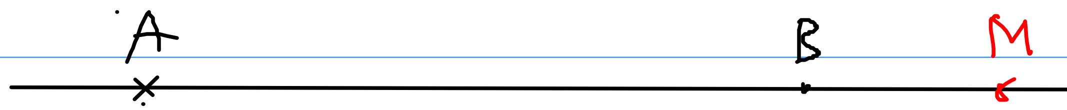
Seminar W2 – 913



ℓ line, $A, B \in \ell$

$\forall M \in \ell$: $\exists!\ \lambda \in \mathbb{R}$ so that:

$$\vec{r_M} = \lambda \vec{r_A} + (1-\lambda)\vec{r_B}$$

(the line equation in vector form)

In the particular case where $M \in [AB]$

$$\frac{AM}{MB} = \alpha \Rightarrow \boxed{\vec{r_M} = \frac{\alpha}{\alpha+1}\vec{r_B} + \frac{1}{\alpha+1}\vec{r_A}}$$

Say we have two lines ℓ_1, ℓ_2 in the plane and $A_1, B_1 \in \ell_1$, $A_2, B_2 \in \ell_2$. Let $\{M\} = \ell_1 \cap \ell_2$. There is a way for us to determine $\vec{r_M}$.

Step 1: Write M as a point on both lines.

$\Rightarrow \exists \lambda, \mu \in \mathbb{R}$ so that:

$$\begin{aligned} \overrightarrow{r_M} &= \lambda \overrightarrow{r_{A_1}} + (1-\lambda) \overrightarrow{r_{B_1}} \\ &= \mu \overrightarrow{r_{A_2}} + (1-\mu) \overrightarrow{r_{B_2}} \qquad (*) \end{aligned}$$

Step 2: Choose two lin. indep. vectors $\vec{u}$ and $\vec{w}$.

Step 3: Write $\overrightarrow{r_{A_1}}, \overrightarrow{r_{B_1}}, \overrightarrow{r_{A_2}}, \overrightarrow{r_{B_2}}$ as linear combination of $\vec{u}$ and $\vec{w}$

Step 4: We have arrived at a relation of the form:

$$\alpha(\lambda,\mu) \cdot \vec{u} + \beta(\lambda,\mu) \cdot \vec{w} = \vec{0}$$

$$\Rightarrow (S) \begin{cases} \alpha(\lambda,\mu) = 0 \\ \beta(\lambda,\mu) = 0 \end{cases}$$

Step 5: Solve the system (S) to get λ, μ.

Step 6: Replace λ in $(*)$

2.1. $\triangle ABC$, G centroid, H orthocenter

I incenter, O circumcenter

For any point P fixed in the plane (as our origin), we have:

(a) $$\vec{r_G} = \frac{\vec{r_A} + \vec{r_B} + \vec{r_C}}{3}$$

(b) $$\vec{r_I} = \frac{a\vec{r_A} + b\vec{r_B} + c\vec{r_C}}{a+b+c},$$ $a = BC$, $b = CA$, $c = AB$

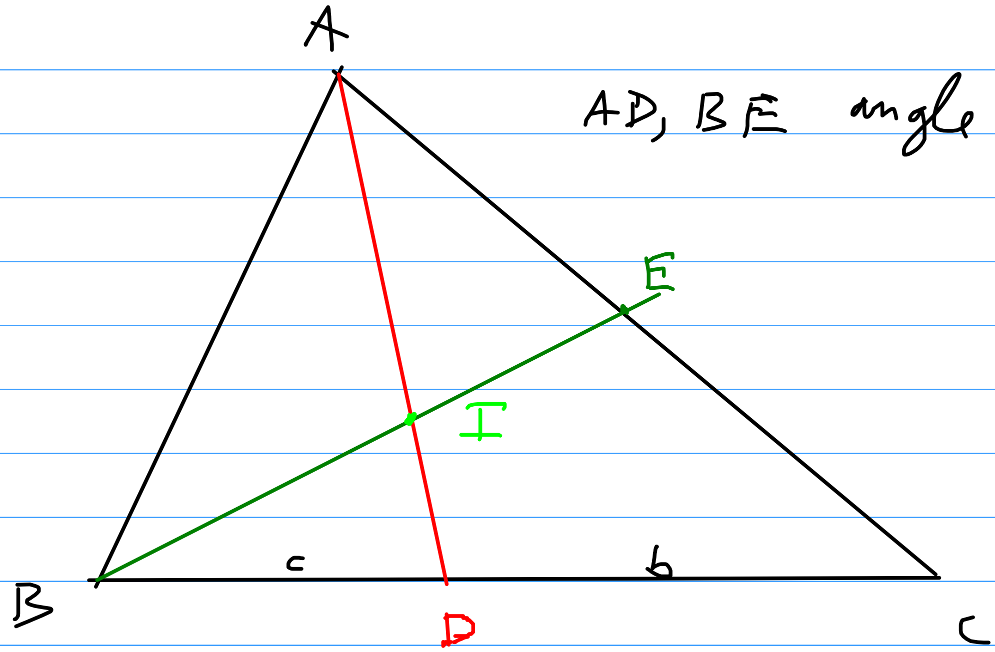
(c) $$\vec{r_H} = \frac{\tan A \cdot \vec{r_A} + \tan B \cdot \vec{r_B} + \tan C \cdot \vec{r_C}}{\tan A + \tan B + \tan C}$$

(d) $$\vec{r_O} = \frac{\sin 2A \cdot \vec{r_A} + \sin 2B \cdot \vec{r_B} + \sin 2C \cdot \vec{r_C}}{\sin 2A + \sin 2B + \sin 2C}$$

(barycentric coordinates)

(a) easy

(b)



AD, BE angle bisectors

bisector thm.: $\dfrac{BD}{DC} = \dfrac{c}{b}$, $\dfrac{AE}{EC} = \dfrac{c}{a} = \dfrac{AB}{BC}$

$$\vec{r_I} = \lambda \vec{r_A} + (1-\lambda)\vec{r_D}$$
$$= \mu \vec{r_B} + (1-\mu)\vec{r_E}$$

$$\vec{r_D} = \frac{c}{b+c}\cdot\vec{r_C} + \frac{b}{b+c}\cdot\vec{r_B}$$

$$\vec{r_E} = \frac{a}{a+c}\vec{r_A} + \frac{c}{a+c}\cdot\vec{r_C}$$

$$\Rightarrow \lambda\vec{r_A} + \frac{c(1-\lambda)}{b+c}\vec{r_C} + \frac{b(1-\lambda)}{b+c}\vec{r_B} =$$

$$= \mu\vec{r_B} + \frac{a(1-\mu)}{a+c}\vec{r_A} + \frac{c(1-\mu)}{a+c}\vec{r_C} \Leftrightarrow$$

$$\left(\lambda - \frac{a(1-\mu)}{a+c}\right)\vec{r_A} + \left(\frac{b(1-\lambda)}{b+c} - \mu\right)\cdot\vec{r_B} +$$

$$+\left(\frac{c(1-\lambda)}{b+c} - \frac{c(1-\mu)}{a+c}\right)\vec{r_C} = \vec{0}$$

But $\vec{r_A}, \vec{r_B}, \vec{r_C}$ are <u>not</u> lin. indp.

We know (because the triangle exists) that

$\vec{u} = \vec{AB}$, $\vec{w} = \vec{BC}$ are linearly independent.

$$\Rightarrow \vec{r_B} = \vec{r_A} + \vec{u}, \quad \vec{r_C} = \vec{r_B} + \vec{w} = \vec{r_A} + \vec{u} + \vec{w}$$

$$\Rightarrow \left(\lambda - \frac{a(1-\mu)}{a+c} + \frac{b(1-\lambda)}{b+c} - \mu + \frac{c(1-\lambda)}{b+c} - \frac{c(1-\mu)}{a+c}\right)\cdot$$

$$\cdot\vec{r_A} + \left(\frac{b(1-\lambda)}{b+c} - \mu\right)\cdot\vec{u} + \left(\frac{c(1-\lambda)}{b+c} - \frac{c(1-\mu)}{a+c}\right)\cdot$$

$$\vec{u} + \left(\frac{c(1-\lambda)}{b+c} - \frac{c(1-\mu)}{a+c}\right)\cdot\vec{w} = \vec{0}$$

$$\lambda - \frac{a(1-\mu)}{a+c} + \frac{b(1-\lambda)}{b+c} - \mu + \frac{c(1-\lambda)}{b+c} - \frac{c(1-\mu)}{a+c} =$$

$$= \lambda\left(1 - \frac{b}{b+c} - \frac{c}{b+c}\right) + \mu\left(\frac{a}{a+c} - 1 + \frac{c}{a+c}\right) +$$

$$-\frac{a}{a+c} + \frac{b}{b+c} + \frac{c}{b+c} - \frac{c}{a+c} = 0$$

$$\Rightarrow \left(\frac{b(1-\lambda)}{b+c} - \mu + \frac{c(1-\lambda)}{b+c} - \frac{c(1-\mu)}{a+c}\right)\cdot \vec{u} +$$

$$+ \left(\frac{c(1-\lambda)}{b+c} - \frac{c(1-\mu)}{a+c}\right)\cdot \vec{w} = \vec{0}$$

$\vec{u}, \vec{w}$ linearly independent, so:

$$\begin{cases} \dfrac{b(1-\lambda)}{b+c} - \mu + \dfrac{c(1-\lambda)}{b+c} - \dfrac{c(1-\mu)}{a+c} = 0 \\ \dfrac{c(1-\lambda)}{b+c} - \dfrac{c(1-\mu)}{a+c} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -\lambda + \mu\left(-1 + \dfrac{c}{a+c}\right) + \overset{\frac{a}{a+c}}{\cancel{1}} = 0 \\ \dfrac{c}{b+c} - \dfrac{c}{a+c} + \lambda\cdot\dfrac{-c}{b+c} + \mu\cdot\dfrac{c}{a+c} = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda = \overset{\frac{a}{a+c}}{\cancel{1}} - \mu + \dfrac{c\mu}{a+c} \\ \dfrac{c(a-b)}{(a+c)(b+c)} \overset{-\frac{ac}{(b+c)(a+c)}}{\cancel{- \dfrac{c}{b+c}}} + \dfrac{c\mu}{b+c} - \dfrac{c^2\mu}{(a+c)(b+c)} + \mu\cdot\dfrac{c}{a+c} = 0 \end{cases}$$

$$\frac{ac}{(b+c)(a+c)}$$

$$\Rightarrow \mu \cdot \left(\frac{c}{b+c} + \frac{c}{a+c} - \frac{c^2}{(a+c)(b+c)} \right) = \cancel{\frac{c}{b+c}} - \frac{c(a-b)}{(a+c)(b+c)} \quad \Big| \cdot (a+c)\cdot(b+c)$$

$$ac$$

$$\mu\left(c(a+c) + c(b+c) - c^2 \right) = \cancel{c(a+c)} - c(a-b) \quad \Big| : c$$

$$a$$

$$\Rightarrow \mu\left(a+b+2c-c\right) = \cancel{a+c} - a + b$$

$$b$$

$$\Rightarrow \mu = \frac{\cancel{c+b}}{a+b+c} \Rightarrow 1-\mu = \frac{a+c}{a+b+c}$$

$$\Rightarrow \vec{r_I} = \mu \vec{r_B} + (1-\mu) \cdot \vec{r_E} =$$

$$= \frac{b+c}{a+b+c} \cdot \vec{r_B} + \frac{a}{a+b+c} \cdot \left(\frac{a}{a+c} \vec{r_A} + \frac{c}{a+c} \vec{r_c} \right)$$

$$\vec{r_I} = \mu \vec{r_B} + \frac{a(1-\mu)}{a+c} \vec{r_A} + \frac{c(1-\mu)}{a+c} \vec{r_c}$$

$$\vec{r_I} = \frac{b}{a+b+c} \cdot \vec{r_B} + \frac{a}{a+b+c} \cdot \vec{r_A} + \frac{c}{a+b+c} \cdot \vec{r_C}$$

1.2. Consider the non zero angle $\widehat{BOB'}$ and the points $A \in [OB]$, $A' \in [OB']$. Show:

$$\overrightarrow{OM} = m \cdot \frac{1-n}{1-mn} \overrightarrow{OA} + n \cdot \frac{1-m}{1-mn} \overrightarrow{OA'}$$

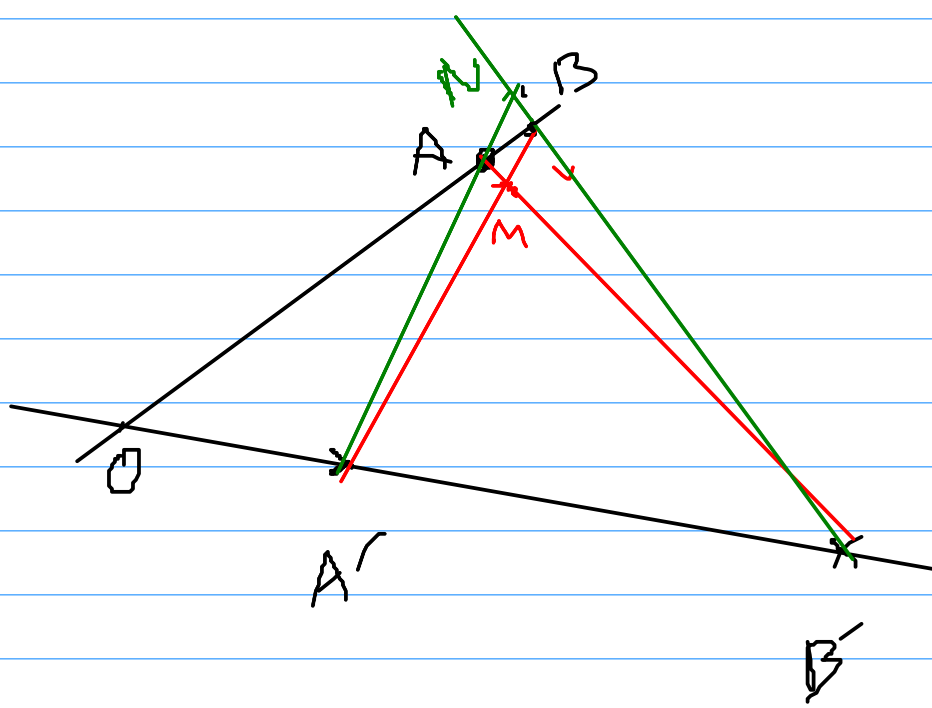
$$\overrightarrow{ON} = m \frac{n-1}{n-m} \overrightarrow{OA} + n \cdot \frac{m-1}{m-n} \overrightarrow{OA'}$$

where $\{M\} = AB' \cap A'B$

$\{N\} = AA' \cap BB'$

$\vec{u} := \overrightarrow{OA}$, $\vec{v} := \overrightarrow{OA'}$

$\overrightarrow{OB} = m \cdot \overrightarrow{OA}$, $\overrightarrow{OB'} = n \cdot \overrightarrow{OA'}$



$$\vec{OM} = \lambda \cdot \vec{OA'} + (1-\lambda)\cdot \vec{OB'} = \lambda \vec{u} + (1-\lambda) n \cdot \vec{v}$$

$$= \mu \vec{OA'} + (1-\mu)\cdot \vec{OB} = \mu \cdot \vec{v} + (1-\mu)\cdot m \cdot \vec{u}$$

$\widehat{BOB'}$ is nonzero $\Rightarrow \vec{u}, \vec{v}$ are linearly indep.

So from $\lambda \vec{u} + (1-\lambda)\cdot n \vec{v} = \mu \cdot \vec{v} + (1-\mu)\cdot m \cdot \vec{u}$

$$\Rightarrow (\lambda - m + \mu m)\cdot \vec{u} + (n - \lambda n - \mu)\cdot \vec{v} = \vec{0}$$

$\vec{u}, \vec{v}$ lin indep. $\Rightarrow \begin{cases} \lambda - m + \mu m = 0 \\ n - \lambda n - \mu = 0 \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} \lambda = m - \mu m \\ n - (m - \mu m)\cdot n - \mu = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda = m - \mu m \\ n - mn + \mu mn - \mu = 0 \end{cases}$$

$$\Rightarrow \mu(mn - 1) = mn - n$$

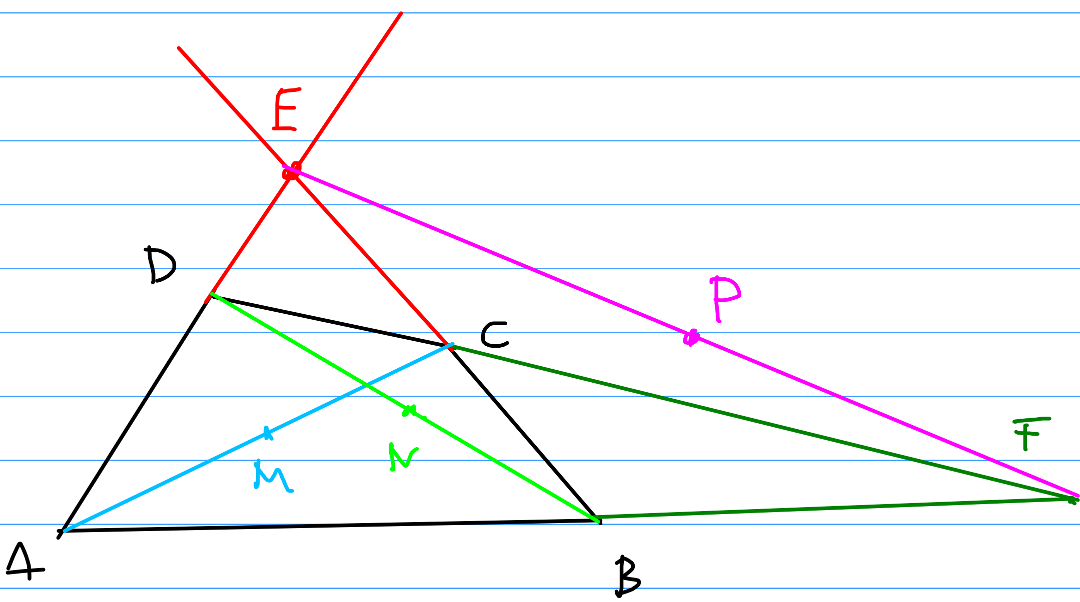
$$\Rightarrow \mu = \frac{n(m-1)}{1-mn} \Rightarrow 1 - \mu = \frac{1 - mn - mn + n}{1-mn}$$

$$\Rightarrow \vec{OM} = \mu \cdot \vec{v} + (1-\mu)\cdot m \cdot \vec{u}$$

$$\vec{OM} = \frac{n(m-1)}{1-mn}\cdot \vec{v} + \frac{m(1+n)}{1-mn}\cdot \vec{u}$$

We can do the same thing for $\overrightarrow{ON}$

3.3. Show that the midpoints of the diagonals of a complete quadrilateral are collinear



Show that M, N, P collinear

Let $\overrightarrow{AB} =: \vec{u}$, $\overrightarrow{AD} = \vec{w}$

$\Rightarrow$ $\overrightarrow{AE} = \alpha \cdot \vec{u}$, $\overrightarrow{AF} = \beta \cdot \vec{w}$

$\overrightarrow{AN} = \frac{\vec{u}+\vec{w}}{2}$, $\overrightarrow{AP} = \frac{\alpha\vec{u}+\beta\vec{w}}{2}$

$\overrightarrow{AM} = \frac{\overrightarrow{AC}}{2}$, $\overrightarrow{AC}$? $\{C\} = DF \cap EB$. We use the same arguments