Seminar W9 - 915

Conics

$\ell: \quad ax + by + c = 0$ line

$\mathscr{C}: \quad a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x +$

$+ 2a_{01}y + a_{00} = 0$

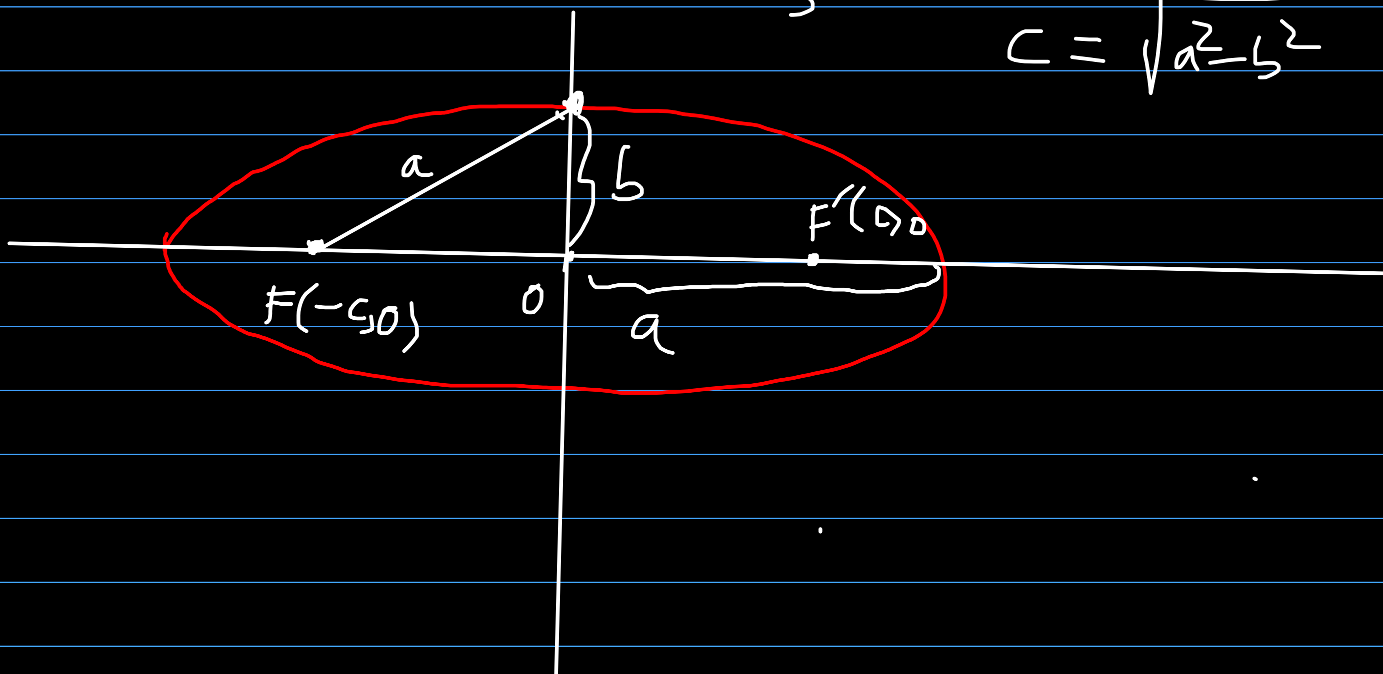
$\rightarrow$ conic

ellipse — hyperbola — parabola

"the good" — "the bad" — "the ugly"

Ellipse: $\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

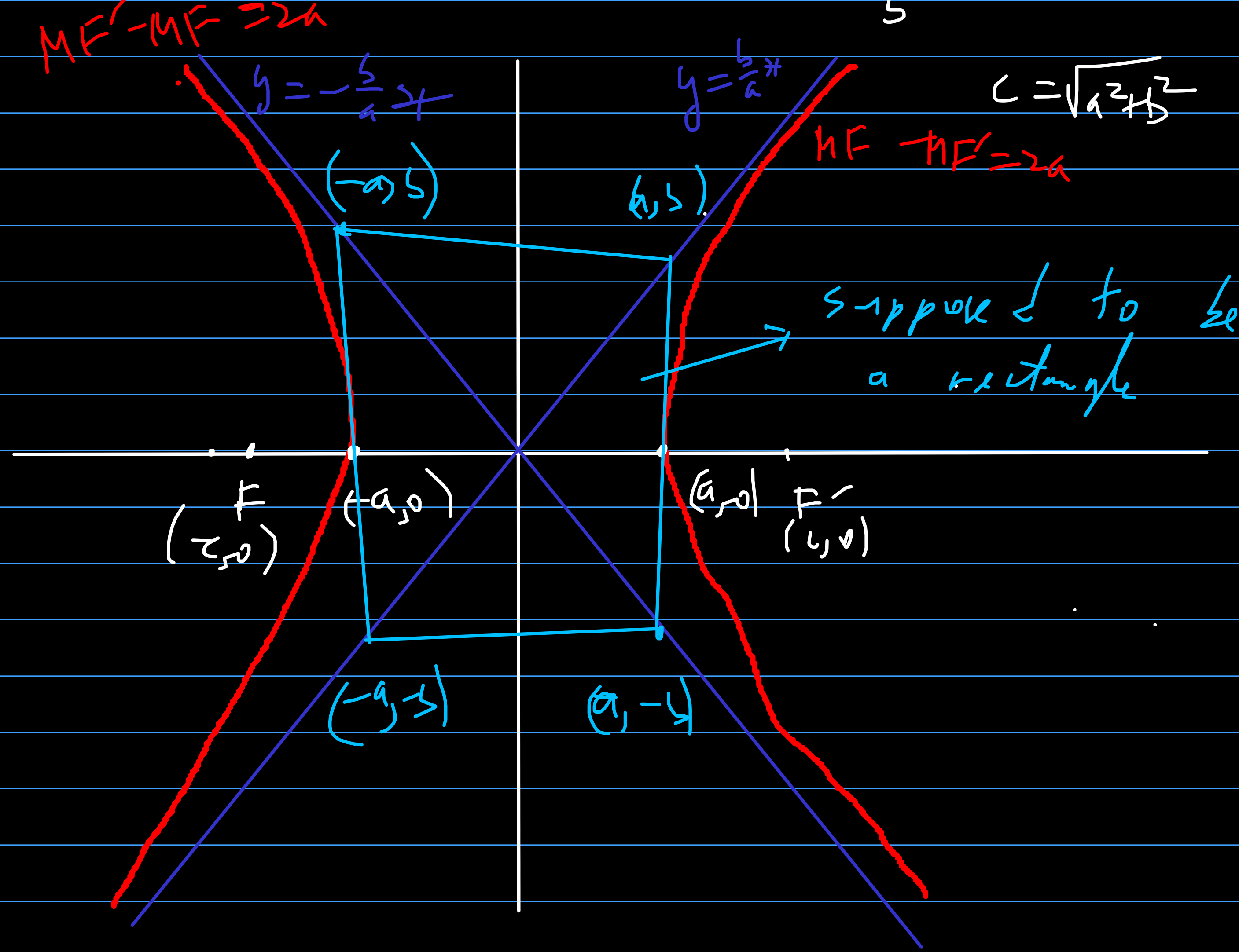
$c = \sqrt{a^2 - b^2}$.



→ locus of points M in the plane so that $MF + MF' = 2a$, where F and F' are fixed points called foci (pl. to focus)

$T_{\mathcal{E}}(x_0, y_0): \frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1$

Hyperbola: $\mathcal{H}: \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



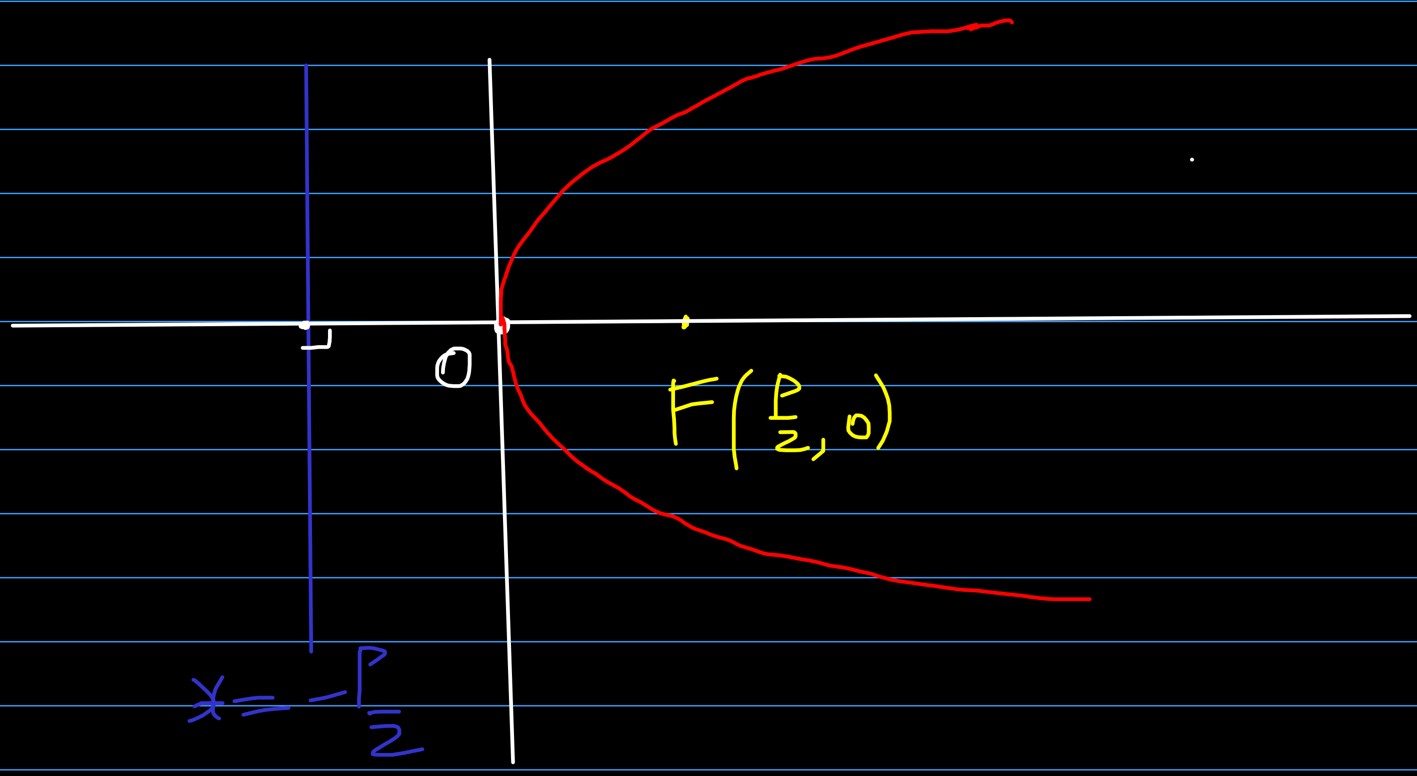
→ locus of points M in the plane so that $|MF - MF'| = 2a$, where F and F′ are two fixed points called foci.

the hyperbola has the oblical asymptotes

$$y = \pm \frac{b}{a} x$$

$$T_{\mathcal{H}}(x_0, y_0): \quad \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$$

Parabola: $\quad \mathcal{P}: y^2 = 2px$



→ locus of points M in the plane that are equidistant to a point F (called the focus) and a line d (called the director line (directrix))

$T_{\mathcal{P}}(x_0, y_0): \quad y y_0 = p(x + x_0)$

5.3. Find the equations of the tangent lines to the ellipse $\mathcal{E}: \frac{x^2}{25} + \frac{y^2}{16} = 1$, passing through $P_0(10, -8)$.

$$T_{\mathcal{E}}(x_0, y_0): \quad \frac{x x_0}{25} + \frac{y y_0}{16} = 1$$

$$P_0 \in T_{\mathcal{E}}(x_0, y_0): \quad \frac{10 \cdot x_0}{25} - \frac{8 y_0}{16} = 1 \iff$$

$$\iff \frac{2 x_0}{5} - \frac{y_0}{2} = 1$$

$$(x_0, y_0) \in \mathcal{E} \Rightarrow \frac{x_0^2}{25} + \frac{y_0^2}{16} = 1$$

$$\Rightarrow \begin{cases} \dfrac{2x_0}{5} - \dfrac{y_0}{2} = 1 \\[2ex] \dfrac{x_0^2}{25} + \dfrac{y_0^2}{16} = 1 \end{cases} \Leftrightarrow \begin{cases} x_0 = \dfrac{5}{2}\left(1 + \dfrac{y_0}{2}\right) \\[2ex] \dfrac{x_0^2}{25} + \dfrac{y_0^2}{16} = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_0 = \dfrac{5}{2}\left(1 + \dfrac{y_0}{2}\right) \\[2ex] \dfrac{1}{4}\left(1 + \dfrac{y_0^2}{4} + y_0\right) + \dfrac{y_0^2}{16} = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_0 = \dfrac{5}{2}\left(1 + \dfrac{y_0}{2}\right) \\[2ex] y_0^2\left(\dfrac{1}{16} + \dfrac{1}{16}\right) + y_0 \cdot \dfrac{1}{4} + \dfrac{1}{4} = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_0 = \dfrac{5}{2}\left(1 + \dfrac{y_0}{2}\right) \\[2ex] y_0^2 + 2y_0 - 6 = 0 \end{cases}$$

$$\Rightarrow (y_0)_{1,2} = \frac{-2 \pm \sqrt{4 + 24}}{2} =$$

$$= -1 \pm \sqrt{7} .$$

$$\Rightarrow T_{\mathcal{E}}(x_0, y_0): \frac{x x_0}{25} + \frac{y y_0}{16} = 1$$

9.6. Find the equations of the tangent lines to the hyperbola

$$\mathcal{H}: \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$$

which are orthogonal to the line

$$d: 4x + 3y - 7 = 0 .$$

$$d: 3y = 7 - 4x$$

$$d: y = -\frac{4}{3}x + \frac{7}{3}$$

$$\Rightarrow m_d = -\frac{4}{3}$$

Let ℓ_n be a line that is perpendicular to d:

$$\Rightarrow m_{\ell_n} = \frac{3}{4}$$

$$\Rightarrow \ell_n : \quad y = \frac{3}{4}x + n$$

ℓ_n tangent to $\mathcal{H} \Leftrightarrow |\ell_n \cap \mathcal{H}| = 1$

$$\ell_n \cap \mathcal{H} : \begin{cases} y = \frac{3}{4}x + n \\ \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = \frac{3}{4}x + n \\ x^2 - 4\cdot\left(\frac{3}{4}x + n\right)^2 - 20 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = \frac{3}{4}x + n \\ x^2 - \left(\frac{9}{4}x^2 + 6nx + 4n^2\right) - 20 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = \frac{3}{4}x + n \\ -\frac{5}{4}x^2 - 6nx - 4n^2 - 20 = 0 \end{cases}$$

$$\iff \begin{cases} y = \frac{3}{4}x + n \\ 5x^2 + 24nx + 16n^2 + 80 = 0 \end{cases}$$

ℓ_n tangent to $\mathcal{H} \iff |\ell_n \cap \mathcal{H}| = 1 \iff$

$\iff$ the equation $E: 5x^2 + 24nx + 16n^2 + 80 = 0$ has a unique solution $\iff$

$$\iff \Delta_E = 0$$

$$\Delta_E = 24^2 n^2 - 20 \cdot 16n^2 - 20 \cdot 80 =$$

$$= 256n^2 - 1600$$

$$\Delta_E = 0 \iff n^2 = \frac{1600}{256} \iff n = \pm\frac{400}{16} \iff$$

$$\iff n = \pm 25$$

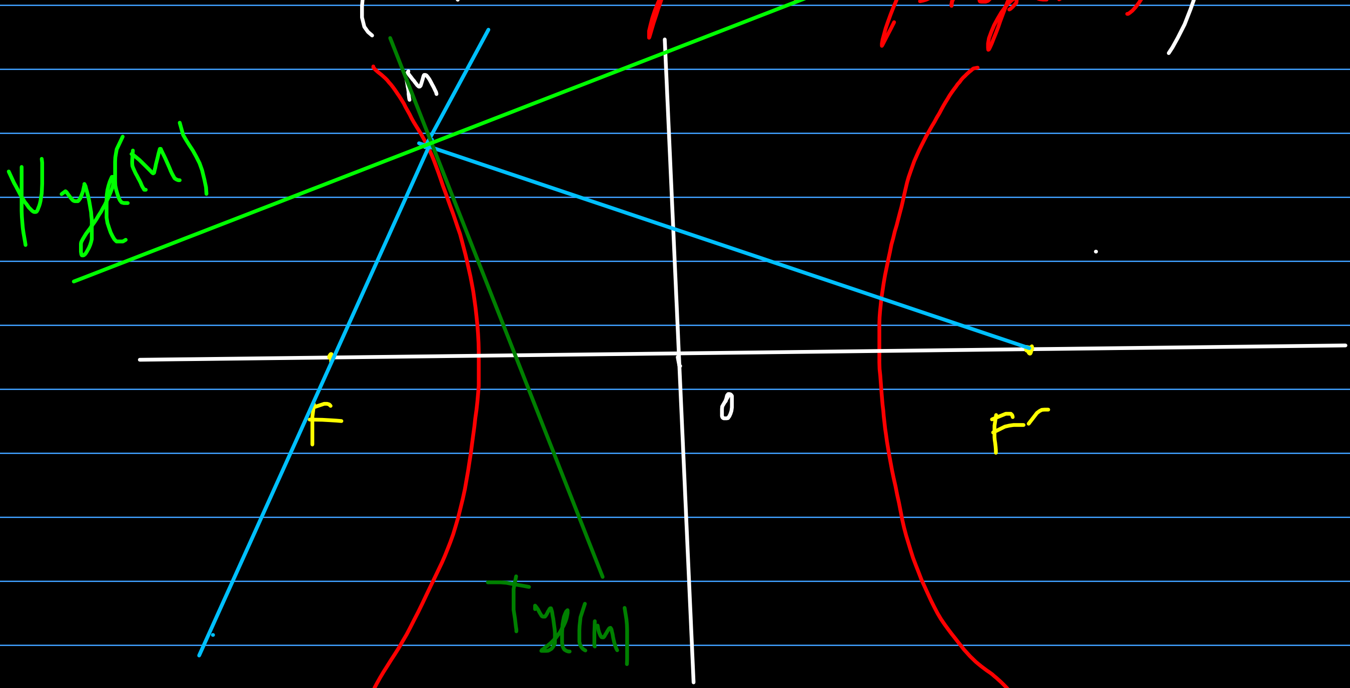
$\Rightarrow$ The tangents are:

$$l_{25} : y = \frac{3}{4}x + 25$$

$$l_{-25} : y = \frac{3}{4}x - 25$$

9.13. Show that a ray of light through a focus of a hyperbola reflects to a ray that passes through the other focus.

(the optical property)



or the tangent line

We have to show that the normal line to the hyperbola in any point M is a external bisector for the angle $\widehat{FMF'}$.

$$\mathcal{H}: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$M(x_0, y_0)$

$$N_{\mathcal{H}}(M): \frac{x - x_0}{\frac{2x_0}{a^2}} = \frac{y - y_0}{\frac{-2y_0}{b^2}}$$

$$\Rightarrow N_{\mathcal{H}}(M): \begin{cases} x = x_0 + \frac{2x_0}{a^2} \cdot \lambda \\ y = y_0 - \frac{2y_0}{b^2} \cdot \lambda \end{cases}, \lambda \in \mathbb{R}$$

$M(x_0, y_0)$, $F(-c, 0)$, $F'(c, 0)$

$$MF: \frac{x+c}{x_0+c} = \frac{y}{y_0} \iff$$

$$\iff y_0 x - y(x_0+c) + c y_0 = 0$$

$$MF': \quad y_0 x - y(x_0-c) - c y_0 = 0$$

Let $A \in N_{\mathcal{H}}(x_0, y_0)$:

$$A: \begin{cases} x = x_0 + \frac{2x_0}{a^2}\lambda \\ y = y_0 - \frac{2y_0}{b^2}\lambda \end{cases}$$

$$\operatorname{dist}(A, MF) = \frac{\left| y_0 \cdot \left(x_0 + \frac{2x_0}{a^2}\lambda\right) - (x_0+c)\cdot\left(y_0 - \frac{2y_0}{b^2}\lambda\right) + c y_0 \right|}{\sqrt{y_0^2 + (x_0+c)^2}}$$

$$= \frac{\left| \frac{2x_0 y_0}{a^2}\lambda + \frac{2x_0 y_0}{b^2}\lambda + \frac{2c y_0}{b^2}\lambda \right|}{\sqrt{y_0^2 + (x_0+c)^2}}.$$

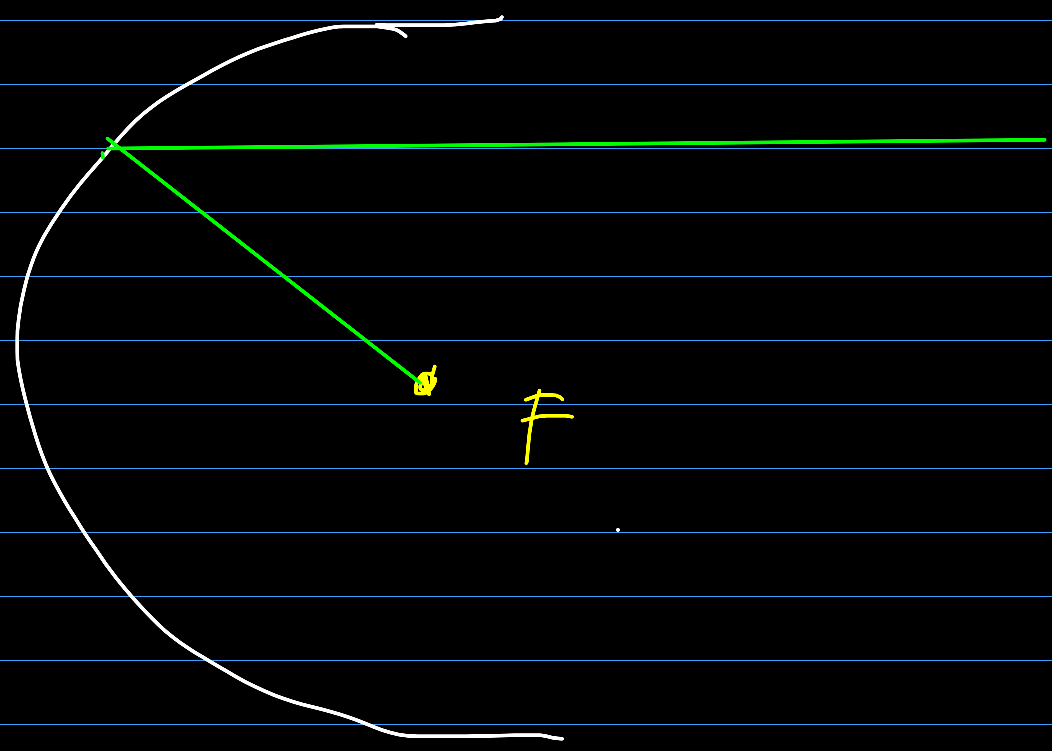
$$\operatorname{dist}(A, MF') = \frac{\left|\frac{2x_0y_0}{a^2}\lambda + \frac{2x_0y_0}{b^2}\lambda - \frac{2cy_0}{b^2}\lambda\right|}{\sqrt{y_0^2+(x_0-c)^2}}$$

All we need to show is that

$$\frac{\left|\frac{2x_0y_0}{a^2}\lambda + \frac{2x_0y_0}{b^2}\lambda + \frac{2cy_0}{b^2}\lambda\right|}{\sqrt{y_0^2+(x_0+c)^2}} = \frac{\left|\frac{2x_0y_0}{a^2}\lambda + \frac{2x_0y_0}{b^2}\lambda - \frac{2cy_0}{b^2}\lambda\right|}{\sqrt{y_0^2+(x_0-c)^2}}$$

(we will need to use $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$ and

$c^2 = a^2 + b^2$)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