Seminar W1 - 916

02 December 2020 12:14

$CC = \max\left(M, S + \frac{M}{2}\right)$

$S$ = solved exercises in class (2p/3p) $S \le 12$

→ hell

$FG = \max\left(\frac{4}{10} CC + \frac{6}{10} E, E\right)$

minimal conditions: $CC \ge 4.5$, $E \ge 4.5$

## Analytic Geometry

↳ points = tuples of numbers "coordinates"

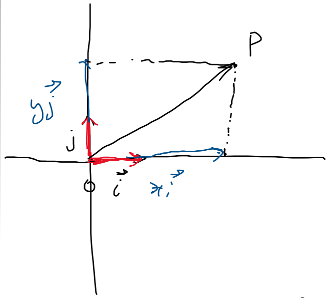
We will work in the Euclidean plane (and space).

In practice, these are identified with $\mathbb{R}^2$ and $\mathbb{R}^3$ ↪ René Descartes

A Cartesian reference system:

$\mathcal{R} = (O; b)$

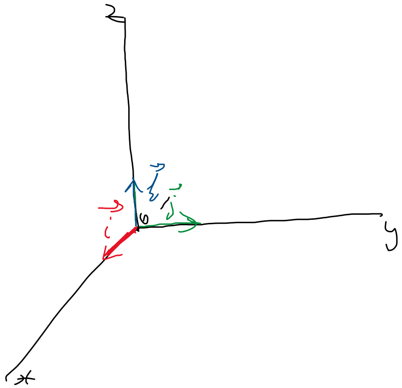
$O$ → point (origin); $b$ → basis of the vector space of the Euclidean plane/space




$\vec{OP} = \vec{r}_P = x\vec{i} + y\vec{j}$

↳ the position vector of P with regards to the reference system

$[P]_{\mathcal{R}} = \begin{pmatrix} x \\ y \end{pmatrix} = [\vec{OP}]_b$



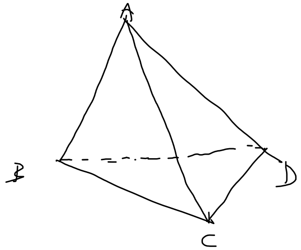

1.1. Consider a tetrahedron ABCD. Find the following sums of vectors

(a) $\vec{AB} + \vec{BC} + \vec{CD}$

(b) $\vec{AB} - \vec{CB} + \vec{DC}$

(c) $\vec{AB} + \vec{BC} + \vec{DA} + \vec{CD}$

(a)




$\vec{AB} + \vec{BC} + \vec{CD} = (\vec{AB} + \vec{BC}) + \vec{CD} =$

$= \vec{AC} + \vec{CD} = \vec{AD}$

(b) $\vec{AD} + \vec{CB} + \vec{DC} = (\vec{AD} + \vec{DC}) + \vec{CB} =$

$= \vec{AC} + \vec{CB} = \vec{AB}$

(c) $\vec{AB} + \vec{BC} + \vec{DA} + \vec{CD} = (\vec{AB} + \vec{BC}) + (\vec{DA} + \vec{CD}) =$

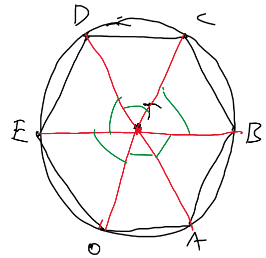
$= \vec{AC} + \vec{CA} = \vec{0}$

1.2. OABCDE regular hexagon

$\vec{OA} = \vec{a}$, $\vec{OE} = \vec{b}$.

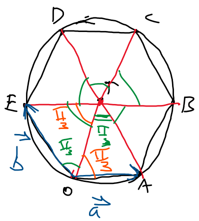
Express the vectors $\vec{OB}, \vec{OC}, \vec{OD}$ in terms of $\vec{a}$ and $\vec{b}$. Show that

$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} + \vec{OE} = 3\vec{OC}$




$\Delta TDC, \Delta TDE, \Delta TEO, \Delta TOA, \Delta TBA, \Delta TCB$ congruent triangles + they are isosceles

$\Rightarrow$ they are equilateral




$\vec{OB} = \vec{a} + \vec{AT} + \vec{TB}$

$m(\widehat{EOT}) = \frac{\pi}{3} = m(\widehat{OTA}) \Rightarrow OE \parallel TA$
$m(\widehat{ETO}) = \frac{\pi}{3} = m(\widehat{TOA}) \Rightarrow ET \parallel OA$ $\Rightarrow$

$\Rightarrow$ TEOA parallelogram

$\Rightarrow \vec{AT} = \vec{OE} = \vec{b}$

We use a similar argument to show that OABT is a parallelogram

$\Rightarrow \vec{TB} = \vec{OA} = \vec{a}$

$\Rightarrow \vec{OB} = \vec{OA} + \vec{AT} + \vec{TB} = \vec{a} + \vec{b} + \vec{a} = 2\vec{a} + \vec{b}$

We show like before that OECB parallelogram $\Rightarrow \vec{OE} = \vec{BC} = \vec{b}$

$\vec{OC} = \vec{OB} + \vec{BC} = 2\vec{a} + \vec{b} + \vec{b} = 2\vec{a} + 2\vec{b}$

$\vec{OD} = \vec{OC} + \vec{CD} = 2\vec{a} + 2\vec{b} + (-\vec{a}) = \vec{a} + 2\vec{b}$

$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} + \vec{OE} = \vec{a} + 2\vec{a} + \vec{b} +$
$+ 2\vec{a} + 2\vec{b} + \vec{a} + 2\vec{b} + \vec{b} = 6\vec{a} + 6\vec{b} =$
$= 3(2\vec{a} + 2\vec{b}) = 3 \cdot \vec{OC}$


M midpoint of [AB]

If we fix a reference system

$\vec{r_M} = \frac{\vec{r_A} + \vec{r_B}}{2}$

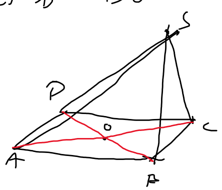

Say we know $\frac{AM}{MB} = k \in \mathbb{R}$

$\vec{r_M} = \frac{k\vec{r_B} + \vec{r_A}}{k+1}$

1.3. Consider a pyramid with the vertex at S and the basis a parallelogram ABCD, whose diagonals are concurrent at O. Show the equality:

$\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4\vec{SO}$

$\vec{SA} = \vec{SO} + \vec{OA}$
$\vec{SB} = \vec{SO} + \vec{OB}$
$\vec{SC} = \vec{SO} + \vec{OC}$
$\vec{SD} = \vec{SO} + \vec{OD}$




ABCD parallelogram $\Rightarrow$ $\vec{OA} = \vec{CO}$, $\vec{OB} = \vec{DO}$

$\Rightarrow$

$\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4\vec{SO} + \vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} =$

$= 4\vec{SO} + \vec{CO} + \vec{DO} + \vec{OC} + \vec{OD} =$

$= 4\vec{SO} + \underbrace{(\vec{CO} + \vec{OC})}_{=\vec{0}} + \underbrace{(\vec{DO} + \vec{OD})}_{=\vec{0}} =$

$= 4\vec{SO}$

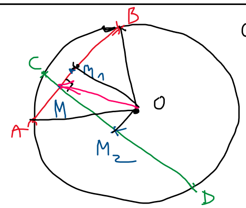
1.7. (3p) Consider two perpendicular chords AB and CD of a given circle and $\{M\} = AB \cap CD$

Show that

$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 2\vec{OM}$

(O is the centre of the circle) center

$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 2\vec{OM}$




Let $M_1$ be the midpoint of [AB]
$M_2$ — " — [CD]

$\Delta$ OAB isosceles, $OM_1$ median $\Rightarrow OM_1$ height

$\Rightarrow OM_1 \perp AB$

We prove in the same way that $OM_2 \perp CD$

$$\left.\begin{array}{l} CD \perp AB \\ OM_1 \perp AB \end{array}\right\} \Rightarrow OM_1 \parallel CD$$

$$\left.\begin{array}{l} CD \perp AB \\ OM_2 \perp CD \end{array}\right| \Rightarrow OM_2 \parallel AB$$

$\Rightarrow OM_1MM_2$ parallelogram

$\Rightarrow \overrightarrow{OM_1} + \overrightarrow{OM_2} = \overrightarrow{OM}$

$\overrightarrow{OM_1} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$

$\overrightarrow{OM_2} = \dfrac{\overrightarrow{OC} + \overrightarrow{OD}}{2}$

$\Rightarrow$ the r