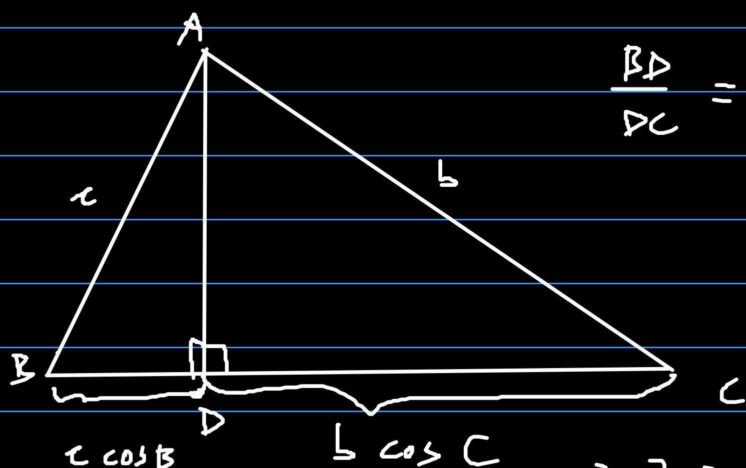
5. In a triangle ABC we consider the height AD from the vertex A ($D \in BC$). Find the decomposition of the vector AD in terms of the vectors $\vec{c} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AC}$.



$$\frac{BD}{DC} = \frac{c \cos B}{b \cos C}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow \frac{BD}{DC} = \frac{\cancel{c} \cdot \frac{c^2+a^2-b^2}{\cancel{2}a\cancel{c}}}{\cancel{b} \cdot \frac{a^2+b^2-c^2}{\cancel{2}a\cancel{b}}} = \frac{c^2+a^2-b^2}{a^2+b^2-c^2}$$

$$\overrightarrow{AD} = \frac{BD}{BC} \cdot \vec{b} + \frac{CD}{BC} \cdot \vec{c} =$$

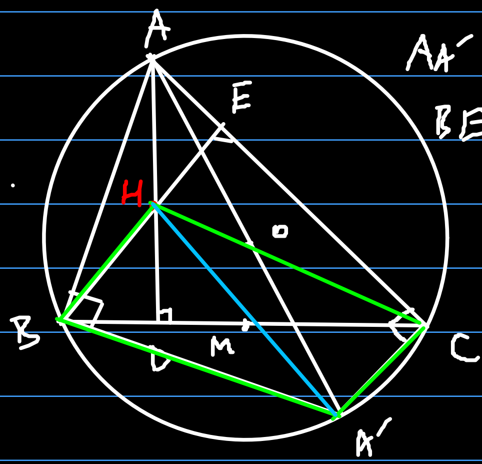
$$= \frac{c^2+a^2-b^2}{2a^2} \cdot \vec{b} + \frac{a^2+b^2-c^2}{2a^2} \cdot \vec{c}$$

9. ([4, Problem 14, p. 4]) Consider the triangle ABC alongside its orthocenter H, its circumcenter O and the diametrically opposed point A' of A on the latter circle. Show that:

(a) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$.

(b) $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$.

(c) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\,\overrightarrow{HO}$.



$$\left.\begin{array}{l} AA' \text{ diameter} \Rightarrow AC \perp CA' \\ BE \text{ height} \Rightarrow BE \perp AC \end{array}\right\} \Rightarrow BH \parallel CA'$$

$$\left.\begin{array}{l} AA' \text{ diameter} \Rightarrow AB \perp BA' \\ CH \text{ height} \Rightarrow CH \perp AB \end{array}\right\} \Rightarrow BA' \parallel CH$$

$\Rightarrow$ BHCA' parallelogram

$\Rightarrow \overrightarrow{HA'} = \overrightarrow{HB} + \overrightarrow{HC}$ (b) ✓

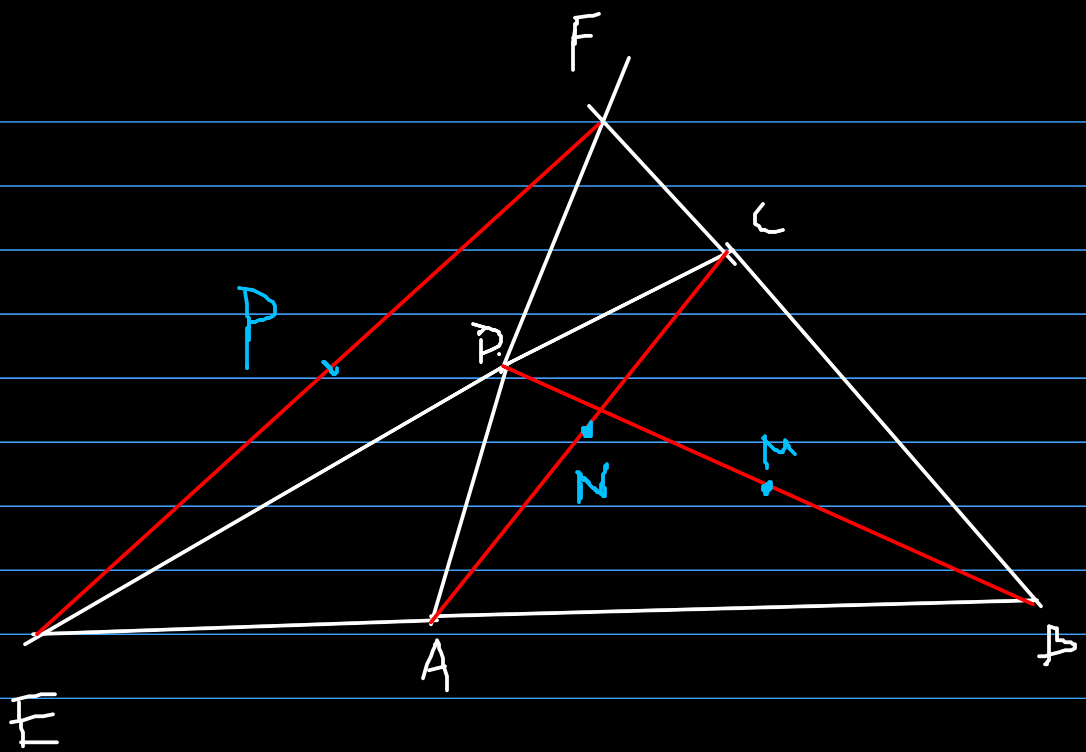
BHCA' parallelogram $\Rightarrow \{M\} = HA' \cap BC$

$\Rightarrow$ AM median in $\Delta AHA'$

$\Rightarrow$ G, centroid of ΔABC is also a centroid for $\Delta AHA'$

HO is another median in $\Delta AHA'$ $\Rightarrow$
$\Rightarrow O, G, H$ collinear, $\overrightarrow{OG} = \frac{1}{3}\overrightarrow{OH}$

Another approach: $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$
$\Rightarrow \overrightarrow{OB} + \overrightarrow{OC} - 2\overrightarrow{OH} = \overrightarrow{OA'} - \overrightarrow{OH}$



$\vec{BA} = \vec{u}$, $\vec{BC} = \vec{w}$, $\vec{u}, \vec{w}$ lin indep

$\vec{BF} = \alpha \cdot \vec{w}$ $\quad \vec{BE} = \beta \cdot \vec{u}$

$\{D\} = AF \cap EC$

$$\Rightarrow \vec{BD} = \lambda \vec{BF} + (1-\lambda) \vec{BA} =$$

$$= \mu \cdot \vec{BE} + (1-\mu) \cdot \vec{BC}$$

$$\Rightarrow \lambda \alpha \vec{w} + (1-\lambda) \cdot \vec{u} = \mu \cdot \beta \cdot \vec{u} + (1-\mu) \cdot \vec{w}$$

$$\Rightarrow \begin{cases} \lambda \alpha = 1-\mu \\ 1-\lambda = \mu\beta \end{cases} \Rightarrow \lambda = 1-\mu\beta$$

$$\Rightarrow \quad \alpha(1-\mu\beta) = 1-\mu$$

$$\Rightarrow \quad \mu(\alpha\beta-1) = \alpha-1$$

$$\Rightarrow \quad \mu = \frac{\alpha-1}{\alpha\beta-1}$$

$$\Rightarrow \overrightarrow{BD} = \frac{\alpha-1}{\alpha\beta-1}\cdot\beta\cdot\vec{u} + \frac{\alpha\beta-\alpha}{\alpha\beta-1}\cdot\vec{w}$$

$$\overrightarrow{BP} = \frac{\alpha\vec{w}+\beta\cdot\vec{u}}{2}$$

$$\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BD} = \frac{\beta(\alpha-1)}{2(\alpha\beta-1)}\cdot\vec{u} + \frac{\alpha(\beta-1)}{2(\alpha\beta-1)}\vec{w}$$

$$\overrightarrow{BN} = \frac{\vec{u}+\vec{w}}{2}$$

$$\overrightarrow{MN} = \overrightarrow{BN}-\overrightarrow{BM} = \vec{u}\left(\frac{1}{2}-\frac{\beta(\alpha-1)}{2(\alpha\beta-1)}\right)+$$
$$+\vec{w}\cdot\left(\frac{1}{2}-\frac{\alpha(\beta-1)}{2(\alpha\beta-1)}\right)$$

$$\overrightarrow{PN} = \overrightarrow{BN}-\overrightarrow{BP} = \vec{u}\left(\frac{1}{2}-\frac{\alpha}{2}\right)+w\left(\frac{1}{2}-\frac{\beta}{2}\right)$$

$\Rightarrow \vec{MN} = k \cdot \vec{PN}$ unde:

$$k = \frac{\frac{1}{2} - \frac{\beta(\alpha-1)}{2(\alpha\beta-1)}}{\frac{1}{2} - \frac{\alpha}{2}} = \frac{\frac{1}{2} - \frac{\alpha(\beta-1)}{2(\alpha\beta-1)}}{\frac{1}{2} - \frac{\beta}{2}}$$

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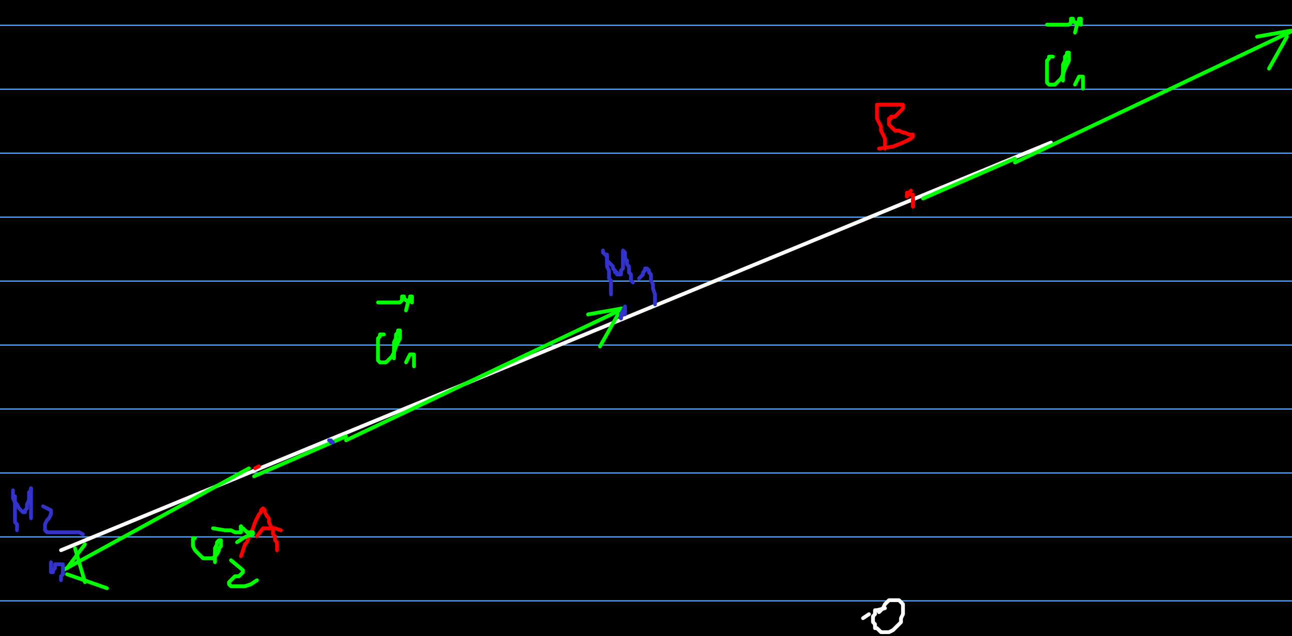
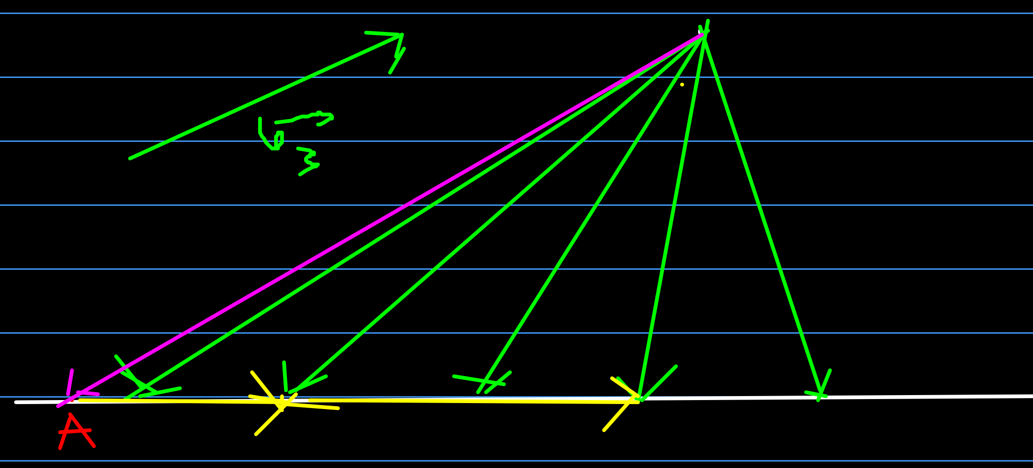
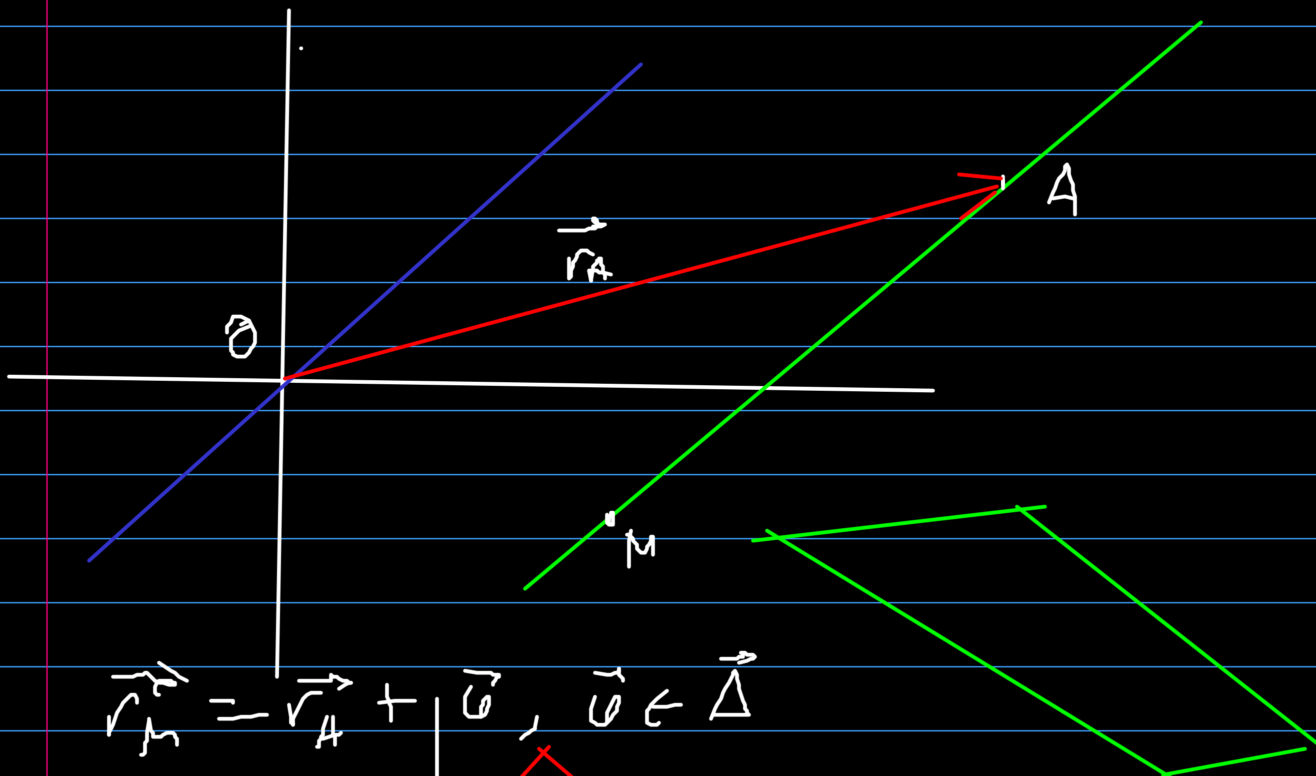


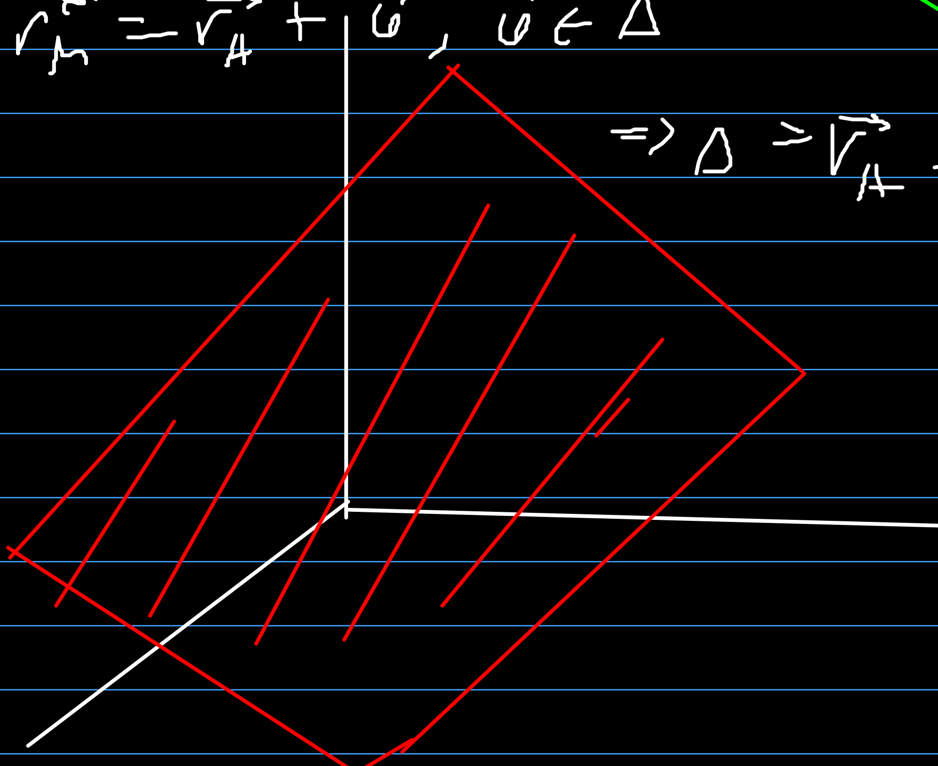


$\mathbb{R}^2$, $\mathbb{R}^3$



$\vec{r}_M = \vec{r}_A + \vec{u}$, $\vec{u} \in \vec{\Delta}$

$\Rightarrow \Delta = \vec{r}_A + \vec{\Delta}$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$