Seminar W3 - 9.15

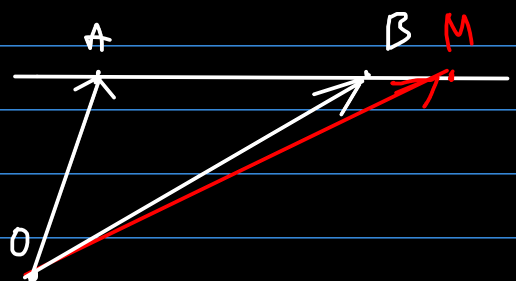
Lines l lin

$A, B \in \ell$, $A \neq B$

Vector equation:

$$\vec{r_M} = \lambda \vec{r_A} + (1-\lambda)\vec{r_B}$$

$\lambda \in \mathbb{R}$



Parametric (Cartesian) equation:

$$\begin{cases} x = \lambda x_A + (1-\lambda) x_B \\ y = \lambda y_A + (1-\lambda) y_B \\ z = \lambda z_A + (1-\lambda) z_B \end{cases}$$

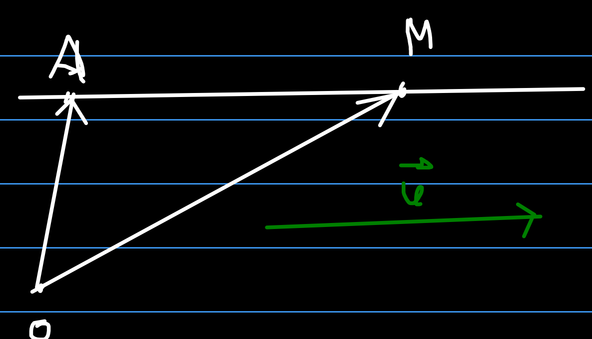
Canonical equation:

$$(\Leftarrow) \frac{x - x_B}{x_A - x_B} = \frac{y - y_B}{y_A - y_B} = \frac{z - z_B}{z_A - z_B}$$

$A \in \ell$, $\vec{u} \parallel \ell$

$$\vec{r_M} = \vec{r_A} + t \cdot \vec{u}$$

$t \in \mathbb{R}$



$$\begin{cases} x = x_A + t \cdot x_{\vec{u}} \\ y = y_A + t \cdot y_{\vec{u}} \\ z = z_A + t \cdot z_{\vec{u}} \end{cases}$$

$$(\Leftarrow) \frac{x - x_A}{x_{\vec{u}}} = \frac{y - y_A}{y_{\vec{u}}} = \frac{z - z_A}{z_{\vec{u}}}$$

Issue: The canonical equation has fringe cases (e.g. $x_A = x_B$, $y_{\vec{v}} = 0$)

For example, if $x_{\vec{v}} = 0$, $y_{\vec{v}} \neq 0$, $z_{\vec{v}} \neq 0$: then the equation becomes:

$$\begin{cases} x = x_A \\ \dfrac{y - y_A}{y_{\vec{v}}} = \dfrac{z - z_A}{z_{\vec{v}}} \end{cases}$$

For example, if $x_{\vec{v}} = 0$, $y_{\vec{v}} = 0$, $z_{\vec{v}} \neq 0$: then the equation becomes:

$$\begin{cases} x = x_A \\ y = y_A \end{cases}$$

Impliait equation

3D $\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$

2D $Ax + By + C = 0$

Explicit equation: for 2D $\rightarrow$ $y = mx + n$ ($m \rightarrow$ slope)

3.2: Write the equation of the line which passes through $A(1,-2,6)$ and is parallel to:

(a) the x-axis

(b) the line (d_1): $\dfrac{x-1}{2} = \dfrac{y+5}{-3} = \dfrac{z-1}{4}$

(c) the vector $\vec{u}(1,0,2)$

(a) $\vec{u} = (1,0,0)$

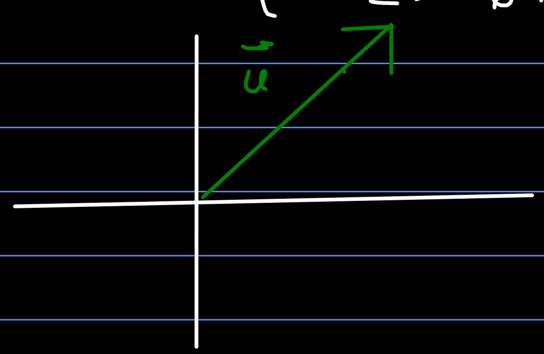
parametric $\qquad\qquad$ canonical

$$\begin{cases} x = 1+t \\ y = -2 \\ z = 6 \end{cases} \iff \begin{cases} y = -2 \\ z = 6 \end{cases}$$

(b)
$$\begin{cases} x = 1+2t \\ y = -2-3t \\ z = 6+4t \end{cases}$$

parametric $\qquad\qquad$ canonical

(c)
$$\begin{cases} x = 1+t \\ y = -2 \\ z = 6+2t \end{cases} \iff \begin{cases} \dfrac{x-1}{1} = \dfrac{z-6}{2} \\ y = -2 \end{cases}$$

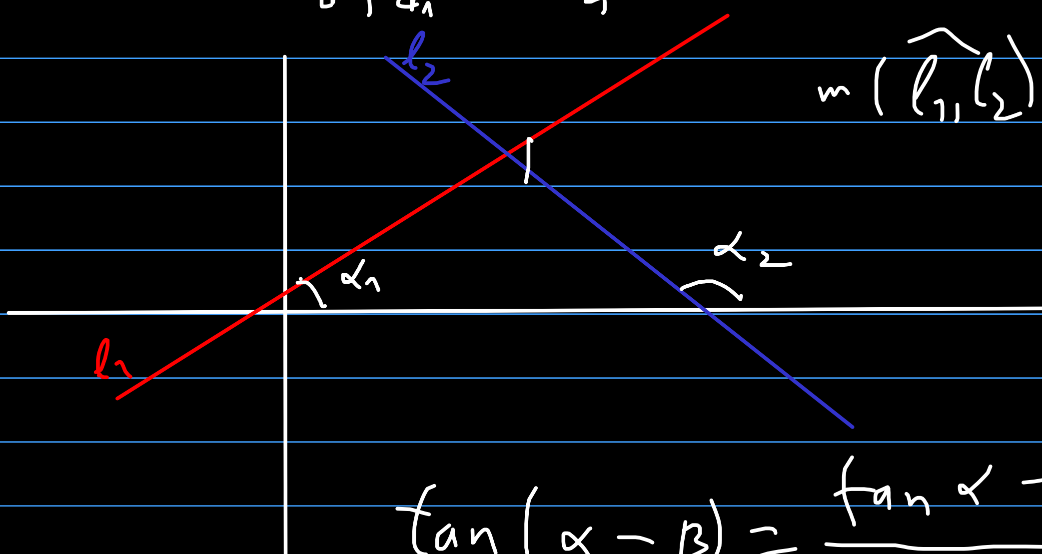


3.17. Given the line $d: 2x+3y+4=0$, find the equation of the line d_1 through the point $M_0(2,1)$, in the following situations:

(a) $d_1 \parallel d$

(b) $d_1 \perp d$

(c) $m(\widehat{d, d_1}) = \frac{\pi}{4}$



$$m(\widehat{l_1, l_2}) = \alpha_2 - \alpha_1$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha \cdot \tan\beta}$$

(a) $d: 2x+3y+4=0 \Leftrightarrow d: y = -\frac{2}{3}x - \frac{4}{3}$

$\Rightarrow m_d = -\frac{2}{3}$

$d \parallel d_1 \Leftrightarrow m_{d_1} = m_d$

$\Rightarrow m_{d_1} = -\frac{2}{3} \Rightarrow d_1: y = -\frac{2}{3}x + n$

$M_0 \in d_1 \Rightarrow 1 = -\frac{2}{3} \cdot 2 + n \Rightarrow n = \frac{7}{3}$

$\Rightarrow d_1: y = -\frac{2}{3}x + \frac{7}{3}$

(b) $m_d = -\frac{2}{3}$

$d \perp d_1 \iff m_d \cdot m_{d_1} = -1$

$\Rightarrow m_{d_1} = \frac{3}{2}$

$d_1 : y - 1 = \frac{3}{2} \cdot (x - 2)$

(c) $m_d = \tan \alpha_d$

$m_{d_1} = \tan \alpha_{d_1}$

$\frac{\pi}{4} = \alpha_d - \alpha_{d_1}$

$$\Rightarrow 1 = \tan\left(\frac{\pi}{4}\right) = \frac{\tan(\alpha_d) - \tan(\alpha_{d_1})}{1 + \tan(\alpha_{d_1}) \cdot \tan(\alpha_d)} =$$

$$= \frac{m_d - m_{d_1}}{1 + m_{d_1} m_d} \Rightarrow -\frac{2}{3} - m_{d_1} = 1 + m_{d_1} \cdot \left(-\frac{2}{3}\right)$$

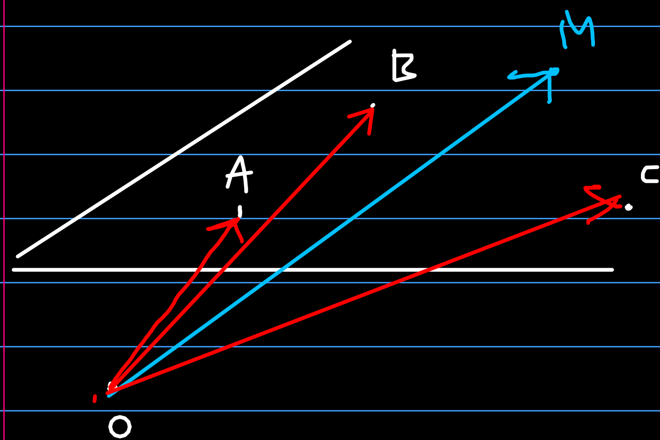
$$\Rightarrow m_{d_1} \cdot \frac{1}{3} = \frac{5}{3} \Rightarrow m_{d_1} = 5$$

$$\Rightarrow d_1 : y - 1 = 5 \cdot (x - 2)$$

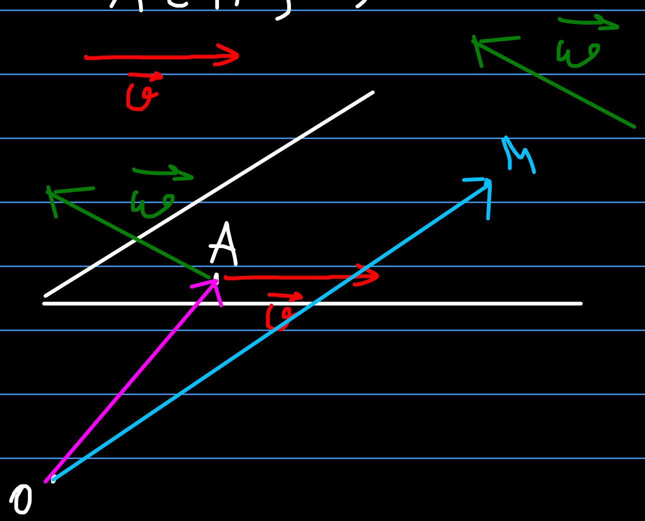
Plans

Π plan

$A, B, C \in \Pi$ noncollinear



$A \in \Pi$, $\vec{u}, \vec{w} \parallel \Pi$



Vector equation

$$\vec{r}_M = (1-\lambda-\mu)\vec{r}_A + \lambda \vec{r}_B + \mu \vec{r}_C$$

$$\lambda, \mu \in \mathbb{R}$$

$$\vec{r}_M = \vec{r}_A + \alpha \vec{u} + \beta \vec{w}$$

$$\alpha, \beta \in \mathbb{R}$$

Parametric equation

$$\Pi: \begin{cases} x = (1-\lambda-\mu)x_A + \lambda x_B + \mu x_C \\ y = (1-\lambda-\mu)y_A + \lambda y_B + \mu y_C \\ z = (1-\lambda-\mu)z_A + \lambda z_B + \mu z_C \end{cases}$$

$$\begin{cases} x = x_A + \alpha x_{\vec{u}} + \beta x_{\vec{w}} \\ y = y_A + \alpha y_{\vec{u}} + \beta y_{\vec{w}} \\ z = z_A + \alpha z_{\vec{u}} + \beta z_{\vec{w}} \end{cases}$$

Canonical equation:

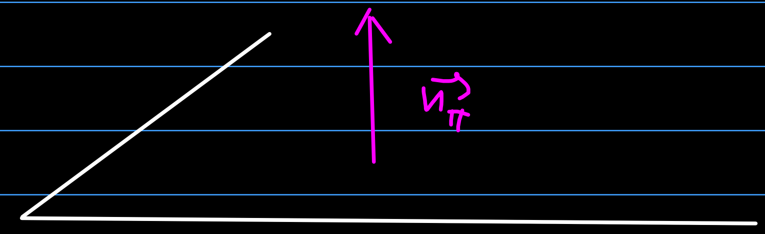
$$\begin{vmatrix} x-x_A & y-y_A & z-z_A \\ x_B-x_A & y_B-y_A & z_B-z_A \\ x_C-x_A & y_C-y_A & z_C-z_A \end{vmatrix} = 0 \qquad \begin{vmatrix} x-x_A & y-y_A & z-z_A \\ x_{\vec{u}} & y_{\vec{u}} & z_{\vec{u}} \\ x_{\vec{w}} & y_{\vec{w}} & z_{\vec{w}} \end{vmatrix} = 0$$

$$\Updownarrow$$

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0$$

Implicit equation $\quad Ax+By+Cz+D=0$

$\rightarrow$ Very useful: $\vec{n}_\pi(A,B,C)$



3.1. Write the equation of the plane which passes through $M_0(2, 5, 7)$ and is parallel to the vectors $\vec{u_1}(1, 2, 0)$, $\vec{u_2}(3, 1, 7)$

$$\begin{vmatrix} x-2 & y-5 & z-7 \\ 1 & 2 & 0 \\ 3 & 1 & 7 \end{vmatrix} = 0 \iff$$

$\iff 14(x-2) + (z-7) + 0 - 6(z-7) -$
$- 0 - 7(y-5) = 0 \iff 14x - 28 + z - 7 - 6z +$
$+ 42 - 7y + 35 = 0 \iff 14x - 7y - 5z + 42 = 0$

$$\begin{cases} x = 2 + \alpha + 3\beta \\ y = 5 + 2\alpha + \beta \\ z = 7 + 7\beta \end{cases}$$

3.3. Write the equation of the plane which contains the line

$$(d_1)\quad \frac{x-3}{2} = \frac{y+4}{1} = \frac{z-2}{-3}$$

and is parallel to the line

$$(d_2)\quad \frac{x+5}{2} = \frac{y-2}{2} = \frac{z-1}{2}$$

$$\left.\begin{array}{l} A(3,-4,2) \in d_1 \subset \Pi \Rightarrow A \in \Pi \\ d_1 \subset \Pi \Rightarrow \vec{d_1}(2,1,-3) \parallel \Pi \\ d_2 \parallel \Pi \Rightarrow \vec{d_2}(2,2,2) \parallel \Pi \end{array}\right\} \Rightarrow$$

$$\Rightarrow \begin{vmatrix} x-3 & y+4 & z-2 \\ 2 & 1 & -3 \\ 2 & 2 & 2 \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow (x-3)\cdot\begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} + (y+4)\cdot\begin{vmatrix} -3 & 2 \\ 2 & 2 \end{vmatrix} + (z-2)\cdot\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 0$$

$$\Rightarrow 8x - 24 + (-10y) - 40 + 2z - 4 = 0$$

$$\Rightarrow 8x - 10y + 2z - 68 = 0$$