Seminar W6 – 917

Cross product (vector product)

- $\vec{u}, \vec{w} \in V$, $\vec{u}, \vec{w}$ linearly dependent

$$\vec{u} \times \vec{w} = \vec{0}$$

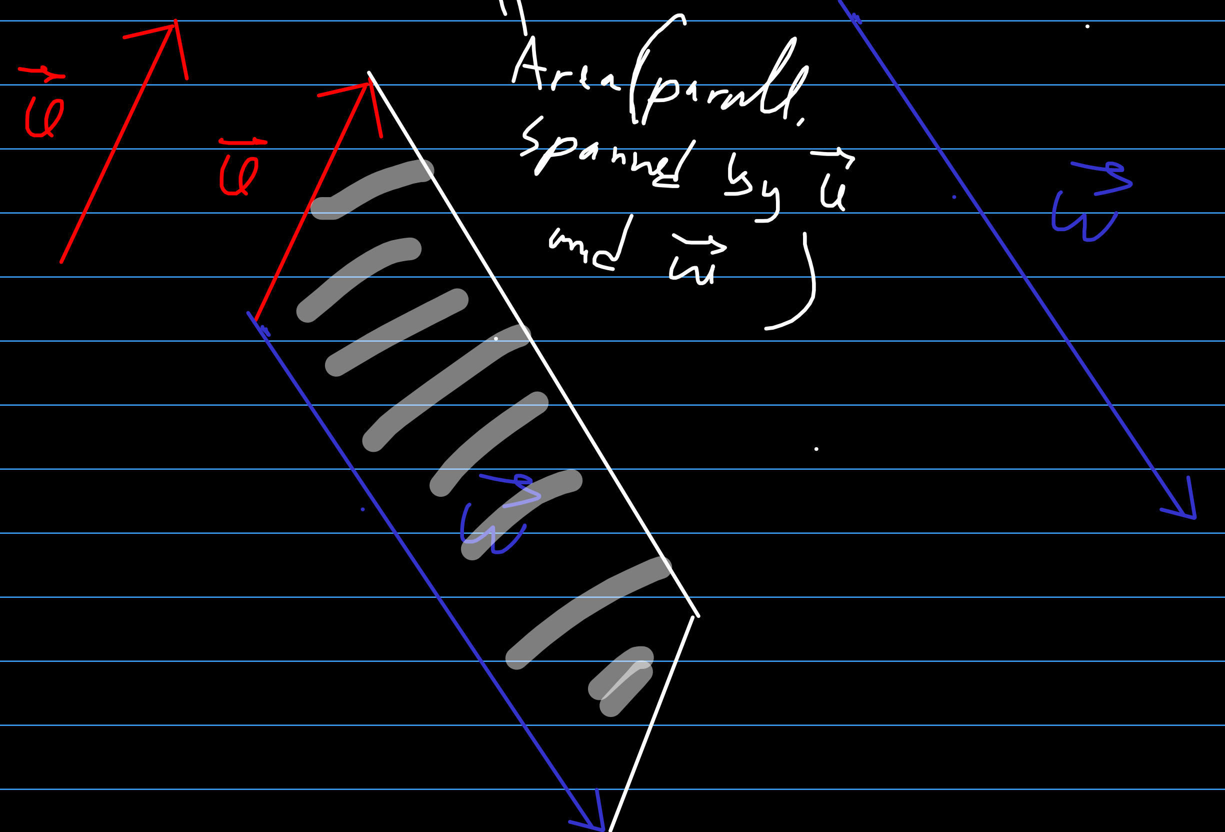
- if $\vec{u}, \vec{w}$ linearly independent:

– direction: perpendicular to both $\vec{u}$ and $\vec{w}$

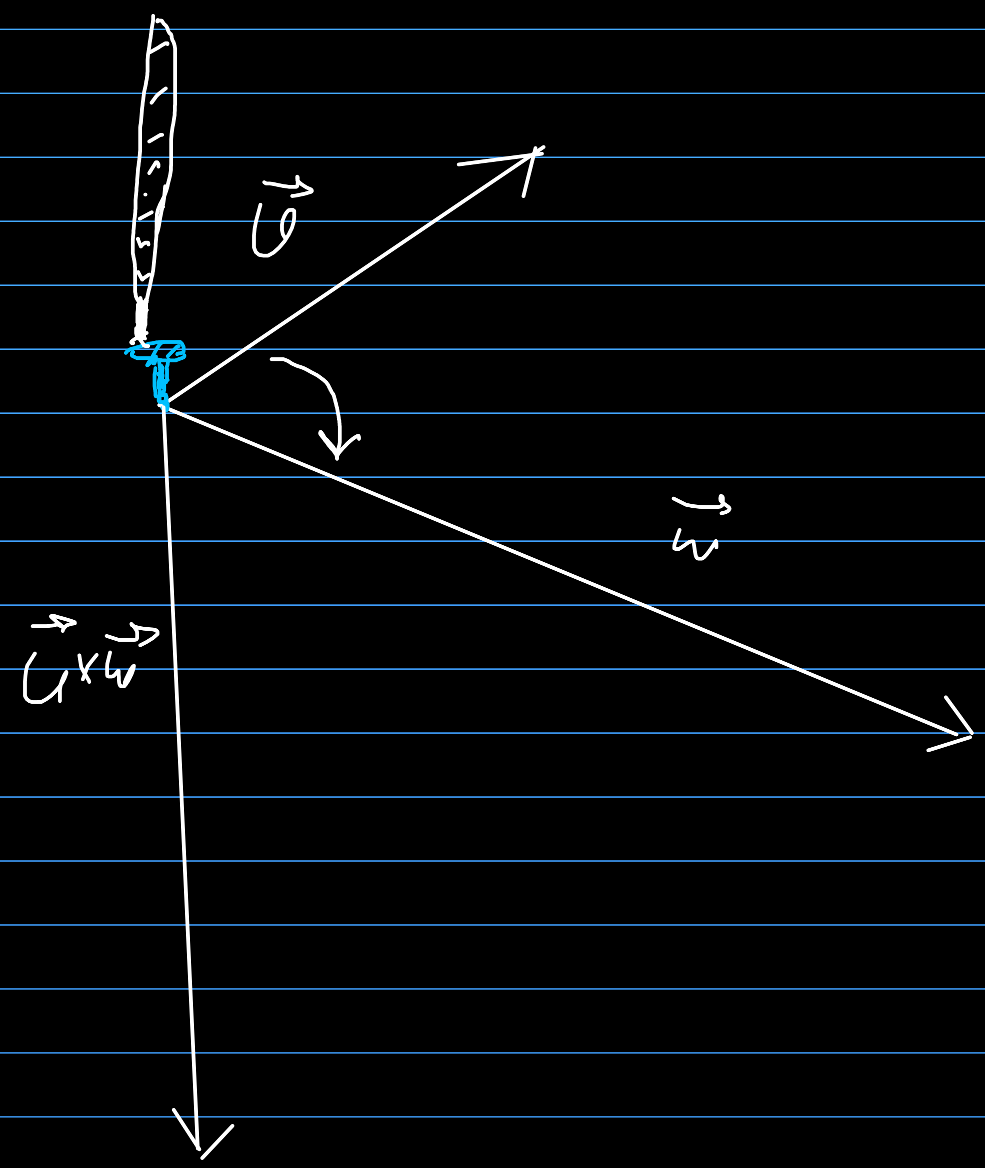
(i.e. $\vec{u} \times \vec{w} \perp \langle \vec{u}, \vec{w} \rangle$)

– norm: $\|\vec{u} \times \vec{w}\| = \|\vec{u}\| \cdot \|\vec{w}\| \cdot \sin(\widehat{\vec{u}, \vec{w}})$

$\|$ Area(parall. spanned by $\vec{u}$ and $\vec{w}$)

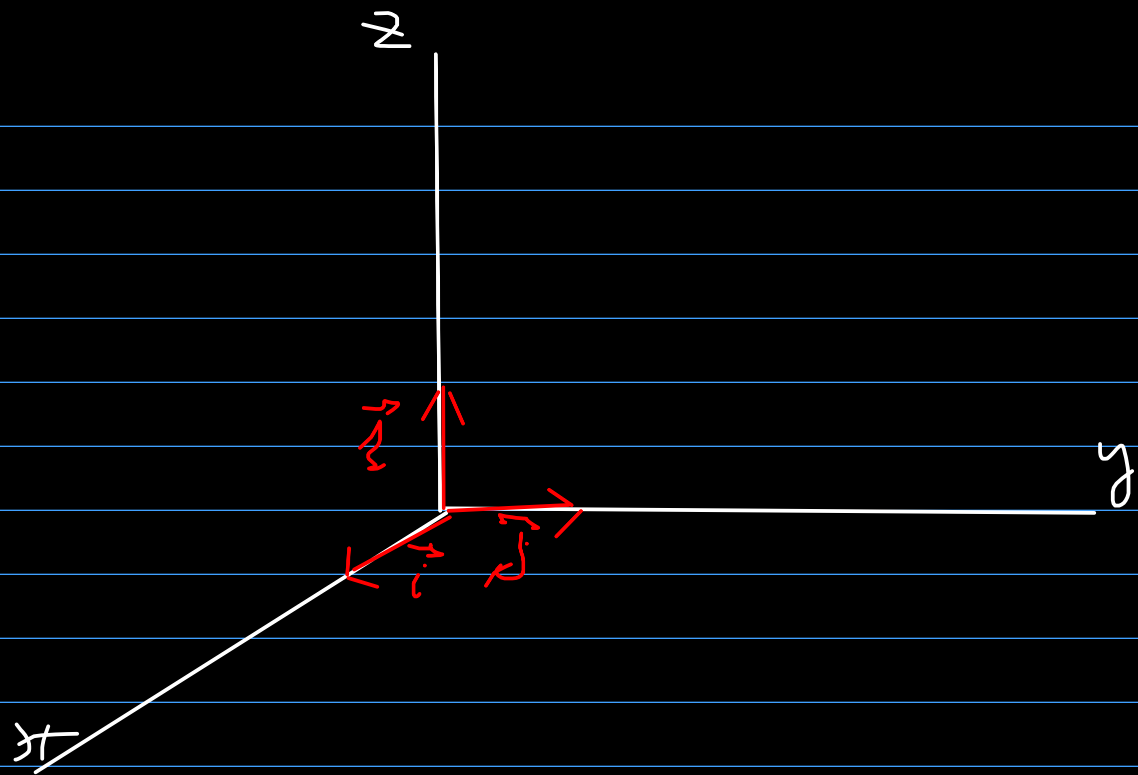


- orientation



3/ the reference system $(O, [\vec{i}, \vec{j}, \vec{k}])$ is orthonormal and <u>direct</u>

$$\vec{i}\times\vec{j} = \vec{k}$$



then the cross product is computed as follows: $\vec{u}(a_1, b_1, c_1)$, $\vec{w}(a_2, b_2, c_2)$

$$\vec{u} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} =$$

$$= \vec{i} \cdot (b_1 c_2 - c_1 b_2) - \vec{j}(a_1 c_2 - a_2 c_1) +$$

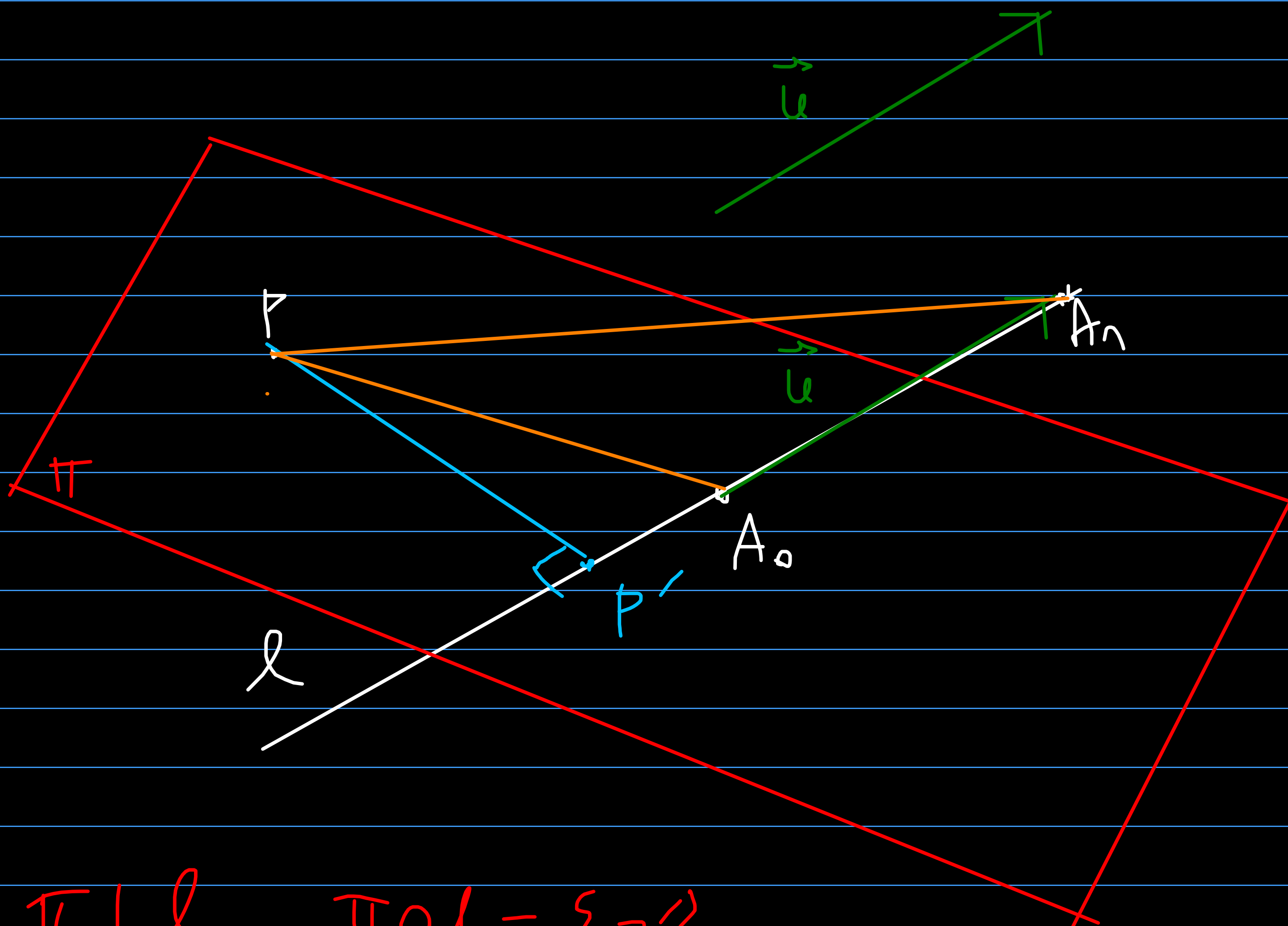
$$+ \vec{k}(a_1 b_2 - a_2 b_1) =$$

$$= (b_1 c_2 - c_1 b_2, a_2 c_1 - a_1 c_2, a_1 b_2 - a_2 b_1)$$

the cross product is anti-commutative:

$$\vec{u} \times \vec{w} = -\vec{w} \times \vec{u}$$

The distance from a point to a line



$\pi \perp l$, $\pi \cap l = \{P'\}$

$\Rightarrow PP'$ is the perpendicular from P to l

Let $A_0 \in \ell$ and $\vec{u} \parallel \ell \Rightarrow \exists A_1 \in \ell$ so that $\overrightarrow{A_0A_1} = \vec{u}$

$\Rightarrow PP'$ height in $\Delta\, PA_0A_1$

$$\Rightarrow \operatorname{dist}(P, \ell) = PP' = \frac{2\, A_{PA_0A_1}}{A_0A_1} =$$

$$= \frac{\|\overrightarrow{PA_0} \times \overrightarrow{A_0A_1}\|}{\|\overrightarrow{A_0A_1}\|} = \frac{\|\overrightarrow{PA_0} \times \vec{u}\|}{\|\vec{u}\|}$$

6.4. Find the distance from $P(1,2,-1)$ to the line $\ell: x = y = z$.

$A, B \in \ell$; $A(1,1,1)$, $B(2,2,2)$

$$\operatorname{dist}(P, \ell) = \frac{\|\overrightarrow{PA} \times \overrightarrow{AB}\|}{\|\overrightarrow{AB}\|}$$

$$\vec{PA} = (0,-1,2) = (1,1,1)-(1,2,-1)$$

$$\vec{AB} = (1,1,1)$$

$$\vec{PA} \times \vec{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -3\vec{i} + 2\vec{j} + \vec{k}$$

$$\Rightarrow \|\vec{PA} \times \vec{AB}\| = \sqrt{9+4+1} = \sqrt{14}$$

$$\Rightarrow \|\vec{AB}\| = \sqrt{3}$$

$$\Rightarrow \operatorname{dist}(P,\ell) = \frac{\sqrt{14}}{\sqrt{3}}$$

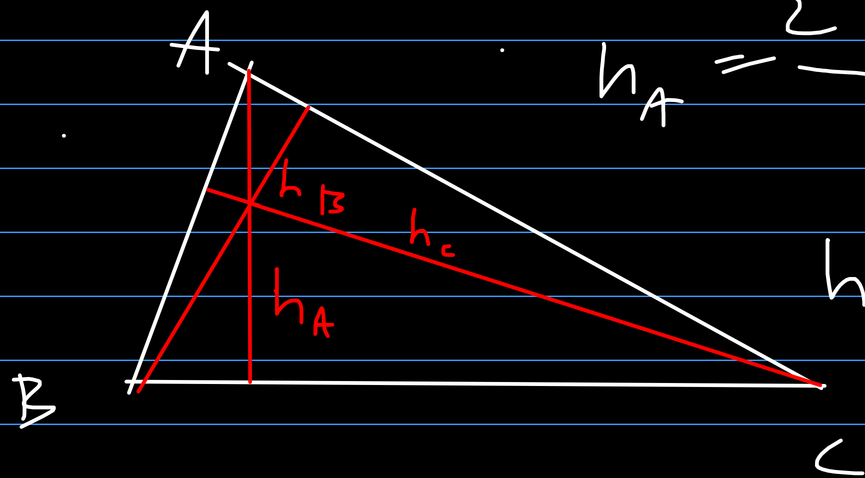
6.5. Find the area of the triangle ABC and the lengths of its heights, where $A(-1,1,2)$, $B(2,-1,1)$, $C(2,-3,-2)$.

Proof: $\overrightarrow{AB}(3,-2,-1)$, $\overrightarrow{BC}(0,-2,-3)$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -1 \\ 0 & -2 & -3 \end{vmatrix} = 4\vec{i} + 9\vec{j} - 6\vec{k}$$

$$A_{ABC} = \frac{1}{2} \cdot \|\overrightarrow{AB} \times \overrightarrow{BC}\| = \frac{1}{2} \cdot \sqrt{16+81+36} =$$

$$= \frac{1}{2} \cdot \sqrt{133}.$$



$$h_A = \frac{2 \cdot A_{ABC}}{\|\overrightarrow{BC}\|} = \frac{\sqrt{133}}{\sqrt{13}}$$

$$h_B = \frac{2 \cdot A_{ABC}}{\|\overrightarrow{AC}\|} = \sqrt{\frac{133}{41}}$$

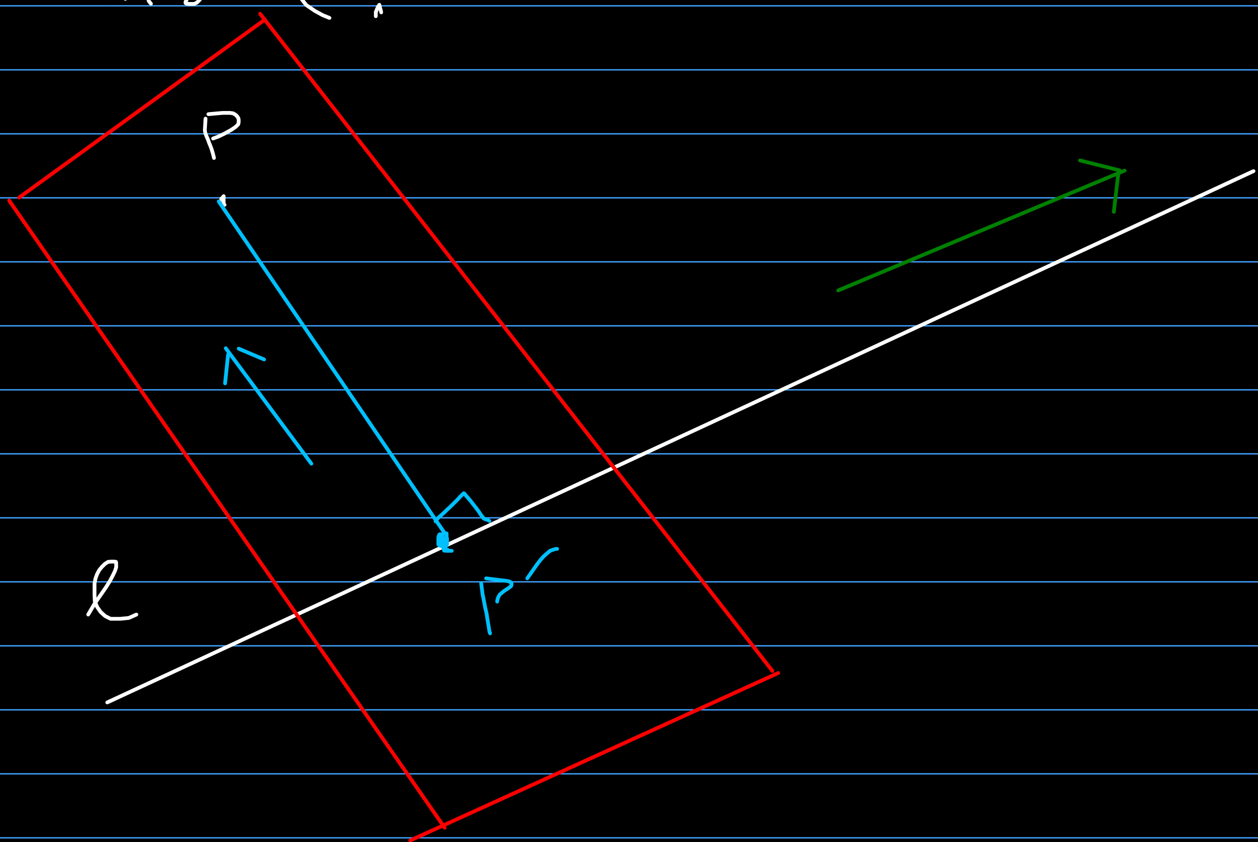
$$h_C = \frac{2 \cdot A_{ABC}}{\|\overrightarrow{AB}\|} = \sqrt{\frac{133}{14}}$$

6.π Consider the line:

$$l : \begin{cases} 2x - y + z - 4 = 0 \\ x + 3y - 6z + 7 = 0 \end{cases}$$

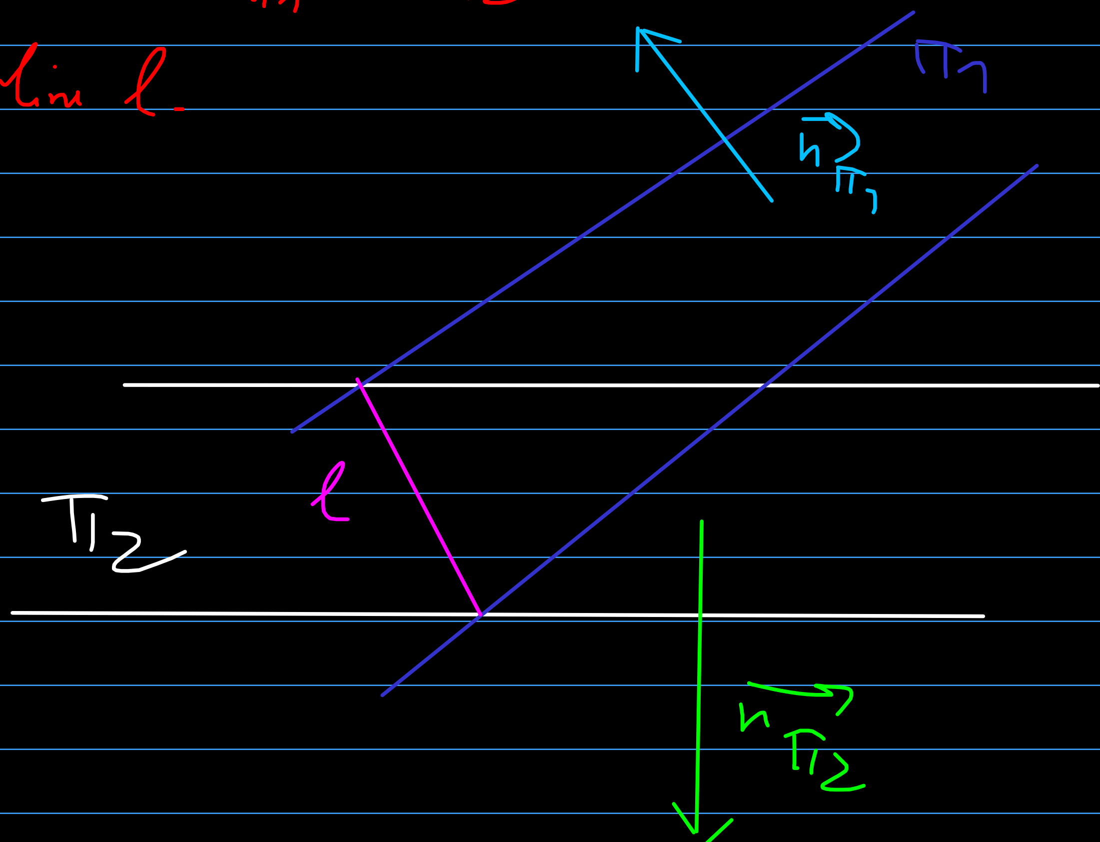
and the point $P(1, 2, 3)$.

Find the equation of the perpendicular from P onto l.



If ℓ is given by $\ell: \begin{cases} \Pi_1: A_1x+B_1y+C_1z+D_1=0 \\ \Pi_2: A_2x+B_2y+C_2z+D_2=0 \end{cases}$

then $\overrightarrow{n_{\Pi_1}} \times \overrightarrow{n_{\Pi_2}}$ is a director vector of the line ℓ.



$$\left.\begin{array}{l} \overrightarrow{n_{\Pi_1}} \perp \Pi_1 \Rightarrow \overrightarrow{n_{\Pi_1}} \perp \ell \\ \overrightarrow{n_{\Pi_2}} \perp \Pi_2 \Rightarrow \overrightarrow{n_{\Pi_2}} \perp \ell \end{array}\right\} \Rightarrow \ell \parallel \left(\overrightarrow{n_{\Pi_1}} \times \overrightarrow{n_{\Pi_2}}\right)$$

$$l: \begin{cases} 2x - y + z - 4 = 0 \\ x + 3y - 6z + 7 = 0 \end{cases}$$

$$\overrightarrow{n_{\pi_1}}(2, -1, 1) \quad , \quad \overrightarrow{n_{\pi_2}}(1, 3, -6)$$

$$\overrightarrow{n_{\pi_1}} \times \overrightarrow{n_{\pi_2}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 3 & -6 \end{vmatrix} =$$

$$= 3\vec{i} + 13\vec{j} + 7\vec{k}$$

$$\Rightarrow \vec{u_l}(3, 13, 7)$$

We will write the equation of the plane π that is perpendicular to l and contains P.

$$\pi: 3x + 13y + 7z + D = 0$$

$$P \in \pi \Rightarrow 3 \cdot 1 + 13 \cdot 2 + 7 \cdot 3 + D = 0$$

$\Rightarrow D = -50$

$\Rightarrow \Pi: \; 3x + 13y + 7z - 50 = 0$

y) we want the plane that has (A, B, C) as a normal vector and contains the point (x_0, y_0, z_0):

$\Pi: A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

$$\{P'\} = \ell \cap \Pi : \begin{cases} 3x + 13y + 7z - 50 = 0 \\ 2x - y + z - 4 = 0 \\ x + 3y - 6z + 7 = 0 \end{cases}$$

$$\begin{pmatrix} 3 & 13 & 7 & 50 \\ 2 & -1 & 1 & 4 \\ 1 & 3 & -6 & -7 \end{pmatrix} \underset{\sim}{L_1 \Leftrightarrow L_3}$$

$$\begin{pmatrix} 1 & 3 & -6 & -7 \\ 2 & -1 & 1 & 4 \\ 3 & 13 & 7 & 50 \end{pmatrix} \sim$$

$$\begin{array}{c} L_2 \leftarrow L_2 - 2L_1 \\ \sim \\ L_3 \leftarrow L_3 - 3L_1 \end{array} \begin{pmatrix} 1 & 3 & -6 & -7 \\ 0 & -7 & 13 & 18 \\ 0 & 4 & 25 & 71 \end{pmatrix} \sim$$

$$\begin{array}{c} L_3 \leftarrow L_3 + \frac{4}{7} L_2 \\ \sim \end{array} \begin{pmatrix} 1 & 3 & -6 & -7 \\ 0 & -7 & 13 & 18 \\ 0 & 0 & 25 + \frac{52}{7} & 71 + \frac{72}{7} \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + 3y - 6z = -7 \\ -7y + 13z = -18 \\ z = \dfrac{71 + \frac{72}{7}}{25 + \frac{52}{7}} = \dfrac{497 + 72}{175 + 52} \end{cases}$$

$\Rightarrow x_{p'}, y_{p'}, z_{p'}$

$$\Rightarrow PP': \frac{x - x_p}{x_{p'} - x_p} = \frac{y - y_p}{y_{p'} - y_p} = \frac{z - z_p}{z_{p'} - z_p}$$

The double cross product.

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \end{vmatrix} =$$

$$= (\vec{a}\cdot\vec{c})\cdot\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{b}\cdot\vec{c} & \vec{a}\cdot\vec{c} \end{vmatrix} =$$

$$= (\vec{a}\cdot\vec{c})\cdot\vec{b} - (\vec{b}\cdot\vec{c})\cdot\vec{a}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b}).$$

$\rightarrow$ Conclusion: The cross product is not associative !!!