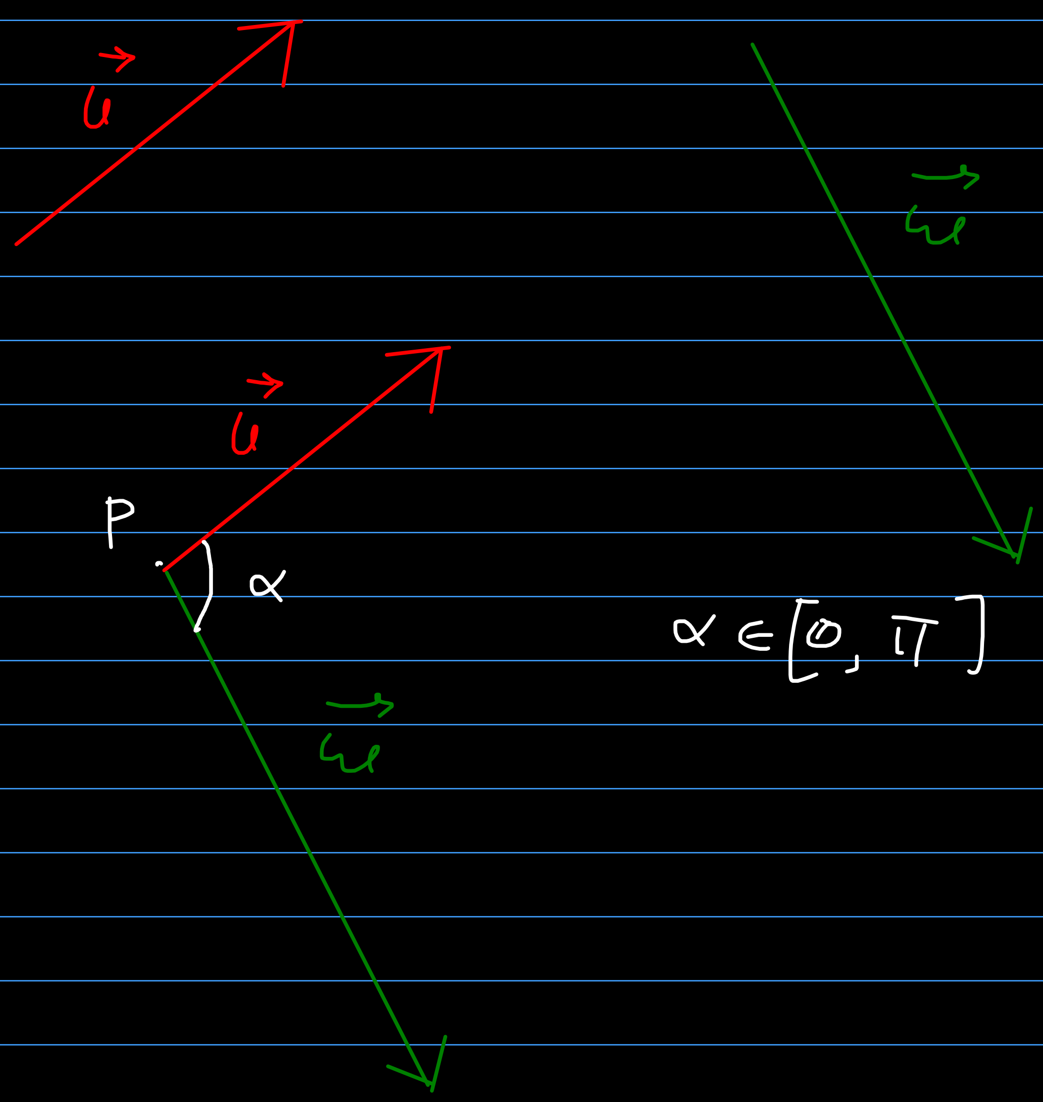
Seminar W5 – 916

Dot product (scalar product)

$\vec{u}, \vec{w} \in \mathcal{U} \Rightarrow \vec{u} \cdot \vec{w} = \|\vec{u}\| \cdot \|\vec{w}\| \cdot \cos(\widehat{\vec{u}, \vec{w}})$



$\alpha \in [0, \pi]$

If we fix a reference system that is orthonormal, then:

$\vec{u}(a_1, b_1, c_1)$, $\vec{w}(a_2, b_2, c_2)$

$\Rightarrow \vec{u} \cdot \vec{w} = a_1 a_2 + b_1 b_2 + c_1 c_2$

$\mathcal{R} = (O, [\vec{i}, \vec{j}, \vec{k}])$

orthonormal = orthogonal + normed

$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$

$\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$

5.3. Find the angle between:

(a) $d_1: \begin{cases} x+2y+z-1=0 \\ x-2y+z+1=0 \end{cases}$

$d_2: \begin{cases} x-y-z-1=0 \\ x-y+2z+1=0 \end{cases}$

(b) $\Pi_1: x+3y+2z+1=0$

$\Pi_2: 3x+2y-z=0$

(c) the plane xoy and the line M_1M_2, $M_1(1,2,3)$, $M_2(-2,1,4)$

(a) $d_1: \begin{cases} x+2y+z-1=0 \\ x-2y+z+1=0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} 2x+2z=0 \\ x-2y+z+1=0 \end{cases} \Leftrightarrow \begin{cases} x=-z \\ -z-2y+z+1=0 \end{cases} \Rightarrow$

$$\Leftrightarrow \begin{cases} x = -z \\ 2y = 1 \end{cases} \Leftrightarrow \begin{cases} y = \frac{1}{2} \\ x = -z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow d_1: \begin{cases} x = t \\ y = \frac{1}{2} + 0 \cdot t \\ z = -t \end{cases} \Rightarrow \vec{d_1}(1, 0, -1)$$

$$d_2: \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} -3z - 2 = 0 \\ x - y + 2z + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = -\frac{2}{3} \\ x - y - \frac{4}{3} + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} z = -\frac{2}{3} \\ x - y = \frac{1}{3} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = y + \frac{1}{3} \\ z = -\frac{2}{3} \end{cases} \Leftrightarrow d_2: \begin{cases} x = t \\ y = t - \frac{1}{3} \\ z = -\frac{2}{3} + t \cdot 0 \end{cases}$$

$\Rightarrow \vec{d_2}\,(1,1,0)$.

$\vec{d_1}\cdot\vec{d_2} = (1,0,-1)\cdot(1,1,0) = 1\cdot 1 + 0\cdot 1 + (-1)\cdot 0$

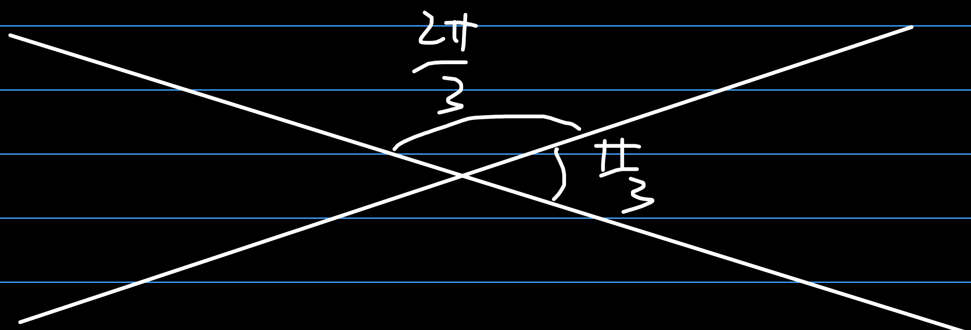
$= 1$

$\|\vec{d_1}\| = \sqrt{1+0+1} = \sqrt{2}$

$\|\vec{d_2}\| = \sqrt{1+1+0} = \sqrt{2}$

$\Rightarrow \cos(\vec{d_1},\vec{d_2}) = \dfrac{\vec{d_1}\cdot\vec{d_2}}{\|\vec{d_1}\|\cdot\|\vec{d_2}\|} = \dfrac{1}{2}$

$\Rightarrow m(\widehat{d_1, d_2}) = \dfrac{\pi}{3}$



(b) $\pi_1: x+3y+2z+1=0$

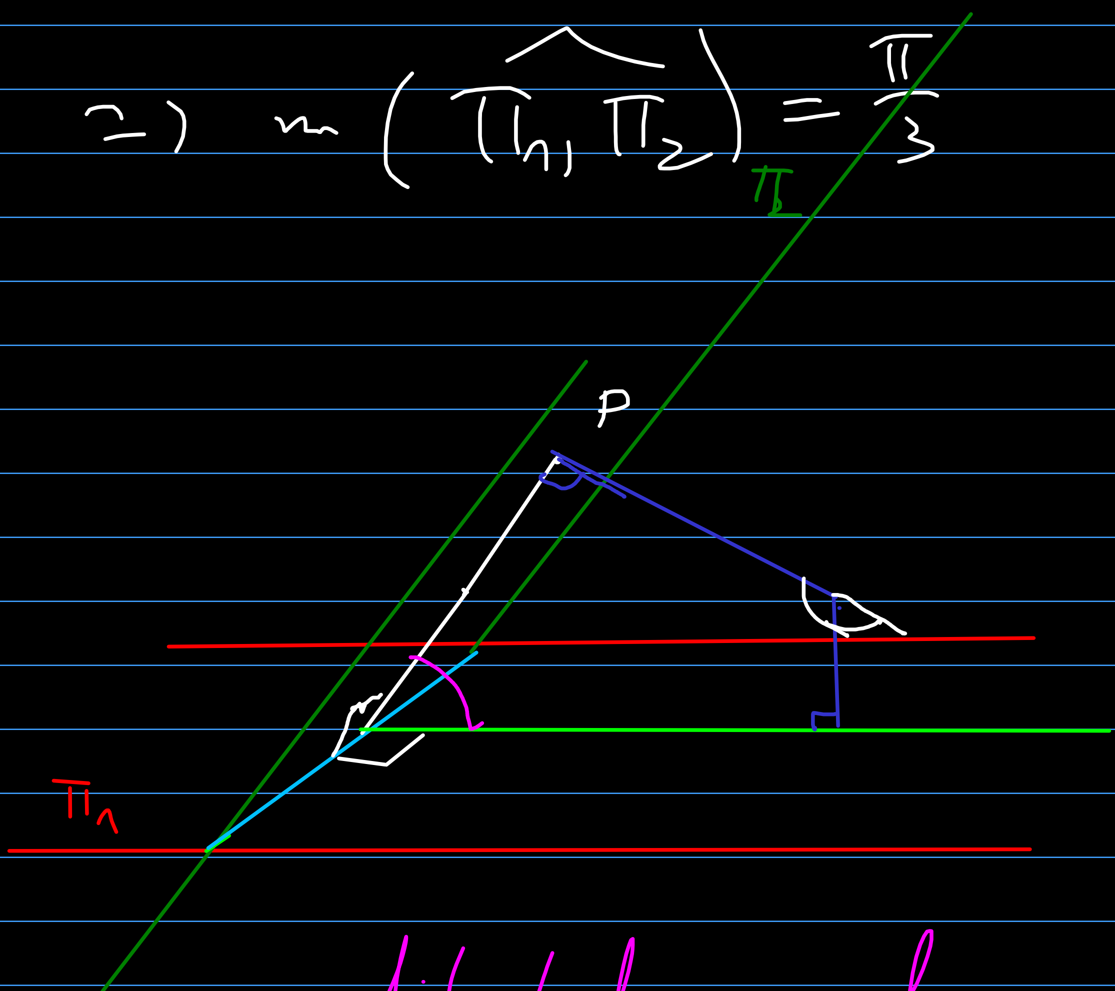
$\pi_2: 3x+2y-z=0$

$\vec{n}_{\pi_1}(1,3,2) \qquad \vec{n}_{\pi_2}(3,2,-1)$

$$\cos\left(\widehat{\vec{n}_{\pi_1}, \vec{n}_{\pi_2}}\right) = \frac{\vec{n}_{\pi_1} \cdot \vec{n}_{\pi_2}}{\|\vec{n}_{\pi_1}\| \cdot \|\vec{n}_{\pi_2}\|} =$$

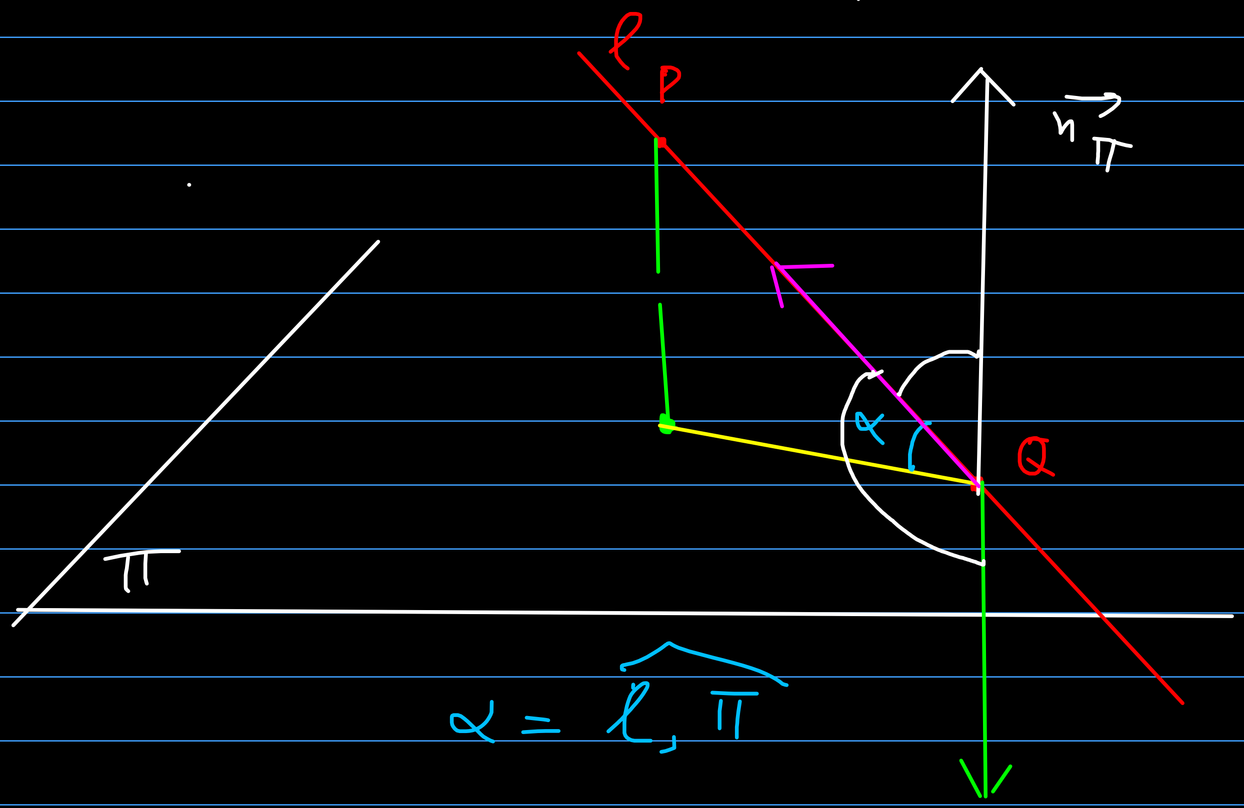
$$= \frac{1 \cdot 3 + 3 \cdot 2 + 2 \cdot (-1)}{\sqrt{1^2+3^2+2^2} \cdot \sqrt{3^2+2^2+(-1)^2}} = \frac{7}{\sqrt{14} \cdot \sqrt{14}} = \frac{1}{2}$$

$$\Rightarrow m\left(\widehat{\pi_1, \pi_2}\right) = \frac{\pi}{3}$$



dihedral angle

(c) the plane xOy and the line M_1M_2, $M_1(1,2,3)$, $M_2(-2,1,4)$



$\alpha = \widehat{\ell, \pi}$

The distance from a point to a plane

$\pi: Ax + By + Cz + D = 0$

$P(x_0, y_0, z_0)$

$$\text{dist}(P, \pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

The distance from a point to a line in 2D

$l: Ax + By + C = 0$

$P(x_0, y_0, z_0)$

$$\text{dist}(P, l) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

5.5. Find the points on the z-axis which are equidistant with respect to the planes

$$\Pi_1: 12x+9y-20z-19=0$$

$$\Pi_2: 16x+12y+15z-9=0$$

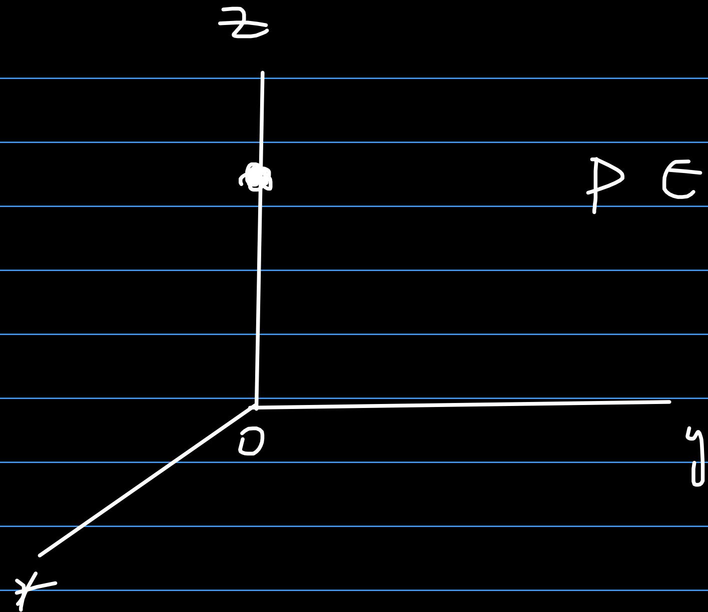
$P(x_0, y_0, z_0)$

$$\text{dist}(P, \Pi_1) = \frac{|12x_0+9y_0-20z_0-19|}{\sqrt{12^2+9^2+20^2}} =$$

$$= \frac{|12x_0+9y_0-20z_0-19|}{25}$$

$$\text{dist}(P, \Pi_2) = \frac{|16x_0+12y_0+15z_0-9|}{\sqrt{16^2+12^2+15^2}} =$$

$$= \frac{|16x_0+12y_0+15z_0-9|}{25}$$



$P \in Oz \Rightarrow x_0 = y_0 = 0$

$$\Rightarrow dist(P, \pi_1) = \frac{|-20z_0 - 19|}{25}$$

$$dist(P, \pi_2) = \frac{|15z_0 - 9|}{25}$$

$$\Rightarrow dist(P, \pi_1) = dist(P, \pi_2)$$

$$\Rightarrow |-20z_0 - 19| = |15z_0 - 9|$$

$$\Rightarrow -20z_0 - 19 = \pm(15z_0 - 9)$$

<u>Case 1:</u> $-20z_0 - 19 = 15z_0 - 9$

$$\Rightarrow z_0 = -\frac{10}{35} = -\frac{2}{7}$$

<u>Case 2:</u> $-20z_0 - 19 = 9 - 15z_0$

$$\Rightarrow z_0 = \frac{-28}{5}$$

→ loc geometric

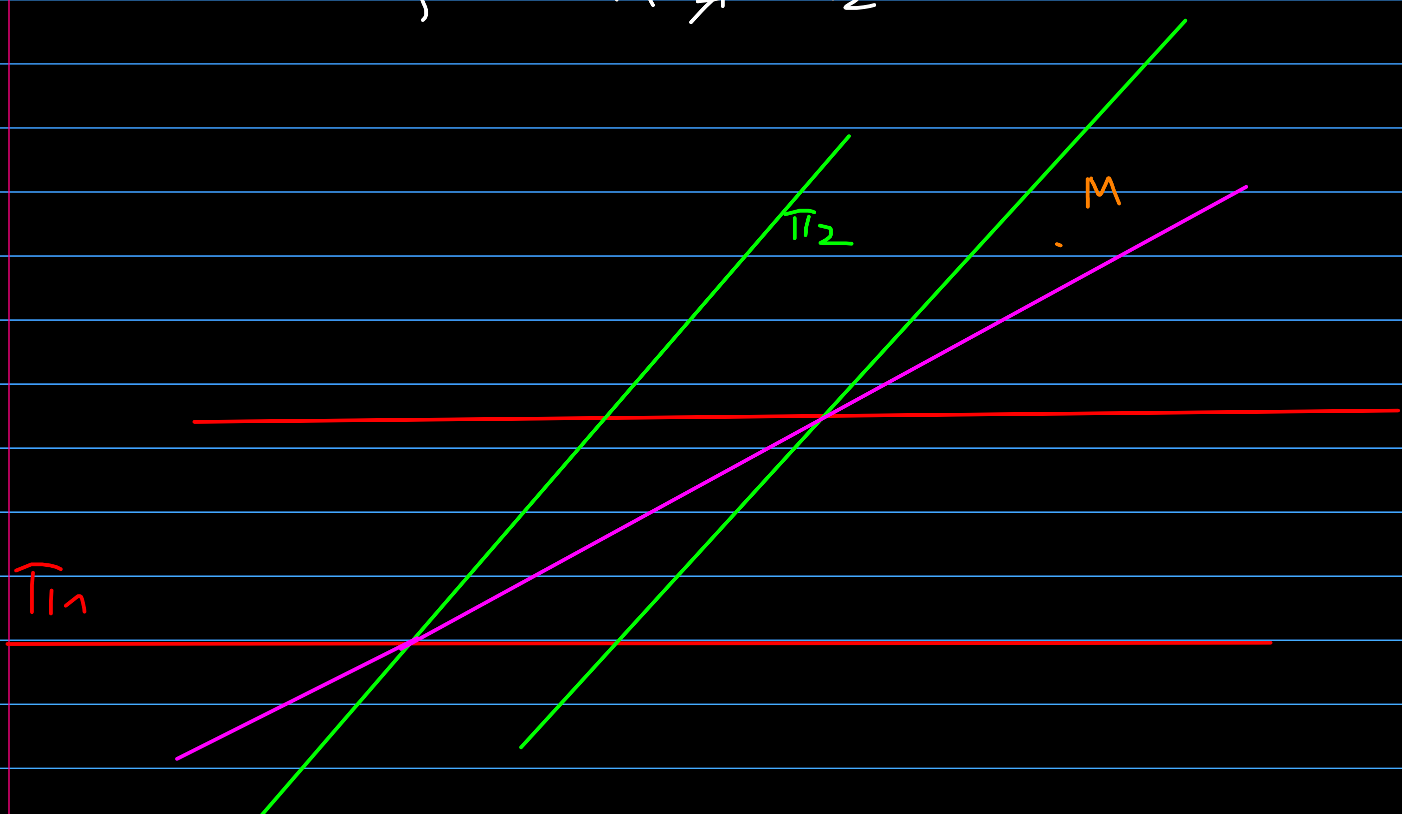
The locus of points equidistant to two non-parallel planes consists of two perpendicular planes called the bisector planes.

Read the text of exercise 5.6.

5.6. $\pi_1: \ A_1x + B_1y + C_1z + D_1 = 0$

$\pi_2: \ A_2x + B_2y + C_2z + D_2 = 0$

$\pi_1 \nparallel \pi_2, \quad \pi_1 \not\perp \pi_2$



$$F_1(x, y, z) = A_1 x + B_1 y + C_1 z + D_1$$

$$F_2(x, y, z) = A_2 x + B_2 y + C_2 z + D_2$$

$M(x_0, y_0, z_0) \in$ acute region $\Leftrightarrow$

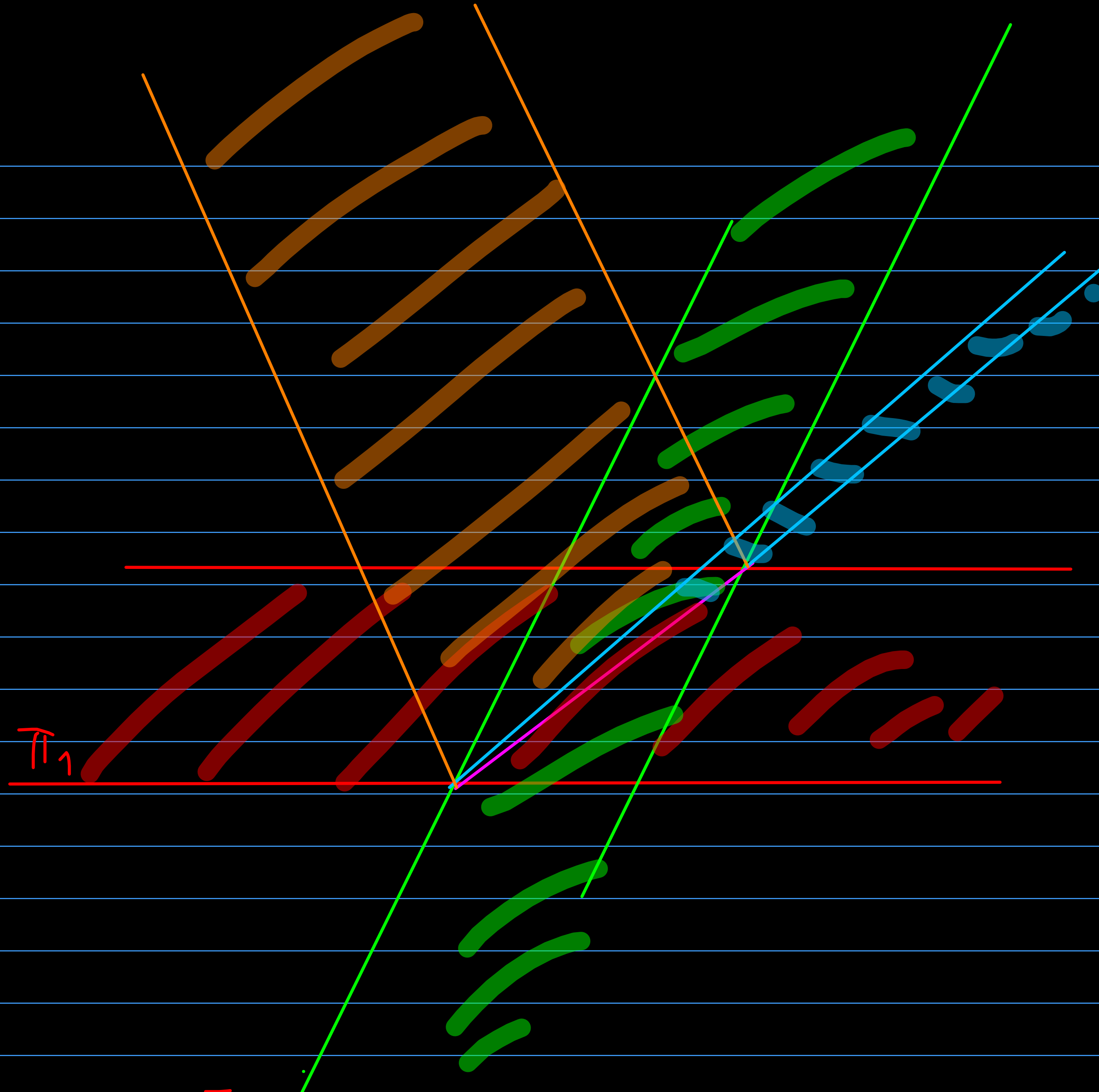
$$\Leftrightarrow F_1(x_0, y_0, z_0) \cdot F_2(x_0, y_0, z_0) \cdot \underbrace{(A_1A_2 + B_1B_2 + C_1C_2)}_{\overrightarrow{n_{\pi_1}} \cdot \overrightarrow{n_{\pi_2}}} < 0$$

5.7. (3p)

$\pi_1 : 2x + y - 3z - 5 = 0$

$\pi_2 : x + 3y + 2z + 1 = 0$

Find the equations of the bisector planes of the dihedral angles formed by the planes π_1 and π_2 and select the one contained in the acute regions.



$$\pi_1 : \overbrace{2x+y-3z-5}^{F_1} = 0$$

$$\pi_2 : \underbrace{x+3y+2z+1}_{F_2} = 0$$

$P(x,y,z)$

$$\operatorname{dist}(P, \pi_1) = \frac{|2x+y-3z-5|}{\sqrt{14}}$$

$$\text{dist}(P, \Pi_2) = \frac{|x+3y+2z+1|}{\sqrt{14}}$$

$$\text{dist}(P, \Pi_1) = \text{dist}(P, \Pi_2) \iff$$

$$\iff |2x+y-3z-5| = |x+3y+2z+1|$$

$$\Rightarrow 2x+y-3z-5 = \pm(x+3y+2z+1)$$

<u>Case 1</u>: $2x+y-3z-5 = -x-3y-2z-1$

$\Pi_3 : 3x+4y-z-4 = 0$

<u>Case 2</u>: $2x+y-3z-5 = x+3y+2z+1$

$\Pi_4 : x-2y-5z-6 = 0$

Let $M(0,-3,0) \in \Pi_4$. We check if M belongs to the acute region

$$F_1(M) = F_1(0,-3,0) =$$
$$= 2\cdot 0 + (-3) + 3\cdot 0 - 5 = -8$$
$$F_2(M) = F_2(0,-3,0) =$$
$$= 0 + 3\cdot(-3) + 2\cdot 0 + 1 = -8$$
$$A_1A_2 + B_1B_2 + C_1C_2 = \overrightarrow{n_{\pi_1}} \cdot \overrightarrow{n_{\pi_2}} =$$
$$= (2,1,-3)\cdot(1,3,2) =$$
$$= 2 + 3 - 6 = -1$$
$$\Rightarrow F_1(M)\cdot F_2(M)\cdot\left(\overrightarrow{n_{\pi_1}} \cdot \overrightarrow{n_{\pi_2}}\right) = -64 < 0$$
$\Rightarrow M \in$ acute region $\Rightarrow \Pi_4 \subset$ acute regions

5.17. (3p) Let M be a point whose coordinates satisfy:

$$\frac{4x+2y+8}{3x-y+1} = \frac{5}{2}.$$

(a) Prove that M belongs to a fixed line ℓ.

(b) Find the minimum of x^2+y^2, where $M \in \ell \setminus \{M_0(-1,-2)\}$

(a) $$\frac{4x+2y+8}{3x-y+1} = \frac{5}{2}$$

$$\iff 2\cdot(4x+2y+8) = 5(3x-y+1)$$

$$\iff 8x+4y+16 = 15x-5y+5$$

$$\iff 7x-9y-11 = 0$$

This the line ℓ

(b) $x^2+y^2 \geq 2xy$ with equality if $x=y$.

We will look at the point on the line that satisfies this, which is $Q(a,a)$

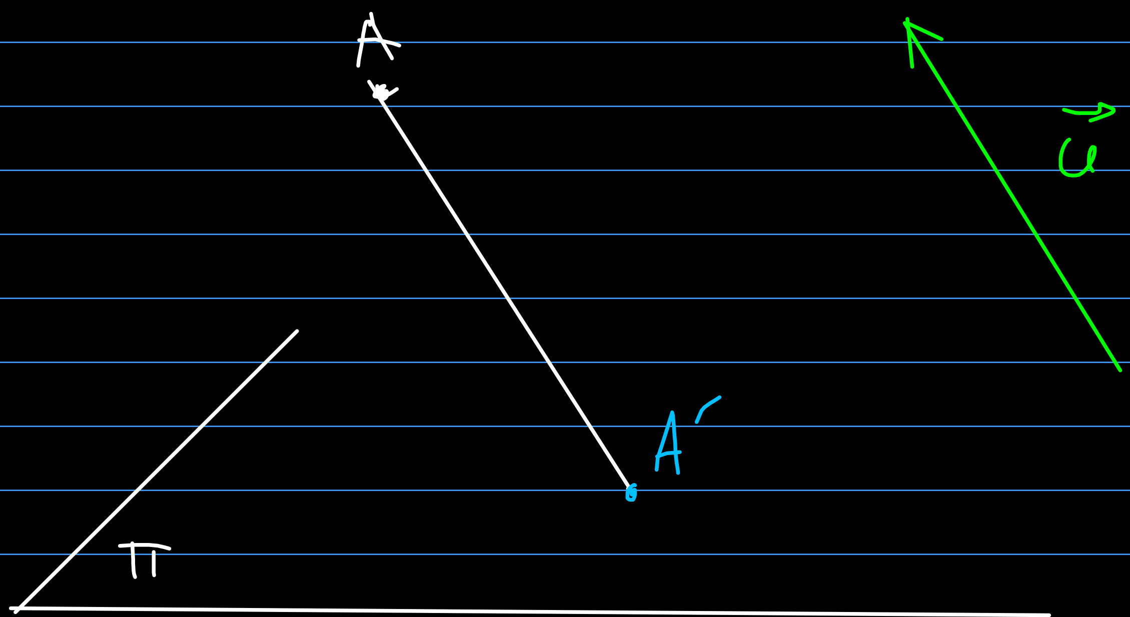
$$\Rightarrow \quad 7a - 9a - 11 = 0$$

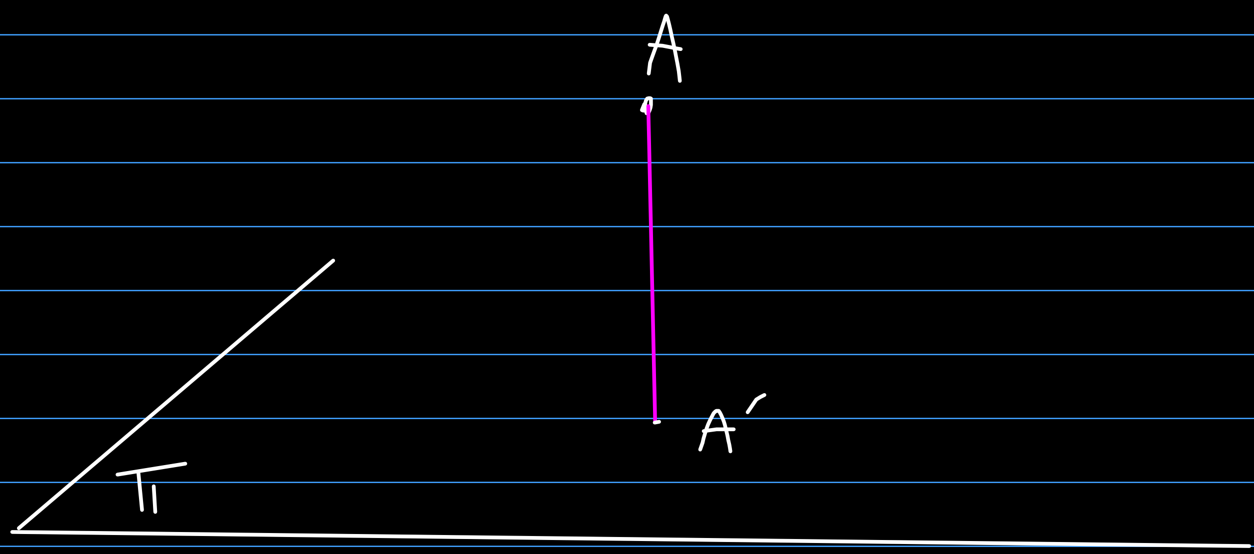
$$\Rightarrow a = -\frac{11}{2}$$

$$\Rightarrow \min_{(x,y)\neq(-1,-2)} (x^2+y^2) = 2 \cdot \frac{121}{4} = \frac{121}{2}$$

5.11. (a) Find the orthogonal projection of the point $A(1,2,1)$ on the plane $\pi : x+y+3z+5=0$

$$\vec{r}_{p_{\pi,u}(A)} = \vec{r}_A - \frac{F(A)}{\vec{n}_\pi \cdot \vec{u}} \cdot \vec{u}$$





Since the projection is orthogonal, we have

$$\vec{u} = \overrightarrow{n_\pi}$$

$$\overrightarrow{r_{P_\pi(A)}} = \overrightarrow{r_A} - \frac{F(A)}{\|\overrightarrow{n_\pi}\|^2} \cdot \overrightarrow{n_\pi}$$

$A(1,2,1) \quad F(x,y,z)$

$\Pi: \overbrace{x+y+3z}+5=0 \quad \Rightarrow \vec{n_\Pi}(1,1,3)$

$$\vec{r_{P_\Pi(A)}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{1+2+3\cdot 1+5}{1^2+1^2+3^2} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \vec{r_{P_\Pi(A)}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{11}{11} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$\Rightarrow$ $P_\Pi(A)$ has the coords.

$(0, 1, -2)$