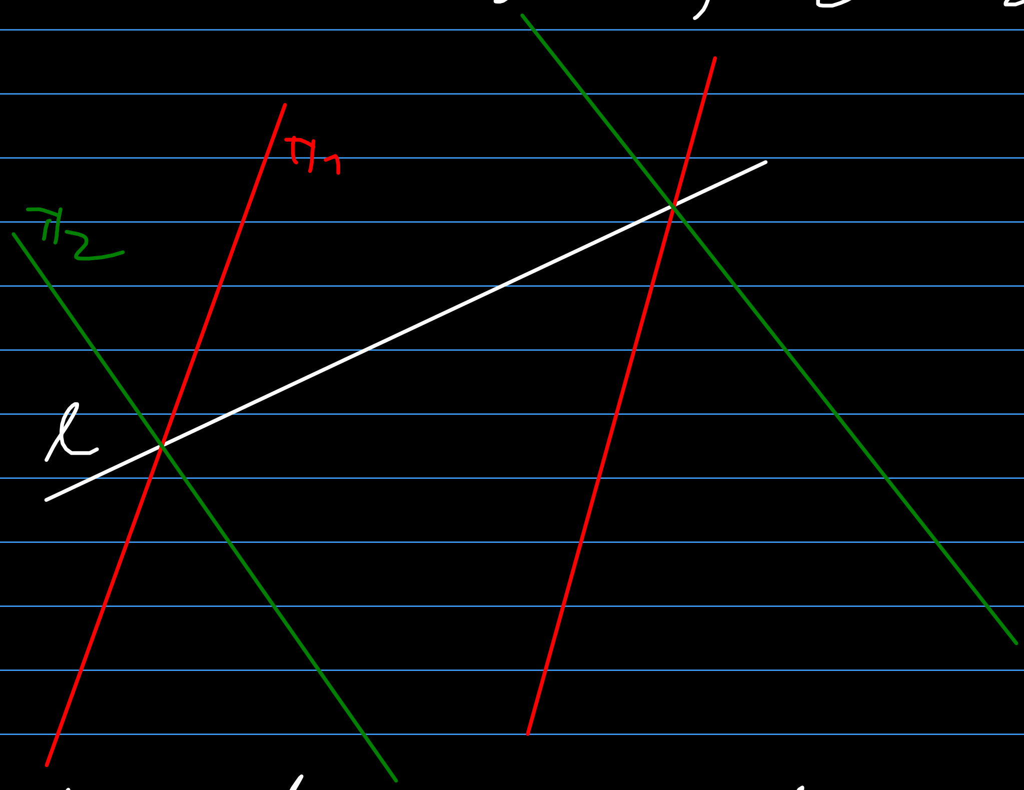
Seminar W4 - 917

Pencil of planes

$$\ell: \begin{cases} \pi_1: A_1 x + B_1 y + C_1 z + D_1 = 0 \\ \pi_2: A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$



Any plane containing ℓ is of the form:

$$\pi_{\alpha,\beta}: \alpha \cdot (A_1 x + B_1 y + C_1 z + D_1) + \beta \cdot (A_2 x + B_2 y + C_2 z + D_2) = 0$$

4.1. Write the equation of the plane determined by the line:

$$\ell : \begin{cases} x - 2y + 3z = 0 \\ 2x + z - 3 = 0 \end{cases}$$

and the point $A(-1, 2, 6)$

We write the equation of a plane that contains ℓ.

$$\pi_{\alpha,\beta} : \alpha \cdot (x - 2y + 3z) + \beta \cdot (2x + z - 3) = 0$$

$$\pi_{\alpha,\beta} : x(\alpha + 2\beta) + y(-2\alpha) + z(3\alpha + \beta) - 3\beta = 0$$

$$A \in \pi_{\alpha,\beta} \Rightarrow (-1)(\alpha + 2\beta) + 2 \cdot (-2\alpha) + 6(3\alpha + \beta) - 3\beta = 0$$

$$\Rightarrow -\alpha - 2\beta - 4\alpha + 18\alpha + 6\beta - 3\beta = 0 \Leftrightarrow$$

$$\Leftrightarrow 13\alpha + \beta = 0 \Leftrightarrow \beta = -13\alpha$$

Then the planes that we want are $\pi_{\alpha,-13\alpha}$

$$\pi_{\alpha,-13\alpha} : -25\alpha x - 2\alpha y - 10\alpha z + 39\alpha = 0$$

$\Rightarrow \Pi_{\alpha,-13\alpha}: \alpha(-25x - 2y - 10z + 39) = 0$

$\alpha \neq 0 \Rightarrow$ a unique solution

$\Pi_{1,-13}: -25x - 2y - 10z + 39 = 0$

$\Pi: Ax + By + Cz + D = 0$

$\Rightarrow \vec{n_\Pi}(A, B, C)$ normal vector for Π

$\left(\vec{n_\Pi} \perp \vec{\omega},\ \forall \vec{\omega} \parallel \Pi \right)$

$\Updownarrow$

$\vec{\omega} \in \vec{\Pi}$

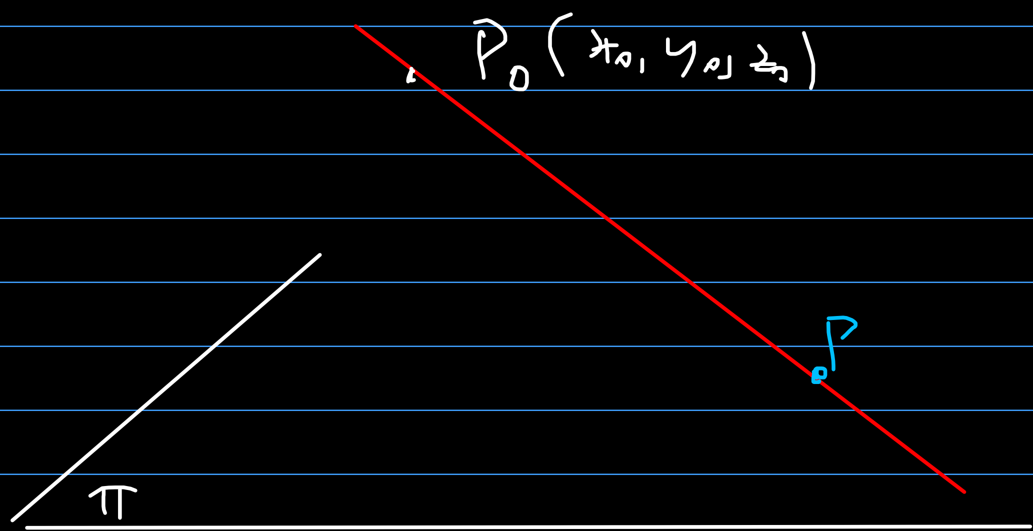
if l line, $l: \begin{cases} x = x_0 + \lambda x_{\vec{\omega}} \\ y = y_0 + \lambda y_{\vec{\omega}} \\ z = z_0 + \lambda z_{\vec{\omega}} \end{cases}$

$l \parallel \Pi \Leftrightarrow A x_{\vec{\omega}} + B y_{\vec{\omega}} + C z_{\vec{\omega}} = 0$

$\Updownarrow$

$\vec{n_\Pi} \cdot \vec{\omega} = 0$

3) $\ell \nparallel \pi$ (i.e. $Ax_{\vec{v}} + By_{\vec{v}} + Cz_{\vec{v}} \neq 0$),
then $\exists \{P\} = \ell \cap \pi$



Then:

$$\begin{cases} x_P = x_0 - \dfrac{Ax_0 + By_0 + Cz_0 + D}{Ax_{\vec{v}} + B \cdot y_{\vec{v}} + Cz_{\vec{v}}} \cdot x_{\vec{v}} \\[2ex] y_P = y_0 - \dfrac{Ax_0 + By_0 + Cz_0 + D}{Ax_{\vec{v}} + B \cdot y_{\vec{v}} + Cz_{\vec{v}}} \cdot y_{\vec{v}} \\[2ex] z_P = z_0 - \dfrac{Ax_0 + By_0 + Cz_0 + D}{Ax_{\vec{v}} + B \cdot y_{\vec{v}} + Cz_{\vec{v}}} \cdot z_{\vec{v}} \end{cases}$$

Ex.: $\Pi: x+2y-3z+6=0$

$$l: \frac{x-1}{3} = \frac{y+2}{2} = \frac{z-3}{-6}$$

Find the coordinates of $\{P\} = l \cap \Pi$ (without using the formulas).

$$l: \begin{cases} x = 1+3\lambda \\ y = -2+2\lambda \\ z = 3-6\lambda \end{cases}$$

$$P: \begin{cases} x = 1+3\lambda \\ y = -2+2\lambda \\ z = 3-6\lambda \\ x+2y-3z+6=0 \end{cases} \Leftrightarrow$$

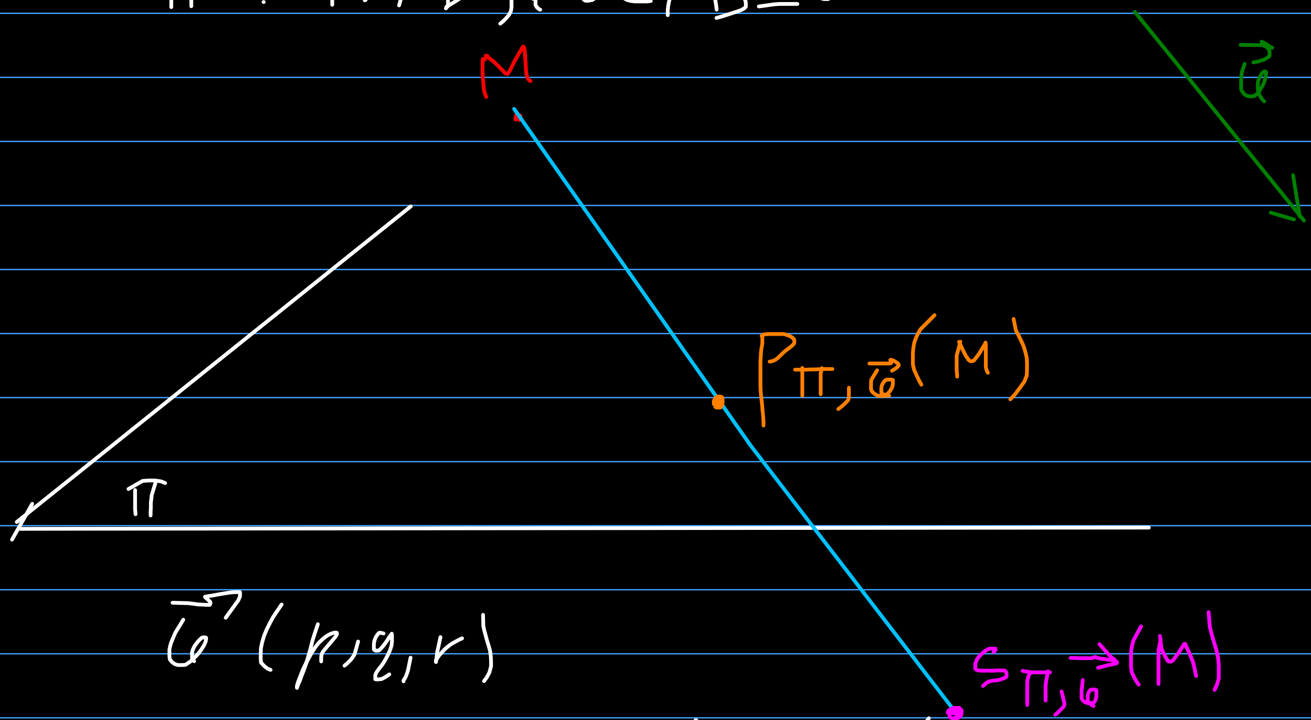
$$\Leftrightarrow \begin{cases} x = 1+3\lambda \\ y = -2+2\lambda \\ z = 3-6\lambda \\ 1+3\lambda+(-4)+4\lambda-9+18\lambda+6=0 \end{cases} \Leftrightarrow$$

$$(\Leftarrow) \quad P: \begin{cases} x = 1 + 3\lambda \\ y = -2 + 2\lambda \\ z = 3 - 6\lambda \\ -6 + 25\lambda = 0 \Rightarrow \lambda = \dfrac{6}{25} \end{cases}$$

$$\Rightarrow P: \begin{cases} x = \dfrac{43}{25} \\ y = \dfrac{-38}{25} \\ z = \dfrac{39}{25} \end{cases}$$

Projections and reflections

$\Pi : Ax+By+Cz+D=0$



$\vec{u}(p,q,r)$

We need to have $\vec{u} \nparallel \Pi$ (i.e. $Ap+Bq+Cr\neq 0$)

We define the projection onto Π, parallel with $\vec{u}$, as the function:

$$P_{\Pi,\vec{u}} : \mathbb{R}^3 \longrightarrow \Pi$$

$$(x,y,z) \longmapsto (x',y',z')$$

$$S_{\Pi,\vec{u}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x,y,z) \longmapsto (x'',y'',z'')$$

$$\begin{cases} x' = x - \dfrac{Ax+B\cdot y+Cz+D}{Ap+Bq+Cr}\cdot p \\[2ex] y' = y - \dfrac{Ax+B\cdot y+Cz+D}{Ap+Bq+Cr}\cdot q \\[2ex] z' = z - \dfrac{Ax+B\cdot y+Cz+D}{Ap+Bq+Cr}\cdot r \end{cases}$$

$$Ap+Bq+Cr = \vec{n_\pi}\cdot\vec{u}$$

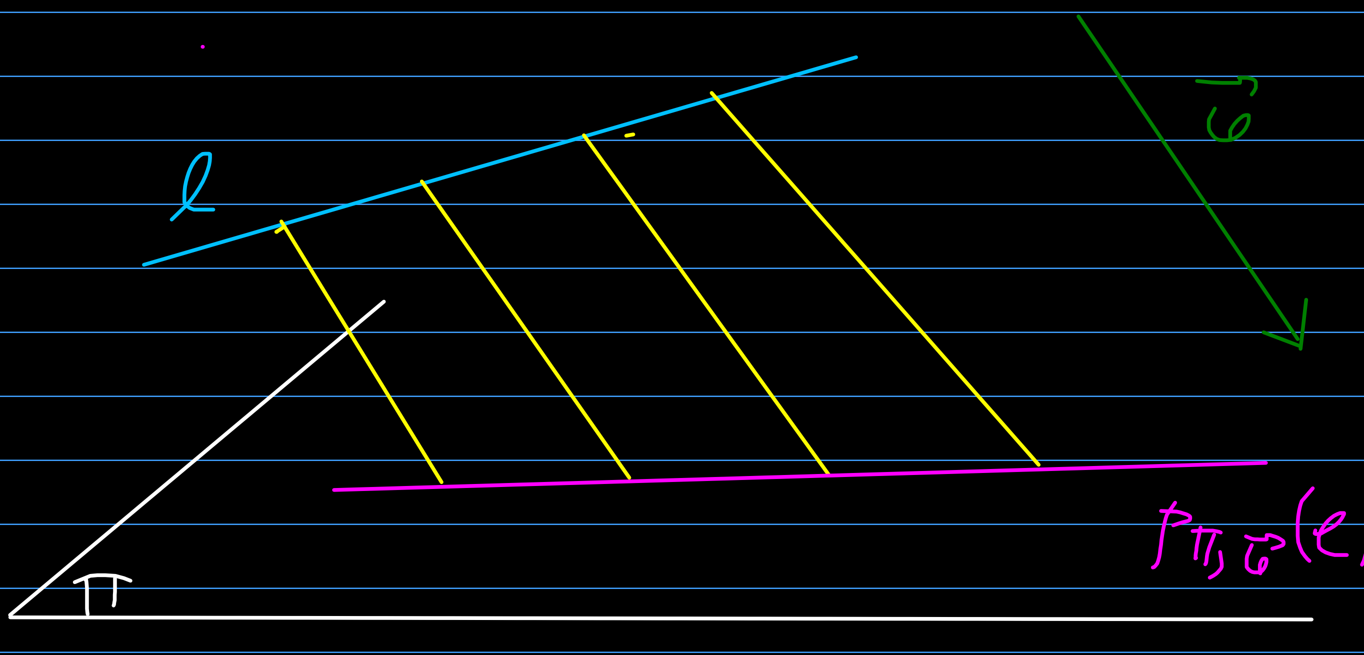
$$\begin{cases} x'' = x - 2\cdot\dfrac{Ax+B\cdot y+Cz+D}{Ap+Bq+Cr}\cdot p \\[2ex] y'' = y - 2\cdot\dfrac{Ax+B\cdot y+Cz+D}{Ap+Bq+Cr}\cdot q \\[2ex] z'' = z - 2\cdot\dfrac{Ax+B\cdot y+Cz+D}{Ap+Bq+Cr}\cdot r \end{cases}$$

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

9.3. Write the equations of the projection of the line

$$\ell : \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

on the plane $\Pi : x + 2y - z = 0$ parallel to the direction $\vec{u}(1, 1, -2)$.



$$\ell : \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

$$\ell : \begin{cases} 2x - y = 1 - z \\ x + y = z - 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x + y = z - 1 \\ 3x = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = z - 1 \\ z = z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = \lambda - 1 \\ z = \lambda \end{cases}$$

$$\begin{cases} x' = x - \dfrac{Ax + B.y + Cz + D}{Ap + Bq + Cr} \cdot p \\ y' = y - \dfrac{Ax + B.y + Cz + D}{Ap + Bq + Cr} \cdot q \\ z' = z - \dfrac{Ax + B.y + Cz + D}{Ap + Bq + Cr} \cdot r \end{cases}$$

$$\vec{u} = (p, q, r) = (1, 1, -2)$$

$$(A, B, C, D) = (1, 2, -1, 0)$$

$$\frac{Ax+By+Cz+D}{Ap+Bs+Cr} = \frac{1\cdot 0+2\cdot(\lambda-1)+(-1)\lambda}{1\cdot 1+2\cdot 1+(-1)(-2)} =$$

$$= \frac{\lambda-2}{5}$$

$$\Rightarrow \begin{cases} x' = 0 - \frac{\lambda-2}{5}\cdot 1 \\ y' = \lambda-1-\frac{\lambda-2}{5}\cdot 1 \\ z' = \lambda - \frac{\lambda-2}{5}\cdot(-2) \end{cases}$$

$$\Rightarrow P_{\pi,\vec{v}}(\ell): \begin{cases} x' = -\frac{1}{5}\lambda + \frac{2}{5} \\ y' = \frac{4}{5}\lambda - \frac{3}{5} \\ z' = \frac{7}{5}\lambda - \frac{4}{5} \end{cases}, \lambda \in \mathbb{R}$$

This the parametric equation of a line.

We will now do the same for the reflection. $s_{\pi,\vec{b}}$

$$\begin{cases} x'' = -\frac{2}{5}\lambda + \frac{4}{5} \\ y'' = \frac{3}{5}\lambda - \frac{1}{5} \\ z'' = \frac{9}{5}\lambda - \frac{8}{5} \end{cases}$$

$$\begin{cases} x = 0 \\ y = \lambda - 1 \\ z = \lambda \end{cases}$$