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1st Semester, 2020-2021

Algebra (Computer Science)

Bonus Exercises : Week 4

Exercise 1. Consider V to be the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and for every $k \in \mathbb{R}$ and $x, y \in V$ we define:

$$x \text{ €} \; y := \arctan\left(\tan(x) + \tan(y)\right)$$

$$k \; \$ \; x := \arctan\left(k \cdot \tan(x)\right),$$

Show that $(V, \text{€})$ is an $\mathbb{R}$-vector space with external operation \$.

Exercise 2. Which of the following sets are $\mathbb{R}$-subspaces of $\mathbb{R}^4$? Justify your answer.

(i) $A = \{(x, y, z, t) \in \mathbb{R}^4 |\ x = 0 \text{ and } z = 0\}$;

(ii) $B = \{(x, y, z, t) \in \mathbb{R}^4 |\ x^2 - y = 5\}$;

(iii) $C = \{(x, y, z, t) \in \mathbb{R}^4 |\ x - y + 2z = 0\}$;

(iv) $D = \{(x, y, z, t) \in \mathbb{R}^4 |\ x + y + 1 = z\}$;

(v) $E = \{(x, y, z, t) \in \mathbb{R}^4 |\ e^{x+y} = e^z \cdot e^y\}$;

(vi) $F = \{(x, y, z, t) \in \mathbb{R}^4 |\ x^2 + y^2 + z^2 + t^2 = 0\}$

<u>Note:</u> To receive points for this exercise, you must discuss all of the sets A, B, C, D, E, F.

Exercise 3. Consider two fields K, L and a field homomorphism

$$\phi : K \to L$$

between them. Show that ϕ must be injective.

Denote by $+$ and $\cdot$ the operations in L. We define for every $k \in K$ and $x \in L$:

$$k \odot x := \phi(k) \cdot x$$

Show that $(L, +)$ is a K-vector space, with external operation $\odot$.

Exercise 4. A **real inner product space** $(V, \langle \cdot, \cdot \rangle)$ consists of an $\mathbb{R}$-vector space V, endowed with a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

that satisfies the properties:

1. $\forall u, v, w \in V,\ \forall a, b \in \mathbb{R} :\ \langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$
2. $\forall u, v \in V$: $\langle u, v \rangle = \langle v, u \rangle$
3. $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \iff u = 0$

The **dot product** (otherwise called the **scalar product**) on $\mathbb{R}^3$ is defined for any $v_1, v_2 \in \mathbb{R}^3$, $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$ by:

$$\langle v_1, v_2 \rangle := x_1 x_2 + y_1 y_2 + z_1 z_2$$

Show that $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ is a real inner product space.

Exercise 5. For every $a \in \mathbb{R}^3$ we define the **translation by** a:

$$\begin{aligned} t_a : \mathbb{R}^3 &\to \mathbb{R}^3 \\ v &\mapsto v + a \end{aligned}$$

Consider the set:

$$\pi = \{(x, y, z) \in \mathbb{R}^3 |\ x + 2y + 3z = 4\}$$

Show that π is not an $\mathbb{R}$-subspace of $\mathbb{R}^3$, but there exists a vector $a \in \mathbb{R}^3$, so that

$$t_a(\pi) = a + \pi := \{a + v |\ v \in \pi\}$$

is an $\mathbb{R}$-subspace of $\mathbb{R}^3$. If you wish, you can also give a geometrical interpretation.

Exercise 6. Let $a, b \in \mathbb{R}$ with $a \neq b$ and consider:

$C([a, b], \mathbb{R}) := \{f : [a, b] \to \mathbb{R} |\ f$ is continuous$\}$

$C^1([a, b], \mathbb{R}) := \{f : [a, b] \to \mathbb{R} |\ f$ is differentiable with continuous derivative$\}$

Let $P, Q \in C([a, b], \mathbb{R})$. We consider a **1st order ordinary linear differential equation**:

$$E :\ y' + P(x) \cdot y = Q(x)$$

We denote the **set of solutions of** E by:

$$S_E := \{y \in C^1([a, b], \mathbb{R}) |\ y' + P(x) \cdot y = Q(x)\}$$

Show that these solutions form an $\mathbb{R}$-subspace of $C^1([a, b], \mathbb{R})$, that is:

$$S_E \leq_{\mathbb{R}} C^1([a, b], \mathbb{R})$$

Exercise 7. We are in the same context as in Exercise 6. Consider the **homogeneous equation associated to** E:

$$E_{hom} :\ y' + P(x) \cdot y = 0$$

Show that if $S_E \neq \emptyset$ (that is, the equation E has solutions), then:

$$S_E = S_{E_{hom}} + y_p = \{y_o + y_p |\ y_o \in S_{E_{hom}}\},$$

where $y_p \in S_E$.

<u>Note:</u> *What this means is that if such a differential equation has solutions, then any one of its solutions can be written as the sum between a solution of the homogeneous equation E_{hom} and a particular solution of E.*

Remember that the unknown in these equations is the function y and it is a function whose argument is x, so instead of y', it would have, perhaps, been more rigorous to write $\frac{dy}{dx}$.

Trivia: *While now is a bit early for you to encounter differential equations, after all, they will be properly studied in the 2nd Semester, in the course **Dynamical Systems**, the Exercises 6 and 7 do not involve anything more than simple Linear Algebra and the fact that differentiation is linear. By this we mean that for any functions f, g and $\alpha \in \mathbb{R}$ we have:*

$$(f+g)' = f' + g'$$
$$(\alpha f)' = \alpha f',$$

which can be combined for $\alpha, \beta \in \mathbb{R}$ into:

$$(\alpha f + \beta g)' = \alpha f' + \beta g'$$