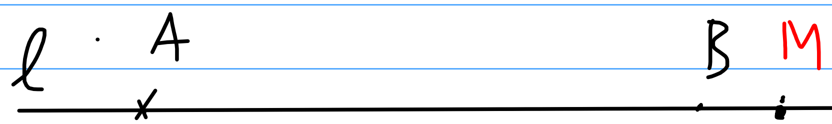
Seminar W2 – 9 15

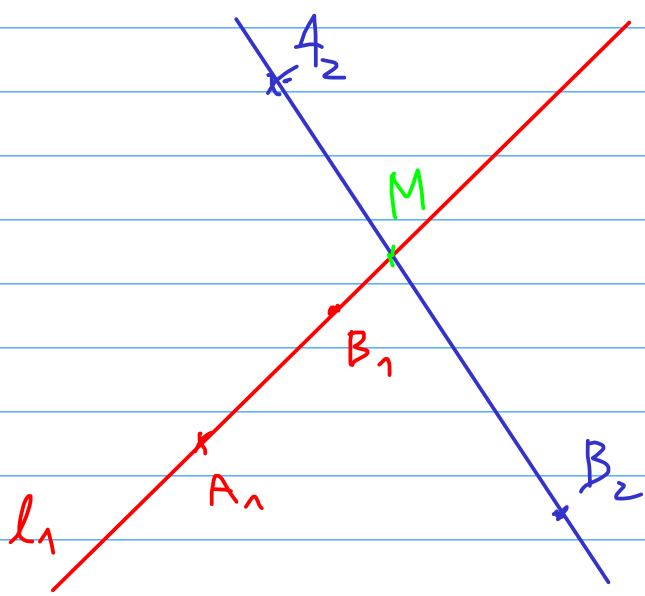


ℓ line, $A, B \in \ell$

$\forall M \in \ell \ \exists! \lambda \in \mathbb{R} : \boxed{\vec{r}_M = \lambda \vec{r}_A + (1-\lambda)\vec{r}_B}$

In the particular case $M \in [AB]$ and

$$\frac{AM}{MB} = \alpha \in \mathbb{R} \Rightarrow \vec{r}_M = \frac{\alpha}{\alpha+1}\vec{r}_B + \frac{1}{\alpha+1}\vec{r}_A$$



ℓ_1, ℓ_2 lines in the plane

$A_1, B_1 \in \ell_1$

$A_2, B_2 \in \ell_2$

$\{M\} = \ell_1 \cap \ell_2$

$\vec{r}_M = ?$

Template for proofs

Step 1: Write M as a point on l_1 and l_2

$$M \in l_1 \Rightarrow \exists! \lambda \in \mathbb{R} : \vec{r_M} = \lambda \vec{r_{A_1}} + (1-\lambda)\vec{r_{B_1}} \quad (1)$$

$$M \in l_2 \Rightarrow \exists! \mu \in \mathbb{R} : \vec{r_M} = \mu \vec{r_{A_2}} + (1-\mu)\vec{r_{B_2}} \quad (2)$$

$$\Rightarrow \underline{\lambda \vec{r_{A_1}} + (1-\lambda)\vec{r_{B_1}} = \mu \vec{r_{A_2}} + (1-\mu)\vec{r_{B_2}}}$$

Step 2: We find two vectors $\vec{u}$ and $\vec{w}$ in the plane that are linearly independent (always, no matter the reference point)

Step 3: Write $\vec{r_{A_1}}, \vec{r_{B_1}}, \vec{r_{A_2}}, \vec{r_{B_2}}$ as linear combinations of $\vec{u}$ and $\vec{w}$

Step 4: We have obtained:

$$\alpha(\lambda, \mu) \cdot \vec{u} + \beta(\lambda, \mu) \cdot \vec{w} = \vec{0}$$

$\vec{\omega}, \vec{w}$ lin. dep. $\Rightarrow$ $\begin{cases} \alpha(\lambda, \mu) = 0 \\ \beta(\lambda, \mu) = 0 \end{cases}$

Step 5: Solve the system, by eliminating λ and μ.

Step 6: Replace λ (or μ) in the expression (1) or (2) of $\overrightarrow{r_M}$

Step 7: Rejoice!

1.1. ΔABC, G centroid, H orthocenter

I incenter, O circumcenter $\downarrow$ inters. point of the perpendicular bisectors

We fix a point P as our origin. Then:

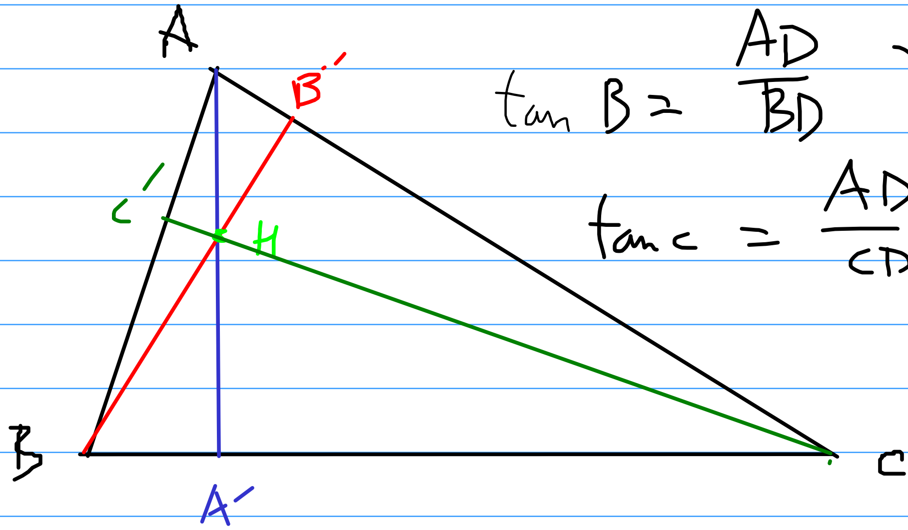
(a) $\overrightarrow{r_G} = \dfrac{\overrightarrow{r_A} + \overrightarrow{r_B} + \overrightarrow{r_C}}{3}$

(b) $\overrightarrow{r_I} = \dfrac{a\overrightarrow{r_A} + b\overrightarrow{r_B} + c\overrightarrow{r_C}}{a+b+c}$, $a = BC$, $b = CA$, $c = AB$

(c) $$\vec{r_H} = \frac{(\tan A)\vec{r_A} + (\tan B)\vec{r_B} + (\tan C)\vec{r_C}}{\tan A + \tan B + \tan C}$$

(d) $$\vec{r_O} = \frac{(\sin 2A)\vec{r_A} + (\sin 2B)\vec{r_B} + (\sin 2C)\vec{r_C}}{\sin 2A + \sin 2B + \sin 2C}$$

(c)



$$\left.\begin{aligned}\tan B &= \frac{AD}{BD} \\ \tan C &= \frac{AD}{CD}\end{aligned}\right\} \Rightarrow \frac{BD}{CD} = \frac{\tan C}{\tan B}$$

P

Let $\frac{AH}{HA'} = h$

Using <u>Van Aubel's thm</u>: $\frac{AC'}{C'B} + \frac{AB'}{B'C} = \frac{AH}{HA'}$

$$\frac{AC'}{C'B} = \frac{\tan B}{\tan A} \qquad \frac{AB'}{B'C} = \frac{\tan C}{\tan A} \qquad \frac{BA'}{CA'} = \frac{\tan C}{\tan B}$$

$$\Rightarrow \frac{\tan B + \tan C}{\tan A} = \frac{AH}{HA'}$$

$$\Rightarrow \vec{r_H} = \frac{(\tan B + \tan C)\cdot \vec{r_{A'}} + (\tan A)\cdot \vec{r_A}}{\tan A + \tan B + \tan C}$$

We fill now find $\vec{r}_{A'}$

$$\frac{BA'}{CA'} = \frac{\tan C}{\tan B} \Rightarrow \vec{r}_{A'} = \frac{\tan B \cdot \vec{r}_B + \tan C \cdot \vec{r}_C}{\tan B + \tan C}$$

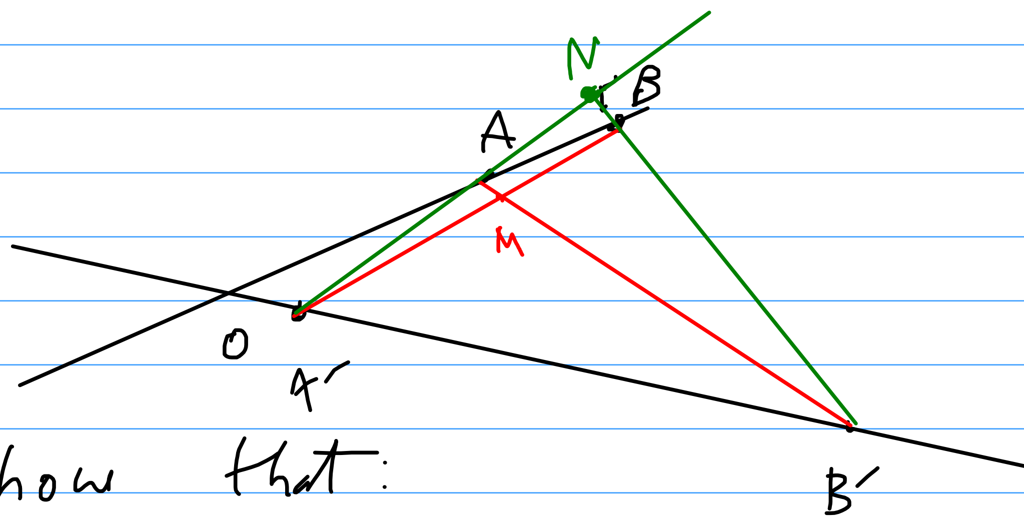
$$\vec{r}_H = \frac{(\tan B + \tan C) \cdot \vec{r}_{A'} + (\tan A) \cdot \vec{r}_A}{\tan A + \tan B + \tan C}$$

$$\Rightarrow \vec{r}_H = \frac{\tan A \cdot \vec{r}_A + \tan B \cdot \vec{r}_B + \tan C \cdot \vec{r}_C}{\tan A + \tan B + \tan C}$$

2.2. Consider the nonzero angle $\widehat{BOB'}$ and the points $A \in [OB]$, $A' \in [OB']$

$\{M\} = AB' \cap A'B$, $\{N\} = AA' \cap BB'$

$\vec{u} := \overrightarrow{OA}$, $\vec{u'} := \overrightarrow{OA'}$, $\overrightarrow{OB} = m \cdot \overrightarrow{OA}$, $\overrightarrow{OB'} = n \cdot \overrightarrow{OA'}$



Show that:

$$\overrightarrow{OM} = m \cdot \frac{1-n}{1-mn} \overrightarrow{OA} + n \cdot \frac{1-m}{1-mn} \overrightarrow{OA'}$$

$$\overrightarrow{ON} = m \cdot \frac{n-1}{n-m} \overrightarrow{OA} + n \cdot \frac{m-1}{m-n} \overrightarrow{OA'}$$

<u>Proof</u>: We fix O as our origin

$N \in AA' \Rightarrow \exists \lambda \in \mathbb{R}: \vec{r_N} = \lambda \vec{r_A} + (1-\lambda)\vec{r_{A'}}$ (1)

$N \in BB' \Rightarrow \exists \mu \in \mathbb{R}: \vec{r_N} = \mu \vec{r_B} + (1-\mu)\vec{r_{B'}}$ (2)

$$(1) \Rightarrow \quad \vec{r_N} = \lambda \vec{u} + (1-\lambda)\vec{v}$$

$$(2) \Rightarrow \quad \vec{r_N} = \mu \cdot m \cdot \vec{u} + (1-\mu) \cdot n \cdot \vec{v}$$

$$\Rightarrow \lambda \vec{u} + (1-\lambda) \cdot \vec{v} = \mu m \vec{u} + (1-\mu) \cdot n \cdot \vec{v}$$

$$\Rightarrow (\lambda - m\mu) \cdot \vec{u} + (1 - \lambda - n + n\mu) \cdot \vec{v} = \vec{0}$$

$\vec{u}, \vec{v}$ linearly independent because the angle is nonzero

$$\Rightarrow \begin{cases} \lambda - m\mu = 0 \\ 1 - \lambda - n + n\mu = 0 \end{cases} \Rightarrow \begin{cases} \lambda = m\mu \\ 1 - m\mu - n + n\mu = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \lambda = m\mu \\ 1 - n + \mu(n-m) = 0 \end{cases} \Rightarrow \mu = \frac{n-1}{n-m} = \frac{1-n}{m-n}$$

$$\Rightarrow \vec{r_N} = \frac{1-n}{m-n} \cdot m \cdot \vec{u} + \left(1 - \frac{1-n}{m-n}\right) \cdot n \cdot \vec{v} =$$

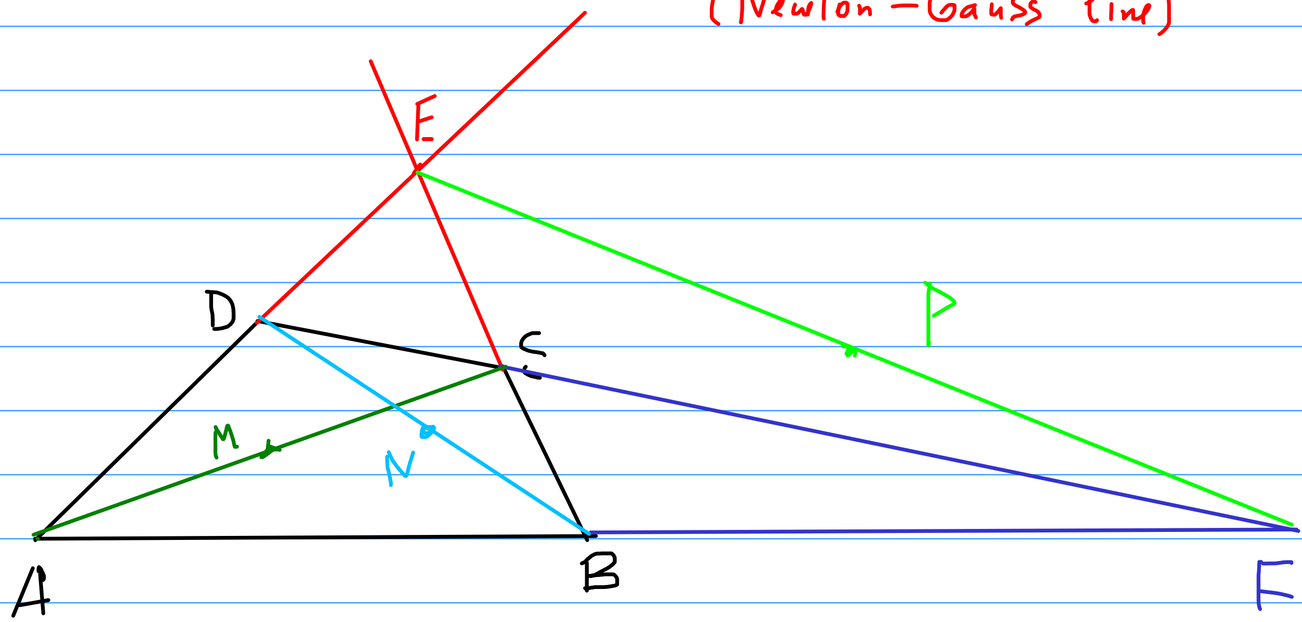
$$= \frac{m(1-n)}{m-n} \cdot \vec{u} + \frac{(m-1)n}{m-n} \cdot \vec{v} = \vec{ON}$$

$$\Rightarrow \vec{ON} = \frac{m(n-1)}{n-m} \cdot \vec{u} + \frac{(m-1)n}{m-n} \cdot \vec{v}$$

$\vec{OM}$ is homework.

2.3. Show that the midpoints of the diagonals of a complete quadrilateral are collinear.

(Newton – Gauss line)



M midpoint of $[AC]$, N midpoint of $[BD]$

P midpoint of $[EF]$

We fix $\vec{u} := \overrightarrow{AB}$, $\vec{w} = \overrightarrow{BC}$. Due to the non-degeneracy of the quadrilateral, we have that $\vec{u}$ and $\vec{w}$ are linearly independent.

$$\overrightarrow{MN} = \frac{\overrightarrow{MD} + \overrightarrow{MB}}{2} = \frac{-\overrightarrow{BM} + (-\overrightarrow{DM})}{2} = \frac{-\frac{\overrightarrow{BA}+\overrightarrow{BC}}{2} - \frac{\overrightarrow{DA}+\overrightarrow{DC}}{2}}{2} =$$

$$= \frac{\overrightarrow{AB} + \overrightarrow{CB} + \overrightarrow{AD} + \overrightarrow{CD}}{4} = \frac{\vec{u} - \vec{w} + \vec{u} + \vec{w} + 2\overrightarrow{CD}}{4}$$

$$\overrightarrow{BM} = \frac{\overrightarrow{BA} + \overrightarrow{BC}}{2} = \frac{\vec{w} - \vec{u}}{2}$$

$$\overrightarrow{BN} = \frac{\overrightarrow{BD}}{2}$$

$$\overrightarrow{BP} = \frac{\overrightarrow{BE} + \overrightarrow{BF}}{2}$$

Let $\overrightarrow{BE} = \alpha \cdot \vec{w}$, $\overrightarrow{BF} = \beta \cdot \vec{u}$

$$\Rightarrow \overrightarrow{BP} = \frac{\alpha \vec{w} + \beta \cdot \vec{u}}{2}$$

We describe D as the intersection between the lines AE and FC

By using the template we will find $\overrightarrow{BD}$ in terms of $\overrightarrow{BA}, \overrightarrow{BE}, \overrightarrow{BF}, \overrightarrow{BC}$ and then in terms of $\vec{u}$ and $\vec{w}$