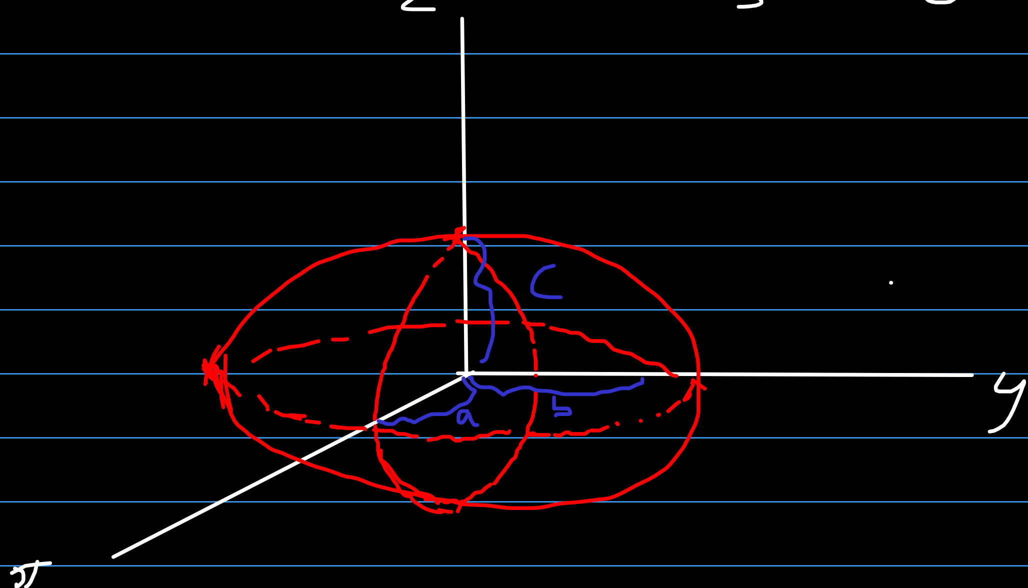
Seminar W10 - 9.1.6

$\mathcal{C}: f(x,y)=0, \quad f \in \mathbb{R}[x,y] \Rightarrow$ conics

$\mathcal{Y}: f(x,y,z)=0, \quad f \in \mathbb{R}[x,y,z] \Rightarrow$ quadrics

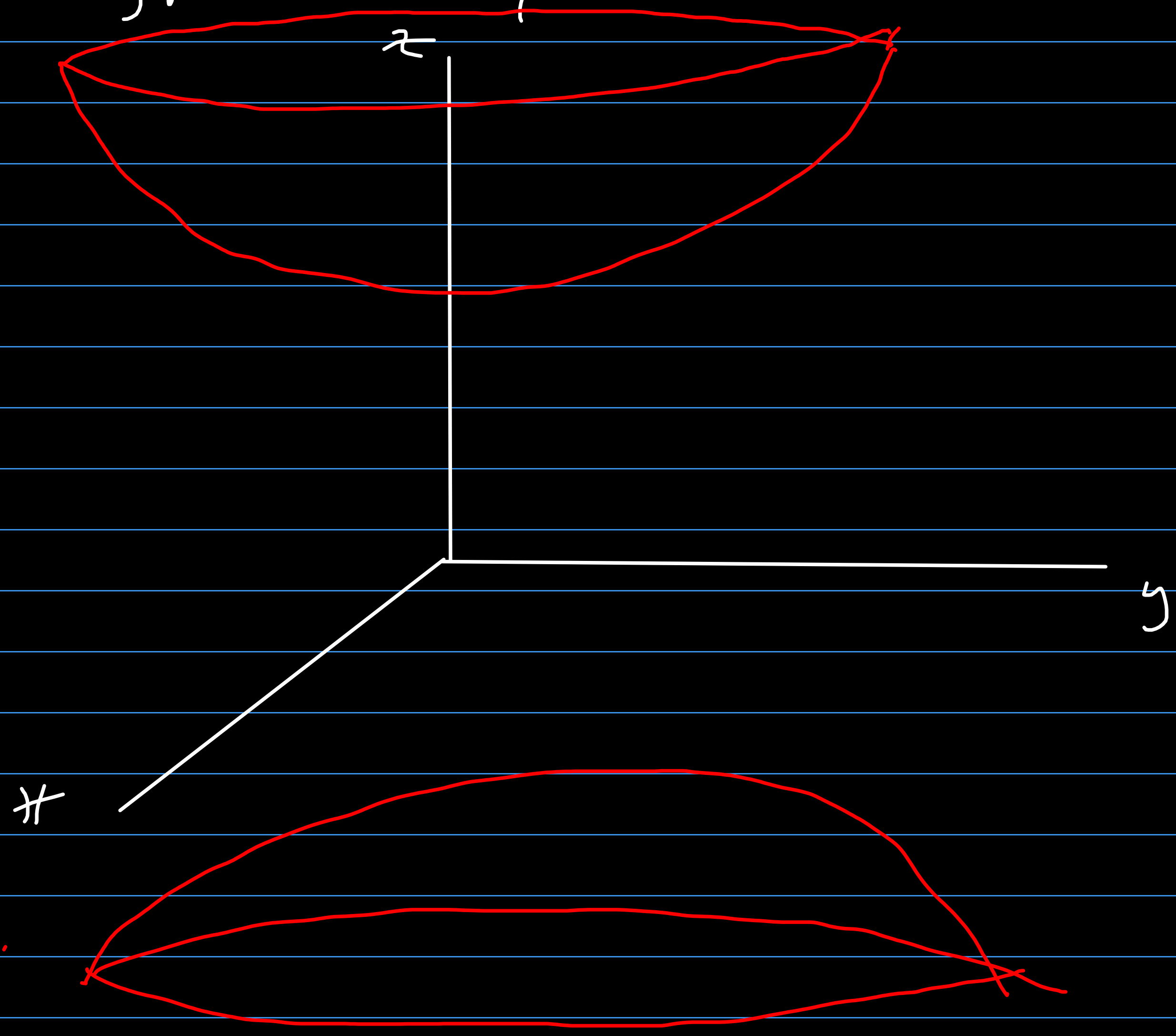
- **Ellipsoid**: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$



- <u>Hyperboloid of one sheet</u>

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

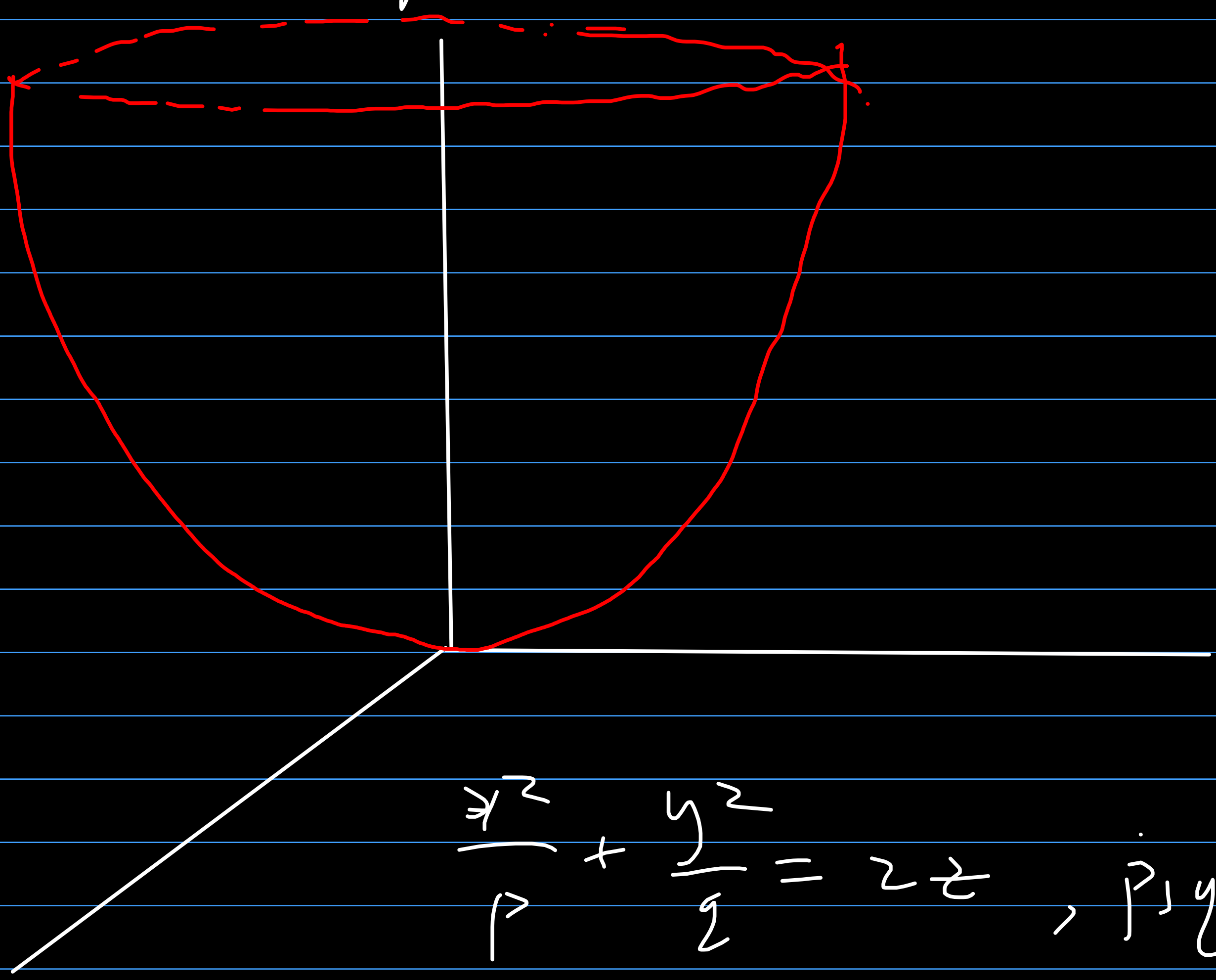
· Hyperboloid of two sheets



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

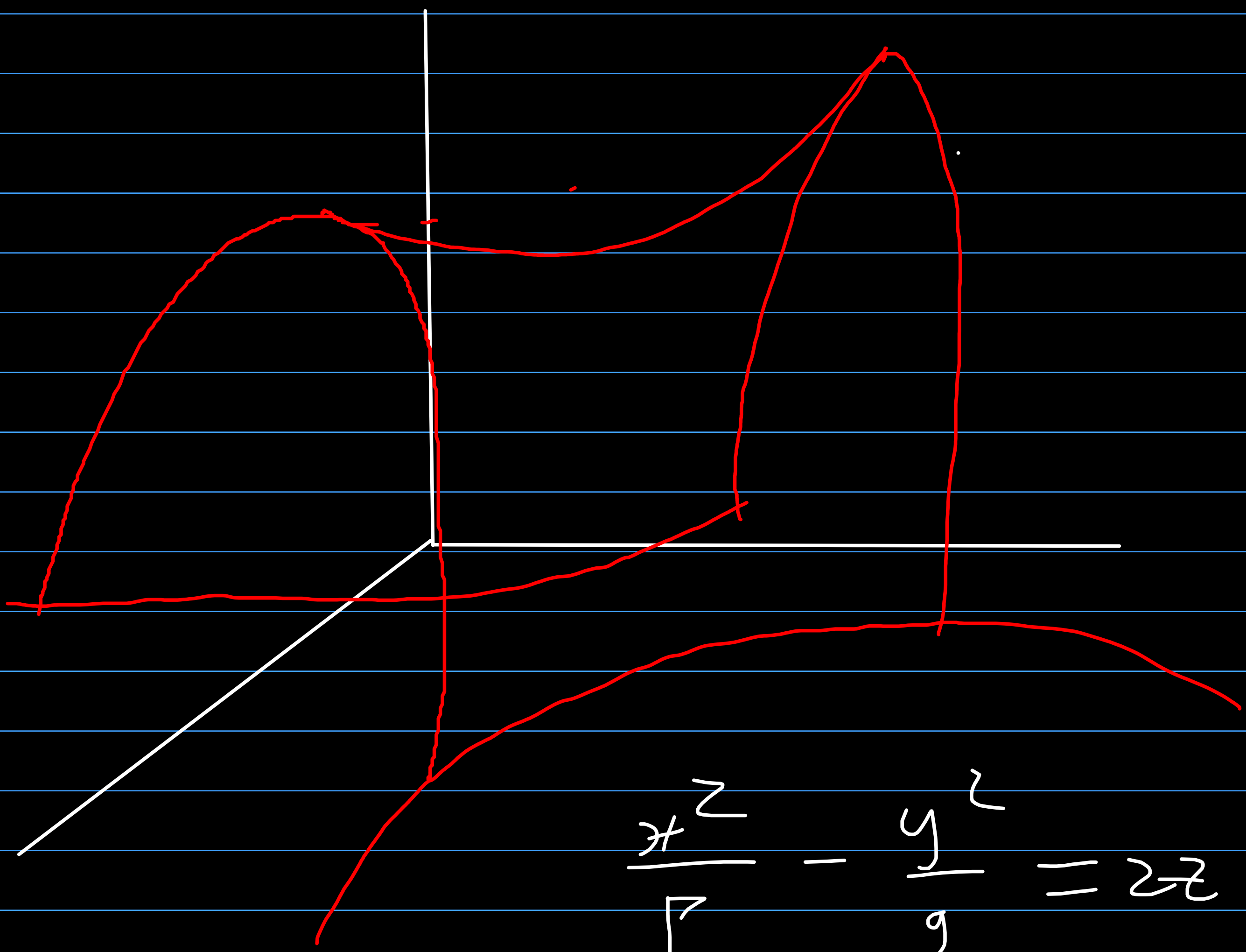
$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

- Elliptic paraboloid



$$\frac{x^2}{p} + \frac{y^2}{q} = 2z \quad , \; p, q > 0$$

· Hyperbolic paraboloid



$$\frac{x^2}{p} - \frac{y^2}{q} = 2z$$

Rectilinear generatrices → only exist on hyperboloids of one sheet and hyperbolic paraboloids

I) $\mathcal{S}: f(x,y,z)=0$

$$T_{\mathcal{S}}(x_0,y_0,z_0): f'_x(x_0,y_0,z_0)\cdot(x-x_0)+f'_y(x_0,y_0,z_0)\cdot(y-y_0)+f'_z(x_0,y_0,z_0)\cdot(z-z_0)=0$$

$\hookrightarrow$ tangent plane

$$N_{\mathcal{S}}(x_0,y_0,z_0): \frac{x-x_0}{f'_x(x_0,y_0,z_0)}=\frac{y-y_0}{f'_y(x_0,y_0,z_0)}=\frac{z-z_0}{f'_z(x_0,y_0,z_0)}$$

10.1. Find the intersection points of the ellipsoid

$$\mathcal{E}: \frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$$

with the line

$$\frac{x-4}{2}=\frac{y+6}{-3}=\frac{z+2}{-2}$$

and write the equations of the tangent planes, as well as the equations of the normal lines to the ellipsoid at the intersection points.

$$l: \begin{cases} x = 2t+4 \\ y = -3t-6 \\ z = -2t-2 \end{cases}$$

$$l \cap \mathcal{E}: \begin{cases} \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} = 1 \\ x = 2t+4 \\ y = -3t-6 \\ z = -2t-2 \end{cases}$$

$$\Rightarrow \frac{(2t+4)^2}{4} + \frac{(-3t-6)^2}{3} + (-2t-2)^2 - 4 = 0$$

$$\Rightarrow (t+2)^2 + 3\cdot(t+2)^2 + (2t+2)^2 - 4 = 0$$

$$\Rightarrow t^2+4t+4+3t^2+12t+12+$$

$$+4t^2+8t+4-4 = 0 \Rightarrow$$

$$\Rightarrow 8t^2+24t+16=0$$

$$\Rightarrow t^2+3t+2=0 \Rightarrow t_{1,2} = \frac{-3 \pm \sqrt{9-8}}{2}$$

$\Rightarrow t_1 = -2$ and $t_2 = -1$

$\Rightarrow P(2,-3,0)$ and $Q(0,0,2)$ are the intersection points of $\mathcal{E}$ and ℓ

$$f(x,y,z) = \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} - 1$$

$$\Rightarrow f'_x(x_0,y_0,z_0) = \frac{x_0}{8}$$

$$f'_y(x_0,y_0,z_0) = \frac{y_0}{6}$$

$$f'_z(x_0,y_0,z_0) = \frac{z_0}{2}$$

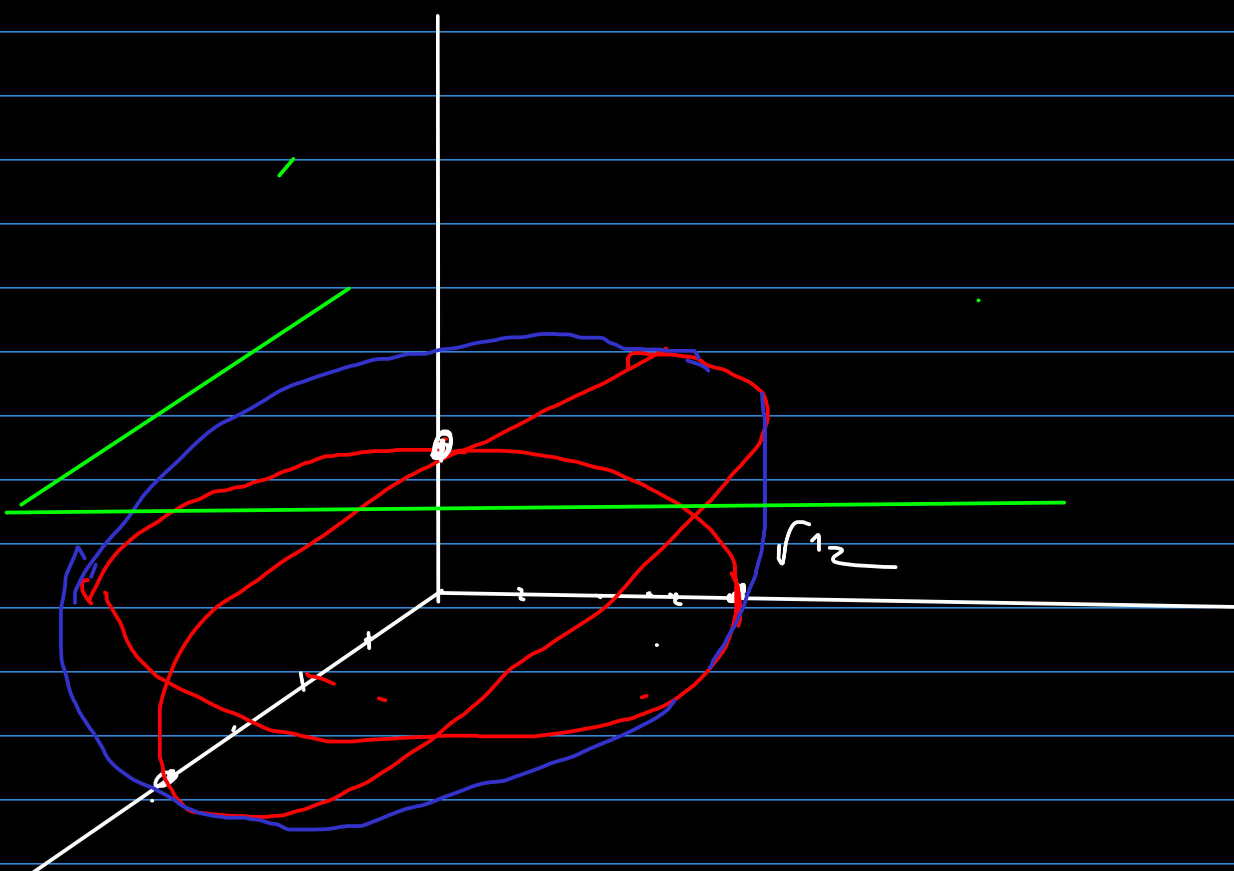
$$\Rightarrow T_{\mathcal{E}}(x_0,y_0,z_0): \frac{x_0}{8}\cdot(x-x_0) + \frac{y_0}{6}(y-y_0) + \frac{z_0}{2}(z-z_0) = 0$$

$$\Rightarrow T_{\mathcal{E}}(2,-3,0): \frac{1}{4}\cdot(x-2) - \frac{1}{2}\cdot(y+3) = 0$$

$$N_{\varphi}(2,-3,0): \begin{cases} \dfrac{x-2}{\frac{1}{4}} = \dfrac{y+3}{-\frac{1}{2}} \\ z = 0 \end{cases}$$

$$T_{\varphi}(0,0,2): \quad z = 2$$

$$N_{\varphi}(0,0,2): \begin{cases} x = 0 \\ y = 0 \end{cases}$$



Rectilinear generatrices

→ **hyperboloid of one sheet**

$\mathcal{H}: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

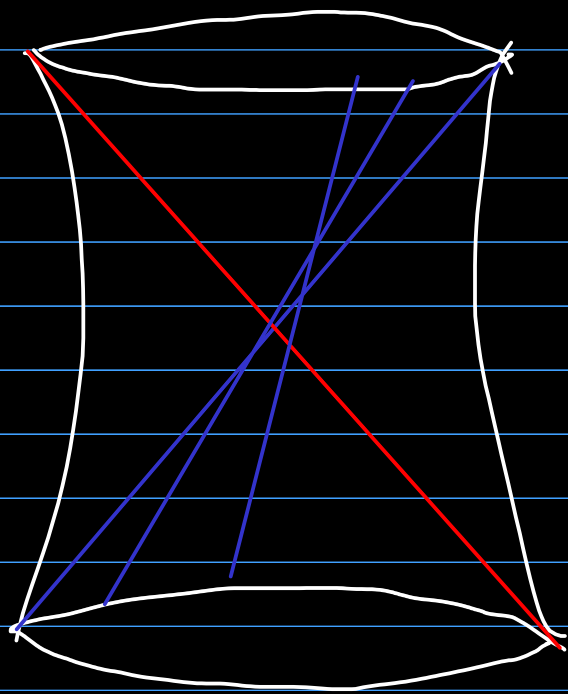
$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}$$

$$\left(\frac{x}{a} - \frac{z}{c}\right)\cdot\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right)\cdot\left(1 + \frac{y}{b}\right)$$

$$d_\lambda : \begin{cases} \dfrac{x}{a} - \dfrac{z}{c} = \lambda\cdot\left(1 - \dfrac{y}{b}\right) \\ \lambda\cdot\left(\dfrac{x}{a} + \dfrac{z}{c}\right) = 1 + \dfrac{y}{b} \end{cases}$$

(every point on every line d_λ is a point on the hyperboloid)

$$d'_\mu \begin{cases} \dfrac{x}{a} - \dfrac{z}{c} = \mu\cdot\left(1 + \dfrac{y}{b}\right) \\ \mu\cdot\left(\dfrac{x}{a} + \dfrac{z}{c}\right) = 1 - \dfrac{y}{b} \end{cases}$$



→ hyperbolic paraboloid

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z$$

$$\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right)\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = 2z$$

$$d_\lambda : \begin{cases} \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = 2\lambda \\ \lambda\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = z \end{cases}$$

$$d_\mu' : \begin{cases} \mu\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right) = z \\ \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} = 2\lambda \end{cases}$$

10.2. Find the rectilinear generatrices of the quadric $4x^2 - 9y^2 = 36z$ which pass through the point $P(3\sqrt{2}, 2, 1)$

$$(2x - 3y)\cdot(2x + 3y) = 36z$$

$$\Rightarrow d_\lambda : \begin{cases} 2x - 3y = 36\lambda \\ \lambda\cdot(2x + 3y) = z \end{cases}$$

$$P \in d_\lambda \Rightarrow \begin{cases} 6\sqrt{2} - 6 = 36\lambda \\ \lambda\cdot(6\sqrt{2} + 6) = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \lambda = \frac{6\sqrt{2} - 6}{36} = \frac{\sqrt{2} - 1}{6} = \frac{1}{6(\sqrt{2} + 1)} \\ \lambda = \frac{1}{6(\sqrt{2} + 1)} \end{cases}$$

$\Rightarrow \quad \lambda = \dfrac{\sqrt{2}-1}{6}$ is the right value

$$\Rightarrow \quad d_{\frac{\sqrt{2}-1}{6}} : \begin{cases} 2x - 3y = 6(\sqrt{2}-1) \\ \dfrac{\sqrt{2}-1}{6} \cdot (2x+3y) = z \end{cases}$$

$$d'_{\mu} : \begin{cases} \mu \cdot (2x - 3y) = z \\ 2x + 3y = 36\mu \end{cases}$$

$$p \in d'_{\mu} \Rightarrow \begin{cases} \mu \cdot (6\sqrt{2} - 6) = 1 \\ 6\sqrt{2} + 6 = 36\mu \end{cases} \Rightarrow$$

$$\Leftrightarrow \begin{cases} \mu = \dfrac{1}{6(\sqrt{2}-1)} \\ \mu = \dfrac{6(\sqrt{2}+1)}{36} = \dfrac{\sqrt{2}+1}{6} \end{cases}$$

$\Rightarrow \quad d'_{\frac{\sqrt{2}+1}{6}}$ is another solution

10.3. Find the rectilinear generatrices of the hyperboloid of one sheet

$$(\mathcal{H}_1): \frac{x^2}{36} + \frac{y^2}{9} - \frac{z^2}{4} = 1$$

which are parallel to the plane

$$(\pi)\ x + y + z = 0$$

$$(\mathcal{H}_1)\ \frac{x^2}{36} - \frac{z^2}{4} = 1 - \frac{y^2}{9}$$

$$\left(\frac{x}{6} - \frac{z}{2}\right) \cdot \left(\frac{x}{6} + \frac{z}{2}\right) =$$

$$= \left(1 - \frac{y}{3}\right) \cdot \left(1 + \frac{y}{3}\right)$$

$$d_\lambda : \begin{cases} \frac{x}{6} - \frac{z}{2} = \lambda \cdot \left(1 - \frac{y}{3}\right) \\ \lambda \cdot \left(\frac{x}{6} + \frac{z}{2}\right) = 1 + \frac{y}{3} \end{cases}$$

$$d'_\mu : \begin{cases} \frac{x}{6} - \frac{z}{2} = \lambda \cdot \left(1 + \frac{y}{3}\right) \\ \lambda \cdot \left(\frac{x}{6} + \frac{z}{2}\right) = 1 - \frac{y}{3} \end{cases}$$

$$\vec{d}_\lambda = \left(\frac{1}{6}, -\frac{\lambda}{3}, -\frac{1}{2}\right) \times \left(\frac{\lambda}{6}, -\frac{1}{3}, \frac{\lambda}{2}\right)$$

$$\vec{d}_\lambda \parallel \pi \iff \vec{d}_\lambda \perp \vec{n}_\pi \iff$$

$$\iff \vec{d}_\lambda \cdot \vec{n}_\pi = 0$$