Seminar W12 — 9.14

Affine transformations (plane)

$y = mx + n$ affine function

$y = mx$ linear function

$\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ affine transformation.

$$\varphi([Q]_E) = M \cdot [Q]_E + \underbrace{[Q_0]_E}_{\text{translation vector}}$$

$M \in \mathcal{M}_{2,2}(\mathbb{R})$

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

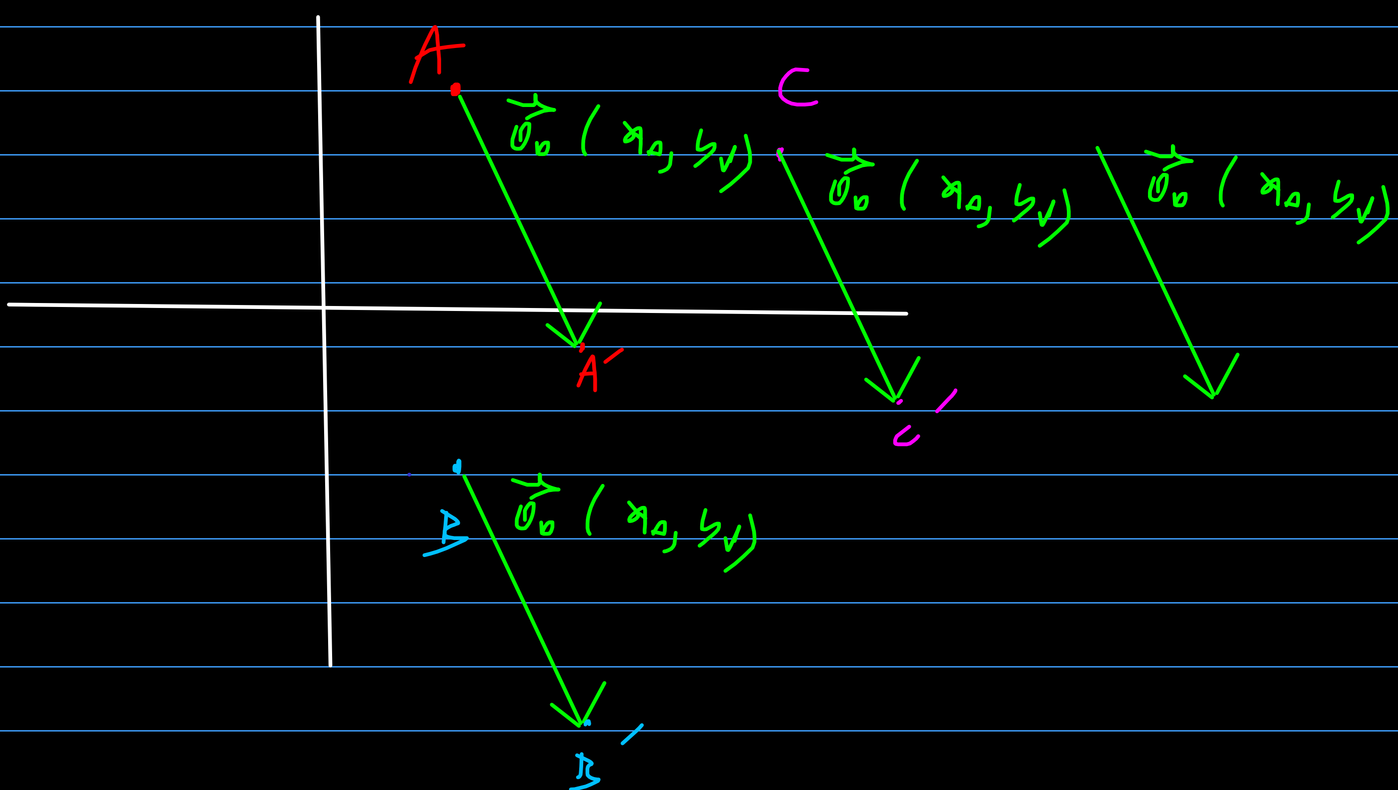
$\longrightarrow$ affine transformations preserve parallelism and lines. (but not necessarily distances and angles)

Translation

$M = I_2 \qquad v_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$T(x_0, y_0) : \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) \mapsto (x + x_0, y + y_0)$



How to recognize it: $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$

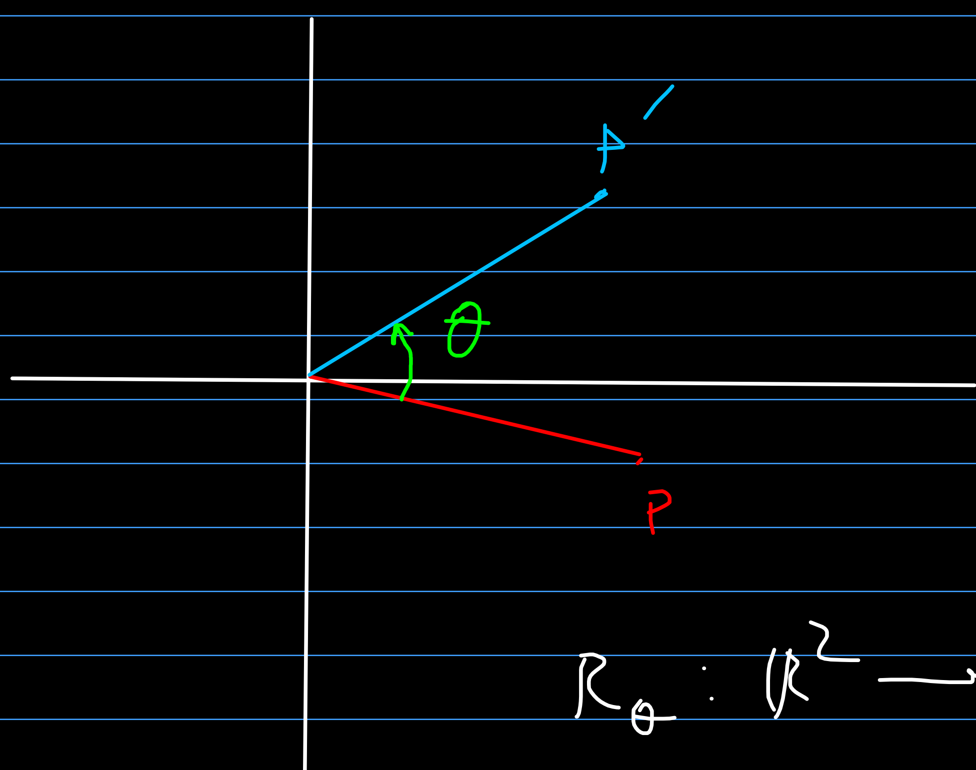
- choose A
- choose B

- if $\overrightarrow{AA'} = \overrightarrow{BB'} \Rightarrow \overrightarrow{AA'}$ is the translation vector

(you still need to check this)

$\rightarrow T(x_0, y_0)$ is invertible (non-singular)

<u>Rotations (around the origin)</u>



$$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$R_\theta([v]_E) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot [v]_E$$

How to recognize it $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$

- does it fix the origin? $Fix(\varphi) = \{0\}$

Yes

No

Not a rotation!

- Is it true that $OP = OP'$, $\forall P$?

Yes

No

Maybe a rotation

Not a rotation

- Is $m(\widehat{POP'}) = \theta$?

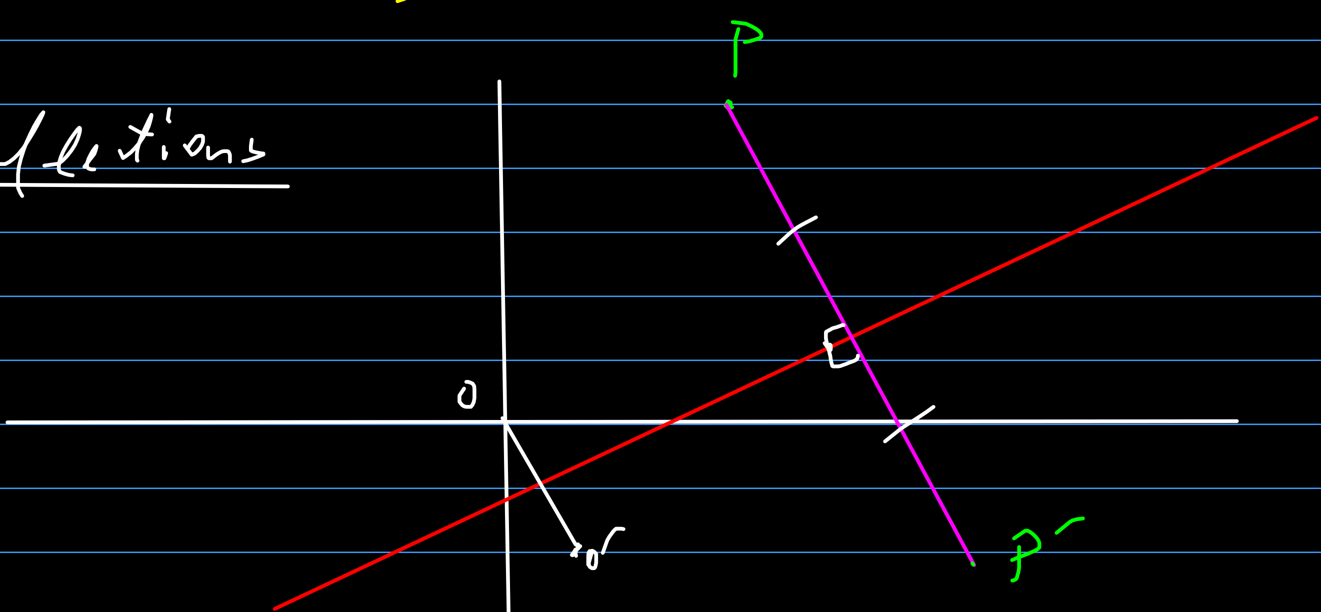
Yes

No

A rotation with angle θ

Not a rotation

Reflections



$$\ell: \quad ax+by+c=0$$

$$r_\ell: \mathbb{R}^2 \to \mathbb{R}^2$$

$$r_\ell([v]_E) = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{pmatrix} \cdot [v]_E +$$

$$+ \begin{pmatrix} \frac{2ac}{a^2+b^2} \\ \\ \frac{2bc}{a^2+b^2} \end{pmatrix}$$

$$r_x \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$r_y \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

How to recognize it: $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$

$\to$ $Fix(\varphi) = \ell$ line?

$$\left[\begin{array}{l} f: A \to B \\ Fix(f) = \{x \in A \mid f(x) = x\} \end{array} \right]$$

How to recognize it: $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$

→ Is $\mathrm{Fix}(\varphi) = \ell$ line?

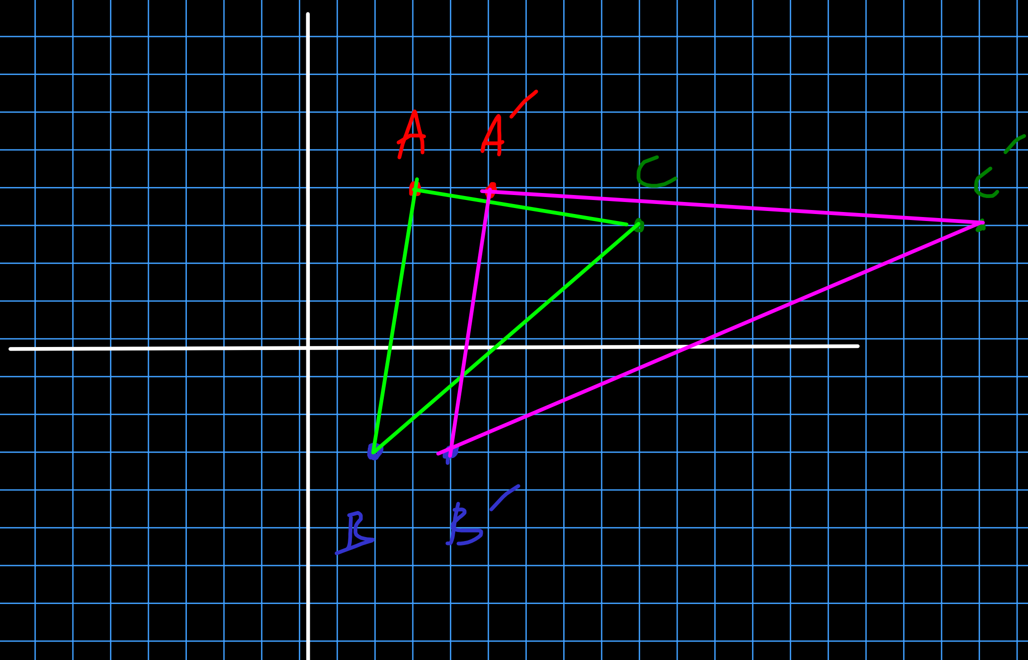
No → Not a reflection

Yes → Is a reflection

Scalings

$S(s_x, s_y)$, $s_x, s_y \neq 0$

$S(2,1)$



$$S(s_x, s_y)\left([v]_E\right) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot [v]_E$$

($s_x = s_y \Rightarrow$ "homothety")

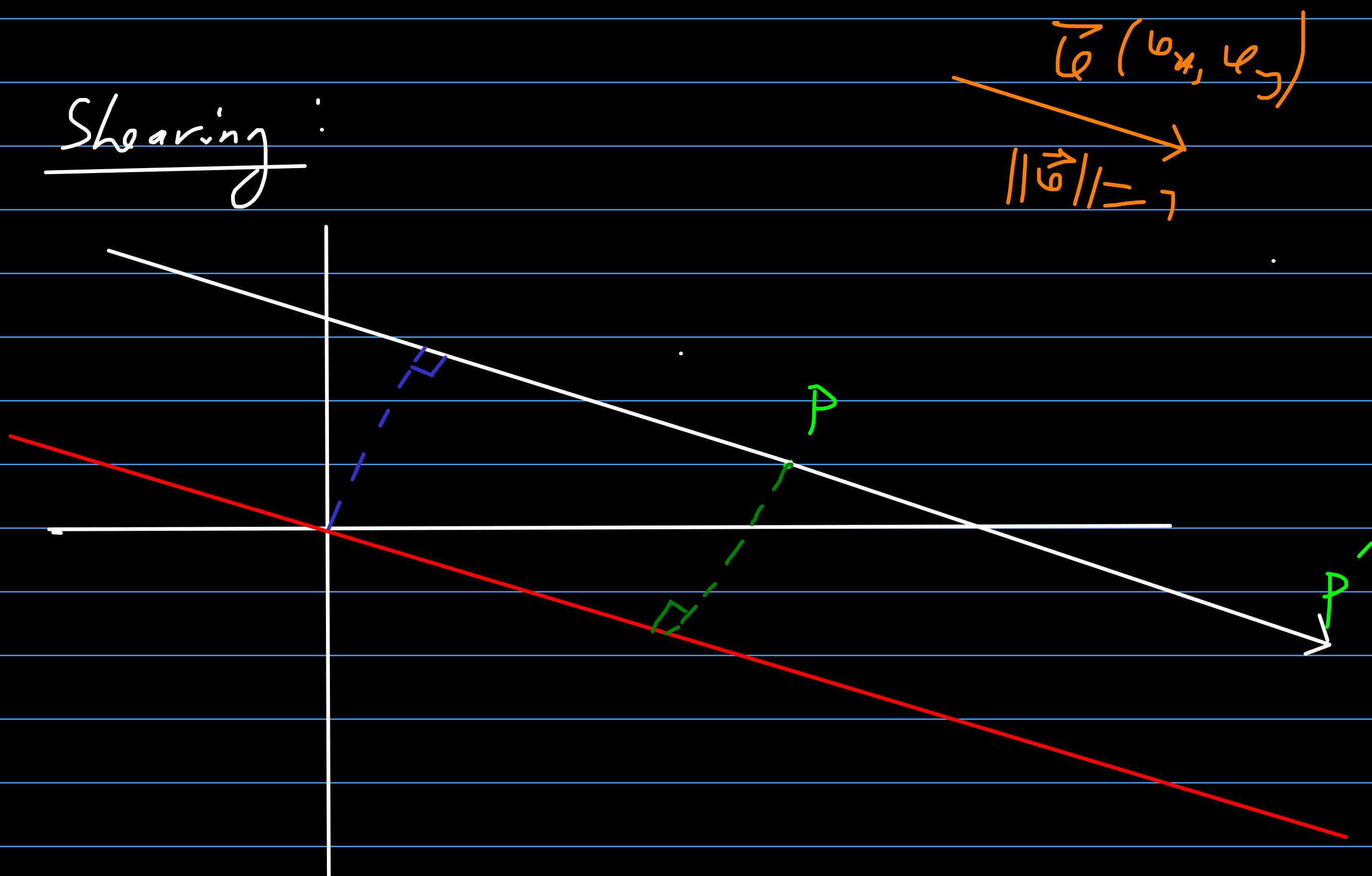
How to recognize them:

- take a point A
- find $\frac{x_{A'}}{x_A}$ and $\frac{y_{A'}}{y_A}$, so

these are the candidates for the scaling factors

- check this against the other points.

Shearing:



$$\mathrm{Sh}(\vec{u}, r) : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\mathrm{Sh}(\vec{u}, r)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{r \cdot \delta(0, \ell_p) \cdot \vec{u}}$$

ℓ_p = the line through p that is parallel to $\vec{u}$

$$\left[\quad p(x,y), \quad \ell : ax+by+c=0 \qquad \delta(p, \ell) = \frac{ax+by+c}{\sqrt{a^2+b^2}} \quad \right]$$

$$\mathrm{Sh}(\vec{u}, r)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - r \cdot \delta(P, \ell_0) \cdot \vec{u}$$

$$\mathrm{Sh}(u, r)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - r u_x u_y & r u_x^2 \\ -r u_y^2 & 1 + r u_x u_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

How do we recognize it?

- Pick a point P. Is $\overrightarrow{PP'} \parallel \vec{w}$?

Yes

No

Not a shear

Do a little more work to find $\vec{w}$ and r.

12.1. Find the image of the triangle ABC through r_d, where

$$(d): x - y = 2$$

$A(-1,2)$, $B(-2,-1)$, $C(3,3)$

$a = 1$, $b = 1$, $c = -2$

$$\begin{pmatrix} \underbrace{\frac{b^2-a^2}{a^2+b^2}}_{} & \frac{-2ab}{a^2+b^2} \\ \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{pmatrix} =$$

$= \cos\theta$ $\quad$ $\sin\theta$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{2ac}{a^2+b^2} \\ \frac{2bc}{a^2+b^2} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} y-1 \\ x+1 \end{pmatrix}$$

$\varphi(A) = (1, 0)$, $\quad \varphi(B) = (-2, -1)$

$\varphi(C) = (2, 4)$

11.2. Find the image of the ΔABC through the clockwise rotation of angle $\frac{\pi}{6}$ where $A(6,4)$, $B(6,2)$, $C(10,6)$.

$$[R_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$[R_\theta(A)]_E = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 3\sqrt{3} - 2 \\ 3 + 2\sqrt{3} \end{pmatrix}$$

$$[R_\theta(B)]_E = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 3\sqrt{3} - 1 \\ 3 + \sqrt{3} \end{pmatrix}$$

$$\left[R_\theta(C)\right]_E = \begin{pmatrix} 5\sqrt{3} - 3 \\ 3 - \sqrt{3} \end{pmatrix}$$

11.3. ABCD quadrilateral

$A(1,1)$, $B(3,1)$, $C(2,2)$, $D(\frac{3}{2}, 3)$

Find the images of this quadrilateral through the translation $T(1,2)$, the scaling $S(2, \frac{5}{2})$, r_x, r_y, the clockwise and anticlockwise rotations through the angle $\frac{\pi}{2}$ and the shear $sh\left(\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \frac{3}{2}\right)$

→ T and S → one person

→ r_x, r_y, rotations → one person

→ shear → one person

$$T(1,2)\,(1,1) = (2,3)$$

$$T(1,2)\,(3,1) = (4,3)$$

$$T(1,2)\,(2,2) = (3,4)$$

$$T(1,2)\left(\frac{3}{2},\ 3\right) = \left(\frac{5}{2},\ 5\right)$$

$$S\left(2,\frac{5}{2}\right)(1,1) = \left(2,\ \frac{5}{2}\right)$$

$$S\left(2,\frac{5}{2}\right)(3,1) = \left(6,\ \frac{5}{2}\right)$$

$$S\left(2,\frac{5}{2}\right)(2,2) = (4,5)$$

$$S\left(2,\frac{5}{2}\right)\left(\frac{3}{2},3\right) = \left(5,\ \frac{15}{2}\right)$$

$$r_x\,(1,1) = (1,-1)$$

$$r_x(3,1) = (3,-1)$$

$$r_x(2,2) = (2,-2)$$

$$r_x\left(\frac{3}{2},3\right) = \left(\frac{3}{2},-3\right)$$

$$r_y(1,1) = (-1,1)$$
$$r_y(3,1) = (-3,1)$$
$$r_y(2,2) = (-2,2)$$
$$r_y\left(\frac{3}{2},3\right) = \left(-\frac{3}{2},3\right)$$

$$[R_{\frac{\pi}{2}}] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow R_{\frac{\pi}{2}}(1,1) = (-1,1)$$
$$R_{\frac{\pi}{2}}(3,1) = (-1,3)$$
$$R_{\frac{\pi}{2}}(2,2) = (-2,2)$$
$$R_{\frac{\pi}{2}}\left(\frac{3}{2},3\right) = \left(-3,\frac{3}{2}\right)$$

$$[R_{\frac{\pi}{2}}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$R_{\frac{\pi}{2}}(1,1) = (1,-1)$$

$$R_{\frac{\pi}{2}}(3,1) = (1,-3)$$

$$R_{\frac{\pi}{2}}(2,2) = (2,-2)$$

$$R_{\frac{\pi}{2}}\left(\frac{3}{2},3\right) = \left(3,-\frac{3}{2}\right)$$

$$Sh(u,r)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - r u_x u_y & r u_x^2 \\ -r u_y^2 & 1 + r u_x u_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$u_x = \frac{2}{\sqrt{5}} \qquad u_y = \frac{1}{\sqrt{5}} \qquad r = \frac{3}{2}$$

$$[Sh(\vec{u},r)] = \begin{pmatrix} 1 - \frac{2}{5} \cdot \frac{3}{2} & \frac{3}{2} \cdot \frac{4}{5} \\ -\frac{3}{2} \cdot \frac{1}{5} & 1 + \frac{3}{2} \cdot \frac{2}{5} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{2}{5} & \frac{6}{5} \\ -\frac{3}{10} & \frac{8}{5} \end{pmatrix}$$

$$sh(\vec{v}, r)(1,1) = \left(\frac{8}{5}, \frac{13}{10}\right)$$

$$sh(\vec{v}, r)(3,1) = \left(\frac{12}{5}, \frac{7}{10}\right).$$

$$sh(\vec{v}, r)(2,2) = \left(\frac{16}{5}, \frac{26}{10}\right)$$

$$sh(\vec{v}, r)\left(\frac{3}{2}, 3\right) = \left(\frac{21}{5}, \frac{39}{10}\right)$$