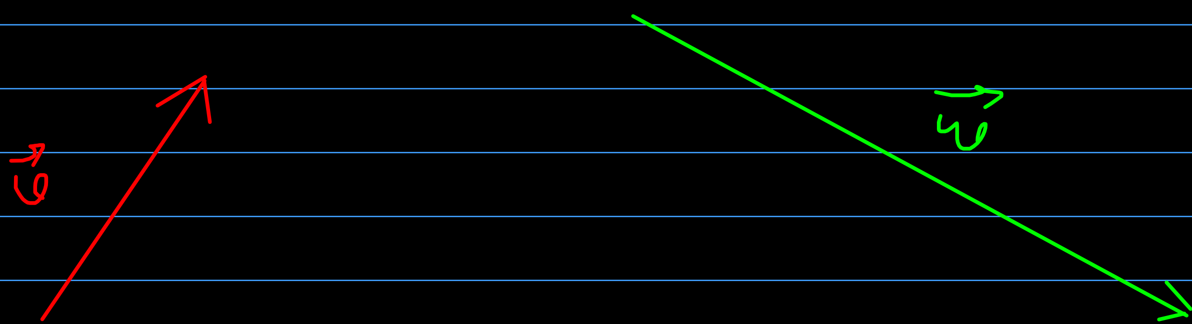
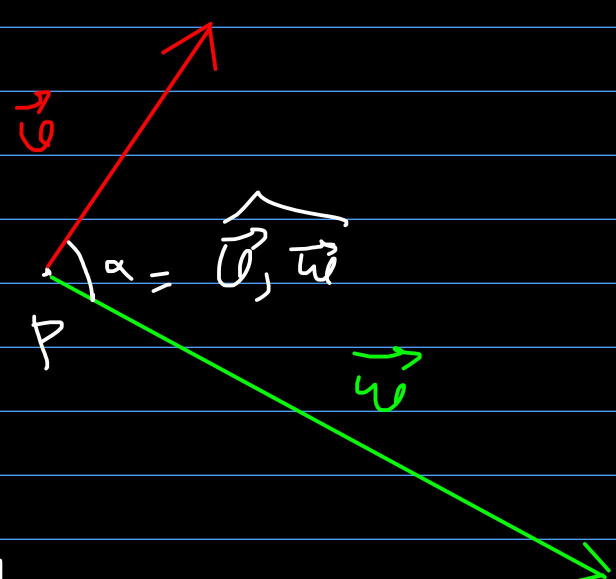
Seminar W5 - 975

The dot product (scalar product):

$$\vec{u} \cdot \vec{w} = \|\vec{u}\| \cdot \|\vec{w}\| \cdot \cos\left(\widehat{\vec{u}, \vec{w}}\right)$$





$\alpha \in [0, \pi]$

If we have fixed an orthonormal reference system, then:

$\vec{v}(a_1, b_1, c_1)$, $\vec{w}(a_2, b_2, c_2)$

$\vec{v} \cdot \vec{w} = a_1 a_2 + b_1 b_2 + c_1 c_2$

orthonormal system = orthogonal system thormed

$\mathcal{R} = (O, [\vec{i}, \vec{j}, \vec{k}])$

$1 = \|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\|$

$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$

5.3. Find the angle between

(a) $d_1 : \begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases}$

$d_2 : \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0 \end{cases}$

(b) $\pi_1 : x + 3y + 2z + 1 = 0$

$\pi_2 : 3x + 2y - z = 6$

(c) the plane xOy and the line M_1M_2, where $M_1(1,2,3)$, $M_2(-2,1,4)$

(a) $d_1: \begin{cases} x+2y+z-1=0 \\ x-2y+z+1=0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x=-2y-z+1 \\ -2y-z+1-2y+z+1=0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x=-2y-z+1 \\ -4y+2=0 \end{cases} \Leftrightarrow \begin{cases} y=\frac{1}{2} \\ x=-1-z+1=-z \end{cases} \Leftrightarrow$

$\Leftrightarrow d_1: \begin{cases} x=t \\ y=\frac{1}{2} \\ z=-t \end{cases} \Rightarrow \vec{d_1}(1,0,-1)$

$$d_2: \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = y + z + 1 \\ y + z + 1 - y + 2z + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = y + z + 1 \\ 3z + 2 = 0 \end{cases} \Leftrightarrow \begin{cases} z = -\frac{2}{3} \\ x = y - \frac{2}{3} + 1 \\ \quad = y + \frac{1}{3} \end{cases} \Leftrightarrow$$

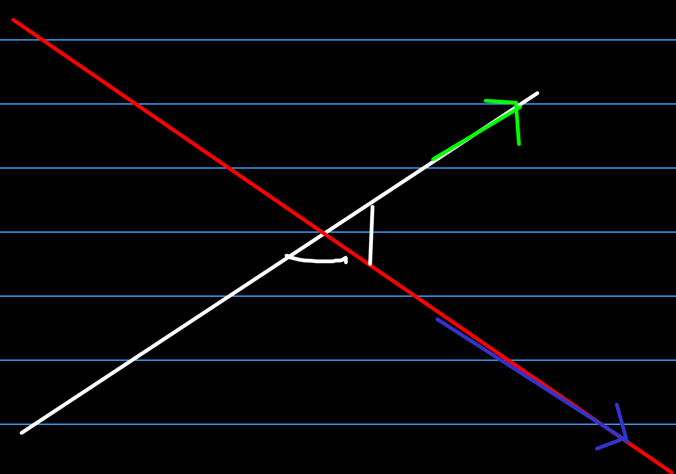
$$\Leftrightarrow d_2: \begin{cases} x = t \\ y = t - \frac{1}{3} \\ z = -\frac{2}{3} \end{cases} \Rightarrow \vec{d_2}(1,1,0)$$

$\vec{d_1}(1,0,-1)$, $d_2(1,1,0)$,

$$\cos(\widehat{d_1, d_2}) = \cos(\widehat{\vec{d_1}, \vec{d_2}}) =$$

$$= \frac{\vec{d_1} \cdot \vec{d_2}}{\|\vec{d_1}\| \cdot \|\vec{d_2}\|} = \frac{1 \cdot 1 + 0 \cdot 1 + (-1) \cdot 0}{\sqrt{2} \cdot \sqrt{2}} =$$

$$= \frac{1}{2} \Rightarrow m(\widehat{d_1, d_2}) = \frac{\pi}{3}$$



(b) $\pi_1: x + 3y + 2z + 1 = 0$

$\pi_2: 3x + 2y - z = 6$

$$m(\widehat{\pi_1, \pi_2}) = m(\widehat{\vec{n}_{\pi_1}, \vec{n}_{\pi_2}})$$

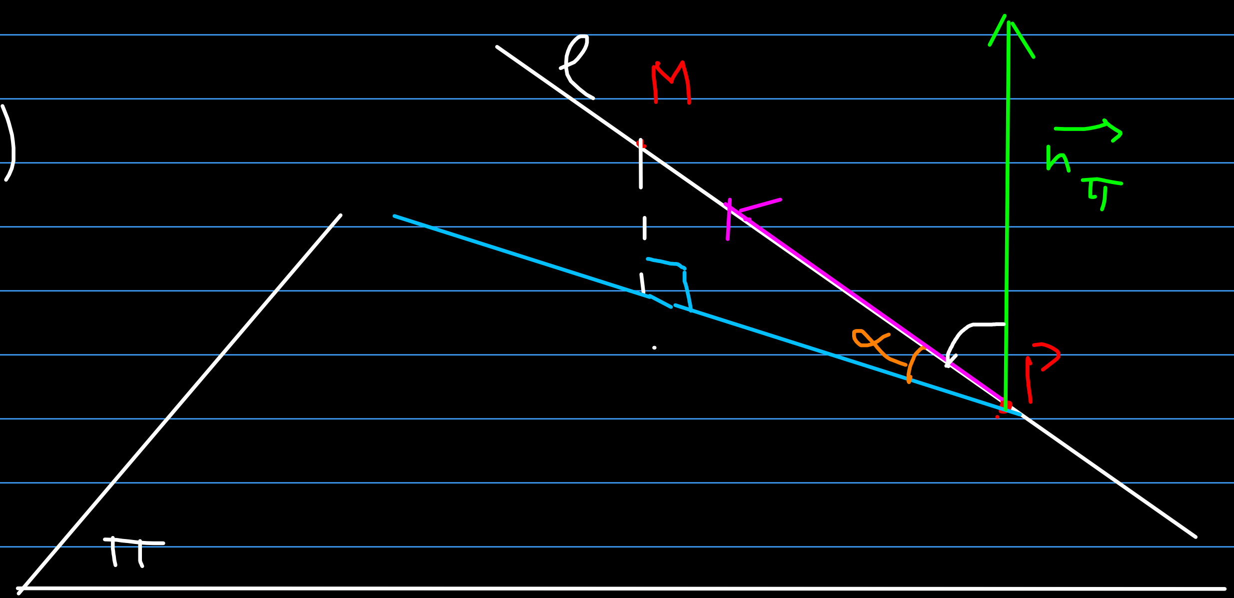
$\vec{n}_{\pi_1} = (1, 3, 2)$, $\vec{n}_{\pi_2}(3, 2, -1)$

$$\cos(\widehat{n_{\pi_1}, n_{\pi_2}}) = \frac{\vec{n_{\pi_1}} \cdot \vec{n_{\pi_2}}}{\|\vec{n_{\pi_1}}\| \cdot \|\vec{n_{\pi_2}}\|} =$$

$$= \frac{1\cdot 3 + 3\cdot 2 + 2\cdot(-1)}{\sqrt{1^2+3^2+2^2}\cdot\sqrt{3^2+2^2+(-1)^2}} = \frac{7}{\sqrt{14}\cdot\sqrt{14}} = \frac{1}{2}$$

$$\Rightarrow m(\widehat{\pi_1, \pi_2}) = \frac{\pi}{3}$$

(c)



$M_1(1,2,3)$, $M_2(-2,1,4)$

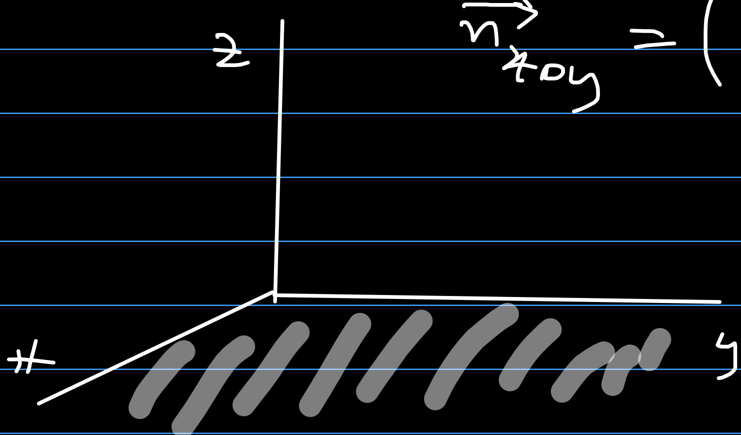
$$d: \begin{cases} x = \lambda x_{M_1} + (1-\lambda) x_{M_2} \\ y = \lambda y_{M_1} + (1-\lambda) y_{M_2} \\ z = \lambda z_{M_1} + (1-\lambda) z_{M_2} \end{cases}$$

$$\Rightarrow d: \begin{cases} x = \lambda - 2 + 2\lambda = 3\lambda - 2 \\ y = 2\lambda + 1 - \lambda = \lambda + 1 \\ z = 3\lambda + 4 - 4\lambda = 4 - \lambda \end{cases}$$

$$\Rightarrow \vec{d}(3, 1, -1)$$

xOy: $\quad z = 0 \Leftrightarrow 0 \cdot x + 0 \cdot y + 1 \cdot z + 0 = 0$

$$\vec{n}_{xOy} = (0, 0, 1)$$



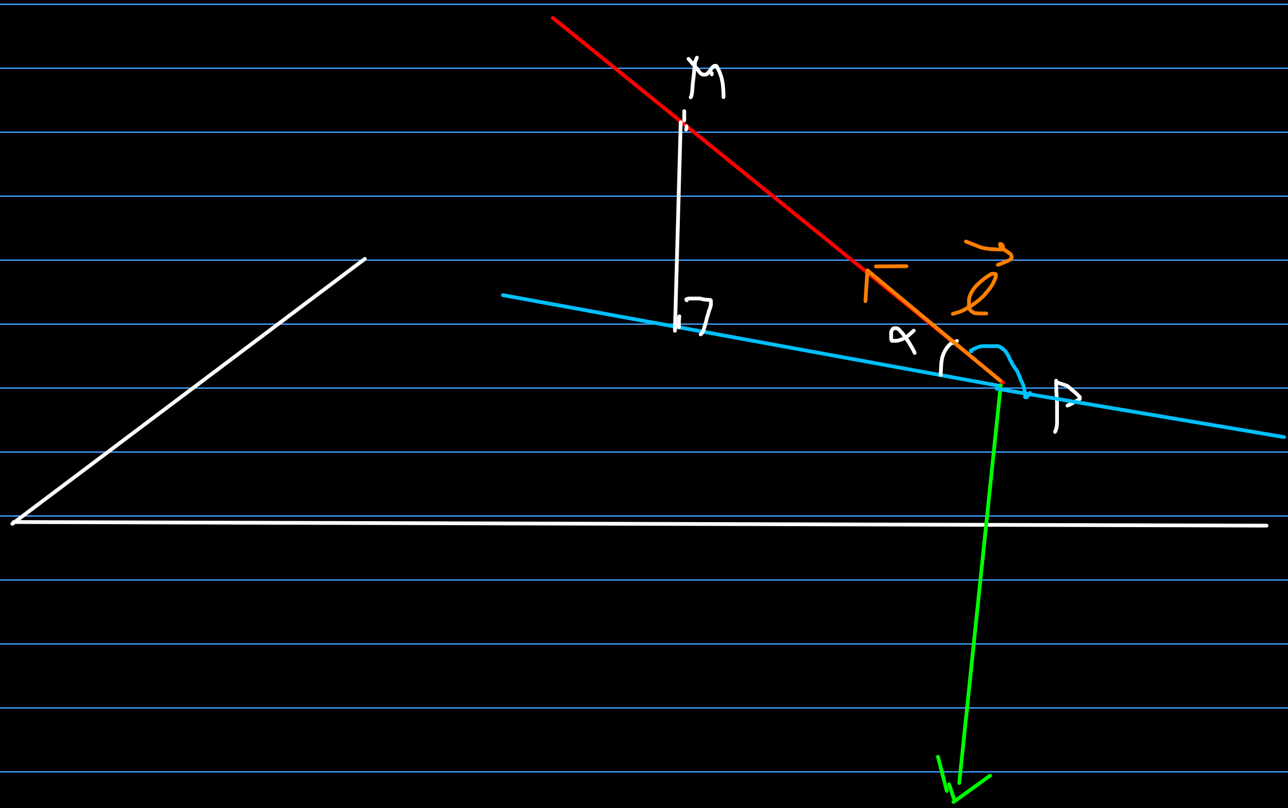
$$\cos(\widehat{\vec{n}_{xOy}, \vec{d}}) = \frac{\vec{n}_{xOy} \cdot \vec{d}}{\|\vec{n}_{xOy}\| \cdot \|\vec{d}\|} =$$

$$= \frac{(0,0,1) \cdot (3,1,-1)}{\|(0,0,1)\| \cdot \|(3,1,-1)\|} = \frac{-1}{\sqrt{11}}$$

$$\Rightarrow m(\widehat{n_{xOy}, d}) = \arccos\left(-\frac{1}{\sqrt{11}}\right)$$

Because $m(\widehat{\vec{n}_{xOy}, \vec{d}}) > \frac{\pi}{2} \Rightarrow$

$$\Rightarrow m(xOy, d) = \arccos\left(-\frac{1}{\sqrt{11}}\right) - \frac{\pi}{2}$$



$$\Rightarrow m(\widehat{xOy, d}) = \pi - \arccos\left(\frac{1}{\sqrt{11}}\right) - \frac{\pi}{2} =$$
$$= \frac{\pi}{2} - \arccos\left(\frac{1}{\sqrt{11}}\right)$$

The distance from a point to a plane

$\Pi : Ax + By + Cz + D = 0$

$P(x_0, y_0, z_0)$

$$\operatorname{dist}(P, \Pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

In the 2D case $\rightarrow$ distance from a point to line

$\ell : Ax + By + C = 0$

$P(x_0, y_0)$

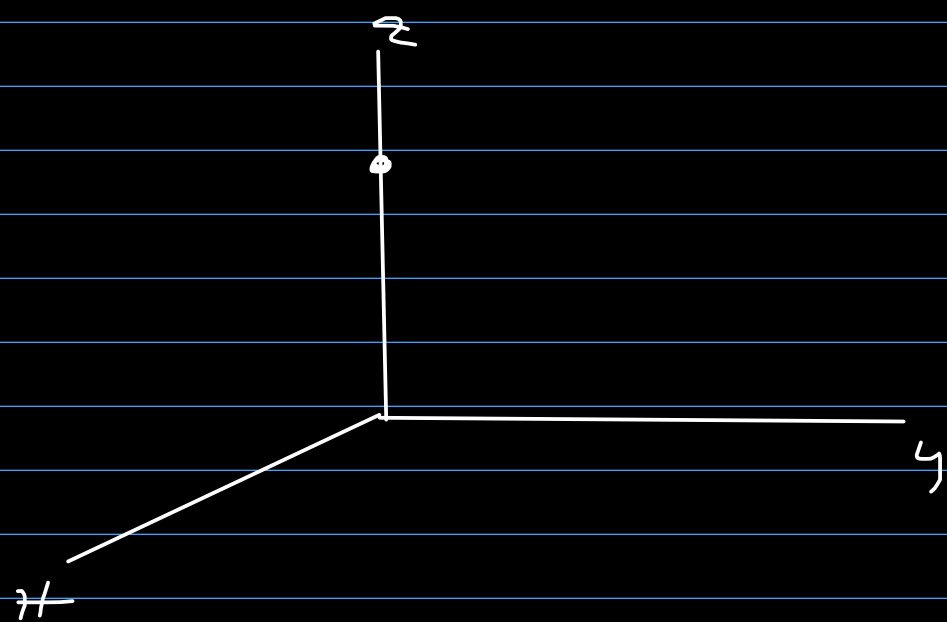
$$\operatorname{dist}(P, \ell| = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

5.5. Find the points on the z-axis which are equidistant with respect to the planes

$\pi_1: 12x + 9y - 20z - 19 = 0$

$\pi_2: 16x + 12y + 15z - 9 = 0$

$A(0, 0, z_A)$, because $A \in Oz$



$$dist(A, \pi_1) = dist(A, \pi_2)$$

$$\frac{|12 \cdot 0 + 9 \cdot 0 - 20 z_A - 19|}{\sqrt{12^2 + 9^2 + 20^2}} = \frac{|16 \cdot 0 + 12 \cdot 0 + 15 z_A - 9|}{\sqrt{16^2 + 12^2 + 15^2}}$$

$$\frac{|-20 z_A - 19|}{\sqrt{625}} = \frac{|15 z_A - 9|}{\sqrt{625}}$$

$$\Rightarrow |20 z_A + 19| = |15 z_A - 9|$$

$$\Rightarrow 20 z_A + 19 = \pm (15 z_A - 9)$$

<u>Case 1</u>: $20 z_A + 19 = 15 z_A - 9$

$$\Rightarrow z_A = -\frac{28}{5}$$

$$\Rightarrow A\left(0, 0, -\frac{28}{5}\right)$$

<u>Case 2</u>: $20 z_A + 19 = -15 z_A + 9$

$$\Rightarrow z_A = -\frac{10}{35} = -\frac{2}{7}$$

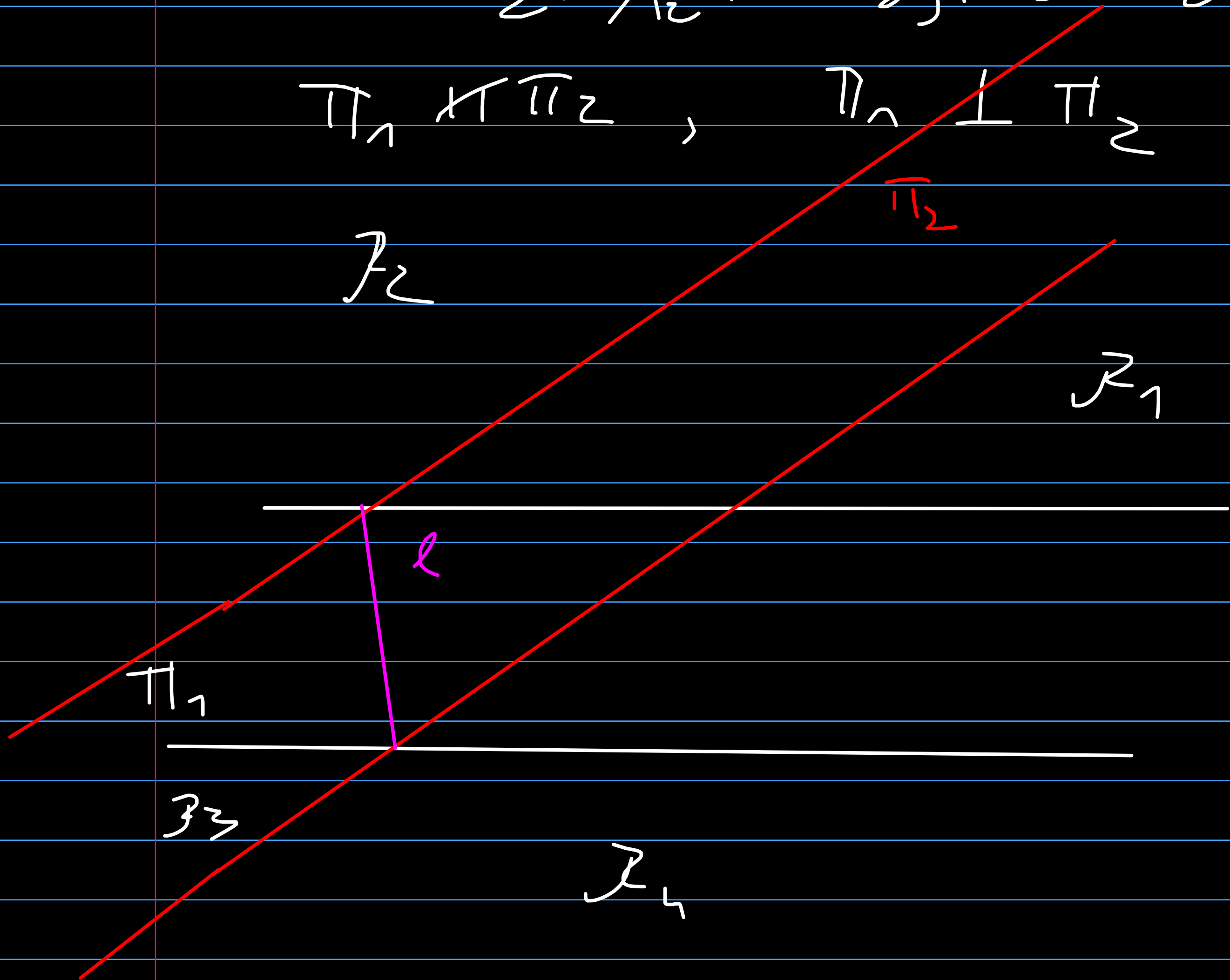
$$\Rightarrow A\left(0, 0, -\frac{2}{7}\right)$$

Read exercise 5.6.

5.10. $\Pi_1: A_1x + B_1y + C_1z + D_1 = 0$

$\Pi_2: A_2x + B_2y + C_2z + D_2 = 0$

$\Pi_1 \not\parallel \Pi_2, \quad \Pi_1 \not\perp \Pi_2$



$\mathcal{R}_1$ and $\mathcal{R}_3$ correspond to the acute angle $\left(< \frac{\pi}{2}\right)$

$F_1(x,y,z) := A_1x + B_1y + C_1z + D_1$

$F_2(x,y,z) := A_2x + B_2y + C_2z + D_2$

If $M(x_M, y_M, z_M)$ then:

$M \in$ acute regions $\Leftrightarrow F_1(x_M, y_M, z_M) \cdot F_2(x_M, y_M, z_M) \cdot \underbrace{(A_1A_2 + B_1B_2 + C_1C_2)}_{= \overrightarrow{n_{\pi_1}} \cdot \overrightarrow{n_{\pi_2}}} < 0$

5.7. (3 p.) $\pi_1 : 2x + y - 3z - 5 = 0$

$\pi_2 : x + 3y + 2z + 1 = 0$

Find the equations of the bisector planes of the dihedral angles formed by π_1 and π_2 and select the one contained in the acute regions of the dihedral angles.

$M \in$ bisector planes $\Leftrightarrow \mathrm{dist}(M, \pi_1) = \mathrm{dist}(M, \pi_2)$

$$dist(M, \Pi_1) = \frac{|2x + y - 3z - 5|}{\sqrt{14}}$$

$$dist(M, \Pi_2) = \frac{|x + 3y + 2z + 1|}{\sqrt{14}}$$

$$dist(M, \Pi_1) = dist(M, \Pi_2) \Leftrightarrow |2x + y - 3z - 5| =$$

$$= |x + 3y + 2z + 1|$$

$$\Rightarrow 2x + y - 3z - 5 = \pm (x + 3y + 2z + 1)$$

<u>Case 1</u>: $2x + y - 3z - 5 = x + 3y + 2z + 1$

$$\Rightarrow \beta_1 : x - 2y - 5z - 6 = 0$$

<u>Case 2</u>: $2x + y - 3z - 5 = -x - 3y - 2z - 1$

$$\Rightarrow \beta_2 : 3x + 4y - z - 4 = 0$$

$\Rightarrow$ bisector planes β_1 and β_2

We pick the point $P(6, 0, 0) \in \beta_1$

We check if it's in the acute region.

We do that by plugging the coordinates of P into the relation at exercise 5.6.

$$F_1(x,y,z) = 2x+y-3z-5$$

$$F_2(x,y,z) = x+3y+2z+1$$

$$F_1(6,0,0) = 7, \quad F_2(6,0,0) = 7$$

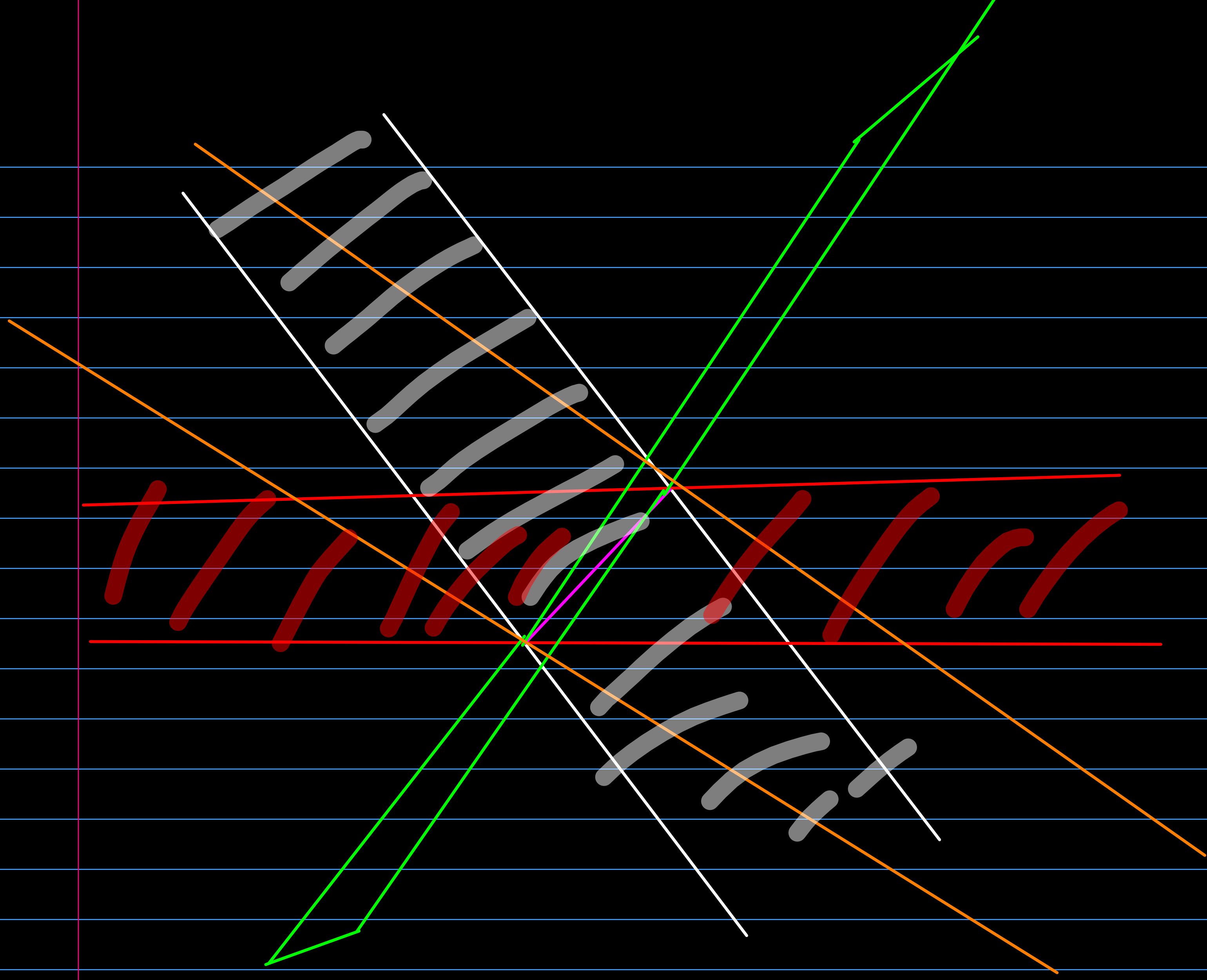
$$\vec{n_{\pi_1}} \cdot \vec{n_{\pi_2}} = 2+3-6 = -1$$

$$\Rightarrow F_1(x_P, y_P, z_P) \cdot F_2(x_P, y_P, z_P) \cdot$$

$$\cdot (A_1A_2 + B_1B_2 + C_1C_2) =$$

$$= 7 \cdot 7 \cdot (-1) = -49 < 0$$

$\Rightarrow P \in$ acute region $\Rightarrow B_1 \subseteq$ acute region



5.8(3p). $a, b \in \mathbb{R}$, $a^2 \neq b^2$

$\alpha_1 : \quad ax + by - (a+b)z = 0$

$\alpha_2 : \quad ax - by - (a-b)z = 0$

and the quadric

$(\mathscr{C}) : \quad a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 -$

$- 2a^2xz + 2b^2yz - a^2b^2 = 0$

$a^2 < b^2 \Leftrightarrow \mathscr{C} \subseteq$ acute regions

$a^2 > b^2 \Leftrightarrow \mathscr{C} \subseteq$ obtuse regions

$M(x, y, z) \in$ acute region $\Leftrightarrow$

$\Leftrightarrow F_1(x, y, z) \cdot F_2(x, y, z) \cdot \left(\overrightarrow{n_{\pi_1}} \cdot \overrightarrow{n_{\pi_2}}\right) < 0 \Leftrightarrow$

$\Leftrightarrow \left(ax + by - (a+b)z\right) \cdot \left(ax - by - (a-b)z\right) \cdot$

$\cdot \left((a, b, -a-b) \cdot (a, -b, -a+b)\right) < 0$

$\Leftrightarrow$

$$\Leftarrow \underbrace{(ax+by-(a+b)z)(ax-by-(a-b)z)\cdot(a^2-b^2+a^2-b^2)}_{=\varphi} < 0$$

$$(\mathcal{C}):\quad a^2x^2-b^2y^2+(a^2-b^2)z^2-$$

$$-2a^2xz+2b^2yz-a^2b^2=0$$

$$(\mathcal{C}):\quad a^2(x^2-2xz)-b^2(y^2-2yz)$$

$$+(a^2-b^2)z^2-a^2b^2=0$$

$$a^2(x^2-2xz+z^2)-b^2(y^2-2yz+z^2)-$$

$$-a^2b^2=0$$

$$(\mathcal{C}):\quad \underline{a^2(x-z)^2-b^2(y-z)^2}-a^2b^2=0$$

$$\varphi=2(a^2-b^2)(ax+by-(a+b)z)\cdot$$

$$\cdot(ax-by-(a-b)z)$$

$\varphi = 2(a^2 - b^2)\,(a(x-z) + b(y-z))\cdot$
$\cdot(a(x-z) - b(y-z)) =$

$= 2(a^2 - b^2)\cdot\left(a^2(x-z)^2 - b^2(y-z)^2 -\right.$
$\left. - ab(x-z)(y-z) + ab(x-z)(y-z)\right)$

$M \in \mathcal{C} \Rightarrow a^2(x-z)^2 - b^2(y-z)^2 =$
$= a^2b^2$

$\Rightarrow M \in \mathcal{C},\ M \in$ acute region $\Leftrightarrow$

$\Leftrightarrow \varphi = 2(a^2 - b^2)\cdot a^2b^2 < 0 \Leftrightarrow$

$\Leftrightarrow a^2 < b^2$