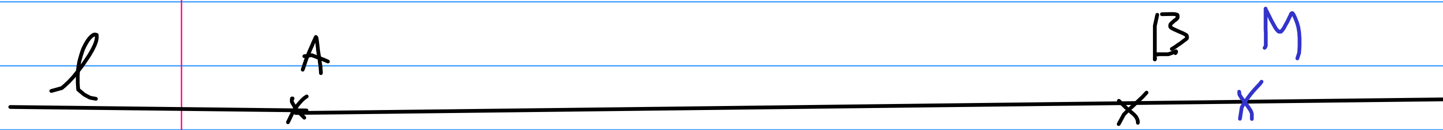
# Seminar W2 - 914




$l$ line $A, B \in l$

$$\boxed{\forall M \in l \ \exists!\ \lambda \in \mathbb{R} : \vec{r}_M = \lambda \vec{r}_A + (1-\lambda)\vec{r}_B}$$

If $M \in [AB]$, $\frac{AM}{MB} = \alpha \in \mathbb{R}$

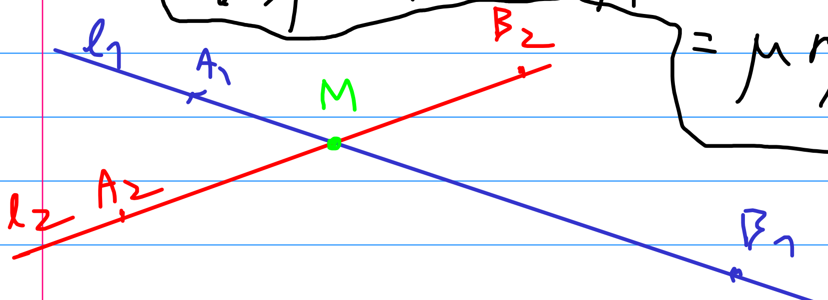
$$\Rightarrow \vec{r}_M = \frac{\alpha}{\alpha+1}\vec{r}_B + \frac{1}{\alpha+1}\vec{r}_A$$

Template for proofs

Say we have $\{M\} = l_1 \cap l_2$

Say we have $A_1, B_1 \in l_1$, $A_2, B_2 \in l_2$

$$\Rightarrow \exists!\ \lambda, \mu \in \mathbb{R} : \vec{r}_M = \lambda \vec{r}_{A_1} + (1-\lambda)\vec{r}_{B_1} = \mu \vec{r}_{A_2} + (1-\mu)\vec{r}_{B_2} \quad (**)$$




$$\overset{(*)}{\Rightarrow} \lambda \vec{r}_{A_1} + (1-\lambda)\vec{r}_{B_1} = \mu \vec{r}_{A_2} + (1-\mu)\vec{r}_{B_2}$$

- We write $\vec{r_{A_1}}, \vec{r_{B_1}}, \vec{r_{A_2}}, \vec{r_{B_2}}$ in terms of two vectors $\vec{u}$ and $\vec{w}$ that we choose so that they are linearly indep.
- We replace them in $(*)$
- We obtain smth like

$$\alpha(\lambda,\mu)\cdot\vec{u}+\beta(\lambda,\mu)\cdot\vec{w}=\vec{0}$$

- By linear indep., we deduce that:

$$\begin{cases}\alpha(\lambda,\mu)=0\\ \beta(\lambda,\mu)=0\end{cases}$$

- This enables us to eliminate $\lambda, \mu$ from $(**)$
- This gives us $\vec{r_M}$ in terms of $\vec{u}$ and $\vec{w}$.

2.1. $\triangle ABC$, $G$ centroid, $H$ orthocenter, $I$ incenter, $O$ circumcenter

We fix $P$ the origin of the reference system

Show that:

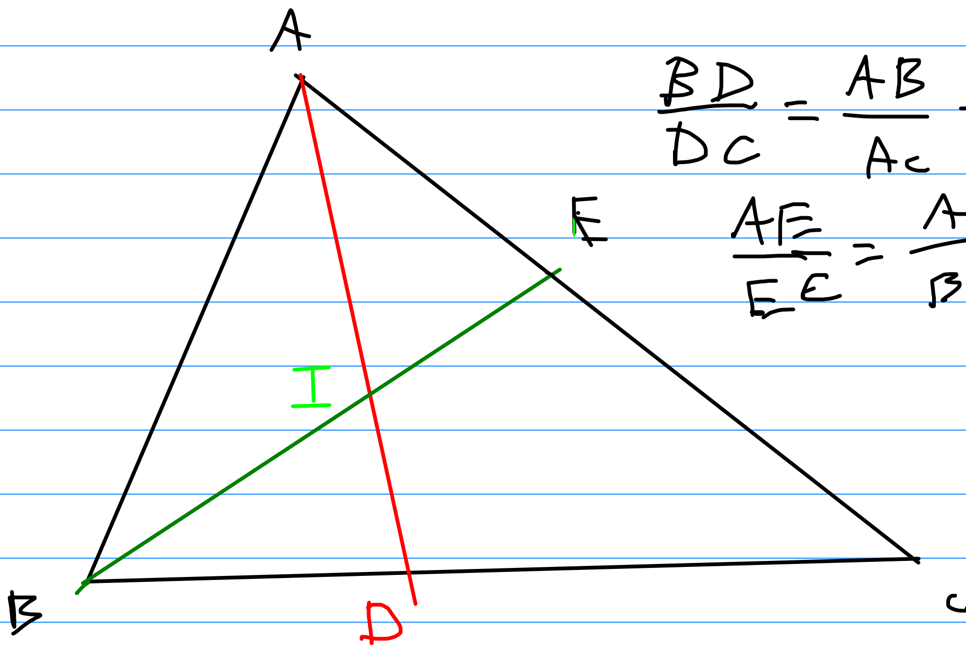
(a) $$\vec{r}_G = \frac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}$$ easy

(b) $$\vec{r}_I = \frac{a\vec{r}_A + b\vec{r}_B + c\vec{r}_C}{a+b+c}$$

(c) $$\vec{r}_H = \frac{\tan A \cdot \vec{r}_A + \tan B \cdot \vec{r}_B + \tan C \cdot \vec{r}_C}{\tan A + \tan B + \tan C}$$

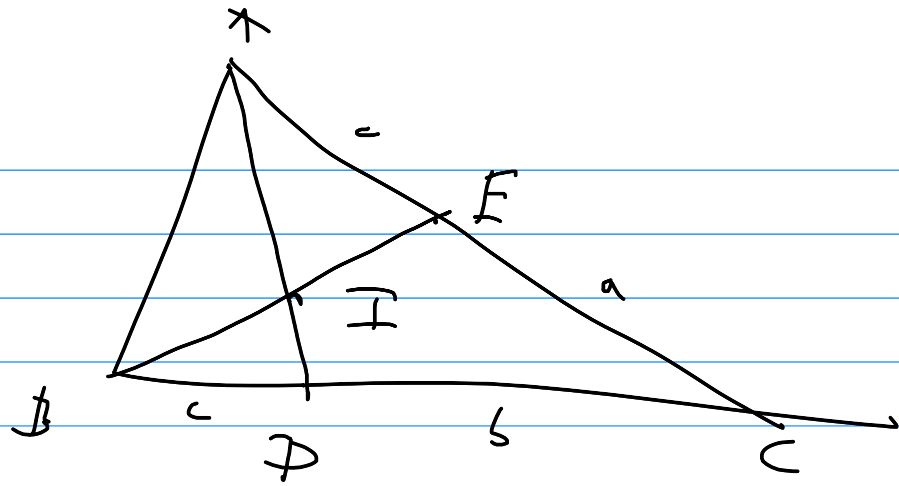
(d) $$\vec{r}_O = \frac{\sin 2A \cdot \vec{r}_A + \sin 2B \cdot \vec{r}_B + \sin 2C \cdot \vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}$$

(b)




$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$$

$$\frac{AE}{EC} = \frac{AB}{BC} = \frac{c}{a}$$




$$\vec{r}_D = \frac{DC}{BC}\cdot\vec{r}_B + \frac{BD}{BC}\cdot\vec{r}_C =$$

$$= \frac{b}{b+c}\cdot\vec{r}_B + \frac{c}{b+c}\cdot\vec{r}_C$$

$$\vec{r}_E = \frac{a}{a+c}\cdot\vec{r}_A + \frac{c}{a+c}\cdot\vec{r}_C$$

$$\vec{r}_I = \lambda\,\vec{r}_D + (1-\lambda)\,\vec{r}_A = \mu\cdot\vec{r}_E + (1-\mu)\cdot\vec{r}_B$$

$$\frac{b\lambda}{b+c}\vec{r}_B + \frac{\lambda c}{b+c}\vec{r}_C + (1-\lambda)\,\vec{r}_A =$$

$$= \frac{\mu a}{a+c}\vec{r}_A + \frac{\mu c}{a+c}\vec{r}_C + (1-\mu)\,\vec{r}_B$$

$$\Rightarrow \overbrace{\left(1-\lambda-\frac{\mu a}{a+c}\right)}^{=:X}\vec{r}_A + \overbrace{\left(\frac{b\lambda}{b+c}+\mu-1\right)}^{=:Y}\cdot\vec{r}_B +$$

$$+ \overbrace{\left(\frac{\lambda c}{b+c}-\frac{\mu c}{a+c}\right)}^{=:Z}\vec{r}_C = \vec{0}$$

$$\Rightarrow \quad X\vec{r_A} + Y\vec{r_B} + Z\vec{r_C} = \vec{0}$$

We write $\vec{u} = \vec{AB}$, $\vec{w} = \vec{AC}$. We know that they are linearly independent (if not, then the $\Delta ABC$ is degenerate)

$$\vec{r_B} = \vec{r_A} + \vec{u}, \qquad \vec{r_C} = \vec{r_A} + \vec{w}$$

$$\Rightarrow \vec{r_A}(X+Y+Z) + Y\vec{u} + Z\vec{w} = \vec{0}$$

$$X+Y+Z = \left(1 - \lambda - \frac{\mu a}{a+c}\right) + \left(\frac{b\lambda}{b+c} + \mu - 1\right) + \left(\frac{\lambda c}{b+c} - \frac{\mu c}{a+c}\right) = 0$$

$$\left.\begin{array}{l}\Rightarrow Y\vec{u} + Z\vec{w} = \vec{0} \\ \vec{u}, \vec{w} \text{ lin. indep.}\end{array}\right\} \Rightarrow \begin{cases} Y = 0 \\ Z = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \dfrac{b\lambda}{b+c} + \mu - 1 = 0 \\ \dfrac{\lambda c}{b+c} - \dfrac{\mu c}{a+c} = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \mu = 1 - \dfrac{b\lambda}{b+c} \\ \dfrac{\lambda c}{b+c} - \dfrac{c}{a+c} + \dfrac{bc\lambda}{(b+c)(a+c)} = 0 \end{cases}$$

$$\lambda\left(\frac{c}{b+c}+\frac{bc}{(b+c)(a+c)}\right)=\frac{c}{a+c}$$

$(b+c)(a+c)$

$$\Rightarrow \lambda = \left(\frac{\frac{c}{b+c}+\frac{bc}{(b+c)(a+c)}}{\frac{c}{a+c}}\right)^{-1}$$

$$\Rightarrow \lambda = \left(\frac{c(a+c)+bc}{c(b+c)}\right)^{-1} =$$

$$= \frac{b+c}{a+b+c}.$$

$$\Rightarrow \vec{r_I} = \lambda\vec{r_D} + (1-\lambda)\vec{r_A} =$$

$$= \frac{b+c}{a+b+c}\left(\frac{c}{b+c}\vec{r_C}+\frac{b}{b+c}\vec{r_B}\right)+\frac{a}{a+b+c}\cdot\vec{r_A} =$$

$$= \frac{a}{a+b+c}\vec{r_A}+\frac{b}{d+b+c}\vec{r_B}+\frac{c}{a+b+c}\vec{r_C}$$

2.2. Consider the angle $\widehat{BOB'}$ and the points $A \in [OB]$, $A' \in [OB']$. Show that:

$$\overrightarrow{OM} = m \cdot \frac{1-n}{1-mn}\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

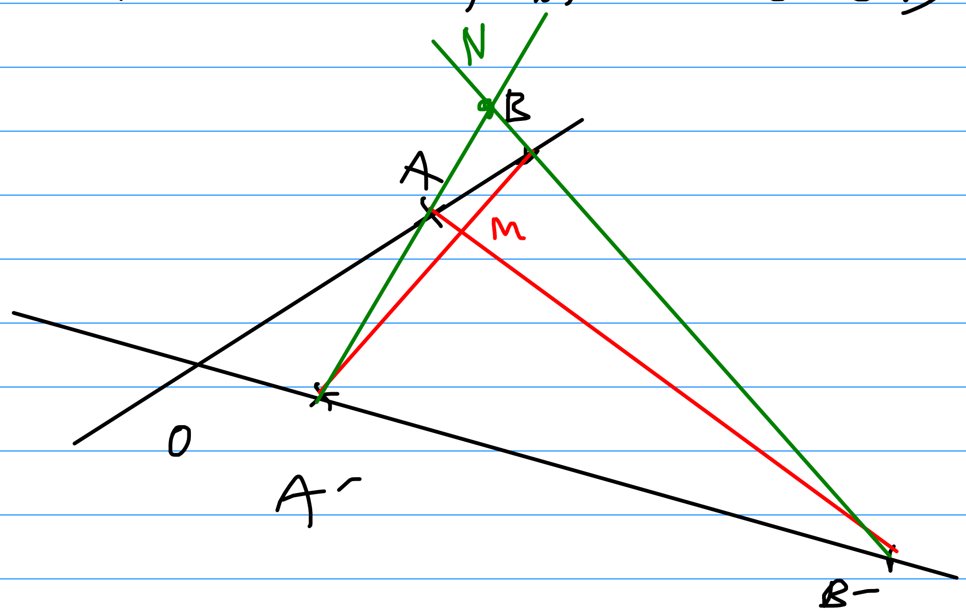
$$\overrightarrow{ON} = m \cdot \frac{n-1}{n-m}\overrightarrow{OA} + n\frac{m-1}{m-n}\overrightarrow{OA'}$$

where $\{M\} = AB' \cap A'B$, $\{N\} = AA' \cap BB'$

$\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\,\overrightarrow{OA}$

$\overrightarrow{OB'} = n\,\overrightarrow{OA'}$

We assume that $m(\widehat{BOB'}) \neq 0$



$$\overrightarrow{OM} = \lambda \overrightarrow{OA} + (1-\lambda)\overrightarrow{OB'} = \mu \overrightarrow{OA'} + (1-\mu)\overrightarrow{OB}$$

$$\overrightarrow{OA} = \vec{u}, \quad \overrightarrow{OA'} = \vec{v}, \quad \overrightarrow{OB} = m\vec{u}, \quad \overrightarrow{OB'} = n\vec{v}$$

$$\Rightarrow \overrightarrow{OM} = \lambda \vec{u} + (1-\lambda)\cdot n \cdot \vec{v} = \mu \cdot \vec{v} + (1-\mu)\cdot m \cdot \vec{u}$$

$$\Rightarrow (\lambda - m + m\mu)\cdot \vec{u} + (n - n\lambda - \mu)\cdot \vec{v} = \vec{0}$$

Because $\vec{u}$ and $\vec{v}$ are linearly independent

$$\Rightarrow \begin{cases} \lambda - m + m\mu = 0 \\ n - n\lambda - \mu = 0 \end{cases} \Leftrightarrow \begin{cases} \mu = n - n\lambda \\ \lambda - m + m\mu = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu = n - n\lambda \\ \lambda - m + mn - mn\lambda = 0 \end{cases} \Leftrightarrow \begin{cases} \mu = n - n\lambda \\ \lambda(1-mn) = m - mn \end{cases}$$

$$\Rightarrow \lambda = \frac{m(1-n)}{1-mn}$$

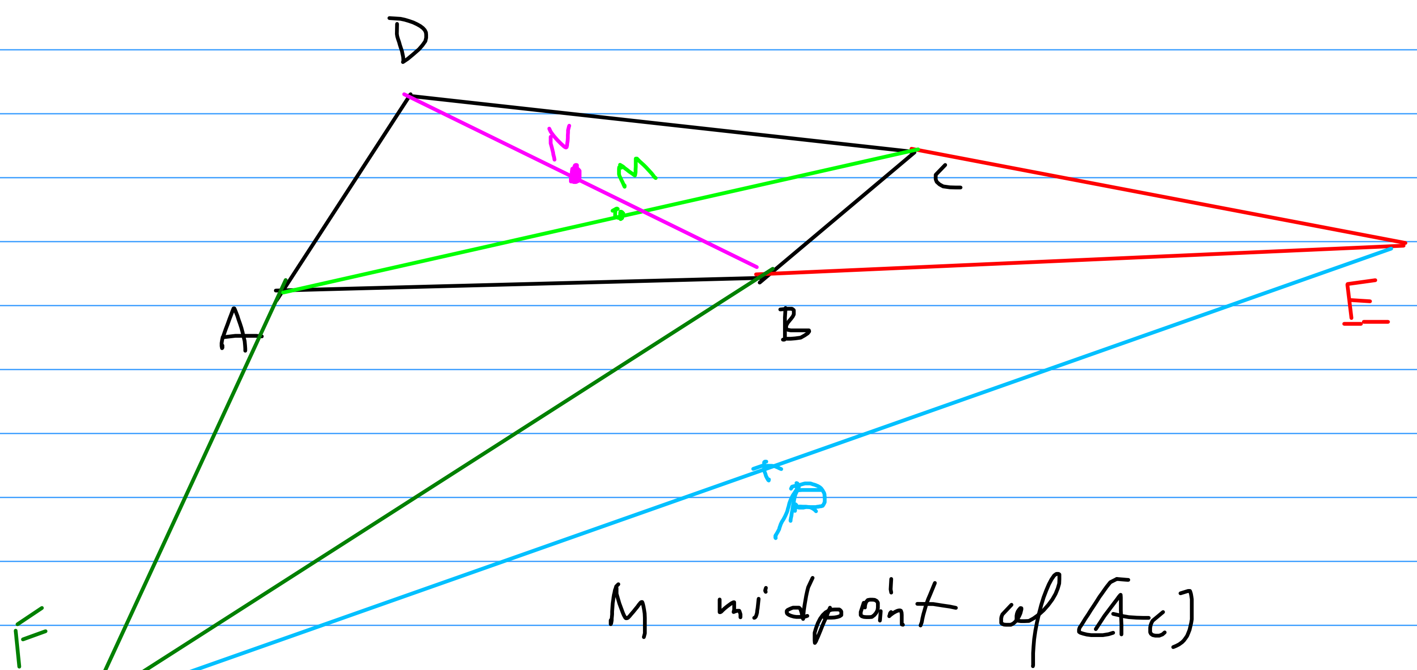
$$\Rightarrow \overrightarrow{OM} = \lambda \vec{u} + (1-\lambda) n \vec{v} =$$

$$= \frac{m(1-n)}{1-mn}\cdot \vec{u} + \frac{1-mn-m+mn}{1-mn}\cdot n \cdot \vec{v} =$$

$$= \frac{m(1-n)}{1-mn}\cdot \overrightarrow{OA} + \frac{n(1-m)}{1-mn}\cdot \overrightarrow{OA'}$$

Same for $\overrightarrow{ON}$

2.3. Show that the midpoints $\underbrace{\phantom{x}}_{}$ of the diagonals of a complete quadrilateral are collinear.




M midpoint of [AC]

N midpoint of [BD]

P midpoint of [EF]

Show that N, M, P collinear.

Sketch (of a proof): We choose $\vec{u} := \overrightarrow{DF}$, $\vec{w} := \overrightarrow{DE}$

$\Rightarrow \overrightarrow{DA} = \alpha \cdot \vec{u}$, $\overrightarrow{DC} = \beta \cdot \vec{w}$, $\alpha, \beta \in \mathbb{R}$

$$\overrightarrow{DM} = \frac{\overrightarrow{DA}+\overrightarrow{DC}}{2} = \frac{\alpha\vec{u}+\beta\vec{w}}{2}$$

$$\overrightarrow{DP} = \frac{\overrightarrow{DE}+\overrightarrow{DF}}{2} = \frac{\vec{u}+\vec{w}}{2}$$

$$\overrightarrow{DN} = \frac{1}{2}\overrightarrow{DB}$$

$$\overrightarrow{DB} = ?\cdot\vec{u} + ?\cdot\vec{w}$$

$$\{B\} = AE \cap FC$$

the next step: find $\overrightarrow{DB}$ in terms of $\vec{u}$ and $\vec{w}$

$$\overrightarrow{DB} = \lambda\overrightarrow{DA} + (1-\lambda)\overrightarrow{DE} = \lambda\alpha\vec{u} + (1-\lambda)\vec{w}$$

$$= \mu\overrightarrow{DF} + (1-\mu)\overrightarrow{DC} = \mu\vec{u} + (1-\mu)\beta\vec{w}$$

Eliminate $\lambda$ and $\mu$ and we're good to go.