Seminar W12 – 915

Affine transformations (2D)

$y = mx + n$ affine function

$y = mx$ linear function

$f : \mathbb{R} \to \mathbb{R}$

$x \mapsto mx$

$g : \mathbb{R} \to \mathbb{R}$

$x \mapsto mx + n$

$g(x_1 + x_2) \neq g(x_1) + g(x_2)$

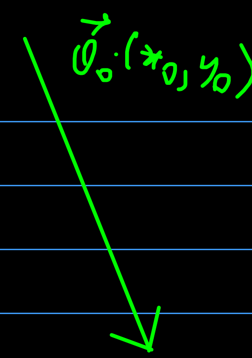
preserves parallelism and lines

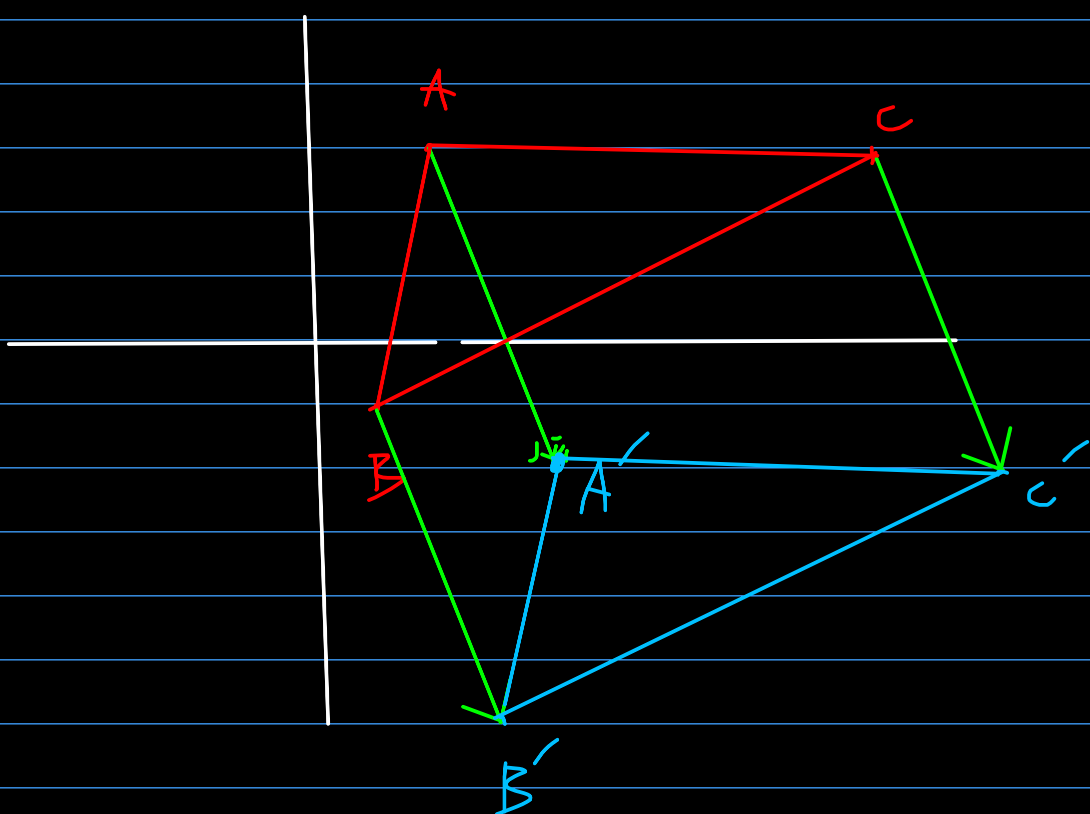
$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ affine transformation

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{M}_{\in M_{2,2}(\mathbb{R})} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}_{= \vec{v_0}}$$

$$\varphi(x \; y) = (x \; y) \cdot M^t + (x_0 \; y_0)$$

Translations: $T(x_0, y_0)$





If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transf., how do we tell if it is a translation?

- Pick a point A and let $A' = \varphi(A)$
- If φ is a translation, then $\varphi = T(\overrightarrow{AA'})$
- Check if, indeed, for any $B \in \mathbb{R}^2$

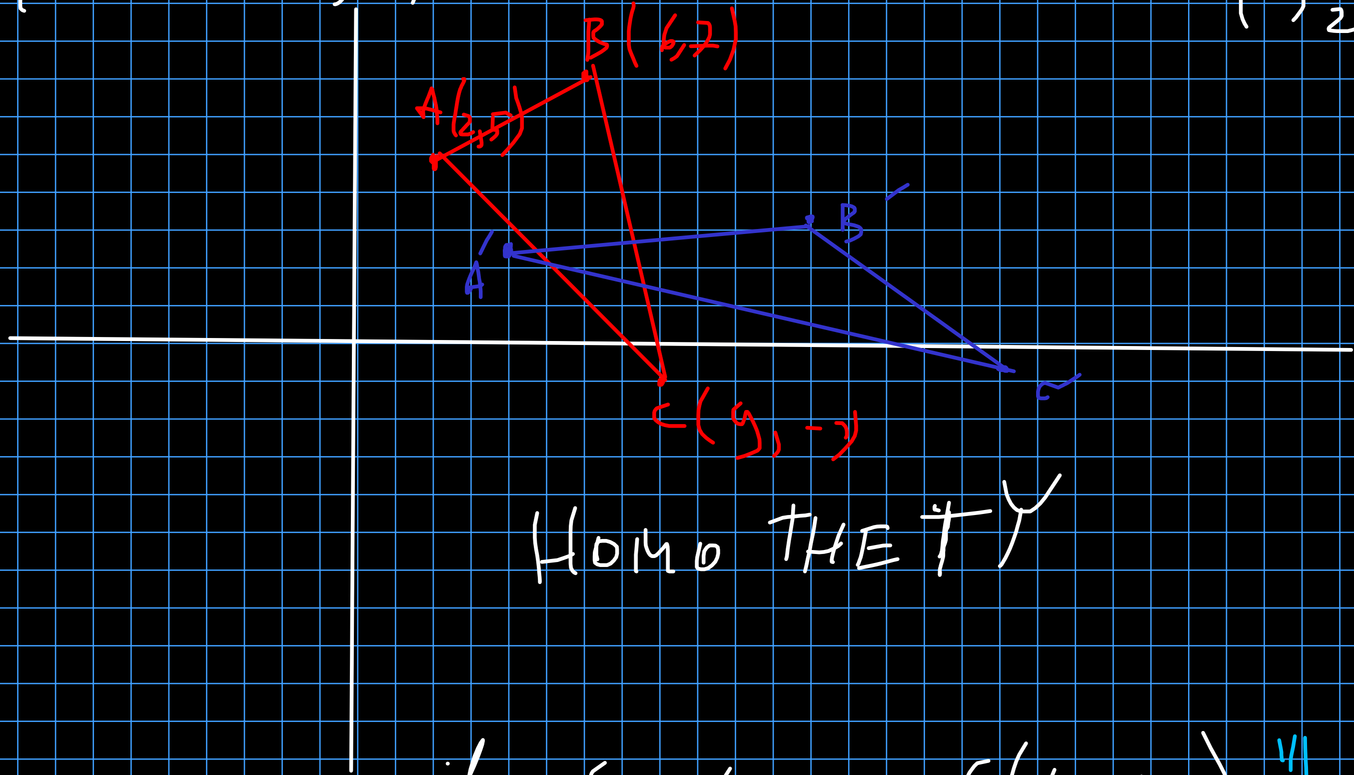
$$\overrightarrow{B\varphi(B)} = \overrightarrow{AA'}$$

$T(x_0, y_0)$ corresponds to $M = Id_2$

Scalings: $S(s_x, s_y)$

(about the origin)

$S(2, \frac{1}{2})$



HOMOTHETY

if $s_x = s_y \Rightarrow S(s_x, s_y)$ "homothety"

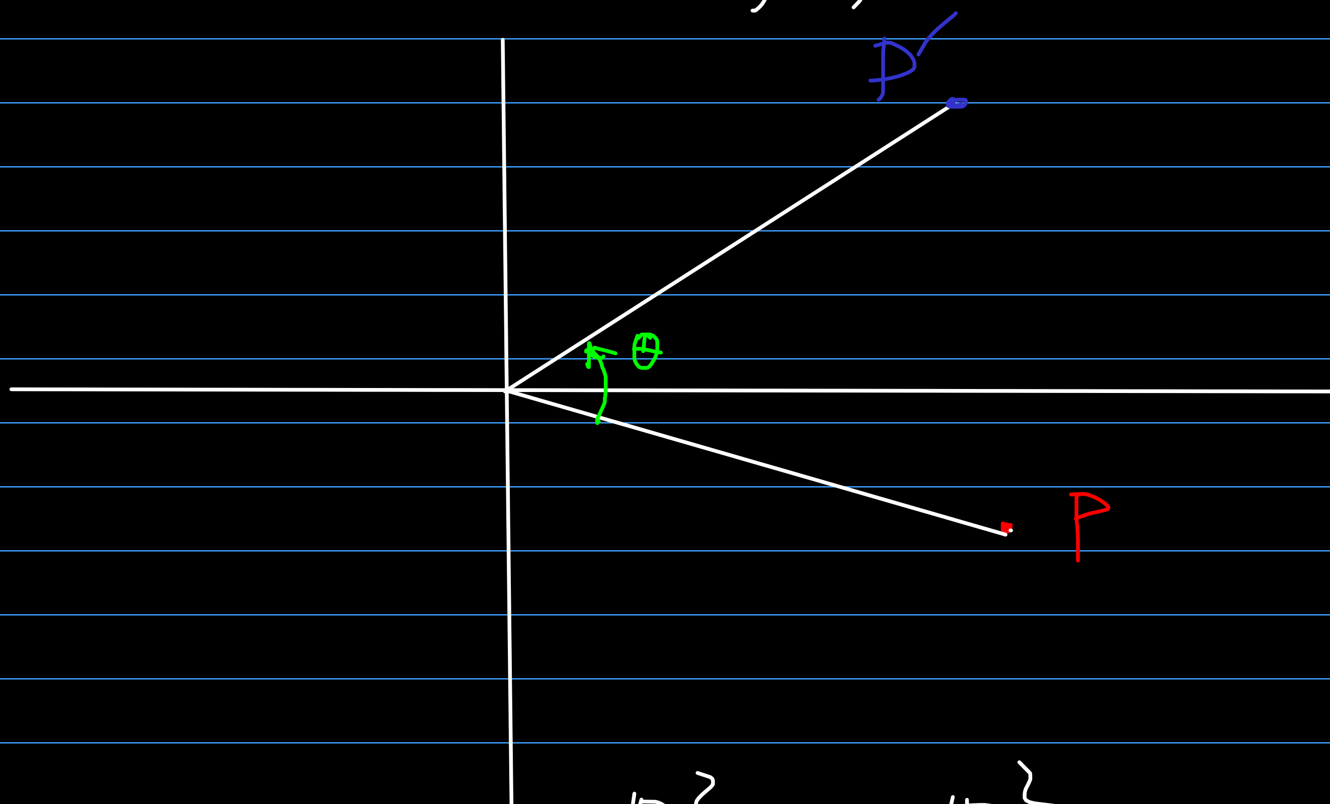
If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transf., how do we tell if it is a scaling?

- Pick a point A, $A' = \varphi(A)$
- If φ is to be a scaling, then the scaling factors should be

$$s_x = \frac{x_{A'}}{x_A}, \quad s_y = \frac{y_{A'}}{y_A}$$

- Check this against another point.

Rotations (about the origin) R_θ



$$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transf., how do we tell if it is a rotation?

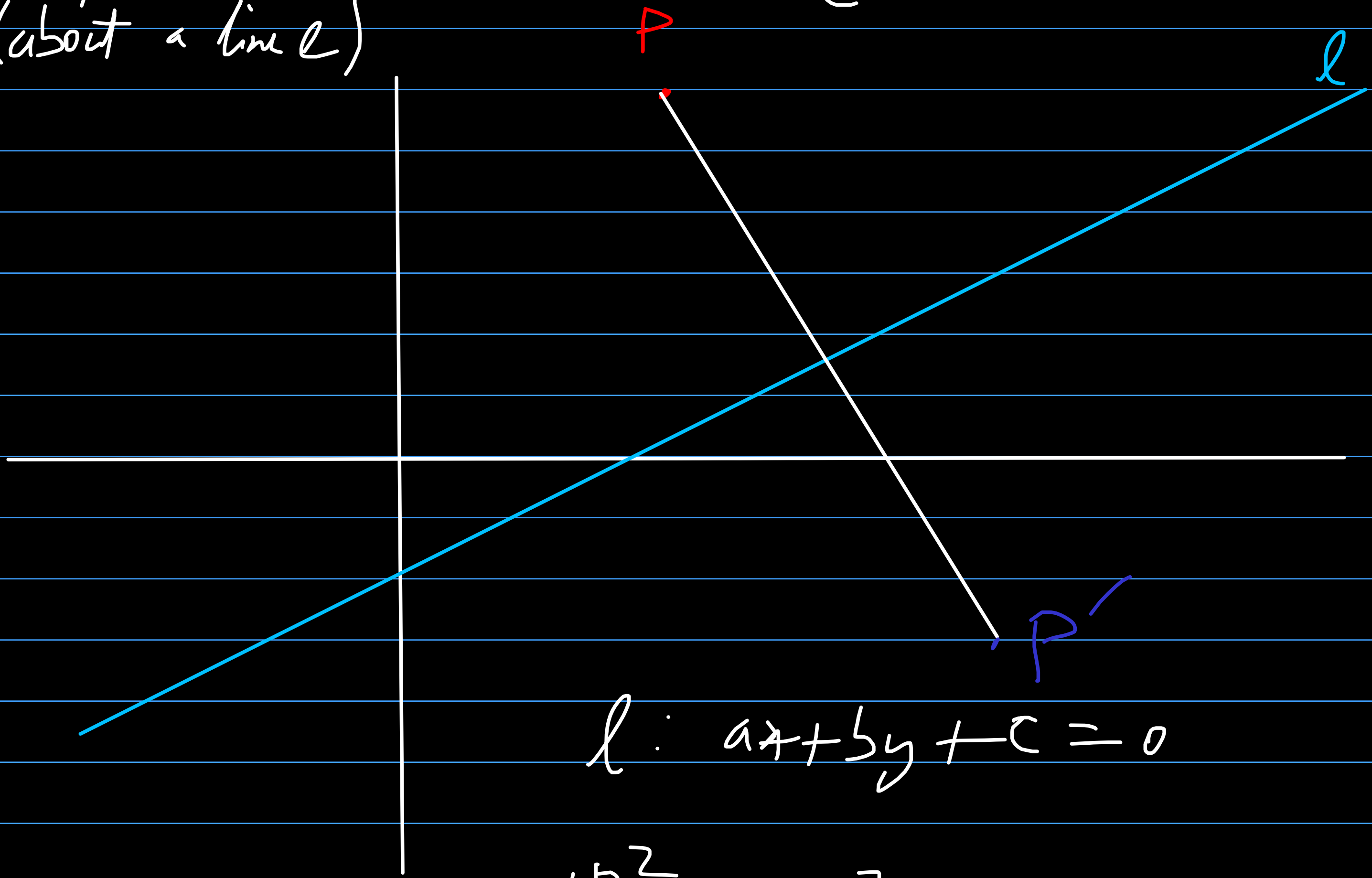
- Check if $\mathrm{Fix}(\varphi) = \{(0,0)\}$

$\Big[\ f : A \to B,\ \mathrm{Fix}(f) = \{x \in A \mid f(x) = x\}\ \Big]$

- Check if $\forall P \in \mathbb{R}^2$, $OP = O\varphi(P)$
- check if $\forall P \in \mathbb{R}^2$, $m(\widehat{POP'}) = \theta$

Reflections (about a line ℓ)

r_ℓ



$\ell : ax + by + c = 0$

$r_\ell : \mathbb{R}^2 \to \mathbb{R}^2$

$$r_\ell \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} +$$

$$+ \frac{-2c}{a^2+b^2} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

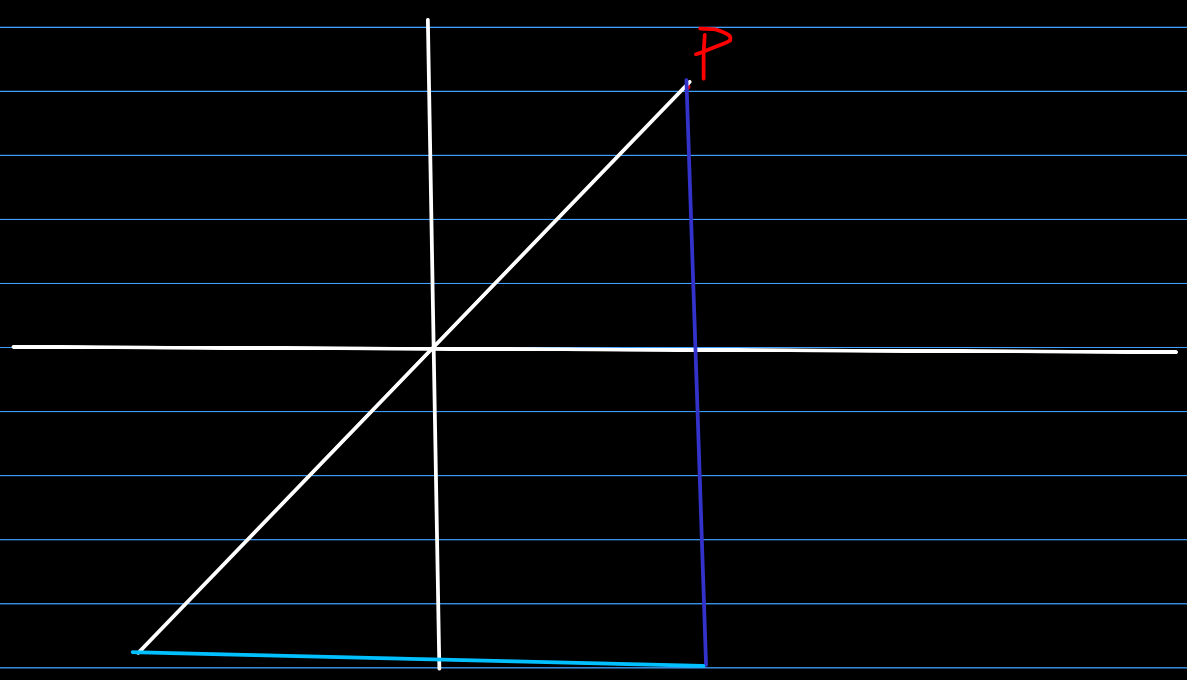
If $c=0$ (i.e. $0 \in \ell$), then

r_ℓ is a linear transformation

$r_x := r_{0x}$, $r_y := r_{0y}$

$$r_x\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$r_y\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$



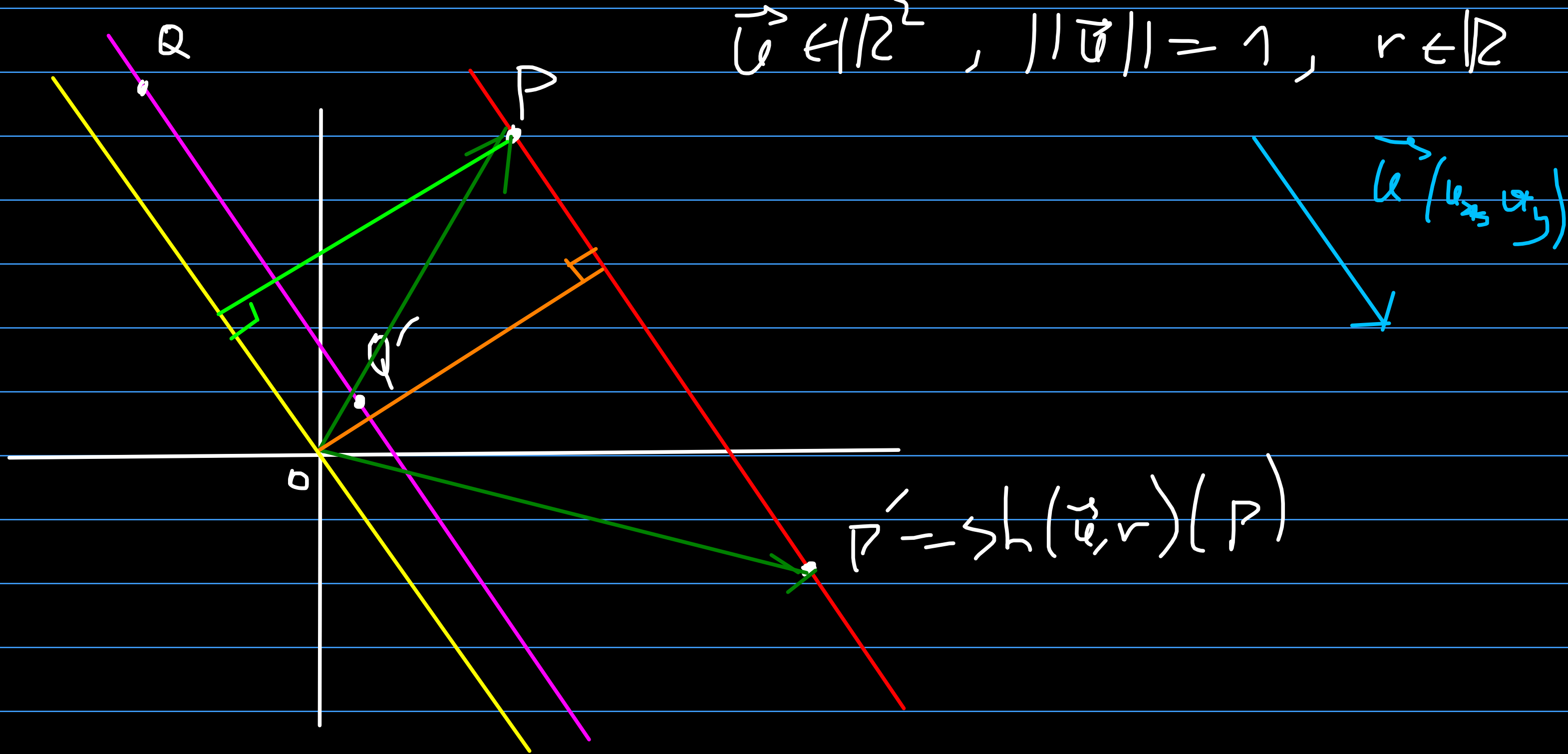
If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transf., how do we tell if it is a reflection?

- We check that $Fix(\varphi) = \ell$ line.
- If so, then it is possibly r_ℓ
- Check if $\forall A \in \mathbb{R}^2$, $A' = \varphi(A)$ we have that ℓ is the perpendicular bisector of AA'.

Shears (transvections)

$sh(\vec{u}, r)$

$\vec{u} \in \mathbb{R}^2$, $\|\vec{u}\| = 1$, $r \in \mathbb{R}$



$$sh(\vec{v}, r)(P) = sh(\vec{v}, r)\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} + r \cdot \underline{\delta(O, \ell_P)} \cdot \vec{v}$$

ℓ_P = line through P that is parallel to $\vec{v}$

ℓ line, $ax+by+c=0$, $P(x_P, y_P)$ point

$$\delta(P, \ell) = \frac{ax_P + by_P + c}{\sqrt{a^2+b^2}}$$

$\hookrightarrow$ oriented distance

$$sh(\vec{v}, r)(P) = sh(\vec{v}, r)\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} - r \cdot \underline{\delta(P, \ell_O)} \cdot \vec{v}$$

ℓ_O = line through O with the direction $\vec{v}$

$$sh(\vec{v}, r) : \mathbb{R}^2 \to \mathbb{R}^2$$

$$sh(\vec{v}, r)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - r v_x v_y & r v_x^2 \\ -r v_y^2 & 1 + r v_x v_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

12.1. Find the image of the triangle ABC through the reflection w.r.t. the line $d: x - y = 2$, where $A(-1, 2)$, $B(-2, -1)$, $C(3, 3)$

$$d: x - y = 2$$

$$a = 1, \quad b = -1, \quad c = -2$$

$$r_\ell\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2}\begin{pmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} +$$

$$+ \frac{-2c}{a^2+b^2} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow r_d\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\cdot\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} +$$

$$+ \frac{4}{2}\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$r_d(A) = r_d(-1,2) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix}\right)^t =$$

$$= \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix}\right)^t = \begin{pmatrix} 4 \\ -3 \end{pmatrix}^t = (4,-3)$$

$$r_d(B) = r_d(-2,-1) = \left(\begin{pmatrix} -1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix}\right)^t =$$

$$= (1,-4)$$

$$r_d(C) = r_d(3\,3) = \left(\begin{pmatrix} 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix}\right)^t =$$

$$= \begin{pmatrix} 5 \\ 1 \end{pmatrix}^t = (5,1)$$

12.2. Find the image of the triangle ABC through the clockwise rotation of angle $\frac{\pi}{6}$ where $A(6,4)$, $B(6,2)$, $C(10,6)$.

$$[R_{-\frac{\pi}{6}}] = \begin{pmatrix} \cos\left(-\frac{\pi}{6}\right) & -\sin\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$R_{-\frac{\pi}{6}}(A) = R_{-\frac{\pi}{6}}(6,4) = \left(\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 4 \end{pmatrix}\right)^t =$$

$$= \begin{pmatrix} 3\sqrt{3}+2 \\ -3+2\sqrt{3} \end{pmatrix}^t = (3\sqrt{3}+2, -3+2\sqrt{3})$$

$$R_{-\frac{\pi}{6}}(B) = R_{-\frac{\pi}{6}}(6,2) = (3\sqrt{3}+1, -3+\sqrt{3})$$

$$R_{-\frac{\pi}{6}}(C) = R_{-\frac{\pi}{6}}(10,6) = (5\sqrt{3}+3, -5+3\sqrt{3})$$

12.3. ABCD quadrilateral

$A(1,1)$, $B(3,1)$, $C(2,2)$, $D(\frac{3}{2},3)$

Find the images of ABCD through the following transformations:

(a) $T(1,2)$, $S(2,\frac{5}{2})$, r_x

(b) r_y, $R_{-\frac{\pi}{2}}$, $R_{\frac{\pi}{2}}$

(c) $sh\left(\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right),\frac{3}{2}\right)$

$$sh(\vec{u},r)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1-ru_xu_y & ru_x^2\\ -ru_y^2 & 1+ru_xu_y\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$[sh(\vec{u},r)]=\begin{pmatrix}1-\frac{3}{2}\cdot\frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}} & \frac{3}{2}\cdot\frac{4}{5}\\ -\frac{3}{2}\cdot\frac{1}{5} & 1+\frac{3}{2}\cdot\frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{2}{5} & \frac{6}{5}\\ -\frac{3}{10} & \frac{8}{5}\end{pmatrix}$$

$$sh(\vec{v}, r)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{6}{5} \\ -\frac{3}{10} & \frac{8}{5} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{2}{5}x + \frac{6}{5}y \\ -\frac{3}{10}x + \frac{8}{5}y \end{pmatrix}$$

$$sh(\vec{v}, r)(A) = sh(\vec{v}, r)(1,1) =$$

$$= \left(\frac{8}{5}, \frac{13}{10}\right)$$