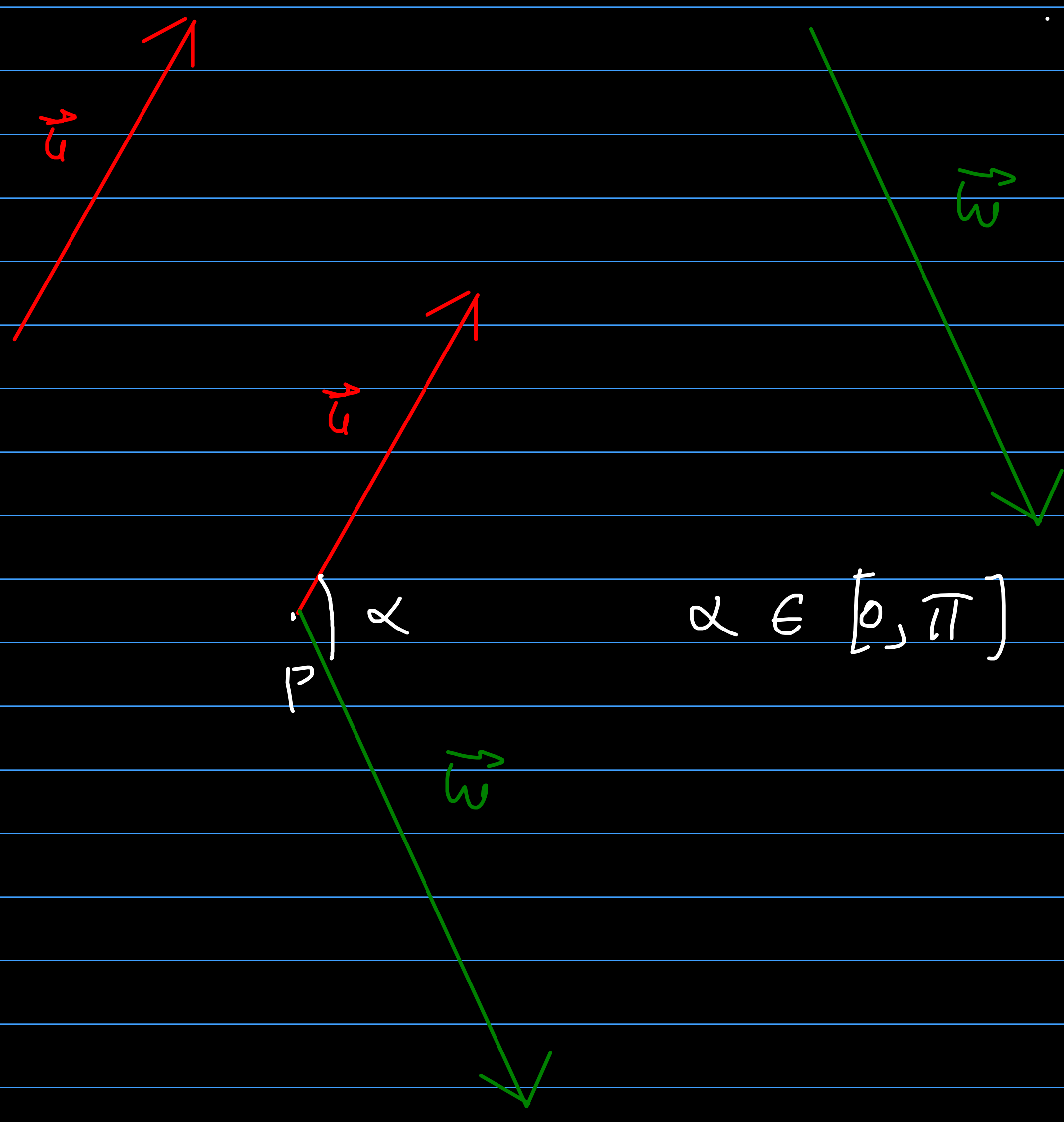
Seminar W5 – 917

Dot product (scalar product):

$\vec{u}, \vec{w} \in V$, $\vec{u} \cdot \vec{w} = \|\vec{u}\| \cdot \|\vec{w}\| \cdot \cos(\widehat{\vec{u}, \vec{w}})$



$\alpha \in [0, \pi]$

3) the reference system is orthonormal, then if $\vec{u}(a_1, b_1, c_1)$, $\vec{w}(a_2, b_2, c_2)$

$$\boxed{\vec{u} \cdot \vec{w} = a_1 \cdot a_2 + b_1 b_2 + c_1 c_2}$$

$\mathcal{R} = (O, [\vec{i}, \vec{j}, \vec{k}])$

orthonormal = orthogonal + normed

$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0 \qquad \|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\|$

5.3. Find the angle between

(a) $d_1 : \begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases}$

$d_2 : \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0 \end{cases}$

(b) $\pi_1 : x + 3y + 2z + 1 = 0$

$\pi_2 : 3x + 2y - z = 6$

(c) the plane xOy and the line M_1M_2, where $M_1(1,2,3)$, $M_2(-2,1,4)$

(a) $d_1: \begin{cases} x+2y+z-1=0 \\ x-2y+z+1=0 \end{cases}$

$d_1: \begin{cases} 2x+2z=0 \\ x+2y+z-1=0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x=-z \\ -z+2y+z-1=0 \end{cases} \Leftrightarrow \begin{cases} 2y-1=0 \\ x=-z \end{cases} \Leftrightarrow$

$\Leftrightarrow d_1: \begin{cases} x=-\lambda \\ y=\frac{1}{2} \\ z=\lambda \end{cases} \Rightarrow \vec{d_1}(-1,0,1)$

$d_2: \begin{cases} x-y-z-1=0 \\ x-y+2z+1=0 \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} z = x - y - 1 \\ x - y + 2z + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = x - y - 1 \\ x - y + 2x - 2y - 2 + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = x - y - 1 \\ 3x - 3y - 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = x - y - 1 \\ y = x - \frac{1}{3} \end{cases} \Leftrightarrow \begin{cases} z = x - x + \frac{1}{3} - 1 \\ y = x - \frac{1}{3} \end{cases}$$

$$\Leftrightarrow \begin{cases} z = -\frac{2}{3} \\ y = x - \frac{1}{3} \end{cases} \Leftrightarrow d_2: \begin{cases} x = t \\ y = t - \frac{1}{3} \\ z = -\frac{2}{3} \end{cases}$$

$$\Rightarrow \vec{d_2}\,(1, 1, 0)$$

$\vec{d_1}(-1,0,1), \quad \vec{d_2}(1,1,0)$

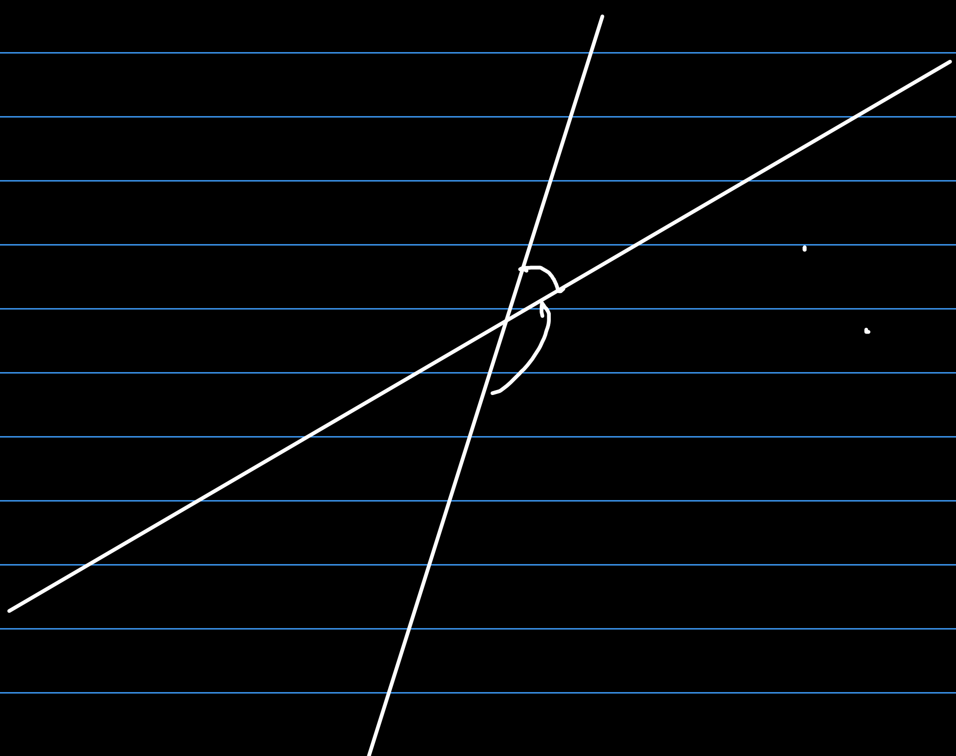
$\vec{d_1} \cdot \vec{d_2} = -1+0+0 = -1$

$\|\vec{d_1}\| = \sqrt{(-1)^2+0^2+1^2} = \sqrt{2}$

$\|\vec{d_2}\| = \sqrt{1^2+1^2+0} = \sqrt{2}$

$\cos(\vec{d_1}, \vec{d_2}) = \dfrac{-1}{\sqrt{2}\cdot\sqrt{2}} = -\dfrac{1}{2}$

$\Rightarrow m(\widehat{d_1, d_2}) = \dfrac{2\pi}{3}$



(b) $\Pi_1 : x + 3y + 2z + 1 = 0$

$\Pi_2 : 3x + 2y - z = 6$

$\overrightarrow{n_{\Pi_1}} = (1, 3, 2)$, $\overrightarrow{n_{\Pi_2}} = (3, 2, -1)$

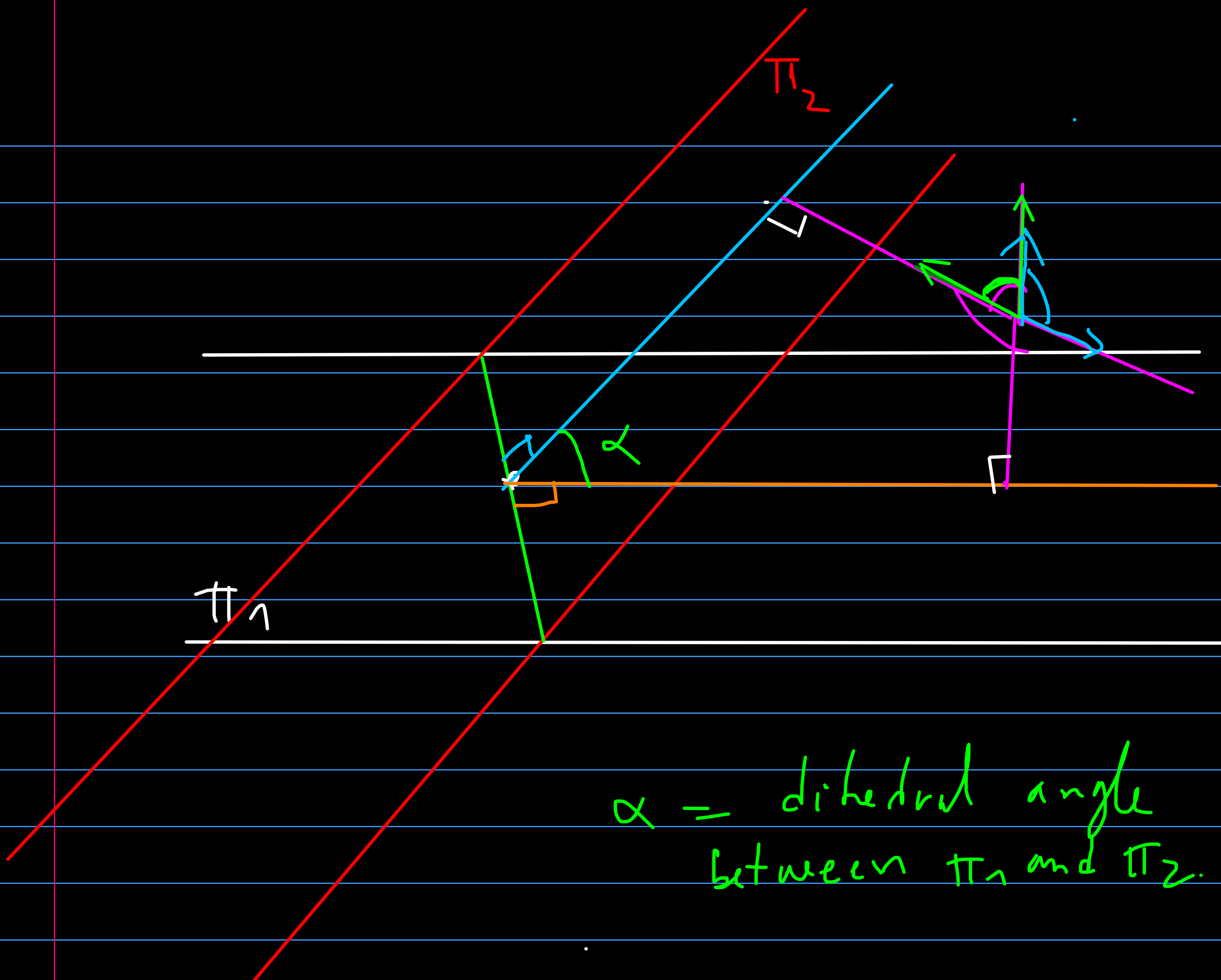
$\overrightarrow{n_{\Pi_1}} \cdot \overrightarrow{n_{\Pi_2}} = 3 + 6 - 2 = 7$

$\|\overrightarrow{n_{\Pi_1}}\| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$

$\|\overrightarrow{n_{\Pi_2}}\| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$

$\cos(\widehat{\overrightarrow{n_{\Pi_1}}, \overrightarrow{n_{\Pi_2}}}) = \frac{7}{\sqrt{14} \cdot \sqrt{14}} = \frac{1}{2}$

$\Rightarrow m(\widehat{\Pi_1, \Pi_2}) = m(\widehat{\overrightarrow{n_{\Pi_1}}, \overrightarrow{n_{\Pi_2}}}) = \frac{\pi}{3}$



α = dihedral angle between π_1 and π_2.

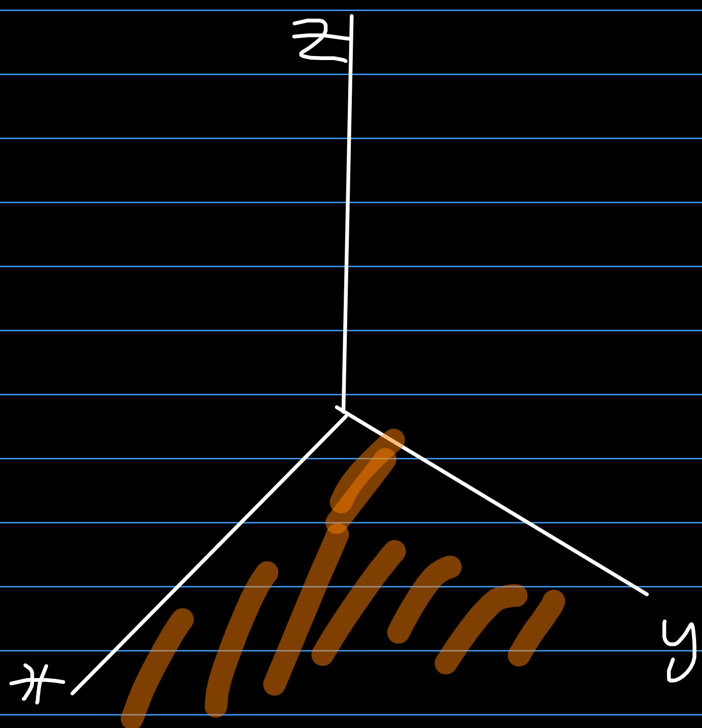
(c) the plane xOy and the line M_1M_2, where $M_1(1,2,3)$, $M_2(-2,1,4)$

$$M_1M_2: \frac{x-1}{-2-1} = \frac{y-2}{1-2} = \frac{z-3}{4-3}$$

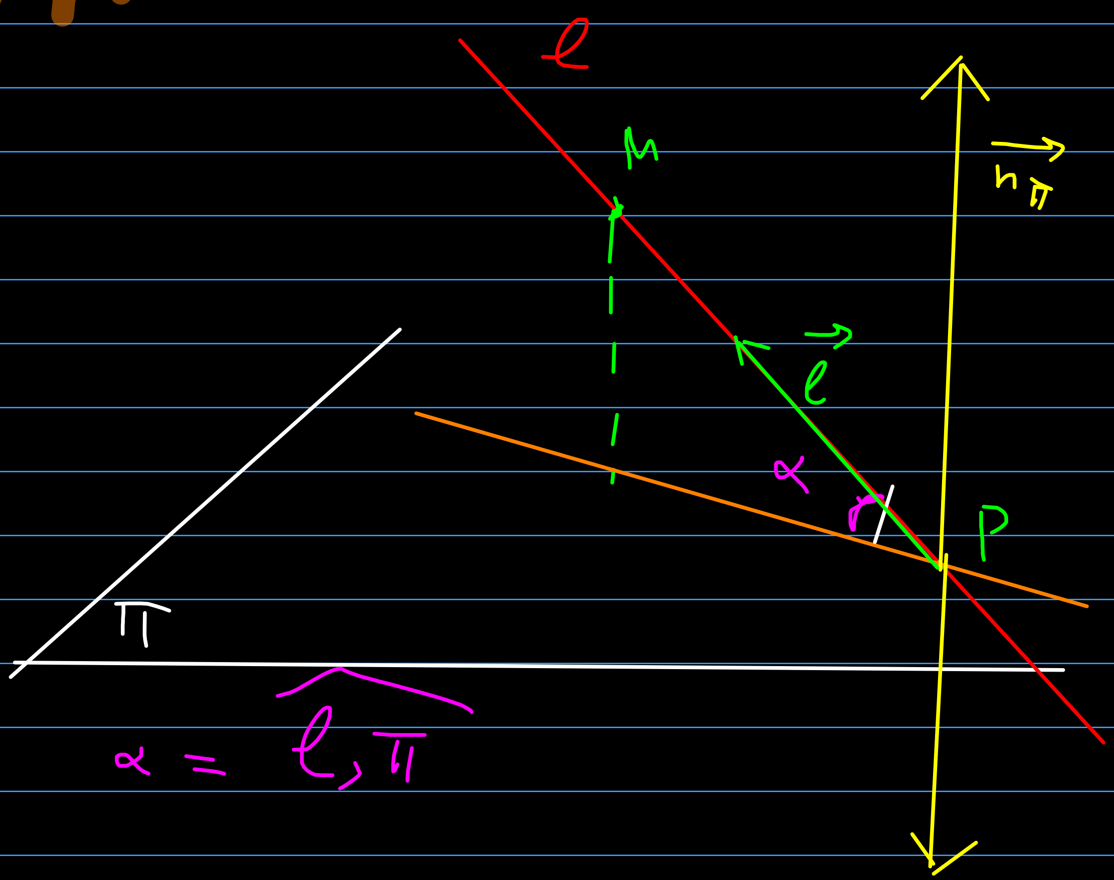
$$\Rightarrow M_1M_2: \frac{x-1}{-3} = \frac{y-2}{-1} = \frac{z-3}{1}$$

$$\overrightarrow{M_1M_2} = (-3, -1, 1)$$

xOy : $z = 0$



$\Rightarrow \overrightarrow{n_{xOy}} = (0, 0, 1)$



$\alpha = \widehat{\ell, \pi}$

$$\vec{n}_{xOy} = (0, 0, 1)$$

$$\overrightarrow{M_1M_2} = (-3, -1, 1)$$

$$\cos\left(\vec{n}_{xOy}, \overrightarrow{M_1M_2}\right) = \frac{0\cdot(-3)+0\cdot(-1)+1\cdot 1}{\sqrt{9+1+1}\cdot\sqrt{1}} =$$

$$= \frac{1}{\sqrt{11}}$$

$$\arccos\left(\frac{1}{\sqrt{11}}\right) \in \left[0, \frac{\pi}{2}\right] \Rightarrow$$

$$\Rightarrow m(M_1M_2, xOy) = \frac{\pi}{2} - \arccos\left(\frac{1}{\sqrt{11}}\right)$$

The distance from a point to a plane

$\pi : Ax + By + Cz + D = 0$

$P(x_0, y_0, z_0)$

$$\text{dist}(P, \pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2+B^2+C^2}}$$

The distance from a point to a line in 2D

$l : Ax + By + C = 0$

$P(x_0, y_0)$

$$\text{dist}(P, l) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}$$

5.5. Find the points on the z-axis which are equidistant with respect to the planes

$$\pi_1 : 12x + 9y - 20z - 19 = 0$$

$$\pi_2 : 16x + 12y + 15z - 9 = 0$$

Read 5.6

$$P(x, y, z) \in Oz \iff x = y = 0, \quad z = g$$

$$\text{dist}(P, \pi_1) = \frac{|12 \cdot 0 + 9 \cdot 0 - 20 \cdot g - 19|}{\sqrt{12^2 + 9^2 + 20^2}}$$

$$\text{dist}(P, \pi_2) = \frac{|16 \cdot 0 + 12 \cdot 0 + 15 \cdot g - 9|}{\sqrt{16^2 + 12^2 + 15^2}}$$

$$\mathrm{dist}(P, \pi_1) = \frac{|-20q - 19|}{25}$$

$$\mathrm{dist}(P, \pi_2) = \frac{|15q - 9|}{25}$$

So $\mathrm{dist}(P, \pi_1) = \mathrm{dist}(P, \pi_2) \iff$

$$\iff |-20q - 19| = |15q - 9|$$

$$\iff -20q - 19 = \pm(15q - 9)$$

<u>Case 1</u> : $-20q - 19 = 15q - 9$

$$\Rightarrow q = -\frac{10}{35} = -\frac{2}{7}$$

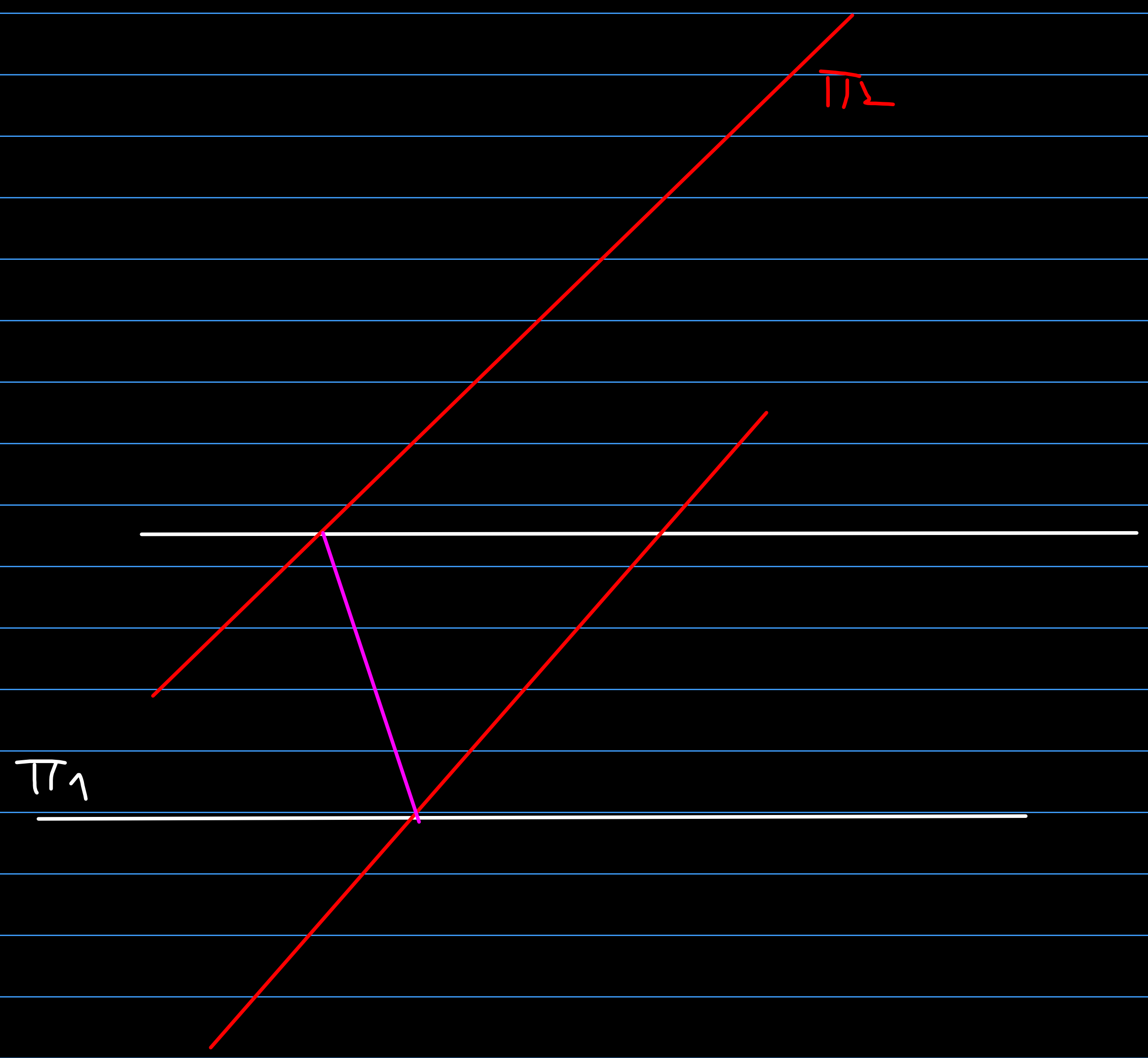
<u>Case 2</u> : $-20q - 19 = 9 - 15q$

$$\Rightarrow q = -\frac{28}{5}$$

5.6. $\Pi_1 : A_1 x + B_1 y + C_1 z + D_1 = 0$

$\Pi_2 : A_2 x + B_2 y + C_2 z + D_2 = 0$

$\Pi_1 \not\parallel \Pi_2$, $\Pi_1 \not\perp \Pi_2$



$F_1(x, y, z) = A_1 x + B_1 y + C_1 z + D_1$

$F_2(x, y, z) = A_2 x + B_2 y + C_2 z + D_2$

$M(x_0, y_0, z_0) \in$ acute region $\Leftrightarrow$

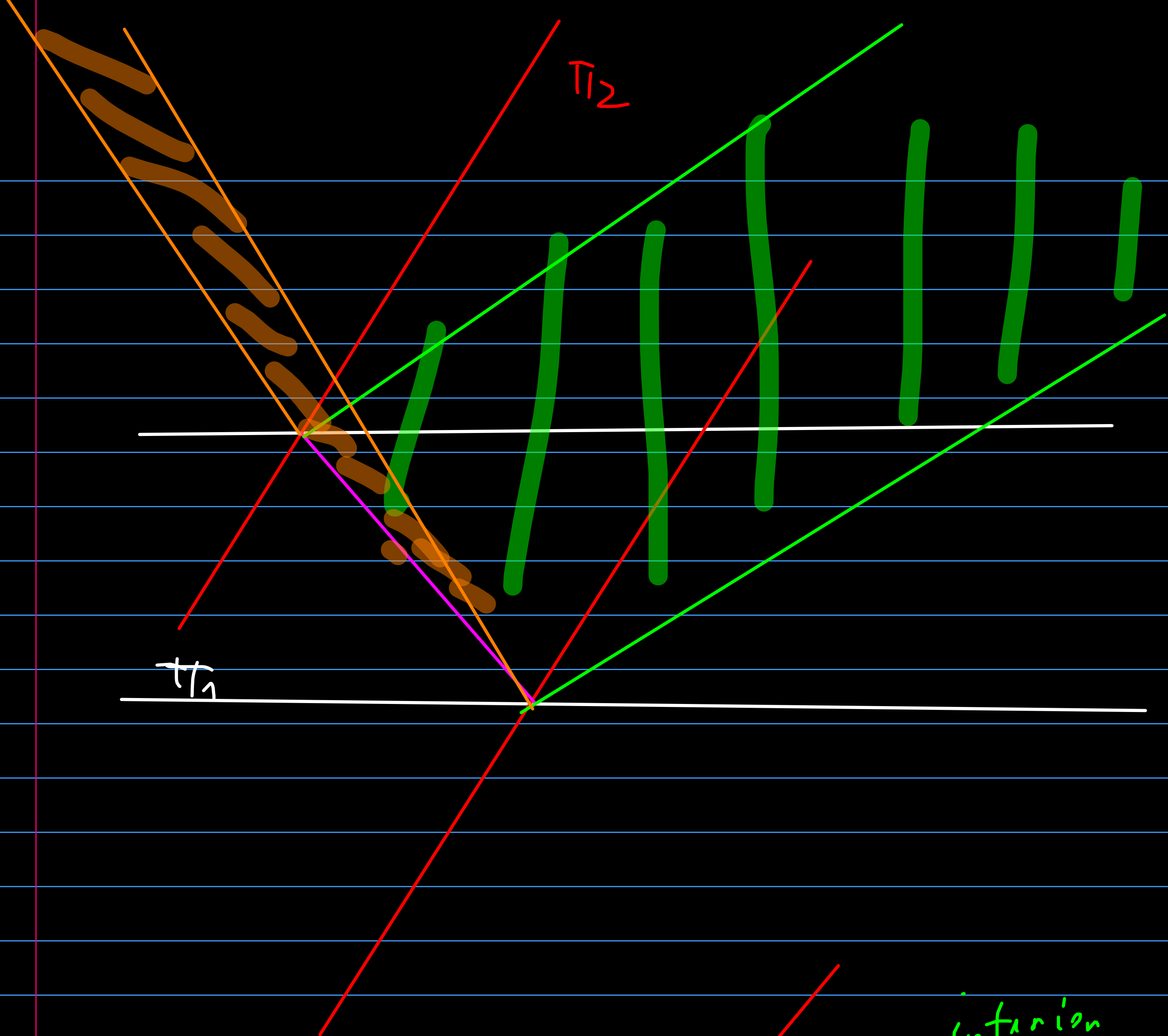
$\Leftrightarrow F_1(x_0, y_0, z_0) \cdot F_2(x_0, y_0, z_0) \cdot$

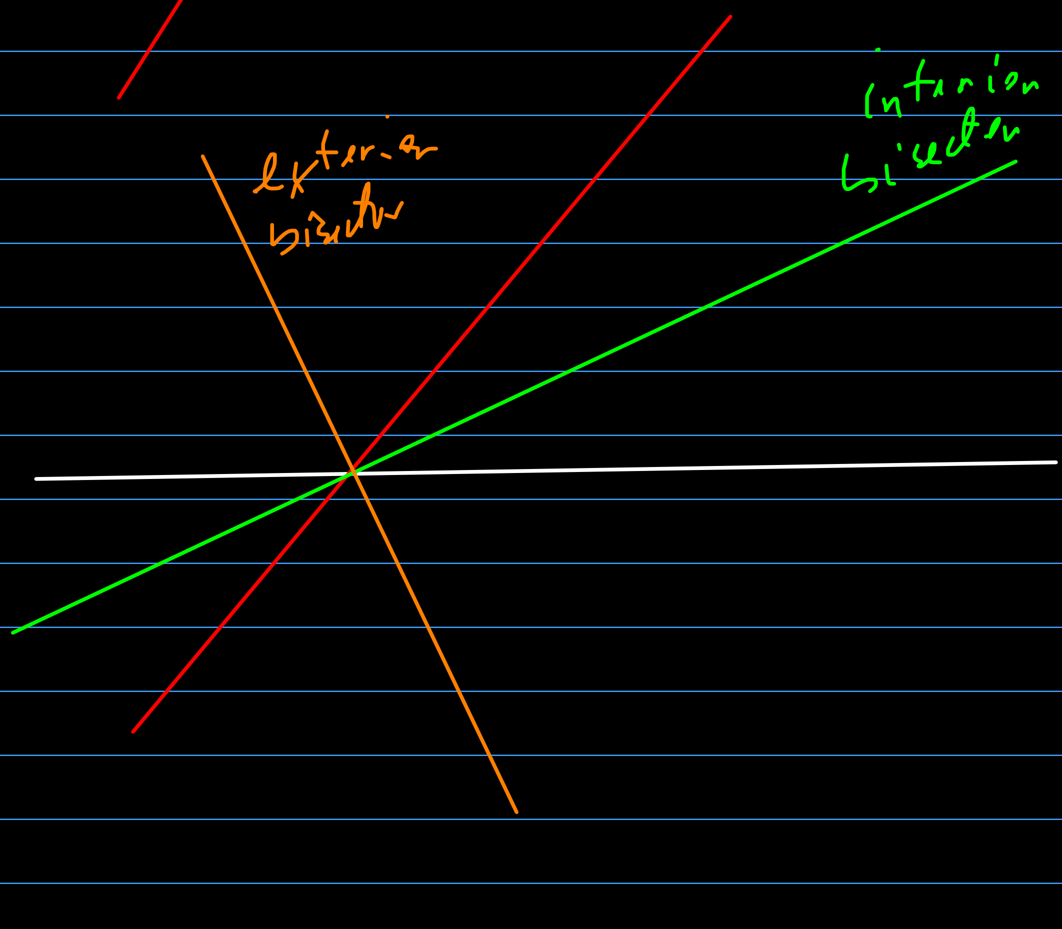
$\cdot \underbrace{(A_1A_2 + B_1B_2 + C_1C_2)}_{\vec{n}_{\Pi_1} \cdot \vec{n}_{\Pi_2}} < 0$

5.7. (3p.) $\Pi_1 : 2x + y - 3z - 5 = 0$

$\Pi_2 : x + 3y + 2z + 1 = 0$

Find the equations of the bisector planes of the dihedral angles formed by the planes Π_1 and Π_2 and select the one contained in the acute regions of the dihedral angles formed by the two planes.


π_2
π_1


exterior bisector
interior bisector

$M(x, y, z)$

$$\text{dist}(M, \Pi_1) = \frac{|2x + y - 3z - 5|}{\sqrt{14}}$$

$$\text{dist}(M, \Pi_2) = \frac{|x + 3y + 2z + 1|}{\sqrt{14}}$$

$$\text{dist}(M, \Pi_1) = \text{dist}(M, \Pi_2) \iff$$

$$\iff |2x + y - 3z - 5| = |x + 3y + 2z + 1|$$

$$\iff 2x + y - 3z - 5 = \pm (x + 3y + 2z + 1)$$

<u>Case 1</u> : $2x + y - 3z - 5 = x + 3y + 2z + 1$

$$\Rightarrow \alpha_1 : x - 2y - 5z - 6 = 0$$

<u>Case 2</u> $2x + y - 3z - 5 = -x - 3y - 2z - 1$

$$\Rightarrow \alpha_2 : 3x + 4y - z - 4 = 0$$

α_1 and α_2 are the bisector planes

$\alpha_1: x - 2y - 5z - 6 = 0$

$\pi_1: 2x + y - 3z - 5 = 0$

$\pi_2: x + 3y + 2z + 1 = 0$

$\cos(\widehat{\pi_1, \alpha_1}) = \cos(\widehat{\vec{n}_{\pi_1}, \vec{n}_{\alpha_1}}) =$

$= \dfrac{2 - 2 + 15}{\sqrt{30} \cdot \sqrt{14}} = \sqrt{\dfrac{15}{28}} = \dfrac{\sqrt{\frac{15}{14}}}{\sqrt{2}} > \dfrac{1}{\sqrt{2}}$

$m(\widehat{\pi_1, \alpha_1}) \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow m(\widehat{\pi_1, \alpha_1}) < \frac{\pi}{4}$

$\Rightarrow$ α_1 is in the acute region

Another approach: $P(0, -3, 0) \in \alpha_1$

We check if P is in the acute region

$F_1(P) = -8 \quad , \quad F_2(P) = -8$

$\overrightarrow{n_{\pi_1}} \cdot \overrightarrow{n_{\pi_2}} = 5 - 6 = -1$

$\Rightarrow F_1(P) \cdot F_2(P) \cdot (\overrightarrow{n_{\pi_1}} \cdot \overrightarrow{n_{\pi_2}}) < 0$

$\Rightarrow P \in$ acute region $\Rightarrow \alpha_1 \subset$ acute region