Seminar W7 - 916

The mixed product (the triple scalar product)

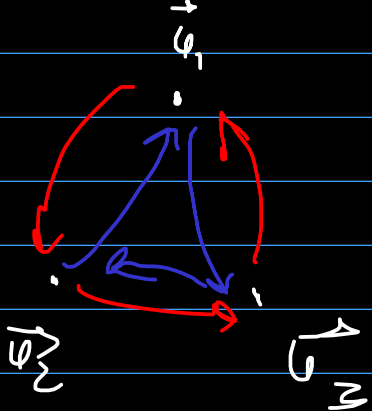
$\vec{a}, \vec{b}, \vec{c} \in V$

$$(\vec{a}, \vec{b}, \vec{c}) := \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

If the reference system $(O, [\vec{i}, \vec{j}, \vec{k}])$ is orthonormal and direct, then:

$\vec{u}_1(a_1, b_1, c_1), \vec{u}_2(a_2, b_2, c_2), \vec{u}_3(a_3, b_3, c_3)$

$$\Rightarrow (\vec{u}_1, \vec{u}_2, \vec{u}_3) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$



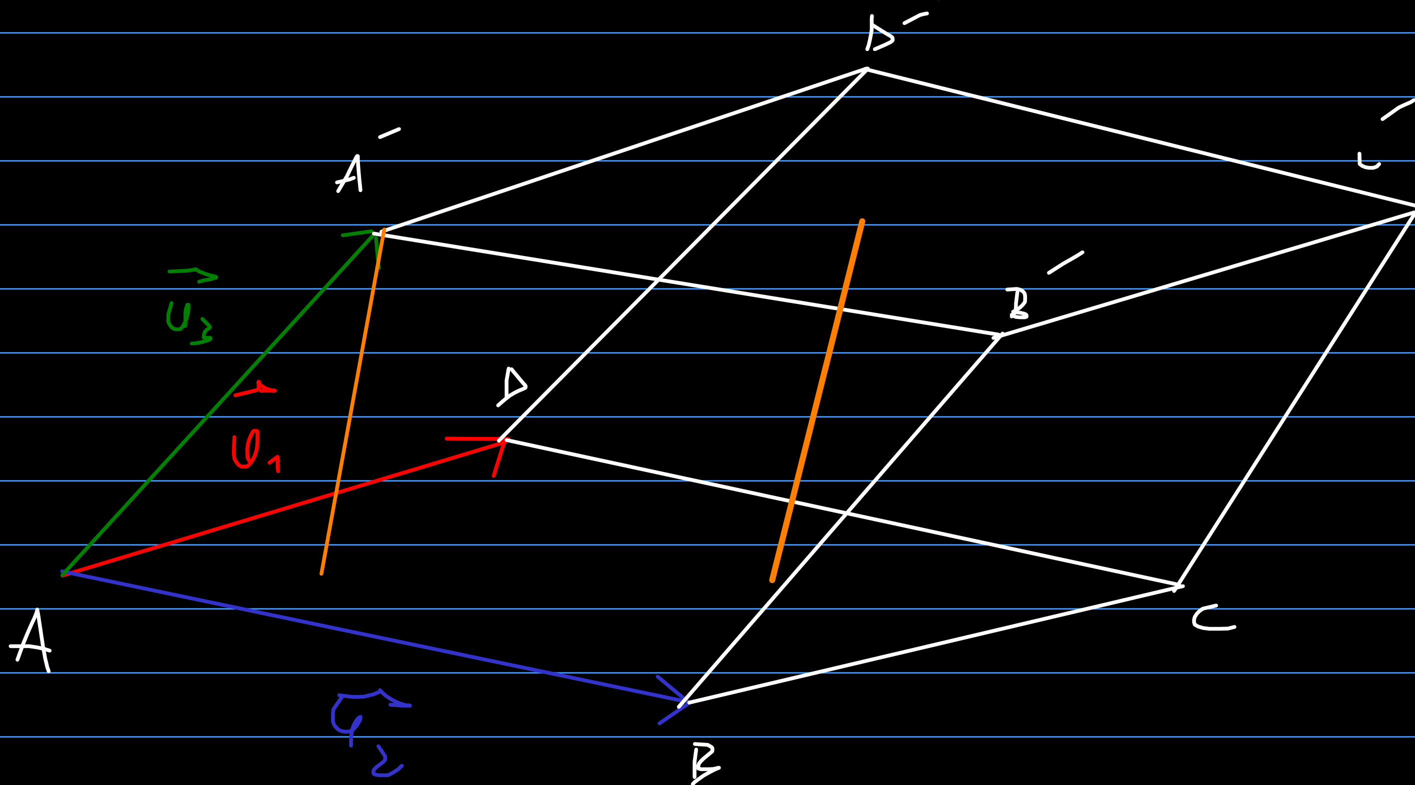
$$(\vec{u}_1, \vec{u}_2, \vec{u}_3) = (\vec{u}_2, \vec{u}_3, \vec{u}_1) =$$

$$= (\vec{u}_3, \vec{u}_1, \vec{u}_2) =$$

$$= -(\vec{u}_1, \vec{u}_3, \vec{u}_2) = -(\vec{u}_2, \vec{u}_1, \vec{u}_3) =$$

$$= -(\vec{u}_3, \vec{u}_2, \vec{u}_1)$$

$$|(\vec{u}_1, \vec{u}_2, \vec{u}_3)| = \text{volume of the parallelepiped built on } \vec{u}_1, \vec{u}_2, \vec{u}_3$$

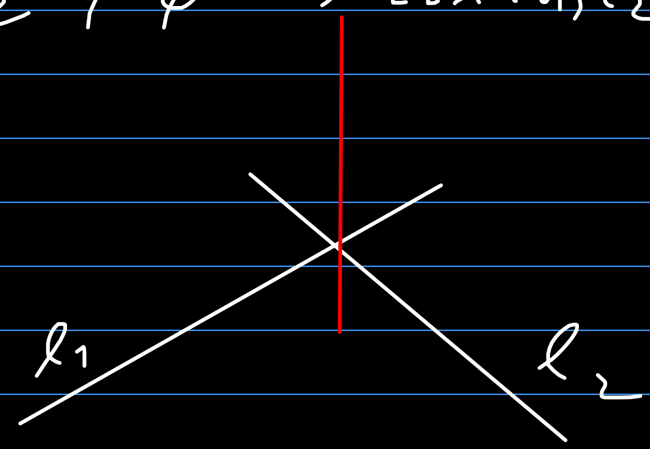


$$\text{dist}(A', (ABCD)) = \frac{|(\vec{u}_1, \vec{u}_2, \vec{u}_3)|}{\|\vec{u}_1 \times \vec{u}_2\|}$$

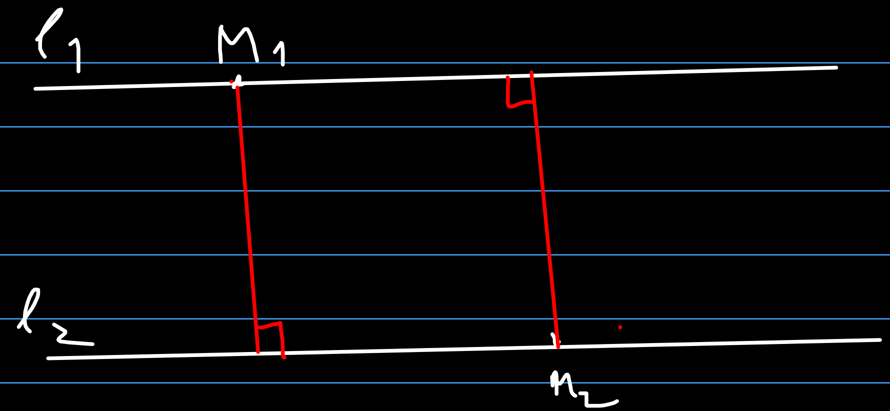
The distance between two lines (and the common perpendicular)

l_1, l_2 lines in space

- $l_1 \cap l_2 \neq \emptyset \Rightarrow dist(l_1, l_2) = 0$



- $l_1 \cap l_2 = \emptyset$, $l_1 \parallel l_2$



$$dist(l_1, l_2) = dist(M_1, l_2) = dist(M_2, l_1)$$

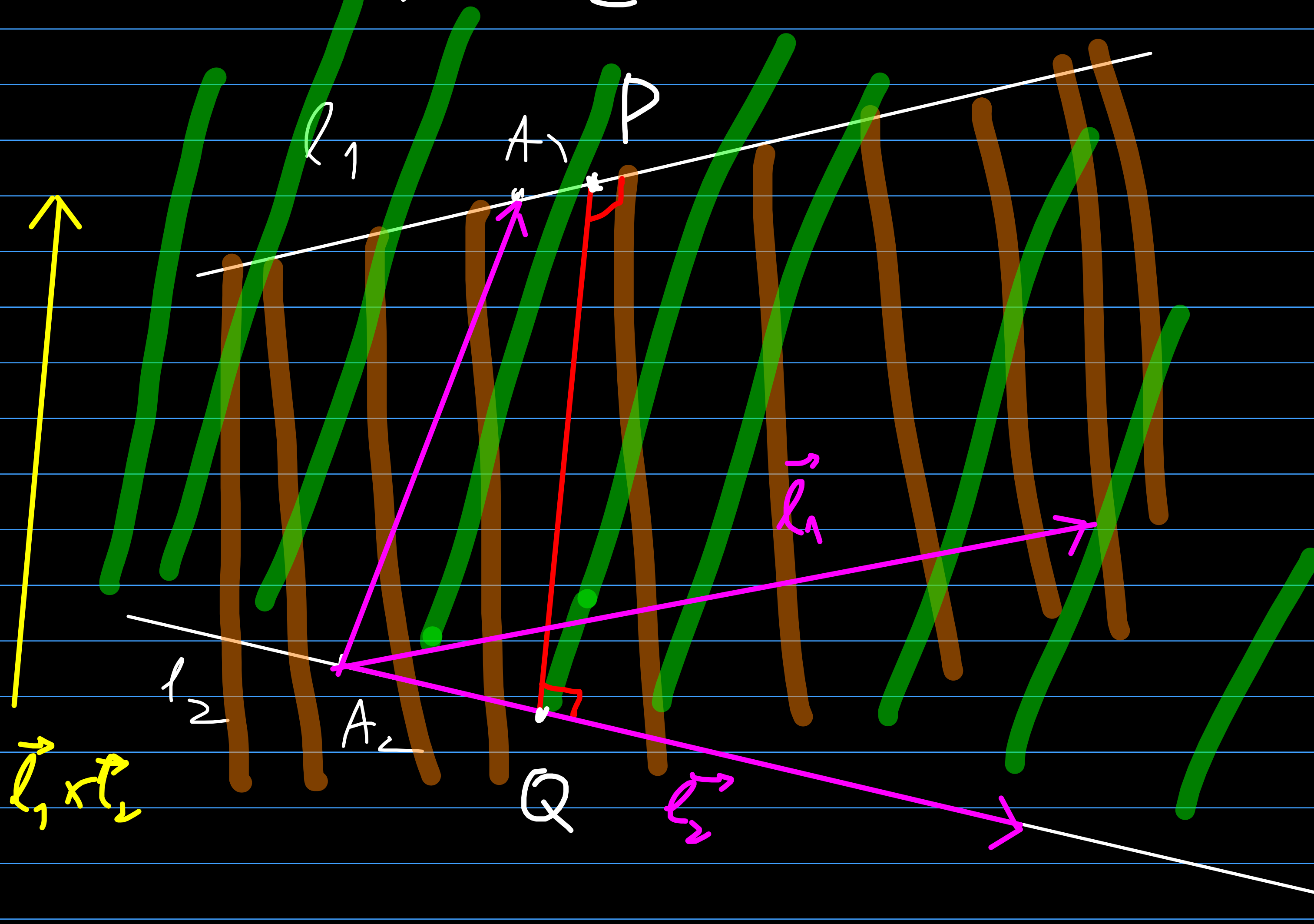
$\forall M_1 \in l_1$, $\forall M_2 \in l_2$

common perp. = perp. from $\forall M_1 \in l_1$
onto l_2

- $l_1 \cap l_2 = \emptyset$, $l_1 \not\parallel l_2$

$\rightarrow$ l_1 and l_2 are skew (noncoplanar)

$A_1 \in l_1$, $A_2 \in l_2$



Π_1 = plane that contains l_1 and is parallel to $\vec{l_1} \times \vec{l_2}$

Π_2 = plane that contains l_2 and is parallel to $\vec{l_1} \times \vec{l_2}$

the common perpendicular is

$$l = \Pi_1 \cap \Pi_2$$

$$\mathrm{dist}(l_1, l_2) = \frac{\mathrm{Vol}(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2})}{\mathrm{Area}(\vec{l_1}, \vec{l_2})} =$$

$$= \frac{|(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2})|}{\|\vec{l_1} \times \vec{l_2}\|}$$

7.8. Find the distance between the lines M_1M_2 and l, where

$M_1(-1,0,1)$, $M_2(-2,1,0)$

$$l: \begin{cases} x+y+z=1 \\ 2x-y-5z=0 \end{cases}$$

as well as the equations of the common perpendicular.

$$\text{dist}(l_1,l_2) = \frac{|(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2})|}{\|\vec{l_1}\times\vec{l_2}\|}$$

$$M_1M_2: \frac{x+1}{-1} = \frac{y}{1} = \frac{z-1}{-1}$$

$$\Rightarrow A_1 := M_1, \quad \vec{l_1}(-1,1,-1)$$

$$l: \begin{cases} x+y+z=1 \\ 2x-y-5z=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=1-y-z \\ 2-2y-2z-y-5z=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=1-y-z \\ -3y-7z+2=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=1-y-z \\ y=\frac{2}{3}-\frac{7}{3}z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=\frac{1}{3}+\frac{4}{3}z \\ y=\frac{2}{3}-\frac{7}{3}t \\ z=t \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = \frac{4}{3}t + \frac{1}{3} \\ y = -\frac{7}{3}t + \frac{2}{3} \\ z = t \end{cases}$$

$$A_2 \in l, \quad A_2\left(\frac{1}{3}, \frac{2}{3}, 0\right)$$

$$\vec{l_2} = \vec{l}\,(4, -7, 3)$$

$$\overrightarrow{A_1A_2} = \vec{r}_{A_2} - \vec{r}_{A_1} = \left(\frac{1}{3}, \frac{2}{3}, 0\right) -$$

$$-(-1, 0, 1) = \left(\frac{4}{3}, \frac{2}{3}, -1\right)$$

$$(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}) = \begin{vmatrix} \frac{4}{3} & \frac{2}{3} & -1 \\ -1 & 1 & -1 \\ 4 & -7 & 3 \end{vmatrix} =$$

$$= 4 - 7 - \frac{8}{3} + 4 + 2 - \frac{28}{3} \ldots$$

$$= -9$$

$$\vec{l_1} \times \vec{l_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & -1 \\ 4 & -7 & 3 \end{vmatrix} =$$

$$= -4\vec{i} - \vec{j} + 3\vec{k}$$

$$\Rightarrow \|\vec{l_1} \times \vec{l_2}\| = \sqrt{16+1+9} = \sqrt{26}$$

$$\Rightarrow dist(l_1, l_2) = \frac{9}{26}$$

π_1: plane that contains l_1 and is parallel to $\vec{l_1} \times \vec{l_2}$

$$\Pi_1: \begin{vmatrix} x+1 & y & z-1 \\ -1 & 1 & -1 \\ -4 & -1 & 3 \end{vmatrix} = 0$$

$$\Pi_1: 3x+3+4y+z-1+$$
$$+4z-4+3y-x-1=0$$

$$\Pi_1: 2x+7y+5z-3=0$$

Π_2: plane that contains l_2 and is parallel to $\vec{l_1} \times \vec{l_2}$

$$\Pi_2: \begin{vmatrix} x-\frac{1}{3} & y-\frac{2}{3} & z \\ 4 & -7 & 3 \\ -4 & -1 & 3 \end{vmatrix} = 0$$

$$\Pi_2: 18x+24y+z-22=0$$

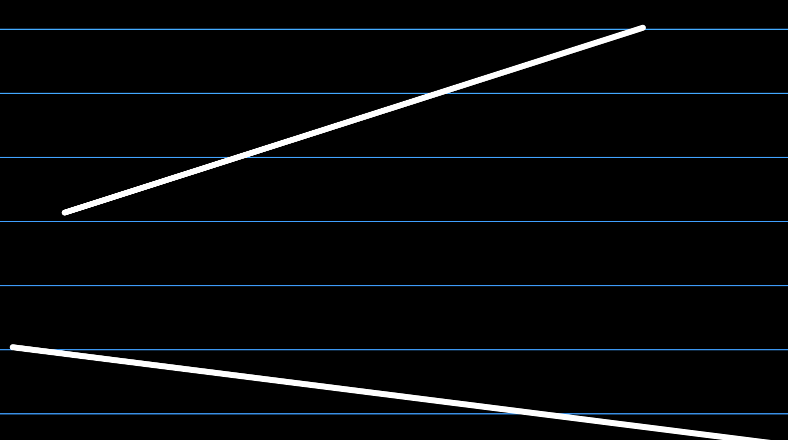
$$\Rightarrow \ell: \begin{cases} 2x + 7y + 5z - 3 = 0 \\ 18x + 24y + z - 22 = 0 \end{cases}$$

<u>Checking for coplanarity</u>

ℓ_1, ℓ_2 lines, $A_1 \in \ell_1$, $A_2 \in \ell_2$

ℓ_1, ℓ_2 coplanar $\Leftrightarrow$ $\overrightarrow{A_1A_2}, \vec{\ell_1}, \vec{\ell_2}$ are linearly dependent $\Leftrightarrow$ $(\overrightarrow{A_1A_2}, \vec{\ell_1}, \vec{\ell_2}) = 0 \Leftrightarrow$

$\Leftrightarrow$ the volume of the parallelepiped built on $\overrightarrow{A_1A_2}, \vec{\ell_1}, \vec{\ell_2}$ is zero



7.6. Find the value of the parameter λ for which the lines

$$l_1: \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1}$$

$$l_2: \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{\lambda}$$

are coplanar.

Show that they are not parallel and find their intersection point.

$A_1(1, -2, 0) \in l_1$

$A_2(-1, 3, 0) \in l_2$

$\overrightarrow{A_1A_2}(-2, 5, 0)$, $\vec{l_1}(3, -2, 1)$, $\vec{l_2}(4, 1, \lambda)$

l_1, l_2 coplanar $\Leftrightarrow (\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}) = 0$

$$(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}) = \begin{vmatrix} -2 & 5 & 0 \\ 3 & -2 & 1 \\ 4 & 1 & \lambda \end{vmatrix} =$$

$$= 4\lambda + 20 + 0 - 0 - 15\lambda + 2 =$$

$$= -11\lambda + 22$$

$\Rightarrow l_1, l_2$ coplanar $\Leftrightarrow 22 - 11\lambda = 0 \Rightarrow \lambda = 2$

$\vec{l_1}(3, -2, 1)$, $\vec{l_2}(4, 1, \lambda)$

Because $\frac{4}{3} \neq \frac{1}{-2} \Rightarrow l_1 \nparallel l_2$

$$l_1 \cap l_2: \begin{cases} \dfrac{x-1}{3} = \dfrac{y+2}{-2} = \dfrac{z}{1} \\[2ex] \dfrac{x+1}{4} = \dfrac{y-3}{1} = \dfrac{z}{\lambda} \end{cases}$$

$$\forall\, \vec a, \vec b, \vec c \in \mathcal{V}$$

7.1. (a) $|(\vec a, \vec b, \vec c)| \le \|\vec a\| \cdot \|\vec b\| \cdot \|\vec c\|$

(b) $(\vec a + \vec b, \vec b + \vec c, \vec c + \vec a) = 2 \cdot (\vec a, \vec b, \vec c)$

(a) $$(\vec a, \vec b, \vec c) = \|\vec a\| \cdot \|\vec b \times \vec c\| \cdot \cos(\vec a, \vec b \times \vec c) =$$

$$= \|\vec a\| \cdot \|\vec b\| \cdot \|\vec c\| \cdot \sin(\vec b, \vec c)$$

$$\Rightarrow |(\vec a, \vec b, \vec c)| = \|\vec a\| \cdot \|\vec b\| \cdot \|\vec c\| \cdot \underbrace{|\cos(\widehat{\vec a, \vec b \times \vec c})|}_{\le 1} \cdot$$

$$\cdot \underbrace{|\sin(\vec b, \vec c)|}_{\le 1} \le \|\vec a\| \cdot \|\vec b\| \cdot \|\vec c\|$$

(b) $$(\vec a + \vec b, \vec b + \vec c, \vec c + \vec a) =$$

$$= (\vec a + \vec b) \cdot [(\vec b + \vec c) \times (\vec c + \vec a)] =$$

$$= (\vec a + \vec b) \cdot (\vec b \times \vec c + \vec b \times \vec a + \underbrace{\vec c \times \vec c}_{= \vec 0} + \vec c \times \vec a) =$$

$$= \vec a \cdot (\vec b \times \vec c) + \overbrace{\vec b \cdot (\vec b \times \vec c)}^{0} + \underbrace{\vec a \cdot (\vec b \times \vec a)}_{0} + \overbrace{\vec b \cdot (\vec b \times \vec a)}^{0} +$$

$$+ \underbrace{\vec{a}\cdot(\vec{c}\times\vec{a})}_{0} + \vec{b}\cdot(\vec{c}\times\vec{a}) =$$

$$= \vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{b}\cdot(\vec{c}\times\vec{a}) =$$

$$= 2\cdot(\vec{a},\vec{b},\vec{c}).$$