Seminar W7 – 975

The triple scalar product
(the mixed product)

$\vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$

$$(\vec{a}, \vec{b}, \vec{c}) := \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

If we use a reference system $\mathcal{R} = (O, [\vec{i}, \vec{j}, \vec{k}])$ that is orthonormal and direct, then:

$\vec{v}_1 (a_1, b_1, c_1)$, $\vec{v}_2 (a_2, b_2, c_2)$, $\vec{v}_3 (a_3, b_3, c_3)$

$$(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$(\vec{v}_1, \vec{v}_2, \vec{v}_3) = (\vec{v}_2, \vec{v}_3, \vec{v}_1) =$$

$$= (\vec{v}_3, \vec{v}_1, \vec{v}_2) =$$

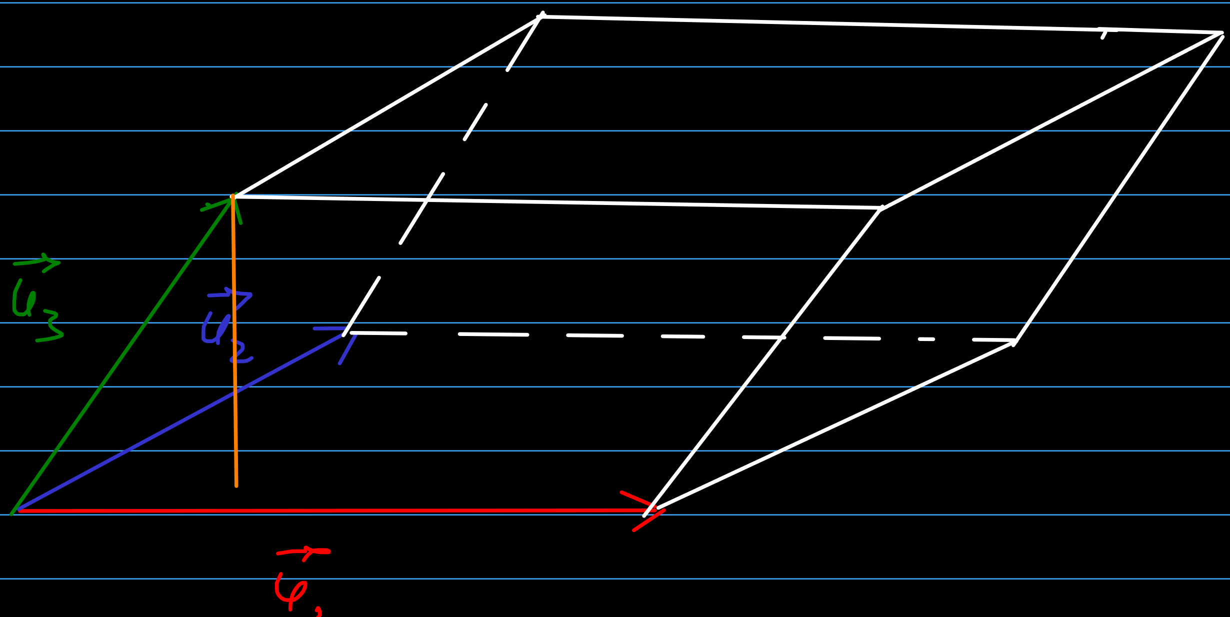
$$= -(\vec{v}_1, \vec{v}_3, \vec{v}_2) = -(\vec{v}_2, \vec{v}_1, \vec{v}_3)$$

$$= -(\vec{v}_3, \vec{v}_2, \vec{v}_1)$$

$\vec{u_1}, \vec{u_2}, \vec{u_3} \in \mathcal{U}$

$$|(\vec{u_1}, \vec{u_2}, \vec{u_3})| = |\vec{u_1} \cdot (\vec{u_2} \times \vec{u_3})| =$$

$=$ Vol(parallelepiped built on the three vectors)

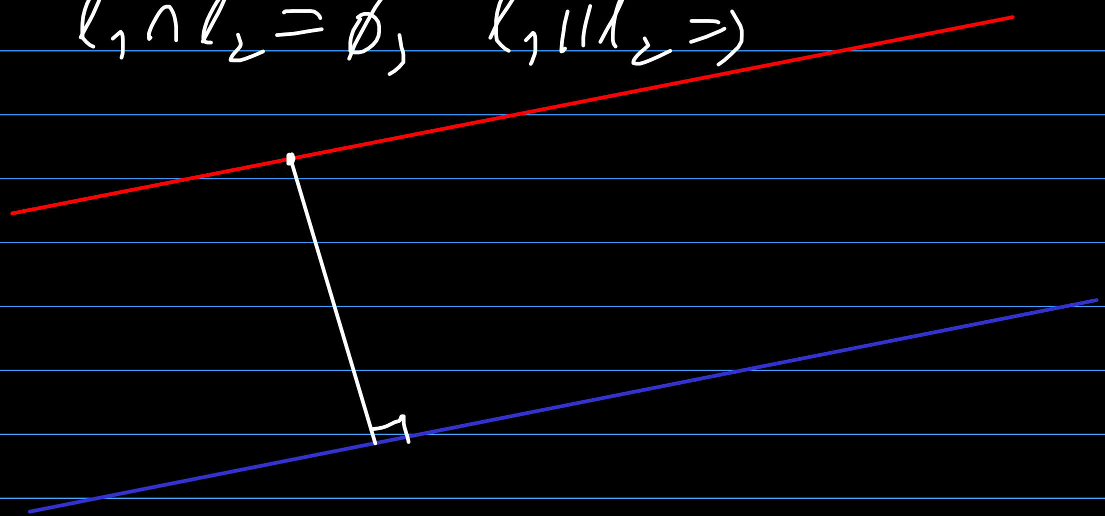


$$h_{parallelepiped} = \frac{|(\vec{u_1}, \vec{u_2}, \vec{u_3})|}{\|\vec{u_1} \times \vec{u_2}\|}$$

The distance between two lines in space and the common perpendicular

l_1, l_2 lines in space

- $l_1 \cap l_2 \neq \emptyset \Rightarrow dist(l_1, l_2) = 0$
- $l_1 \cap l_2 = \emptyset$, $l_1 \parallel l_2 \Rightarrow$



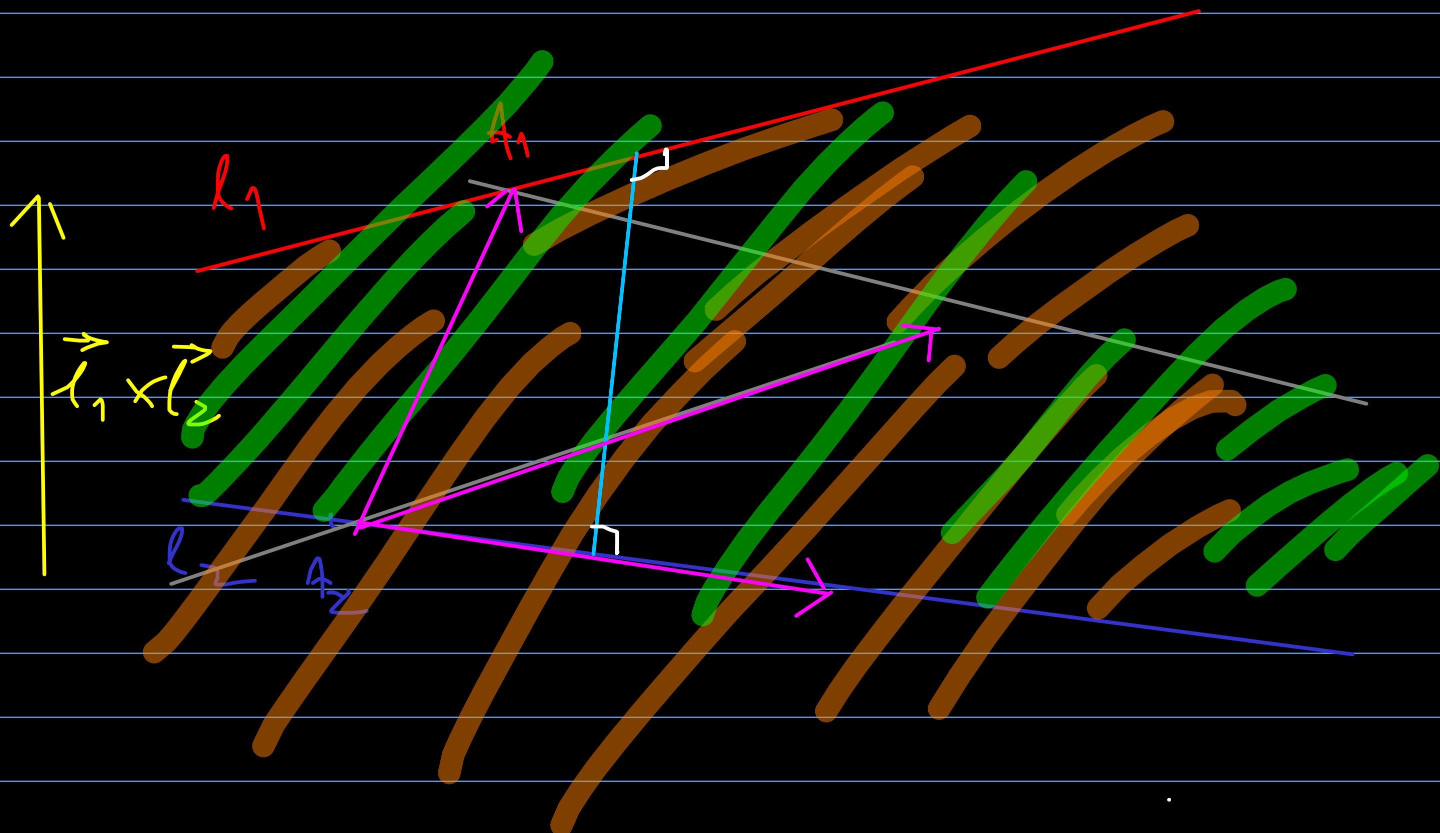
$$dist(l_1, l_2) = dist(M_1, l_2) = dist(M_2, l_1)$$

$\forall M_1 \in l_1$, $\forall M_2 \in l_2$

common perp. = any perpendicular from a point $M_1 \in l_1$ onto l_2.

- $l_1 \cap l_2 = \emptyset$, $l_1 \nparallel l_2$

(i.e. l_1 and l_2 are skew (or noncoplanar)



the common perpendicular $= \Pi_1 \cap \Pi_2$

where $\Pi_1 =$ the plane given by ℓ_1 and $\vec{\ell_1} \times \vec{\ell_2}$

$\Pi_2 =$ the plane given by ℓ_2 and $\vec{\ell_1} \times \vec{\ell_2}$

$\Pi_1 \nparallel \Pi_2 \Rightarrow \exists\, \ell = \Pi_1 \cap \Pi_2$

The common perpendicular is a height in the parallelepiped built on the vectors $\overrightarrow{A_1A_2}$, $\vec{l}_1$, $\vec{l}_2$

$$\Rightarrow \operatorname{dist}(l_1, l_2) = \frac{|(\overrightarrow{A_1A_2}, \vec{l}_1, \vec{l}_2)|}{\|\vec{l}_1 \times \vec{l}_2\|}$$

7.7. Find the distance between the lines $l_1: \frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{1}$

$l_2: \frac{x+1}{3} = \frac{y}{4} = \frac{z-1}{3}$

as well as the equations of the common perpendicular

We can see that $l_1 \nparallel l_2$.

$A_1(1,-1,0) \in l_1$, $A_2(-1,0,1) \in l_2$

$\overrightarrow{A_1A_2}(-2,1,1)$

$$\vec{l_1} \times \vec{l_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 3 & 4 & 3 \end{vmatrix} =$$

$$= 5\vec{i} - 3\vec{j} - \vec{k}$$

$$\|\vec{l_1} \times \vec{l_2}\| = \sqrt{25+9+1} = \sqrt{35}$$

$$(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}) = \begin{vmatrix} -2 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 3 \end{vmatrix} =$$

$$= -18 + 3 + 8 - 9 - 6 + 8 = -14$$

$$dist(l_1, l_2) = \frac{|(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2})|}{\|\vec{l_1} \times \vec{l_2}\|} = \frac{14}{\sqrt{35}}$$

Π_1 : the plane given by A_1, $\vec{l_1}$, $\vec{l_1} \times \vec{l_2}$

$A_1(1,-1,0)$, $\vec{l_1}(2,3,1)$, $\vec{l_1} \times \vec{l_2}(5,-3,-1)$

$$\Rightarrow \Pi_1: \begin{vmatrix} x-1 & y+1 & z \\ 2 & 3 & 1 \\ 5 & -3 & -1 \end{vmatrix} = 0$$

$$\Leftrightarrow \Pi_1: \begin{vmatrix} 3 & 1 \\ -3 & -1 \end{vmatrix} \cdot (x-1) - \begin{vmatrix} 2 & 1 \\ 5 & -1 \end{vmatrix} \cdot (y+1) +$$

$$+ \begin{vmatrix} 2 & 3 \\ 5 & -3 \end{vmatrix} \cdot z = 0 \Leftrightarrow$$

$$\Leftrightarrow \quad \Pi_1: 7y - 21z + 7 = 0 \Leftrightarrow$$

$$\Leftrightarrow \quad \Pi_1: y - 3z + 1 = 0$$

Π_2: plane given by A_2, $\vec{l_2}$, $\vec{l_1} \times \vec{l_2}$

$A_2(-1,0,1)$, $\vec{l_2}(3,4,3)$, $\vec{l_1} \times \vec{l_2}(5,-3,-1)$

$$\Pi_2: \begin{vmatrix} x+1 & y & z-1 \\ 3 & 4 & 3 \\ 5 & -3 & -1 \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{vmatrix} 4 & 3 \\ -3 & -1 \end{vmatrix} \cdot (x+1) - \begin{vmatrix} 3 & 3 \\ 5 & -1 \end{vmatrix} \cdot y +$$

$$+ \begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix} \cdot (z-1) = 0 \Leftrightarrow$$

$$\Leftrightarrow 5x + 18y - 29z + 34 = 0$$

So the common perpendicular is

$$l = \Pi_1 \cap \Pi_2 : \begin{cases} y - 3z + 1 = 0 \\ 5x + 18y - 29z + 34 = 0 \end{cases}$$

$$l: \begin{cases} y = 3z - 1 \\ 5x + 18(3z-1) - 29z + 34 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow l: \begin{cases} y = 3z - 1 \\ 5x + 54z - 18 - 29z + 34 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow l: \begin{cases} y = 3z - 1 \\ 5x + 25z + 16 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow l: \begin{cases} y = 3z - 1 \\ x = -\frac{16}{5} - 5z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow l: \begin{cases} x = -5t - \frac{16}{5} \\ y = 3t - 1 \\ z = t \end{cases}$$

$$\Rightarrow \left(-\frac{16}{5}, -1, 0\right) \in l$$

$$\vec{u}\,(-5, 3, 1) \parallel l$$

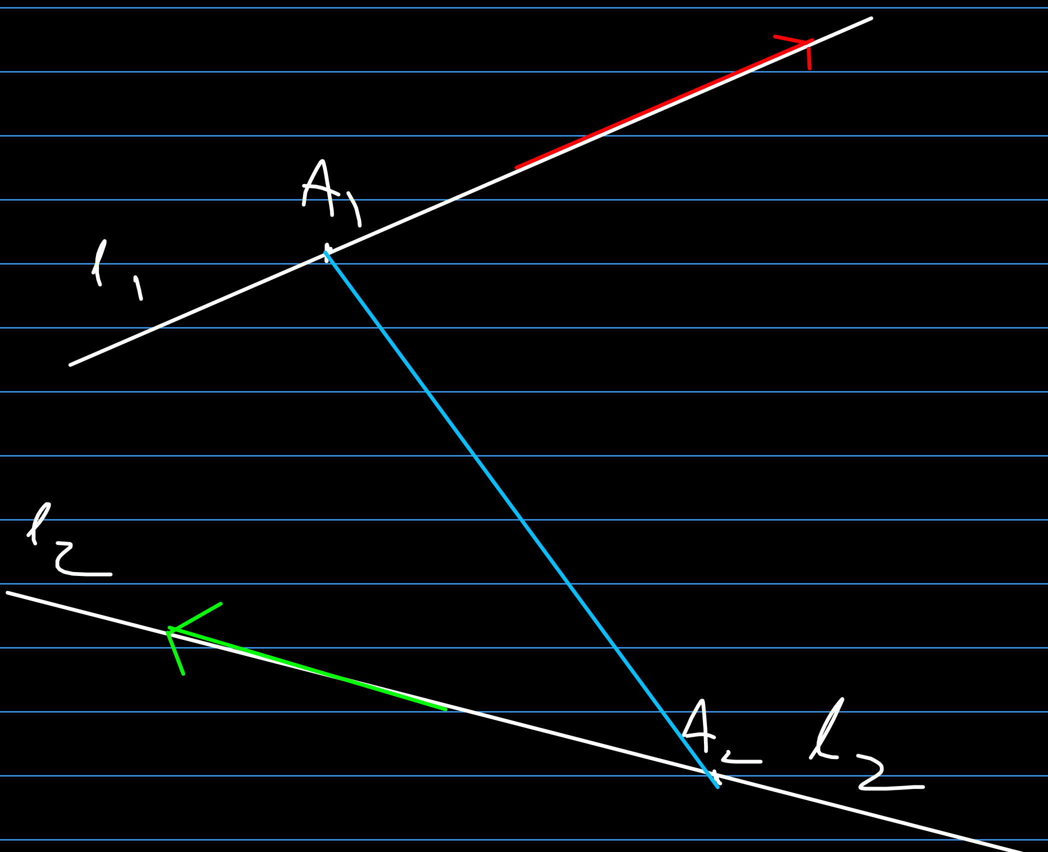
The coplanarity condition for two lines

l_1, l_2 lines, $A_1 \in l_1$, $A_2 \in l_2$

l_1, l_2 coplanar $\Leftrightarrow$ $\overrightarrow{A_1A_2}$, $\vec{l_1}$ and $\vec{l_2}$ are linearly dependent

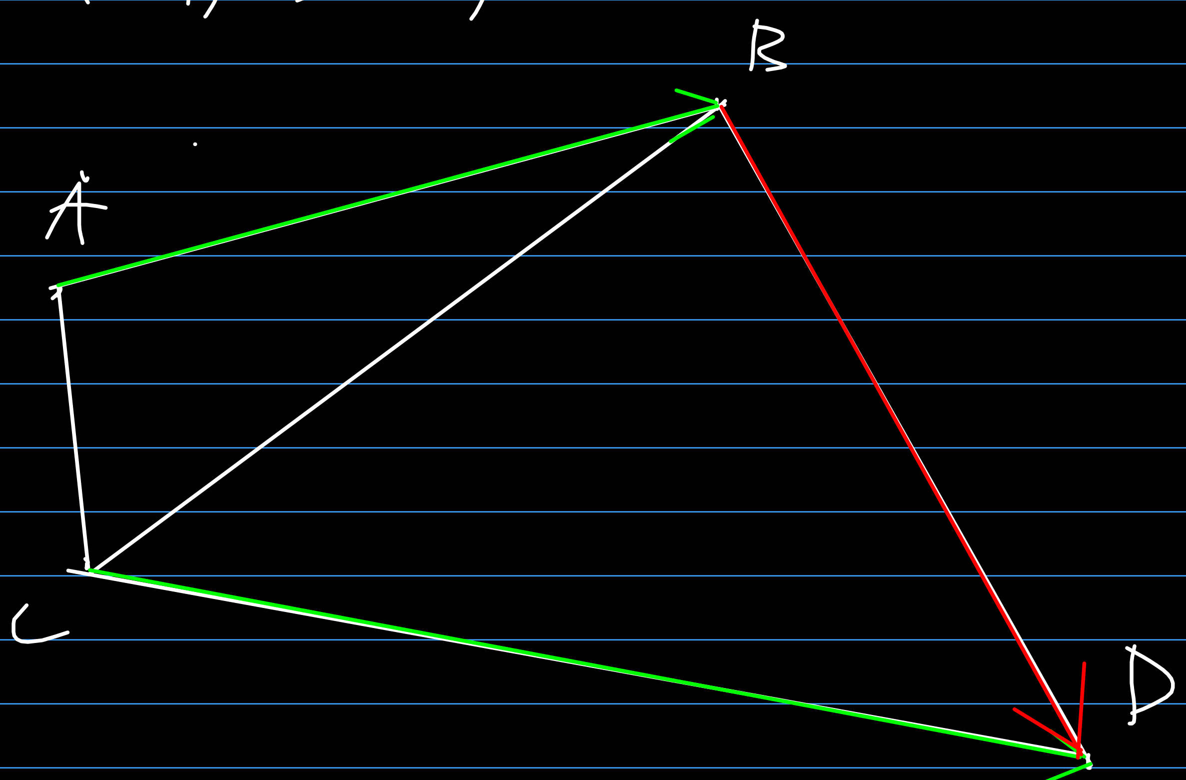
$\Leftrightarrow$ $\left(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}\right) = 0$

$\Leftrightarrow$ vol (parallelepiped) $= 0$



7.5. Find the value of the parameter α for which the pencil of planes through the line AB has a common plane with the pencil of planes through the line CD, where

$A(1, 2\alpha, \alpha)$, $B(3, 2, 1)$, $C(-\alpha, 0, \alpha)$, $D(-1, 3, -3)$



AB and CD are coplanar $\Leftrightarrow$

$\Leftrightarrow V_{A,BCD} = 0 \Leftrightarrow (\vec{BD}, \vec{AB}, \vec{CD}) = 0$

$\vec{BD}(-4, 1, -4)$, $\vec{AB}(2, 2-2\alpha, 1-\alpha)$

$\vec{CD}(\alpha-1, 3, -3-\alpha)$

$$(\vec{BD}, \vec{AB}, \vec{CD}) = \begin{vmatrix} -4 & 1 & -4 \\ 2 & 2-2\alpha & 1-\alpha \\ \alpha-1 & 3 & -3-\alpha \end{vmatrix} =$$

$$= (-4) \cdot \begin{vmatrix} 2-2\alpha & 1-\alpha \\ 3 & -3-\alpha \end{vmatrix} -$$

$$- \begin{vmatrix} 2 & 1-\alpha \\ \alpha-1 & -3-\alpha \end{vmatrix} + (-4) \cdot$$

$$\cdot \begin{vmatrix} 2 & 2-2\alpha \\ \alpha-1 & 3 \end{vmatrix} = (-4) \cdot (-6-2\alpha +$$

$$+ 6\alpha + 2\alpha^2 - 3 + 3\alpha) - (-6 - 2\alpha +$$

$$+ \alpha^2 - 2\alpha + 1) + (-4) \cdot (6 - 2\alpha + 2 + 2\alpha^2 -$$

$$- 2\alpha) =$$

$$= -17\alpha^2 - 8\alpha + 9$$

$$\Rightarrow 17\alpha^2 + 8\alpha - 9 = 0$$

$$\alpha_{1,2} = \frac{-8 \pm \sqrt{64 + 612}}{34} = \frac{-8 \pm \sqrt{676}}{34} = \frac{-4 \pm \sqrt{169}}{17} = \frac{-4 \pm 13}{17}$$

$$\Rightarrow \alpha = -1 \quad \text{or} \quad \alpha = \frac{9}{17}$$

7.1. Show that: $\forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{U}$

(a) $|(\vec{a}, \vec{b}, \vec{c})| \leq \|\vec{a}\| \cdot \|\vec{b}\| \cdot \|\vec{c}\|$

(b) $(\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}) = 2(\vec{a}, \vec{b}, \vec{c})$

(b) $(\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}) = (\vec{a}+\vec{b}) \cdot$

$\cdot \left((\vec{b}+\vec{c}) \times (\vec{c}+\vec{a})\right) = (\vec{a}+\vec{b}) \cdot (\vec{b} \times \vec{c} +$

$+ \vec{b} \times \vec{a} + \underbrace{\vec{c} \times \vec{c}}_{\vec{0}} + \vec{c} \times \vec{a}) =$

$= (\vec{a}+\vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}) =$

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \overbrace{\vec{b} \cdot (\vec{b} \times \vec{c})}^{=0} + \overbrace{\vec{a} \cdot (\vec{b} \times \vec{a})}^{=0} +$

$+ \underbrace{\vec{b} \cdot (\vec{b} \times \vec{a})}_{=0} + \underbrace{\vec{a} \cdot (\vec{c} \times \vec{a})}_{=0} + \vec{b} \cdot (\vec{c} \times \vec{a}) =$

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{c} \times \vec{a}) =$

$= 2 \cdot \vec{a} \cdot (\vec{b} \times \vec{c}) = 2 \cdot (\vec{a}, \vec{b}, \vec{c})$