Seminar W7 - 974

The triple scalar product (the mixed product)

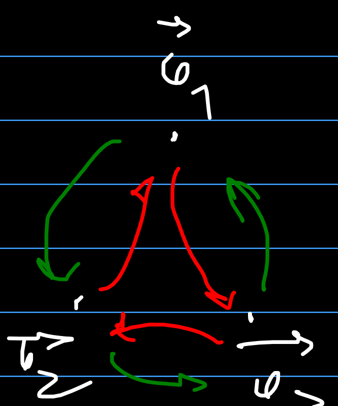
$\vec a, \vec b, \vec c \in V$

$$(\vec a, \vec b, \vec c) := \vec a \cdot (\vec b \times \vec c) = (\vec a \times \vec b) \cdot \vec c$$

$\rightsquigarrow$ If $\mathcal{R} = (O, \vec i, \vec j, \vec k)$ is a orthonormal and direct:

$\vec v_1 (a_1, b_1, c_1)$, $\vec v_2 (a_2, b_2, c_2)$, $\vec v_3 (a_3, b_3, c_3)$

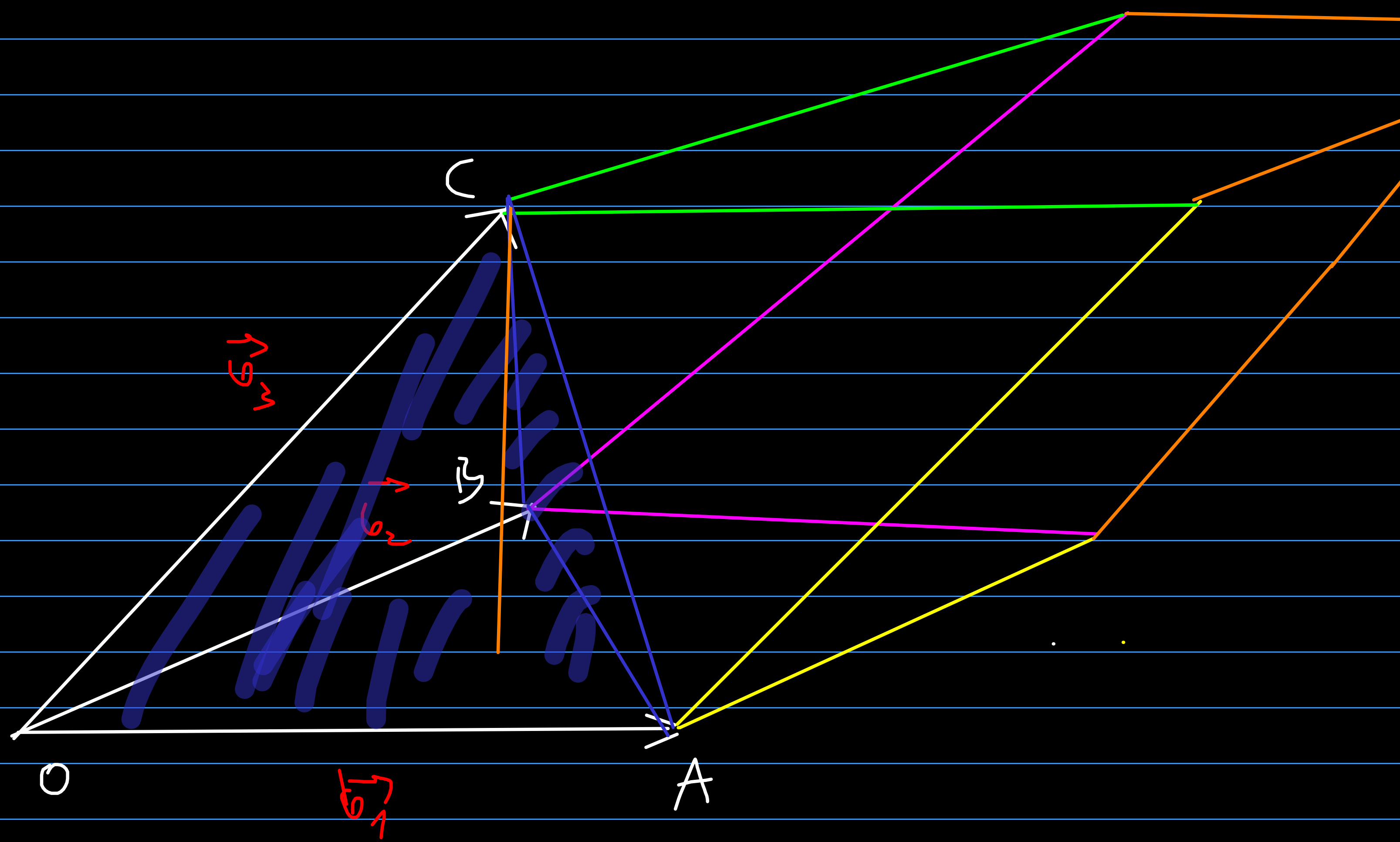
$$(\vec v_1, \vec v_2, \vec v_3) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$



$$(\vec v_1, \vec v_2, \vec v_3) = (\vec v_2, \vec v_3, \vec v_1) =$$

$$= (\vec v_3, \vec v_1, \vec v_2) = -(\vec v_1, \vec v_3, \vec v_2) =$$

$$= -(\vec v_2, \vec v_1, \vec v_3) = -(\vec v_3, \vec v_2, \vec v_1)$$



$$(\vec{u}_1, \vec{u}_2, \vec{u}_3) = \mathrm{Vol}\left(\text{parallelepiped formed by } \vec{u}_1, \vec{u}_2, \vec{u}_3\right)$$

$$\mathrm{Vol} = \frac{1}{6}\left|(\vec{u}_1, \vec{u}_2, \vec{u}_3)\right|$$

$$\mathrm{dist}\left(C, (OBA)\right) = \frac{|(\vec{u}_1, \vec{u}_2, \vec{u}_3)|}{\|\vec{u}_1 \times \vec{u}_2\|}$$

The common perpendicular of two lines in space.

Let l_1, l_2 be two lines

$l_1 \cap l_2 \neq \emptyset \Rightarrow$ common perp. is a line perp. to the common plane
$dist(l_1, l_2) = 0$

$l_1 \parallel l_2 \Rightarrow \forall M \in l_1$, any perp. from M to l_2 is a common perpendicular

$dist(l_1, l_2) = dist(M, l_2)$

$l_1 \nparallel l_2 \Rightarrow$
$l_1 \cap l_2 = \emptyset$
(skew lines)
↓
noncoplanar

the common perpendicular

$\hookrightarrow$ intersection between

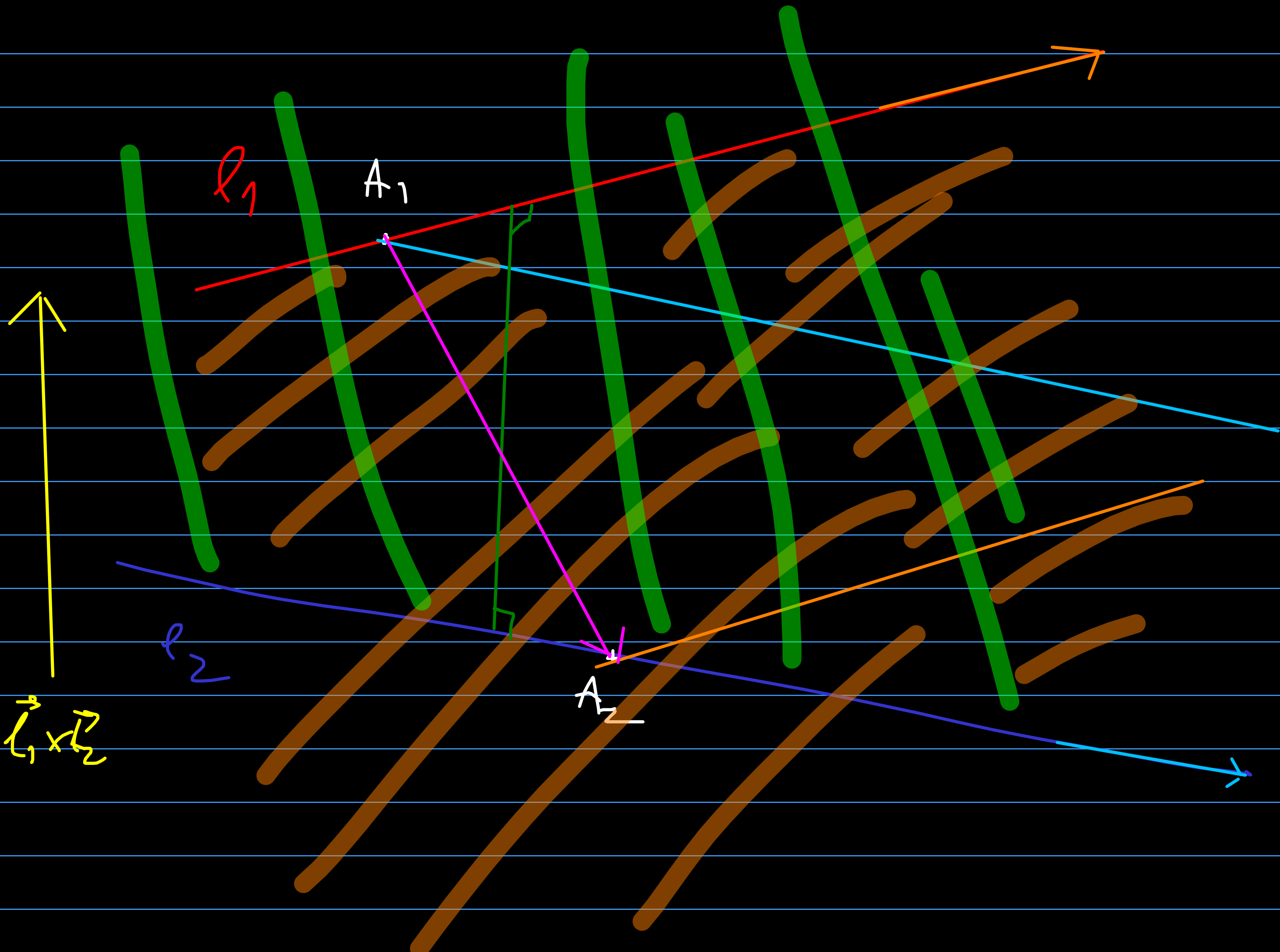
Π_1 = plane given by d_1 and $\vec{d_1} \times \vec{d_2}$

Π_2 = plane given by d_2 and $\vec{d_1} \times \vec{d_2}$

$\Pi_1 \nparallel \Pi_2$, because $\vec{d_1} \nparallel \vec{d_2}$

$\Rightarrow \exists\ \Pi_1 \cap \Pi_2 = \ell$

this line is the <u>common perpendicular</u>



dist(l_1, l_2) = length of the common perpendicular = height in the parallelepiped given by the vectors $\vec{d}_1, \vec{d}_2, \overrightarrow{A_1A_2}$ =

$$= \frac{|(\vec{d_1}, \vec{d_2}, \overrightarrow{A_1A_2})|}{\|\vec{d_1} \times \vec{d_2}\|}$$

l_1, l_2 are coplanar $\iff$ $\forall A_1 \in l_1, \forall A_2 \in l_2$ $\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}$ are linearly dependent $\iff$

$\iff (\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}) = 0$

7.7. Find the distance between the lines

$$(l_1): \frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{1}$$

$$(l_2): \frac{x+1}{3} = \frac{y}{4} = \frac{z-1}{3}$$

as well as the equations of the common perpendicular.

We pick $A_1 \in l_1$, $A_1(1,-1,0)$

$A_2 \in l_2$: $A_2(-1,0,1)$

$\overrightarrow{A_1A_2}(-2,1,1)$

$\vec{l_1}(2,3,1)$, $\vec{l_2}(3,4,3)$

$$\Rightarrow (\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}) = \begin{vmatrix} -2 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 3 \end{vmatrix} =$$

$$= -18 + 3 + 8 - 9 + 8 - 6 = -14$$

$$\vec{l_1} \times \vec{l_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 3 & 4 & 3 \end{vmatrix} =$$

$$= 5\vec{i} - 3\vec{j} - \vec{k}$$

$$\Rightarrow dist(l_1, l_2) = \frac{|(A_1A_2, \vec{l_1}, \vec{l_2})|}{\|\vec{l_1} \times \vec{l_2}\|} =$$

$$= \frac{14}{\sqrt{25+9+1}} = \frac{14}{\sqrt{35}}$$

In order to find the common perpendicular we write:

$\Pi_1 =$ the plane given by l_1 and $\vec{l_1} \times \vec{l_2}$

$\Pi_2 =$ the plane given by l_2 and $\vec{l_1} \times \vec{l_2}$

$\vec{l_1}(2,3,1), \quad \vec{l_2}(3,4,3), A_1(1,-1,0), A_2(-1,0,1)$

$\vec{l_1} \times \vec{l_2}\,(5,-3,-1)$

$$\Pi_1: \begin{vmatrix} x-1 & y+1 & z \\ 2 & 3 & 1 \\ 5 & -3 & -1 \end{vmatrix} = 0 \iff$$

$$\iff 7(y+1) + (-21)z = 0$$

$$\iff 7y - 21z + 7 = 0$$

$$\Pi_2: \begin{vmatrix} x+1 & y & z-1 \\ 3 & 4 & 3 \\ 5 & -3 & -1 \end{vmatrix} = 0$$

$$\Leftrightarrow 5(x+1) + 18y - 29(z-1) = 0$$

$$\Leftrightarrow \pi_2 : 5x + 18y - 29z + 34 = 0$$

$\Rightarrow$ the common perp is the line

$$\begin{cases} 7y - 21z + 7 = 0 \\ 5x + 18y - 29z + 34 = 0 \end{cases}$$

7.6. Find the value of the parameter λ for which the straight lines

$$l_1 : \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1}$$

$$l_2 : \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{\lambda}$$

are coplanar. Find the intersection point in that case.

We choose $A_1 \in l_1$, $A_1(1, -2, 0)$

$A_2 \in l_2$, $A_2(-1, 3, 0)$

$\Rightarrow \overrightarrow{A_1A_2}(-2, 5, 0)$

$$(\overrightarrow{A_1A_2}, l_1, l_2) = \begin{vmatrix} -2 & 5 & 0 \\ 3 & -2 & 1 \\ 4 & 1 & \lambda \end{vmatrix} =$$

$$= (-1) \cdot \begin{vmatrix} -2 & 5 \\ 4 & 1 \end{vmatrix} + \lambda \cdot \begin{vmatrix} -2 & 5 \\ 3 & -2 \end{vmatrix} =$$

$$= 22 + \lambda \cdot (-11)$$

l_1, l_2 coplanar $\Leftrightarrow (\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}) = 0 \Leftrightarrow$

$\Leftrightarrow 22 - 11\lambda = 0 \Leftrightarrow \lambda = 2$

We will now find the intersection point.

$$P: \begin{cases} \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1} \\ \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{2} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = \frac{x-1}{3} \\ z = \frac{y+2}{-2} \\ \frac{z}{2} = \frac{x+1}{4} \\ \frac{z}{2} = \frac{y-3}{1} \end{cases} \Leftrightarrow \begin{cases} x = 3z+1 \\ y = -2z-2 \\ \frac{z}{2} = \frac{3z+2}{4} \\ \frac{z}{2} = \frac{-2z-5}{1} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 3z+1 \\ y = -2z-2 \\ 4z = 6z+4 \\ z = -4z-10 \end{cases} \Leftrightarrow \begin{cases} x = 3z+1 \\ y = -2z-2 \\ z = -2 \\ -2 = 8-10 \end{cases}$$

$$\Rightarrow \quad P(-5, 2, -2)$$

7.1. a. $\left|(\vec{a}, \vec{b}, \vec{c})\right| \leq \|\vec{a}\| \cdot \|\vec{b}\| \cdot \|\vec{c}\|$

b. $(\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}) = 2 \cdot (\vec{a}, \vec{b}, \vec{c})$

b. $(\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}) = (\vec{a}+\vec{b}) \cdot \left[(\vec{b}+\vec{c}) \times (\vec{c}+\vec{a})\right]$

$$= (\vec{a}+\vec{b}) \cdot \left(\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \underbrace{\vec{c} \times \vec{c}}_{\vec{0}} + \vec{c} \times \vec{a}\right)$$

$$= (\vec{a}+\vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}) =$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \overbrace{\vec{a} \cdot (\vec{b} \times \vec{a})}^{0} + \overbrace{\vec{a} \cdot (\vec{c} \times \vec{a})}^{=0} +$$

$$+ \underbrace{\vec{b} \cdot (\vec{b} \times \vec{c})}_{=0} + \underbrace{\vec{b} \cdot (\vec{b} \times \vec{a})}_{=0} + \vec{b} \cdot (\vec{c} \times \vec{a})$$

$$= \vec{a} \cdot (\vec{b} + \vec{c}) + \vec{b} \cdot (\vec{c} \times \vec{a}) = 2 \cdot (\vec{a}, \vec{b}, \vec{c})$$