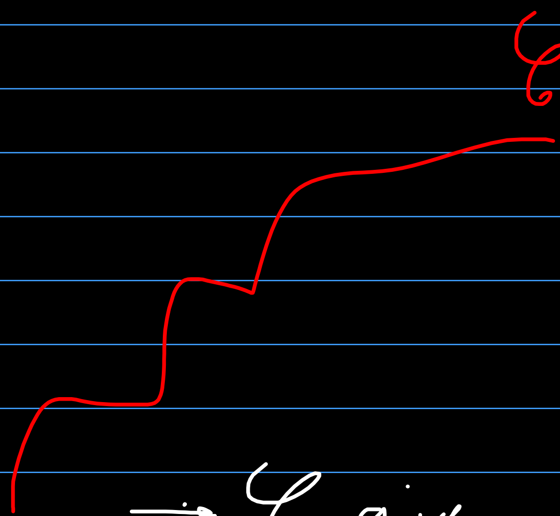
Seminar W8 – 917

Curves and surfaces



→ $\mathcal{C}$ given parametrically:

$$\mathcal{C}: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

→ $\mathcal{C}$ given implicitly:

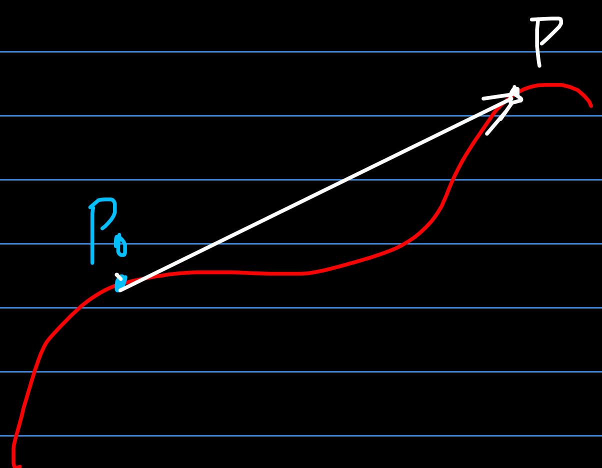
- if $\mathcal{C}$ planar (in 2D)

$$f(x, y) = 0$$

- if $\mathcal{C}$ spatial (in 3D)

$$\begin{cases} f_1(x, y, z) = 0 \\ f_2(x, y, z) = 0 \end{cases}$$

The tangent line to the curve $\mathscr{C}$ at the point P_0 is a line that contains the point P_0, whose direction is given by:



$$\vec{u} = \lim_{\substack{P \to P_0 \\ P \in \mathscr{C}}} \frac{\overrightarrow{P_0P}}{\|\overrightarrow{P_0P}\|}$$

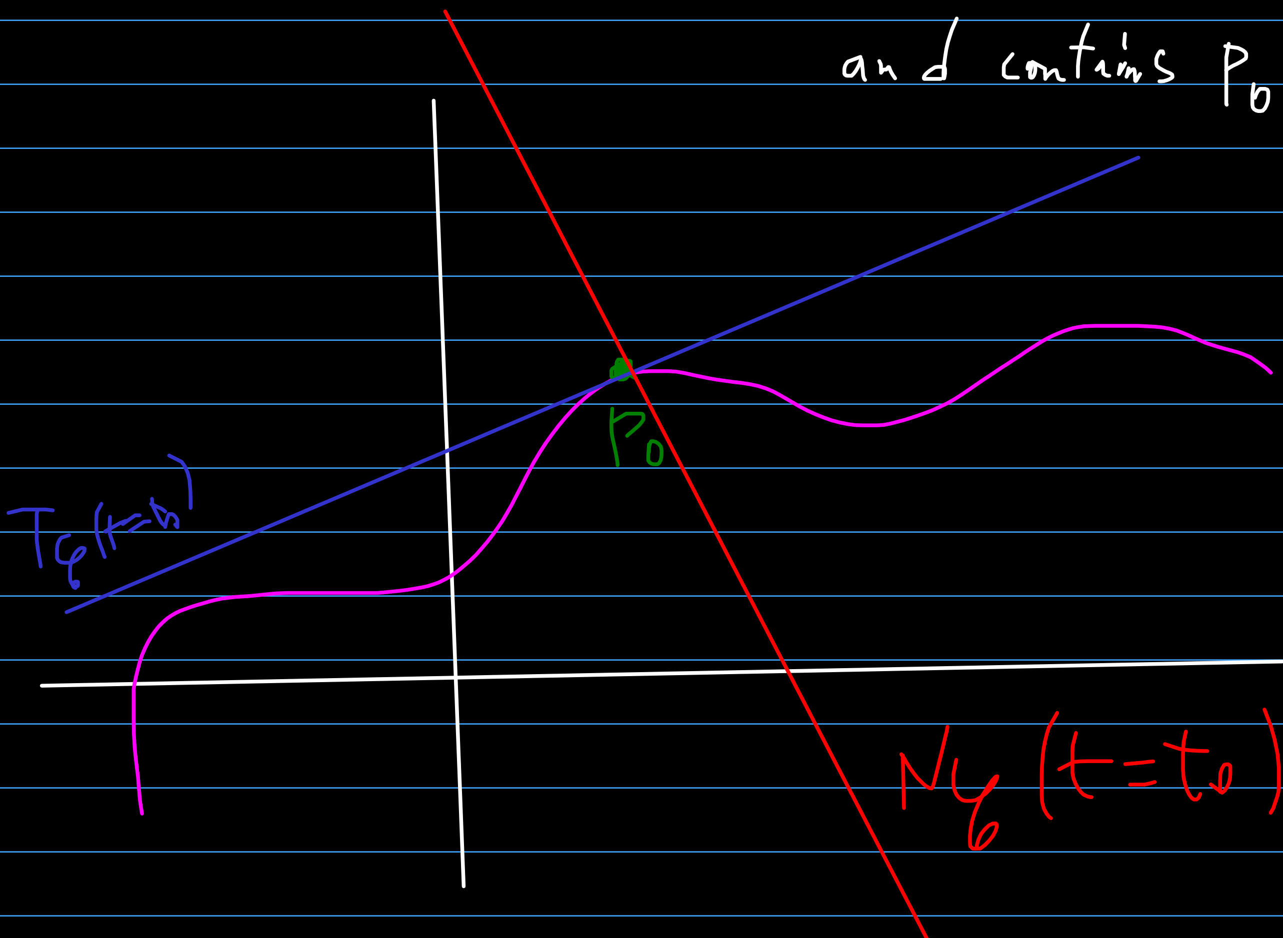
· if $\mathscr{C}$ is given parametrically:

$$\mathscr{C}: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

$$T_{\mathscr{C}}(t = t_0): \quad \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}$$

→ if $\mathcal{C}$ is planar (in 2D)

normal line to $\mathcal{C}$ at $t=t_0$ = the line that is perp. to $T_{\mathcal{C}}(t=t_0)$ and contains $P_0(t=t_0)$

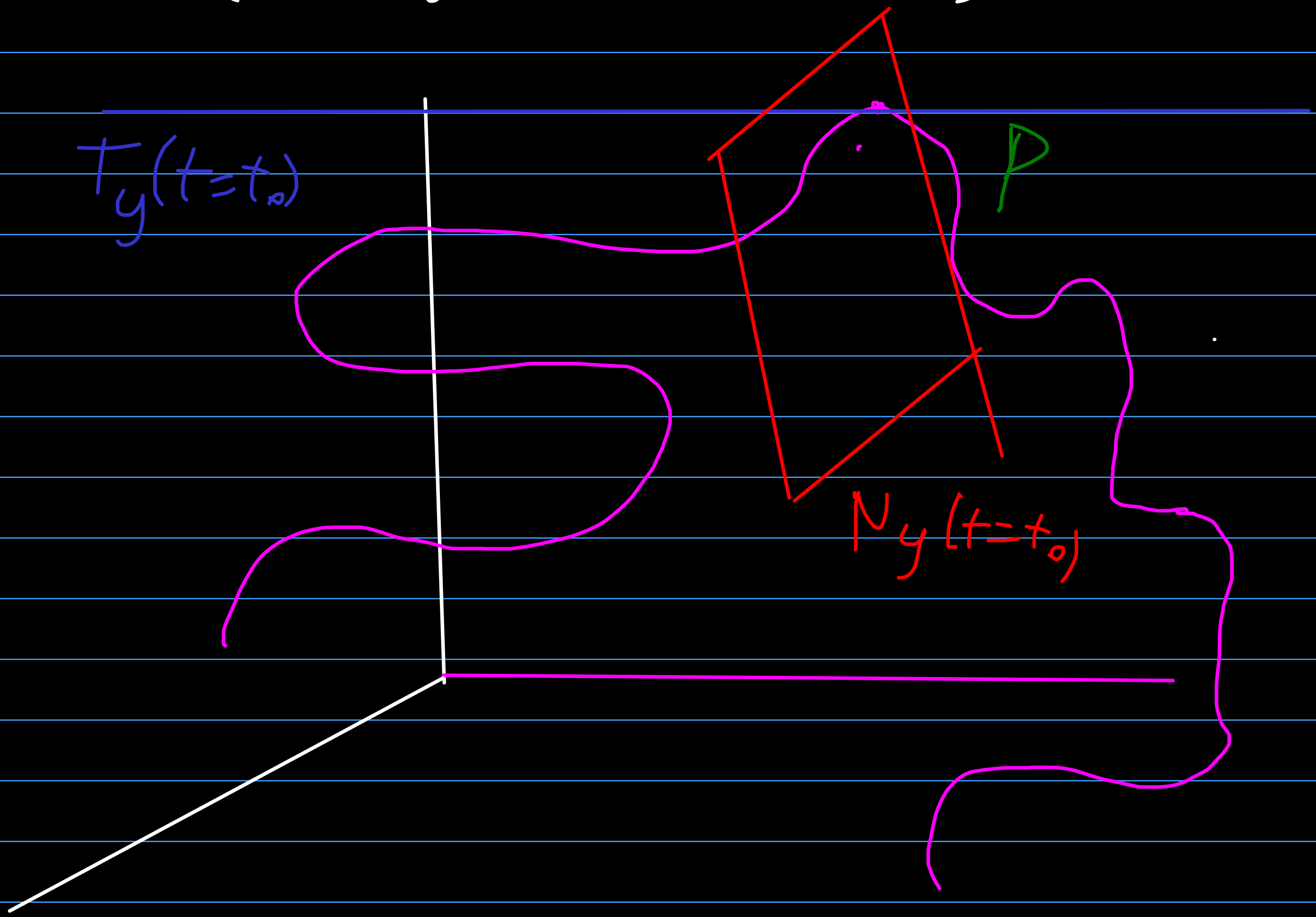


→ if $\mathcal{C}$ is spatial (in 3D)

$$N_{\mathcal{C}}(t=t_0): x'(t_0)\cdot(x-x(t_0))+y'(t_0)\cdot(y-y(t_0))=0$$

the normal plane to $\mathcal{C}$ =
at $t = t_0$

= the plane that is perp to $T_{\mathcal{C}}(t = t_0)$
and contains $P_0(t = t_0)$



$N_y(t = t_0)$:

$x'(t_0) \cdot (x - x(t_0)) + y'(t_0) \cdot (y - y(t_0)) +$
$+ z'(t_0) \cdot (z - z(t_0)) = 0$

- If $\mathcal{C}$ is given implicitly (and is planar):

$$f(x,y)=0$$

$$T_{\mathcal{C}}(x_0,y_0): f'_x(x_0,y_0)\cdot(x-x_0)+$$
$$+f'_y(x_0,y_0)\cdot(y-y_0)=0$$

$$N_{\mathcal{C}}(x_0,y_0): \frac{x-x_0}{f'_x(x_0,y_0)}=\frac{y-y_0}{f'_y(x_0,y_0)}$$

8.1. Show that the angle between the tangent of the circular helix

$$\mathcal{C}: \begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases}$$

and the z-axis is constant.

$$\vec{u}_{Oz} : (0, 0, 1)$$

$$x'(t) = -a\sin t$$

$$y'(t) = a\cos t$$

$$z'(t) = b$$

$$\vec{u}_{T_{\mathcal{C}}(t=t_0)} (-a\sin t_0, a\cos t_0, b)$$

$$\vec{u}_{T_{\mathcal{C}}} \cdot \vec{v}_{Oz} = 0\cdot(-a\sin t_0) + 0\cdot(a\cos t_0) + 1\cdot b =$$

$$m\left(\overbrace{T_{\mathcal{C}(t=t_0)}, O_z}^{=b}\right) =$$

$$= \arccos \frac{\vec{v}_{T_{\mathcal{C}}} \cdot \vec{v}_{O_z}}{\|\vec{v}_{T_{\mathcal{C}}}\| \cdot \|\vec{v}_{O_z}\|} =$$

$$= \arccos \frac{b}{\sqrt{a^2 \sin^2 t_0 + a^2 \cos^2 t_0 + b^2}} =$$

$$= \arccos \frac{b}{\sqrt{a^2+b^2}}$$ does not

depend on t_0 $\Rightarrow$ q.e.d.

8.8. Write the equations of the tangent line and the normal plane for the following curve

$$\begin{cases} x = e^t \cos 3t \\ y = e^t \sin 3t \\ z = e^{-2t} \end{cases}$$

at $t=0$.

$x'(t) = e^t \cos 3t - 3e^t \sin 3t$

$y'(t) = e^t \sin 3t + 3e^t \cos 3t$

$z'(t) = -2e^{-2t}$

$x(0) = 1, \quad y(0) = 0, \quad z(0) = 1$

$x'(0) = 1, \quad y'(0) = 3, \quad z'(0) = -2$

$$T_{\mathscr{C}}(t=0): \quad \frac{x-1}{1} = \frac{y}{3} = \frac{z-1}{-2}$$

$N_{\mathcal{C}}(t=0): \ 1\cdot(x-1)+3\cdot y+(-2)\cdot(z-1)$
$=0.$

8.? Write the equation of the tangent line and the normal line at the point $P_0(0,-1)$ of the curve

$$\mathcal{C}: \ x^4-x^3y+y^2-x-1=0$$

$$f(x,y)=x^4-x^3y+y^2-x-1$$

$$\frac{\partial f}{\partial x}(x,y)=4x^3-3x^2y-1$$

$$\frac{\partial f}{\partial y}(x,y)=-x^3+2y$$

$$T_{\mathcal{C}}(0,-1): \frac{\partial f}{\partial x}(0,-1)\cdot x + \frac{\partial f}{\partial y}(0,-1)\cdot$$

$$\cdot (y+1) = 0$$

$$\frac{\partial f}{\partial x}(0,-1) = -1 \quad , \quad \frac{\partial f}{\partial y}(0,-1) = -2$$

$$\Rightarrow T_{\mathcal{C}}(0,-1): \; -x - 2(y+1) = 0$$

$$\Rightarrow N_{\mathcal{C}}(0,-1): \; \frac{x}{-1} = \frac{y+1}{-2}$$

Surfaces

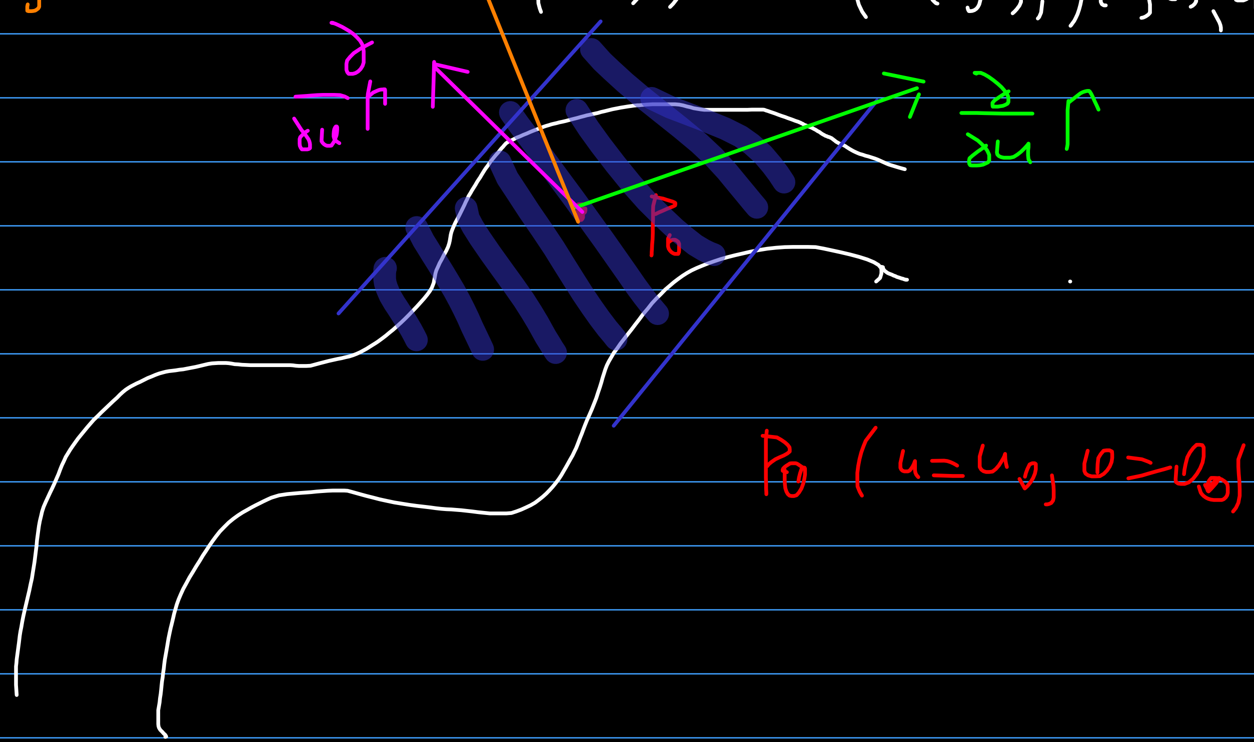
- If S is given parametrically

$$S : \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

$$r : \mathbb{R}^2 \to \mathbb{R}^3$$

$$(u,v) \mapsto (x(u,v), y(u,v), z(u,v))$$

$N_S(u=u_0, v=v_0)$



$$T_y\,(u=u_0,\ v=v_0):$$

$$\begin{vmatrix} x-x(u_0,v_0) & y-y(u_0,v_0) & z-z(u_0,v_0) \\ \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \\ \frac{\partial x}{\partial v}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0) \end{vmatrix} = 0$$

$$\Leftrightarrow \frac{\partial(y,z)}{\partial(u,v)}(u_0,v_0)\cdot\big(x-x(u_0,v_0)\big)+$$

$$+\frac{\partial(z,x)}{\partial(u,v)}(u_0,v_0)\cdot\big(y-y(u_0,v_0)\big)+$$

$$+\frac{\partial(x,y)}{\partial(u,v)}(u_0,v_0)\cdot\big(z-z(u_0,v_0)\big)=0$$

$$N_y(u_0,v_0):\ \frac{x-x(u_0,v_0)}{\frac{\partial(y,z)}{\partial(u,v)}(u_0,v_0)}=\frac{y-y(u_0,v_0)}{\frac{\partial(z,x)}{\partial(u,v)}(u_0,v_0)}=\frac{z-z(u_0,v_0)}{\frac{\partial(x,y)}{\partial(u,v)}(u_0,v_0)}$$

· if γ is given implicitly:

$$f(x,y,z)=0$$

$T_\gamma(x_0, y_0, z_0)$:

$$f'_x(x_0,y_0,z_0)\cdot(x-x_0)+f'_y(x_0,y_0,z_0)\cdot(y-y_0)+$$

$$+f'_z(x_0,y_0,z_0)\cdot(z-z_0)=0$$

$N_\gamma(x_0, y_0, z_0)$:

$$\frac{x-x_0}{f'_x(x_0,y_0,z_0)}=\frac{y-y_0}{f'_y(x_0,y_0,z_0)}=\frac{z-z_0}{f'_z(x_0,y_0,z_0)}$$

8.9. Write the equations of the tangent planes of the hyperboloid of one sheet

$$x^2+y^2-z^2=1$$

at the points of the form $(x_0, y_0, 0)$ and show that they are parallel to the z-axis.

$$f(x,y,z) = x^2+y^2-z^2-1$$

$$f'_x(x,y,z) = 2x$$

$$f'_y(x,y,z) = 2y$$

$$f'_z(x,y,z) = -2z$$

$$T_{\mathcal{G}}(x_0, y_0, 0): \quad 2x_0(x-x_0)+2y_0(y-y_0)=0$$

$$\vec{u}_{Oz}(0,0,1) \quad , \quad \vec{n}_{T_{\mathcal{G}}(x_0,y_0,0)}(2x_0, 2y_0, 0)$$

$$\vec{n}_{T_{\mathcal{Y}}(x_0, y_0, 0)} \cdot \vec{v}_{Oz} = 0$$

$$\Rightarrow \vec{v}_{Oz} \perp \vec{n}_{T_{\mathcal{Y}}(x_0, y_0, 0)} \Rightarrow$$

$$\Rightarrow Oz \parallel T_{\mathcal{Y}}(x_0, y_0, 0)$$

8.?? We consider the following surface:

$$\mathcal{Y}: \begin{cases} x = \cos u \cdot \cos v \\ y = \cos u \cdot \sin v \\ z = \sin u \end{cases}$$

$$v \in [0, 2\pi), \quad u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Find the tangent plane in the point corresponding to $v = 0$, $u = 0$ (also the normal line)

$x(0,0)=1$, $y(0,0)=0$, $z(0,0)=0$

$$\frac{\partial x}{\partial u} = -\cos v \sin u$$

$$\frac{\partial x}{\partial v} = -\cos u \sin v$$

$$\frac{\partial y}{\partial u} = -\sin v \cdot \sin u$$

$$\frac{\partial y}{\partial v} = \cos u \cdot \cos v$$

$$\frac{\partial z}{\partial u} = \cos u$$

$$\frac{\partial z}{\partial v} = 0$$

$$\begin{vmatrix} x-1 & y & z \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow -(x-1)$$

$$\Rightarrow x = 1$$