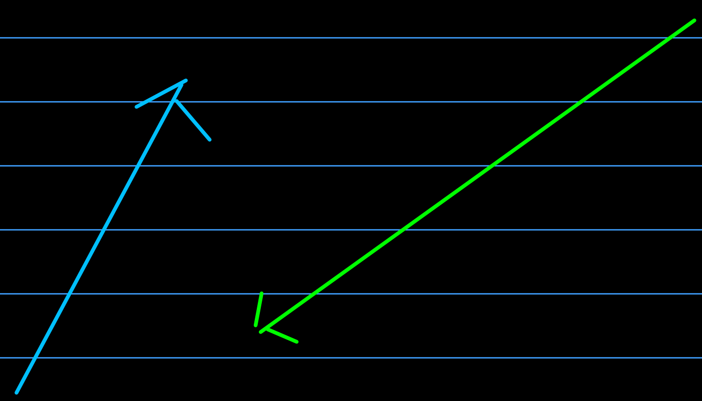
Seminar W5 — 973

Dot product (scalar product)

$\forall\, \vec{v}, \vec{w}$: $\quad \vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\vec{v}, \vec{w})$



$\Rightarrow \quad \alpha \in [0, \pi)$

If the reference system is orthonormal (for us, all the time), then we have

$$\vec{v} = (a_1, a_2, a_3), \quad \vec{w} = (b_1, b_2, b_3)$$

$$\vec{v} \cdot \vec{w} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\mathcal{R} = (O, [\vec{u}, \vec{v}, \vec{w}])$$

$\mathcal{R}$ orthogonal $\iff$ $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{u} = 0$

$\mathcal{R}$ orthonormal $\iff$ $\mathcal{R}$ orthogonal and $\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| = 1$

If $\vec{u}, \vec{v} \neq 0$:

$$\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$$

5.3 Find the angle between:

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(a) $d_1: \begin{cases} x+2y+z-1=0 \\ x-2y+z+1=0 \end{cases}$

$d_2: \begin{cases} x-y-z-1=0 \\ x-y+2z+1=0 \end{cases}$

(b) $\Pi_1: x+3y+2z+1=0$

$\Pi_2: 3x+2y-z=0$

(c) the plane xOy and the straight line M_1M_2, $M_1(1,2,3)$, $M_2(-2,1,4)$

(a) $d_1: \begin{cases} x+2y+z-1=0 \\ x-2y+z+1=0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x=1-z-2y \\ 1-z-2y-2y+z+1=0 \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} x = 1 - z - 2y \\ -4y + z = 0 \end{cases} \Leftrightarrow \begin{cases} y = \frac{1}{2} \\ x = -z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = -t \\ y = \frac{1}{2} \\ z = t \end{cases} \Rightarrow \vec{d_1}\,(-1, 0, 1)$$

$$d_2: \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 1 + y + z \\ 1 + y + z - y + 2z + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 1 + y + z \\ 2 + 3z = 0 \end{cases} \Leftrightarrow \begin{cases} z = -\frac{2}{3} \\ x = y + \frac{1}{3} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = t + \frac{1}{3} \\ y = t \\ z = -\frac{2}{3} \end{cases} \Rightarrow \vec{d_2}(1,1,0)$$

$\vec{d_1}(-1,0,1)$, $\vec{d_2}(1,1,0)$

$$\cos(\widehat{d_1, d_2}) = \frac{\vec{d_1}\cdot\vec{d_2}}{\|\vec{d_1}\|\cdot\|\vec{d_2}\|} =$$

$$= \frac{-1\cdot 1 + 0\cdot 1 + 1\cdot 0}{\sqrt{2}\cdot\sqrt{2}} = -\frac{1}{2}$$

$$\Rightarrow m(\widehat{d_1, d_2}) = \frac{2\pi}{3}$$

(4) $\pi_1: x + 3y + 2z + 1 = 0$

$\pi_2: 3x + 2y - z = 0$

$$\vec{n_{\pi_1}} = (1, 3, 2)$$

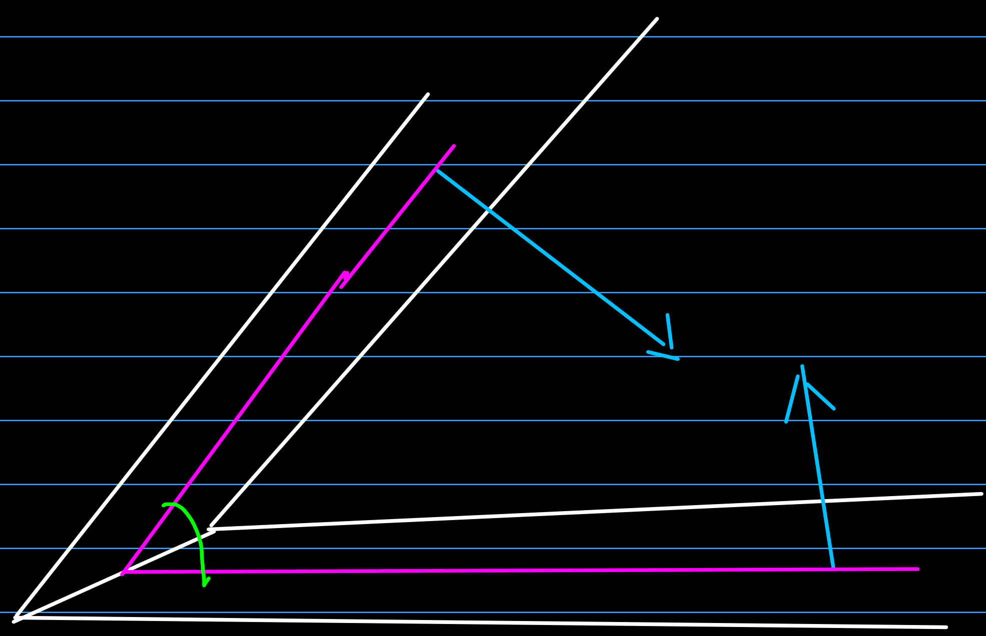
$$\vec{n_{\pi_2}} = (3, 2, -1)$$

$$\vec{n_{\pi_1}} \cdot \vec{n_{\pi_2}} = 1 \cdot 3 + 3 \cdot 2 + 2 \cdot (-1) =$$

$$= 7$$

$$\|\vec{n_{\pi_1}}\| = \sqrt{1+9+4} = \sqrt{14}$$

$$\|\vec{n_{\pi_2}}\| = \sqrt{9+4+1} = \sqrt{14}$$



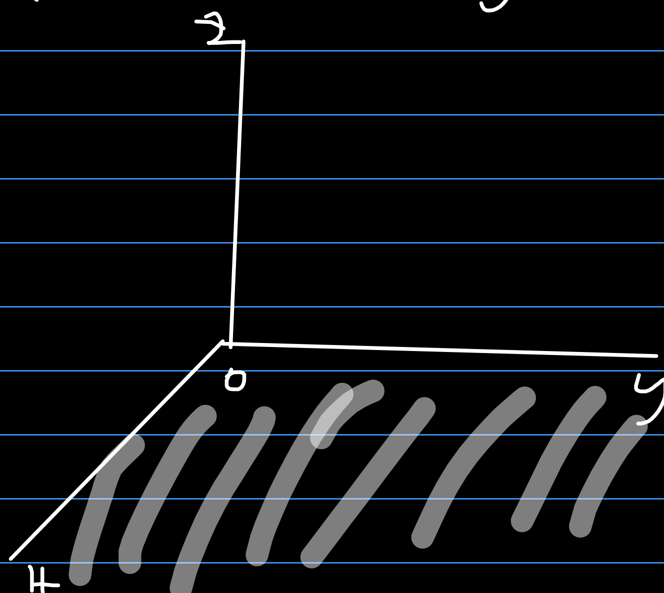
$$m(\widehat{\pi_1, \pi_2}) = m(\widehat{\vec{n_{\pi_1}}, \vec{n_{\pi_2}}})$$

$$\cos\left(\widehat{\vec{n}_{\Pi_1}, \vec{n}_{\Pi_2}}\right) = \frac{7}{\sqrt{14}\cdot\sqrt{14}} = \frac{7}{14} = \frac{1}{2}$$

$$\Rightarrow m\left(\widehat{\vec{n}_{\Pi_1}, \vec{n}_{\Pi_2}}\right) = \frac{\pi}{3}$$

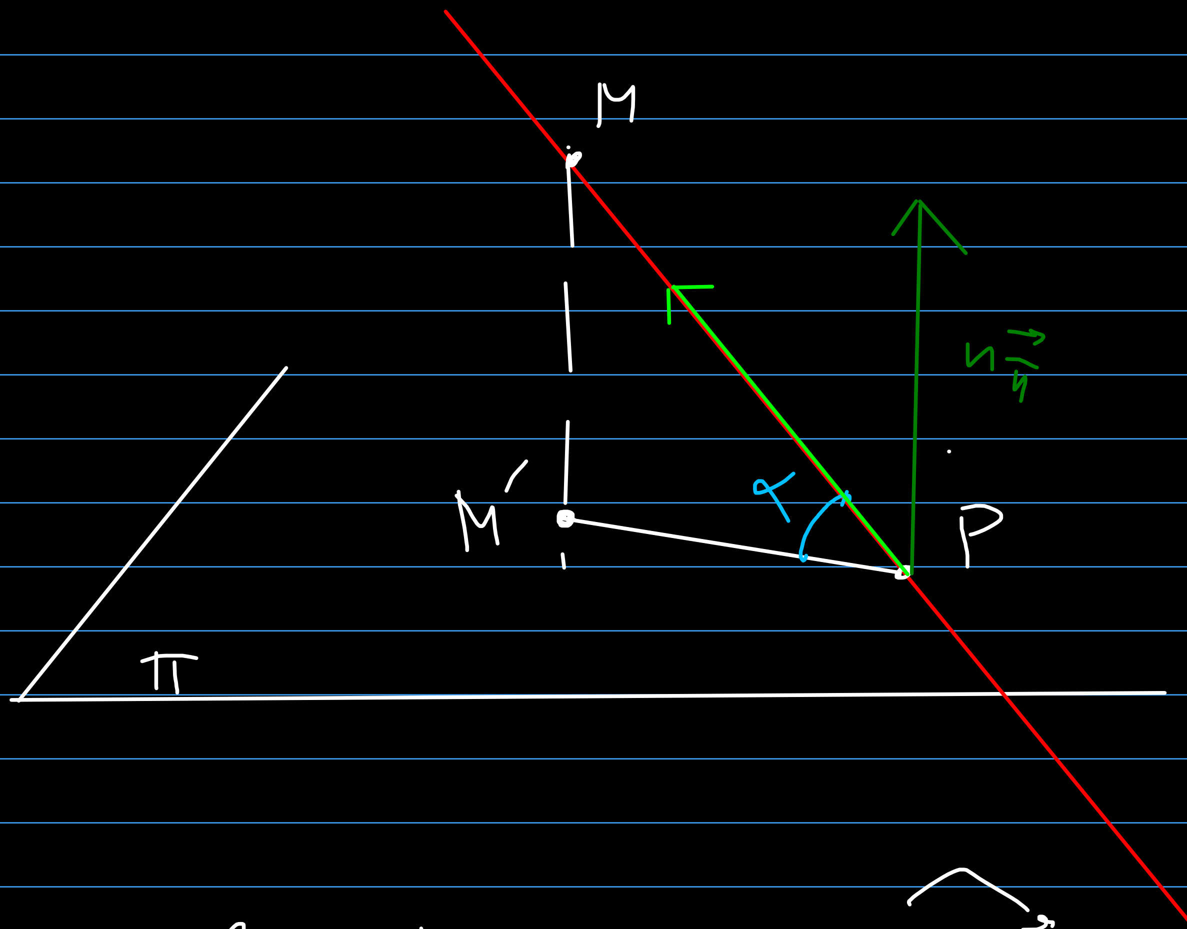
(c) $xOy: \ z = 0$

$$\vec{n}_{xOy} = (0,0,1)$$



$M_1(1,2,3)$, $M_2(-2,1,4)$

$$\Rightarrow \overrightarrow{M_1M_2} = \vec{r}_{M_2} - \vec{r}_{M_1} = (-3,-1,1)$$



$$\Rightarrow m(\ell, \pi) = \frac{\pi}{2} - m(\widehat{\vec{n}_\pi, \vec{\ell}})$$

$$\vec{n}_\pi = (0,0,1), \quad \overrightarrow{M_1M_2} = (-3,-1,1)$$

$$\vec{n}_\pi \cdot \overrightarrow{M_1M_2} = 0\cdot(-3) + 0\cdot(-1) + 1\cdot 1 = 1$$

$$\|\vec{n}_\pi\| = 1, \quad \|\overrightarrow{M_1M_2}\| = \sqrt{11}$$

$$\Rightarrow m(\widehat{\vec{n}_\pi, \vec{\ell}}) = \arccos\left(\frac{1}{\sqrt{11}}\right)$$

$$\Rightarrow m(xOy, M_1M_2) = \frac{\pi}{2} - \arccos\left(\frac{1}{\sqrt{11}}\right)$$

The distance from a point to a plane

$\Pi: Ax + By + Cz + D = 0$

$P(x_0, y_0, z_0)$

$$\text{dist}(P, \Pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

In 2D we have the distance from a point to a line.

$\ell: Ax + By + C = 0$

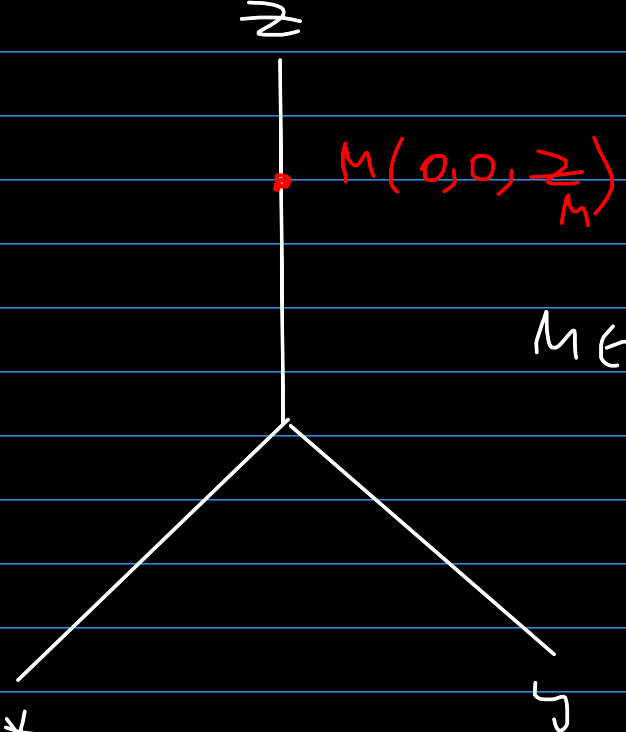
$P(x_0, y_0)$

$$\text{dist}(P, \ell) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

5.5. Find the points on the z-axis which are equidistant with respect to the planes:

$\pi_1: 12x+9y-20z-19=0$

$\pi_2: 16x+12y+15z-9=0$



$M \in Oz \Rightarrow x_M = y_M = 0$

$$\text{dist}(M, \pi_1) = \text{dist}(M, \pi_2)$$

$$\frac{|12 \cdot 0 + 9 \cdot 0 - 20 \cdot z_M - 19|}{\sqrt{144+81+400}} = \frac{|16 \cdot 0 + 12 \cdot 0 + 15 \cdot z_M - 9|}{\sqrt{144+256+225}} \Leftrightarrow$$

$$\Leftrightarrow \frac{|-20 z_M - 19|}{\sqrt{625}} = \frac{|15 z_M - 9|}{\sqrt{625}} \Leftrightarrow$$

$$\Leftrightarrow |-20 z_M - 19| = |15 z_M - 9|$$

$$\Rightarrow -20 z_M - 19 = \pm (15 z_M - 9)$$

$$\Rightarrow 1.\ -20 z_M - 19 = 15 z_M - 9$$

$$\Rightarrow z_M = -\frac{10}{35}$$

$$2.\ -20 z_M - 19 = 9 - 15 z_M$$

$$\Rightarrow z_M = -\frac{28}{5}$$

5.6. Consider two planes,

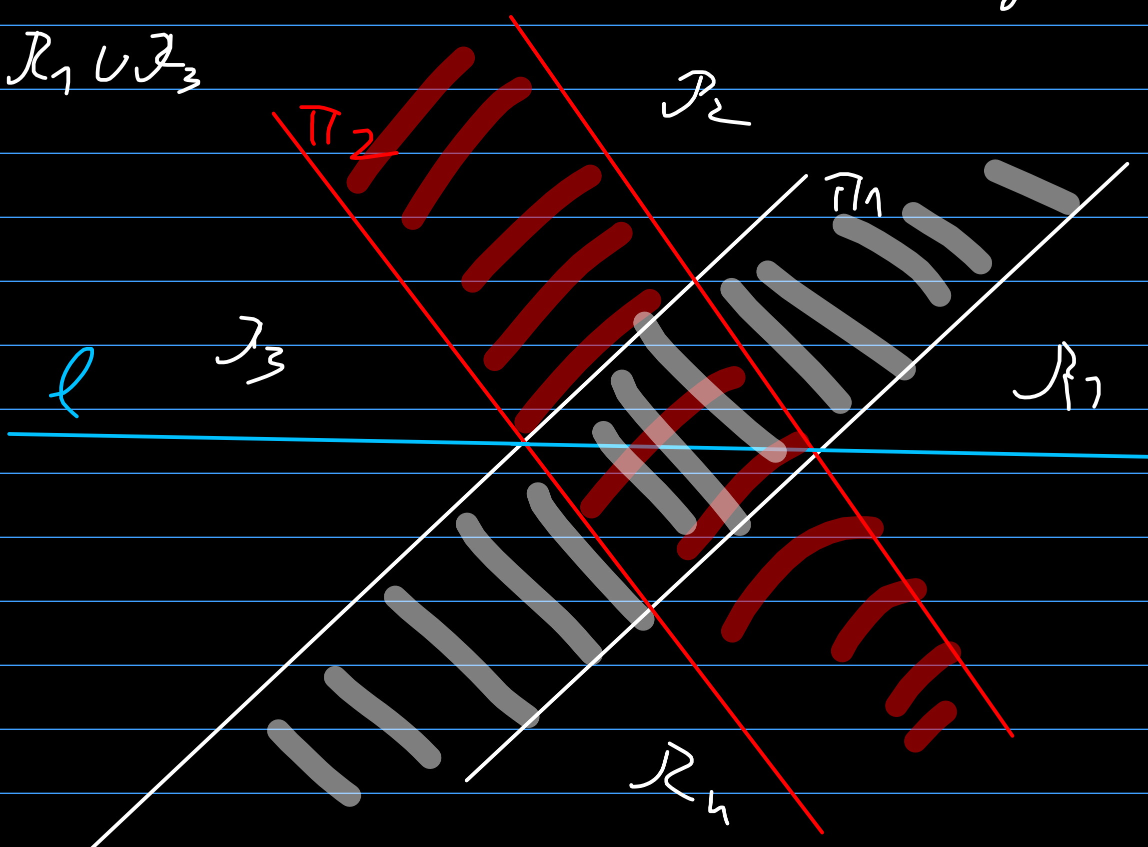
$\Pi_1: A_1x + B_1y + C_1z + D_1 = 0$

$\Pi_2: A_2x + B_2y + C_2z + D_2 = 0$

Assume $\Pi_1 \not\parallel \Pi_2$ and $\Pi_1 \not\perp \Pi_2$

Π_1, Π_2 divide the space into four regions: R_1, R_2, R_3, R_4.

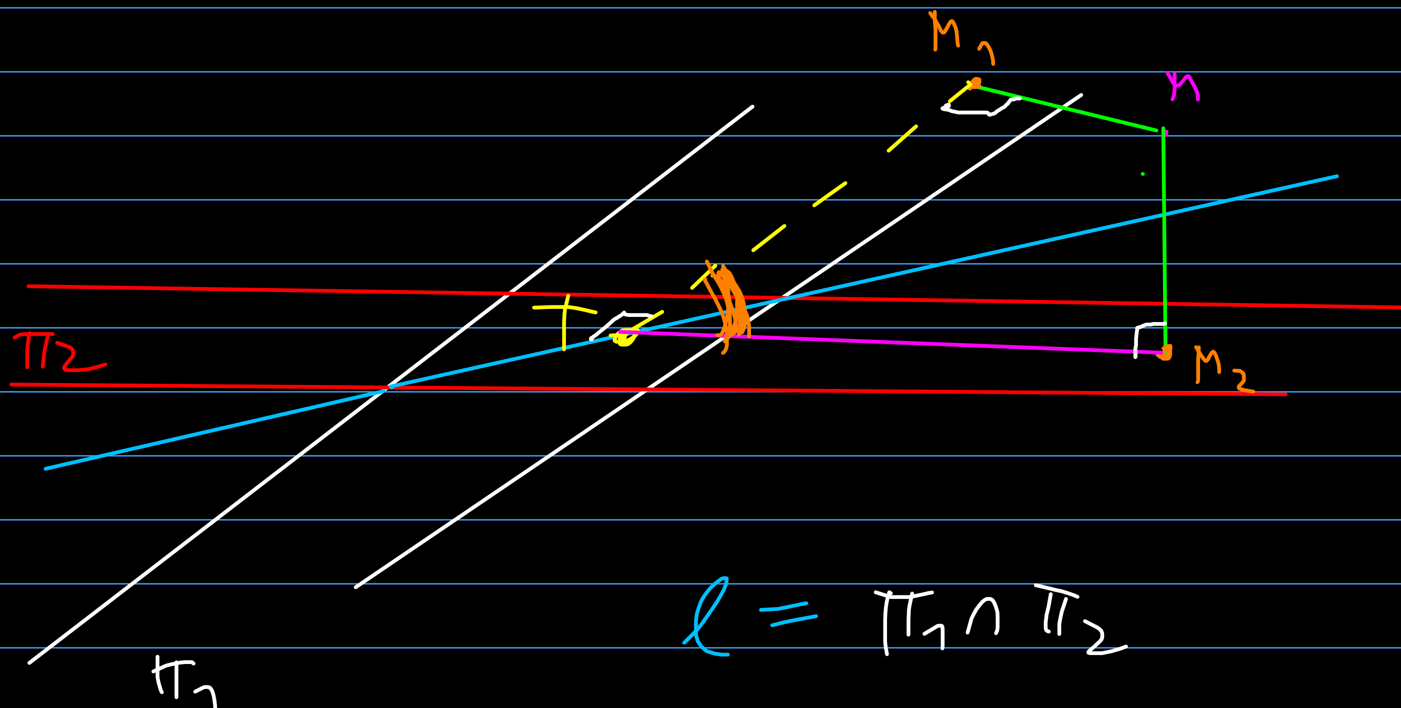
Assume that the acute region is $R_1 \cup R_3$



$$F_1(x,y,z) = A_1x + B_1y + C_1z + D_1$$

$$F_2(x,y,z) = A_2x + B_2y + C_2z + D_2$$

$$M(x,y,z) \in R_1 \cup R_3 \iff F_1(x,y,z)\cdot F_2(x,y,z)\cdot \underbrace{(A_1A_2 + B_1B_2 + C_1C_2)}_{=\vec{n}_{\pi_1}\cdot\vec{n}_{\pi_2}} < 0$$



Let M_1 and M_2 be the orthogonal projections. Let T be the perpendicular from M_1 onto the line l inside the plane π_1

By a geometrical argument we can show

that $M_2 T \perp \ell$

This means that

$$m(\widehat{\pi_1, \pi_2}) = m(\widehat{M_1 T M_2})$$

in the region where M is

MM_1TM_2 is a quadrilateral in the same plane

MM_1TM_2 has two opposite right angles $\Rightarrow$

$$\Rightarrow m(\widehat{M_1 T M_2}) = \pi - m(\widehat{M_1 M M_2})$$

$$m(\widehat{M_1 M M_2}) = m(\widehat{\overrightarrow{MM_1}, \overrightarrow{MM_2}})$$

$$M \in \text{acute region} \Leftrightarrow m(\widehat{M_1 T M_2}) < \frac{\pi}{2} \Leftrightarrow$$

$$\Leftrightarrow m(\widehat{\overrightarrow{MM_1}, \overrightarrow{MM_2}}) > \frac{\pi}{2} \Leftrightarrow \cos(\widehat{\overrightarrow{MM_1}, \overrightarrow{MM_2}}) < 0 \Leftrightarrow$$

$$\Leftrightarrow \overrightarrow{MM_1} \cdot \overrightarrow{MM_2} < 0$$

$$\overrightarrow{MM_1} = \overrightarrow{r_{M_1}} - \overrightarrow{r_M}$$

$$\overrightarrow{r_{M_1}} = \overrightarrow{r_{P_{\pi_1}(M)}} = \overrightarrow{r_M} - \frac{F_1(M)}{\|\overrightarrow{n_{\pi_1}}\|^2} \cdot \overrightarrow{n_{\pi_1}}$$

$$\Rightarrow \overrightarrow{MM_1} = -\frac{F_1(M)}{\|\overrightarrow{n_{\pi_1}}\|^2} \cdot \overrightarrow{n_{\pi_1}}$$

$$\Rightarrow \overrightarrow{MM_2} = -\frac{F_2(M)}{\|\overrightarrow{n_{\pi_2}}\|^2} \cdot \overrightarrow{n_{\pi_2}}$$

$\Rightarrow M \in$ anti-region $\Leftrightarrow \left(-\frac{F_1(M)}{\|\overrightarrow{n_{\pi_1}}\|^2} \cdot \overrightarrow{n_{\pi_1}}\right) \cdot \left(-\frac{F_2(M)}{\|\overrightarrow{n_{\pi_2}}\|^2} \cdot \overrightarrow{n_{\pi_2}}\right) < 0 \Leftrightarrow$

$$\Leftrightarrow \left(-\frac{F_1(M)}{\|\overrightarrow{n_{\pi_1}}\|^2}\right) \cdot \left(-\frac{F_2(M)}{\|\overrightarrow{n_{\pi_2}}\|^2}\right) \cdot \left(\overrightarrow{n_{\pi_1}} \cdot \overrightarrow{n_{\pi_2}}\right) < 0$$

$\Leftrightarrow F_1(M)\cdot F_2(M)\cdot\left(A_1A_2+B_1B_2+C_1C_2\right)<0$

5.7. $\Pi_1: 2x+y-3z-5=0$

$\Pi_2: x+3y+2z+7=0$

Find the eqns. of the bisector planes of the dihedral angles formed by Π_1 and Π_2 and select the one contained in the acute region

The bisector planes of a dihedral angle formed by Π_1 and Π_2 are the solutions of the equations:

$$\text{dist}(M,\Pi_1)=\text{dist}(M,\Pi_2)$$

$$\frac{|A_1x+B_1y+C_1z+D_1|}{\sqrt{A_1^2+B_1^2+C_1^2}}=\frac{|A_2x+B_2y+C_2z+D_2|}{\sqrt{A_2^2+B_2^2+C_2^2}}$$

$$\frac{A_1x+B_1y+C_1z+D_1}{\sqrt{A_1^2+B_1^2+C_1^2}}=\frac{A_2x+B_2y+C_2z+D_2}{\sqrt{A_2^2+B_2^2+C_2^2}}$$

$$\frac{A_1x+B_1y+C_1z+D_1}{\sqrt{A_1^2+B_1^2+C_1^2}}=-\frac{A_2x+B_2y+C_2z+D_2}{\sqrt{A_2^2+B_2^2+C_2^2}}$$

To decide which is the right plane, we use the preceding exercise for a random point in one of the planes