Seminar W6 - 915

Cross product (vector product)

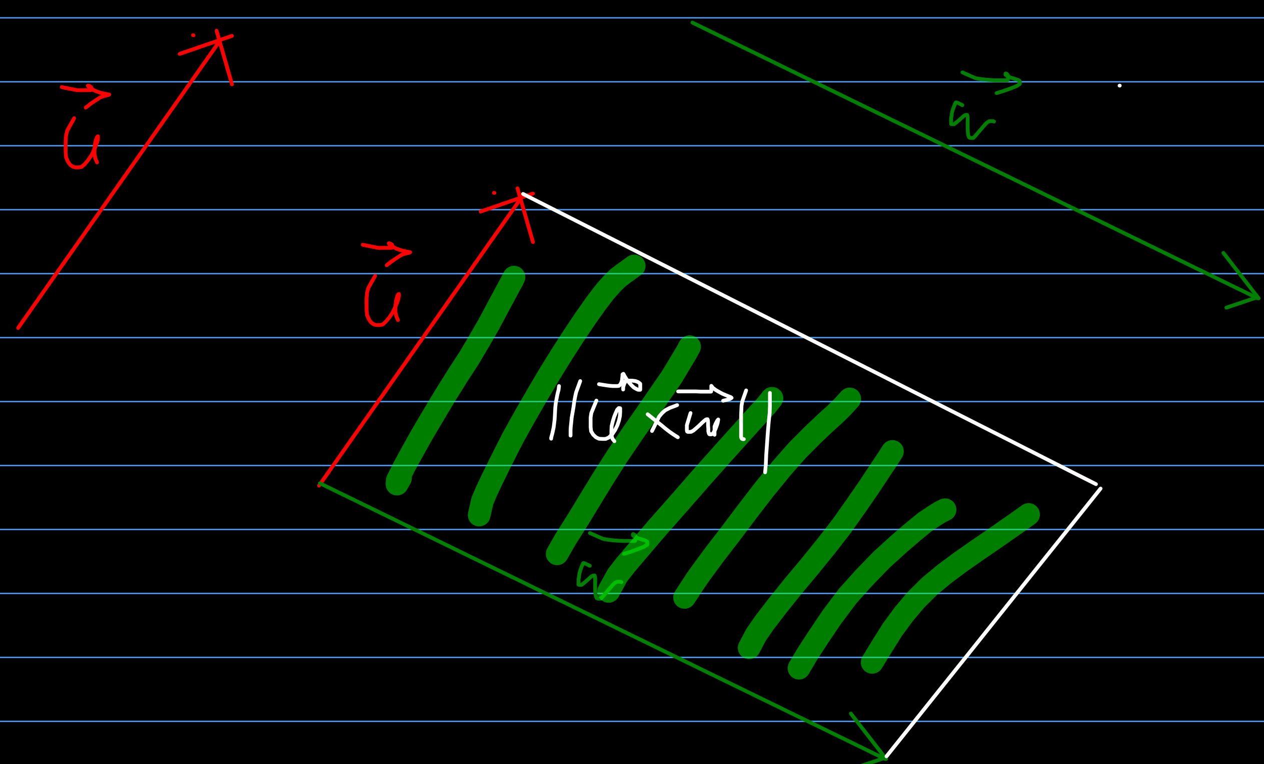
$\vec{u}, \vec{w} \in V$, if $\vec{u}, \vec{w}$ linearly dependent

$$\vec{u} \times \vec{w} = \vec{0}$$

if $\vec{u}, \vec{w}$ lin. independent, then:

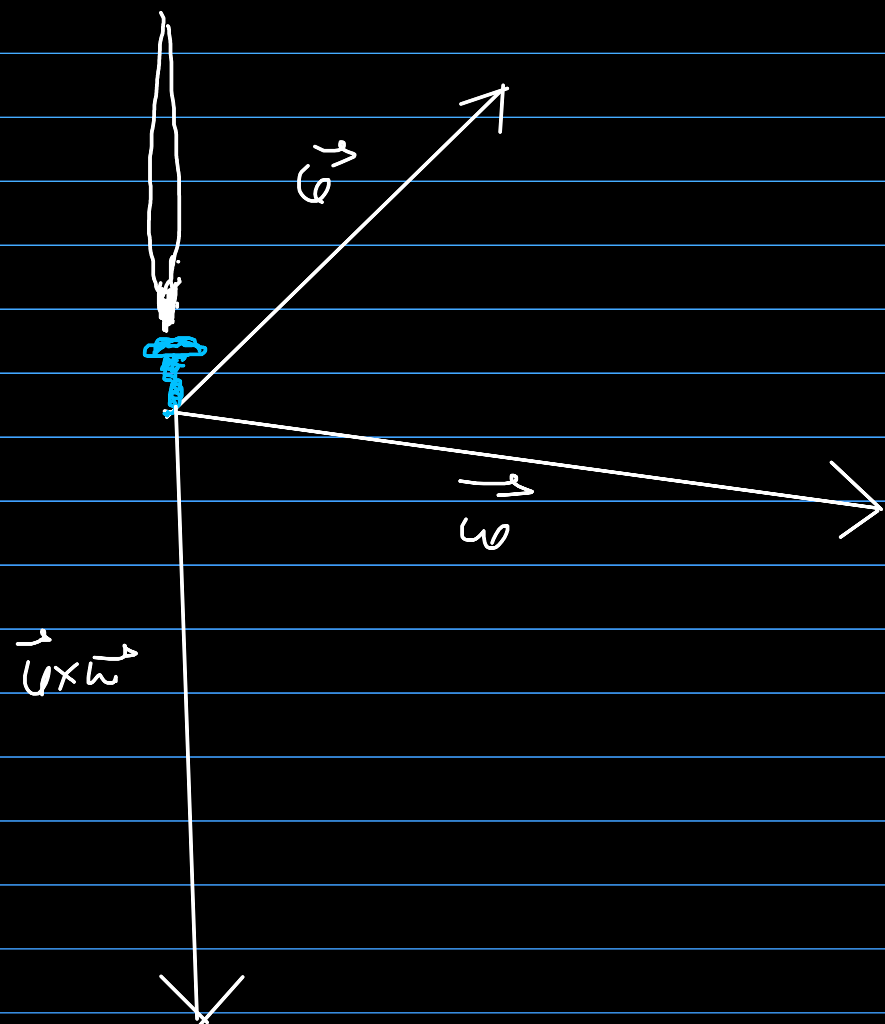
$$\vec{u} \times \vec{w} \in V$$

- **direction**: perpendicular to $\vec{u}$ and $\vec{w}$, perpendicular to $\langle \vec{u}, \vec{w} \rangle$
- **norm**: $\|\vec{u} \times \vec{w}\| = \|\vec{u}\| \cdot \|\vec{w}\| \cdot \sin(\widehat{\vec{u}, \vec{w}})$



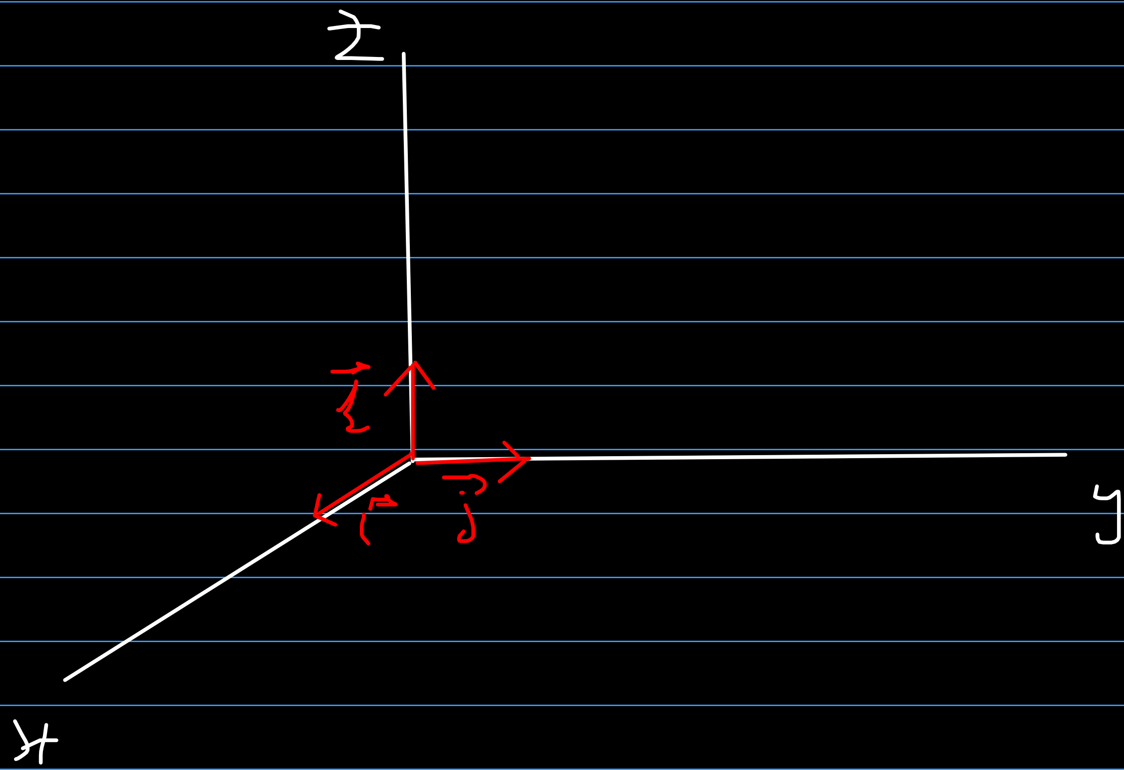
• orientation :

"screw rule"



If the reference system $\mathcal{R} = (O, (\vec{i}, \vec{j}, \vec{k}))$ is orthonormal and direct

$\rightarrow \vec{i} \times \vec{j} = \vec{k}$

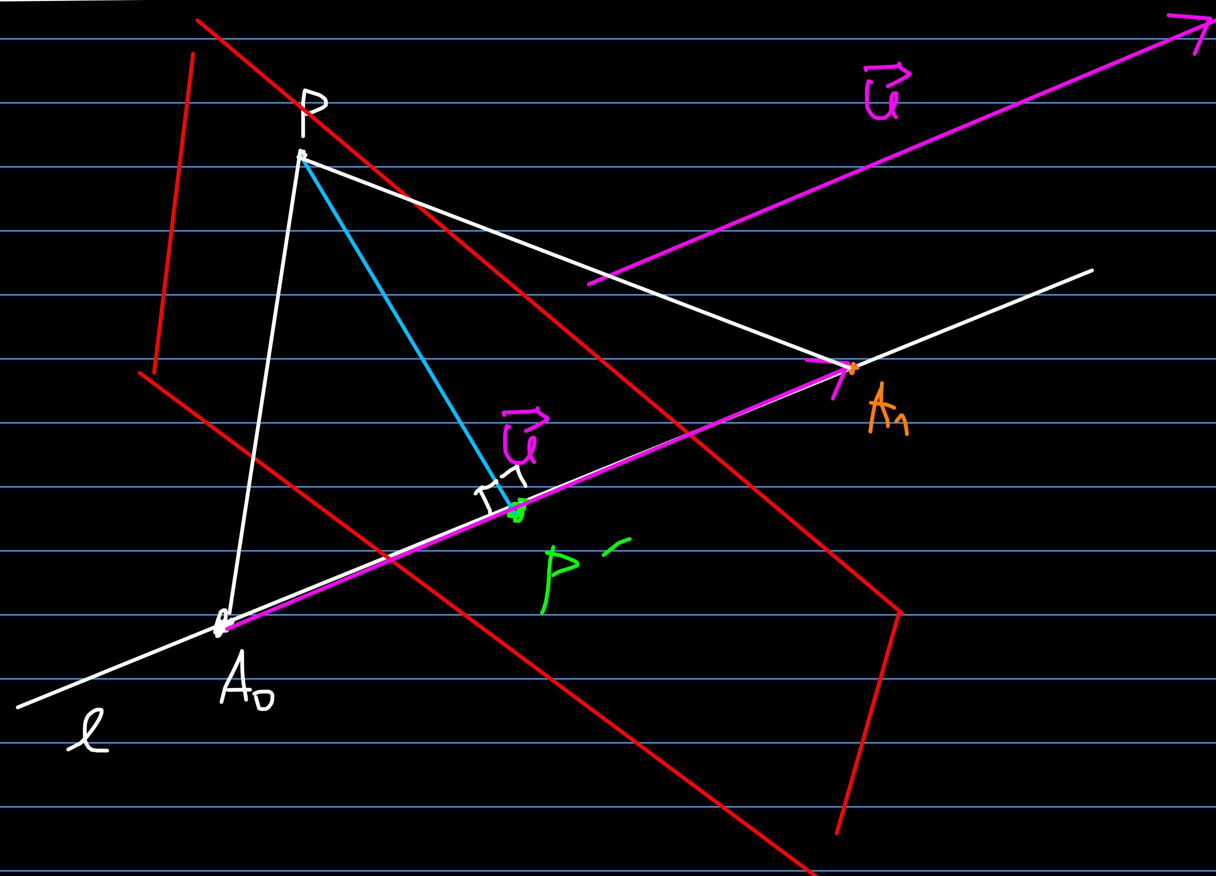


Then the cross product can be computed as follows: $\vec{v}(a_1, b_1, c_1)$, $\vec{w}(a_2, b_2, c_2)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} =$$

$$= \left(b_1c_2 - c_1b_2,\ a_2c_1 - a_1c_2,\ a_1b_2 - a_2b_1 \right)$$

The distance from a point to a line



$A_0 \in \ell$, $\vec{u} \parallel \ell$

$\exists A_1 \in \ell : \overrightarrow{A_0A_1} = \vec{u}$

PP' is a height in the ΔPA_0A_1

$$\Rightarrow PP' = dist(P,\ell) = \frac{2A_{PA_0A_1}}{A_0A_1} =$$

$$= \frac{\|\overrightarrow{PA_0} \times \overrightarrow{A_0A_1}\|}{\|\overrightarrow{A_0A_1}\|} = \frac{\|\overrightarrow{PA_0} \times \vec{u}\|}{\|\vec{u}\|}$$

6.4. Find the distance from the point $P(1, 2, -1)$ to the line $\ell : x = y = z$

Proof: We choose $\vec{u}_\ell = (1,1,1)$ and $A_0(0,0,0) \in \ell$

$$\overrightarrow{PA_0} = (-1, -2, 1)$$

$$\left(\begin{array}{l} A(x_A, y_A, z_A), \quad B(x_B, y_B, z_B) \\ \overrightarrow{AB}(x_B - x_A, y_B - y_A, z_B - z_A) \end{array} \right)$$

$$\operatorname{dist}(P, \ell) = \frac{\|\overrightarrow{PA_0} \times \vec{u}_\ell\|}{\|\vec{u}_\ell\|}$$

$$\overrightarrow{PA_0} \times \vec{u}_\ell = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -3\vec{i} + 2\vec{j} + \vec{k}$$

$$\|\overrightarrow{PA_0} \times \vec{u}_\ell\| = \sqrt{9+4+1} = \sqrt{14}$$

$$\|\vec{u}_\ell\| = \sqrt{3}$$

$$\operatorname{dist}(P, \ell) = \frac{\sqrt{14}}{\sqrt{3}} = \sqrt{\frac{14}{3}}$$

6.5. Find the area of the triangle ABC and the lengths of its heights, where

$A(-1,1,2)$, $B(2,-1,1)$, $C(2,-3,-2)$

$\vec{AC}(3,-4,-4)$, $\vec{AB}(3,-2,-1)$

$\vec{BC}(0,-2,-3)$

$$A_{ABC} = \frac{1}{2} \|\vec{AB} \times \vec{BC}\|$$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -1 \\ 0 & -2 & -3 \end{vmatrix} =$$

$$= 4\vec{i} + 9\vec{j} - 6\vec{k}$$

$$\Rightarrow A_{ABC} = \frac{1}{2} \cdot \sqrt{16+81+36} = \frac{\sqrt{133}}{2}$$

$$h_A = \frac{2\, A_{ABC}}{\|\overrightarrow{BC}\|} = \frac{\sqrt{133}}{\sqrt{13}} = \sqrt{\frac{133}{13}}$$

$$h_B = \frac{2 \cdot A_{ABC}}{\|\overrightarrow{AC}\|} = \frac{\sqrt{133}}{\sqrt{41}} = \sqrt{\frac{133}{41}}$$

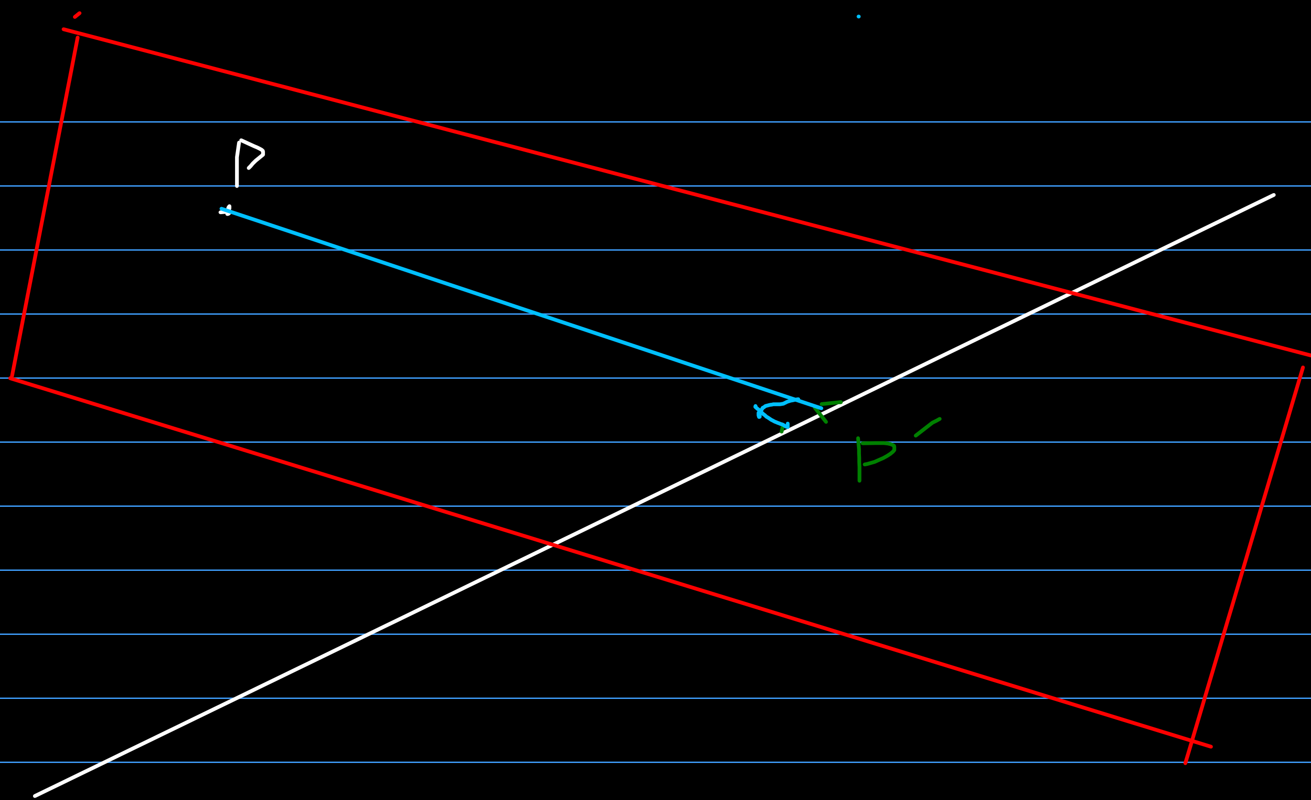
$$h_C = \frac{2 \cdot A_{ABC}}{\|\overrightarrow{AB}\|} = \frac{\sqrt{133}}{\sqrt{14}} = \sqrt{\frac{133}{14}}$$

<u>Ex:</u> $6.\pi$ Consider the line:

$$l: \begin{cases} \pi_1: x - 7y + 5z - 3 = 0 \\ \pi_2: 2x - y + 3z + 5 = 0 \end{cases}$$

and the point $P(1,2,3)$.

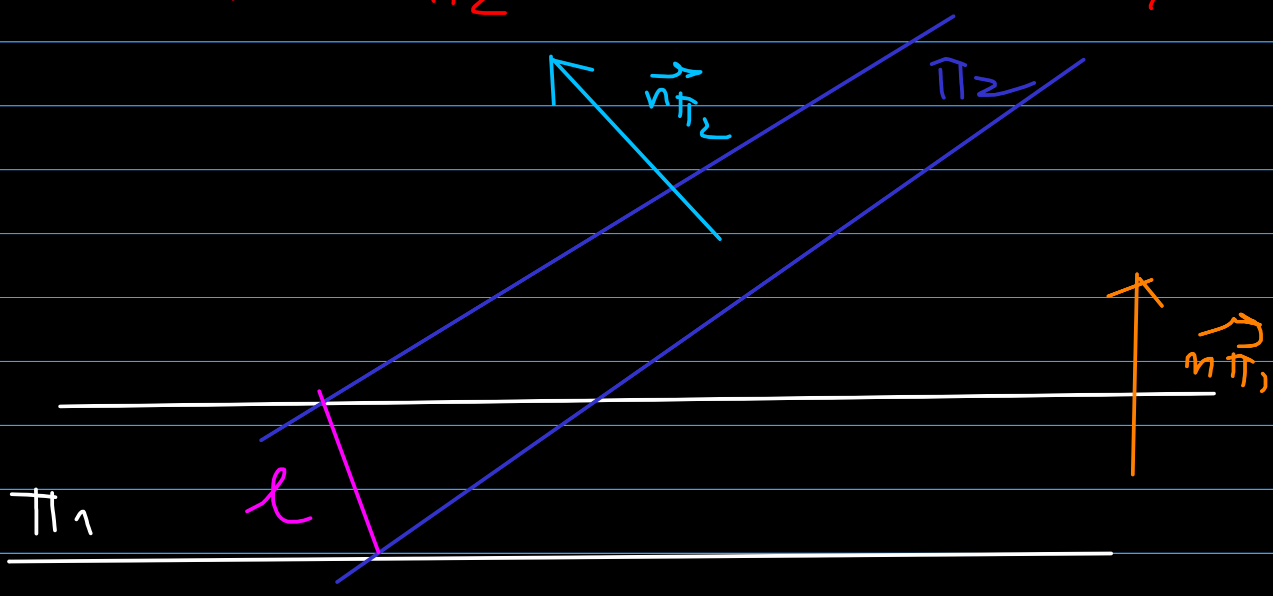
Find the equation of the perpendicular from P onto l.



If we have a line ℓ given by

$$\ell : \begin{cases} \Pi_1 : A_1 x + B_1 \cdot y + C_1 z + D_1 = 0 \\ \Pi_2 : A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$

Then $\vec{n_{\Pi_1}} \times \vec{n_{\Pi_2}}$ is a director vector for ℓ.



$\vec{n}_{\pi_1} \perp \pi_1 \Rightarrow \vec{n}_{\pi_1} \perp \ell$

$\vec{n}_{\pi_2} \perp \pi_2 \Rightarrow \vec{n}_{\pi_2} \perp \ell$

$\Rightarrow \ell \parallel (\vec{n}_{\pi_1} \times \vec{n}_{\pi_2})$.

$$\ell: \begin{cases} \pi_1: x - 7y + 5z - 3 = 0 \\ \pi_2: 2x - y + 3z + 5 = 0 \end{cases}$$

$\vec{n}_{\pi_1} = (1, -7, 5)$, $\vec{n}_{\pi_2} = (2, -1, 3)$

$$\vec{n}_{\pi_1} \times \vec{n}_{\pi_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -7 & 5 \\ 2 & -1 & 3 \end{vmatrix} =$$

$$= -16\vec{i} + 7\vec{j} + 13\vec{k}$$

$\Rightarrow \vec{\ell} = (-16, 7, 13)$

We will now write the equation of a plane Π that is perpendicular to the line ℓ and contains the point P.

$$\Pi : \quad -16x + 7y + 13z + D = 0$$

$$P \in \Pi \Rightarrow \quad -16 + 14 + 39 + D = 0$$

$$\Rightarrow \quad D = -37$$

$$\Rightarrow \Pi : \quad -16x + 7y + 13z - 37 = 0$$

[If we want a plane Π whose normal vector is $\vec{u} = (A, B, C)$ and so that $\Pi \ni P(x_0, y_0, z_0)$, then

$$\Pi : \quad A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$]

$$P' : \begin{cases} x - 7y + 5z - 3 = 0 \\ 2x - y + 3z + 5 = 0 \qquad \Leftrightarrow \\ -16x + 7y + 13z - 37 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 7y - 5z + 3 \\ 14y - 10z + 6 - y + 3z + 5 = 0 \\ -112y + 80z - 48 + 7y + 13z - 37 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 7y - 5z + 3 \\ 13y - 7z + 11 = 0 \\ -105y + 93z - 85 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 7y - 5z + 3 \\ z = \dfrac{13y + 11}{7} \\ -105y + \dfrac{1209y + 1023}{7} - 85 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} y\left(\dfrac{1209}{7} - 105\right) = 85 - \dfrac{1023}{7} \\ z = \dfrac{13y}{7} + \dfrac{11}{7} \\ x = 7y - 5z + 3 \end{cases}$$

$$\Rightarrow \begin{cases} y = \dfrac{595 - 1023}{7} \cdot \dfrac{7}{1209 - 735} \\ z = \dfrac{13}{7}y + \dfrac{11}{7} \\ x = 7y - 5z + 3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y_{p'} = \dfrac{-428}{474} = \dfrac{-214}{237} \\ z_{p'} = \dfrac{13}{7} \cdot \dfrac{-214}{237} + \dfrac{11}{7} \\ x_{p'} = 7y - 5z + 3 \end{cases}$$

$\Rightarrow$ perpendicular: $\dfrac{x - x_P}{x_{P'} - x_P} = \dfrac{y - y_P}{y_{P'} - y_P} = \dfrac{z - z_P}{z_{P'} - z_P}$

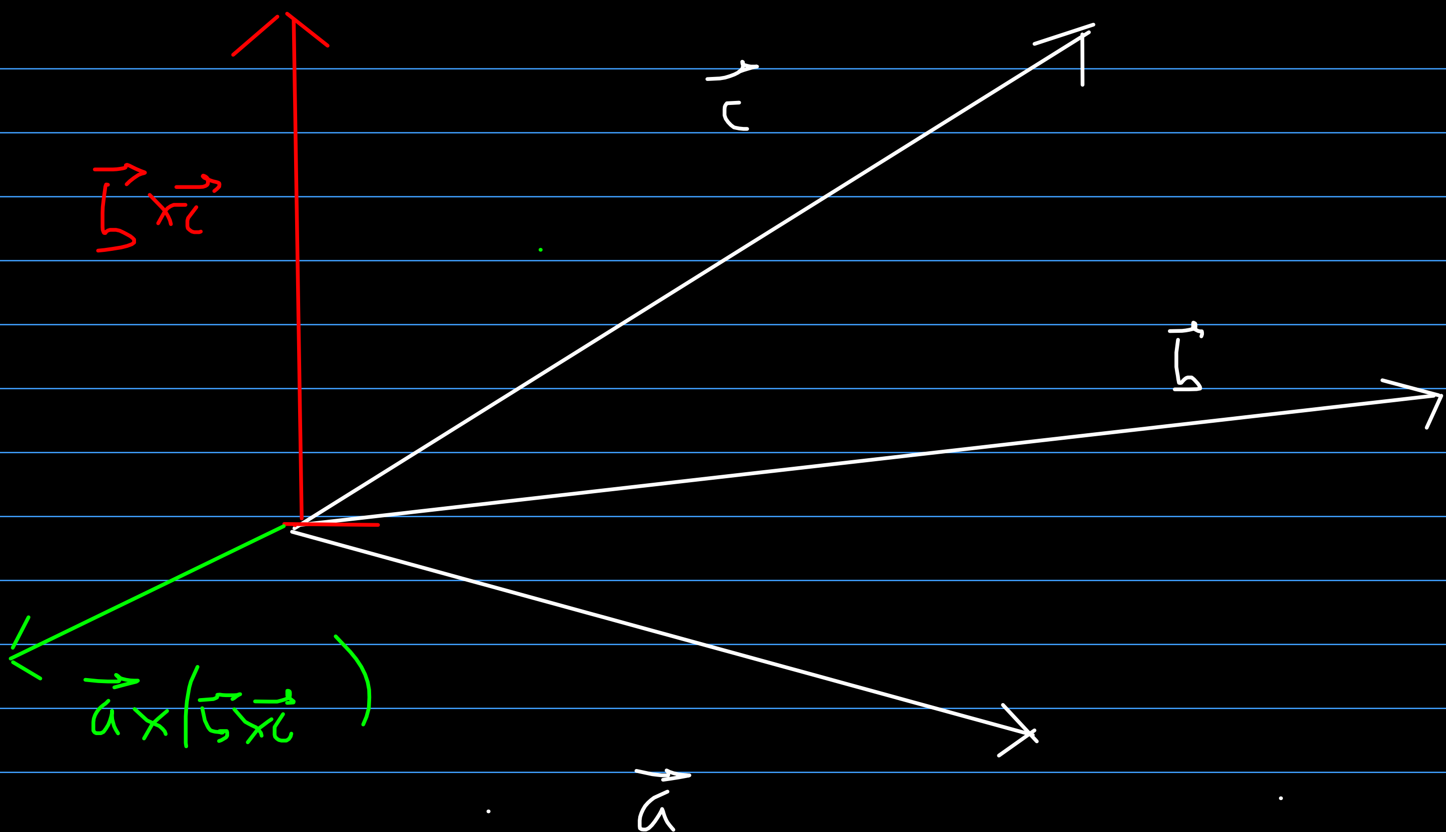
The double cross product

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \end{vmatrix} =$$

$$= (\vec{a}\cdot\vec{c})\cdot\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{b}\cdot\vec{c} & \vec{a}\cdot\vec{c} \end{vmatrix} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$$

$\Rightarrow$ the cross product is not associative!!!



Jacobi:

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$$