Seminar W9 – 916

Conics

$\ell : ax + by + c = 0$ line

$\mathcal{C} : a_{11}x^2 + 2a_{12}xy + a_{22}y^2 +$

$+ 2a_{10}x + 2a_{01}y + a_{00} = 0$

$\hookrightarrow$ conics (nondegenerate)

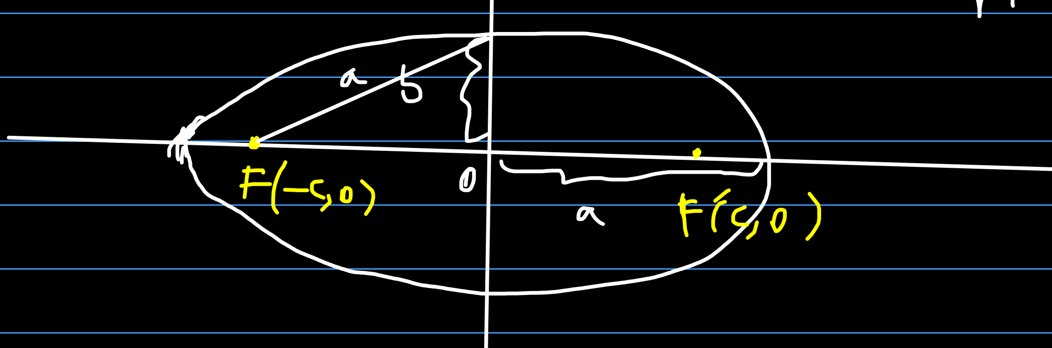
ellipse — hyperbola — parabolas.

"the good" "the bad" "the ugly"

Ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

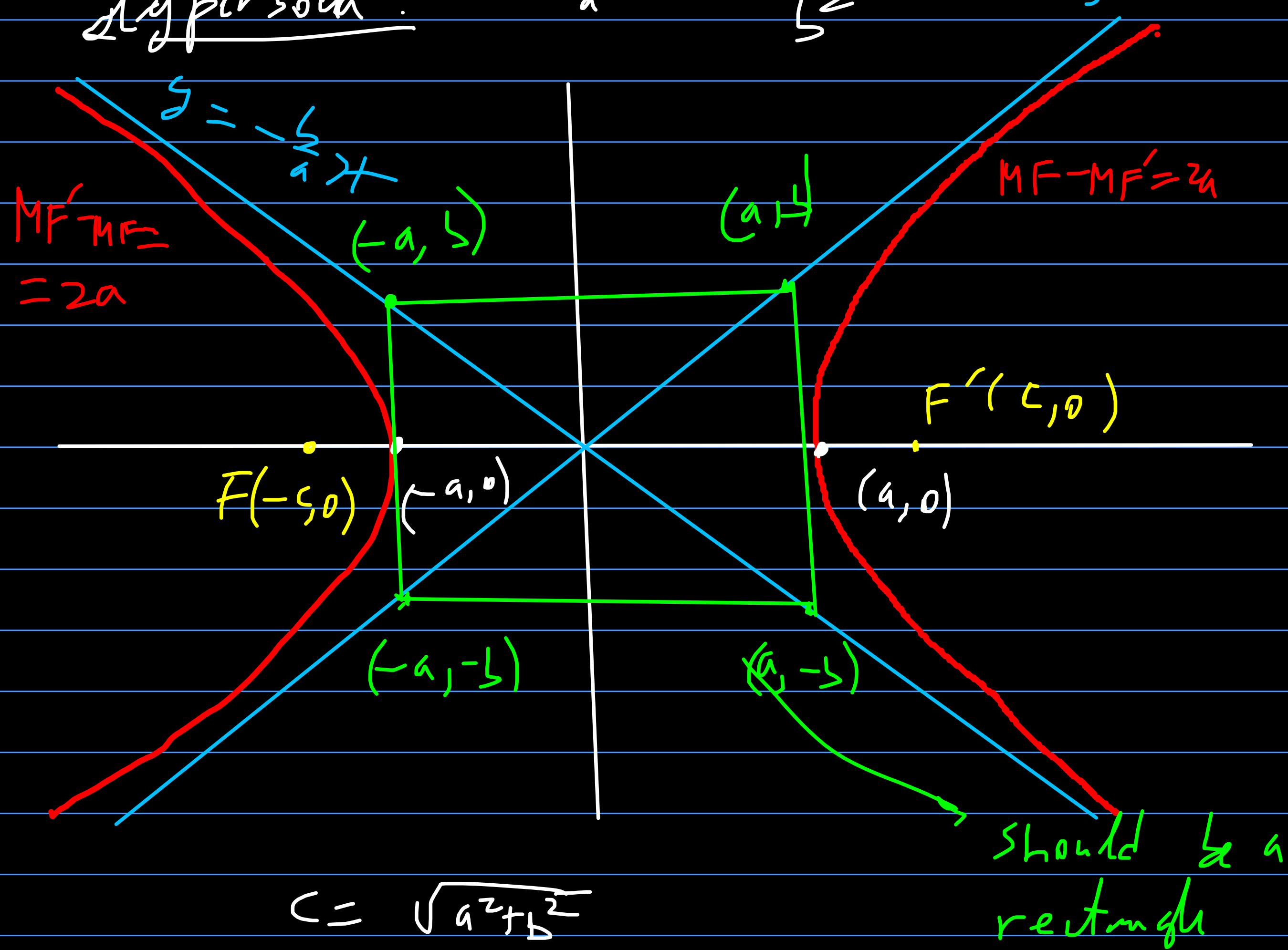
$$c = \sqrt{a^2 - b^2}$$



→ locus of points M in the plane so that $MF + MF' = 2a$, where F and F' are fixed points called the foci of the ellipse.

$$T_{\mathcal{E}}(x_0, y_0) : \frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1$$

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\quad y = \frac{b}{a}x$



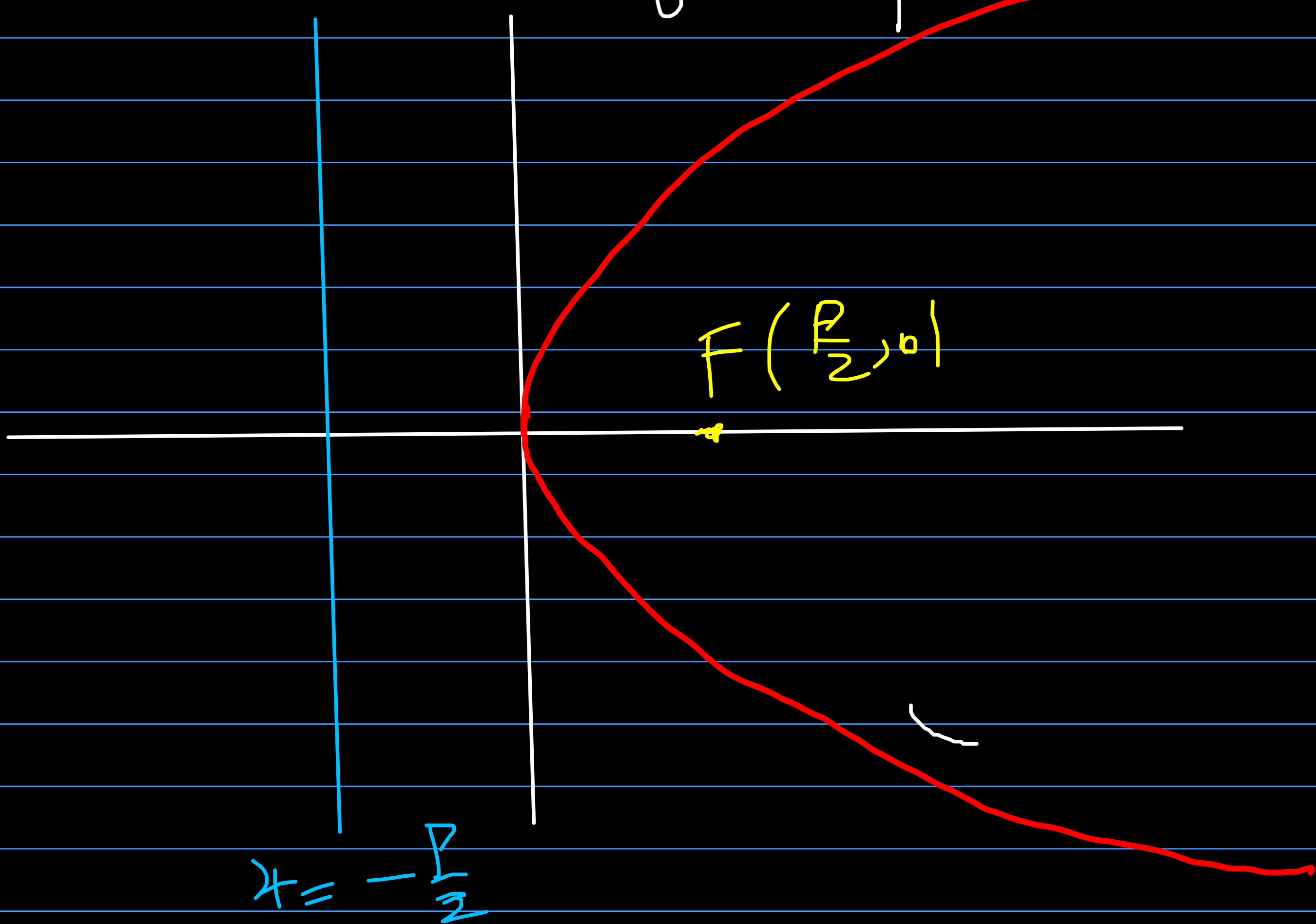
$c = \sqrt{a^2 + b^2}$

→ locus of points M in the plane for which $|MF - MF'| = 2a$ where F and F' are two fixed points called the foci of the hyperbola.

→ the hyperbola has the oblical asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$

$$T_{\mathcal{H}}(x_0, y_0) : \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1 .$$

Parabola $\quad \mathcal{P}: y^2 = 2px$



$\rightarrow$ locus of points M in the plane that are equidistant to a line d called the director line (directrix) and a point F, called the focus.

$T_{\mathcal{P}}(x_0, y_0): \quad y y_0 = p(x + x_0)$

9.6. Find the equations of the tangent lines to the hyperbola

$$\mathcal{H}: \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$$

which are orthogonal to the line

$$d: 4x + 3y - 7 = 0$$

$$T_{\mathcal{H}}(x_0, y_0): \frac{x x_0}{20} - \frac{y y_0}{5} - 1 = 0$$

$$\frac{y y_0}{5} = \frac{x x_0}{20} - 1$$

Assume $y_0 \neq 0$: $y = \frac{x_0}{4 y_0} \cdot x - \frac{5}{y_0}$

$$m_{T_{\mathcal{H}}(x_0, y_0)} = \frac{x_0}{4 y_0}$$

$$d: y = -\frac{4}{3} x + \frac{7}{3}$$

$\Rightarrow m_d = -\frac{4}{3}$

$$T_{\mathcal{H}}(x_0, y_0) \perp d \Rightarrow m_{T_{\mathcal{H}}(x_0, y_0)} \cdot m_d = -1$$

$$\Rightarrow \frac{x_0}{4y_0} \cdot \left(-\frac{4}{3}\right) = -1$$

$$\Rightarrow \frac{x_0}{y_0} = 3 \Rightarrow x_0 = 3y_0$$

We know that $(x_0, y_0) \in \mathcal{H}$

$$\Rightarrow \frac{x_0^2}{20} - \frac{y_0^2}{5} - 1 = 0 \Rightarrow$$

$$\Rightarrow \frac{9y_0^2}{20} - \frac{y_0^2}{5} = 1 \Rightarrow$$

$$\Rightarrow y_0^2\left(\frac{9}{20} - \frac{1}{5}\right) = 1 \Rightarrow$$

$$\Rightarrow y_0^2 = 4 \Rightarrow y_0 = \pm 2$$

$$\Rightarrow x_0 = \pm 6$$

Therefore the tangents that we want are:

$$T_{\mathcal{H}}(6,2): \frac{6x}{20} - \frac{2y}{5} - 1 = 0$$

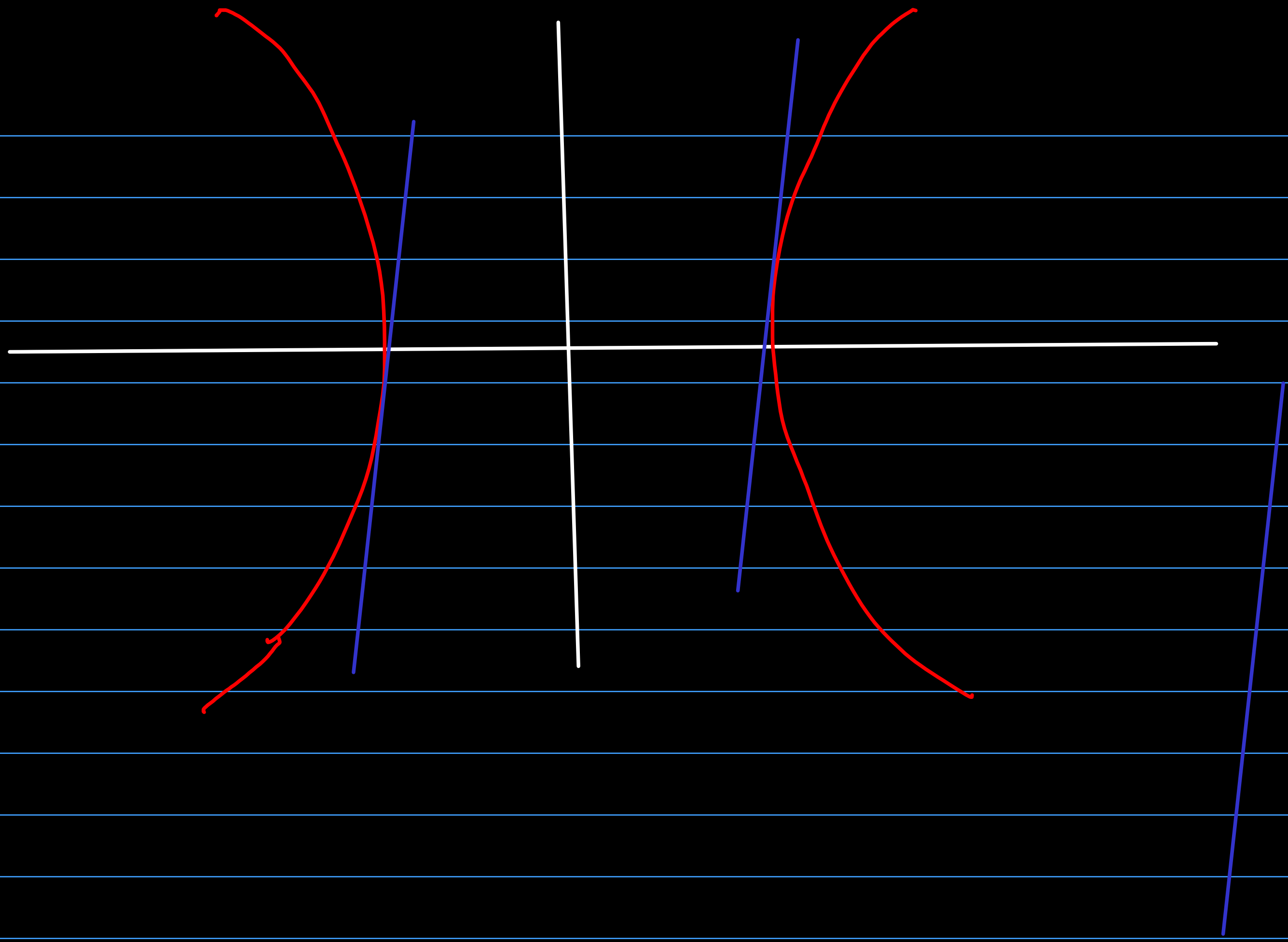
$$T_{\mathcal{H}}(-6,-2): \frac{-6x}{20} + \frac{2y}{5} - 1 = 0$$

$$y/\ y_0 = 0 \Rightarrow T_{\mathcal{H}}(x_0, y_0): \frac{x x_0}{20} - 1 = 0$$

$$\Leftrightarrow T_{\mathcal{H}}(x_0, 0): x = \frac{20}{x_0}$$

$$\Rightarrow m_{T_{\mathcal{H}}(x_0, 0)} = \infty$$

does not work, because it isn't perpendicular to d.



e.g. Find the equation of the tangent line to the parabola $\mathcal{P}: y^2 - 36x = 0$, passing through the point $P(2, 9)$.

$\mathcal{P}: \ y^2 = 36x \implies 2p = 36 \implies p = 18$

$\implies T_{\mathcal{P}}(x_0, y_0): \ yy_0 = 18(x + x_0)$

$$\Leftrightarrow \quad y = \frac{18(x + x_0)}{y_0}, \text{ if } y_0 \neq 0$$

$$\Rightarrow \quad (x_0, y_0) \in \mathcal{P} \Rightarrow y_0^2 = 36 x_0$$

$$P \in T_{\mathcal{P}}(x_0, y_0) \Rightarrow 9 y_0 = 18(x_0 + 2) \Rightarrow$$

$$\Rightarrow y_0 = 2(x_0 + 2)$$

$$y_0^2 = 36 x_0 \Rightarrow 4(x_0 + 2)^2 = 36 x_0 \Rightarrow$$

$$\Rightarrow x_0^2 + 4x_0 + 4 = 9x_0 \Rightarrow$$

$$\Rightarrow x_0^2 - 5x_0 + 4 = 0$$

$$(x_0)_{1,2} = \frac{5 \pm \sqrt{25 - 16}}{2} = \frac{5 \pm 3}{2}$$

$$\Rightarrow x_0 = 4 \text{ or } x_0 = 1$$

$$\Rightarrow y_0 = 12 \text{ or } y_0 = 6$$

$$\Rightarrow T_{\mathcal{P}}(4, 12): 12y = 18(x + 4)$$

$$T_{\mathcal{P}}(1, 6): 6y = 18(x + 1)$$

9.3. Find the equations of the tangent lines to the ellipse

$$\mathcal{E}: \frac{x^2}{25} + \frac{y^2}{16} - 1 = 0,$$

passing through $P_0(10, -8)$

Ex.: $x^2 + 4y^2 = 7$

$$\frac{x^2}{7} + \frac{4}{7}y^2 = 1$$

$$\frac{x^2}{7} + \frac{y^2}{\frac{7}{4}} = 1$$

$$\Longrightarrow a = \sqrt{7}, \quad b = \frac{\sqrt{7}}{2}$$

$$c = \frac{\sqrt{7}}{2}$$

We write the general equation of a line that contains $P_0(10, -8)$

$$\ell : ax + by + c = 0$$

$$P_0 \in \ell \Rightarrow 10a - 8b + c = 0 \Rightarrow$$

$$\Rightarrow c = 8b - 10a$$

$$\Rightarrow \ell : a(x-10) + b(y+8) = 0$$

$$\Leftrightarrow \ell : y+8 = \underbrace{-\frac{a}{b}}_{=: m}(x-10)$$

$$\Rightarrow \ell_m : y+8 = m(x-10)$$

ℓ_m is tangent to $\mathcal{E} \Leftrightarrow |\ell_m \cap \mathcal{E}| = 1$

$$l_m \cap \mathcal{E} : \begin{cases} \frac{x^2}{25} + \frac{y^2}{16} - 1 = 0 \\ y + 8 = m(x - 10) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \frac{x^2}{25} + \frac{(mx - 10m - 8)^2}{16} - 1 = 0 \\ y + 8 = m(x - 10) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 16x^2 + 25 \cdot \overbrace{(mx - 10m - 8)^2}^{m^2x^2 - 2m(10m+8) \cdot x + (10m+8)^2} - 400 = 0 \\ y + 8 = m(x - 10) \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2(16 + 25m^2) + x(-500m^2 - 400m) + 25(10m+8)^2 - 400 = 0 \\ y + 8 = m(x - 10) \end{cases}$$

ℓ_m tangent to $\mathcal{E}$ $\Leftrightarrow$ $|\ell_m \cap \mathcal{E}| = 1$ $\Leftrightarrow$

$\Leftrightarrow$ The equation E has one solution

$$E: x^2(16+25m^2) + x(-500m^2 - 400m) + 25(10m+8)^2 - 400 = 0$$

$$\Leftrightarrow \quad \Delta_E = 0$$

$$\Delta_E = 10^4 \cdot (5m^2+4m)^2 - 4\cdot(16+25m^2)\cdot(25\cdot(10m+8)^2 - 400)$$

$$\Rightarrow m = \pm \frac{4}{15}\cdot\sqrt{7} - \frac{16}{15}$$