Seminar w5 - 914

The dot product (Scalar product):

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(\widehat{\vec{a}, \vec{b}})$$

If we've fixed an orthonormal reference system:

$$\vec{a} = (x_{\vec{a}}, y_{\vec{a}}, z_{\vec{a}}), \quad \vec{b} = (x_{\vec{b}}, y_{\vec{b}}, z_{\vec{b}})$$

$$\vec{a} \cdot \vec{b} = x_{\vec{a}} \cdot x_{\vec{b}} + y_{\vec{a}} \cdot y_{\vec{b}} + z_{\vec{a}} \cdot z_{\vec{b}}$$

$\mathcal{R} = (O, [\vec{i}, \vec{j}, \vec{k}])$

$\mathcal{R}$ orthogonal if: $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$

$\mathcal{R}$ orthonormal if it is orthogonal and

$$\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1.$$

$\forall \vec{v} \in \mathcal{V}: \quad \vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

(if $\vec{w}, \vec{v} \neq \vec{0}$

$\vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0$)

5.3. Find the angle between,

(a)

$$d_1: \begin{cases} x+2y+z-1=0 \\ x-2y+z+1=0 \end{cases}$$

$$d_2: \begin{cases} x-y-z-1=0 \\ x-y+2z+1=0 \end{cases}$$

(b) $\Pi_1: x+3y+2z+1=0$

$\Pi_2: 3x+2y-z=6$

(c) the plane xOy and the straight line M_1M_2, $M_1(1,2,3)$, $M_2(-2,1,4)$

(a)

$$d_1: \begin{cases} x+2y+z-1=0 \\ x-2y+z+1=0 \end{cases}$$

$$d_1: \begin{cases} x=-2y-z+1 \\ -2y-z+1-2y+z+1=0 \end{cases} \Rightarrow$$

$\Leftrightarrow d_1: \begin{cases} x = -2y - z + 1 \\ -4y + 2 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow d_1: \begin{cases} y = \frac{1}{2} \\ x = -1 - z + 1 = -z \end{cases} \Leftrightarrow$

$\Rightarrow d_1: \begin{cases} x = -t \\ y = \frac{1}{2} + t \cdot 0 \Rightarrow \vec{d_1}(-1, 0, 1) \\ z = t \end{cases}$

$d_2: \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x = y + z + 1 \\ y + z + 1 - y + 2z + 1 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x = y + z + 1 \\ 3z + 2 = 0 \end{cases} \Leftrightarrow \begin{cases} z = -\frac{2}{3} \\ x = y + \frac{1}{3} \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x = t + \frac{1}{3} \\ y = t \\ z = 0 \cdot t - \frac{2}{3} \end{cases} \Rightarrow \vec{d_2}(1,1,0)$

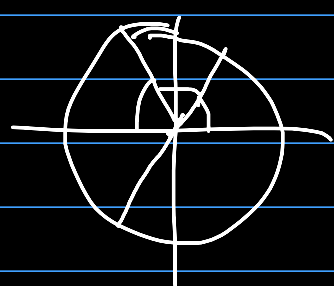
$\vec{d_1}(-1,0,1), \quad \vec{d_2}(1,1,0)$

$\vec{d_1} \cdot \vec{d_2} = \|\vec{d_1}\| \cdot \|\vec{d_2}\| \cdot \cos(\widehat{\vec{d_1}, \vec{d_2}})$

$\Rightarrow \cos(\widehat{\vec{d_1}, \vec{d_2}}) = \dfrac{d_1 \cdot d_2}{\|\vec{d_1}\| \cdot \|\vec{d_2}\|} =$

$= \dfrac{-1 \cdot 1 + 0 \cdot 1 + 0 \cdot 1}{\sqrt{2} \cdot \sqrt{2}} = \dfrac{-1}{2}$

$\Rightarrow m(\widehat{\vec{d_1}, \vec{d_2}}) = \dfrac{2\pi}{3}$



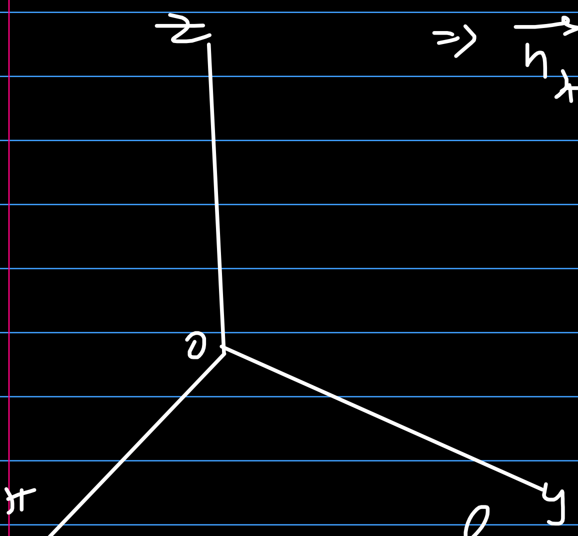
(b) $\Pi_1 : x+3y+2z+1=0$

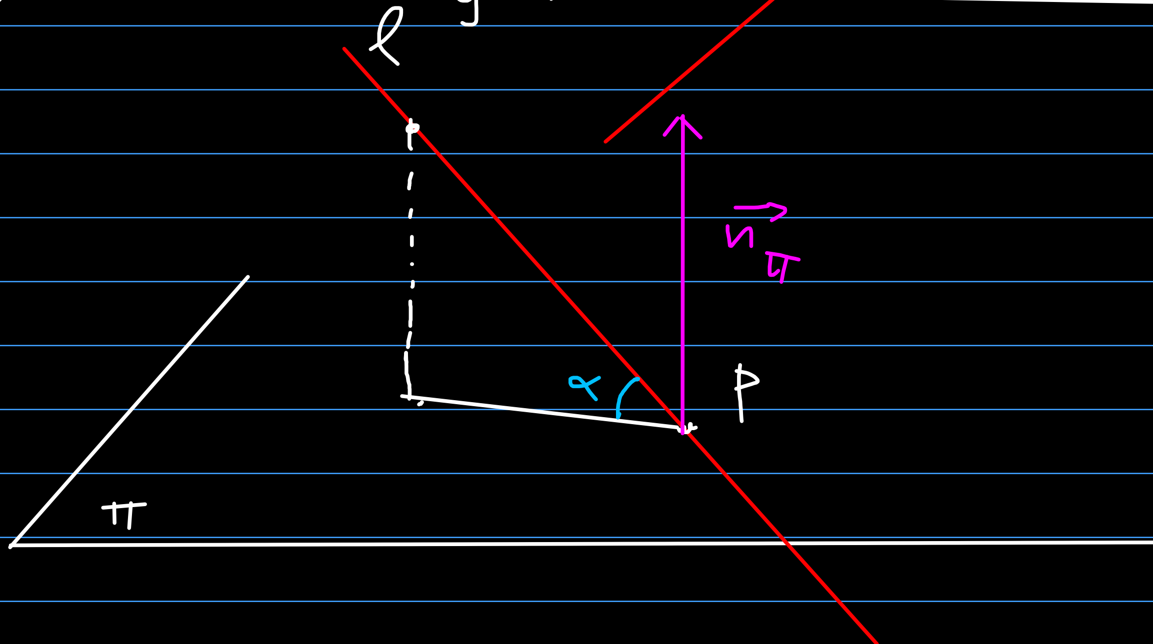
$\Pi_2 : 3x+2y-z \;=6$

(c) the plane xOy and the straight line M_1M_2, $M_1(1,2,3)$, $M_2(-2,1,4)$

xOy : $z=0 \Leftrightarrow 0x+0y+1.z+0=0$

$\Rightarrow \vec{n}_{xOy}=(0,0,1)$





$$m(\widehat{l, \pi}) = \frac{\pi}{2} - m(\widehat{\vec{n_\pi}, \vec{l}})$$

$M_1(1,2,3),\ M_2(-2,1,4)$

$\overrightarrow{M_1M_2} = (-3,-1,1) \qquad \vec{n_\pi} = (0,0,1)$

$$\Rightarrow \cos(\widehat{\overrightarrow{M_1M_2}, \vec{n_\pi}}) = \frac{\overrightarrow{M_1M_2} \cdot \vec{n_\pi}}{\|\overrightarrow{M_1M_2}\| \cdot \|\vec{n_\pi}\|} =$$

$$= \frac{1}{\sqrt{11}}$$

$$\Rightarrow \cos(\widehat{M_1M_2, xOy}) = \frac{\sqrt{10}}{\sqrt{11}} = \sqrt{\frac{10}{11}}$$

By convention the angle between two objects will always be in $[0, \frac{\pi}{2}]$

This means that in our case

$$m(\widehat{M_1M_2, xOy}) = \arccos\left(\sqrt{\frac{10}{11}}\right)$$

$\Pi: Ax + By + Cz + D = 0$, $P(x_0, y_0, z_0)$

$$\text{dist}(P, \Pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

<u>In the plane:</u>

$l: Ax + By + C = 0$

$$\text{dist}(P, l) = \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}}$$

5.5: Find the points on the z axis which are equidistant with respect to the planes:

$\Pi_1: 12x + 9y - 20z - 19 = 0$

$\Pi_2: 16x + 12y + 15z - 9 = 0$

(the intersection between the bisector planes of the dihedral angle and the z - axis)

Let $A(x_A, y_A, z_A)$. If $A \in z$-axis $\Rightarrow$

$\Rightarrow x_A = y_A = 0$

$$\operatorname{dist}(A, \pi_1) = \frac{|-20 \cdot z_A - 19|}{\sqrt{12^2 + 9^2 + 20^2}}$$

$$\operatorname{dist}(A, \pi_2) = \frac{|15 \cdot z_A - 9|}{\sqrt{16^2 + 12^2 + 15^2}}$$

$$\text{dist}(A, \pi_1) = \text{dist}(A, \pi_2) \iff$$

$$\iff \frac{|-20 \cdot z_A - 19|}{\sqrt{12^2+9^2+20^2}} = \frac{|15 \cdot z_A - 9|}{\sqrt{16^2+12^2+15^2}} \iff$$

$$\iff \frac{|20 z_A + 19|}{25} = \frac{|15 z_A - 9|}{25} \iff$$

$$\iff |20 z_A + 19| = |15 z_A - 9|$$

I $20 z_A + 19 = 15 z_A - 9 \Rightarrow 5 z_A = -28 \Rightarrow$

$\Rightarrow z_A = -\frac{28}{5} \Rightarrow A\left(0, 0, -\frac{28}{5}\right)$

II $20 z_A + 19 = -15 z_A + 9 \Rightarrow 35 z_A = -10$

$\Rightarrow z_A = -\frac{2}{7} \Rightarrow A\left(0, 0, -\frac{2}{7}\right)$

$|20 z_A + 19| = |19 - 112| = |-93| = 93$

$|15 z_A - 9| = |-3 \cdot 28 - 9| = |-84 - 9| = 93$

5.6. Consider two planes

$\Pi_1: \ A_1x + B_1y + C_1z + D_1 = 0$

$\Pi_2: \ A_2x + B_2y + C_2z + D_2 = 0$

which are not parallel and not perpendicular.

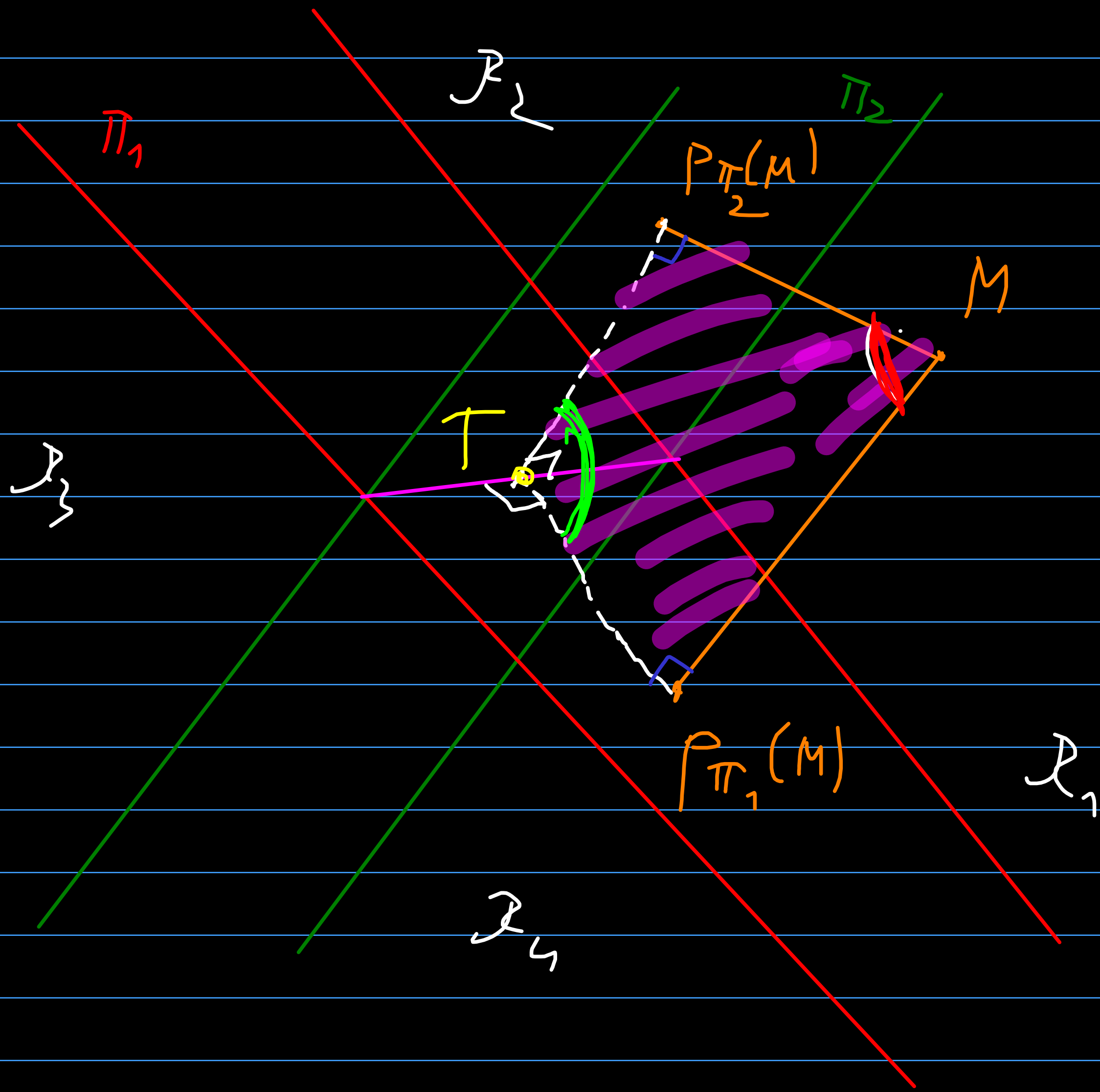
Π_1 and Π_2 divide the space into four regions R_1, R_2, R_3, R_4 so that $R_1 \cup R_3$ is the acute region

show that $M \in R_1 \cup R_3$ iff:

$$F_1(x,y,z) \cdot F_2(x,y,z) \cdot (A_1A_2 + B_1B_2 + C_1C_2) < 0$$

$$F_1(x,y,z) = A_1x + B_1y + C_1z + D_1$$

$$F_2(x,y,z) = A_2x + B_2y + C_2z + D_2$$



Let $p_{\Pi_1}(M)$ and $p_{\Pi_2}(M)$ be the orthogonal projections of M on Π_1 and Π_2, respectively. Let T be the projection of $p_{\Pi_1}(M)$ onto the common line $\{\ell\} = \Pi_1 \cap \Pi_2$

$$M p_{\Pi_1}(M) \perp \Pi_1 \Rightarrow M p_{\Pi_1}(M) \perp \ell$$

By construction, $p_{\Pi_1}(M)T \perp \ell$

Then $\ell \perp p_{\Pi_1}(M)T$ and $\ell \perp Mp_{\Pi_1}(M)$

$\Rightarrow \ell \perp (Mp_{\Pi_1}(M)T)$

$\ell \perp Mp_{\Pi_2}(M) \Rightarrow Mp_{\Pi_2}(M) \in (MTp_{\Pi_1}(M))$

$\Rightarrow \quad p_{\Pi_2}(M)T \in (MTp_{\Pi_1}(M))$

$\Rightarrow \ell \perp p_{\Pi_2}(M)T$

$M\,p_{\Pi_2}(M)\,T\,p_{\Pi_1}(M)$ is a quadrilateral in the same plane.

$$\left.\begin{array}{l} Mp_{\Pi_1}(M) \perp Tp_{\Pi_1}(M) \text{ and} \\ Mp_{\Pi_2}(M) \perp Tp_{\Pi_2}(M) \end{array}\right\} \Rightarrow Mp_{\Pi_1}(M)Tp_{\Pi_2}(M) \text{ inscriptible quadrilateral}$$

$$\Rightarrow m\left(\widehat{\overrightarrow{MP_{\pi_1}(M)},\ \overrightarrow{MP_{\pi_2}(M)}}\right) =$$

$$= \pi - m\left(\widehat{P_{\pi_2}(M)\,T\,P_{\pi_1}(M)}\right)$$

$\Rightarrow M \in$ acute region $\Longleftrightarrow$

$$\Longleftrightarrow m\left(\widehat{\overrightarrow{MP_{\pi_1}(M)},\ \overrightarrow{MP_{\pi_2}(M)}}\right) > \frac{\pi}{2}$$

$$\Longleftrightarrow \overrightarrow{MP_{\pi_1}(M)} \cdot \overrightarrow{MP_{\pi_2}(M)} < 0$$

$$\overrightarrow{MP_{\pi_1}(M)} = \vec{r}_{P_{\pi_1}(M)} - \vec{r}_M =$$

$$= \left(\vec{r}_M - \frac{F_1(M)}{\|\vec{n}_{\pi_1}\|^2}\cdot \vec{n}_{\pi_1}\right) - \vec{r}_M$$

$$= -\frac{F_1(M)}{\|\vec{n}_{\pi_1}\|^2}\cdot \vec{n}_{\pi_1}$$

$$\overrightarrow{M\,p_{\pi_2}(M)} = -\frac{F_2(M)}{\|\vec{n}_{\pi_2}\|^2} \cdot \vec{n}_{\pi_2}$$

$$M \in \text{auto region} \iff \left(-\frac{F_1(M)}{\|\vec{n}_{\pi_1}\|^2} \cdot \vec{n}_{\pi_1}\right) \cdot \left(-\frac{F_2(M)}{\|\vec{n}_{\pi_2}\|^2} \cdot \vec{n}_{\pi_2}\right) < 0 \iff$$

$$\iff F_1(M) \cdot F_2(M) \cdot \left(\vec{n}_{\pi_1} \cdot \vec{n}_{\pi_2}\right) < 0 \iff$$

$$\iff F_1(M) \cdot F_2(M) \cdot (A_1 A_2 + B_1 B_2 + C_1 C_2) < 0$$