Semantic Tableaux Method

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Exercise 2

Prove that the following formulas are tautologies using the semantic tableaux method:

7. distribution of '→' over '↔' :

$$U = (p \to (q \leftrightarrow r)) \leftrightarrow ((p \to q) \leftrightarrow (p \to r))$$

Theoretical results

Decomposition rules for propositional formula:

- conjunctive formulas: which are consistent only if both of its component sub-formulas are satisfied = α rules

$A \wedge B$	$\neg(A \vee B) \equiv \neg A \wedge \neg B$	$\neg(A \to B) \equiv A \wedge \neg B$	$A \leftrightarrow B \equiv (A \to B) \wedge (B \to A)$
|	|	|	|
A	$\neg A$	A	$A \to B$
|	|	|	|
B	$\neg B$	$\neg B$	$B \to A$

- disjunctive formulas: which are satisfied if one of its component sub-formulas is consistent = β rules

$A \vee B$	$\neg(A \wedge B) \equiv \neg A \vee \neg B$	$A \to B \equiv \neg A \vee B$	$\neg(A \leftrightarrow B)$
/ \	/ \	/ \	/ \
A B	$\neg A$ $\neg B$	$\neg A$ B	$\neg(A \to B)$ $\neg(B \to A)$

Construction of a semantic tableau

To a propositional/predicate formula U we can associate a *semantic tableau*, which is a binary tree having formulas in its nodes and it is built as follows:

1. the root of the tree is labeled with the initial formula;
2. every branch of the tree which contains a formula will be extended with a subtree according to the decomposition rule specific to its class;
3. the extension of a branch stops in the following cases:
   a) if that branch contains a formula and its negation, the branch is marked as closed using the symbol $\otimes$;
   b) if all the formulas on that branch are already decomposed or if by decomposing the formulas which are not decomposed yet, no new formulas are obtained.

Theorem 1: Soundness and completeness of the semantic tableaux method

A propositional/predicate formula U is a tautology *if and only* if $\neg U$ has a closed semantic tableau.

Therefore, we build the semantic tableaux for ¬ U.

$\neg U = \neg\, (p\to(q\leftrightarrow r)) \leftrightarrow ((p\to q) \leftrightarrow (p\to r))$ (1)

/ \ β rule for (1)

$\neg(\,(p\to(q\leftrightarrow r)) \to ((p\to q) \leftrightarrow (p\to r))\,)$ (2)

| α rule for (2)

$p\to(q\leftrightarrow r)$ (7)

|

$\neg(\,(p\to q) \leftrightarrow (p\to r)\,)$ (3)

/ \ β rule for (3)

$\neg((p\to q) \to (p\to r))$ (4)

| α rule for (4)

$(p\to q)$ (6)

|

$\neg(p\to r)$ (5)

| α rule for (5)

p

|

$\neg r$

$\neg((p\to r) \to (p\to q))$ (11)

| α rule for (11)

$(p\to r)$ (13)

|

$\neg(p\to q)$ (12)

| α rule for (12)

p

|

$\neg q$

$\neg(\,((p\to q) \leftrightarrow (p\to r)) \to (p\to(q\leftrightarrow r))\,)$ (15)

| α rule for (15)

$(p\to q) \leftrightarrow (p\to r)$ (16)

|

$\neg(p\to(q\leftrightarrow r))$ (17)

| α rule for (16)

$(p\to q) \to (p\to r)$ (18)

|

$(p\to r) \to (p\to q)$ (19)

| α rule for (17)

p

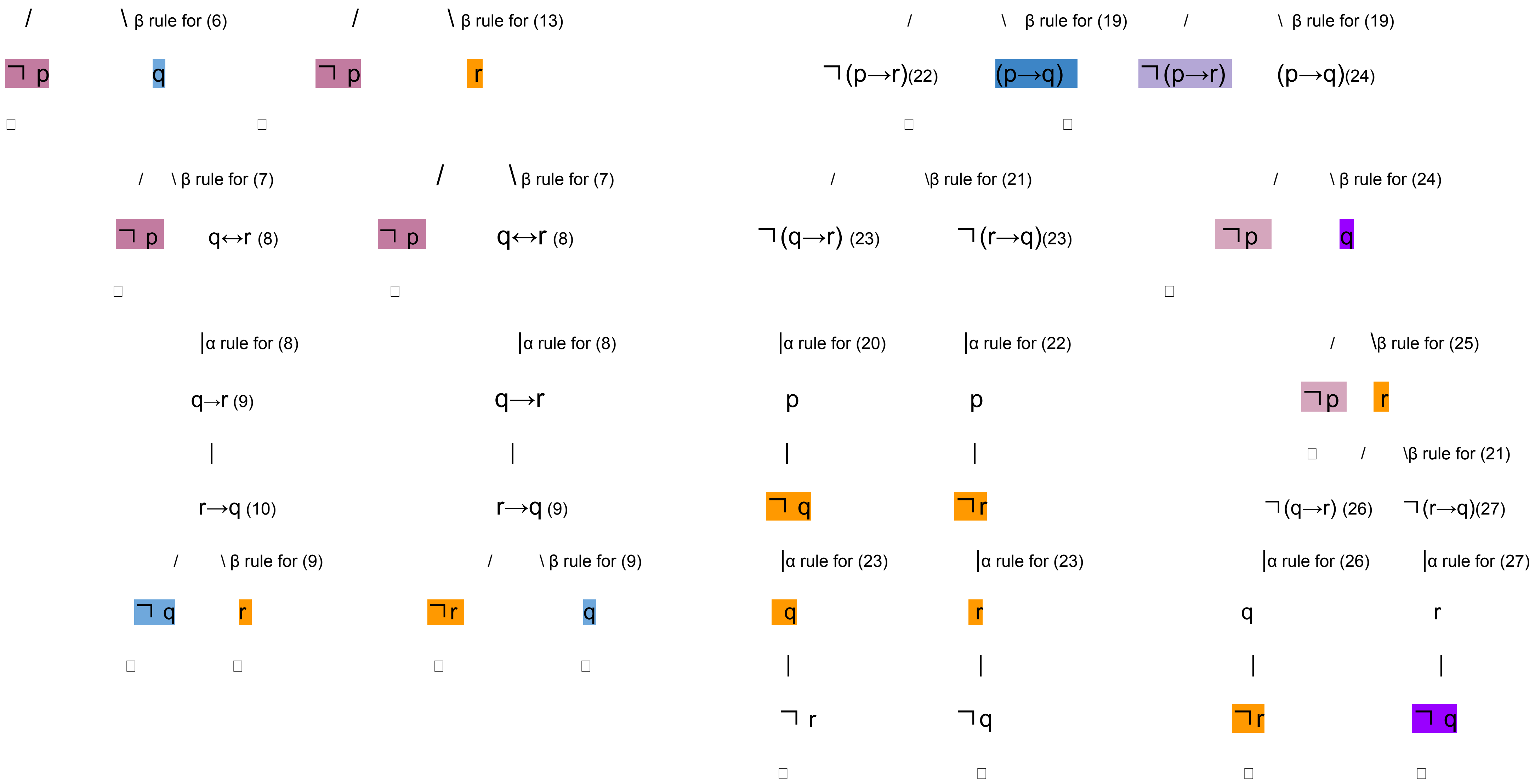
|

$\neg(q\leftrightarrow r)$ (21)

/ \ β rule for (18)

$\neg(p\to q)$ (20)

$(p\to r)$ (25)



All branches of the semantic tableau are closed, containing pairs of opposite literals. Therefore, the formula ¬ U has no models, it is an inconsistent formula. We conclude that U is a tautology.