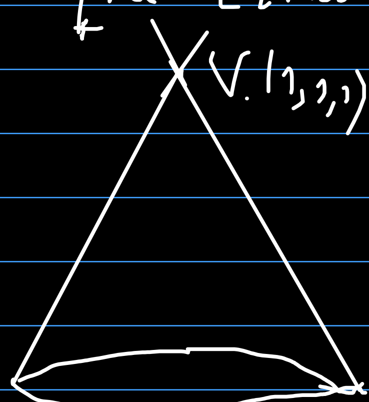
Seminar W11 - 915

Generated surfaces

- Ruled surface
  - conical surface
  - cylindrical — " —
  - conoidal — " —
- Revolution surface

Example 11.2: Determine the equation of the conical surface having the vertex $V(1,1,1)$ and the director curve $\mathcal{C}: \underbrace{\begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases}}_{\text{director curve}}$



Step 1: Write the most general equation possible of a line that satisfies the 1st condition

(in our case: contains V)

generatrix

$$d_{\lambda,\mu} : \frac{x-1}{a} = \frac{y-1}{b} = \frac{z-1}{c} \iff$$

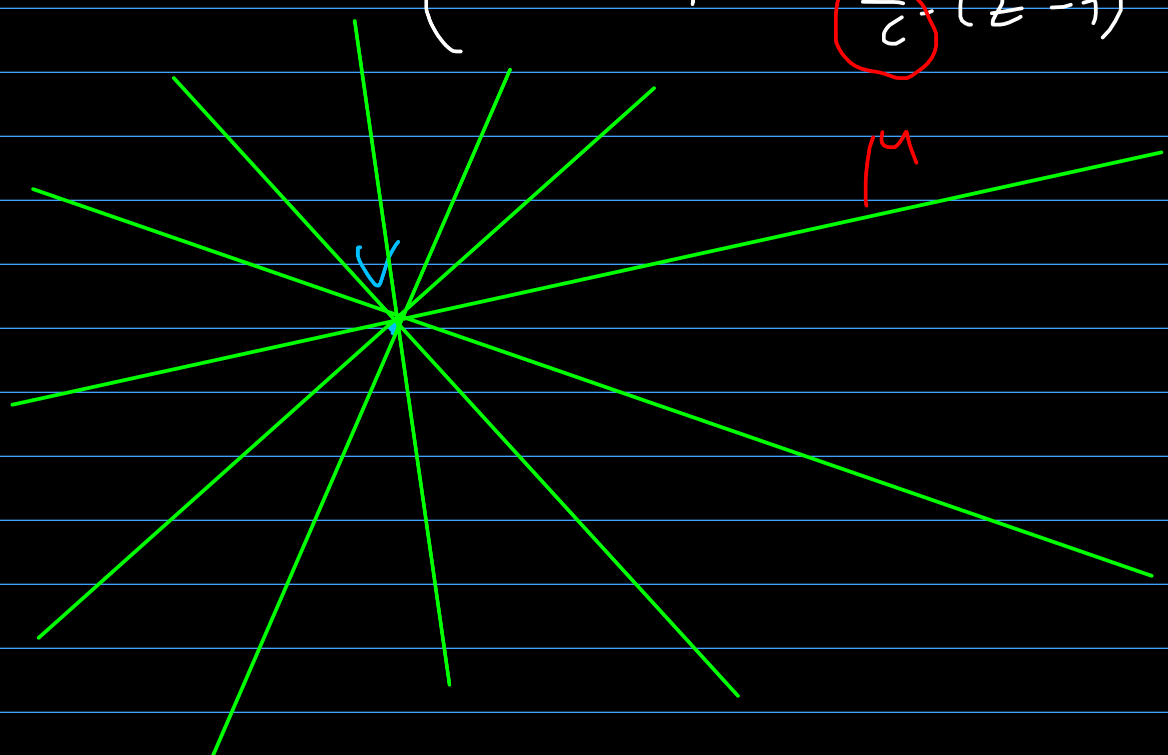
$(a, b, c \neq 0)$

$$\iff d_{\lambda,\mu} : \begin{cases} b(x-1) - a\cdot(y-1) = 0 \\ c\cdot(x-1) - a\cdot(z-1) = 0 \end{cases} \iff$$

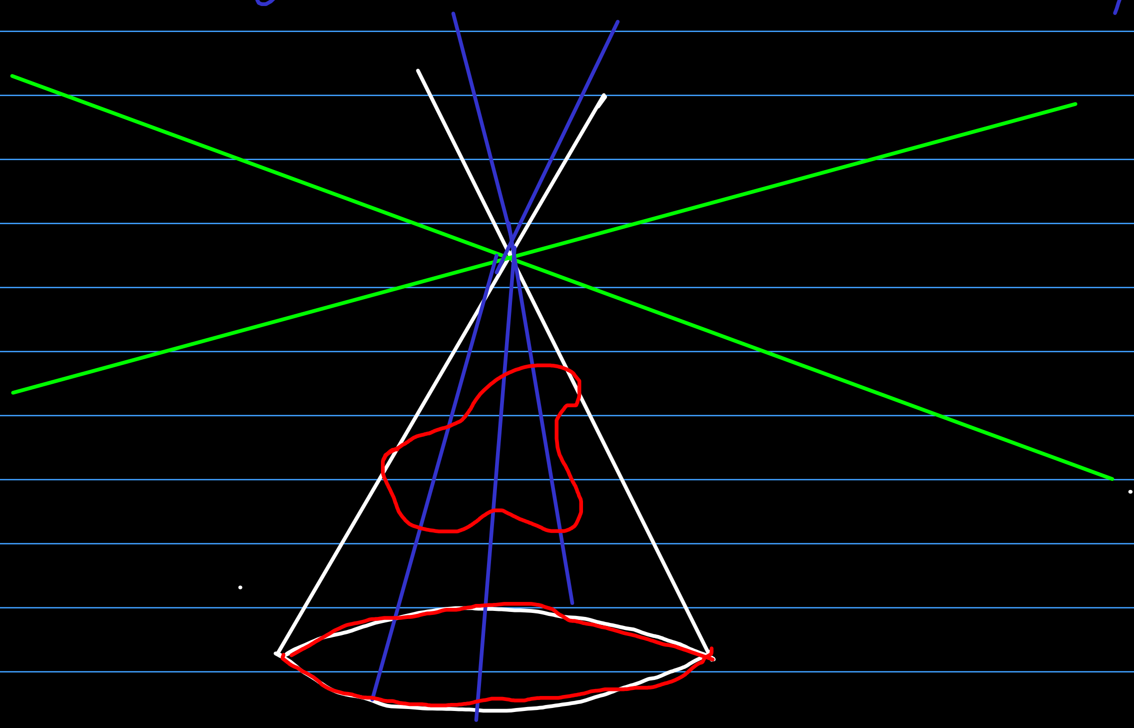
$$\iff d_{\lambda,\mu} : \begin{cases} x-1 = \frac{a}{b}\cdot(y-1) \\ x-1 = \frac{a}{c}\cdot(z-1) \end{cases}$$

$:= \lambda$

μ



Step 2: Choose only those lines among the $d_{\lambda,\mu}$ that satisfy the 2^{nd} condition (always the same: that $d_{\lambda,\mu} \cap \mathcal{C} \neq \emptyset$)



In order to pick the values of λ and μ that give us the lines $d_{\lambda,\mu}$ that intersect the directrix curve $\mathcal{C}$, we will solve the system:

$$\begin{cases} d_{\lambda,\mu} : \begin{cases} x-1 = \lambda(y-1) \\ x-1 = \mu(z-1) \end{cases} \\ \mathcal{C} : \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \end{cases}$$

We eliminate x, y, z from the system, in order to obtain a relation between λ and μ, called the compatibility condition.

$$\begin{cases} x-1 = \lambda(y-1) \\ x-1 = \mu\cdot(z-1) \\ (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} z = 0 \\ x-1 = \lambda\cdot(y-1) \\ x-1 = -\mu \\ (x^2+y^2)^2 - xy = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = 0 \\ x = 1-\mu \\ -\mu = \lambda\cdot(y-1) \\ (x^2+y^2)^2 - xy = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = 0 \\ x = 1 - \mu \\ y = 1 - \dfrac{\mu}{\lambda} \\ (x^2+y^2)^2 - xy = 0 \end{cases}$$

$\Rightarrow$ the compatibility condition is

$$\left((1-\mu)^2 + \left(1-\frac{\mu}{\lambda}\right)^2\right)^2 - (1-\mu)\cdot\left(1-\frac{\mu}{\lambda}\right) = 0$$

Step 3: Replace the values of λ and μ in terms of x, y, z (using the formulas from when λ and μ were introduced)

$$\lambda = \frac{x-1}{y-1} \qquad \mu = \frac{x-1}{z-1}$$

$$1-\mu = \frac{z-x}{z-1} \qquad 1 - \frac{\mu}{\lambda} = 1 - \frac{\frac{y-1}{z-1}}{} = \frac{z-y}{z-1}$$

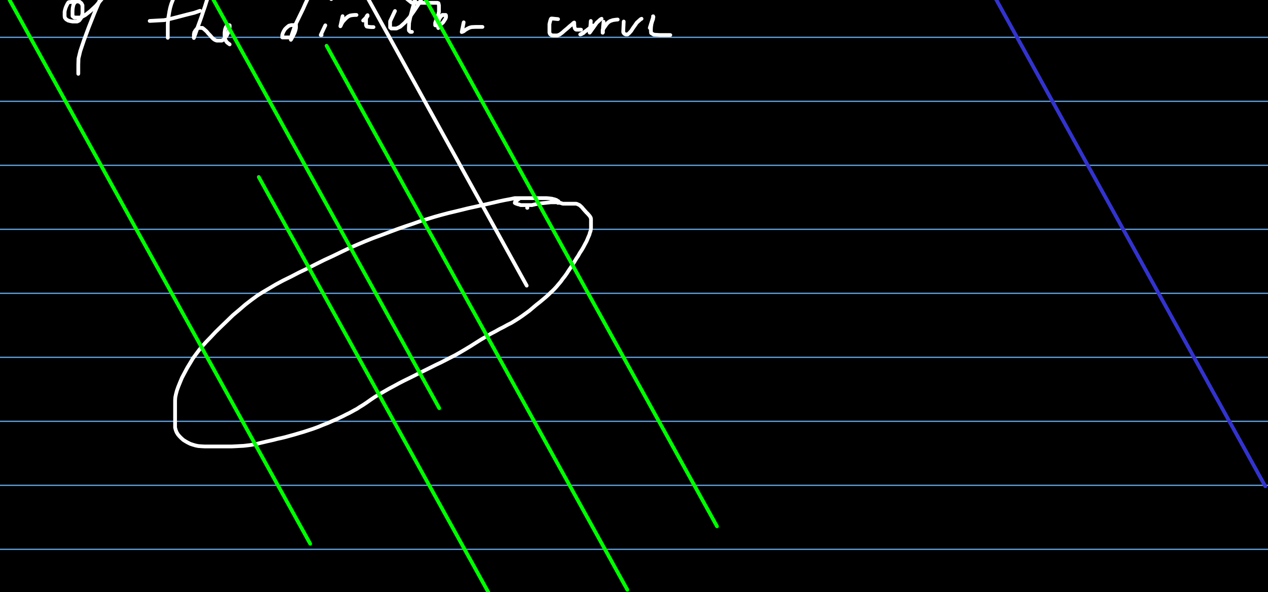
Therefore the equation of the conical surface is:

$$\left(\left(\frac{z-x}{z-1}\right)^2+\left(\frac{z-y}{z-1}\right)^2\right)^2-\frac{(z-x)(z-y)}{(z-1)^2}=0$$

11.1. Find the equation of the cylindrical surface whose director curve is the planar curve

$$(\mathcal{C}):\begin{cases} y^2+z^2=x \\ x=2z \end{cases}$$

and the generatrix is perpendicular to the plane of the director curve



The plan of $\mathcal{C}$ is $\Pi : x = 2z$

(because $\mathcal{C} \subset \Pi$ and $\mathcal{C}$ is planar)

$$d_{\lambda,\mu} \perp \Pi \Rightarrow d_{\lambda,\mu} : \begin{cases} \dfrac{x-a}{1} = \dfrac{z-c}{-2} \\ y = b \end{cases} \Leftrightarrow$$

$$\Leftrightarrow d_{\lambda,\mu} : \begin{cases} -2x - z = \overbrace{-2a - c}^{\lambda} \\ y = \underbrace{b}_{\mu} \end{cases}$$

$$\Leftrightarrow d_{\lambda,\mu} : \begin{cases} -2x - z = \lambda \\ y = \mu \end{cases}$$

$$\begin{cases} -2x - z = \lambda \\ y = \mu \\ y^2 + z^2 = x \\ x = 2z \end{cases} \Leftrightarrow \begin{cases} x = 2z \\ y = \mu \\ -5z = \lambda \\ \mu^2 + z^2 = 2z \end{cases} \Leftrightarrow$$

(=) $\begin{cases} x = 2z \\ y = \mu \\ z = -\dfrac{\lambda}{5} \\ \mu^2 + \dfrac{\lambda^2}{25} = -\dfrac{2\lambda}{5} \end{cases}$

$\Rightarrow$ comp. condition:

$$\mu^2 + \frac{\lambda^2}{25} = -\frac{2\lambda}{5}$$

$\Rightarrow$ the eqn. of the surface:

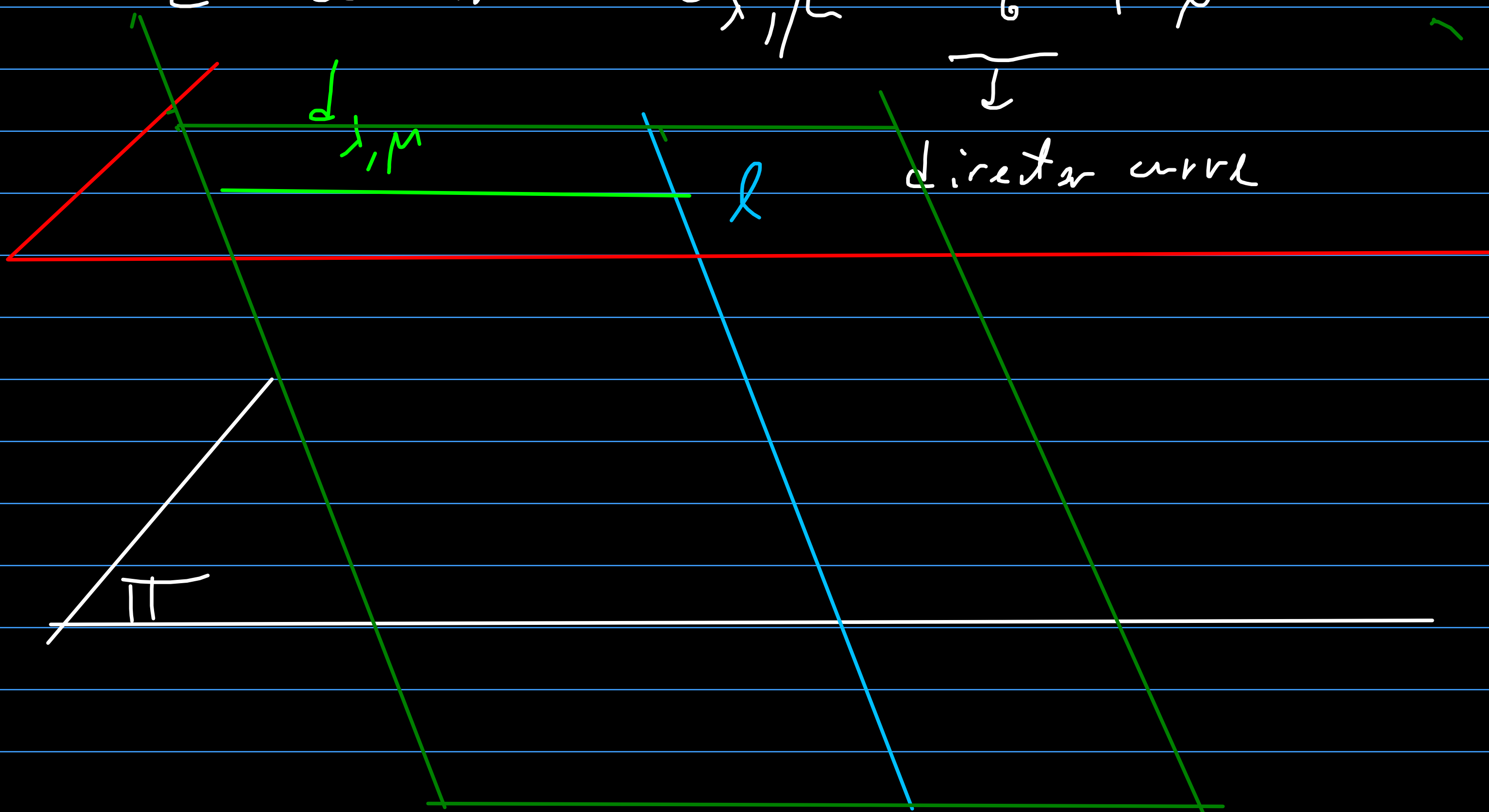
$\lambda = -2x - z$, $\mu = y$

$\Rightarrow$ $\mathcal{Y}$: $y^2 + \dfrac{(-2x-z)^2}{25} = \dfrac{4x+2z}{5}$

Conoidal surfaces

1st condition : $(d_{\lambda,\mu} \parallel \Pi)$ and $(d_{\lambda,\mu} \cap \ell \neq 0)$

2nd condition : $d_{\lambda,\mu} \cap \underbrace{\mathcal{C}}_{\text{director curve}} \neq \emptyset$



$$d_{\lambda,\mu} : \begin{cases} \text{a plane parallel to } \Pi \\ \text{a plane that contains } \ell \end{cases}$$

$\Pi : Ax + By + Cz + D = 0$

$$\ell: \begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$

$$d_{\lambda,\mu}: \begin{cases} Ax + By + Cz + D = \lambda \\ A_1 x + B_1 y + C_1 z + D_1 = \mu \cdot (A_2 x + B_2 y + C_2 z + D_2) \end{cases}$$

<u>Example 11.3</u> Find the conoidal surface, whose generatrices are parallel to xOy and intersect Oz and have the director curve: $\mathcal{C}: \begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}$

$\Pi = xOy: \quad z = 0$

$\ell = Oz: \begin{cases} x = 0 \\ y = 0 \end{cases}$

$d_{\lambda,\mu}: \begin{cases} z = \lambda \\ x = \mu \cdot y \end{cases}$

$$\begin{cases} z = \lambda \\ x = \mu y \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} z = \lambda \\ x = \mu y \\ y^2 - 2\lambda + 2 = 0 \\ \mu^2 y^2 - 2\lambda + 1 = 0 \end{cases} \Leftrightarrow$$

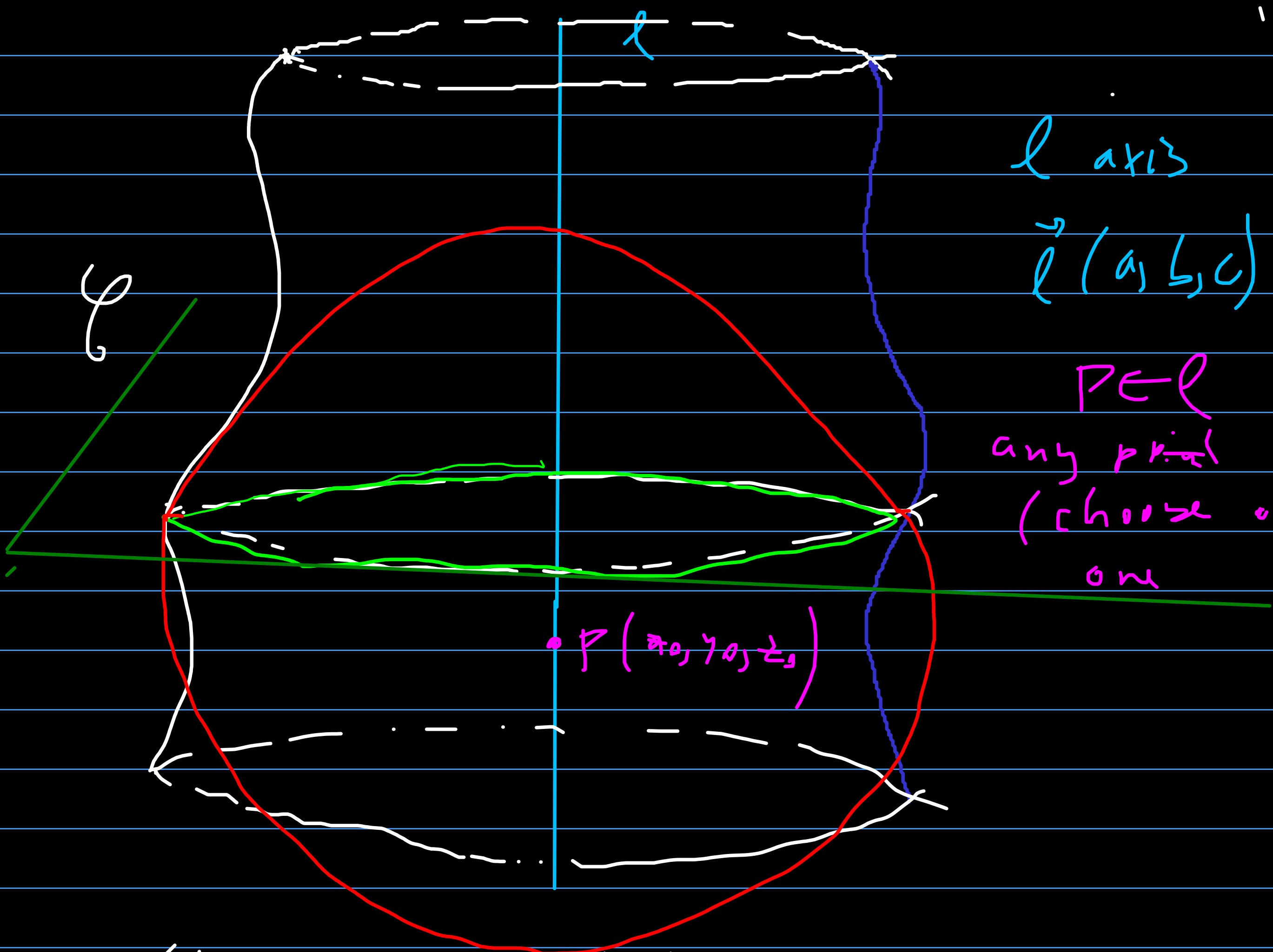
$$\Leftrightarrow \begin{cases} z = \lambda \\ x = \mu y \\ y^2 = 2\lambda - 2 \\ \mu^2 \cdot (2\lambda - 2) - 2\lambda + 1 = 0 \end{cases}$$

$$\Rightarrow \mu^2 \cdot (2\lambda - 2) - 2\lambda + 1 = 0$$

$$\Rightarrow \left(\frac{x}{y}\right)^2 \cdot (2z - 2) - 2z + 1 = 0$$

$$\Rightarrow x^2 (2z - 2) - 2zy^2 + y^2 = 0$$

Revolution surfaces



ℓ axis

$\vec{\ell}(a, b, c)$

$P \in \ell$ any point (choose a simple one)

Step 1: We don't have generatrices (generating lines), but rather generating circles

$C_{\lambda,\mu}$: $\begin{cases} \text{sphere that is centered in } P \\ \text{plane that is perpendicular to } \ell \end{cases}$

$$\begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ ax + by + cz = \mu \end{cases}$$

This was step 1

Step 2: We find the compatibility condition between λ and μ by solving the system.

$$\begin{cases} (x-x_0)^2+(y-y_0)^2+(z-z_0)^2=\lambda \\ ax+by+cz=\mu \\ \varphi: \begin{cases} f_1(x,y,z)=0 \\ f_2(x,y,z)=0 \end{cases} \end{cases}$$

Step 3: replace λ and μ in the compatibility condition