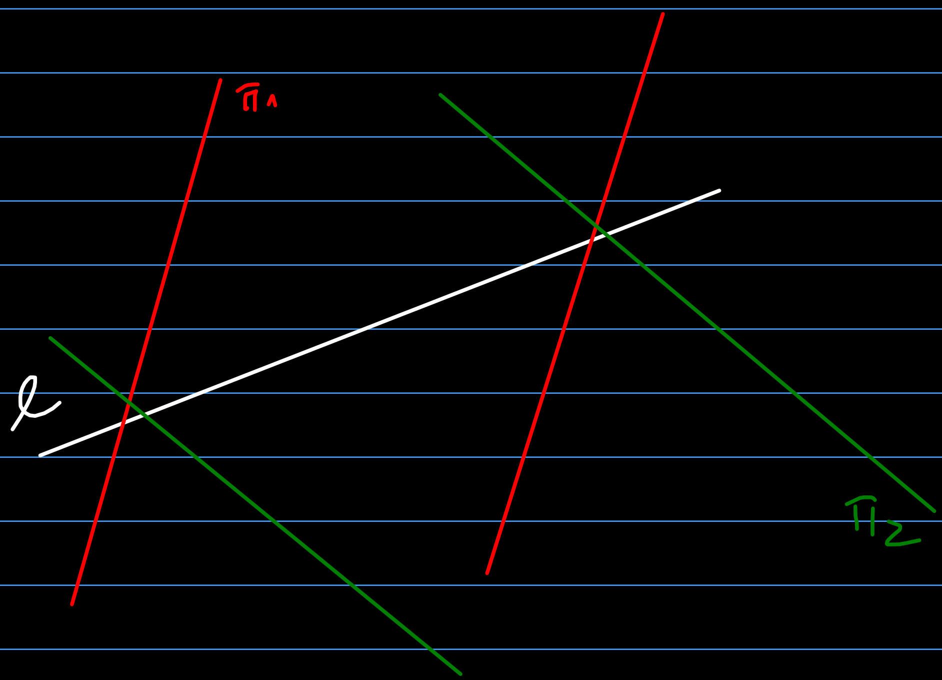
Seminar W4 - 915

Pencils of planes

$$\ell : \begin{cases} \pi_1 : A_1x + B_1y + C_1z + D_1 = 0 \\ \pi_2 : A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

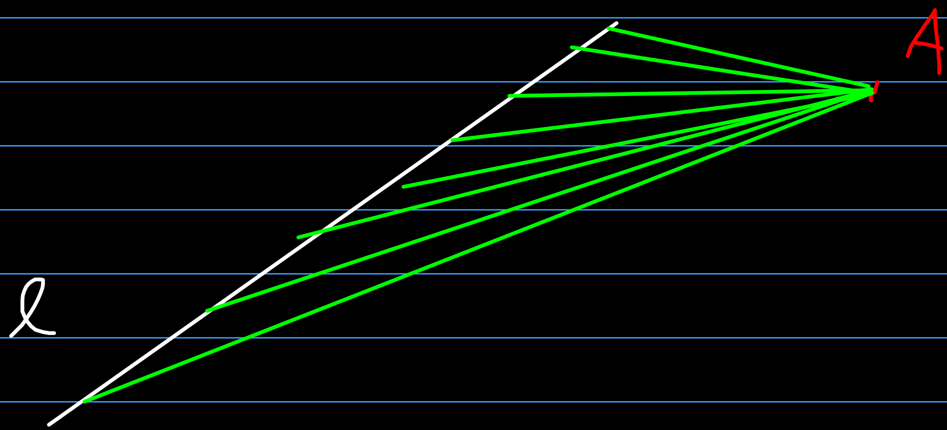


$$\pi_{\alpha,\beta} : \alpha \cdot (A_1x + B_1y + C_1z + D_1) + \beta \cdot (A_2x + B_2y + C_2z + D_2) = 0$$

4.1. Write the equation of the plane determined by the line

$$(d) : \begin{cases} x - 2y + 3z = 0 \\ 2x + z - 3 = 0 \end{cases}$$

and the point $A(-1, 2, 6)$.



We write the pencil of planes corresponding to l:

$$\Pi_{\alpha,\beta} : \alpha(x-2y+3z)+\beta(2x+z-3)=0$$

$$\Pi_{\alpha,\beta} : x(\alpha+2\beta)+y\cdot(-2\alpha)+z\cdot(3\alpha+\beta)-3\beta=0$$

$$A \in \Pi_{\alpha,\beta} \Rightarrow (-1)(\alpha+2\beta)+2\cdot(-2\alpha)+6\cdot(3\alpha+\beta)-3\beta=0$$

$$\Rightarrow 13\alpha+\beta=0 \Rightarrow \beta=-13\alpha$$

$\Rightarrow$ The planes that we need are $\Pi_{\alpha,-13\alpha}$

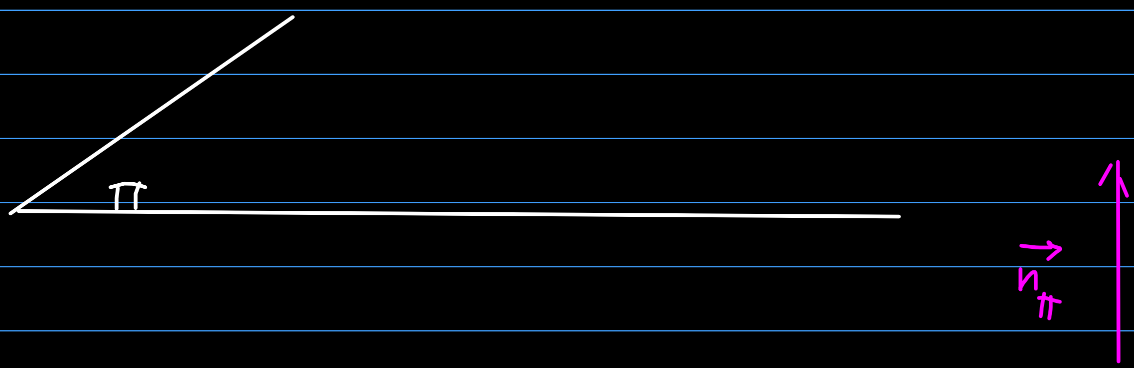
$$\Pi_{\alpha,-13\alpha} : \alpha(x-2y+3z)-13\alpha(2x+z-3)=0$$

$\Rightarrow$ this is, in fact, just one plane : $\Pi_{1,-13} : -25x-2y-10z+39=0$

- $\Pi : Ax + By + Cz + D = 0$

  $\vec{n}_\pi \, (A, B, C)$ normal vector of the plane

$(\text{i.e. } \vec{n}_\pi \perp \vec{u}, \ \forall \vec{u} \parallel \pi)$



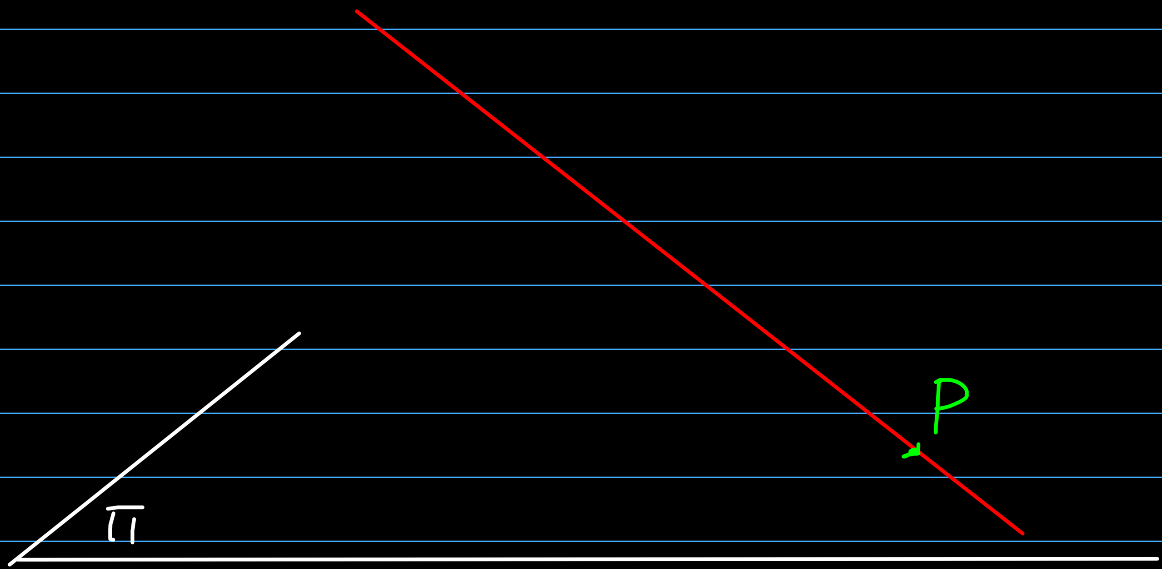
- $\Pi : Ax + By + Cz + D = 0$

$$\ell : \begin{cases} x = x_0 + \lambda u_x \\ y = y_0 + \lambda u_y \\ z = z_0 + \lambda u_z \end{cases}, \quad \lambda \in \mathbb{R}$$

$$\ell \parallel \Pi \iff \vec{\ell} \parallel \Pi \iff \vec{n}_\Pi \cdot \vec{\ell} = \vec{0}$$

$$\iff A u_x + B u_y + C u_z = 0$$

Assume that $\ell \nparallel \pi$, i.e. $Av_x + Bv_y + Cv_z \neq 0$



The coordinates of the intersection point $\{P\} = \ell \cap \pi$ are:

$$\begin{cases} x_P = x_0 - \dfrac{Ax_0 + By_0 + Cz_0 + D}{Av_x + Bv_y + Cv_z} \cdot v_x \\[2ex] y_P = y_0 - \dfrac{Ax_0 + By_0 + Cz_0 + D}{Av_x + Bv_y + Cv_z} \cdot v_y \\[2ex] z_P = z_0 - \dfrac{Ax_0 + By_0 + Cz_0 + D}{Av_x + Bv_y + Cv_z} \cdot v_z \end{cases}$$

$(x_0, y_0, z_0) \in \ell$, $(v_x, v_y, v_z) = \vec{\ell}$

Ex.: Consider the line ℓ:

$$\ell: \frac{x-2}{3} = \frac{y+1}{7} = \frac{z}{2}$$

$$\Pi: x + 2y - 3z + 6 = 0$$

Find the intersection point $\{P\} = \ell \cap \Pi$

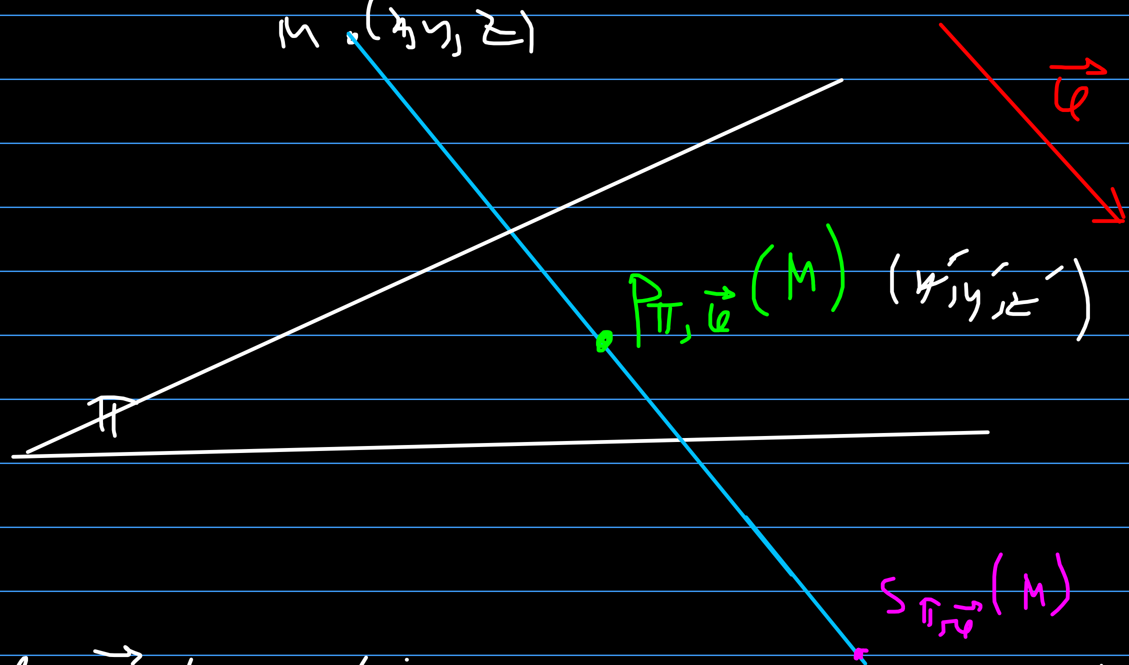
$$\ell: \begin{cases} x = 2 + 3\lambda \\ y = -1 + 7\lambda \\ z = 2\lambda \end{cases}$$

$$P: \begin{cases} x = 2+3\lambda \\ y = -1+7\lambda \\ z = 2\lambda \\ x+2y-3z+6=0 \end{cases} \iff \begin{cases} x = 2+3\lambda \\ y = -1+7\lambda \\ z = 2\lambda \\ 2+3\lambda - 2 + 14\lambda - 6\lambda + 6 = 0 \end{cases} \iff$$

$$\iff \begin{cases} x = 2+3\lambda \\ y = -1+7\lambda \\ z = 2\lambda \\ 11\lambda + 6 = 0 \end{cases} \iff \begin{cases} \lambda = -\frac{6}{11} \\ x = 2 - \frac{18}{11} = \frac{4}{11} \\ y = -1 - \frac{42}{11} = \frac{-53}{11} \\ z = \frac{-12}{11} \end{cases}$$

The projection onto a plane, parallel to a given direction

$\Pi: Ax+By+Cz+D=0$, $\vec{u}(p,q,r)$



If $\vec{u} \nparallel \Pi$ (i.e. $Ap+Bq+Cr \neq 0$), then we can define the projection:

$$P_{\Pi,\vec{u}} : \mathbb{R}^3 \to \Pi$$

$$(x,y,z) \longrightarrow (x',y',z')$$

$$\begin{cases} x' = x - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr}\cdot p \\ y' = y - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr}\cdot q \\ z' = z - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr}\cdot r \end{cases}$$

4.3. Write the equation of the reflection of the line

$$(\ell): \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

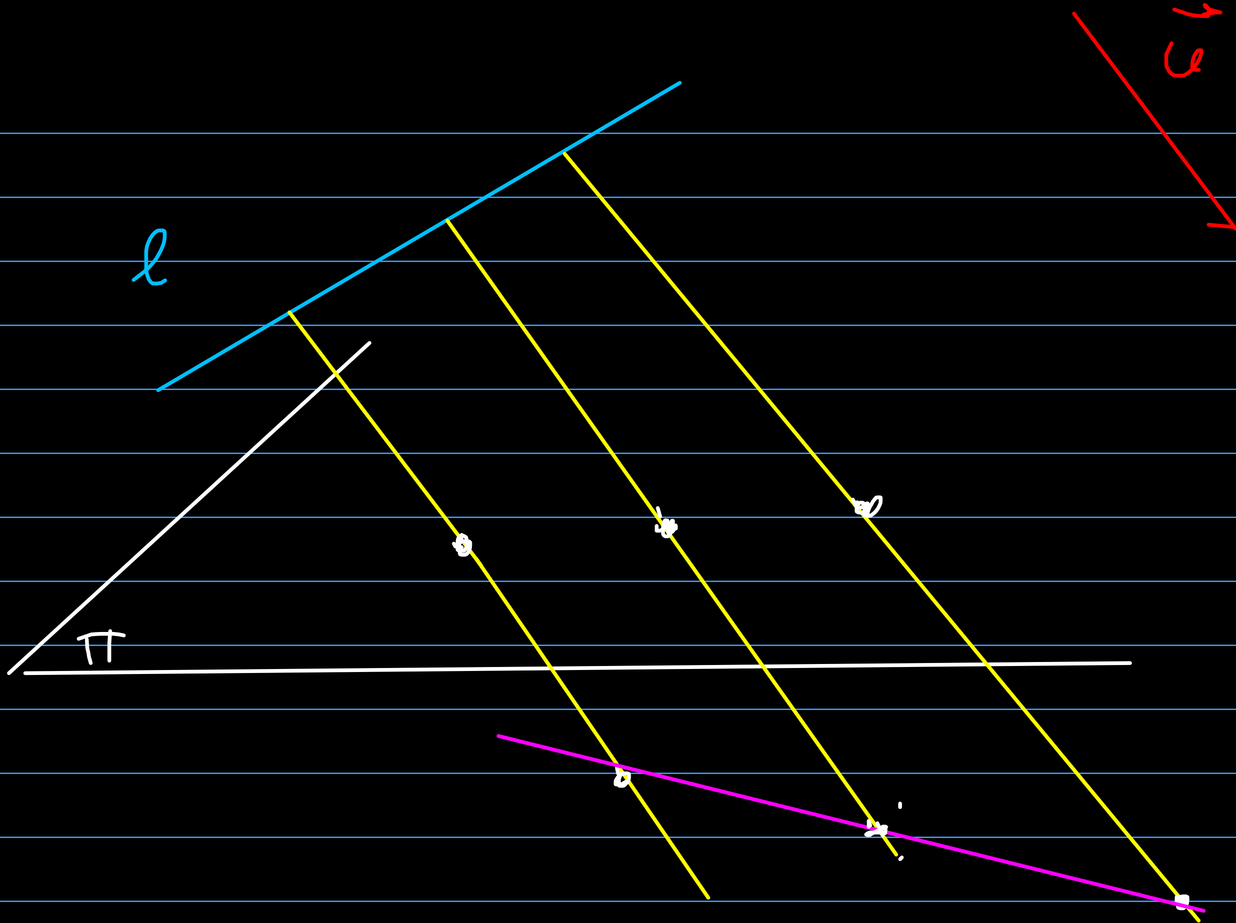
with regards to the plane $\Pi: x + 2y - z = 0$ parallel to the direction $\vec{u}(1, 1, -2)$

(Homework: do the same for the projection)

$$S_{\Pi, \vec{u}} : \mathbb{R}^3 \to \Pi$$

$$(x, y, z) \mapsto (x'', y'', z'')$$

$$\begin{cases} x'' = x - 2 \cdot \dfrac{Ax + By + Cz + D}{Ap + Bq + Cr} \cdot p \\ y'' = y - 2 \cdot \dfrac{Ax + By + Cz + D}{Ap + Bq + Cr} \cdot q \\ z'' = z - 2 \cdot \dfrac{Ax + By + Cz + D}{Ap + Bq + Cr} \cdot r \end{cases}$$



$$\ell : \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases} \iff$$

$$\iff \begin{cases} (2x - y + z - 1) + (x + y - z + 1) = 0 \\ x + y - z + 1 = 0 \end{cases} \iff$$

$$\iff \begin{cases} x = 0 \\ y = z - 1 \end{cases} \iff \begin{cases} x = 0 \\ y = y \iff \\ z = y + 1 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y = t \\ z = t + 1 \end{cases}$$

$$\begin{cases} x'' = x - 2 \cdot \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot p \\ y'' = y - 2 \cdot \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot q \\ z'' = z - 2 \cdot \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot r \end{cases}$$

$(p, q, r) = (1, 1, -2)$

$(A, B, C, D) = (1, 2, -1, 0)$

$$\lambda = \frac{Ax+By+Cz+D}{Ap+Bq+Cr} = \frac{x+2y-z}{p+2q-r} =$$

$$= \frac{0+2t-(t+1)}{1+2+2} = \frac{t-1}{5}$$

$$\Rightarrow \begin{cases} x'' = 0 - 2 \cdot \dfrac{t-1}{5} \cdot 1 \\ y'' = t - 2 \cdot \dfrac{t-1}{5} \cdot 1 \\ z'' = t+1 - 2 \cdot \dfrac{t-1}{5} \cdot (-2) \end{cases}$$

$$\Rightarrow \begin{cases} x'' = -\dfrac{2}{5}t + \dfrac{2}{5} \\ y'' = \dfrac{3}{5}t + \dfrac{2}{5} \\ z'' = \dfrac{9}{5}t + \dfrac{1}{5} \end{cases}$$

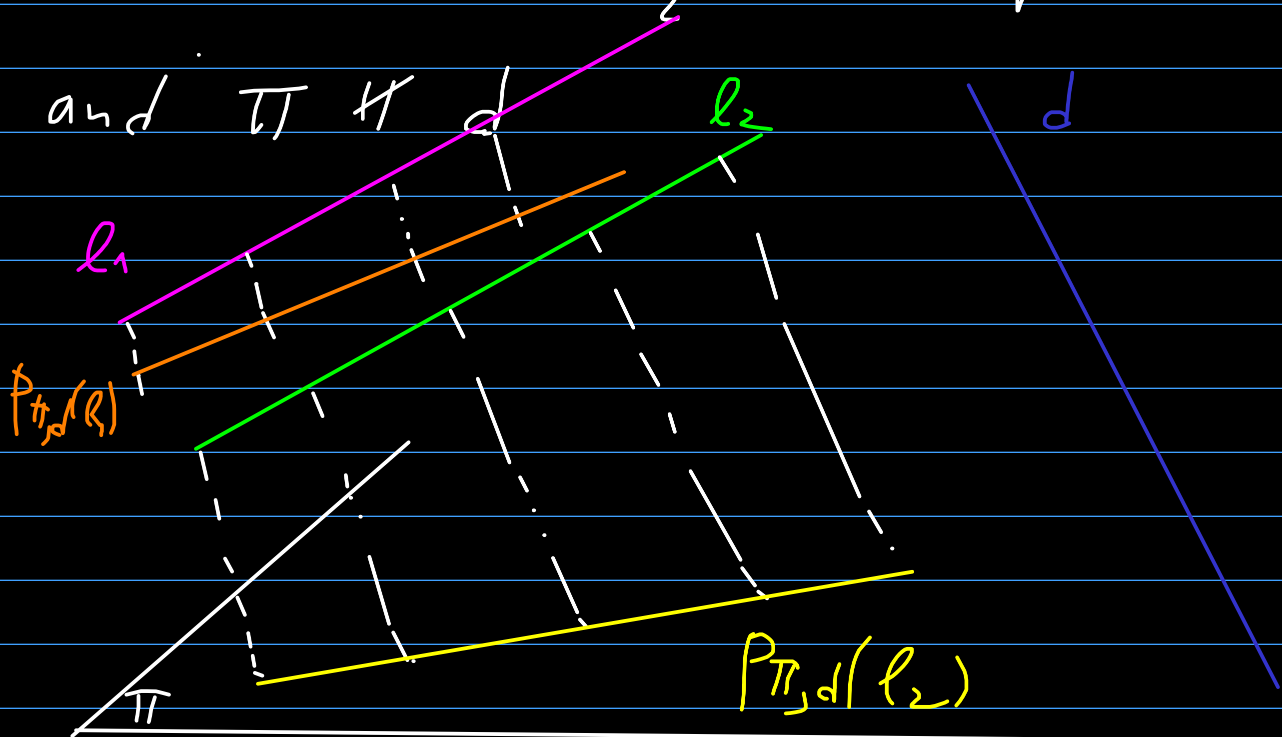
$\Rightarrow$ this is the equation of a line.

4. b(3p) Show that two different parallel lines are either projected onto parallel lines or onto two points by a projection $p_{\pi,d}$

$\pi : Ax + By + Cz + D = 0$

$d : \dfrac{x - x_0}{p} = \dfrac{y - y_0}{q} = \dfrac{z - z_0}{r}$

and $\pi \nparallel d$



Let $\ell_1 : \dfrac{x - x_1}{a_x} = \dfrac{y - y_1}{a_y} = \dfrac{z - z}{a_z}$

$$\Rightarrow \begin{cases} x = \lambda a_x + x_1 \\ y = \lambda a_y + y_1 \\ z = \lambda a_z + z_1 \end{cases}$$

$$\begin{cases} x' = x - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr}\cdot p \\ y' = y - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr}\cdot q \\ z' = z - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr}\cdot r \end{cases}$$

$$Ax+By+Cz+D = A(\lambda u_x + x_1) + B(\lambda u_y + y_1)$$

$$+ C(\lambda u_z + z_1) + D = \lambda \underbrace{(Au_x + Bu_y + Cu_z)}_{=k} +$$

$$+ \underbrace{Ax_1 + By_1 + Cz_1 + D_1}_{m}$$

$$\Rightarrow \quad P_{\pi,d}(l_1): \begin{cases} x' = \lambda u_x + x_1 - \dfrac{\lambda k + m}{Ap+Bq+Cr}\cdot p \\ y' = \lambda u_y + y_1 - \dfrac{\lambda k + m}{Ap+Bq+Cr}\cdot q \\ z' = \lambda u_z + z_1 - \dfrac{\lambda k + m}{Ap+Bq+Cr}\cdot r \end{cases}$$

We do the same thing for l_2:

$$P_{\pi,d}(l_2): \begin{cases} x' = \lambda u_x + x_2 - \dfrac{\lambda \ell_2 + m}{Ap+Bq+cr} \cdot p \\[2ex] y' = \lambda u_y + y_2 - \dfrac{\lambda \ell_2 + m}{Ap+Bq+cr} \cdot q \\[2ex] z' = \lambda u_z + z_2 - \dfrac{\lambda \ell_2 + m}{Ap+Bq+cr} \cdot r \end{cases}$$

The projections are lines, if and only if the coefficients of λ are not all zero.

$$\overrightarrow{P_{\pi,d}(l_1)} = \begin{pmatrix} u_x - \dfrac{\ell_2}{Ap+Bq+cr} \cdot p \\[2ex] u_y - \dfrac{\ell_2}{Ap+Bq+cr} \cdot q \\[2ex] u_z - \dfrac{\ell_2}{Ap+Bq+cr} \cdot r \end{pmatrix}$$

We see that $\overrightarrow{p_{\pi,d}(l_1)} = \overrightarrow{p_{\pi,d}(l_2)}$

$\Rightarrow p_{\pi,d}(l_1) \parallel p_{\pi,d}(l_2)$

$p_{\pi,d}(l_1)$ is a point $\Leftrightarrow \overrightarrow{p_{\pi,d}(l_1)} = \vec{0} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} u_x = \dfrac{p\xi}{Ap+Bq+Cr} \\[2ex] u_y = \dfrac{q\xi}{Ap+Bq+Cr} \\[2ex] u_z = \dfrac{r\xi}{Ap+Bq+Cr} \end{cases} \Leftrightarrow l_1 \parallel d$$