Seminar W6 — 916

Cross product (vector product)

$\vec{u}, \vec{w}$ vectors

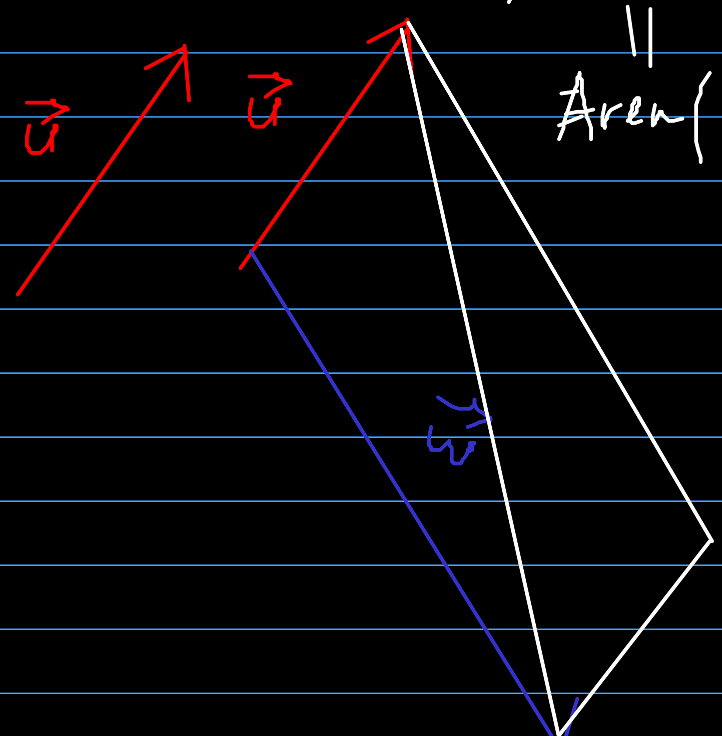
- if $\vec{u}, \vec{w}$ lin. dep. $\Rightarrow \vec{u} \times \vec{w} = \vec{0}$
- if $\vec{u}, \vec{w}$ lin. indep.:

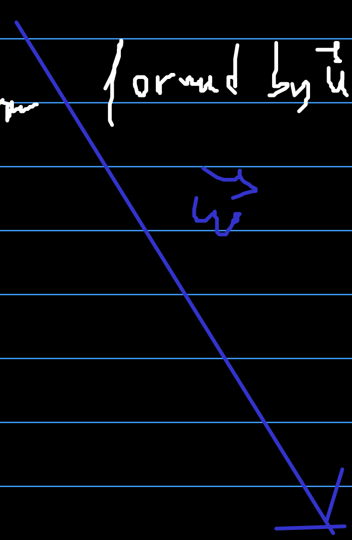
  $\vec{u} \times \vec{w} \in V$

  $\rightarrow$ direction: perp. to $\vec{u}$ and $\vec{w}$, it is actually perp. to $\langle \vec{u}, \vec{w} \rangle$

  $\rightarrow$ norm: $\|\vec{u} \times \vec{w}\| = \|\vec{u}\| \cdot \|\vec{w}\| \cdot \sin(\widehat{\vec{u}, \vec{w}})$

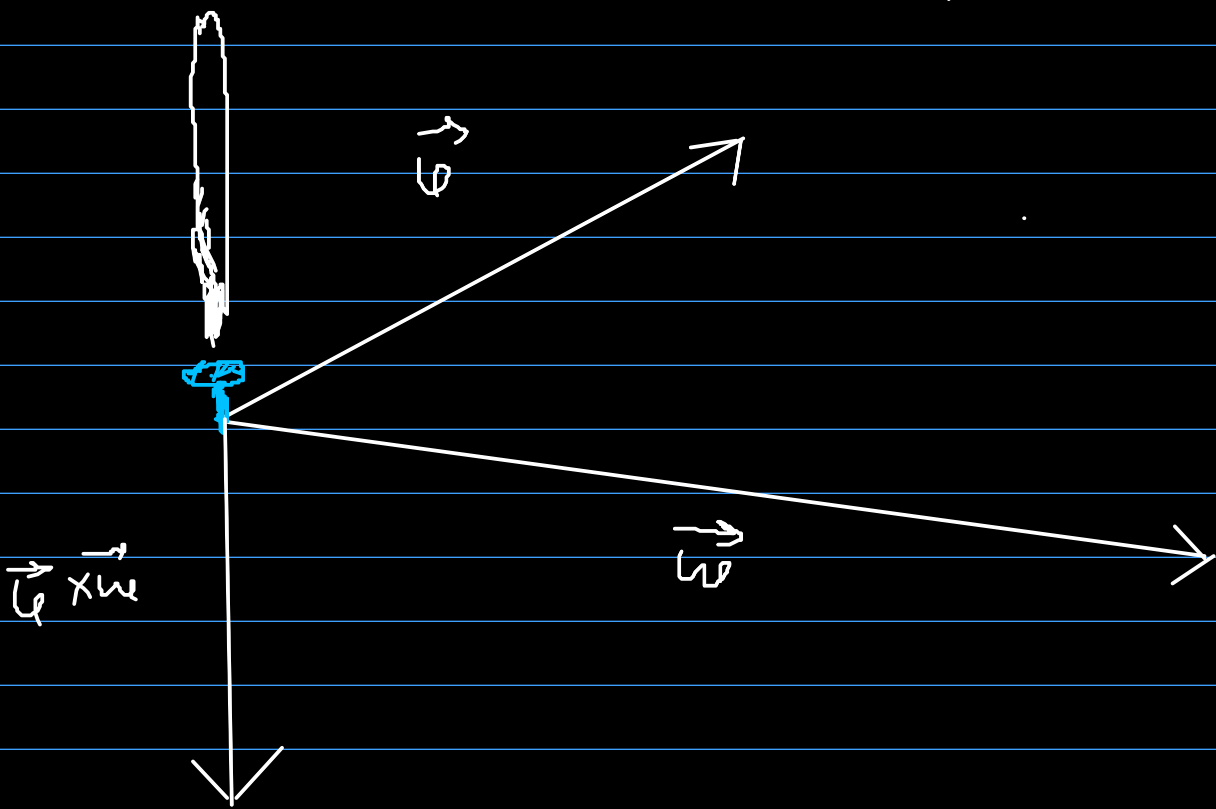
  $\|$ Area(parallelogram formed by $\vec{u}$ and $\vec{w}$)





→ orientation

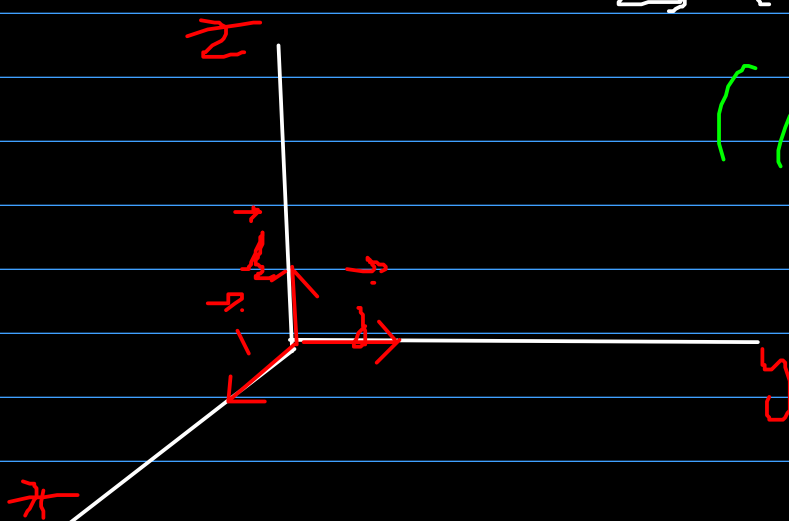
"screw rule"



3/ the reference system $(O, [\vec{i}, \vec{j}, \vec{k}])$ is orthonormal and <u>direct</u>

↳ $\vec{i} \times \vec{j} = \vec{k}$

(for us: all the time!)



Then the cross product is computed as follows: $\vec{u}(a_1, b_1, c_1)$, $\vec{w}(a_2, b_2, c_2)$

$$\vec{u} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} =$$

$$= (b_1 c_2 - b_2 c_1) \cdot \vec{i} - (a_1 c_2 - a_2 c_1) \vec{j} + $$
$$+ (a_1 b_2 - a_2 b_1) \cdot \vec{k} = (b_1 c_2 - a_2 c_1, a_2 c_1 - a_1 c_2, a_1 b_2 - a_2 b_1).$$

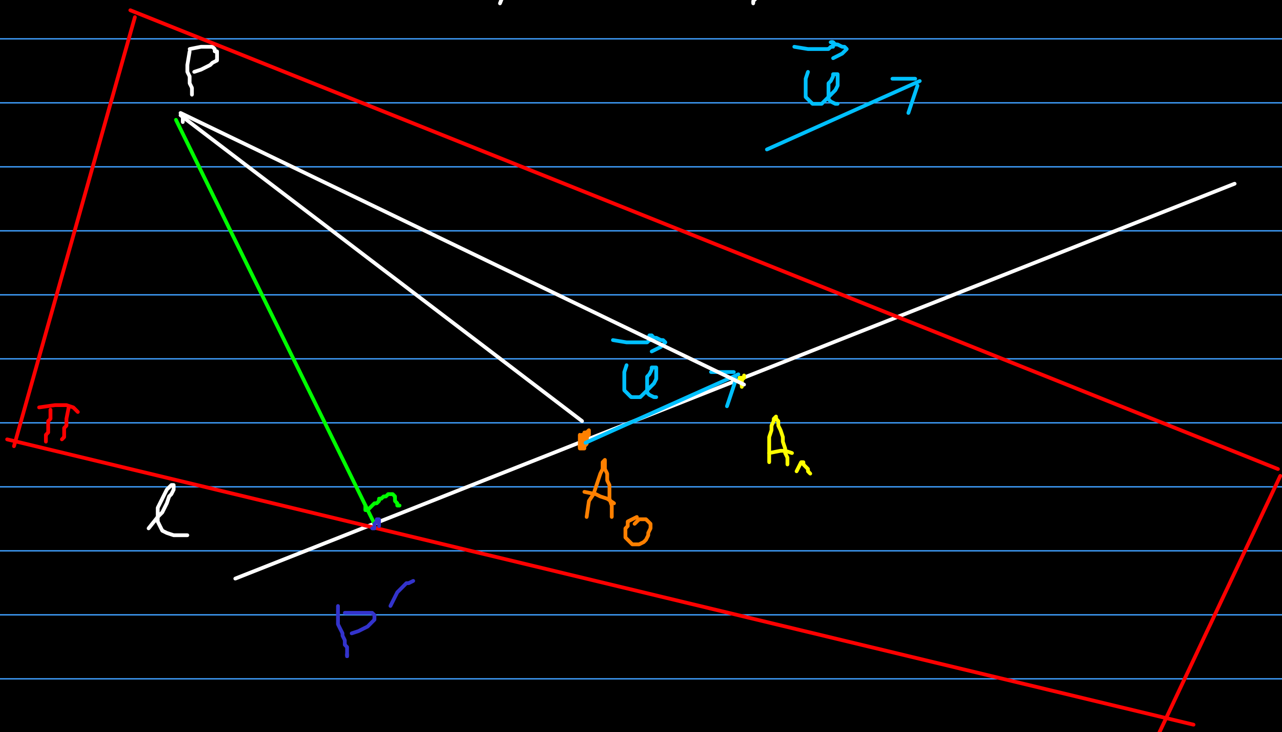
$\rightarrow$ The cross product is $\rightarrow$ anti-commutative

- bilinear.

$\forall \alpha, \beta \in \mathbb{R}$, $\forall u_1, u_2, w \in V$:

$$(\alpha u_1 + \beta u_2) \times \vec{w} = \alpha \vec{u_1} \times \vec{w} + \beta \cdot \vec{u_2} \times \vec{w}$$

The distance from a point to a line



π plane so that $\pi \perp \ell$, $\pi \ni P$

$\{P'\} = \ell \cap \pi \Rightarrow \ell \perp PP'$

$\Rightarrow PP'$ is the perpendicular from P to the line ℓ.

Take $A_0 \in \ell$, $\vec{u} \parallel \ell \Rightarrow \exists A_1 \in \ell$ so that:

$$\overrightarrow{A_0A_1} = \vec{u}$$

$\Rightarrow PP'$ height in $\Delta\, PA_0A_1$

$$\Rightarrow PP' = \frac{2A_{PA_0A_1}}{A_0A_1} = \frac{\|\overrightarrow{PA_0} \times \overrightarrow{A_0A_1}\|}{\|\overrightarrow{A_0A_1}\|} =$$

$$= \frac{\|\overrightarrow{PA_0} \times \vec{u}\|}{\|\vec{u}\|}$$

6.4. Find the distance from $P(1,2,-1)$ to the line ℓ: $x = y = z$.

<u>Proof</u>: $\overrightarrow{v_\ell}(1,1,1)$, $A_0(1,1,1) \in \ell$

$\overrightarrow{PA_0}(0,-1,2)$

$$\overrightarrow{PA_0} \times \overrightarrow{v_\ell} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} =$$

$$= -\vec{i} + 2\vec{j} + \vec{k} - 2\vec{i} = -3\vec{i} + 2\vec{j} + \vec{k}$$

$$\Rightarrow \|\overrightarrow{PA_0} \times \overrightarrow{v_\ell}\| = \sqrt{9+4+1} = \sqrt{14}$$

$$\|v_\ell\| = \sqrt{3}$$

$$\Rightarrow dist(P, \ell) = \sqrt{\frac{14}{3}}.$$

6.5. Find the area of the triangle ABC and the lengths of its heights, where $A(-1,1,2)$, $B(2,-1,1)$, $C(2,-3,-2)$.

$$A_{ABC} = \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\|$$

$$\overrightarrow{AB}(3,-2,-1), \quad \overrightarrow{AC}(3,-4,-4)$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -1 \\ 3 & -4 & -4 \end{vmatrix} =$$

$$= 4\vec{i} + 9\vec{j} - 6\vec{k}$$

$$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{16 + 81 + 36} = \sqrt{133}$$

$$A_{ABC} = \frac{\sqrt{133}}{2}$$

$\overrightarrow{BC}(0; -2, -3)$

$$h_A = \frac{2 \cdot A_{ABC}}{\|\overrightarrow{BC}\|} = \frac{\sqrt{133}}{\sqrt{13}}.$$

$$h_B = \frac{2 \cdot A_{ABC}}{\|\overrightarrow{AC}\|} = \frac{\sqrt{133}}{\sqrt{41}}.$$

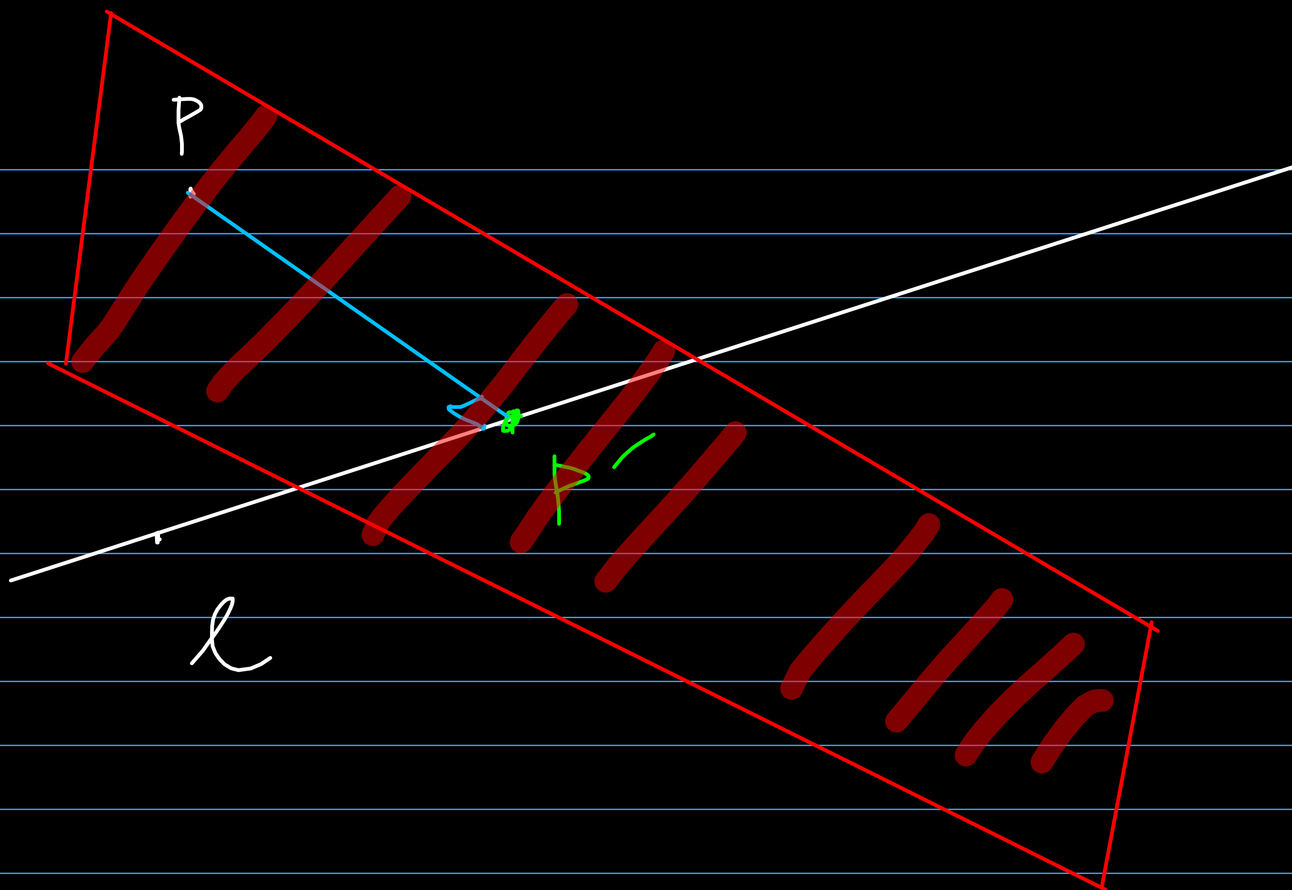
$$h_C = \frac{2 \cdot A_{ABC}}{\|\overrightarrow{AB}\|} = \frac{\sqrt{133}}{\sqrt{14}}$$

6.π. Consider the line:

$$\ell : \begin{cases} \pi_1 : x + 2y - 8z + 5 = 0 \\ \pi_2 : 2x + y + z + 1 = 0 \end{cases}$$

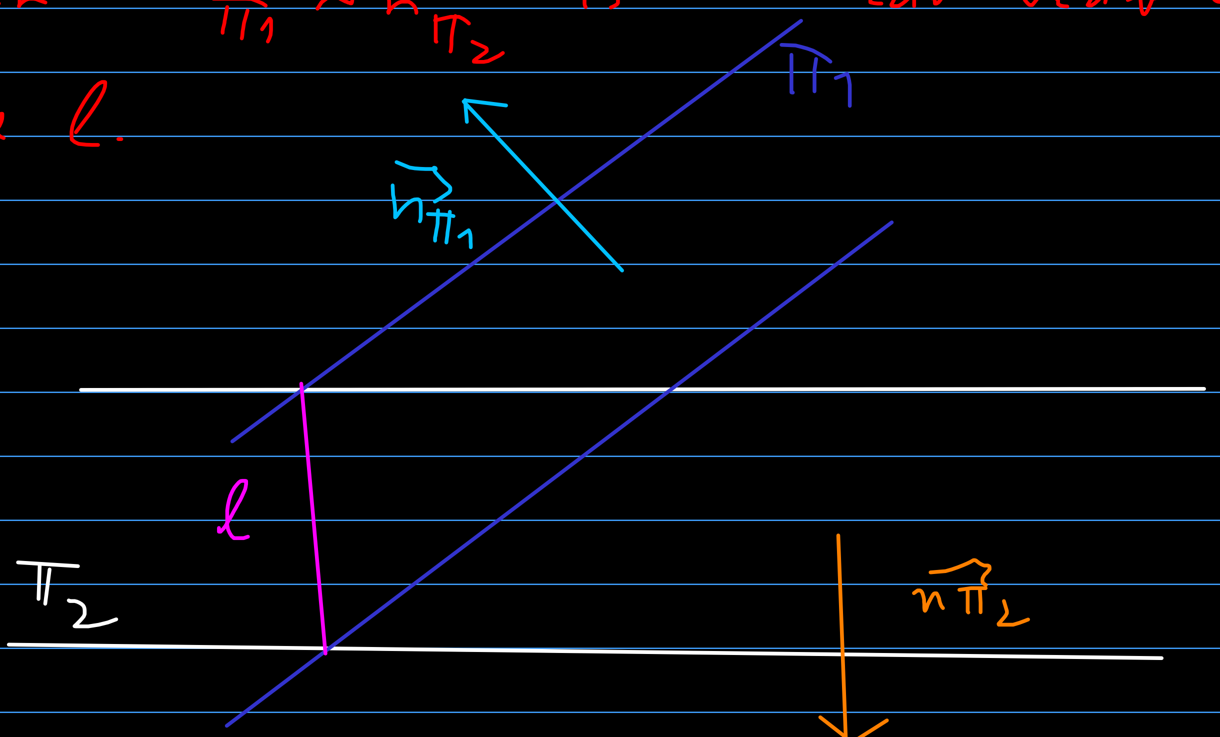
and the point $P(1, 2, 3)$

Find the equation of the perpendicular from P onto the line ℓ.



If l : $\begin{cases} \Pi_1 : A_1x + B_1y + C_1z + D_1 = 0 \\ \Pi_2 : A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$

then $\overrightarrow{n_{\Pi_1}} \times \overrightarrow{n_{\Pi_2}}$ is a director vector of the line l.



$$\left.\begin{array}{ll}\vec{n}_{\pi_1} \perp \pi_1 & \Rightarrow \vec{n}_{\pi_1} \perp l \\ \vec{n}_{\pi_2} \perp \pi_2 & \Rightarrow \vec{n}_{\pi_2} \perp l\end{array}\right\} \Rightarrow l \parallel \vec{n}_{\pi_1} \times \vec{n}_{\pi_2}$$

$$l: \begin{cases} \pi_1: x+2y-8z+5=0 \\ \pi_2: 2x+y+z+1=0 \end{cases}$$

$$\vec{n}_{\pi_1}(1,2,-8), \quad \vec{n}_{\pi_2}(2,1,1)$$

$$\vec{u}_l = \vec{n}_{\pi_1} \times \vec{n}_{\pi_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -8 \\ 2 & 1 & 1 \end{vmatrix} =$$

$$= 2\vec{i} - 16\vec{j} + \vec{k} - 4\vec{k} + 8\vec{i} - \vec{j} =$$

$$= 10\vec{i} - 17\vec{j} - 3\vec{k}$$

$$\Rightarrow \vec{u}_l(10, -17, -3)$$

We now write the equation of the plane π that is perpendicular to l and contains P.

$\vec{u_\ell} \perp \Pi \Rightarrow \vec{u_\ell} \parallel \vec{n_\Pi}$

$\Rightarrow \Pi : \ 10x - 17y - 3z + D = 0$

$P \in \Pi \Rightarrow 10 \cdot 1 - 17 \cdot 2 - 3 \cdot 3 + D = 0$

$\Rightarrow D = 33$

$\Rightarrow \Pi : \ 10x - 17y - 3z + 33 = 0$

$$P' = \Pi \cap \ell : \begin{cases} 10x - 17y - 3z + 33 = 0 \\ x + 2y - 8z + 5 = 0 \\ 2x + y + z + 1 = 0 \end{cases}$$

$$\begin{pmatrix} 10 & -17 & -3 & -33 \\ 1 & 2 & -8 & -5 \\ 2 & 1 & 1 & -1 \end{pmatrix} \underset{\sim}{L_1 \leftrightarrow L_2}$$

$$\sim \begin{pmatrix} 1 & 2 & -8 & -5 \\ 10 & -17 & -3 & -33 \\ 2 & 1 & 1 & -1 \end{pmatrix} \begin{matrix} L_2 \leftarrow L_2 - 10L_1 \\ \sim \\ L_3 \leftarrow L_3 - 2L_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -8 & -5 \\ 0 & -37 & 77 & 17 \\ 0 & -3 & 17 & 9 \end{pmatrix} \sim$$

$L_3 \leftarrow L_3 - \frac{3}{37} L_2$

$$\sim \begin{pmatrix} \cdot & \cdot & \cdot \end{pmatrix}$$

$$\longrightarrow \begin{cases} x_p' = \ldots \\ y_p' = \ldots \\ z_p' = \ldots \end{cases}$$

$$\Rightarrow \frac{x - x_p}{x_p' - x_p} = \frac{y - y_p}{y_p' - y_p} = \frac{z - z_p}{z_p' - z_p}$$