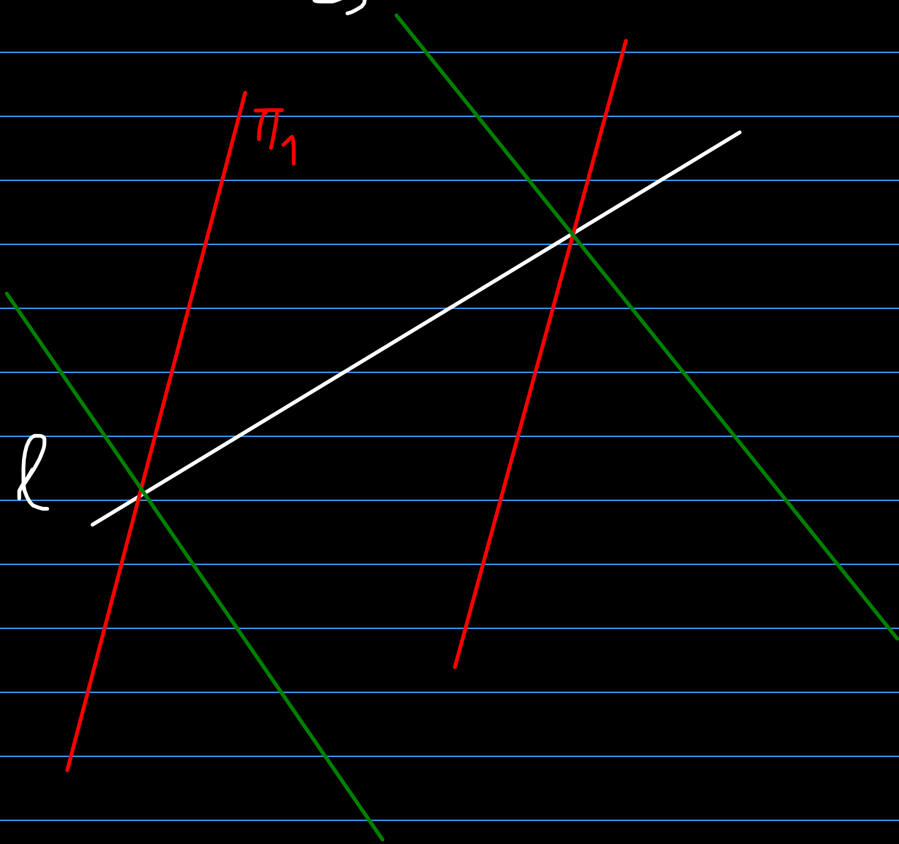
Seminar W4 - 911

Pencils of planes

$$\ell : \begin{cases} \pi_1 : A_1 x + B_1 y + C_1 z + D_1 = 0 \\ \\ \pi_2 : A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$

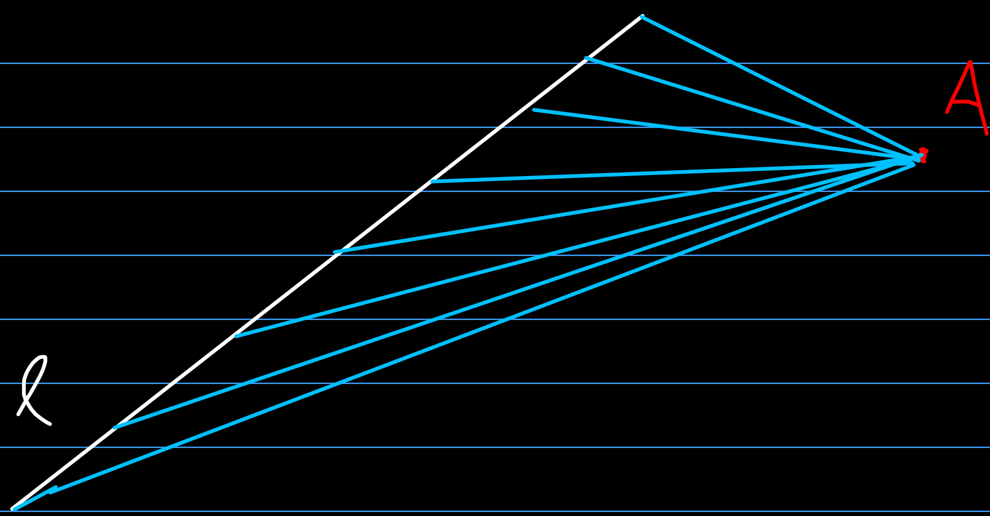
$$\pi_{\alpha,\beta} : \alpha \cdot (A_1 x + B_1 y + C_1 z + D_1) + \beta (A_2 x + B_2 y + C_2 z + D_2) = 0$$



9.1. Write the equation of the plane determined by the line:

$$l: \begin{cases} x - 2y + 3z = 0 \\ 2x + z - 3 = 0 \end{cases}$$

and the point $A(-1, 2, 6)$.



$$\pi_{\alpha,\beta}: \alpha(x - 2y + 3z) + \beta \cdot (2x + z - 3) = 0$$

$$\pi_{\alpha,\beta}: x(\alpha + 2\beta) + y \cdot (-2\alpha) + z \cdot (3\alpha + \beta) - 3\beta = 0$$

$$A \in \pi_{\alpha,\beta} \Rightarrow -1(\alpha + 2\beta) + 2 \cdot (-2\alpha) + 6(3\alpha + \beta) - 3\beta = 0 \Leftrightarrow$$

$\Leftrightarrow 13\alpha + \beta = 0 \Leftrightarrow \beta = -13\alpha$

So the planes that we want are

$\Pi_{\alpha,-13\alpha}: -25\alpha x - 2\alpha y - 10\alpha z + 39\alpha = 0$

$\Pi_{\alpha,-13\alpha}: \alpha(-25x - 2y - 10z + 39) = 0$

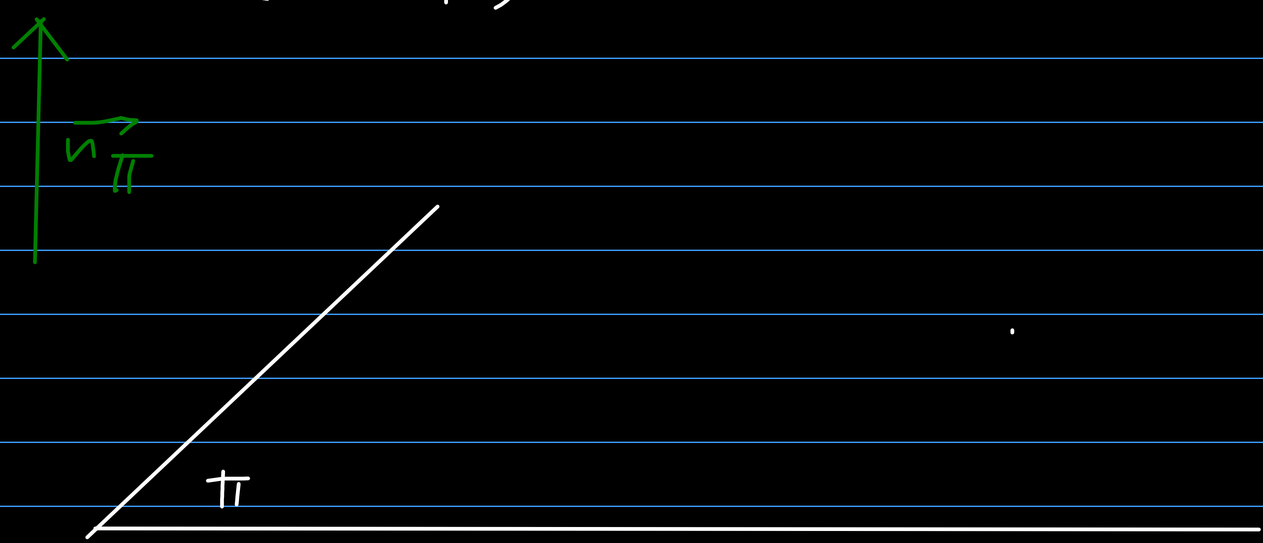
$\alpha \neq 0 \Rightarrow$ we have a unique plane:

$\Pi_{1,-13}: -25x - 2y - 10z + 39 = 0$

· $\Pi : Ax+By+Cz+D=0$

$\Rightarrow \vec{n}_{\Pi}\,(A, B, C)$ normal vector of the plane Π

$$\left(\forall \vec{u} \parallel \Pi \ \ (\vec{u} \in \vec{\Pi}) : \ \vec{n}_{\Pi} \perp \vec{u} \ \ (\Leftrightarrow \vec{n}_{\Pi} \cdot \vec{u} = 0) \right)$$

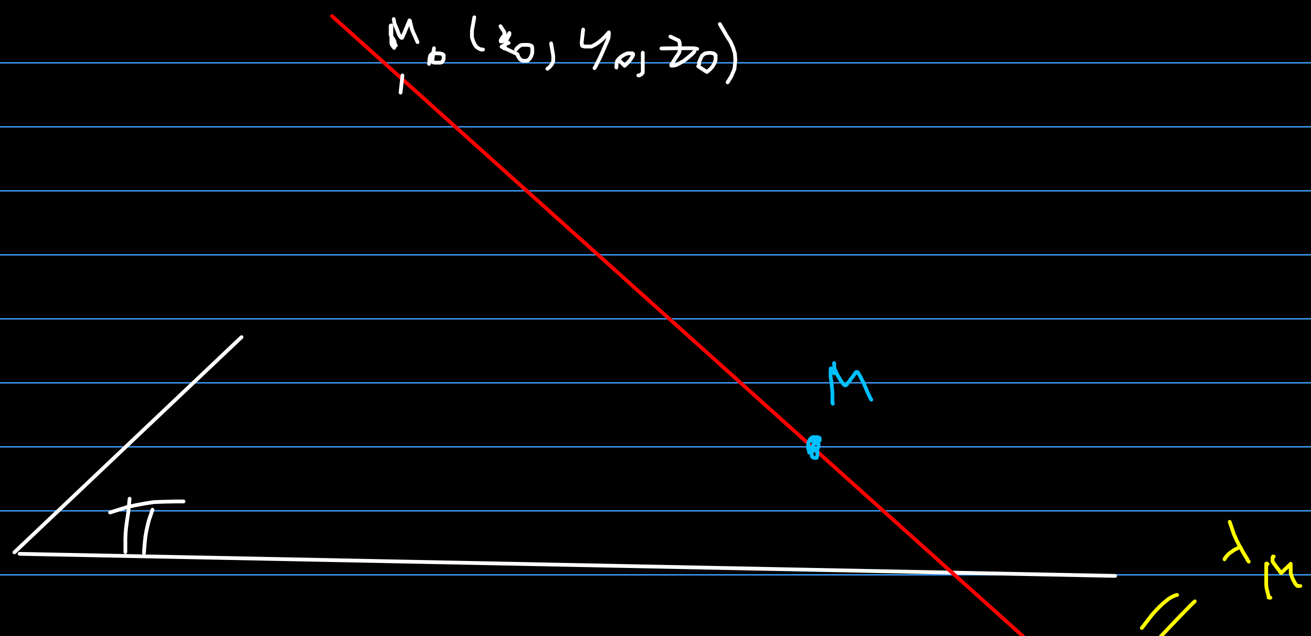


· if ℓ line, $\ell : \begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases}$

$\ell \parallel \Pi \Leftrightarrow Ap + Bq + Cr = 0 \Leftrightarrow \vec{n}_{\Pi} \cdot \vec{\ell} = 0$

• if $\ell \nparallel \pi$ (i.e. if $Ap+Bq+Cr \neq 0$)

then $\exists\, M \;:\; \{M\} = \ell \cap \pi$



$$\begin{cases} x_M = x_0 - \dfrac{Ax_0+By_0+Cz_0+D}{Ap+Bq+Cr}\cdot p \\[2ex] y_M = y_0 - \dfrac{Ax_0+By_0+Cz_0+D}{Ap+Bq+Cr}\cdot q \\[2ex] z_M = z_0 - \dfrac{Ax_0+By_0+Cz_0+D}{Ap+Bq+Cr}\cdot r \end{cases}$$

Ex.: $\Pi: x+2y-5z=0$

$$l: \frac{x-2}{3} = \frac{y+1}{7} = \frac{z}{-2}$$

Find the intersection point $\{M\} = l \cap \Pi$ (without using the formulas above)

$$l: \begin{cases} x = 3t+2 \\ y = 7t-1 \\ z = -2t \end{cases}$$

$$M: \begin{cases} l \\ \Pi \end{cases} \Leftrightarrow \begin{cases} x = 3t+2 \\ y = 7t-1 \\ z = -2t \\ x+2y-5z=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 3t+2 \\ y = 7t-1 \\ z = -2t \\ 3t+2+2(7t-1)-5(-2t)=0 \end{cases}$$

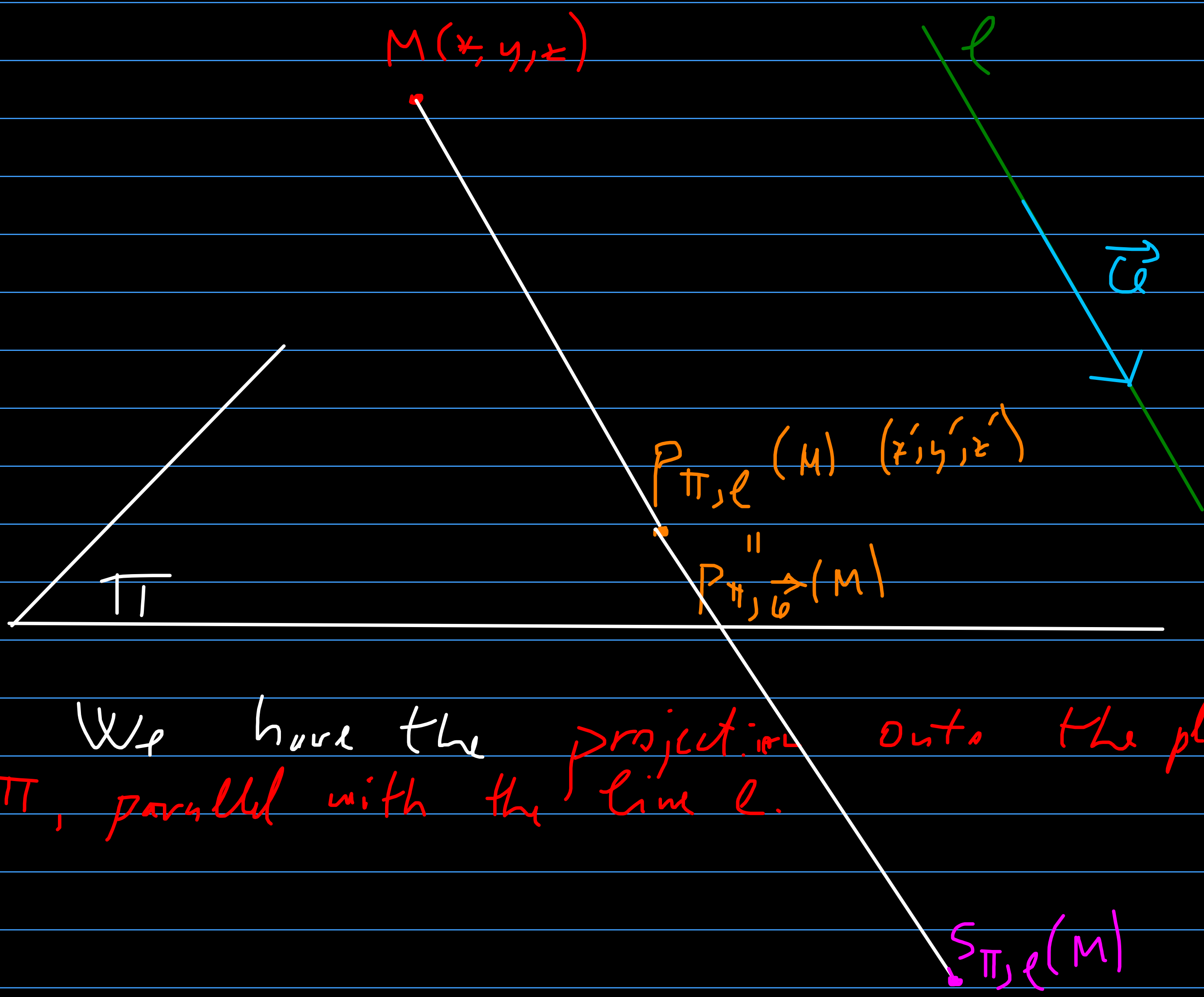
$$\Leftrightarrow \begin{cases} x = 3t+2 \\ y = 7t-1 \\ z = -2t \\ 27t = 0 \end{cases} \Leftrightarrow \begin{cases} x = 2 \\ y = -1 \\ z = 0 \end{cases}$$

Projections and reflections

$\Pi: Ax + By + Cz + D = 0$

$$l: \begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases}$$

$l \nparallel \Pi$ (i.e. $Ap + Bq + Cr \neq 0$)



We have the projection onto the plane Π, parallel with the line l.

$$P_{\pi,\ell} : \mathbb{R}^3 \longrightarrow \pi$$
$$(x, y, z) \longrightarrow (x', y', z')$$

$$\begin{cases} x' = x - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot p \\ y' = y - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot q \\ z' = z - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot r \end{cases}$$

$$S_{\pi,\ell} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x, y, z) \longmapsto (x'', y'', z'')$$

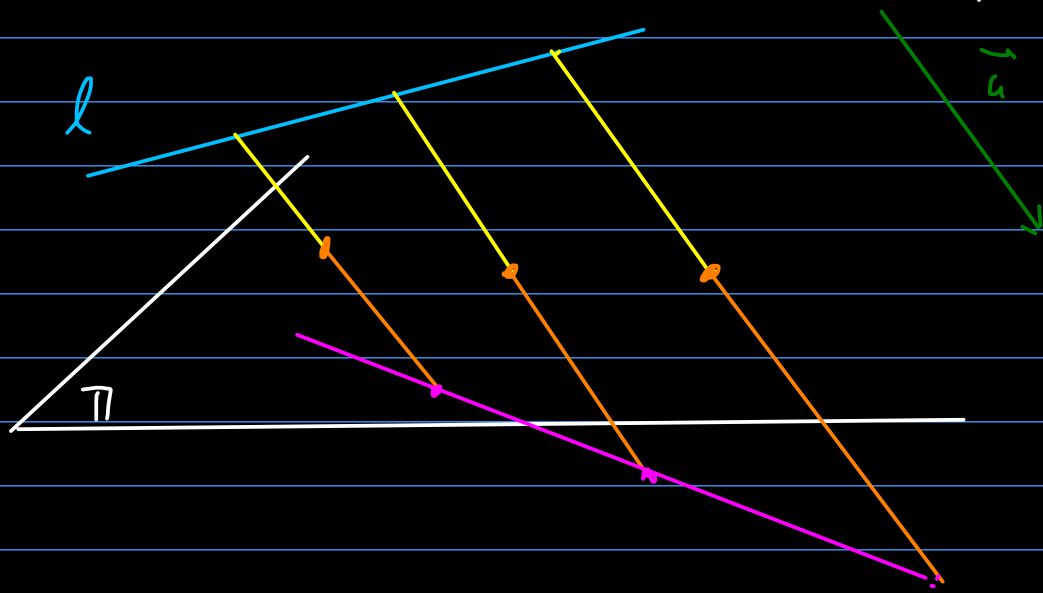
$$\begin{cases} x'' = x - 2 \cdot \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot p \\ y'' = y - 2 \cdot \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot q \\ z'' = z - 2 \cdot \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot r \end{cases}$$

4.3. Write the equations of the reflection of the line ℓ given by

$$\ell : \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

on the plane $\pi : x + 2y - z = 0$

parallel to the direction $\vec{u}(1, 1, -2)$



$$\ell : \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ x + y - z + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = z - 1 \\ z = z \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = t \\ z = t + 1 \end{cases}$$

$$\begin{cases} x'' = x - 2\cdot\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr}\cdot p \\[2ex] y'' = y - 2\cdot\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr}\cdot q \\[2ex] z'' = z - 2\cdot\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr}\cdot r \end{cases}$$

$$\vec{u} = (p, q, r) = (1, 1, -2)$$

$$(A, B, C, D) = (1, 2, -1, 0)$$

$$\frac{Ax+By+Cz+D}{Ap+Bq+Cr} = \frac{1\cdot 0 + 2\cdot t + (-1)\cdot(t+1)}{1\cdot 1 + 1\cdot 2 + (-1)\cdot(-2)} =$$

$$= \frac{t-1}{5}$$

$$\Rightarrow \begin{cases} x'' = 0 - 2\cdot\dfrac{t-1}{5}\cdot 1 = -\dfrac{2}{5}t + \dfrac{2}{5} \\[2ex] y'' = t - 2\cdot\dfrac{t-1}{5}\cdot 1 = \dfrac{3}{5}t + \dfrac{2}{5} \\[2ex] z'' = t+1 - 2\cdot\dfrac{t-1}{5}(-2) = \dfrac{9t}{5} + \dfrac{1}{5} \end{cases}$$

This is the equation of a line.

(3p)

4.8. Assume that $\mathcal{R} = (O, b)$, $b = [e_1, e_2, e_3]$ is the Cartesian reference system behind the equations of a plane:

$$\pi : Ax + By + Cz + D = 0$$

and a line:

$$\ell : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

If $\pi \nparallel d$ show that:

(a) $\overrightarrow{P_{\pi,d}(M)\, P_{\pi,d}(N)} = p(\overrightarrow{MN})$,

for all M, N points in space, where

$$p : \mathcal{V} \to \mathcal{V}$$

is the linear transformation whose matrix representation is:

$$[P]_b = \frac{1}{Ap + Bq + Cr} \cdot \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix}$$

$$\ell : \begin{cases} x = x_0 + t p \\ y = y_0 + t q \\ z = z_0 + t \cdot r \end{cases}, \quad F(x,y,z) = Ax + By + Cz + D$$

$$\vec{r}_{P_{\pi,d}(M)} = \vec{r}_M - \frac{F(M)}{Ap + Bq + Cr} \cdot \vec{\ell}$$

$$\overrightarrow{P_{\pi,d}(M) P_{\pi,d}(N)} = \vec{r}_{P_{\pi,d}(N)} - \vec{r}_{P_{\pi,d}(M)} =$$

$$= \vec{r}_N - \frac{F(N)}{Ap+Bq+Cr} \cdot \vec{\ell} - \vec{r}_M + \frac{F(M)}{Ap+Bq+Cr} \cdot \vec{\ell} =$$

$$= \overrightarrow{MN} - \frac{1}{Ap+Bq+Cr} \cdot \vec{\ell} \left(F(N) - F(M) \right)$$

$$F(N) - F(M) = (Ax_N + By_N + Cz_N + D) -$$

$$- (Ax_M + By_M + Cz_M + D) = A(x_N - x_M) +$$

$$+ B \cdot (y_N - y_M) + C \cdot (z_N - z_M) =$$

$$= \vec{n}_\pi \cdot \overrightarrow{MN}$$

$$\overrightarrow{p_{\pi,d}(M)\,p_{\pi,d}(N)} = \overrightarrow{MN} - \frac{\vec{n_\pi}\cdot\overrightarrow{MN}}{\vec{n_\pi}\cdot\vec{\ell}}\cdot\vec{\ell} =$$

$$= f(\overrightarrow{MN})$$

$$f : \mathcal{U} \to \mathcal{U} \qquad (*)$$

$$\vec{u} \longmapsto \vec{u} - \frac{\vec{n_\pi}\cdot\vec{u}}{\vec{n_\pi}\cdot\vec{\ell}}\cdot\vec{\ell}$$

$$f(e_1) = e_1 - \frac{\vec{n_\pi}\cdot\vec{e_1}}{\vec{n_\pi}\cdot\vec{\ell}}\cdot\vec{\ell} =$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{A}{Ap+Bq+Cr}\cdot\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\Rightarrow [f(e_1)]_b = \begin{pmatrix} 1 - \frac{Ap}{Ap+Bq+Cr} \\ -\frac{Aq}{Ap+Bq+Cr} \\ -\frac{Ar}{Ap+Bq+Cr} \end{pmatrix}$$

$$[f(e_2)]_b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{B}{Ap+Bq+Cr} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$[f(e_3)]_b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{C}{Ap+Bq+Cr} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\Rightarrow [f]_b = \left([f(e_1)]_b \middle| [f(e_2)]_b \middle| [f(e_3)]_b \right)$$

In (*) we should have proven that f is, indeed, a linear map.

$$f : \mathcal{U} \to \mathcal{U}$$

$$\vec{u} \longmapsto \vec{u} - \frac{\vec{n_\pi} \cdot \vec{u}}{\vec{n_\pi} \cdot \vec{\ell}} \cdot \ell$$

Let $\alpha, \beta \in \mathbb{R}$, $\vec{u}, \vec{w} \in \mathbb{R}$

$$f(\alpha \vec{u} + \beta \vec{w}) = \alpha \vec{u} + \beta \vec{w} -$$

$$- \frac{\vec{n_\pi} \cdot (\alpha \vec{u} + \beta \vec{w})}{\vec{n_\pi} \cdot \vec{l}} \cdot \vec{l} =$$

$$= \alpha \vec{u} + \beta \vec{w} - \frac{\alpha \cdot (\vec{n_\pi} \cdot \vec{u}) + \beta \cdot (\vec{n_\pi} \cdot \vec{w})}{\vec{n_\pi} \cdot \vec{l}} \cdot \vec{l}$$

$$= \left(\alpha \vec{u} - \frac{\alpha \; \vec{n_\pi} \cdot \vec{u}}{\vec{n_\pi} \cdot \vec{l}} \cdot \vec{l} \right) +$$

$$+ \left(\beta \vec{w} - \frac{\beta \cdot \vec{n_\pi} \cdot \vec{w}}{\vec{n_\pi} \cdot \vec{l}} \cdot \vec{l} \right) =$$

$$= \alpha f(\vec{u}) + \beta \cdot f(\vec{w})$$