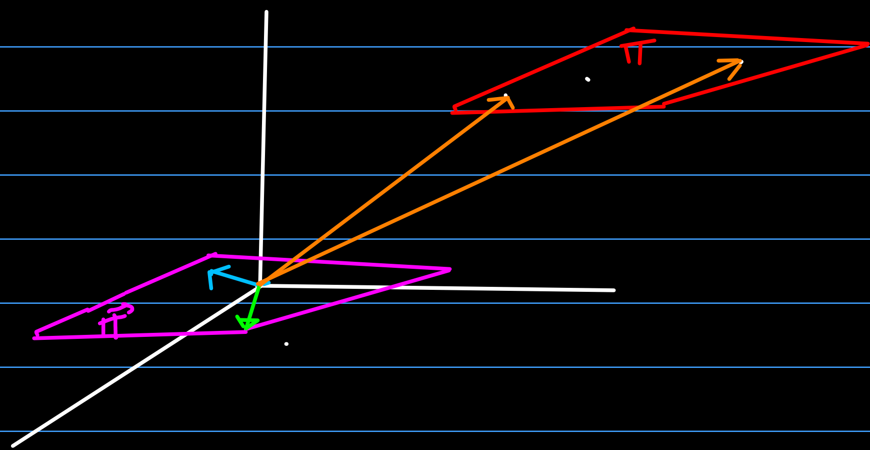
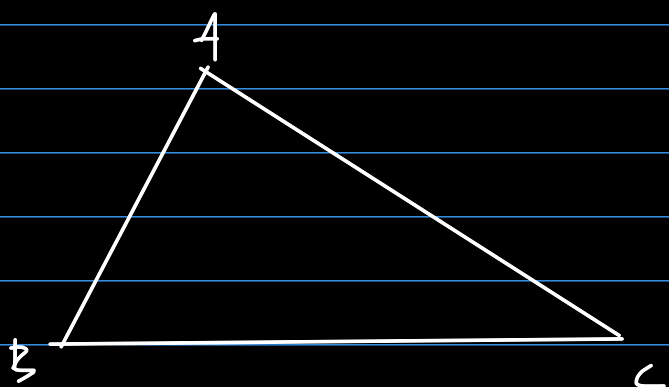




$\exists\, \Delta ABC \iff \vec{AB}$ și $\vec{AC}$ sunt lin. indep.

Proposition 4.7. *If $R = (O, b)$ is the Cartesian reference system behind the equations of the line*

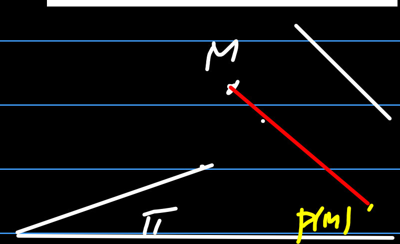
$$(d)\ \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and the plane $(\pi)\ Ax + By + Cz + D = 0$, concurrent with (d), then

$$[p_{\pi,d}(M)]_R = \frac{1}{Ap + Bq + Cr}\begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix}[M]_R - \frac{D}{Ap + Bq + Cr}[\vec{d}]_b,$$

where $\vec{d}\ (p, q, r)$ stands for the director vector of the line (d).

$$\begin{cases} x_{p(M)} = x_M - p\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ y_{p(M)} = y_M - q\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ z_{p(M)} = z_M - r\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr}, \end{cases}$$



$$\vec{r}_{p_{\pi,d}(M)} = \vec{r}_M - \frac{F(M)}{\vec{d}\cdot\vec{n_\pi}}\cdot\vec{d}$$

$$\vec{r}_{P_{\pi,d}(M)} = \vec{r}_M - \frac{Ax_M + By_M + Cz_M + D}{Ap + Bq + Cr} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$= \begin{pmatrix} x_M - \frac{Ax_M + By_M + Cz_M + D}{Ap + Bq + Cr} \cdot p \\ y_M - \frac{Ax_M + By_M + Cz_M + D}{Ap + Bq + Cr} \cdot q \\ z_M - \frac{Ax_M + By_M + Cz_M + D}{Ap + Bq + Cr} \cdot r \end{pmatrix}$$

$$= \frac{1}{Ap + Bq + Cr} \begin{pmatrix} x_M (Ap + Bq + Cr - Ap) + y_M(-Bp) + z_M(-Cp) - Dp \\ y_M (Ap + Bq + Cr - Bq) + x_M \cdot (-Aq) + z_M (-Cq) - Dq \\ z_M (Ap + Bq + Cr - Cr) + x_M \cdot (-Ar) + y_M (-rB) - Dr \end{pmatrix}$$

$$= \frac{1}{Ap + Bq + Cr} \cdot \begin{pmatrix} x_M (Bq + Cr) + y_M (-Bp) + z_M(-Cp) \\ x_M \cdot (-Aq) + y_M \cdot (Ap + Cr) + z_M \cdot (-Cq) \\ x_M \cdot (-Ar) + y_M \cdot (-Br) + z_M \cdot (Ap + Bq \end{pmatrix} -$$

$$- \frac{D}{Ap + Bq + Cr} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} =$$

$$= \frac{1}{Ap+Bq+Cr} \cdot \begin{pmatrix} Bq+Cr & -Bp & -Cp \\ -Aq & Ap+Cr & -Cq \\ -Ar & -Br & Ap+Bq \end{pmatrix} \cdot \underbrace{\begin{pmatrix} x_M \\ y_M \\ z_M \end{pmatrix}}_{[M]_R}$$

$$- \frac{D}{Ap+Bq+Cr} \cdot [d]_b$$

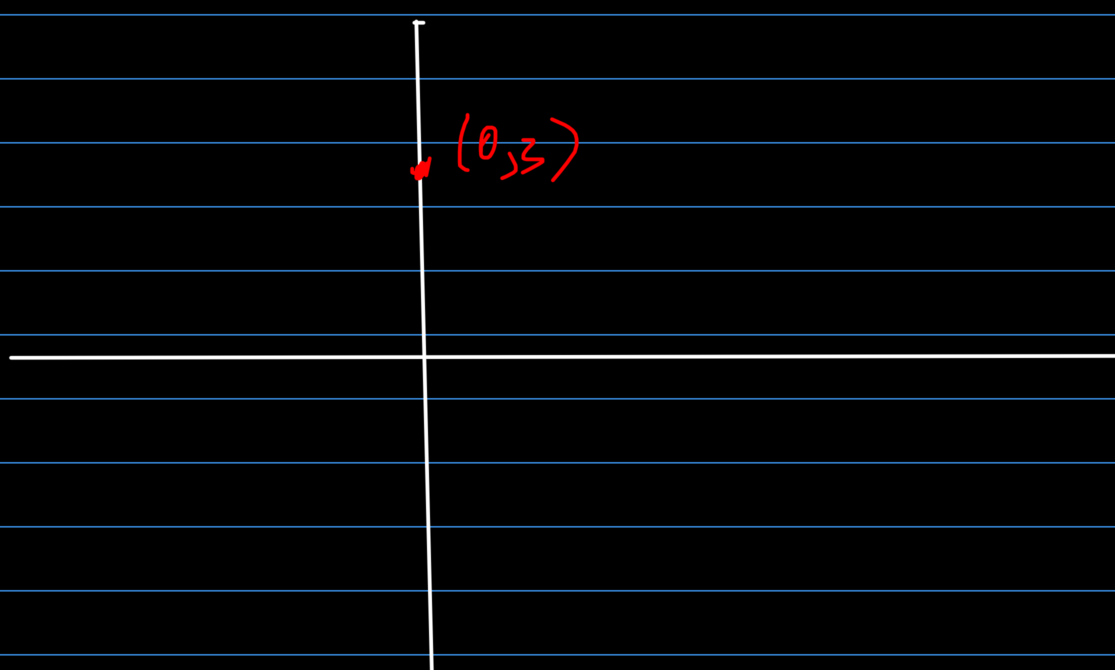
$$[M]_R = [\overrightarrow{OM}]_b$$

12. Find the equation of the line passing through the intersection point of

$$d_1 : 3x - 2y + 5 = 0, \quad d_2 : 4x + 3y - 1 = 0$$

and crossing the positive half axis of Oy at the point A with $OA = 3$.

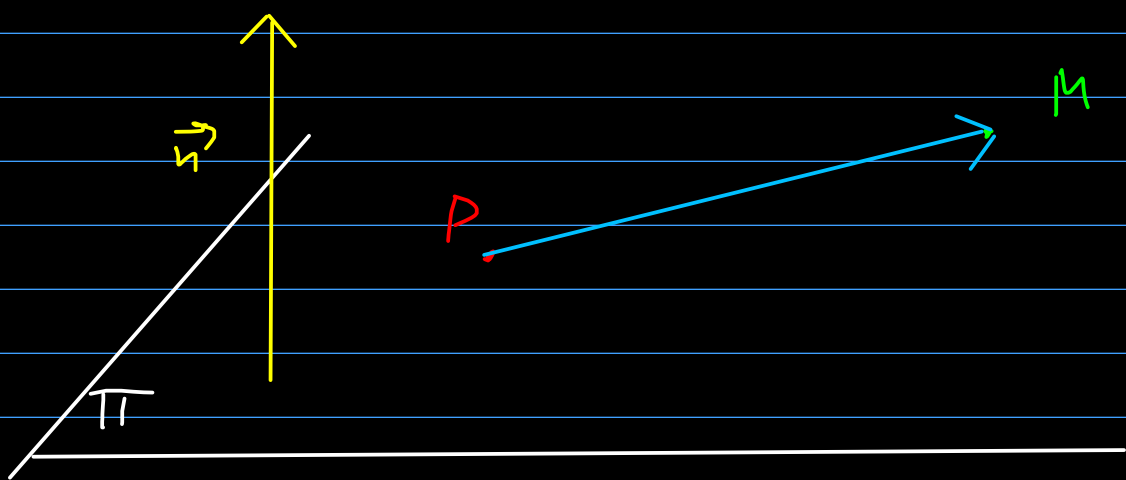
SOLUTION.



- **The normal vector of a plane.** Consider the plane $\pi\,:\,Ax+By+Cz+D=0$ and the point $P(x_0,y_0,z_0)\in\pi$. The equation of π becomes

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0. \tag{5.8}$$

If $M(x,y,z)\in\pi$, the coordinates of $\overrightarrow{PM}$ are $(x-x_0,y-y_0,z-z_0)$ and the equation (5.8) tells us that $\vec{n}\cdot\overrightarrow{PM}=0$, for every $M\in\pi$, that is $\vec{n}\perp\overrightarrow{PM}=0$, for every $M\in\pi$, which is equivalent to $\vec{n}\perp\vec{\pi}$, where $\vec{n}\,(A,B,C)$. This is the reason to call $\vec{n}\,(A,B,C)$ *the normal vector* of the plane π.



$\pi : Ax+By+Cz+D=0$

Vrem $\rho \parallel \pi$ cu $\rho \ni M(x_0,y_0,z_0)$

$\rho \parallel \pi \iff \rho : Ax+By+Cz+D'=0$

$M\in\rho \iff Ax_0+By_0+Cz_0+D'=0 \iff$

$\iff D'=-Ax_0-By_0-Cz_0$.

$p: A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

6. Show that two different parallel lines are either projected onto parallel lines or on two points by a projection $p_{\pi,d}$, where

$$\pi : Ax+By+Cz+D=0,\ \ d : \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$$

and $\pi \not\parallel d$.

$$l_1 : \begin{cases} x = x_1 + \lambda\alpha \\ y = y_1 + \lambda\beta \\ z = z_1 + \lambda\gamma \end{cases}$$

$$l_2 : \begin{cases} x = x_2 + \lambda\alpha \\ y = y_2 + \lambda\beta \\ z = z_2 + \lambda\gamma \end{cases}$$

$$\overrightarrow{r_{P_{\pi,d}(M)}} = \vec{r}_M - \frac{F(M)}{\vec{n}_\pi \cdot \vec{d}} \cdot \vec{d} =$$

$$= \begin{pmatrix} x_1+\lambda\alpha \\ y_1+\lambda\beta \\ z_1+\lambda\gamma \end{pmatrix} - \frac{A(x_1+\lambda\alpha)+B(y_1+\lambda\beta)+C(z_1+\lambda\gamma)+D}{Ap+Bq+Cr}\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$= \Bigg[\overbrace{\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \frac{Ax_1+By_1+Cz_1+D}{Ap+Bq+Cr} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}}^{\omega_1}\Bigg] +$$

$$+\lambda\Bigg[\underbrace{\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} - \frac{A\alpha+B\beta+C\gamma}{Ap+Bq+Cr}\cdot\begin{pmatrix}p\\ q\\ r\end{pmatrix}}_{w}\Bigg]$$

$$\Rightarrow \overrightarrow{r_{p_{\pi,d}(M)}} = w_1 + \lambda\cdot w$$

$\Rightarrow p_{\pi,d}(l_1)$ este o dreaptă

daca $w \neq 0$

Analog $\overrightarrow{r_{p_{\pi,d}(N)}} = w_2 + \lambda\cdot w$

$\forall\ N \in l_2$

$$w_1 = \begin{pmatrix}x_1\\ y_1\\ z_1\end{pmatrix} - \frac{Ax_1+By_1+Cz_1+D}{Ap+Bq+Cr}\cdot\begin{pmatrix}p\\ q\\ r\end{pmatrix}$$

$$w_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \frac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$w = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} - \frac{A\alpha + B\beta + C\gamma}{Ap + Bq + Cr} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

<u>Concluzie</u>: $p_{\pi,d}(l_1)$ și $p_{\pi,d}(l_2)$ sunt drepte paralele, sub presupunerea că $w \neq 0$

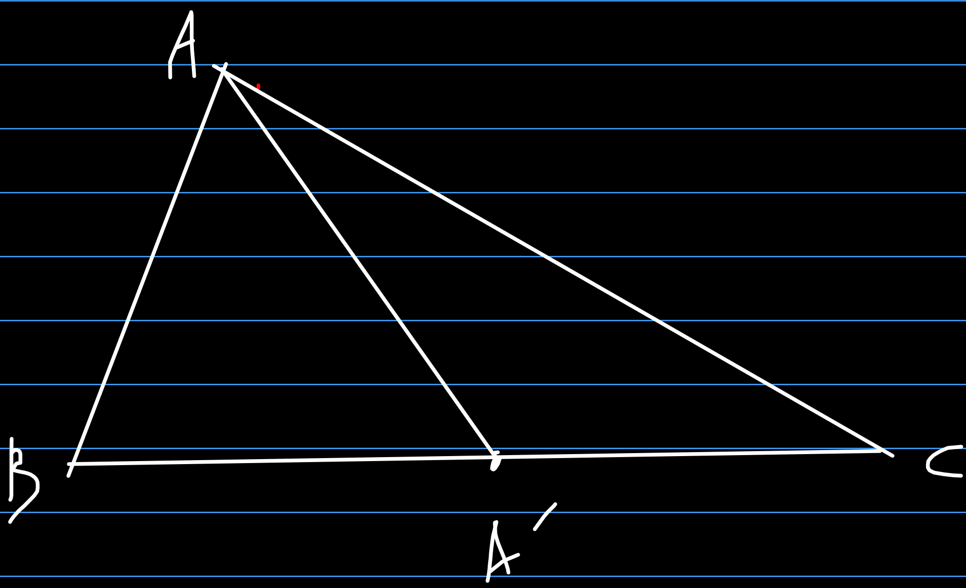
$$w = 0 \iff \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{\vec{n_\pi} \cdot \vec{l_1}}{\vec{n_\pi} \cdot \vec{d}} \cdot \vec{d} \iff$$

$$\iff \vec{l_1} = \frac{\vec{n_\pi} \cdot \vec{l_1}}{\vec{n_\pi} \cdot \vec{d}} \cdot \vec{d} \iff l_1 \parallel d$$

1. **(2p)** Consider the triangle ABC and the midpoint A' of the side $[BC]$. Show that

$$4\,\overrightarrow{AA'}^{\,2} - \overrightarrow{BC}^{\,2} = 4\,\overrightarrow{AB}\cdot\overrightarrow{AC}\,.$$

Solution.



$$\overrightarrow{AA'} = \frac{\overrightarrow{AB}+\overrightarrow{AC}}{2}$$

$$\overrightarrow{AA'}\cdot\overrightarrow{AA'} = \left(\frac{\overrightarrow{AB}+\overrightarrow{AC}}{2}\right)\cdot\left(\frac{\overrightarrow{AB}+\overrightarrow{AC}}{2}\right)$$

$$AA'^2 = \frac{AB^2+\overrightarrow{AB}\cdot\overrightarrow{AC}+\overrightarrow{AC}\cdot\overrightarrow{AB}+AC^2}{4} =$$

$$= \frac{AB^2+AC^2+2\cdot\overrightarrow{AB}\cdot\overrightarrow{AC}}{4}$$

$$BC^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos\alpha$$

$$\vec{BC}^2 = \vec{AB}^2 + \vec{AC}^2 - 2 \cdot \vec{AB} \cdot \vec{AC}$$

$$4\vec{AA'}^2 - \vec{BC}^2 = AB^2 + AC^2 + 2 \cdot \vec{AB} \cdot \vec{AC} -$$

$$- AB^2 - AC^2 + 2 \cdot \vec{AB} \cdot \vec{AC}$$

$$\Rightarrow 4\vec{AA'} - \vec{BC}^2 = 4 \cdot \vec{AB} \cdot \vec{AC}$$