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1st Semester, 2020-2021

Algebra (Computer Science)

Bonus Exercises : Week 5

Exercise 1. Let $V = \{x \in \mathbb{R} |\ x > 0\}$ with the operations:

$$x \perp y = xy$$

$$k \top x = x^k,$$

$\forall k \in \mathbb{R}$ and $\forall x, y \in V$. A few weeks ago, we have shown that $(V, \perp)$ is an $\mathbb{R}$-vector space with external operation $\top$.

Show that V and $\mathbb{R}$ are isomorphic as $\mathbb{R}$-vector spaces.

<u>Note:</u> *Here the vector addition on $\mathbb{R}$ is regular addition and the external operation is regular multiplication. In order to show that two sets are isomorphic as $\mathbb{R}$-vector spaces, it suffices to find a bijective $\mathbb{R}$-linear map between them (an $\mathbb{R}$-isomorphism).*

Exercise 2. A square complex matrix $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{C})$ is called *Hermitian* if it is equal to its *conjugate transpose*:

$$A^H := \overline{A^t} = (\overline{a_{ji}})_{1 \leq i,j \leq n}$$

We know that the set $\mathcal{M}_n(\mathbb{C})$ is a $\mathbb{C}$-vector space, but also an $\mathbb{R}$-vector space.

(i) Show that the set

$$H_n(\mathbb{C}) = \{A \in \mathcal{M}_n(\mathbb{C}) |\ A = A^H\}$$

of Hermitian matrices is an $\mathbb{R}$-subspace of $\mathcal{M}_n(\mathbb{C})$.

(ii) Prove that:

$$H_2(\mathbb{C}) = \langle \sigma_0, \sigma_1, \sigma_2, \sigma_3 \rangle,$$

where

$$\sigma_0 := I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

<u>Trivia:</u> σ_0, σ_x, σ_y and σ_z are called the ***Pauli matrices****.*

Exercise 3. Consider the following subspaces of the real vector space $\mathcal{M}_2(\mathbb{R})$:

(i) $A = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) \mid \ x + y = z + 2t \right\}$

(ii) $B = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) \mid \ x = z + t \text{ and } y = 0 \right\}$

(iii) $C = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) \mid \ x = y = z \right\}$

(iv) $D = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) \mid \ (x - y)^2 + (z + t)^2 = 0 \right\}$

Write A, B, C, D as generated subspaces. Argue whether the generating sets that you have found are minimal.

<u>Note:</u> To receive points for this exercise, you must discuss all of the sets A, B, C and D.

Exercise 4. We consider the following subsets of $\mathcal{M}_2(\mathbb{R})$:

$$S = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

$$T = \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

Show that S and T are $\mathbb{R}$-subspaces of $\mathcal{M}_2(\mathbb{R})$ and $\mathcal{M}_2(\mathbb{R}) = S \oplus T$.

Exercise 5. Consider the function

$$\begin{aligned} f : \mathbb{R}^4 &\to \mathbb{R}^3 \\ (x, y, z, t) &\mapsto (x + y, t, z + 2x) \end{aligned}$$

Show that $f \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^4, \mathbb{R}^3)$ (i.e. f is an $\mathbb{R}$-linear map), determine $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ and write them as generated subspaces.

Exercise 6. Consider the function

$$\begin{aligned} g : \mathbb{R}^3 &\to \mathbb{R}^4 \\ (x, y, z) &\mapsto (x - z, y - 2z, 4x - y - 2z, x - y + z) \end{aligned}$$

Show that $g \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^4)$ (i.e. g is an $\mathbb{R}$-linear map), determine $\mathrm{Ker}(g)$ and $\mathrm{Im}(g)$ and write them as generated subspaces.

Exercise 7. Show that for any field K and $n, m \in \mathbb{N}^*$, the vector spaces K^{nm} and $\mathcal{M}_{n,m}(K)$ are isomorphic.

Exercise 8. Show that for any field K and $n \in \mathbb{N}^*$, the vector spaces K^n and $K_{n-1}[X] = \{P \in K[X] \mid \deg(P) \leq n - 1\}$ are isomorphic.