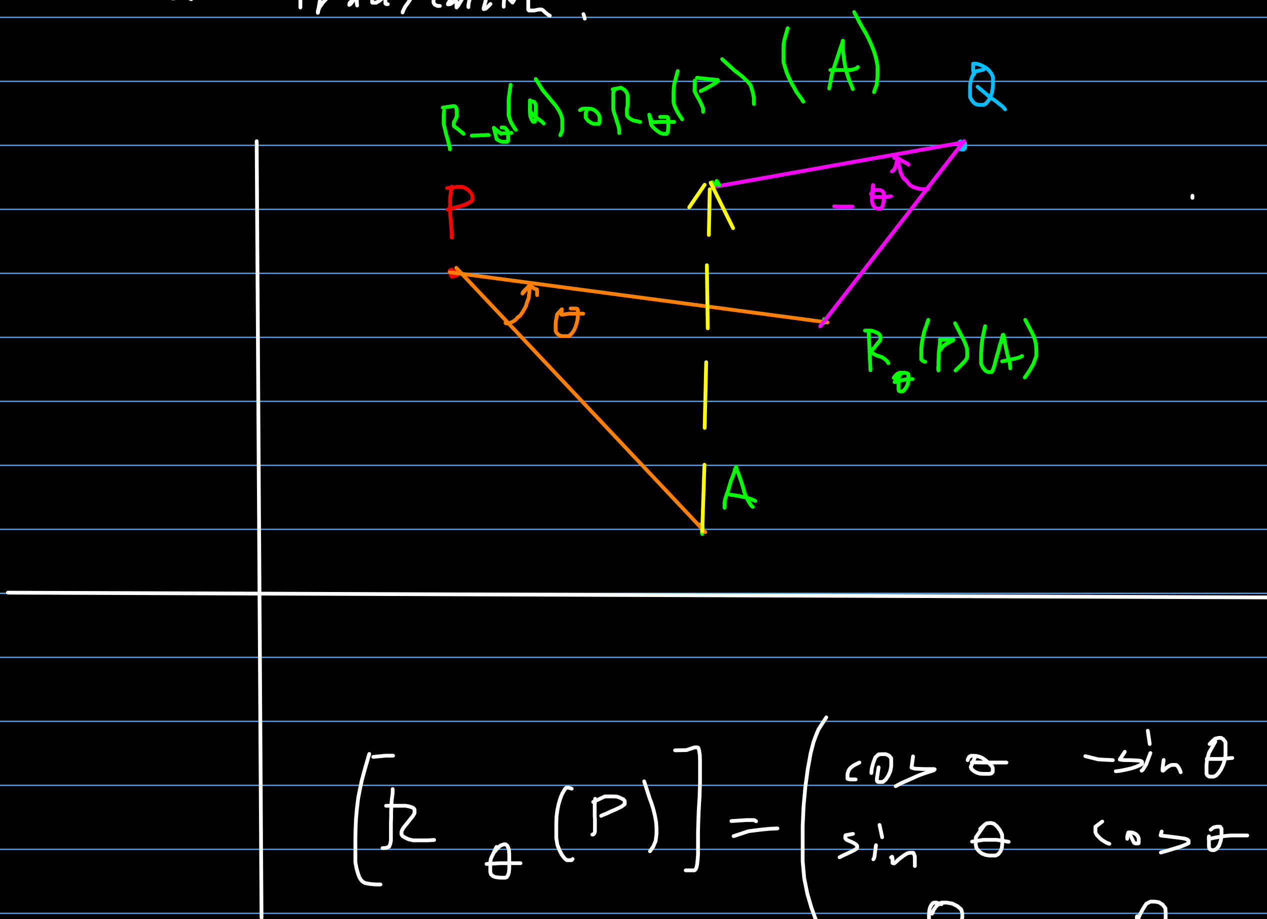
$$\left(r_{l_1} \circ r_{l_2}\right)\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + \frac{2a}{a^2+b^2}\left(c_2-c_1\right) \\ y + \frac{2b}{a^2+b^2}\left(c_2-c_1\right) \\ 1 \end{bmatrix}$$

$$(r_{\ell_1} \circ r_{\ell_2})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \frac{2a}{a^2+b^2}(c_2 - c_1) \\ y + \frac{2b}{a^2+b^2}(c_2 - c_1) \end{pmatrix}$$

13.4. $P(x_0, y_0)$, $Q(x_1, y_1)$, $P \neq Q$

$\theta \in \mathbb{R}$

Show that $R_{-\theta}(Q) \circ R_{\theta}(P)$ is a translation.



$$[R_{\theta}(P)] = \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[R_{-\theta}(Q)] = \begin{pmatrix} \cos\theta & \sin\theta & \alpha_1 \\ -\sin\theta & \cos\theta & \beta_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[R_{-\theta}(Q) \circ R_\theta(P)] = M$$

$$m_{11} = \cos^2\theta + \sin^2\theta = 1$$

$$m_{12} = -\cos\theta\sin\theta + \cos\theta\sin\theta = 0$$

$$m_{21} = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$$

$$m_{22} = \cos^2\theta + \sin^2\theta = 1$$

$$m_{31} = m_{32} = 0 \qquad m_{33} = 1$$

$$\Rightarrow [R_{-\theta}(Q) \circ R_\theta(P)] =$$

$$= \begin{pmatrix} 1 & 0 & m_{13} \\ 0 & 1 & m_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow R_{-\theta}(Q) \circ R_\theta(P) = T(m_{13}, m_{23})$$