Resolution Predicate

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Exercise 4

Using a refinement of predicate resolution prove:

7. the distributivity of ‘ $\exists$ ’ over ‘ $\vee$ ’:
$\vdash (\exists x)(P(x) \vee Q(x)) \leftrightarrow (\exists x)P(x) \vee (\exists x)Q(x)$;

Theoretical Part

Its basic aim is to check the ***consistency/inconsistency*** of a set of clauses.

The ***validity*** of a formula **is proved by contradiction**=>

=> ***refutation method***

Definition

A predicate formula U is in ***prenex normal form*** if it has the form: $(Q_1x_1)...(Q_nx_n)M$, where $Q_i, i=1,\ldots,n$ are quantifiers, and M is quantifier-free. The sequence $(Q_1x_1)...(Q_nx_n)$ is called the ***prefix*** *of the formula* U, and M is called the ***matrix*** *of the formula* U. A predicate formula is in ***conjunctive prenex normal form*** if it is in prenex normal form and the matrix is in CNF.

Theorem:

A predicate formula admits a logical equivalent conjunctive prenex normal form.

The **prenex normal form** is obtained by applying transformations which preserve the logical equivalence, according to the following steps:

Step 1: The connectives '$\rightarrow$' and '$\leftrightarrow$' are replaced using the connectives: $\neg, \wedge, \vee$

Step 2: The bound variables are renamed such that they will be distinct.

Step 3: Application of infinitary DeMorgan's laws.

Step 4: The extraction of quantifiers in front of the formula.

Step 5: The matrix is transformed into CNF using DeMorgan's laws and the distributive laws.

Definitions:

Let U be a first-order formula, and $U^p = (Q_1x_1)...(Q_nx_n)M$ be one of its conjunctive prenex normal form. A formula in ***Skolem normal form***, denoted by U^S corresponds to U and it is obtained as follows:

- For each existential quantifier Q_r from the prefix we apply the transformation:
  - if on the left side of Q_r there are no universal quantifiers, then we introduce a new constant ***a***, and we replace in M all the occurrences of x_r by ***a***. (Q_rx_r) is deleted from the prefix.
  - if $Q_{s_1},\ldots,Q_{s_m}, 1 \leq s_1 < \ldots < s_m < r$, are all the universal quantifiers on the left side of Q_r in the prefix, then we introduce a new m-place function symbol, f, and we replace in M all the occurrences of x_r by $f(x_{s_1},\ldots,x_{s_m})$. (Q_rx_r) is deleted from the prefix.
- The constants and functions used to replace the existentially quantified variables are called ***Skolem constants*** and ***Skolem functions***. The prefix of the formula U^S contains only universal quantifiers, and the matrix is in conjunctive normal form.

A formula in ***clausal normal form*** denoted by U^c is obtained by deleting the prefix of U^S.

$$U \leftrightarrow V \equiv (U \to V) \wedge (V \to U)$$

$$U \to V \equiv \neg U \vee V$$

$$\neg(A \to B) \equiv A \wedge \neg B$$

$$\begin{aligned}
\neg(X \leftrightarrow Y) &\Longleftrightarrow \neg\Big((X \to Y) \wedge (Y \to X)\Big) \\
&\Longleftrightarrow \neg(X \to Y) \vee \neg(Y \to X) \\
&\Longleftrightarrow (X \wedge \neg Y) \vee (Y \wedge \neg X).
\end{aligned}$$

infinitary DeMorgan's law: $\neg(\forall x)A(x) \equiv (\exists x)\neg A(x)$

Solution

$$\neg U = \neg((\exists x)(P(X) \lor Q(X)) \leftrightarrow (\exists x)P(X) \lor (\exists x)Q(X))$$

Replace $\neg(x \leftrightarrow y)$

$$\equiv ((\exists x)(P(X) \lor Q(X)) \land \neg((\exists x)P(X) \lor (\exists x)Q(X))) \lor$$
$$((\exists x)P(X) \lor (\exists x)Q(X)) \land \neg((\exists x)(P(X) \lor Q(X)))$$

Apply infinitary DeMorgan Law

$$\equiv((\exists x)(P(X) \lor Q(X)) \land (\forall x)\neg P(X) \land (\forall x)\neg Q(X)) \lor ((\exists x)P(X) \lor (\exists x)Q(X)) \land (\forall x)\neg P(X) \land \neg Q(X))$$

Split in two

$$\equiv((\exists x)(P(X) \lor Q(X)) \land (\forall x)\neg P(X) \land (\forall x)\neg Q(X)) \lor$$
$$((\exists x)P(X) \lor (\exists x)Q(X)) \land (\forall x)(\neg P(X) \land \neg Q(X)))$$

$$\neg U1 = (\exists x)(P(X) \lor Q(X)) \land (\forall x)\neg P(X) \land (\forall x)\neg Q(X)$$
$$\neg U2 = ((\exists x)P(X) \lor (\exists x)Q(X)) \land (\forall x)(\neg P(X) \land \neg Q(X))$$

Renaming the bound variables

$$\neg U1 = (\exists x)(P(X) \lor Q(X)) \land (\forall x)\neg P(X) \land (\forall x)\neg Q(X)$$

$$\neg U1 = (\exists x)(P(X) \lor Q(X)) \land (\forall y)\neg P(y) \land (\forall z)\neg Q(z)$$

Prenex form

Extraction of quantifiers in front, extracting first the existential quantifier

$(\neg U1)^p = (\exists x)(\forall y)(\forall z)(P(X) \lor Q(X)) \land \neg P(y) \land \neg Q(z)$

Skolem form

$(\neg U1)^p = (\exists x)(\forall y)(\forall z)((P(x) \lor Q(x)) \land \neg P(y) \land \neg Q(z))$

$(\neg U1)^s = (\forall y)(\forall z)((P(a) \lor Q(a)) \land \neg P(y) \land \neg Q(z))$

$[x \leftarrow a]$ a-Skolem constant

Clausal form

$(\neg U1)^c = (P(a) \lor Q(a)) \land \neg P(y) \land \neg Q(z))$

Set of clauses

$(\neg U1)^c = (P(a) \lor Q(a)) \land \neg P(y) \land \neg Q(z)$

$S_1 = \{C_1 = P(a) \lor Q(a), C_2 = \neg P(y), C_3 = \neg Q(z)\}$

$C_1 = P(a) \lor Q(a)$
$C_2 = \neg P(y)$
$C_3 = \neg Q(z)$

Resolvents

$C_1 = P(a) \vee Q(a)$

$C_2 = \neg P(y)$

$C_3 = \neg Q(z)$

$C_4 = Res(C_{1,} C_2) = Q(a)$, $\Theta 1 = [y <\!\!- a] = mgu(y,a)$

$C_5 = Res(C_{3,} C_4) = \square$, $\Theta 2 = [z <\!\!- a] = mgu(z,a)$

We proved that $(\neg U_1)^c \vdash^{\mathrm{Pr}}_{\mathrm{Res}} \square;$

Therefore $\vdash U1$

Renaming the bound variables

$$\neg U2 = ((\exists x)P(X) \vee (\exists x)Q(X)) \wedge (\forall x)(\neg P(X) \wedge \neg Q(X))$$

$$\neg U2 = ((\exists x)P(x) \vee (\exists y)Q(y)) \wedge (\forall z)(\neg P(z) \wedge \neg Q(z))$$

Prenex form

Extraction of quantifiers in front, extracting first the existential quantifier

$(\neg U2)^p = (\exists x)(\exists y)(\forall z)((P(X) \vee Q(y)) \wedge \neg P(z) \wedge \neg Q(z))$

Skolem form

$(\neg U2)^s = (\forall z)((P(a) \lor Q(b)) \land \neg P(z) \land \neg Q(z))$

$[x \leftarrow a]$ a-Skolem constant

$[y \leftarrow b]$ b-Skolem constant

Clausal form

$(\neg U2)^c = (P(a) \lor Q(b)) \land \neg P(z) \land \neg Q(z)$

Set of clauses

$(\neg U2)^c = (P(a) \vee Q(b)) \wedge \neg P(z) \wedge \neg Q(z)$

$S_2 = \{C_1' = P(a) \vee Q(b),\ C_2' = \neg P(z),\ C_3' = \neg Q(z)\}$

$C_1' = P(a) \vee Q(b)$

$C_2' = \neg P(z)$

$C_3' = \neg Q(z)$

Resolvents

$C_1' = P(a) \lor Q(b)$

$C_2' = \neg P(z)$

$C_3' = \neg Q(t)$, renaming the free variable z

$C_4' = Res(C_1', C_2') = Q(b)$, $\Theta 1 = [z<-a] = mgu(z,a)$

$C_5' = Res(C_3', C_4') = \square$, $\Theta 2 = [t<-a] = mgu(t,b)$

We proved that $(\neg U_2)^c \vdash^{\mathrm{Pr}}_{\mathrm{Res}} \square$;

Therefore $\vdash U2$

Conclusion

We proved that $(\neg U_1)^c \vdash^{\mathrm{Pr}}_{\mathrm{Res}} \square$;

Therefore ⊢U1

We proved that $(\neg U_2)^c \vdash^{\mathrm{Pr}}_{\mathrm{Res}} \square$;

Therefore ⊢U2

So ⊢U1 and ⊢U2 are theorems

So we proved the distributivity of "∃" over "∨"