Seminar V7 — 913

The triple scalar product
(the mixed product)

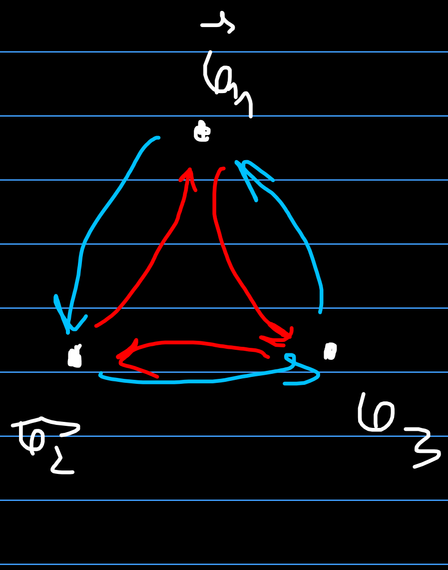
$\vec{a}, \vec{b}, \vec{c} \in V$, $(\vec{a}, \vec{b}, \vec{c}) := \vec{a} \cdot (\vec{b} \times \vec{c}) =$

$= (\vec{a} \times \vec{b}) \cdot \vec{c}$

If $R = (O, [\vec{i}, \vec{j}, \vec{k}])$ reference system that is orthonormal and direct, then:

$\vec{v}_1(a_1, b_1, c_1)$, $\vec{v}_2(a_2, b_2, c_2)$, $\vec{v}_3(a_3, b_3, c_3)$

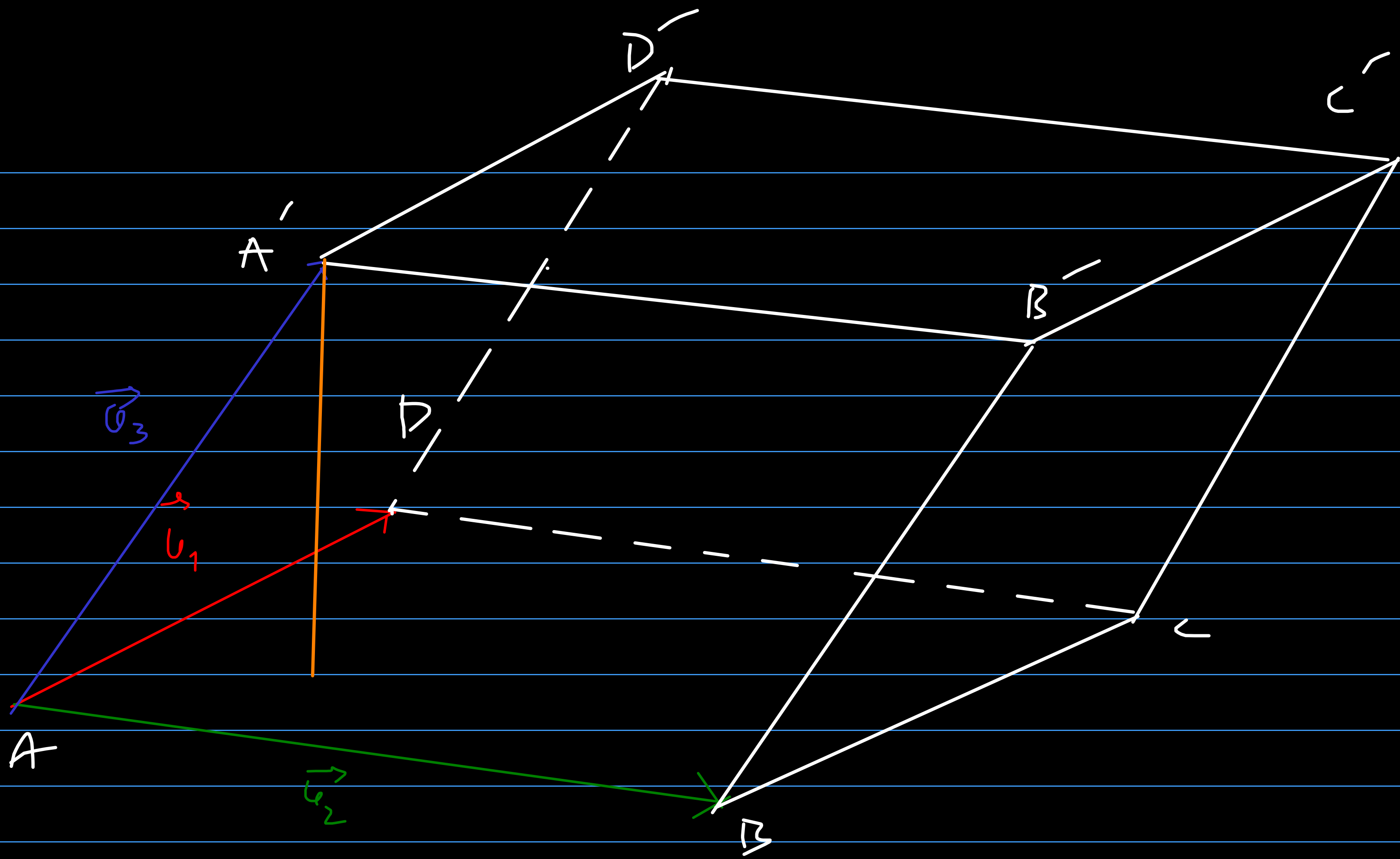
$$(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$



$(\vec{v}_1, \vec{v}_2, \vec{v}_3) = (\vec{v}_2, \vec{v}_3, \vec{v}_1) =$

$= (\vec{v}_3, \vec{v}_1, \vec{v}_2) = -(\vec{v}_1, \vec{v}_3, \vec{v}_2) =$

$= -(\vec{v}_2, \vec{v}_1, \vec{v}_3) = -(\vec{v}_3, \vec{v}_2, \vec{v}_1)$



$$\left|(\vec{u}_1, \vec{u}_2, \vec{u}_3)\right| = Vol(\text{parallelepiped built on } \vec{u}_1, \vec{u}_2, \vec{u}_3)$$

$$dist\left(A', (ABD)\right) = \frac{\left|(\vec{u}_1, \vec{u}_2, \vec{u}_3)\right|}{\|\vec{u}_1 \times \vec{u}_2\|}$$

$$V_{A'ADB} = \frac{1}{6} V_{ABCDA'B'C'D'} =$$

$$= \frac{1}{6} \left|(\vec{u}_1, \vec{u}_2, \vec{u}_3)\right|$$

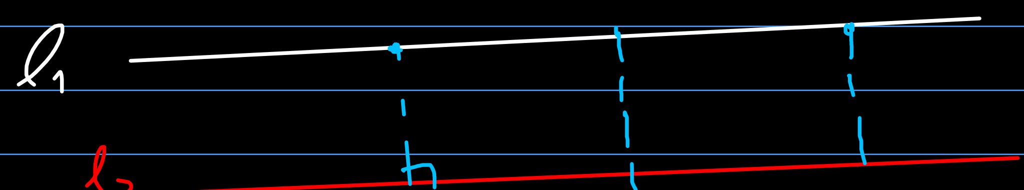
The distance between two lines in space and the common perpendicular

Let l_1, l_2 be lines in space

- If $l_1 \cap l_2 \neq \emptyset$, then the common perpendicular is a line perpendicular to the common plane of l_1 and l_2 that contains their intersection point.

  $$dist(l_1, l_2) = 0$$

- If $l_1 \cap l_2 = \emptyset$ and $l_1 \parallel l_2$:
  - the common perpendicular is any perpendicular from a point on l_1 to l_2



· If $l_1 \cap l_2 = \emptyset$ and $l_1 \nparallel l_2$

(i.e. the lines are skew or noncoplanar)

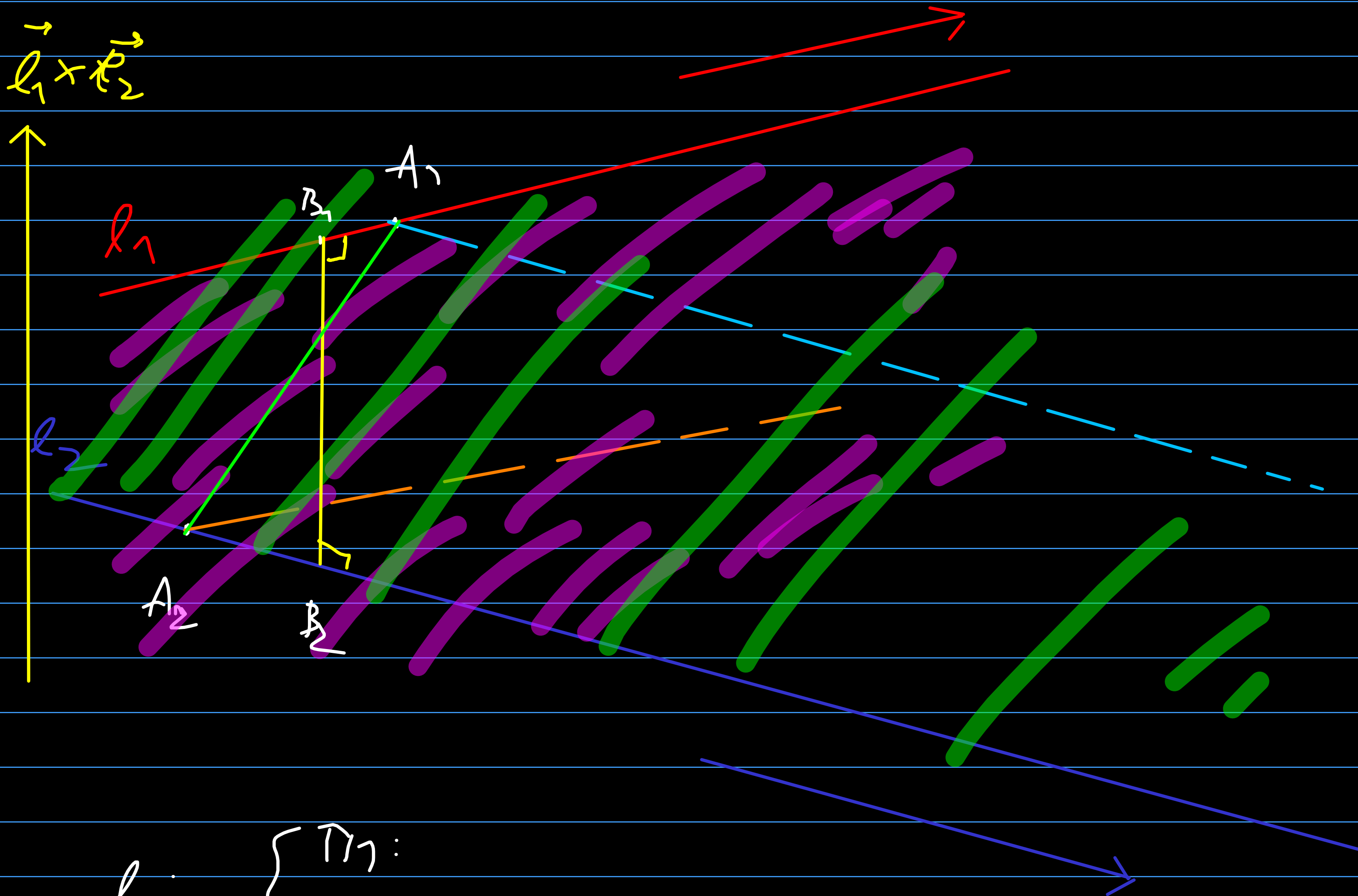
→ the common perpendicular is

$$l = \Pi_1 \cap \Pi_2 \ , \text{ where}$$

Π_1 = the plane given by l_1 and $\vec{l_1} \times \vec{l_2}$

Π_2 = the plane given by l_2 and $\vec{l_1} \times \vec{l_2}$

$\Pi_1 \nparallel \Pi_2$, because $\vec{\Pi_1} \neq \vec{\Pi_2}$ (because $\vec{l_1} \nparallel \vec{l_2}$)



$$l : \begin{cases} \Pi_1 : \\ \Pi_2 : \end{cases}$$

$$l \cap l_1 = \{B_1\}, \quad l \cap l_2 = \{B_2\}$$

$$\text{dist}(l_1, l_2) = B_1B_2 = \frac{\text{Vol}\left(\begin{array}{c}\text{parallelepiped} \\ \text{built on } \overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2\end{array}\right)}{\|\vec{d}_1 \times \vec{d}_2\|}$$

$$= \frac{|(\overrightarrow{A_1A_2}, \vec{l}_1, \vec{l}_2)|}{\|\vec{l}_1 \times \vec{l}_2\|}$$

7.7. Find the distance between the lines: $l_1: \frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{1}$

$$l_2: \frac{x+1}{3} = \frac{y}{4} = \frac{z-1}{3}$$

as well as the equations of the common perpendicular

We take $A(1,-1,0) \in l_1$

$B(-1,0,1) \in l_2$

$\vec{AB}(-2,1,1)$, $\vec{l_1}(2,3,1)$, $\vec{l_2}(3,4,3)$

$$\vec{l_1} \times \vec{l_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 3 & 4 & 3 \end{vmatrix} = 5\vec{i} - 3\vec{j} - \vec{k}$$

$$\| \vec{l_1} \times \vec{l_2} \| = \sqrt{25+9+1} = \sqrt{35}$$

$$(\vec{AB}, \vec{l_1}, \vec{l_2}) = \begin{vmatrix} -2 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 3 \end{vmatrix} =$$

$$= -18 + 3 + 8 - 9 + 8 - 6 = -14$$

$$\Rightarrow \operatorname{dist}(l_1, l_2) = \frac{|(\vec{A_1A_2}, \vec{l_1}, \vec{l_2})|}{\|\vec{l_1} \times \vec{l_2}\|} =$$

$$= \frac{14}{\sqrt{35}}$$

$\vec{l_1}\ (2,3,1)$, $\vec{l_2}(3,4,3)$, $\vec{l_1} \times \vec{l_2}\ (5,-3,-1)$

$A(1,-1,0) \in l_1$, $B(-1,0,1) \in l_2$

$$\Pi_1 : \begin{vmatrix} x-1 & y+1 & z \\ 2 & 3 & 1 \\ 5 & -3 & -1 \end{vmatrix} = 0$$

$$-3(x-1) - 6z + 5(y+1) - 15z +$$

$$3(x-1) + 2\cdot(y+1) = 7y - 21z + 7$$

$$\pi_2 : \begin{vmatrix} x+1 & y & z-1 \\ 3 & 4 & 3 \\ 5 & -3 & -1 \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow -4(x+1) - 9(z-1) + 15y - 20(z-1) + 9\cdot(x+1) + 3y = 0 \Leftrightarrow 5x + 18y - 29z + 34 = 0$$

$$\Rightarrow l : \begin{cases} 7y - 21z + 7 = 0 \\ 5x + 18y - 29z + 34 = 0 \end{cases}$$

This is the common perpendicular.

Two lines l_1 and l_2 are coplanar

$\Leftrightarrow \forall A_1 \in l_1, \forall A_2 \in l_2$:

$$(\overrightarrow{A_1A_2}, \vec{l}_1, \vec{l}_2) = 0 \Leftrightarrow$$

$\Leftrightarrow \overrightarrow{A_1A_2}, \vec{l}_1, \vec{l}_2$ linearly dependent.

7.6. Find the value of the parameter λ for which the lines

$$l_1 : \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1}$$

$$l_2 : \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{\lambda}$$

are coplanar. In that case, find their intersection.

$\vec{l_1}(3,-2,1)$, $\vec{l_2}(4,1,\lambda)$

$A(1,-2,0) \in l_1$ $B(-1,3,0) \in l_2$

$\vec{AB} = (-2,5,0)$

$$(\vec{AB}, \vec{l_1}, \vec{l_2}) = \begin{vmatrix} -2 & 5 & 0 \\ 3 & -2 & 1 \\ 4 & 1 & \lambda \end{vmatrix} =$$

$$= 4\lambda + 20 + 2 - 15\lambda = 22 - 11\lambda$$

l_1, l_2 coplanar $\Leftrightarrow (\vec{AB}, \vec{l_1}, \vec{l_2}) = 0 \Leftrightarrow$

$\Leftrightarrow 22 - 11\lambda = 0 \Leftrightarrow \lambda = 2$

$$l_1 \cap l_2 : \begin{cases} \dfrac{x-1}{3} = \dfrac{y+2}{-2} = \dfrac{z}{1} \\ \\ \dfrac{x+1}{4} = \dfrac{y-3}{1} = \dfrac{z}{2} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x-1=3z \\ 2x+y=4z \\ y+2=-2z \\ 2y-6=z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=3z+1 \\ 2(3z+1)+y=4z \\ y=-2z-2 \\ 2(-2z-2)-6=z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=3z+1 \\ 2z+4=0 \\ y=-2z-2 \\ -5z=10 \end{cases} \Leftrightarrow \begin{cases} z=-2 \\ y=2 \\ x=-5 \end{cases}$$

$$\Rightarrow l_1 \cap l_2 = \{(-5, 2, -2)\}$$

7.1. $\vec{a}, \vec{b}, \vec{c} \in V$

(a) $|(\vec{a}, \vec{b}, \vec{c})| \leq \|\vec{a}\| \cdot \|\vec{b}\| \cdot \|\vec{c}\|$

(b) $(\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}) = 2 \cdot (\vec{a}, \vec{b}, \vec{c})$

(b) $(\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}) = (\vec{a}+\vec{b}) \cdot \left((\vec{b}+\vec{c}) \times (\vec{c}+\vec{a}) \right) =$

$= (\vec{a}+\vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \underbrace{\vec{c} \times \vec{c}}_{\vec{0}} + \vec{c} \times \vec{a}) =$

$= (\vec{a}+\vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}) =$

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \overbrace{\vec{a} \cdot (\vec{b} \times \vec{a})}^{0} + \overbrace{\vec{a} \cdot (\vec{c} \times \vec{a})}^{0} +$

$+ \underbrace{\vec{b} \cdot (\vec{b} \times \vec{c})}_{0} + \underbrace{\vec{b} \cdot (\vec{b} \times \vec{a})}_{0} + \vec{b} \cdot (\vec{c} \times \vec{a}) =$

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{c} \times \vec{a}) = 2 \cdot (\vec{a}, \vec{b}, \vec{c})$