Seminar W6 — 914

Cross product (vector product)

$V \times V \to V$

$(\vec u, \vec w) \mapsto \vec u \times \vec w$, if $\vec u, \vec w$ linearly dependent $\Rightarrow \vec u \times \vec w = \vec 0$

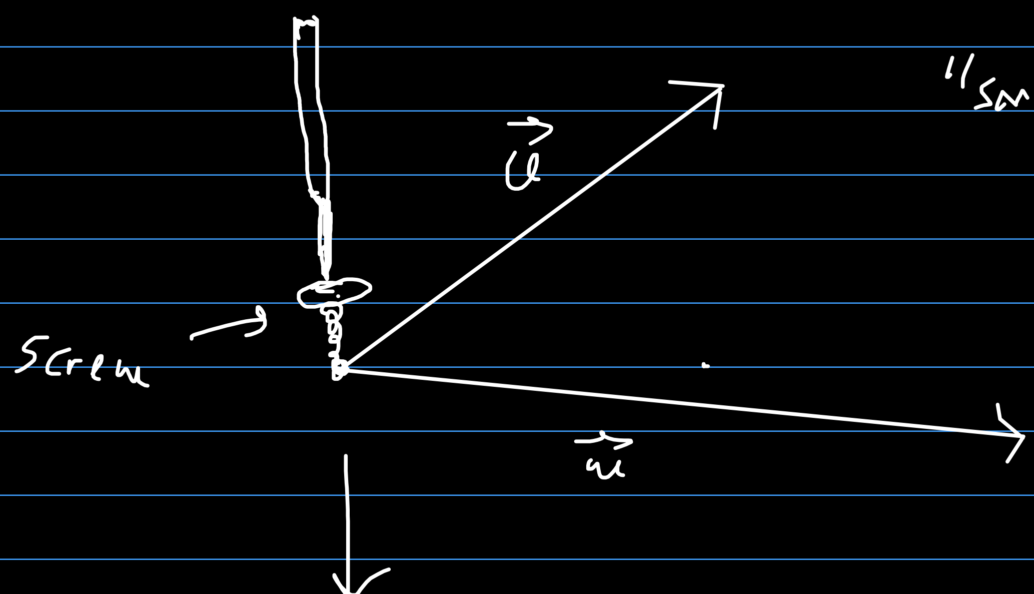
→ the direction : perpendicular to $\langle \vec u, \vec w \rangle$

→ the norm : $\|\vec u \times \vec w\| = \|\vec u\| \, \|\vec w\| \cdot \sin(\widehat{\vec u, \vec w}) =$

$$= 2 \cdot S_{\substack{\text{triangle} \\ \text{formed by} \\ \vec u, \vec w}} = S_{\substack{\text{parallelogram} \\ \text{formed by} \\ \vec u \text{ and } \vec w}}$$

→ the orientation

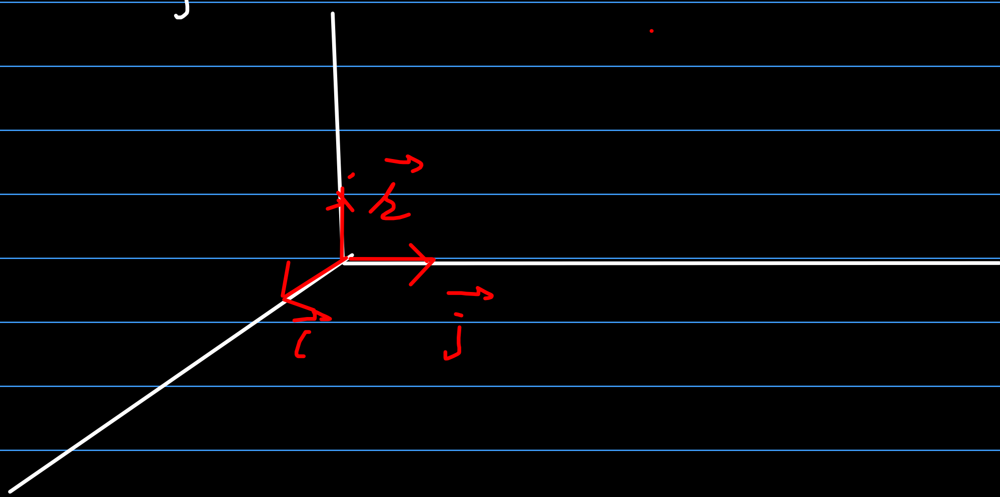
"screw rule"



× is anti-commutative

If the reference system is orthonormal and <u>direct</u> $\mathcal{R} = (O, [\vec{i}, \vec{j}, \vec{k}])$

$\to \vec{i} \times \vec{j} = \vec{k}$



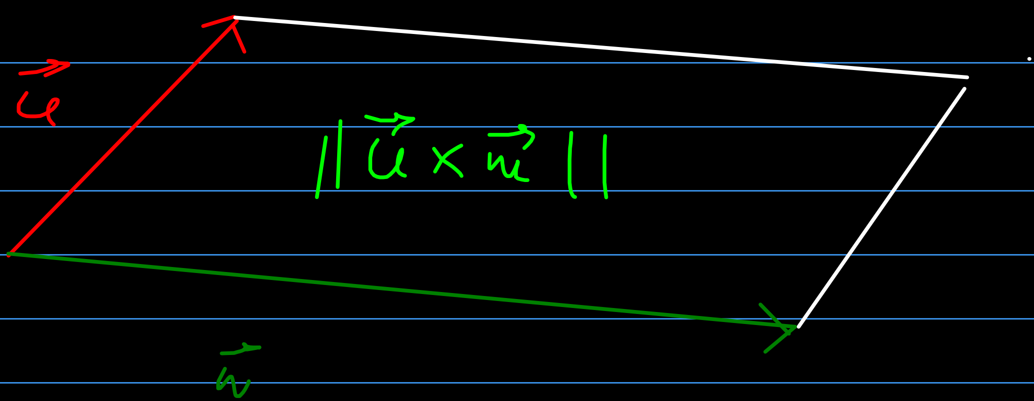
Then we compute the cross product as follows:

$\vec{u}(a_1, b_1, c_1)$, $\vec{w}(a_2, b_2, c_2)$

$$\vec{u} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} =$$

$$= (b_1 c_2 - c_1 b_2 , \; a_2 c_1 - a_1 c_2 , \; a_1 b_2 - a_2 b_1)$$

$$\|\vec{u} \times \vec{w}\| = \|\vec{u}\| \cdot \|\vec{w}\| \cdot \sin(\vec{u}, \vec{w})$$

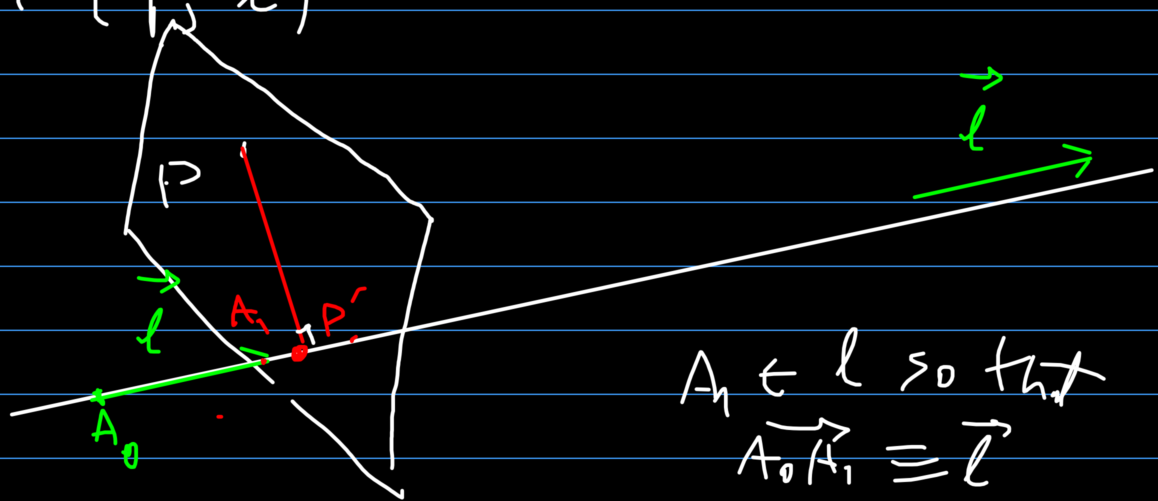


$\forall \alpha, \beta \in \mathbb{R}$:

$$(\alpha \vec{u}_1 + \beta \vec{u}_2) \times \vec{w} = \alpha \cdot \vec{u}_1 \times \vec{w} + \beta \cdot \vec{u}_2 \times \vec{w}$$

ℓ line, $P(x_0, y_0, z_0)$ point

$\mathrm{dist}(P, \ell)$

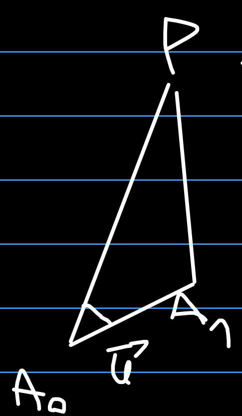


$A_1 \in \ell$ so that

$\overrightarrow{A_0 A_1} = \vec{\ell}$

$$A_{PA_0A_1} = \frac{1}{2} \|\overrightarrow{PA_0} \times \overrightarrow{A_0A_1}\| = \frac{1}{2} \cdot PP' \cdot A_0A_1$$

$$dist(P, \ell) = \frac{2A_{PA_0A_1}}{\|\overrightarrow{A_0A_1}\|} = \frac{\|\overrightarrow{PA_0} \times \vec{u}\|}{\|\vec{u}\|}$$



$$\|\overrightarrow{PA_0} \times \overrightarrow{A_0A_1}\| =$$

$$= \|\overrightarrow{PA_0}\| \cdot \|\overrightarrow{A_0A_1}\| \cdot$$

$$\cdot \sin\left(\widehat{\overrightarrow{PA_0}, \overrightarrow{A_0A_1}}\right)$$

6.4. Find $dist(P, \ell)$

$P(1, 2, -1)$, $\ell : x = y = z$

<u>Proof:</u>

$\vec{\ell}(1,1,1)$; $A_0(1,1,1) \in \ell$

$\overrightarrow{PA_0}(0, -1, 2)$

$\|\vec{\ell}\| = \sqrt{3}$

$$\vec{PA_0} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -3\vec{i} + 2\vec{j} + \vec{k}$$

$$\Rightarrow \quad \vec{PA_0} \times \vec{u} \ (-3, 2, 1)$$

$$\Rightarrow \quad \|\vec{PA_0} \times \vec{u}\| = \sqrt{9+4+1} = \sqrt{14}$$

$$\operatorname{dist}(P, \ell) = \frac{\sqrt{14}}{\sqrt{3}} = \sqrt{\frac{14}{3}} \simeq 2, \ldots$$

6.5. Find the area of the triangle ABC and the lengths of its heights, where $A(-1, 1, 2)$, $B(2, -1, 1)$, $C(2, -3, -2)$

Proof: $\vec{AB}\,(3, -2, -1) \Rightarrow \|\vec{AB}\| = \sqrt{9+4+1} = \sqrt{14}$

$\vec{AC}\,(3, -4, -4) \Rightarrow \|\vec{AC}\| = \sqrt{9+16+16} = \sqrt{41}$

$\vec{BC}\,(0, -2, -3) \Rightarrow \|\vec{BC}\| = \sqrt{4+9} = \sqrt{13}$

$$A_0 = B(2,-1,7)$$

$$S_{ABC} = \frac{1}{2}\|\vec{AB} \times \vec{BC}\| = \frac{1}{2}\left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -1 \\ 0 & -2 & -3 \end{vmatrix} \right\| =$$

$$= \frac{1}{2}\| 4\vec{i} + 9\vec{j} - 6\vec{k} \| = \frac{1}{2}\|(4, 9, -6)\| =$$

$$= \frac{1}{2}\sqrt{16+81+36} = \frac{1}{2}\sqrt{133}$$

$$h_A = \frac{2S_{ABC}}{\|\vec{BC}\|} = \frac{\sqrt{133}}{\sqrt{13}} = \frac{\sqrt{133}}{\sqrt{13}}.$$

$$h_B = \frac{2S_{ABC}}{\|\vec{AC}\|} = \frac{\sqrt{133}}{\sqrt{41}}$$

$$h_C = \frac{2S_{ABC}}{\|\vec{AB}\|} = \frac{\sqrt{133}}{\sqrt{14}}$$

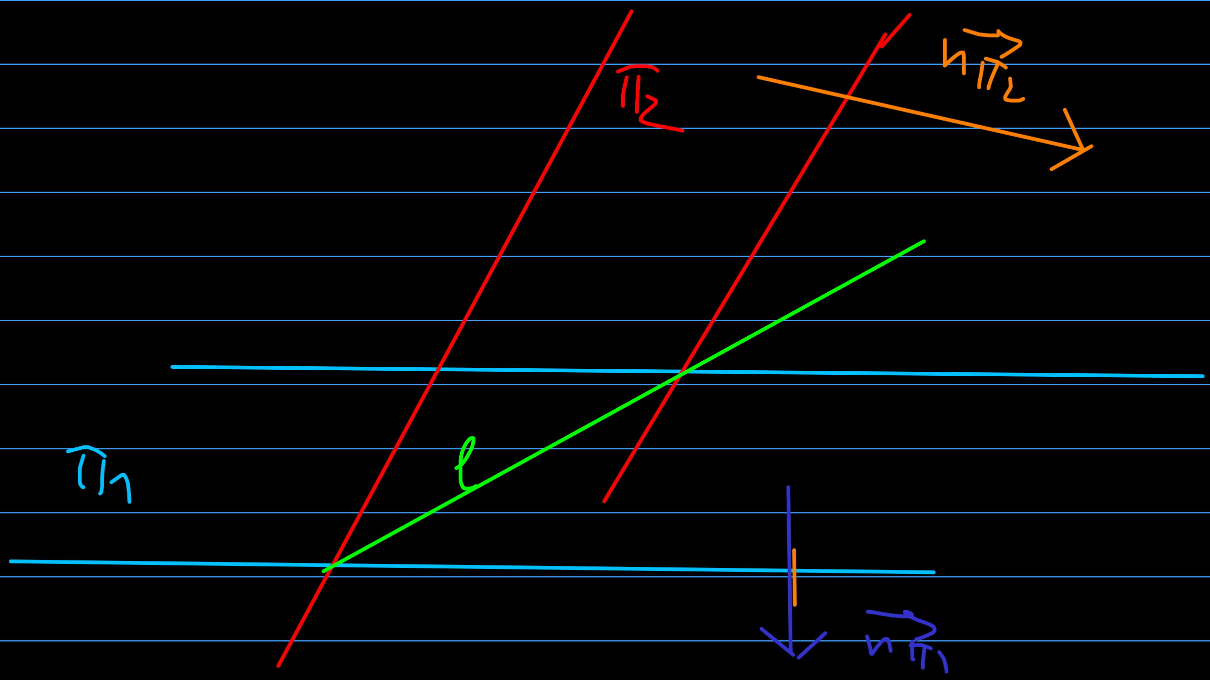
Ex. 6.II : Consider the line.

$$\ell : \begin{cases} 2x - y + z + 5 = 0 \\ x - y + 3z + 1 = 0 \end{cases}$$

Find the equation of the perpendicular line from $P(1,2,3)$ to ℓ.

If $\ell : \begin{cases} \Pi_1 : A_1 x + B_1 y + C_1 z + D_1 = 0 \\ \Pi_2 : A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$

then $\vec{n}_{\Pi_1} \times \vec{n}_{\Pi_2} \parallel \ell$



We have that $\vec{n_{\pi_1}} \perp \ell$ and $\vec{n_{\pi_2}} \perp \ell$

$\Rightarrow \ell \parallel (\vec{n_{\pi_1}} \times \vec{n_{\pi_2}})$

Conclusion: We can take $\vec{\ell} = \vec{n_{\pi_1}} \times \vec{n_{\pi_2}}$

$$\ell : \begin{cases} 2x - y + z + 5 = 0 \\ x - y + 3z + 1 = 0 \end{cases}$$

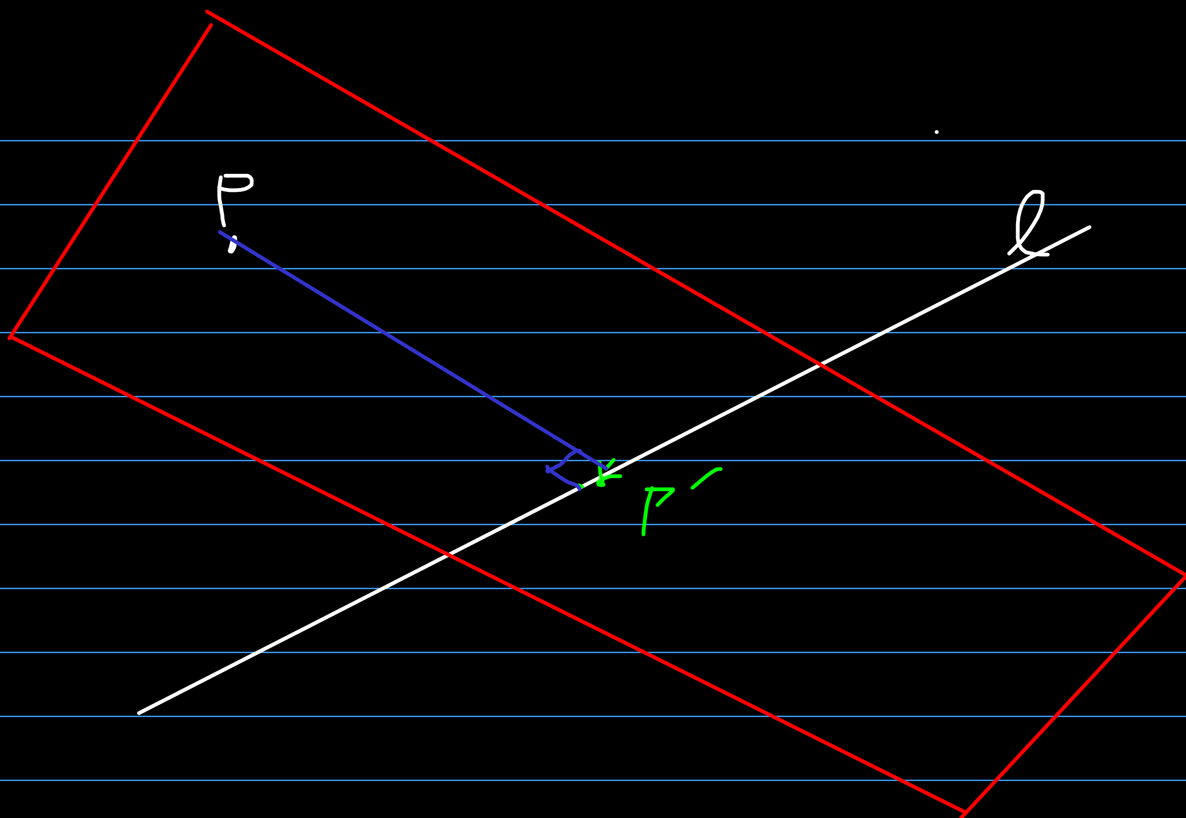
$\vec{n_{\pi_1}}(2, -1, 1)$, $\vec{n_{\pi_2}}(1, -1, 3)$

$$\vec{n_{\pi_1}} \times \vec{n_{\pi_2}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & -1 & 3 \end{vmatrix} =$$

$$= -2\vec{i} - 5\vec{j} - \vec{k}$$

$\Rightarrow \vec{\ell}(2, 5, 1)$

The perpendicular from $P(1,2,3)$ to ℓ



We find the equation of the plane π that is perpendicular to l and contains P.

We make it so that $\vec{n_\pi} = \vec{l}\ (2,5,1)$

$$\Rightarrow \pi : x_{\vec{l}} \cdot (x - x_P) + y_{\vec{l}} \cdot (y - y_P) + z_{\vec{l}} (z - z_P) = 0$$

$$\Rightarrow \pi : 2 \cdot (x - 1) + 5 \cdot (y - 2) + z - 3 = 0$$

$$\Rightarrow \pi : 2x + 5y + z - 2 - 10 - 3 = 0$$

$$\Rightarrow \pi : 2x + 5y + z - 15 = 0$$

$$P' = \Pi \cap l : \begin{cases} 2x+5y+z-15=0 \\ 2x-y+z+5=0 \\ x-y+3z+1=0 \end{cases} \Leftrightarrow$$

$$\begin{pmatrix} 2 & 5 & 1 & 15 \\ 2 & -1 & 1 & -5 \\ 1 & -1 & 3 & -1 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & -1 & 3 & -1 \\ 2 & 5 & 1 & 15 \\ 2 & -1 & 1 & -5 \end{pmatrix} -$$

$$\underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 7 & -5 & 17 \\ 0 & 1 & -5 & -3 \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim}$$

$$\sim \begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 1 & -5 & -3 \\ 0 & 7 & -5 & 17 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 - 7L_2}{\sim} \begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 30 & 38 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x - y + 3z = -1 \\ y - 5z = -3 \\ 30z = 38 \end{cases}$$

$$\Rightarrow z = \frac{19}{15} \qquad y = -3 + 5z =$$

$$= -3 + \frac{19}{3} =$$

$$= \frac{10}{3}$$

$$\Rightarrow x = y - 3z - 1 = \frac{10}{3} - \frac{19}{5} - 1 =$$

$$= \frac{-7}{15} - 1 = -\frac{22}{15}$$

$$\Rightarrow P'\left(-\frac{22}{15}, \frac{10}{3}, \frac{19}{15}\right)$$

$\boxed{\text{perp. from } P \text{ to } \ell}$ $$\frac{x - 1}{-\frac{22}{15} - 1} = \frac{y - 2}{\frac{10}{3} - 2} = \frac{z - 3}{\frac{19}{15} - 3}$$

The double vector/cross product

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \end{vmatrix}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{b}\cdot\vec{c} & \vec{a}\cdot\vec{c} \end{vmatrix}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = -(\vec{b} \times \vec{c}) \times \vec{a} =$$
$$= (\vec{c} \times \vec{b}) \times \vec{a}$$

$\Rightarrow$ the cross product is not associative!