Asist.dr. Tudor Micu

1st Semester, 2020-2021

Algebra (Computer Science)

Bonus Exercises : Week 2

Exercise 1. Let $\mathcal{R} = (A, A, R)$ be a homogeneous binary relation. We say that $\mathcal{R}$ is *antisymmetric* if for any $x, y \in A$ we have the implication:

$$x\mathcal{R}y \text{ and } y\mathcal{R}x \Rightarrow x = y$$

Give 3 examples of antisymmetric relations, defined on different sets and find all the homogeneous binary relations on a set A that are both symmetric and antisymmetric.

Exercise 2. Let $(G, +)$ be an abelian group and $H \leq G$ a subgroup of G. We define the following relation on G:

$$x \ \rho_H \ y \overset{\text{def}}{\iff} x - y \in H$$

Prove that ρ_H is an equivalence relation on G and describe its quotient set G/ρ_H. What happens if G is not abelian?

Exercise 3. Let $G = (V, E)$ be a directed graph, where V is the set of vertices and $E \subseteq \{(x, y)|\ x, y \in V\}$ is the set of edges (loops are allowed). We define on V the binary relation:

$$\forall x, y \in V : x \sim y \overset{\text{def}}{\iff} (x, y) \in E$$

For each of the properties: **reflexivity**, **symmetry**, **transitivity**, **anti-symmetry**, draw a directed graph with 5 vertices, for which $\sim$ satisfies the respective property, but not the other 3.
Draw a directed graph G with 10 vertices, for which $\sim$ is an equivalence relation and explain/describe what the quotient set $G/\sim$ is.

Exercise 4. We define on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ the following relation:

$$(a, b) \sim (c, d) \overset{\text{def}}{\iff} ad = bc$$

Show that it is an equivalence relation and describe its quotient set $\dfrac{\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})}{\sim}$

Exercise 5. Let $\mathbb{E}$ be the set of points in the geometric plane. On $\mathbb{E}^2$ (the set of pairs of points) we define the following relation (called **equipollence**):

$$(A, B) \sim (C, D) \overset{\text{def}}{\iff} \text{the segments } [AD] \text{ and } [BC] \text{ have the same midpoint}$$

Prove that $\sim$ is an equivalence relation and find its quotient set $\mathbb{E}^2/\sim$. What do the elements of this quotient set represent (geometrical interpretation)?

Exercise 6. Let $\mathbb{R}^3$ be the three-dimensional Euclidean space. We define on it the following relation:

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \overset{\text{def}}{\iff} \exists \alpha \in \mathbb{R} : (x_1, y_1, z_1) = \alpha \cdot (x_2, y_2, z_2)$$

Show that $\sim$ is an equivalence relation on $\mathbb{R}^3$ and describe the quotient set $\dfrac{\mathbb{R}^3 \setminus \{0\}}{\sim}$ (geometrical interpretation).