Seminar W11 - 916

Generated surfaces:

- **Ruled surfaces** (generated by a line)
  - → conical surfaces
  - → cylindrical surfaces
  - → Conoidal surfaces
- **Revolution surfaces** (generated by a curve rotating around a line)

Example 11.2: Determine the equation of the conical surface having the vertex $V(1,1,1)$ and the director curve

$$\mathcal{C}: \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases}$$

Step 1: We find all the lines that satisfy condition 1 (called the generatrices)

- conical: $d_{\lambda,\mu} \ni V$, V point
- cylindrical: $d_{\lambda,\mu} \parallel \ell$, ℓ line
- conoidal: $d_{\lambda,\mu} \parallel \Pi$, Π plane

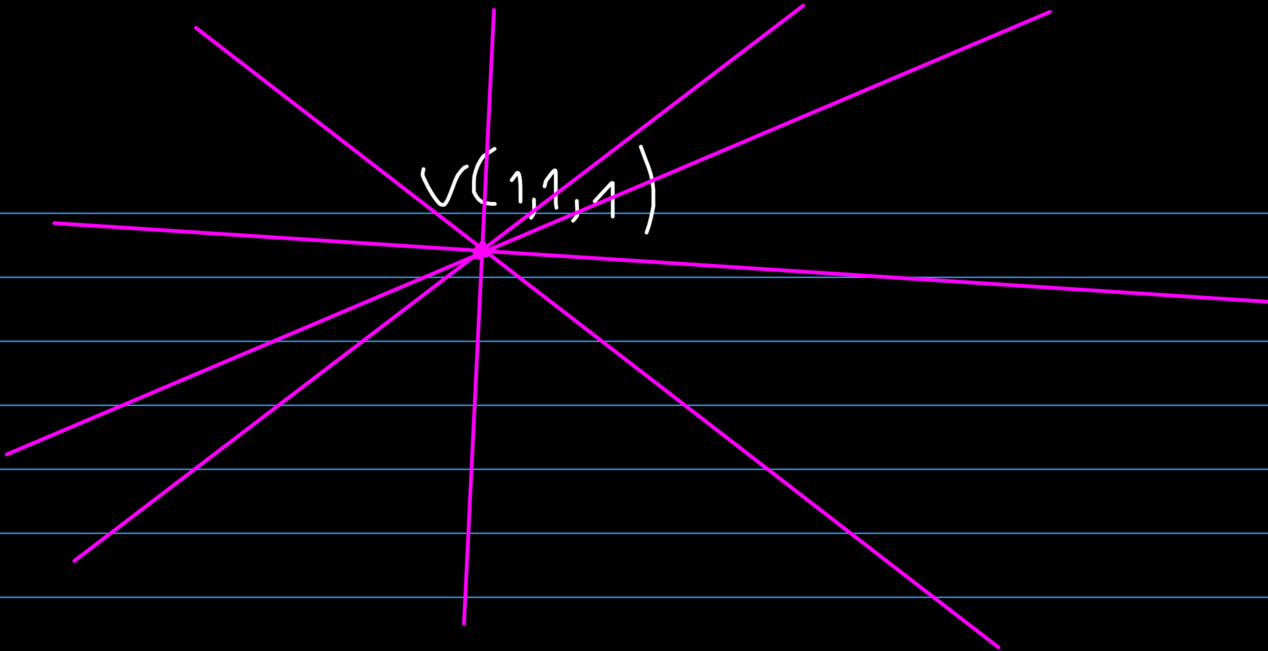
  $d_{\lambda,\mu} \cap \ell \neq \emptyset$, ℓ line

In our example $V(1,1,1)$

$$d_{\lambda,\mu}: \frac{x-1}{a} = \frac{y-1}{b} = \frac{z-1}{c} \iff$$

$$\iff d_{\lambda,\mu}: \begin{cases} b\cdot(x-1) = a\cdot(y-1) \\ c\cdot(x-1) = a\cdot(z-1) \end{cases} \iff$$

$$\iff d_{\lambda,\mu}: \begin{cases} x-1 = \underbrace{\frac{a}{b}}_{=\lambda}\cdot(y-1) \\ x-1 = \underbrace{\frac{a}{c}}_{\mu}\cdot(z-1) \end{cases}$$



Step 2: Out of the generatrices in step 1, we will now select only the ones that satisfy condition 2

$$\hookrightarrow \quad d_{\lambda,\mu} \cap \underbrace{\mathcal{C}}_{\text{director curve}} \neq \emptyset$$

The generatrices that we choose will correspond to values of the parameters λ and μ so that the following system is compatible:

$$\begin{cases} d_{\lambda,\mu}: \begin{cases} x-1 = \lambda\,(y-1) \\ x-1 = \mu\cdot(z-1) \end{cases} \iff \\ \mathcal{C}: \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \end{cases}$$

$$\iff \begin{cases} x-1 = \lambda\cdot(y-1) \\ x-1 = \mu\cdot(z-1) \\ (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \iff \begin{cases} z = 0 \\ x-1 = \lambda\cdot(y-1) \\ x-1 = -\mu(\\ (x^2+y^2)^2 - xy = 0 \end{cases}$$

$$\iff \begin{cases} z = 0 \\ x = 1-\mu \\ -\mu = \lambda\cdot(y-1) \\ (x^2+y^2)^2 - xy = 0 \end{cases} \iff$$

$$\iff \begin{cases} z = 0 \\ x = 1-\mu \\ y = 1 - \dfrac{\mu}{\lambda} \\ (x^2+y^2)^2 - xy = 0 \end{cases}$$

We now obtain the compatibility condition

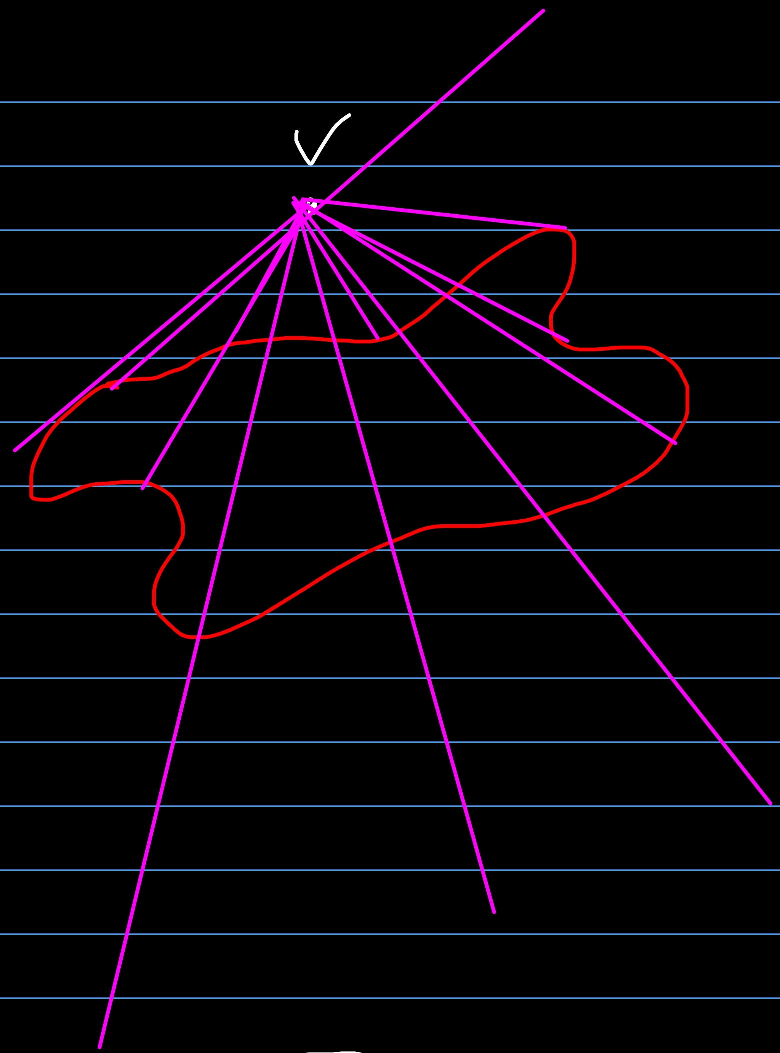
$$\left((1-\mu)^2+\left(1-\frac{\mu}{\lambda}\right)^2\right)^2-(1-\mu)\cdot\left(1-\frac{\mu}{\lambda}\right)=0$$

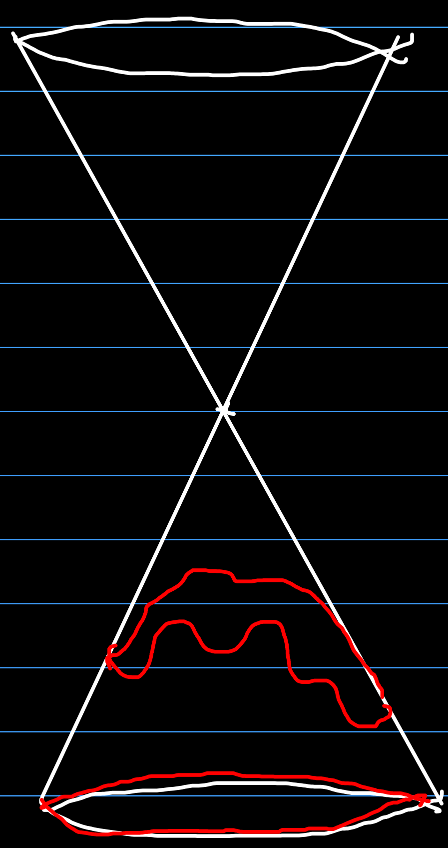
Step 3: We replace λ and μ by their expressions in x, y, z from the beginning (from when they were introduced)

$$\lambda=\frac{x-1}{y-1} \qquad \mu=\frac{x-1}{z-1}$$

Therefore the equation of the conical surface is:

$$\left(\left(1-\frac{x-1}{z-1}\right)^2+\left(1-\frac{y-1}{z-1}\right)^2\right)^2-\left(1-\frac{x-1}{z-1}\right)\cdot\left(1-\frac{y-1}{z-1}\right)$$
$$=0$$





11.1. Find the equation of the cylindrical surface whose director curve is the planar curve

$$(\mathcal{C}): \begin{cases} y^2+z^2 = x \\ x = 2z \end{cases}$$

and the generatrix is perpendicular to the plane of the director curve

The plane of the curve $\mathcal{C}$ is

$$\pi : x = 2z$$

We know that $d_{\lambda,\mu} \perp \pi$

Step 1: $d_{\lambda,\mu}: \begin{cases} \dfrac{x-x_0}{1} = \dfrac{z-z_0}{-2} \\ y = y_0 \end{cases}$

$$\Leftrightarrow d_{\lambda,\mu}: \begin{cases} -2x - z = \underbrace{-2x_0 - z_0}_{=\lambda} \\ y = y_0 = \mu \end{cases}$$

Step 2 :

$$\begin{cases} -2x - z = \lambda \\ y = \mu \\ y^2 + z^2 = x \\ x = 2z \end{cases} \iff \begin{cases} y = \mu \\ -5z = \lambda \\ x = 2z \\ y^2 + z^2 = x \end{cases} \iff$$

$$\iff \begin{cases} y = \mu \\ z = -\frac{\lambda}{5} \\ x = -\frac{2\lambda}{5} \\ y^2 + z^2 = x \end{cases}$$

$\Rightarrow$ the compatibility condition:

$$\mu^2 + \frac{\lambda^2}{25} + \frac{2\lambda}{5} = 0$$

$\Rightarrow$ The equation:

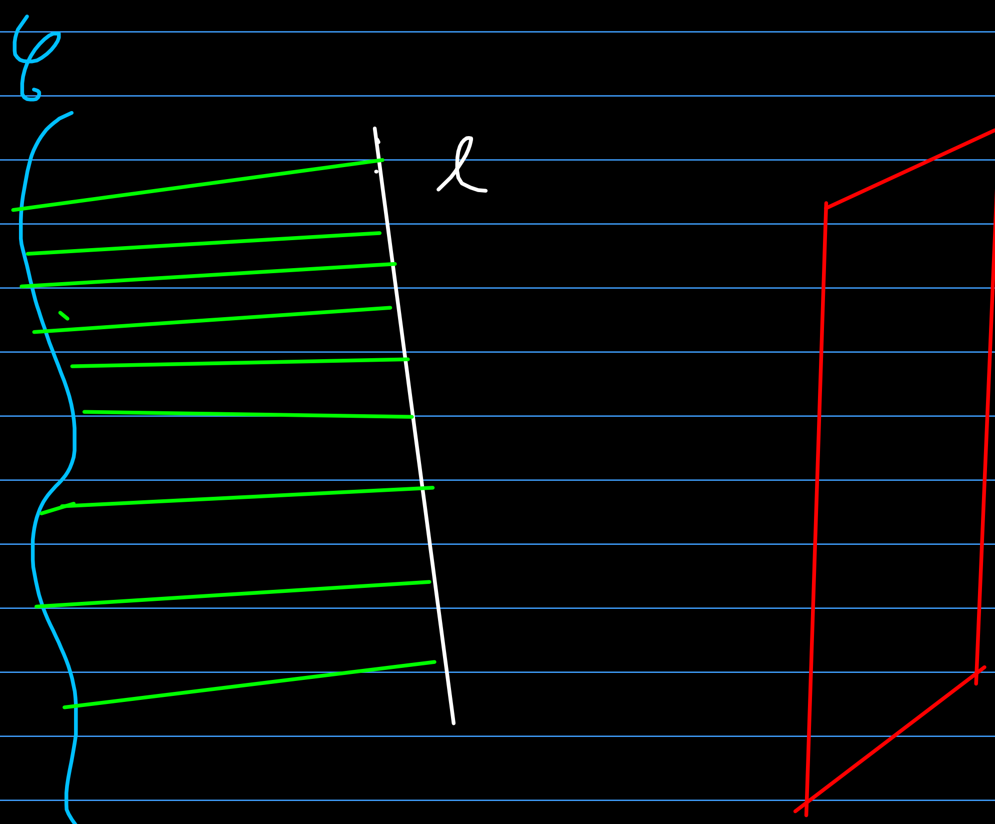
$$25\, y^2 + (-2x - z)^2 + 10\,(-2x - z) = 0$$

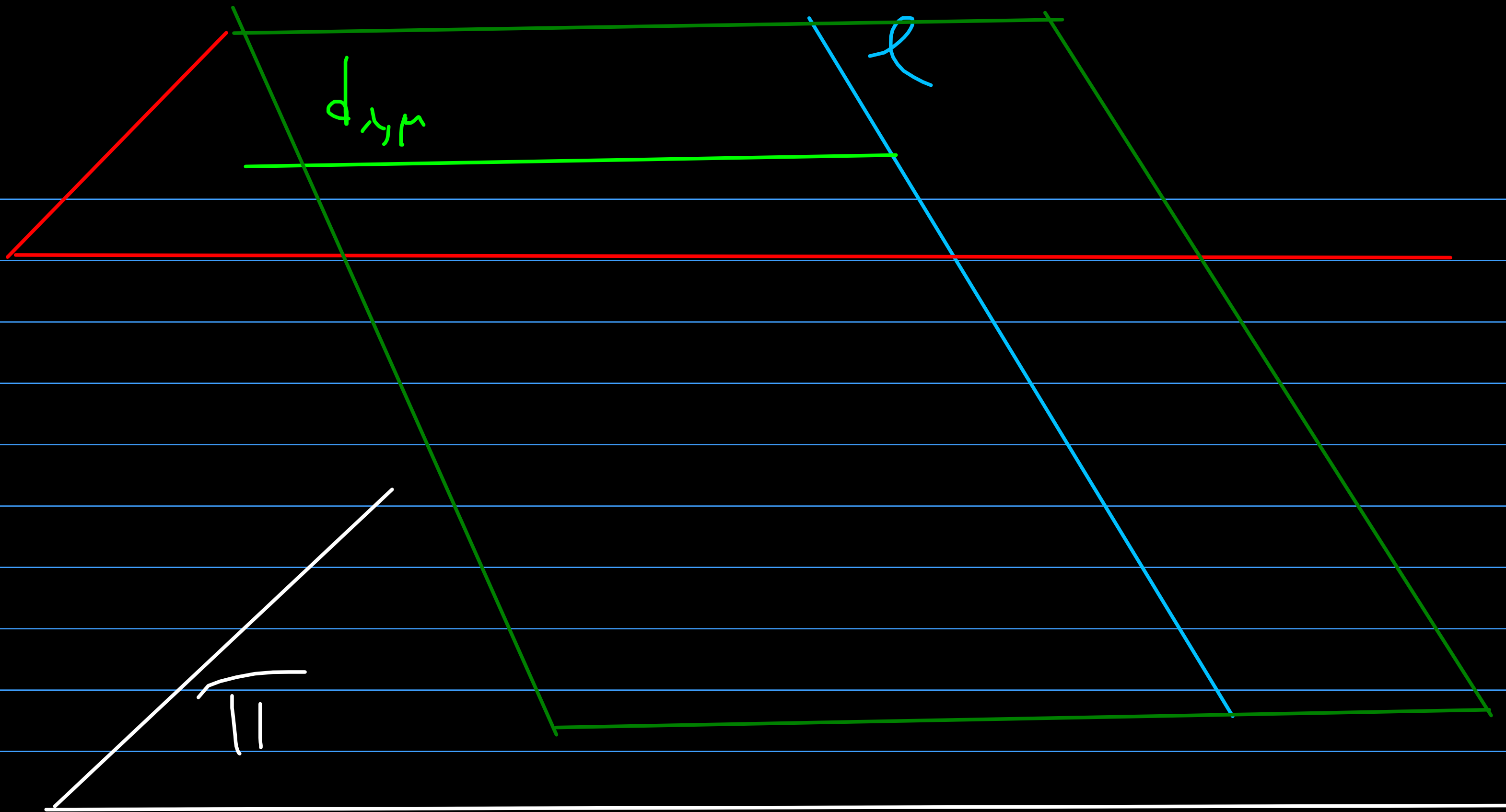
Conoidal surfaces (conoids)

Condition 1 : $d_{\lambda,\mu} \parallel \pi$, π plane

$d_{\lambda,\mu} \cap \ell \neq \emptyset$, ℓ line

Condition 2 : $d_{\lambda,\mu} \cap \mathcal{C} \neq \emptyset$





$d_{\lambda,\mu}$: $\begin{cases} \text{a plane parallel to } \pi \\ \text{a plane that contains } \ell \end{cases}$ $\Leftrightarrow$

$\Leftrightarrow$ $\begin{cases} \text{a plane parallel to } \pi \\ \text{a plane from the (reduced) pencil of planes of } \ell . \end{cases}$

$\pi: \; Ax+By+Cz+D=0$

$\ell: \begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$

$$d_{\lambda,\mu}: \begin{cases} Ax + By + Cz + D = \lambda \\ A_1x + B_1y + C_1z + D_1 + \mu \cdot (A_2x + B_2y + C_2z + D_2) = 0 \end{cases}$$

<u>Example 11.3:</u>

Find the equation of the conoidal surface whose generatrices are parallel to xOy and intersect Oz and have the director curve

$$\mathcal{C}: \begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}$$

$\Pi = xOy : \; z = 0$

$$l = Oz : \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\Rightarrow d_{\lambda,\mu} : \begin{cases} z = \lambda \\ x = \mu \cdot y \end{cases}$$

$$\begin{cases} z = \lambda \\ x = \mu \cdot y \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} z = \lambda \\ x = \mu y \\ y^2 - 2\lambda + 2 = 0 \\ \mu^2 y^2 - 2\lambda + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = \lambda \\ x = \mu y \\ y^2 = 2\lambda - 2 \\ \mu^2 \cdot (2\lambda - 2) - 2\lambda + 1 = 0 \end{cases}$$

$\Rightarrow$ compatibility condition

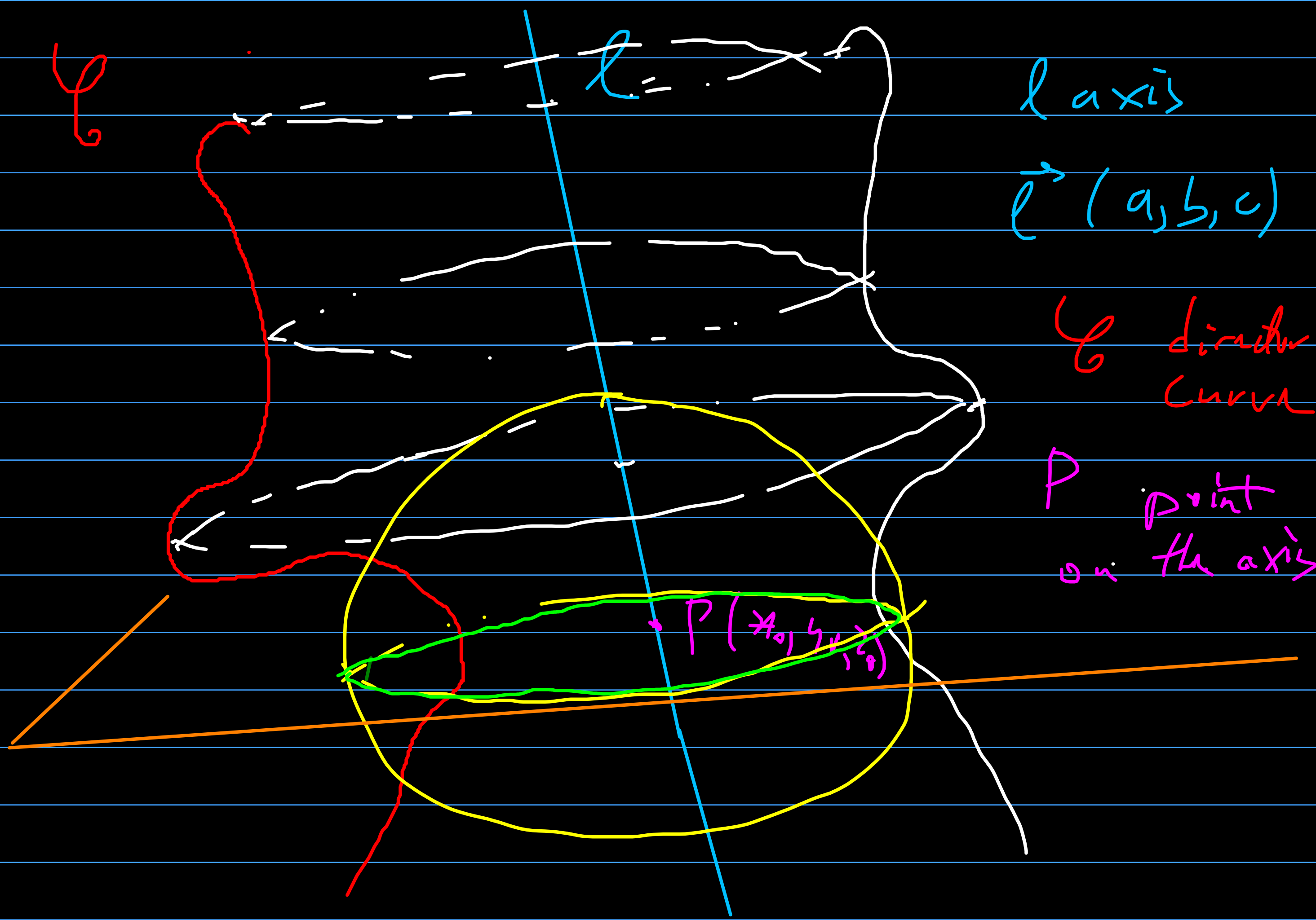
$$\mu^2 \cdot (2\lambda - 2) - 2\lambda + 1 = 0$$

$$\lambda = z, \quad \mu = \frac{x}{y}$$

$\Rightarrow$ the equation of the surface.

$$\frac{x^2}{y^2} \cdot (2z - 2) - 2z + 1 = 0$$

Revolution surfaces



Step 1: Write all the possible circles (here we don't have generating lines, but rather generating circles) whose center lies on the axis

$$C_{\lambda,\mu}: \begin{cases} (x-x_0)^2+(y-y_0)^2+(z-z_0)^2=\lambda \\ ax+by+cz=\mu \end{cases}$$

Step 2 & Step 3 THE SAME!