Seminar W10 - 5.1.7

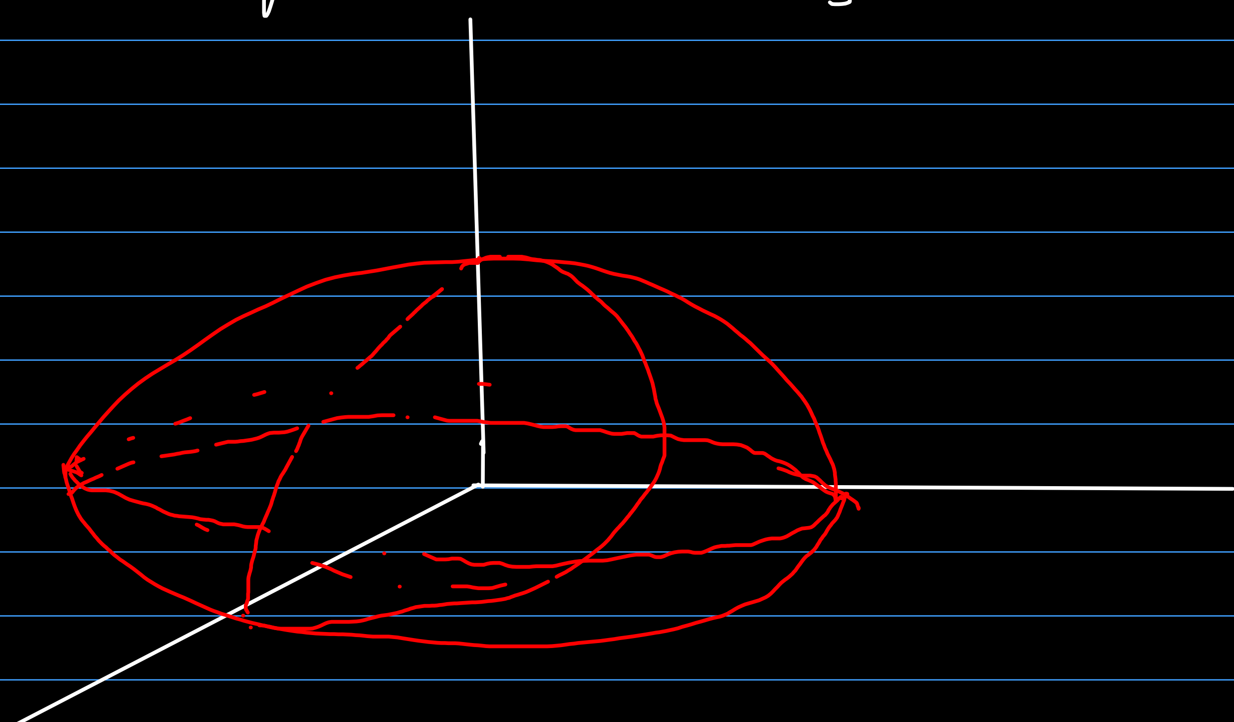
$\mathscr{C}$: $f(x,y)=0$, $f\in\mathbb{R}[x,y]$, $\deg f=2$

↳ conics

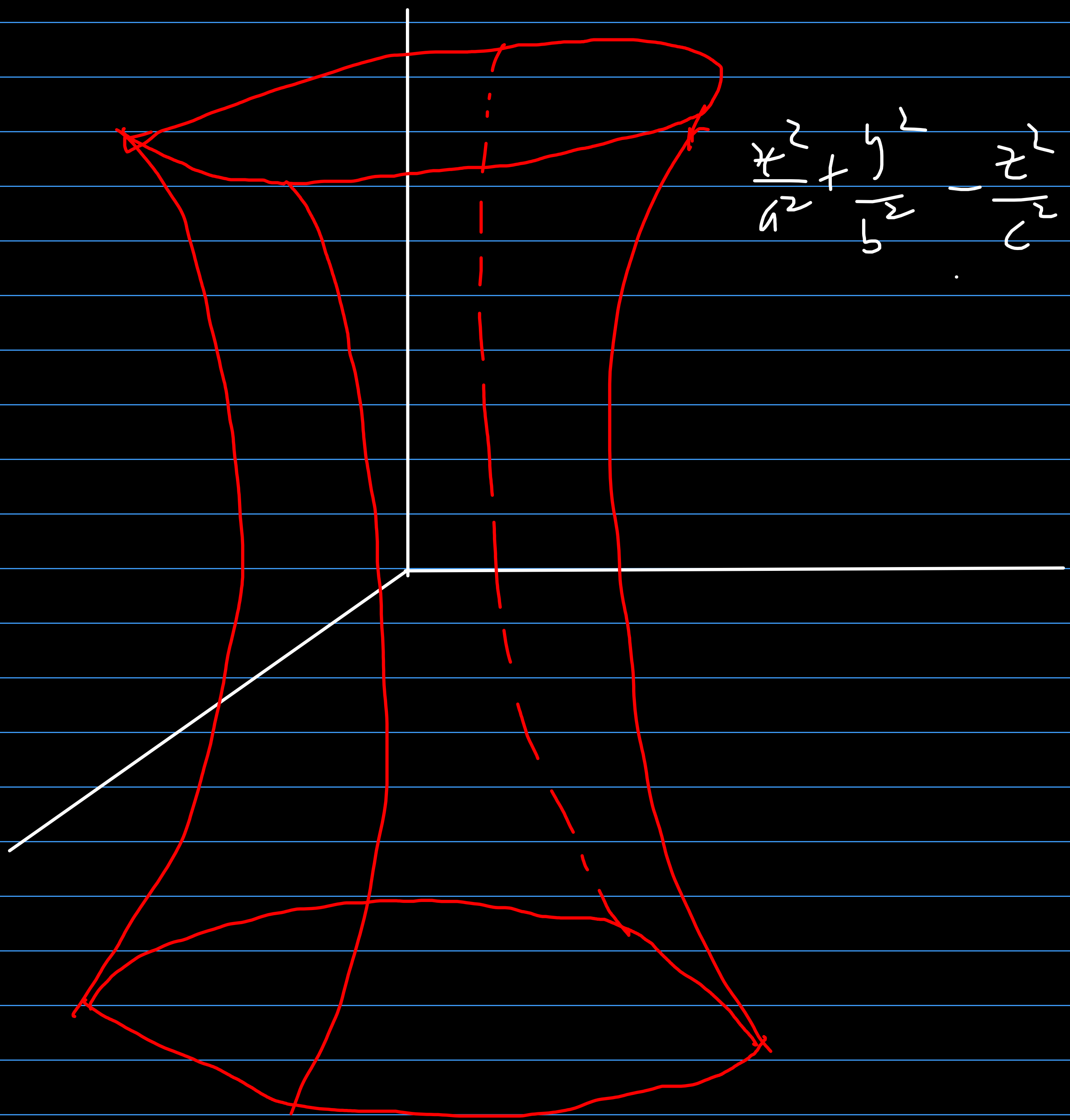
$\mathscr{Y}$: $f(x,y,z)=0$, $f\in\mathbb{R}[x,y,z]$, $\deg f=2$

↳ quadrics

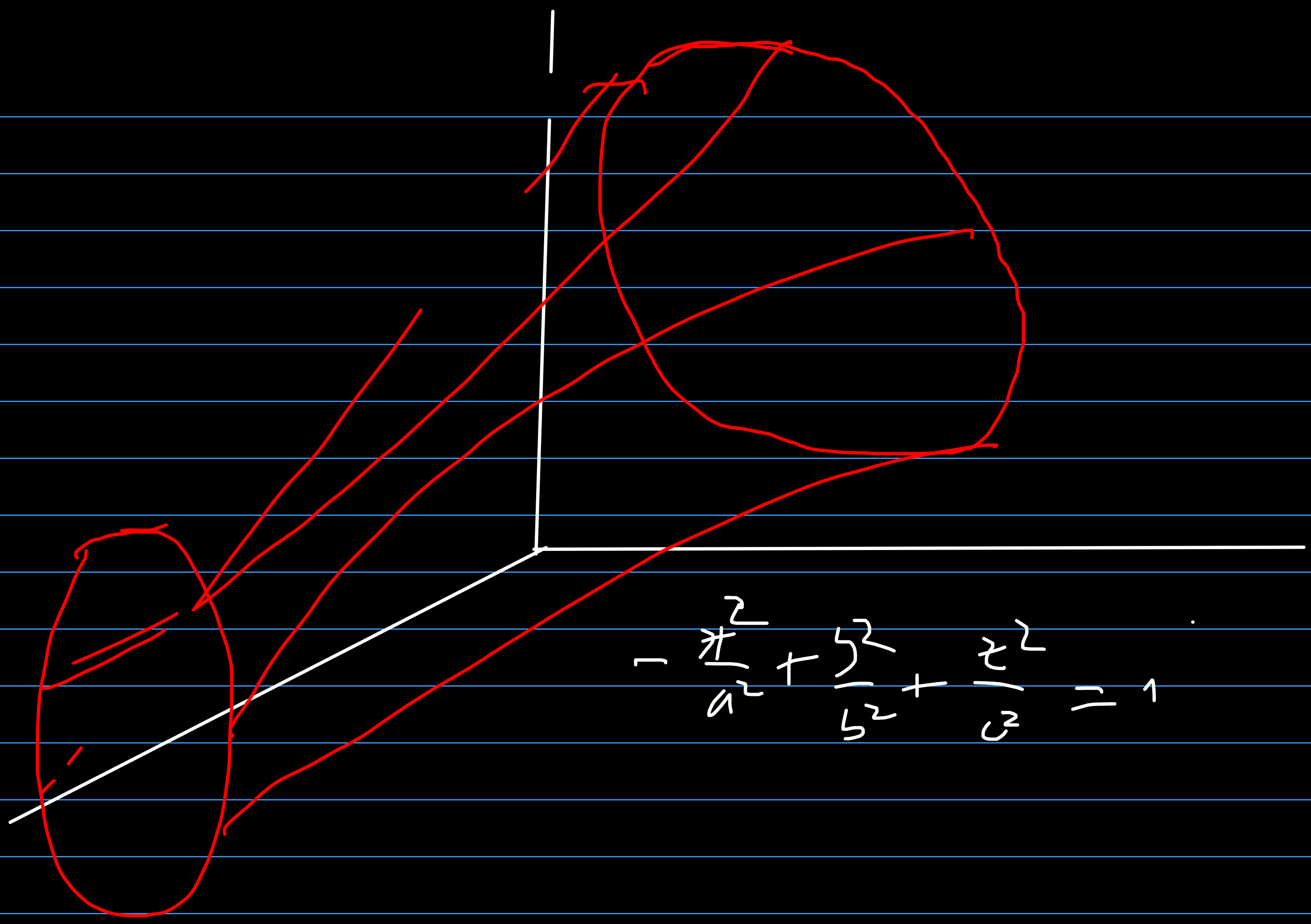
- Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



- Hyperboloid of one sheet

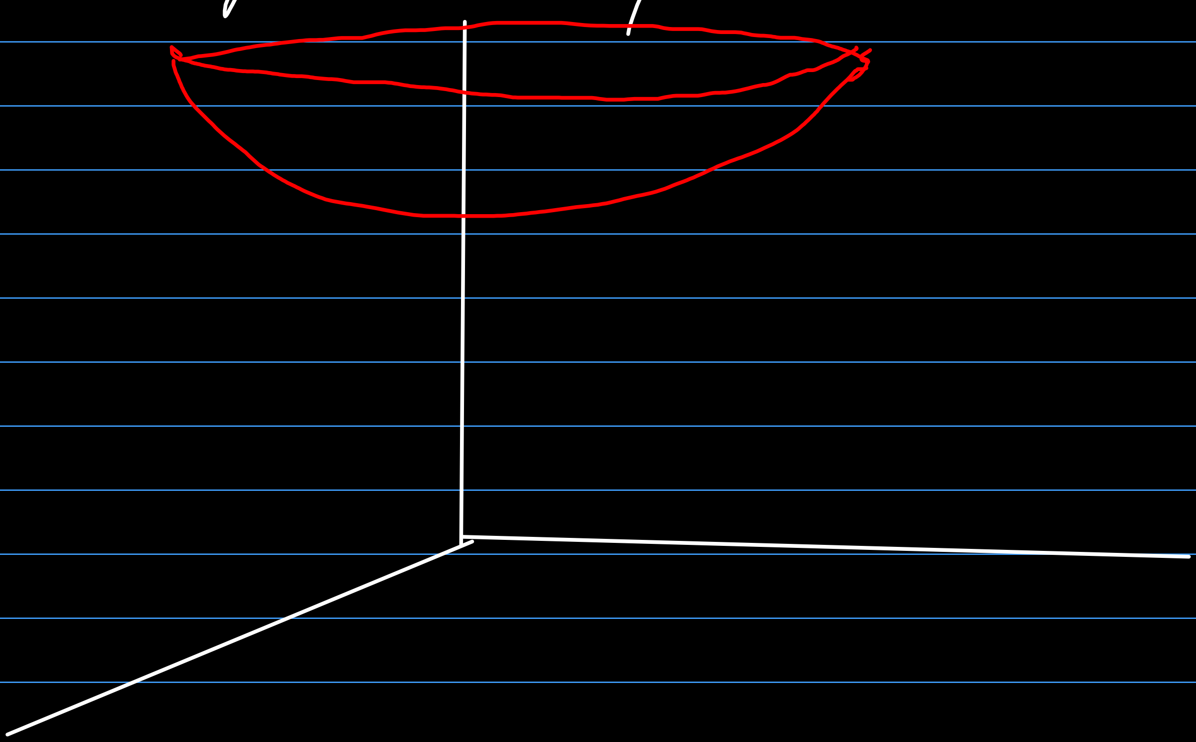


$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

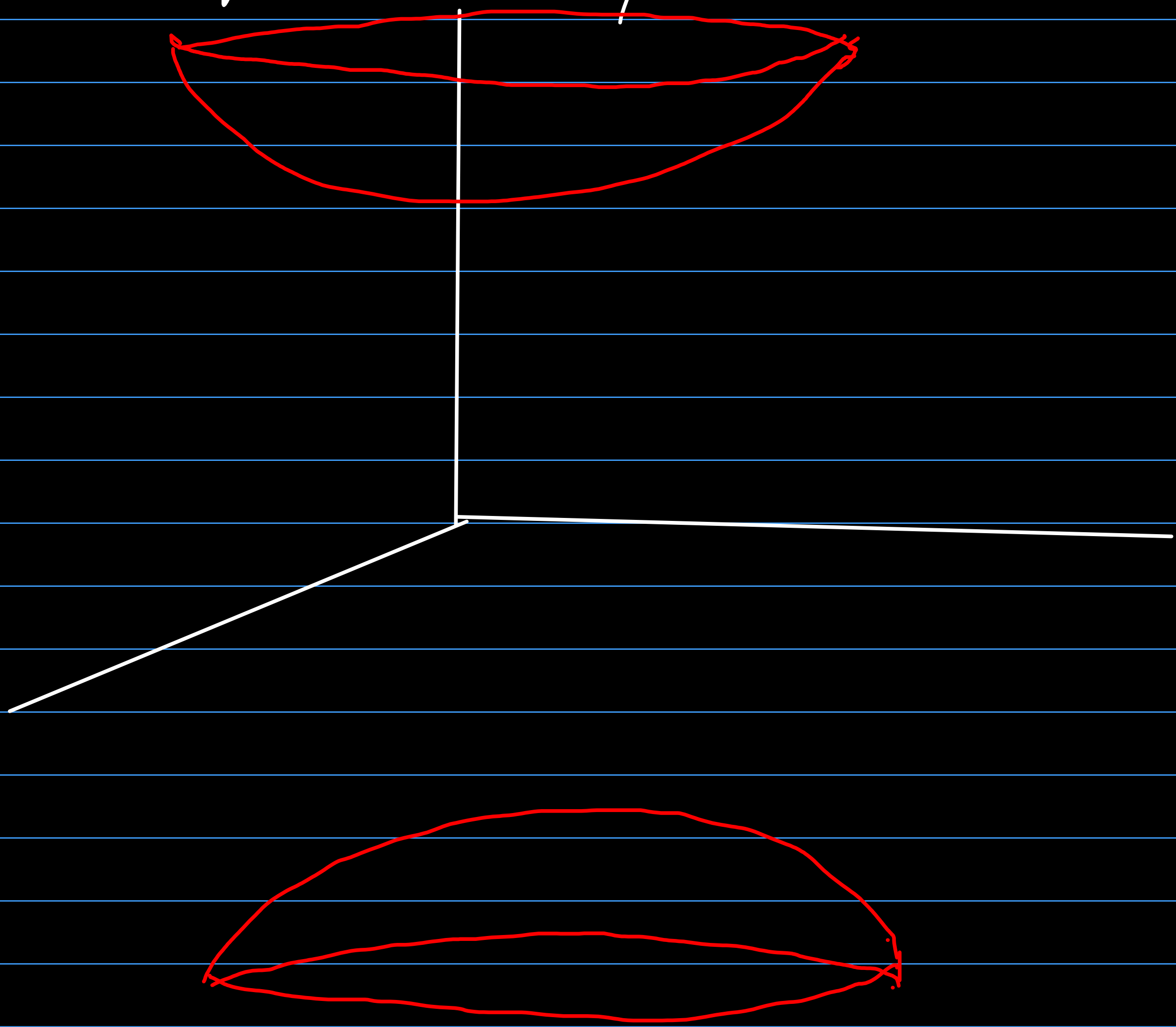


$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

- Hyperboloid of two sheets



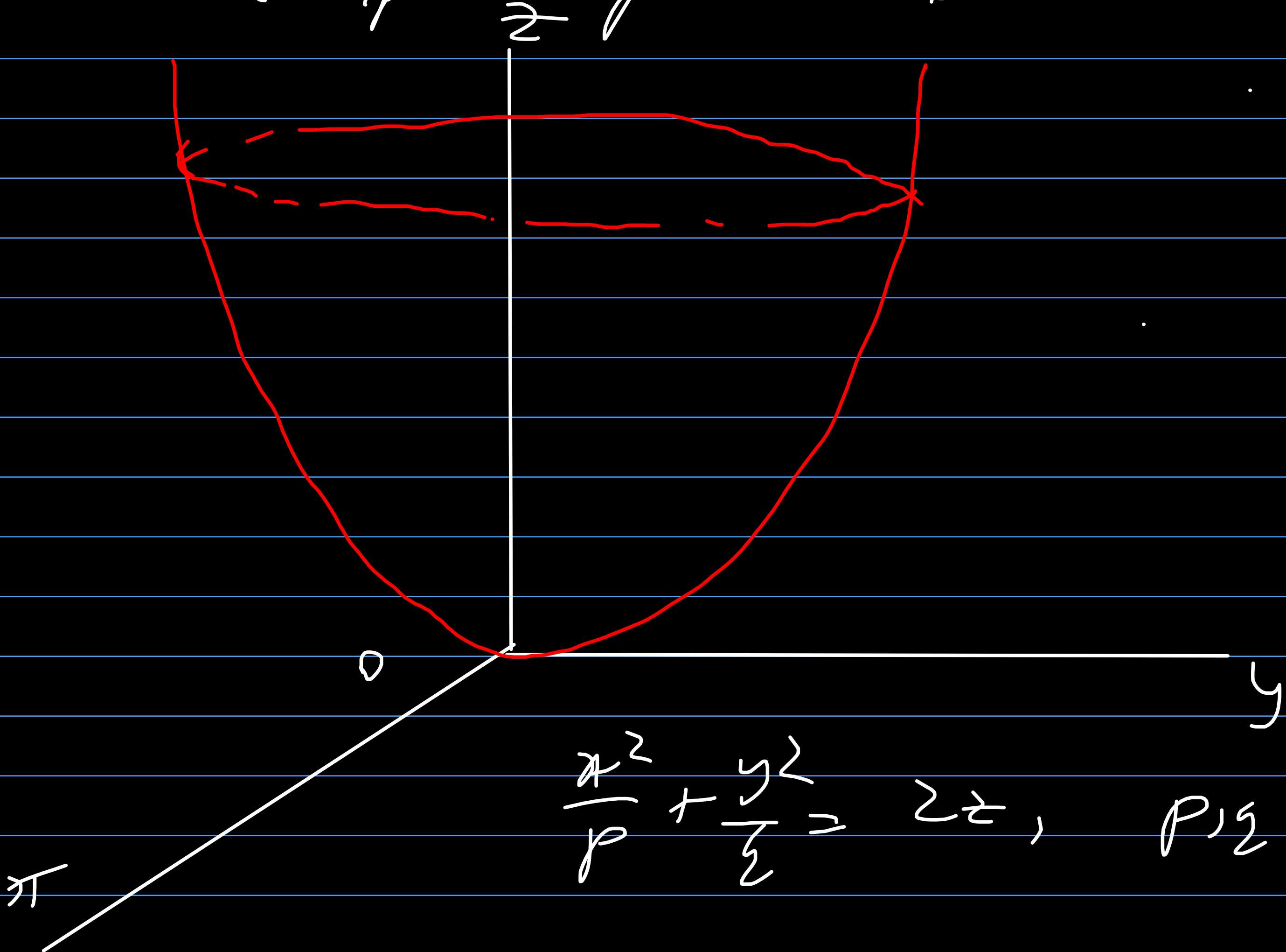
• Hyperboloid of two sheets



$$\mathcal{H}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

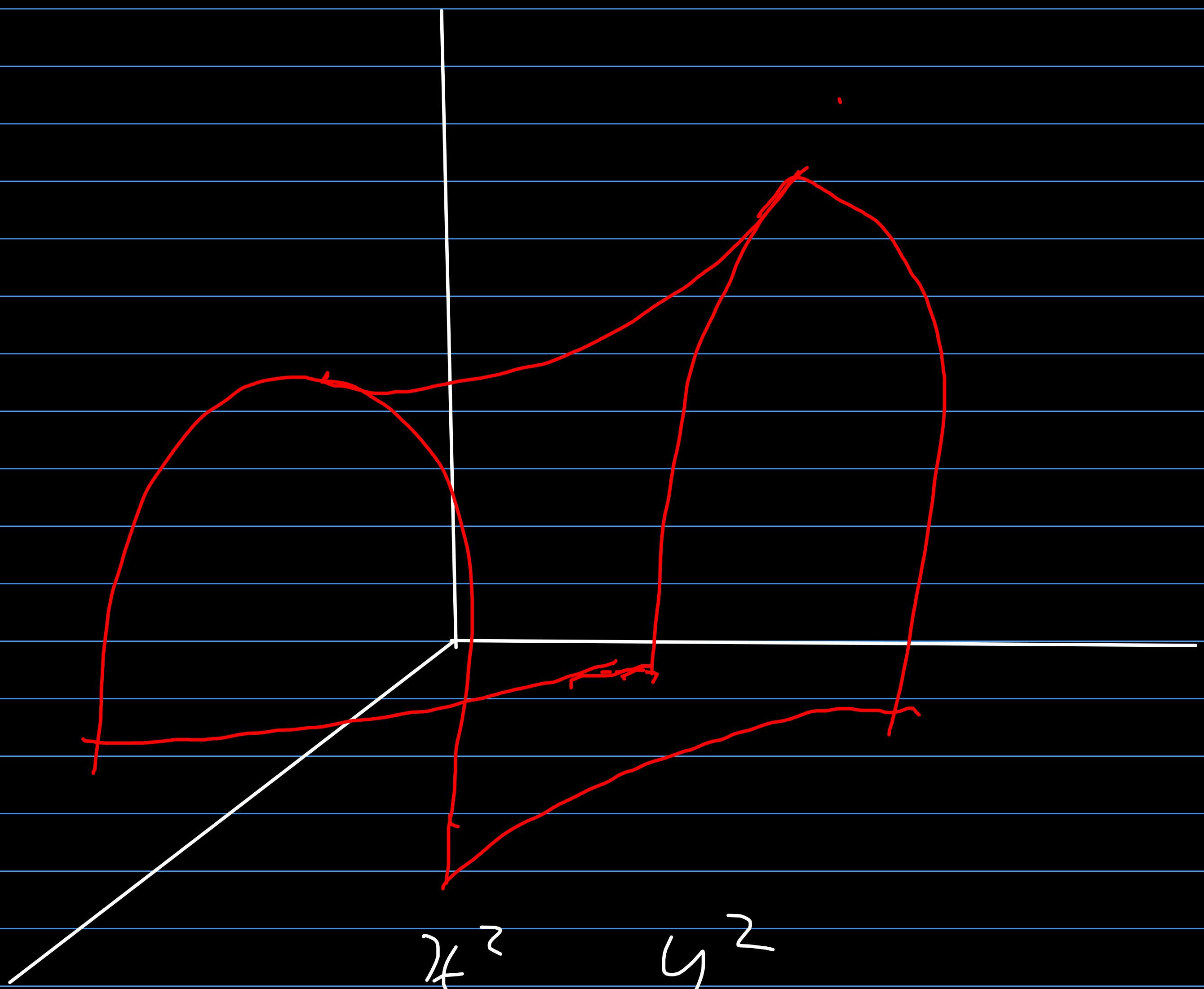
$$\mathcal{H}_2 : -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Elliptic paraboloid



$$\frac{x^2}{p} + \frac{y^2}{q} = 2z, \quad p, q > 0$$

· Hyperbolic paraboloid



$$\frac{x^2}{p} - \frac{y^2}{q} = 2z$$

– y : $f(x,y,z)=0$

$T_y(x_0,y_0,z_0)$: $f'_x(x_0,y_0,z_0)\cdot(x-x_0)+$

$+f'_y(x_0,y_0,z_0)\cdot(y-y_0)+f'_z(x_0,y_0,z_0)\cdot(z-z_0)=0$

$$N_f(x_0,y_0,z_0): \quad \frac{x-x_0}{f'_x(x_0,y_0,z_0)} = \frac{y-y_0}{f'_y(x_0,y_0,z_0)} = \frac{z-z_0}{f'_z(x_0,y_0,z_0)}$$

10.1 Find the intersection points of the ellipsoid

$$\frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} = 1$$

with the line

$$\frac{x-4}{2} = \frac{y+6}{-3} = \frac{z+2}{-2}$$

and write the equations of the tangent planes, as well as the equations of the normal lines in these intersection points.

$$l: \begin{cases} x = 2t+4 \\ y = -3t-6 \\ z = -2t-2 \end{cases}$$

$$l \cap \mathcal{E}: \begin{cases} \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} = 1 \\ x = 2t+4 \\ y = -3t-6 \\ z = -2t-2 \end{cases}$$

$$\Rightarrow \frac{(2t+4)^2}{16} + \frac{(-3t-6)^2}{12} + \frac{(-2t-2)^2}{4} = 1$$

$$\Rightarrow 3(2t+4)^2 + 4(3t+6)^2 + 12(2t+2)^2 - 48 = 0$$

$$\Rightarrow 12t^2 + 48t + 48 + 36t^2 + 144t + 144 +$$

$$+ 48t^2 + 96t + 48 - 48 = 0$$

$$\Rightarrow 96t^2 + 288t + 192 = 0$$

$$\Rightarrow 16t^2 + 48t + 32 = 0$$

$$\Rightarrow 2t^2 + 6t + 4 = 0$$

$$\Rightarrow t^2 + 3t + 2 = 0$$

$$t_{1,2} = \frac{-3 \pm 1}{2} \Rightarrow t \in \{-2, -1\}$$

$\Rightarrow$ For $t = -2$: $P(0,0,2)$

For $t = -1$: $Q(2,-3,0)$

$$f(x,y,z) = \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} - 1$$

$$f'_x(x_0, y_0, z_0) = \frac{x_0}{8}$$

$$f'_y(x_0, y_0, z_0) = \frac{y_0}{6}$$

$$f'_z(x_0, y_0, z_0) = \frac{z_0}{2}$$

$$T_{\xi}(x_0, y_0, z_0): \frac{x_0}{8}\cdot(x-x_0)+\frac{y_0}{6}\cdot(y-y_0)+$$
$$+\frac{z_0}{2}\cdot(z-z_0)=0$$

$$T_{\xi}(0,0,2): \; z-2=0$$

$$T_{\xi}(2,-3,0): \; \frac{1}{4}(x-2)-\frac{1}{2}\cdot(y+3)=0$$

$$\Rightarrow N_{\xi}(0,0,2): \begin{cases} x=0 \\ y=0 \end{cases}$$

$$N_{\xi}(2,-3,0): \begin{cases} \frac{x-2}{1/4}=\frac{y+3}{-1/2} \\ z=0 \end{cases}$$

Rectilinear generatrices

→ hyperboloid of one sheet

$$(\mathcal{H}_1): \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

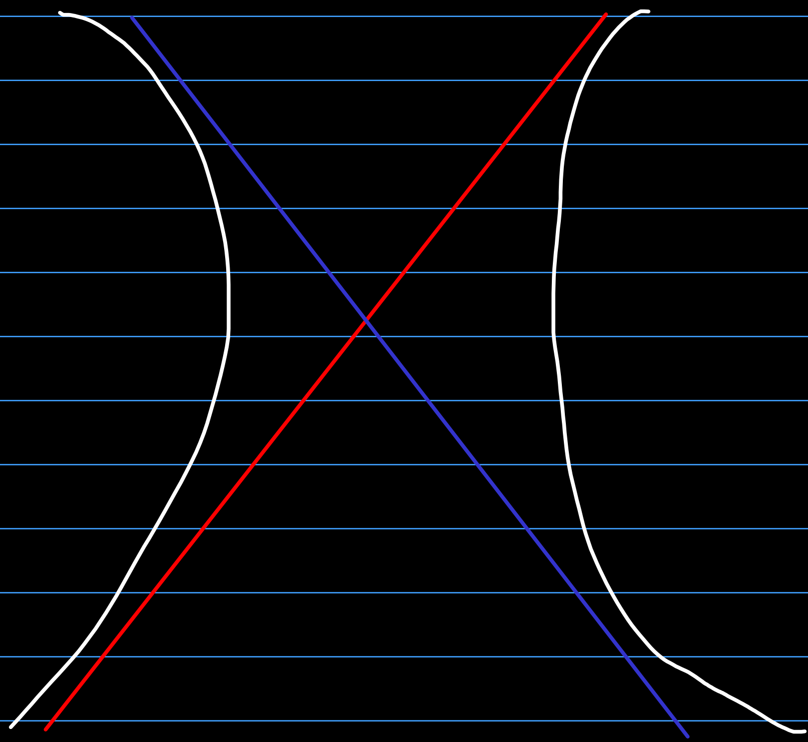
$$(\mathcal{H}_1): \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}$$

$$(\mathcal{H}_1): \left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right) \cdot \left(1 + \frac{y}{b}\right)$$

$$d_\lambda : \begin{cases} \frac{x}{a} - \frac{z}{c} = \lambda \cdot \left(1 - \frac{y}{b}\right) \\ \lambda\left(\frac{x}{a} + \frac{z}{c}\right) = 1 + \frac{y}{b} \end{cases}$$

$\forall \lambda \in \mathbb{R}$

$$d'_\mu : \begin{cases} \frac{x}{a} - \frac{z}{c} = \mu \cdot \left(1 + \frac{y}{b}\right) \\ \mu\left(\frac{x}{a} + \frac{z}{c}\right) = 1 - \frac{y}{b} \end{cases}$$



$\rightarrow$ hyperbolic paraboloid

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z \quad , \; p, q > 0$$

$$\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right) \cdot \left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = 2z$$

$$d_\lambda : \begin{cases} \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = 2\lambda \\ \lambda \cdot \left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = z \end{cases}$$

$$d'_\mu : \begin{cases} \lambda\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right) = z \\ \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} = 2\lambda \end{cases}$$

10.2. Find the rectilinear generatrices of the quadric $4x^2 - 9y^2 = 36z$ which pass through the point $P(3\sqrt{2}, 2, 1)$.

$$4x^2 - 9y^2 = 36z$$

$$(2x-3y)(2x+3y) = 36z$$

$$d_\lambda : \begin{cases} 2x - 3y = 2\lambda \\ \lambda\cdot(2x+3y) = 18z \end{cases}$$

$$P \in d_\lambda \Rightarrow \begin{cases} 6\sqrt{2} - 6 = 2\lambda \\ \lambda \cdot (6\sqrt{2} + 6) = 18 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \lambda = 3\sqrt{2} - 3 = 3(\sqrt{2} - 1) \\ \lambda = \dfrac{18}{6\sqrt{2}+6} = \dfrac{3}{\sqrt{2}+1} = \dfrac{3}{\sqrt{2}+1} \end{cases}$$

$$\Rightarrow \lambda = 3 \cdot (\sqrt{2} - 1)$$

$\Rightarrow$ the rectilinear generatrix from (d_λ) that contains P is $d_{3(\sqrt{2}-1)}$

$$d'_\mu : \begin{cases} \mu(2x - 3y) = 18z \\ 2x + 3y = 2\mu \end{cases}$$

$$P \in d'_\mu \Rightarrow \begin{cases} \mu \cdot (6\sqrt{2} - 6) = 18 \\ 6\sqrt{2} + 6 = 2\mu \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \mu = \dfrac{18}{6\sqrt{2}-6} = \dfrac{3}{\sqrt{2}-1} \\ \mu = 3\sqrt{2} + 3 \end{cases}$$

$\Rightarrow$ $d'_{3\sqrt{2}+3}$ is the rectilinear generatrix that contains P.

10.3. Find the rectilinear generatrices of the hyperboloid of one sheet

$$(\mathcal{H}_1): \frac{x^2}{36} + \frac{y^2}{9} - \frac{z^2}{4} = 1$$

which are parallel to the plane $(\Pi): x+y+z=0$

$$\frac{x^2}{36} - \frac{z^2}{4} = 1 - \frac{y^2}{9}$$

$$\left(\frac{x}{6} + \frac{z}{2}\right)\cdot\left(\frac{x}{6} - \frac{z}{2}\right) = \left(1 - \frac{y}{3}\right)\cdot\left(1 + \frac{y}{3}\right)$$

$$d_\lambda : \begin{cases} \frac{x}{6} - \frac{z}{2} = \lambda\cdot\left(1 - \frac{y}{3}\right) \\ \lambda\cdot\left(\frac{x}{6} + \frac{z}{2}\right) = \left(1 + \frac{y}{3}\right) \end{cases}$$

$$d_\lambda : \begin{cases} \frac{x}{6} + \lambda \cdot \frac{y}{3} - \frac{z}{2} - \lambda = 0 \\ \lambda \cdot \frac{x}{6} - \frac{y}{3} + \lambda \cdot \frac{z}{2} - 1 = 0 \end{cases}$$

$$\Rightarrow \vec{d_\lambda} = \left(\frac{1}{6}, \frac{\lambda}{3}, -\frac{1}{2} \right) \times \left(\frac{\lambda}{6}, -\frac{1}{3}, \frac{\lambda}{2} \right) =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{6} & \frac{\lambda}{3} & -\frac{1}{2} \\ \frac{\lambda}{6} & -\frac{1}{3} & \frac{\lambda}{2} \end{vmatrix} = \vec{i} \cdot \frac{\lambda^2 - 1}{6} +$$

$$+ \vec{j} \cdot \frac{-\lambda}{6} + \vec{k} \cdot \frac{-1-\lambda^2}{18}$$

$$\Rightarrow \vec{d_\lambda} \left(3(\lambda^2 - 1), -3\lambda, -1-\lambda^2 \right)$$

$$d_\lambda \parallel \pi \Leftrightarrow d_\lambda \perp \vec{n_\pi} \Leftrightarrow \vec{d_\lambda} \cdot \vec{n_\pi} = 0$$

$$\vec{d_\lambda} \cdot \vec{n_\pi} = 3(\lambda^2 - 1) - 3\lambda - 1 - \lambda^2 =$$

$$= 2\lambda^2 - 3\lambda - 4$$

$$\Rightarrow \lambda_{1,2} = \frac{3 \pm \sqrt{9+32}}{4} = \frac{3 \pm \sqrt{41}}{4}$$

$$d'_\mu : \begin{cases} \frac{x}{6} - \frac{z}{2} = \mu \cdot \left(1 + \frac{y}{3}\right) \\ \mu\left(\frac{x}{6} + \frac{z}{2}\right) = 1 - \frac{y}{3} \end{cases}$$

$$\vec{d'_\mu} = \left(\frac{1}{6}, -\frac{\mu}{3}, -\frac{1}{2}\right) \times \left(\frac{\mu}{6}, \frac{1}{3}, \frac{\mu}{2}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{6} & -\frac{\mu}{3} & -\frac{1}{2} \\ \frac{\mu}{6} & \frac{1}{3} & \frac{\mu}{2} \end{vmatrix} = \vec{i} \cdot \left(-\frac{\mu^2 - 1}{6}\right) +$$

$$+ \vec{j} \cdot \left(-\frac{\mu}{6}\right) + \vec{k} \cdot \left(\frac{\mu^2 + 1}{18}\right)$$

$$d'_\mu \parallel \pi \Leftrightarrow \vec{d'_\mu} \cdot \vec{n_\pi} = 0$$

We do the same thing.