Seminar W9 – 977

Conics

$\ell : ax + by + c = 0$ line

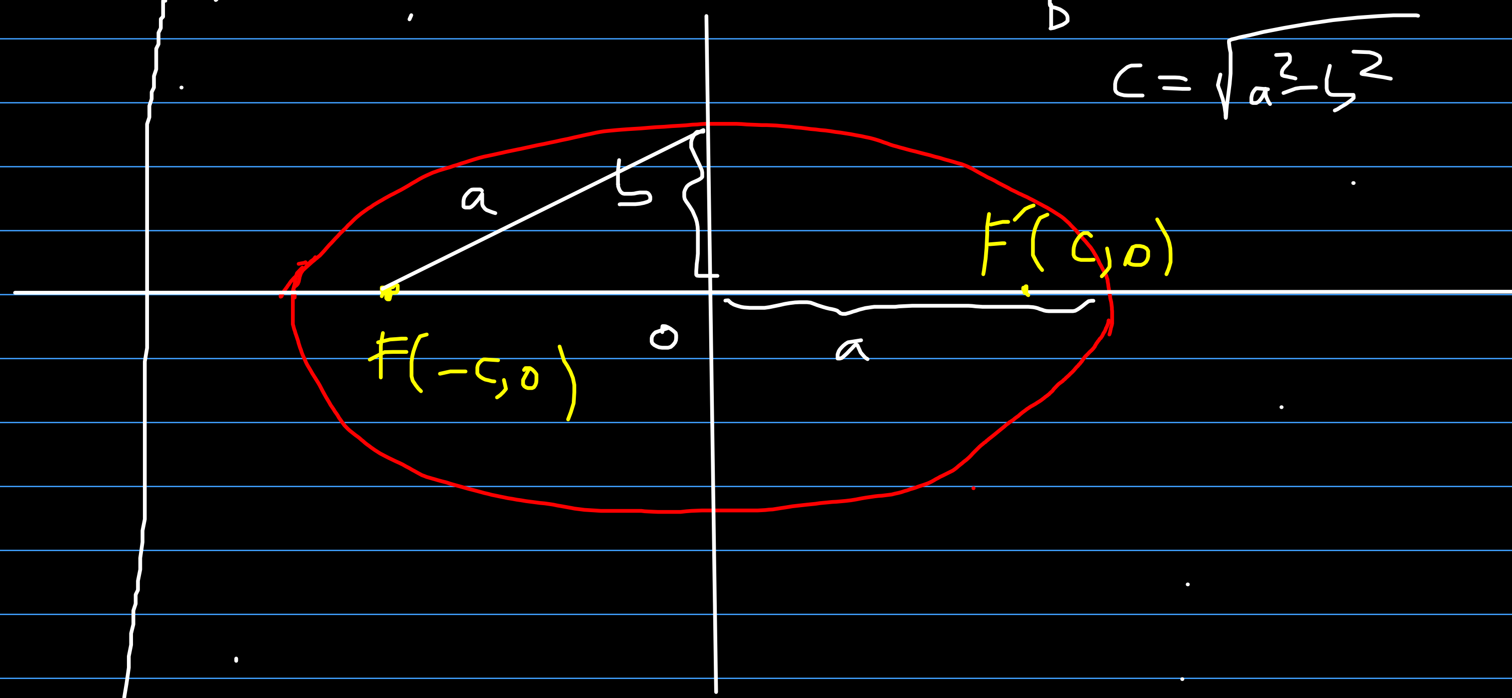
$\mathcal{C} : a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$

$\to$ conics

- ellipse $\to$ "the good"
- hyperbola $\to$ "the bad"
- parabola $\to$ "the ugly"

Ellipse : $\mathcal{E}: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$c=\sqrt{a^2-b^2}$

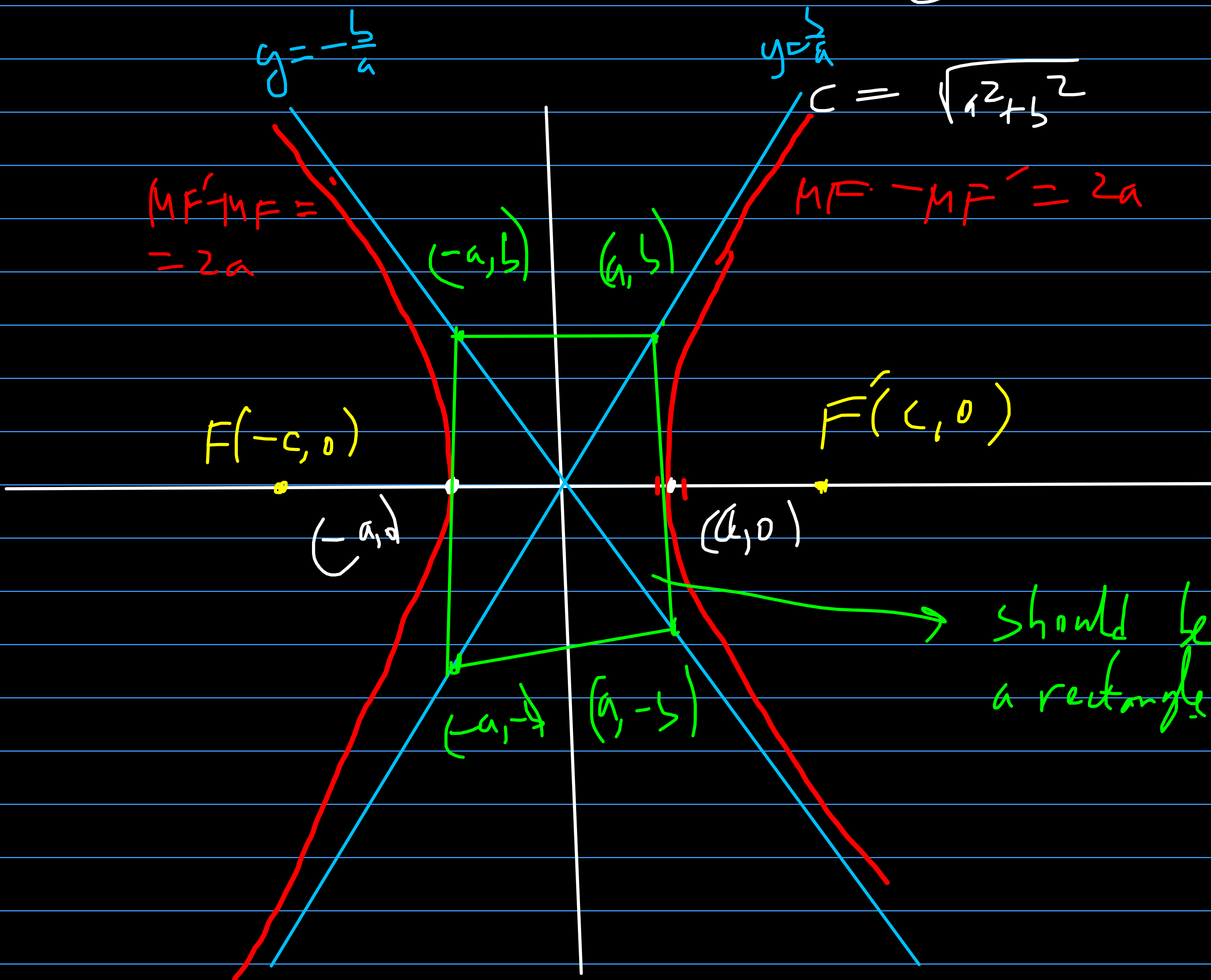


→ locus of points M in the plane so that $MF + MF' = 2a$, where F and F' are two fixed points called foci (pl. focus)

$T_{\mathcal{E}}(x_0,y_0): \dfrac{xx_0}{a^2}+\dfrac{yy_0}{b^2}=1$

Hyperbola :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

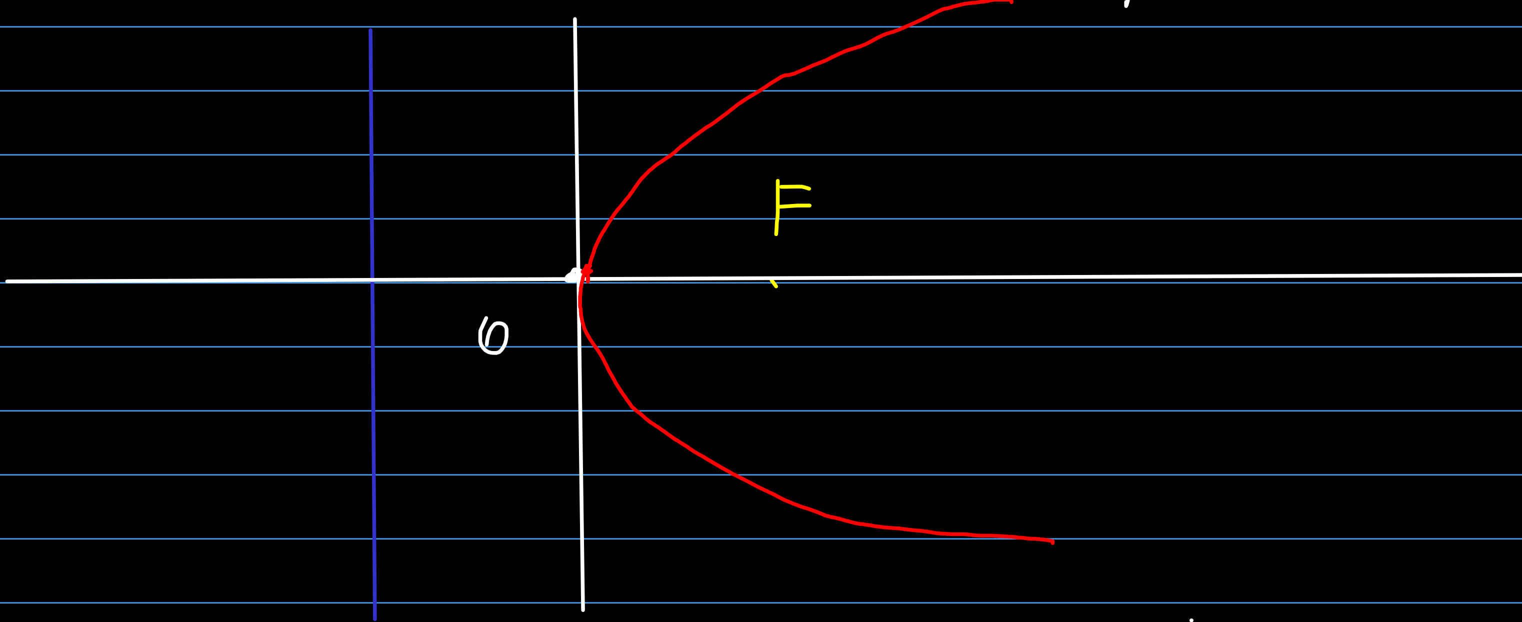


→ the locus of points M in the plane so that $|MF - MF'| = 2a$ where F and F' are fixed points called the foci of the hyperbola

$\mathcal{H}$ has oblical asymptotes $y = \pm \frac{b}{a}$

$$T_{\mathcal{H}}(x_0, y_0) : \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$$

<u>Parabola</u> $\mathcal{P}: y^2 = 2px$



→ locus of points M in the plane that are equidistant to a line d called the director line (or directrix) and a point F called the focus

$T_P(x_0, y_0): \quad y y_0 = p(x + x_0)$

63. Find the equations of the tangent lines to the ellipse $\mathcal{E}: \frac{x^2}{25} + \frac{y^2}{16} - 1 = 0$, passing through $P_0(10, -8)$

$a = 5, \ b = 4$

$$T_{\mathcal{E}}(x_0, y_0): \quad \frac{x x_0}{25} + \frac{y y_0}{16} = 1$$

$$P_0 \in T_{\mathcal{E}}(x_0, y_0): \quad \frac{10 x_0}{25} + \frac{-8 y_0}{16} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{2x_0}{5} - \frac{y_0}{2} = 1 \Leftrightarrow 4x_0 - 5y_0 = 10$$

We also know that $\frac{x_0^2}{25} + \frac{y_0^2}{16} = 1$

and $y_0 = \frac{4x_0 - 10}{5}$

$$\Rightarrow \frac{x_0^2}{25} + \frac{16x_0^2 - 80x_0 + 100}{16 \cdot 25} = 1$$

$$\Rightarrow 32x_0^2 - 80x_0 + 100 - 400 = 0$$

$$\Rightarrow 8x_0^2 - 20x_0 - 75 = 0$$

$$\Rightarrow (x_0)_{1,2} = \frac{20 \pm \sqrt{400 + 2400}}{16} =$$

$$= \frac{20 \pm 10\sqrt{28}}{16} = \frac{20 \pm 20\sqrt{7}}{16}$$

$$\Rightarrow x_0 = \frac{5}{4} \cdot (1 \pm \sqrt{7})$$

$$\Rightarrow y_0 = 1 \pm \sqrt{7} - 2 = -1 \pm \sqrt{7}$$

Therefore the tangents are:

$$T_{\xi}\left(\frac{5}{4}(1+\sqrt{7}), -1+\sqrt{7}\right)$$

$$T_{\xi}\left(\frac{5}{4}(1-\sqrt{7}), -1-\sqrt{7}\right)$$

9.8. Find the equation of the tangent line to the parabola $\mathcal{P}: y^2 - 8x = 0$, parallel to the line $d: 2x + 2y - 3 = 0$.

$$\mathcal{P}: y^2 = 8x \Rightarrow p = 4$$

$$T_{\mathcal{P}}(x_0, y_0): \quad y y_0 = 4(x + x_0)$$

$$\text{if } y_0 \neq 0 \Rightarrow m_{T_{\mathcal{P}}(x_0, y_0)} = \frac{4}{y_0}$$

$$d: 2x + 2y - 3 = 0$$

$$m_d = -1$$

$$T_{\mathcal{P}}(x_0, y_0) \parallel d \Rightarrow \frac{4}{y_0} = -1 \Rightarrow$$

$$\Rightarrow y_0 = -4$$

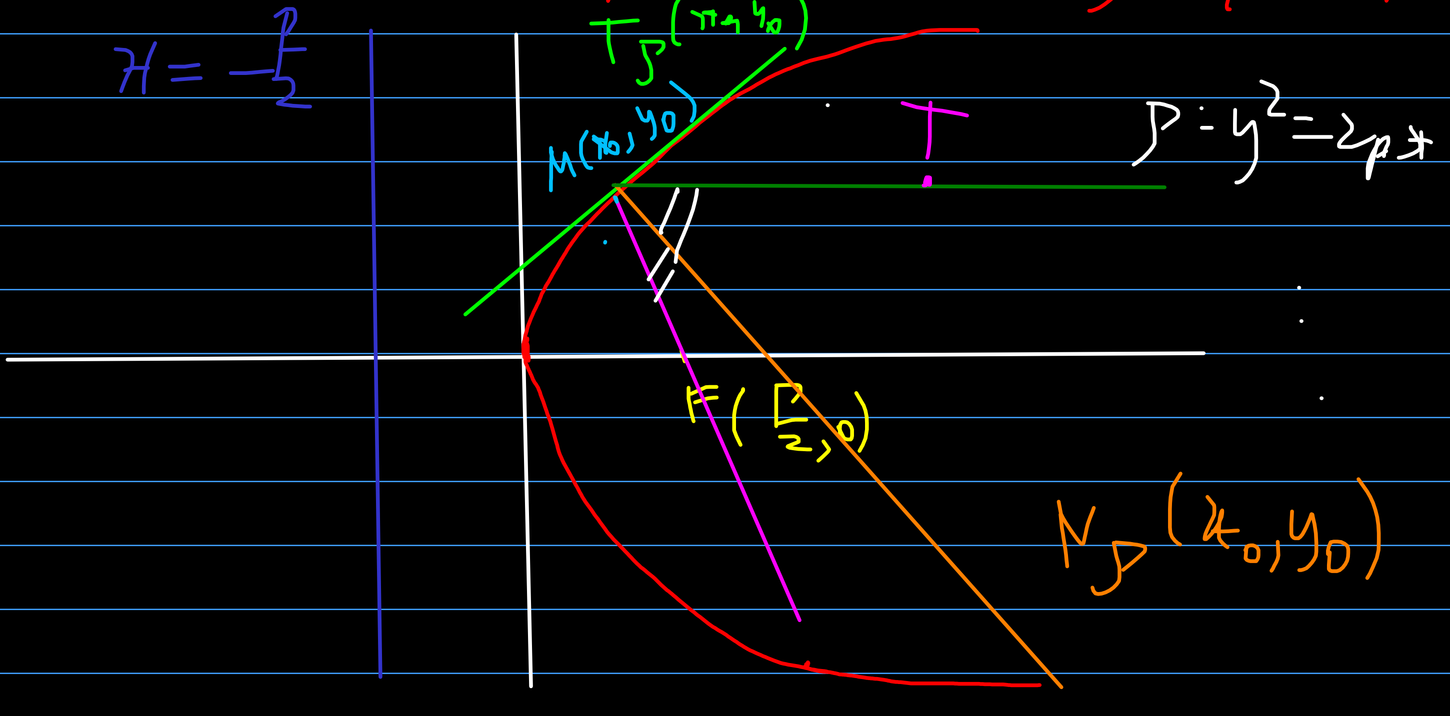
Because $y_0^2 = 8x_0 \Rightarrow x_0 = \dfrac{y_0^2}{8} = 2$

$\Rightarrow y_0 = -4$

$T_{\mathcal{P}}(2,-4):\quad -4y = 4(x+2)$

$\Leftrightarrow x + y + 2 = 0$

9.14. Show that a ray of light through the focus of a parabola reflects to a ray parallel to the axis of the parabola (the optical property of the parabola)



This is equivalent to showing that for any $M(x_0, y_0) \in \mathcal{P}$ the normal line to $\mathcal{P}$ in M is the bisector of the angle $\widehat{FMT}$, where T is on the parallel through M to Ox

Therefore, if we show that $\forall A \in N_{\mathcal{P}}$

$$dist(A, MF) = dist(A, MT)$$

$M(x_0, y_0)$, $T(x_T, y_0)$, $F(\frac{p}{2}, 0)$
$A(x_A, y_A)$

$$MF: \quad \frac{x - x_0}{\frac{p}{2} - x_0} = \frac{y - y_0}{0 - y_0}$$

$$MF: \quad -y_0 x - \left(\frac{p}{2} - x_0\right)(y - y_0) + x_0 y_0 = 0$$

$MT: \quad y = y_0$

$N_J(x_0, y_0): \quad \frac{x - x_0}{-2p} = \frac{y - y_0}{2y_0}$

I had gotten this sign wrong initially

$\Rightarrow A \in N_J \Rightarrow A: \begin{cases} x_A = x_0 - 2p\cdot\lambda \\ y_A = y_0 + 2y_0\lambda \end{cases}$

$$\mathrm{dist}(A, MF) = \frac{\left| -y_0 x_A - \left(\frac{p}{2} - x_0\right)(y_A - y_0) + x_0 y_0 \right|}{\sqrt{y_0^2 + \left(\frac{p}{2} - x_0\right)^2}} =$$

$$= \frac{\left| -y_0\cdot(x_0 - 2p\lambda) - \left(\frac{p}{2} - x_0\right)\cdot 2y_0\lambda + x_0 y_0 \right|}{\sqrt{y_0^2 + \left(\frac{p}{2} - x_0\right)^2}}$$

$$= \frac{\left| -x_0 y_0 + 2p y_0 \lambda - y_0 \lambda p + 2x_0 y_0 \lambda + x_0 y_0 \right|}{}$$

$$= \frac{|2x_0 y_0 \lambda + p y_0 \lambda|}{\sqrt{y_0^2 + \left(\frac{p}{2} - x_0\right)^2}} = \frac{|y_0| \cdot |\lambda| \cdot |2x_0 + p|}{\sqrt{y_0^2 + \left(\frac{p}{2} - x_0\right)^2}}$$

$$\text{dist}(A, MT) = \frac{|y_A - y_0|}{1} =$$

$$= 2|y_0| \cdot |\lambda|$$

We need to show that:

$$\frac{|y_0| \cdot |\lambda| \cdot |2x_0 + p|}{\sqrt{y_0^2 + \left(\frac{p}{2} - x_0\right)^2}} = 2|y_0| \cdot |\lambda|$$

We need to show that

$$y_0^2 + \left(\frac{p}{2} - x_0\right)^2 = \frac{(2x_0 + p)^2}{4}$$

$$y_0^2 + \left(\frac{p}{2} - x_0\right)^2 = 2px_0 + \frac{p^2}{4} - px_0 +$$

$$+ x_0^2 = \frac{p^2}{4} + px_0 + x_0^2 =$$

$$= \frac{4x_0^2 + 4px_0 + p^2}{4} = \frac{(2x_0 + p)^2}{4}$$