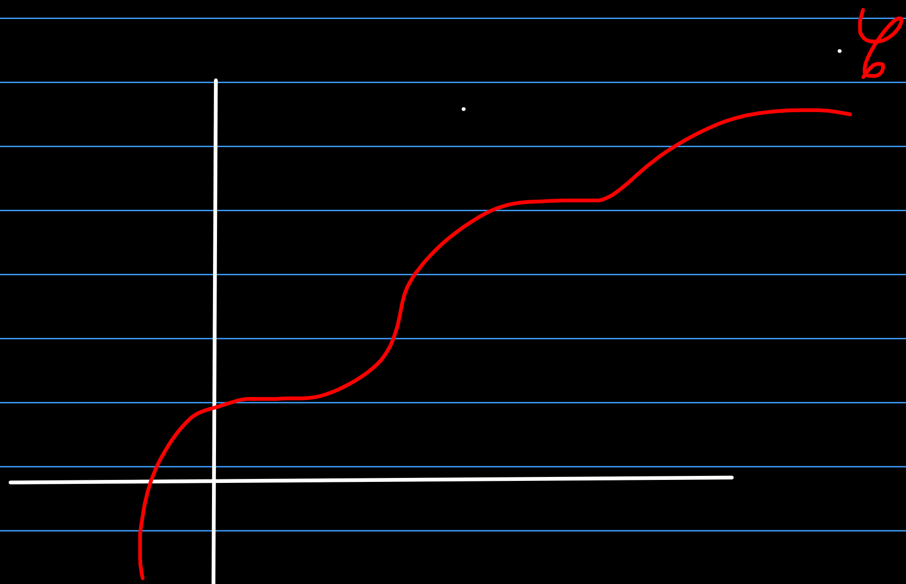
Seminar W8 - 914

Curves:



$\mathcal{C}$ curve (2D, 3D)

parametric:

$$\begin{cases} x = x(t) \\ y = y(t) \\ (z = z(t)) \end{cases}$$

implicit:

$$f(x,y) = 0$$

$$(f(x,y,z) = 0)$$

circle in 2D: (centered in 0, radius 1)

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}, \quad t \in [0, 2\pi]$$

$$x^2 + y^2 = 1$$

The director vector of the tangent to $\mathcal{C}$ in the point M_0 is

$$\lim_{M \to M_0} \overrightarrow{M_0 M}$$

The tangent to a curve in M_0 = the line going through M_0 that has the director vector from above

If the curve is given:

- parametrically: $\gamma : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$

$$T_\gamma(t_0) : \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}$$

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \gamma'(t_0)$$

· implicitly $\mathcal{C}: f(x, y) = 0$

$T_{\mathcal{C}}(x_0, y_0)$:

$$f'_x(x_0, y_0) \cdot (x - x_0) + f'_y(x_0, y_0) \cdot (y - y_0) = 0$$

· $\boxed{\text{In the plane}}$.

The normal line to a curve =

= perp. to the tangent and goes through the same point.

– $\mathcal{C}$ given parametrically:

$$N_\gamma : x'(t_0) \cdot (x - x_0) + y'(t_0) \cdot (y - y_0) = 0$$

Explanation: $\vec{u}(a,b)$

$\Rightarrow$ the equation of a line perpendicular to $\vec{u}$ (that contains (x_0, y_0)):

$$a(x-x_0)+b(y-y_0)=0$$

- if given implicitly:

$$N_{\mathcal{C}}(x_0,y_0): \frac{x-x_0}{f'_x(x_0,y_0)} = \frac{y-y_0}{f'_y(x_0,y_0)}$$

$$\Leftrightarrow f'_y(x_0,y_0)(x-x_0) - f'_x(x_0,y_0)\cdot(y-y_0)=0.$$

$\boxed{\text{In space}}$

↳ normal plane

$\Rightarrow$ the plane perpendicular to the tangent and containing the point

– $\mathcal{C}$ parametric: $\mathcal{C}: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$

$$N_{\mathcal{C}}(t_0): \ x'(t_0)\cdot(x - x(t_0)) + y'(t_0)\cdot(y - y(t_0)) + z'(t_0)\cdot(z - z(t_0)) = 0$$

8.1. Show that the angle between the tangent of the circular helix

$$\mathcal{C}: \begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases} \quad t \in \mathbb{R}$$

and the z-axis is constant.

Let $P(t_0)$.

$x'(t) = -a\sin t \qquad x_0 = x(t_0)$

$y'(t) = a\cos t \qquad y_0 = y(t_0)$

$z'(t) = b \qquad z_0 = z(t_0)$

$\Rightarrow T_{\mathscr{C}}(t_0): \quad \dfrac{x - x_0}{-a\sin t_0} = \dfrac{y - y_0}{a\cos t_0} = \dfrac{z - z_0}{b}$

$\vec{u}_{T_{\mathscr{C}}(t_0)} = (-a\sin t_0, a\cos t_0, b)$

$\vec{u}_{Oz} = (0, 0, 1)$

$\vec{u}_{T_{\mathscr{C}}(t_0)} \cdot \vec{u}_{Oz} = b$

$\Rightarrow \cos(T_{\mathscr{C}}(t_0), Oz) = \dfrac{b}{\sqrt{a^2\sin^2 t_0 + a^2\cos^2 t_0 + b^2}} =$

$= \dfrac{b}{\sqrt{a^2 + b^2}}$

$\Rightarrow m\left(\widehat{T_\gamma(t_0), Oz}\right) = \arccos \frac{b}{\sqrt{a^2+b^2}}$

8.8. Eqns. of the tangent line and the normal plane for the curve

$$\begin{cases} x = e^t \cos 3t \\ y = e^t \sin 3t \\ z = e^{-2t} \end{cases}$$ at

the points $t=0$ and $t = \frac{\pi}{4}$

$x'(t) = e^t \cdot \cos 3t - 3 \cdot e^t \sin(3t)$

$y'(t) = e^t \cdot \sin 3t + 3e^t \cos(3t)$

$z'(t) = -2 \cdot e^{-2t}$

For $t=0$: $x'(0) = 1$ $z'(0) = -2$

$y'(0) = 3$

$\Rightarrow \quad x(0) = 1 \quad , \; y(0) = 0 \; , \quad z(0) = 1$

$$\Rightarrow \; T_{\mathcal{C}}(0) : \; \frac{x-1}{1} = \frac{y}{3} = \frac{z-1}{-2}$$

$$N_{\mathcal{C}}(0) : \; (x-1) + 3 \cdot y + (-2)(z-1) = 0$$

For $t = \frac{\pi}{4}$ we do the same thing.

8.7 Write the equations of the tangent line and the normal line in the point $(2, 0)$ of the ellipse:

$$\mathcal{E} : \; \frac{x^2}{4} + y^2 = 1$$

$$\frac{\partial f}{\partial x} = \frac{x}{2} \qquad \frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial f}{\partial x}(2,0) = 1 \qquad \frac{\partial f}{\partial y}(2,0) = 0$$

$$T_{\mathcal{C}}(2,0): \quad (x-2) = 0 \Leftrightarrow x = 2$$

$$N_{\mathcal{C}}(2,0): \quad y - 0 = 0 \Leftrightarrow y = 0$$

<u>Surfaces</u>:

→ parametric equations

$$\mathcal{S}: \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

<u>Example</u>: $\mathcal{S}: \begin{cases} x = \cos u \cdot \cos v \\ y = \cos u \cdot \sin v \\ z = \sin u \end{cases}$

→ sphere centered in $O(0,0,0)$ with radius r

$$T_\gamma(P(t_0)) : \begin{vmatrix} x - x(t_0) & y - y(t_0) & z - z(t_0) \\ \frac{\partial}{\partial u} x(t_0) & \frac{\partial}{\partial u} y(t_0) & \frac{\partial}{\partial u} z(t_0) \\ \frac{\partial}{\partial v} x(t_0) & \frac{\partial}{\partial v} y(t_0) & \frac{\partial}{\partial v} z(t_0) \end{vmatrix} = 0$$

$$N_{\varphi}(P(t_0)) : \frac{x - x(t_0)}{x} = \frac{y - y(t)}{y} = \frac{z - z(t_0)}{z}$$

$$\frac{\partial}{\partial u} \times \frac{\partial}{\partial v} \quad \frac{\partial}{\partial u} \times \frac{\partial}{\partial v} \quad \frac{\partial}{\partial u} \times \frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial u} \times \frac{\partial}{\partial v} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \times$$

$$\times \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

- If the surface is implicit:

$$\varphi : f(x,y,z) = 0$$

$$T_{\varphi}(x_0, y_0, z_0) : f'_x \cdot (x - x_0) + f'_y (y - y_0) +$$

$$+ f'_z (z - z_0) = 0$$

$$N_{\mathcal{H}}(x_0, y_0, z_0): \quad \frac{x-x_0}{f'_x} = \frac{y-y_0}{f'_y} = \frac{z-z_0}{f'_z}$$

8.9. Write the equations of the tangent planes of the hyperboloid of one sheet:

$$\mathcal{H}: x^2+y^2-z^2 = 1$$

at the points of the form $(x_0, y_0, 0)$ and show that they are parallel to the z-axis.

$$T_{\mathcal{H}}: f'_x(x_0, y_0, 0)\cdot(x-x_0) +$$

$$+ f'_y(x_0, y_0, 0)\cdot(y-y_0) + f'_z(x_0, y_0, 0)\cdot z = 0$$

$$f'_x(x,y,z) = 2x \Rightarrow f'_x(x_0,y_0,0) = 2x_0$$

$$f'_y(x,y,z) = 2y \Rightarrow f'_y(x_0,y_0,0) = 2y_0$$

$$f'_z(x,y,z) = -2z \Rightarrow f'_z(x_0,y_0,0) = 0$$

$$\Rightarrow T_{\mathcal{H}}(x_0,y_0,0): \; 2x_0(x-x_0) +$$
$$+ 2y_0(y-y_0) = 0$$

$$T_{\mathcal{H}} \parallel Oz \Leftrightarrow \overrightarrow{n_{T_{\mathcal{H}}}} \perp Oz \Leftrightarrow$$

$$\Leftrightarrow (2x_0, 2y_0, 0) \perp (0,0,1) \Leftrightarrow$$

$$\Leftrightarrow (2x_0, 2y_0, 0) \cdot (0,0,1) = 0$$

which is true.