Seminar W11 - 917

Generated surface

- → **Ruled surfaces**
  - → conical surfaces
  - → cylindrical surfaces
  - → conoidal surfaces
- → **Revolution surfaces**

Example 11.2: Determine the equation of the conical surface having the vertex $V(1,1,1)$ and the director curve

$$\mathscr{C}: \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases}$$

Step 1: Write all the possible lines $d_{\lambda,\mu}$ that satisfy condition 1

- conical surfaces: $d_{\lambda,\mu} \ni V$, V point ("vertex")
- cylindrical surfaces: $d_{\lambda,\mu} \parallel \ell$, ℓ line
- Conoidal surfaces: $d_{\lambda,\mu} \parallel \Pi$, Π plane
  $d_{\lambda,\mu} \cap \ell \neq \emptyset$, ℓ line

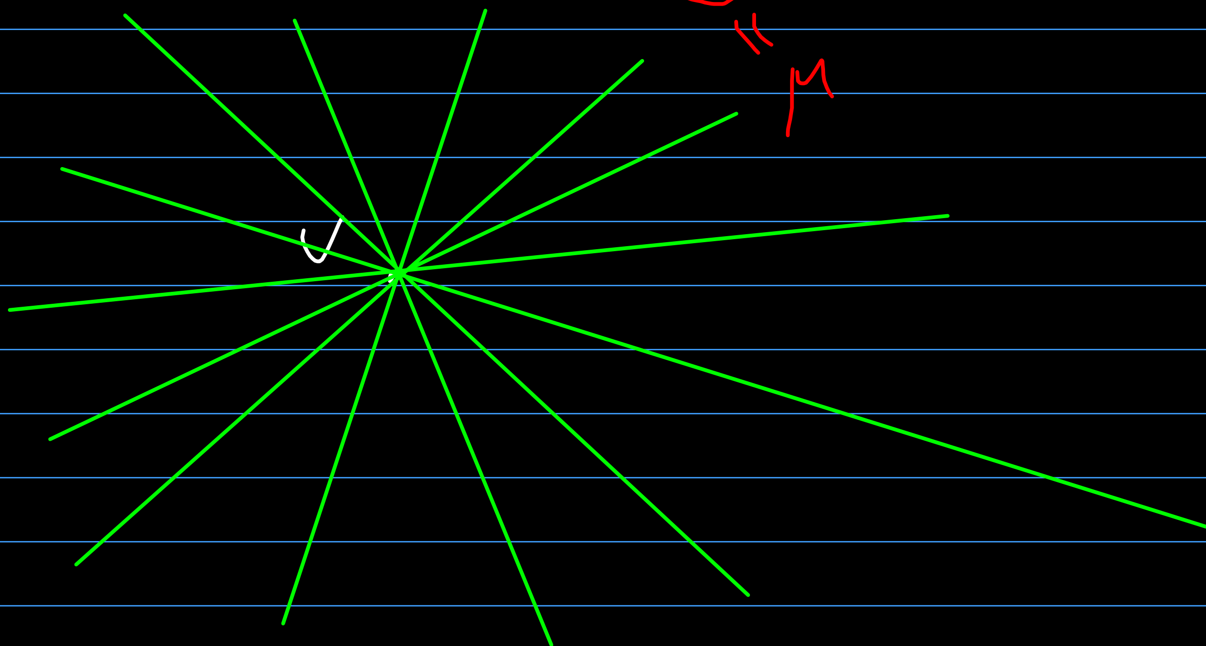
In our case: $V(1,1,1) \in d_{\lambda,\mu}$

We write the most general line equation possible that satisfies this condition

$$d_{\lambda,\mu}: \frac{x-1}{a} = \frac{y-1}{b} = \frac{z-1}{c} \iff$$

$$\iff \begin{cases} \frac{x-1}{a} = \frac{y-1}{b} \\ \frac{x-1}{a} = \frac{z-1}{c} \end{cases} \iff \begin{cases} b\cdot(x-1) = a(y-1) \\ c\cdot(x-1) = a\cdot(z-1) \end{cases}$$

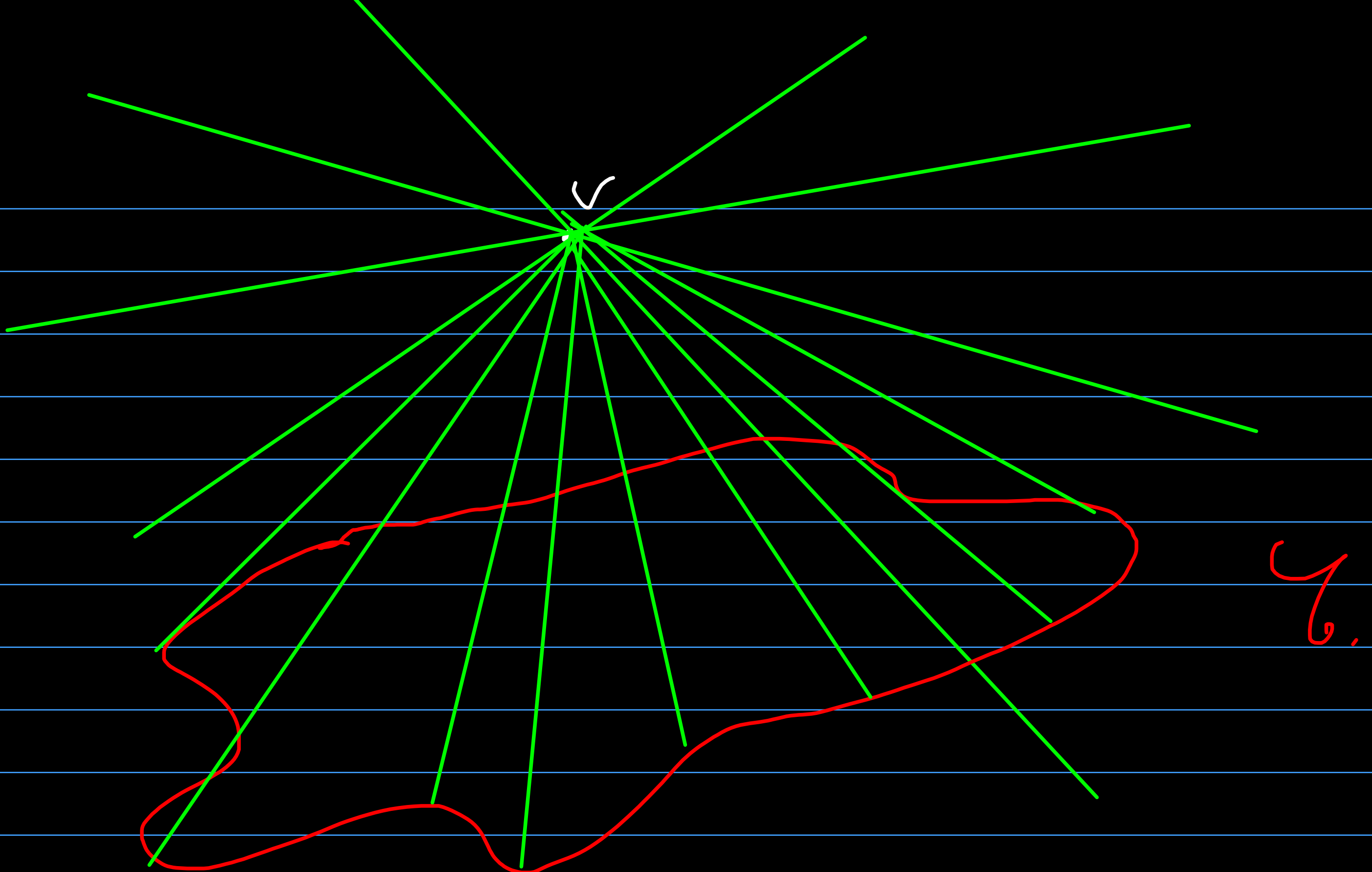
$$\Leftrightarrow \begin{cases} x-1 = \underbrace{\frac{a}{b}}_{=\lambda} \cdot (y-1) \\ x-1 = \underbrace{\frac{a}{c}}_{=\mu} \cdot (z-1) \end{cases}$$



Step 2: We choose the lines $d_{\lambda,\mu}$ that satisfy condition 2

$\hookrightarrow d_{\lambda,\mu} \cap \mathcal{C} \neq \emptyset$

these lines are called generatrices



We perform this selection by finding a relation between λ and μ that ensures the compatibility of the following system:

$$
(S)\quad \begin{cases} d_{\lambda,\mu} : \begin{cases} x-1 = \lambda\cdot(y-1) \\ x-1 = \mu\cdot(z-1) \end{cases} \\ \\ \mathcal{C} : \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \end{cases}
$$

$$\Leftrightarrow \begin{cases} x-1 = \lambda \cdot (y-1) \\ y-1 = \mu \cdot (z-1) \\ (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = 0 \\ x-1 = \lambda \cdot (y-1) \\ y-1 = -\mu \\ (x^2+y^2)^2 - xy = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = 0 \\ y = 1-\mu \\ x-1 = \lambda \cdot (y-1) \\ (x^2+y^2)^2 - xy = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = 0 \\ y = 1-\mu \\ x = 1 - \frac{\mu}{\lambda} \\ (x^2+y^2)^2 - xy = 0 \end{cases}$$

We have obtained the compatibility condition:

$$\left((1-\mu)^2+\left(1-\frac{\mu}{\lambda}\right)^2\right)^2 -$$

$$-(1-\mu)\cdot\left(1-\frac{\mu}{\lambda}\right)=0$$

Step 3: We obtain the final equation by replacing λ and μ by their expressions in terms of x, y, z (FROM WHERE THEY WERE DEFINED!) into the compatibility condition.

$$\lambda=\frac{x-1}{y-1} \qquad \mu=\frac{x-1}{z-1}$$

$\Rightarrow$ The final equation is:

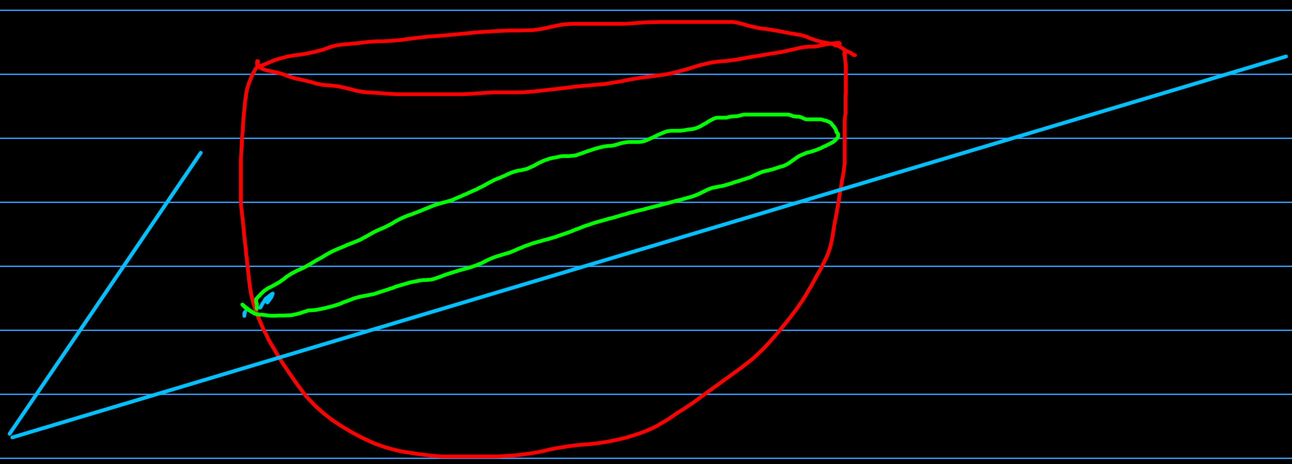
$$\left(\left(1-\frac{x-1}{z-1}\right)^2+\left(1-\frac{y-1}{z-1}\right)^2\right)^2-$$

$$\left(1-\frac{x-1}{z-1}\right)\cdot\left(1-\frac{y-1}{z-1}\right)=0$$

11.1. Find the equation of the cylindrical surface whose director curve is the planar curve

$$\mathcal{C}:\begin{cases} y^2+z^2=4 \\ x=2z \end{cases}$$

and the generatrix is perpendicular to the plane of the director curve.



The plane that $\mathcal{C}$ is included in is

$\pi: \quad x = 2z$

We know that $d_{\lambda,\mu} \perp \pi$

$$d_{\lambda,\mu}: \begin{cases} \dfrac{x - x_0}{1} = \cancel{\dfrac{y - y_0}{0}} = \dfrac{z - z_0}{-2} \\ y = y_0 \end{cases}$$

$$d_{\lambda,\mu}: \begin{cases} -2x - z = \underbrace{-2x_0 - z_0}_{=\lambda} \\ y = \underbrace{y_0}_{=\mu} \end{cases}$$

$$d_{\lambda,\mu}: \begin{cases} -2x - z = \lambda \\ y = \mu \end{cases}$$

$$\begin{cases} -2x - z = \lambda \\ y = \mu \\ x = 2z \\ y^2 + z^2 = x \end{cases} \Leftrightarrow \begin{cases} y = \mu \\ x = 2z \\ -5z = \lambda \\ y^2 + z^2 = 2z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = \mu \\ x = 2z \\ z = -\frac{\lambda}{5} \\ y^2 + z^2 = 2z \end{cases}$$

$\Rightarrow$ the compatibility condition:

$$\mu^2 + \frac{1}{25}\lambda^2 = -\frac{2}{5}\lambda$$

The final equation; $\lambda = -2x - z$

$\mu = y$

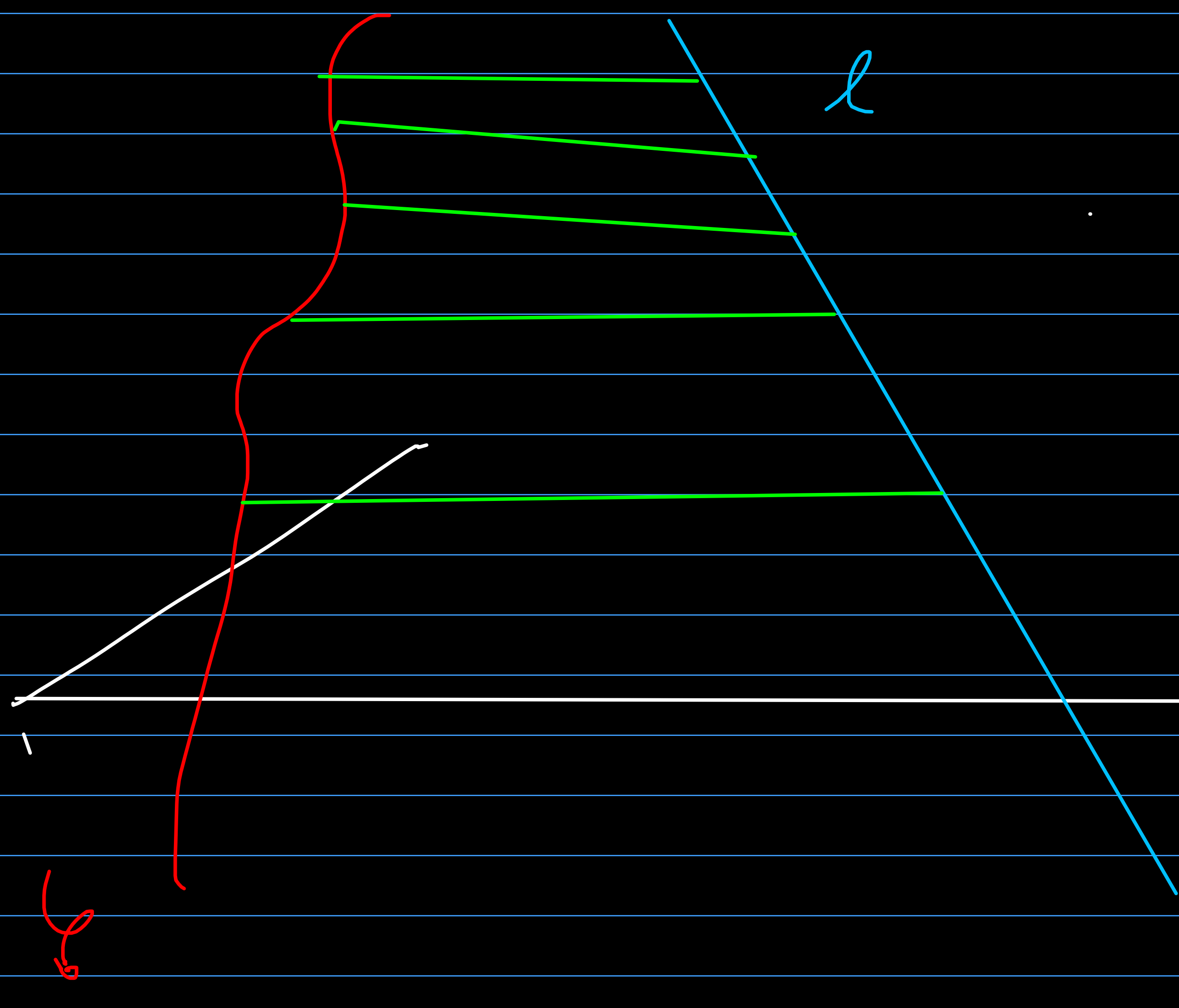
$$\Rightarrow y^2 + \frac{1}{25}\cdot(-2x-z)^2 = -\frac{2}{5}\cdot(-2x-z)$$

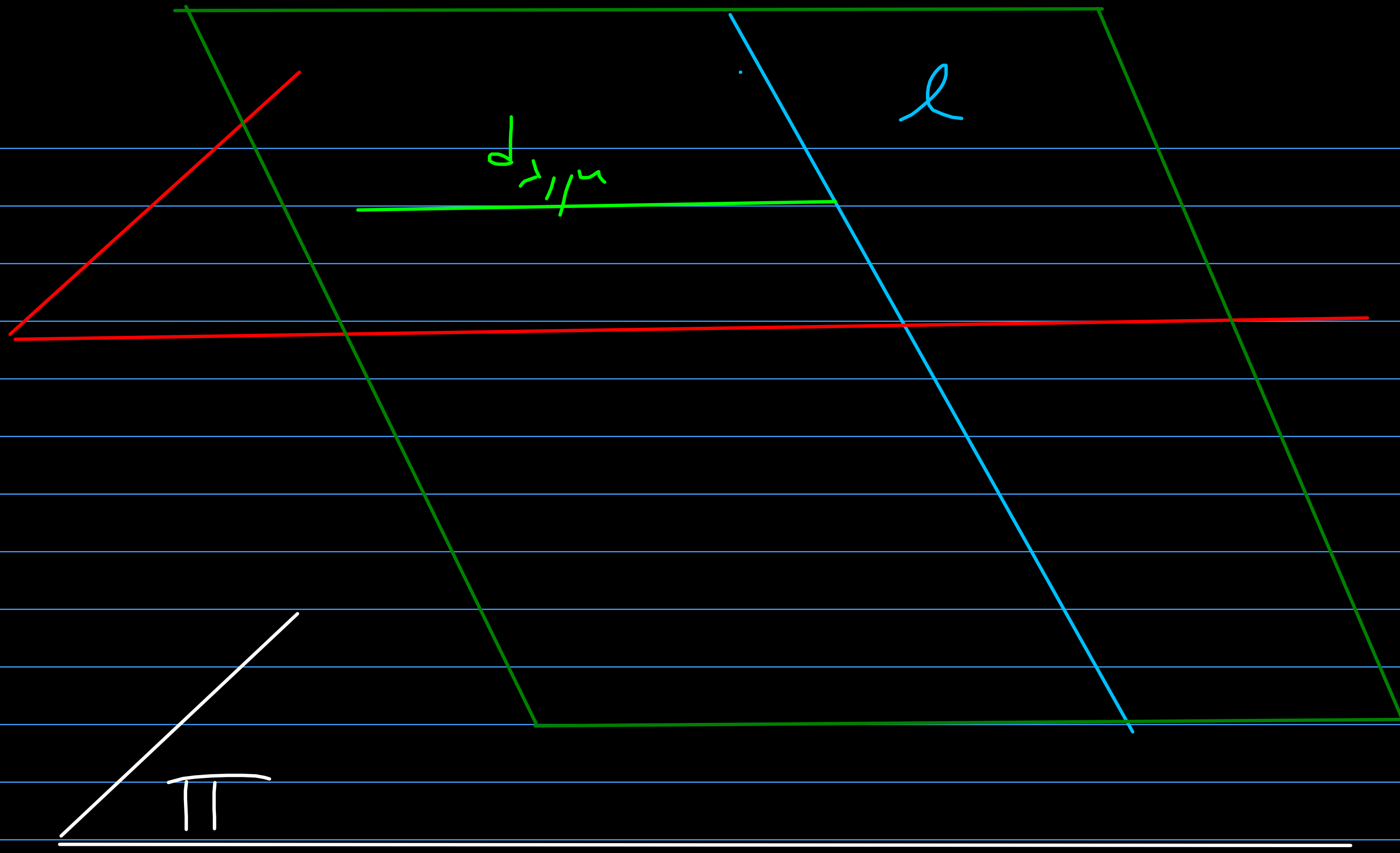
The conoidal surfaces

condition 1 : $-\ d_{\lambda,\mu} \parallel \pi$

$d_{\lambda,\mu} \cap \ell \neq \emptyset$

condition 2 : $-\ d_{\lambda,\mu} \cap \mathcal{C} \neq \emptyset$





$$d_{\lambda,\mu} : \begin{cases} \text{plane that is parallel to } \pi \\ \text{plane that contains } \ell \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \text{plane that is parallel to } \pi \\ \text{plane from the reduced pencil of planes} \end{cases}$$

$$\pi : Ax + By + Cz + D = 0$$

$$\ell : \begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$

$$d_{\lambda,\mu} : \begin{cases} Ax + By + Cz + D = \lambda \\ A_1 x + B_1 y + C_1 z + D_1 + \mu \cdot (A_2 x + B_2 y + C_2 z + D_2) = \\ = 0 \end{cases}$$

<u>Example 11.3</u>: Find the conoidal surface whose generatrices are parallel to xOy and intersect Oz and have the director curve

$$\mathcal{C} : \begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}$$

$\pi = xOy : \quad z = 0$

$\ell = Oz : \begin{cases} x = 0 \\ y = 0 \end{cases}$

$d_{\lambda,\mu} : \begin{cases} z = \lambda \\ x + \mu \cdot y = 0 \end{cases}$

We now find the compatibility condition:

$$\begin{cases} z = \lambda \\ x + \mu y = 0 \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} z = \lambda \\ x = -\mu y \\ y^2 - 2\lambda + 2 = 0 \\ \mu^2 y^2 - 2\lambda + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = \lambda \\ x = -\mu y \\ y^2 = 2\lambda - 2 \\ \mu^2 (2\lambda - 2) - 2\lambda + 1 = 0 \end{cases}$$

$\Rightarrow$ the compatibility condition

$$\mu^2 (2\lambda - 2) - 2\lambda + 1 = 0$$

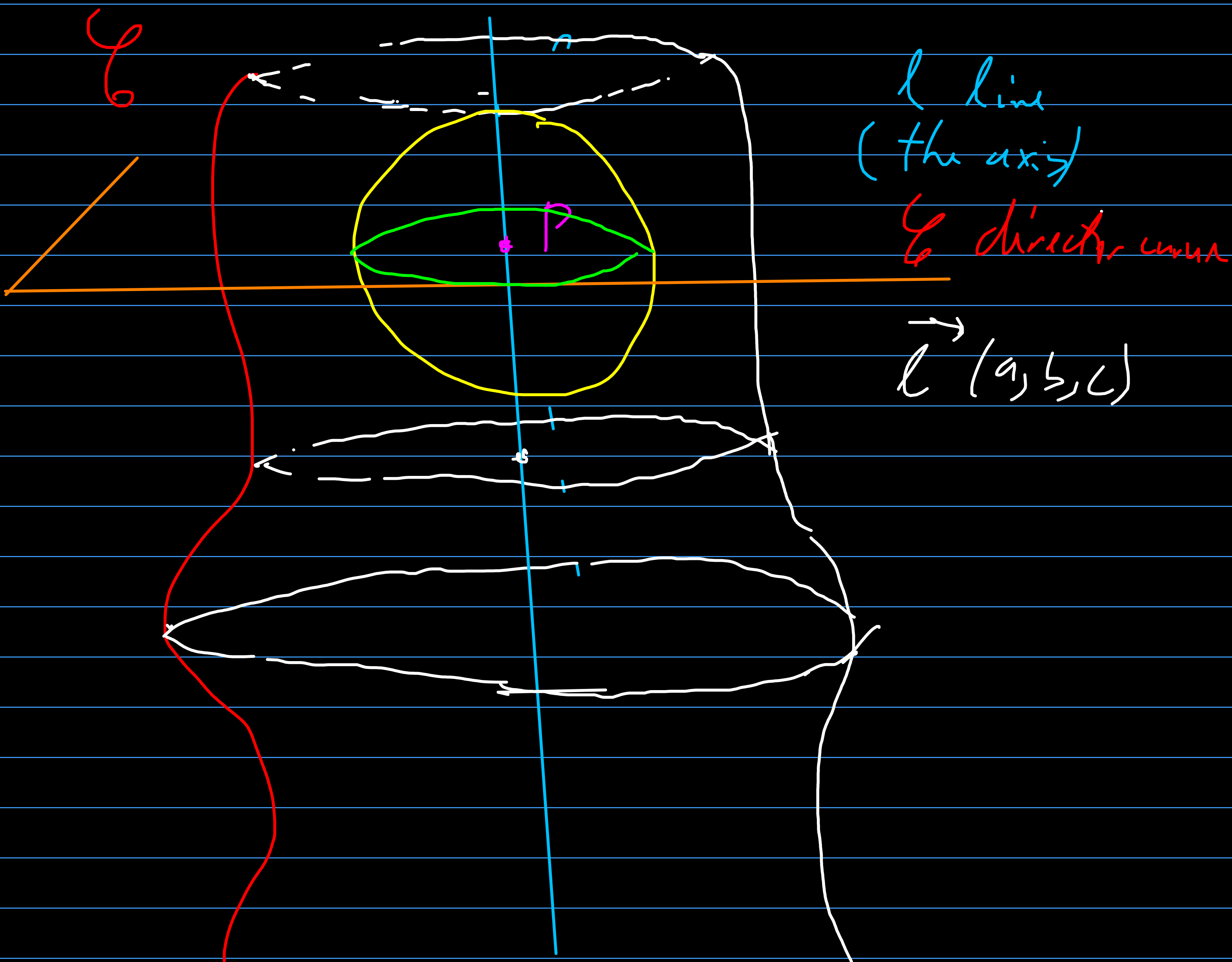
$\therefore \lambda = z \ , \ \mu = -\frac{x}{y}$

$\Rightarrow$ the final equation is:

$$\frac{x^2}{y^2} (2z - 2) - 2z + 1 = 0$$

$$2x^2 z - 2x^2 - 2y^2 z + y^2 = 0$$

Revolution surfaces



Step 1: Instead of having generating lines l like for ruled surfaces, we have generating circles whose center lies on the axis

We choose a point P on the axis. $P(x_0, y_0, z_0)$

$$C_{\lambda,\mu} : \begin{cases} (x-x_0)^2+(y-y_0)^2+(z-z_0)^2=\lambda \\ ax+by+cz=\mu \end{cases}$$

Steps 2 and 3 are the same.