$$\begin{bmatrix} 2 & 4 & 2 \\ 4 & -10 & 2 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \\ 9 \end{bmatrix}$$

$$\tilde{A} = \left(\begin{array}{ccc|c} \boxed{2} & 4 & 2 & 6 \\ 4 & -10 & 2 & -10 \\ 1 & 2 & 4 & 9 \end{array}\right) \begin{array}{l} (R_2 - 2 \cdot R_1) \to (R_2) \\ (R_3 - \frac{1}{2} R_1) \to (R_3) \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ 0 & \boxed{-18} & -2 & -22 \\ 0 & 0 & 3 & 6 \end{array}\right) \Rightarrow \begin{cases} 2x_1 + 4x_2 + 2x_3 = 6 \\ -18x_2 - 2x_3 = -22 \\ 3x_3 = 6 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = \frac{1}{2}(6 - 2x_3 - 4x_2) = \frac{1}{2}(6 - 4 - 4) = -1 \\ x_2 = \frac{1}{-18}(-22 + 2x_3) = 1 \\ x_3 = 2 \end{cases} \qquad X = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}^T = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$(R_1) \leftrightarrow (R_2)$$

$$\tilde{A} = \left(\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ \boxed{4} & -10 & 2 & -10 \\ 1 & 2 & 4 & 9 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{4} & -10 & 2 & -10 \\ 2 & 4 & 2 & 6 \\ 1 & 2 & 4 & 9 \end{array}\right)$$

$$(R_2 - \frac{1}{2} R_1) \to R_2$$

$$(R_3 - \frac{1}{4} R_1) \to R_3$$

$$\tilde{A} \sim \left(\begin{array}{ccc|c} 4 & -10 & 2 & -10 \\ 0 & 9 & 1 & 11 \\ 0 & \frac{9}{2} & \frac{7}{2} & \frac{23}{2} \end{array}\right)$$

$$(2R_3) \to (R_3) \qquad \tilde{A} = \left(\begin{array}{ccc|c} 4 & -10 & 2 & -10 \\ 0 & \boxed{9} & 1 & 11 \\ 0 & 9 & 7 & 23 \end{array}\right) (R_3 - R_2)$$

$$\tilde{A} \sim \left(\begin{array}{ccc|c} 4 & -10 & 2 & -10 \\ 0 & 9 & 1 & 11 \\ 0 & 0 & 6 & 12 \end{array}\right)$$

$s_i = \max_{j=\overline{1,n}} |a_{ij}|$ or $s_i = \sum_{j=1}^{n} |a_{ij}|$, $i=\overline{1,n}$

$$\tilde{A} = \left(\begin{array}{ccc|c} \boxed{2} & 4 & 2 & 6 \\ 4 & -10 & 2 & -10 \\ 1 & 2 & 4 & 9 \end{array}\right) \qquad s=[8, 16, 7]$$

$$\left[\frac{2}{8}, \frac{4}{16}, \frac{1}{7}\right] = \left[\frac{|a_{j,1}|}{s_j}\right]$$

$$=\left[\boxed{\frac{1}{4}}, \frac{1}{4}, \frac{1}{7}\right]$$

$r=1$

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ 0 & -18 & -2 & -22 \\ 0 & 4 & 3 & 6 \end{array}\right) \qquad \cancel{\frac{+18}{-16}}, \cancel{\frac{4}{7}}$$

$s=[20, 7]$

$$\left[\frac{|a_{j,1}|}{s_j}\right] = \left[\frac{9}{10}, \frac{4}{7}\right]$$

$$\tilde{A} = \left(\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ 4 & \boxed{-10} & 2 & -10 \\ 1 & 2 & 4 & 9 \end{array}\right) \qquad (R_1)\leftrightarrow(R_2)$$

$(C_1)\leftrightarrow(C_2)$ $\quad R_3+\frac{1}{5}R_1$

$$\tilde{A} = \left(\begin{array}{ccc|c} 4 & 2 & 2 & 6 \\ -10 & 4 & 2 & -10 \\ 2 & 1 & 4 & 9 \end{array}\right) \sim \left(\begin{array}{ccc|c} -10 & 4 & 2 & -10 \\ 4 & 2 & 2 & 6 \\ 2 & 1 & 4 & 9 \end{array}\right)$$

$X' = [x_2 \quad x_1 \quad x_3]$

$$\sim \left(\begin{array}{ccc|c} -10 & 4 & 2 & -10 \\ 0 & \frac{18}{5} & \frac{14}{5} & 2 \\ 0 & \frac{9}{5} & \boxed{\frac{22}{5}} & 7 \end{array}\right)$$

$X'' = [x_2 \quad x_3 \quad x_1]$

$x^{\prime\prime} = [1 \quad 2 \quad -1]$

$x^{\prime\prime} = [x_2 \quad x_3 \quad x_1] \Rightarrow X = [-1 \quad 1 \quad 2]^T$

$A = L \cdot U$

$A = \left[\begin{array}{c|cc} \boxed{2} & 4 & 2 \\ \hline 4 & -10 & 2 \\ 1 & 2 & 4 \end{array}\right]$ $\quad a_{11} = 2 \quad v = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$w^* = [4 \ , \ 2]$

$A^1 = \begin{bmatrix} -10 & 2 \\ 2 & 4 \end{bmatrix}$

$\tilde{A} \sim \left[\begin{array}{c|cc} 2 & 4 & 2 \\ \hline 2 & -18 & -2 \\ 1/2 & 0 & 3 \end{array}\right]$

A^1 Schur I:

$A^1 - v \cdot w^* / a_{11} =$

$= \begin{bmatrix} -10 & 2 \\ 2 & 4 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix} [4 \quad 2]$

$\tilde{A} \sim \left[\begin{array}{c|c|c} 2 & 4 & 2 \\ \hline 2 & \boxed{-18} & -2 \\ \hline 1/2 & 0 & 3 \end{array}\right]$ $= \begin{bmatrix} -10 & 2 \\ 2 & 4 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 16 & 8 \\ 4 & 2 \end{bmatrix}$

$\tilde{A} \sim \left[\begin{array}{c|c|c} 2 & 4 & 2 \\ \hline 2 & -18 & -2 \\ \hline 1/2 & 0 & 3 \end{array}\right]$ $= \begin{bmatrix} -10 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$

Schur II:

$3 + \frac{1}{18} \cdot 0 \cdot (-2) = 3$ $= \begin{bmatrix} -18 & -2 \\ 0 & 3 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & -18 & -2 \\ 0 & 0 & 3 \end{bmatrix}$

$$LU=\begin{bmatrix}1&0&0\\2&1&0\\\frac{1}{2}&0&1\end{bmatrix}\begin{bmatrix}2&4&2\\0&-18&-2\\0&0&3\end{bmatrix}\Rightarrow$$

$$=\begin{bmatrix}2&4&2\\4&-10&2\\1&2&4\end{bmatrix}=A \checkmark$$

$$Ax=b \Leftrightarrow L\underbrace{Ux}_{y}=b \Leftrightarrow \begin{cases}Ly=b\\Ux=y\end{cases}$$

$$Ly=b \Leftrightarrow \begin{bmatrix}1&0&0\\2&1&0\\1/2&0&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}6\\-10\\9\end{bmatrix}\Rightarrow$$

$$\Leftrightarrow\begin{cases}y_1=6\\2y_1+y_2=-10\\\frac{1}{2}y_1+y_3=9\end{cases}\Rightarrow\begin{cases}y_1=6\\y_2=-22\\y_3=6\end{cases}$$

$$Ux=y \Leftrightarrow \begin{bmatrix}2&4&2\\0&-18&-2\\0&0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}6\\-22\\6\end{bmatrix}\Rightarrow$$

$$\Leftrightarrow\begin{cases}2x_1+4x_2+2x_3=6\\-18x_2-2x_3=-22\\3x_3=6\end{cases}\Rightarrow\begin{cases}x_1=-1\\x_2=1\\x_3=2\end{cases}$$

$P = (1 \quad 2 \quad 3)^T \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$PA = LU$

$A \sim \begin{bmatrix} 2 & 4 & 2 \\ \boxed{4} & -10 & 2 \\ 1 & 2 & 4 \end{bmatrix} \xrightarrow{(R_1) \leftrightarrow (R_2)} \left[\begin{array}{c|cc} 4 & -10 & 2 \\ \hline 2 & 4 & 2 \\ 1 & 2 & 4 \end{array}\right], \quad P \sim \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Schur I: $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \frac{1}{4} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -10 & 2 \end{bmatrix} =$

$= \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \frac{1}{4} \cdot \begin{bmatrix} -20 & 4 \\ -10 & 2 \end{bmatrix} =$

$= \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -5 & 1 \\ -5/2 & 1/2 \end{bmatrix} =$

$= \begin{bmatrix} 9 & 1 \\ \frac{9}{2} & \frac{7}{2} \end{bmatrix}$

$A \sim \left[\begin{array}{c|cc} 4 & -10 & 2 \\ \hline 1/2 & 9 & 1 \\ 1/4 & 9/2 & 7/2 \end{array}\right] \sim \left[\begin{array}{c|c|c} 4 & -10 & 2 \\ \hline 1/2 & \boxed{9} & 1 \\ \cline{2-3} 1/4 & 9/2 & 7/2 \end{array}\right] \sim$

$\sim \left[\begin{array}{c|c|c} 4 & -10 & 2 \\ \hline 1/2 & 9 & 1 \\ \cline{2-3} 1/4 & 1/2 & 3 \end{array}\right]$ Schur II: $\frac{7}{2} - \frac{1}{9} \cdot \frac{9}{2} \cdot 1 = 3$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 1/2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 4 & -10 & 2 \\ 0 & 9 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

$PA = LU$

~~P.1) $Ax=b$~~ Suppose $\frac{9}{2} > 9$: $(R_2) \leftrightarrow (R_3)$,

$P = [2\ 3\ 1]^T$

$$\tilde{A} = \left(\begin{array}{c|cc} 4 & -10 & 2 \\ \hline 1/4 & 9/2 & 7/2 \\ \hline 1/2 & 9 & 1 \end{array}\right) \qquad s_{II} : 1 - \frac{2}{9} \cdot 9 \cdot \frac{7}{2} = -6$$

$$\tilde{A} \approx \left(\begin{array}{c|cc} 4 & -10 & 2 \\ \hline 1/4 & 9/2 & 7/2 \\ \hline 1/2 & 2 & -6 \end{array}\right)$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & -10 & 2 \\ 0 & 9/2 & 7/2 \\ 0 & 0 & -6 \end{pmatrix}$$

~~$LU = \begin{pmatrix} 1 & 0 \end{pmatrix}$~~

$$LU = \begin{bmatrix} 4 & -10 & 2 \\ 1 & 2 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

$$PA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ 4 & -10 & 2 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} \cancel{4} & \cancel{-10} & 2 \\ \cancel{2} & \cancel{4} & 2 \\ \cancel{1} & \cancel{2} & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -10 & 2 \\ 1 & 2 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

$$P.1\ Ax = b \Rightarrow PAx = Pb \Leftrightarrow L\underbrace{Ux}_{y} = \begin{bmatrix} -10 \\ 9 \\ 6 \end{bmatrix}$$

$$\begin{cases} Ly = [-10\ \ 9\ \ 6]^T \\ Ux = y \end{cases}$$

$f: \mathbb{R} \to \mathbb{R},\ f \in C^2[-1, 1]$

a) $(Pf)(x) = ?$

$x_0 = -1,\ f(-1)$ $\qquad (Pf)(-1) = f(-1)$

$x_1 = 1,\ f(1)$ $\qquad (Pf)(1) = f(1)$

$n = 1$

$(Pf)(x) = (Lf)(x) = \ell_0(x) \cdot f(-1) + \ell_1(x) \cdot f(1)$

$\ell_i = \prod\limits_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \Rightarrow \ell_0 = \frac{x - x_1}{x_0 - x_1} = \frac{x-1}{-2} = \frac{1-x}{2}$

$\ell_1 = \frac{x - x_0}{x_1 - x_0} = \frac{x+1}{2}$

$$\boxed{(Pf)(x) = \frac{1-x}{2} \cdot f(-1) + \frac{x+1}{2} \cdot f(1)}$$

$(Pf)(-1) = \frac{2}{2} \cdot f(-1) + 0 \cdot f(1) = f(-1)\ \checkmark$

$(Pf)(1) = 0 \cdot f(-1) + \frac{2}{2} \cdot f(1) = f(1)\ \checkmark$

b) $\int\limits_{-1}^{1} (Pf)(x)\,dx \sim \int\limits_{-1}^{1} f(x)\,dx$

$\int\limits_{-1}^{1} (Pf)(x)\,dx = \int\limits_{-1}^{1} \left(\frac{1-x}{2} f(-1) + \frac{x+1}{2} f(1)\right) dx =$

$= \frac{1}{2} f(-1) \underbrace{\int\limits_{-1}^{1} (1-x)\,dx}_{2} + \frac{1}{2} f(1) \underbrace{\int\limits_{-1}^{1} (x+1)\,dx}_{2} =$

$= f(-1) + f(1)$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (Pf)(x)\,dx + Rf$$

$$Rf = \int_{-1}^{1} f(x)\,dx - \int_{-1}^{1} (Pf)(x)\,dx = \boxed{\int_{-1}^{1} f(x)\,dx - f(-1) - f(1)}$$

$$Re_k \qquad e_k = \{x^k\}_{k=0,1,\dots}$$

$$Re_0 = \int_{-1}^{1} x^0\,dx \cancel{- f(-1) - f(1)} - (-1)^0 - 1^0$$

$$= 2 - 1 - 1 = 0$$

$$Re_1 = \int_{-1}^{1} x^1\,dx - (-1)^1 - 1^1 = 0 + 1 - 1 = 0$$

$$\int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

$$Re_2 = \int_{-1}^{1} x^2\,dx - (-1)^2 - (1)^2 =$$

$$\frac{x^3}{3}\Big|_{-1}^{1} = \frac{1^3}{3} - \frac{(-1)^3}{3} = \frac{2}{3}$$

$$= \frac{2}{3} - 1 - 1 = -\frac{4}{3} \neq 0 \Rightarrow$$ The degree of prec

is $d = 1$

$\mathbb{P}$ $\quad \underset{\sim}{C^n}[a,b] \qquad f \in C^2[-1,1]$

c) $L : H^n[a,b] \to \mathbb{R}$, if $\operatorname{Ker} L = \mathbb{P}_{n-1}$,

In this case, $L = R$, if $\operatorname{Ker} R = \mathbb{P}_1$,

$$Lf = \int_a^b K_n(t) f^{(n)}(t)\,dt \qquad \boxed{z_+ = \begin{cases} z, & z \geq 0 \\ 0, & z < 0 \end{cases}}$$

In this case: $Rf = \int_{-1}^{1} K_2(t) f''(t)\,dt$

$$K_n(t) = \frac{1}{(n-1)!} L\left[(\cdot - t)_+^{n-1}\right]$$

~~In this case:~~ $\cancel{K_2(t) = \frac{1}{2} R((\cdot - t)_+) =}$

$= \frac{1}{2}$

$$K_n(x,t) = \frac{1}{(n-1)!} L\left[(x-t)_+^{n-1}\right]$$

$$K_2(x,t) = \frac{1}{1} R((x-t)_+) =$$

$$= \int_{-1}^{1} (\cdot - t)_+\,dt - (-(-1))_+ - (-0)_+$$