Seminar W11 - 914

Generated surfaces

- → ruled surfaces
  - → conical (CL)
  - → cylindrical (CY)
  - → conoid (CD)
- → revolution surfaces (R)

Task: Find an implicit equation for a generated surface

Algorithm: → if ruled surface:

↳ generated by a moving line (generatrix) subject to condition C and this moving line should always intersect a curve called a director curve

Example 112: Determine the eqn. of the conical surface with vertex $V(1,1,1)$ and director curve

$$\mathcal{C}: \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases}$$

$\rightarrow$ Condition on the generatrix:

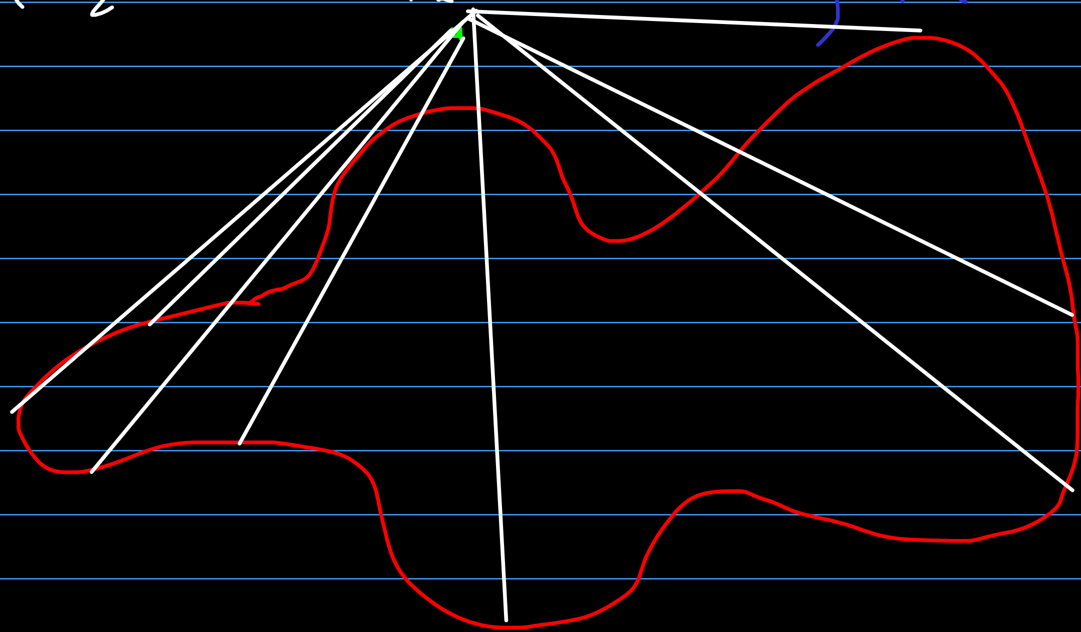
$$d_{\lambda,\mu} \ni V$$

$$d_{\lambda,\mu}: \begin{cases} x = 1 + at \\ y = 1 + bt \\ z = 1 + ct \end{cases}$$

$$\boxed{d_{\lambda,\mu}: \begin{cases} x - 1 = \underset{\frac{b}{a}\,/\!/}{\lambda} \cdot (y-1) \\ z - 1 = \underset{\hookrightarrow \frac{c}{a}}{\mu} (x-1) \end{cases}}$$

this was the 1st step: writing the candidates for the honorable position of generatrix

2nd step: find the generatrices that satisfy the 2nd condition: intersecting the curve $\mathscr{C}$



We check the intersection condition by solving the system:

$$(S): \begin{cases} d_{\lambda,\mu}: \begin{cases} x-1 = \lambda(y-1) \\ z-1 = \mu\cdot(x-1) \end{cases} \\ \mathscr{C}: \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \end{cases}$$

$$(S)': \begin{cases} x-1 = \lambda\cdot(y-1) \\ z-1 = \mu\cdot(x-1) \\ (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \iff$$

$$\iff \begin{cases} x-1 = \lambda\cdot(y-1) \\ -1 = \mu\cdot(x-1) \\ (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \iff$$

$$\iff \begin{cases} x = 1 - \dfrac{1}{\mu} \\ y = \dfrac{1}{\lambda}\cdot(x-1) + 1 \\ z = 0 \\ (x^2+y^2)^2 - xy = 0 \end{cases} \iff \begin{cases} x = 1 - \dfrac{1}{\mu} \\ y = 1 - \dfrac{1}{\lambda\mu} \\ z = 0 \\ (x^2+y^2)^2 - xy = 0 \end{cases}$$

$\Rightarrow$ compatibility condition

$$\left(\left(1-\frac{1}{\mu}\right)^2+\left(1-\frac{1}{\lambda\mu}\right)^2\right)^2-\left(1-\frac{1}{\mu}\right)\left(1-\frac{1}{\lambda\mu}\right)=0$$

<u>Step 3</u> : Replace λ and μ by their <u>original</u> formulas involving x, y, z

$\Rightarrow$ The final equation is:

(with) $\lambda=\dfrac{x-1}{y-1}$, $\mu=\dfrac{z-1}{x-1}$)

$$1-\frac{1}{\mu}=1-\frac{x-1}{z-1}=\frac{z-x}{z-1}$$

$$1-\frac{1}{\lambda\mu}=1-\frac{1}{\frac{z-1}{y-1}}=\frac{z-y}{z-1}$$

$$\left(\left(1-\frac{1}{\mu}\right)^2+\left(1-\frac{1}{\lambda\mu}\right)^2\right)^2-\left(1-\frac{1}{\mu}\right)\left(1-\frac{1}{\lambda\mu}\right)=0$$

$$\left(\left(\frac{z-x}{z-1}\right)^2+\left(\frac{z-y}{z-1}\right)^2\right)^2-\frac{z-x}{z-1}\cdot\frac{z-y}{z-1}=0$$

11.1. Find the equation of the cylindrical surface whose director curve is the planar curve

$$\mathcal{C}:\begin{cases} y^2+z^2=x \\ x=2z \end{cases}$$

and the generatrix is perpendicular to the plane of the director curve.

<u>Cylindrical surface</u>: the main difference is that $d_{\lambda,\mu}$ need not contain a certain point, but rather it needs to be parallel with a direction.

In our case, $d_{\lambda,\mu} \perp \Pi$, where Π is the plane of $\mathcal{C}$

$$\Pi : x = 2z$$

$$\Rightarrow d_{\lambda,\mu} : \begin{cases} \dfrac{x - x_0}{1} = \dfrac{z - z_0}{-2} \\ y = y_0 \end{cases}$$

$$\Rightarrow d_{\lambda,\mu} : \begin{cases} -2x - z = \underbrace{-2x_0 - z_0}_{=\lambda} \\ y = y_0 = \mu \end{cases}$$

$$(5) : \begin{cases} -2x - z = \lambda \\ y = \mu \\ y^2 + z^2 = x \\ x = 2z \end{cases} \Leftrightarrow \begin{cases} -5z = \lambda \\ y = \mu \\ y^2 + z^2 = 2z \\ x = 2z \end{cases} \Leftrightarrow$$

c)
$$\begin{cases} z = -\frac{\lambda}{5} \\ y = \mu \\ y^2 + z^2 = 2z \\ x = 2z \end{cases}$$

$\Rightarrow$ compatibility condition:

$$\mu^2 + \left(-\frac{\lambda}{5}\right)^2 + \frac{2\lambda}{5} = 0$$

From the beginning $\quad \lambda = -2x - z$

$\mu = y$

$\Rightarrow$ the eqn is:

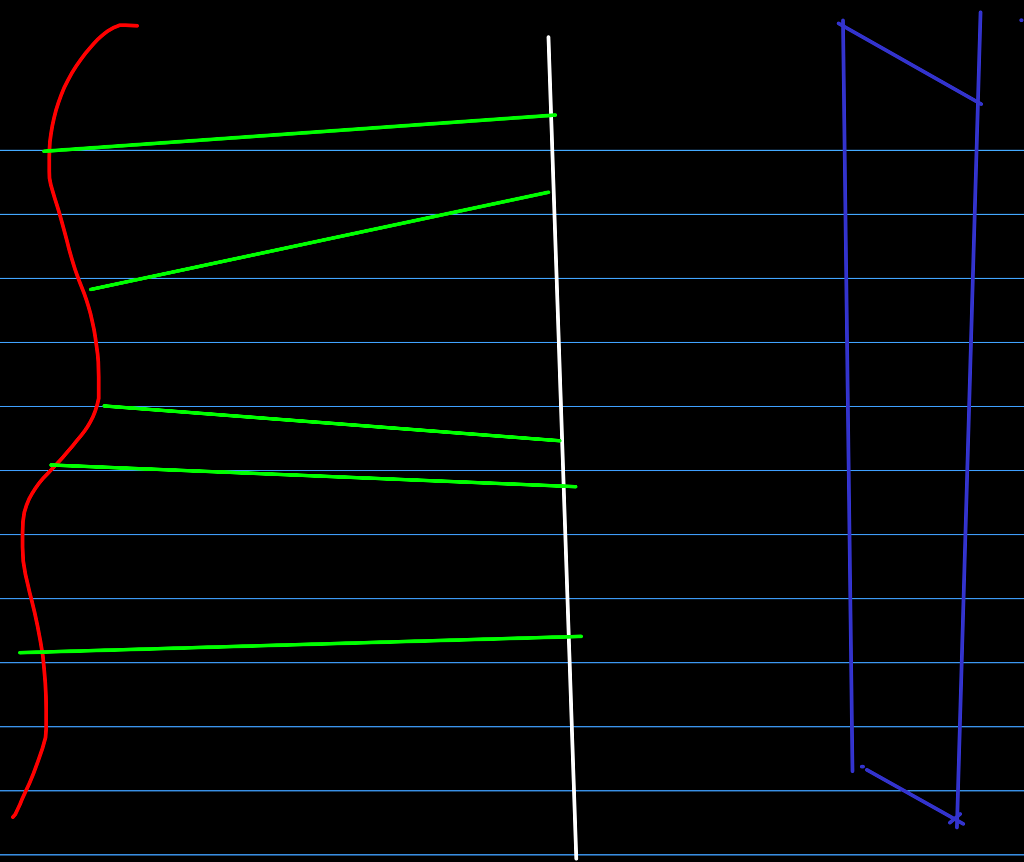
$$y^2 + \left(\frac{2x+z}{5}\right)^2 - \frac{4x+2z}{5} = 0$$

Surface	Condition
Conical surfaces	$V \in d_{\lambda,\mu}$ (vertex)
Cylindrical surfaces	$d_{\lambda,\mu} \parallel \ell$
Conoidal surfaces	$\begin{cases} d_{\lambda,\mu} \parallel \pi \\ d_{\lambda,\mu} \cap \ell \neq \emptyset \end{cases}$ → $d_{\lambda,\mu}$ belongs to the pencil of planes of ℓ

Example 11.3 Conoidal surface:

- generatrices are parallel to xOy, intersect the line Oz
- director curve:

$$\mathcal{C} : \begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}$$



Step 1 $\quad d_{\lambda,\mu} \parallel xOy \;,\; d_{\lambda,\mu} \cap Oz \neq \emptyset$

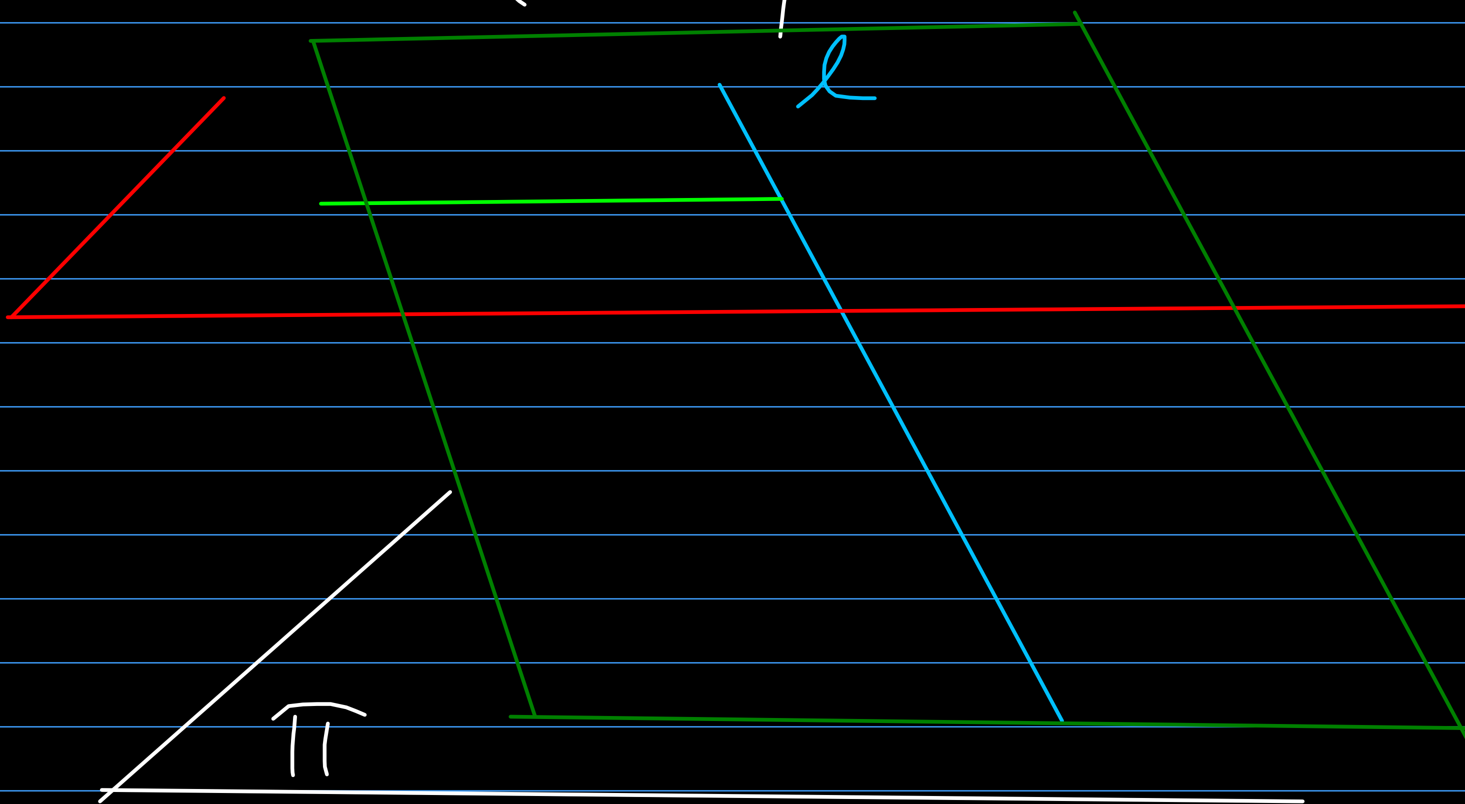
$xOy: \quad z = 0$

$Oz: \quad \begin{cases} x = 0 \\ y = 0 \end{cases}$

$d_{\lambda,\mu}: \begin{cases} \text{Pencil of planes for } Oz : \alpha x + \beta y = 0 \\ z = \mu \end{cases}$

$$-\frac{B}{\alpha}$$

$$d_{\lambda\mu} : \begin{cases} x = \lambda y \\ z = \mu \end{cases}$$



$$\Pi : Ax + By + Cz + D = 0$$

$$\ell : \begin{cases} \Pi_1 : A_1x + B_1y + C_1z + D_1 = 0 \\ \Pi_2 : A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

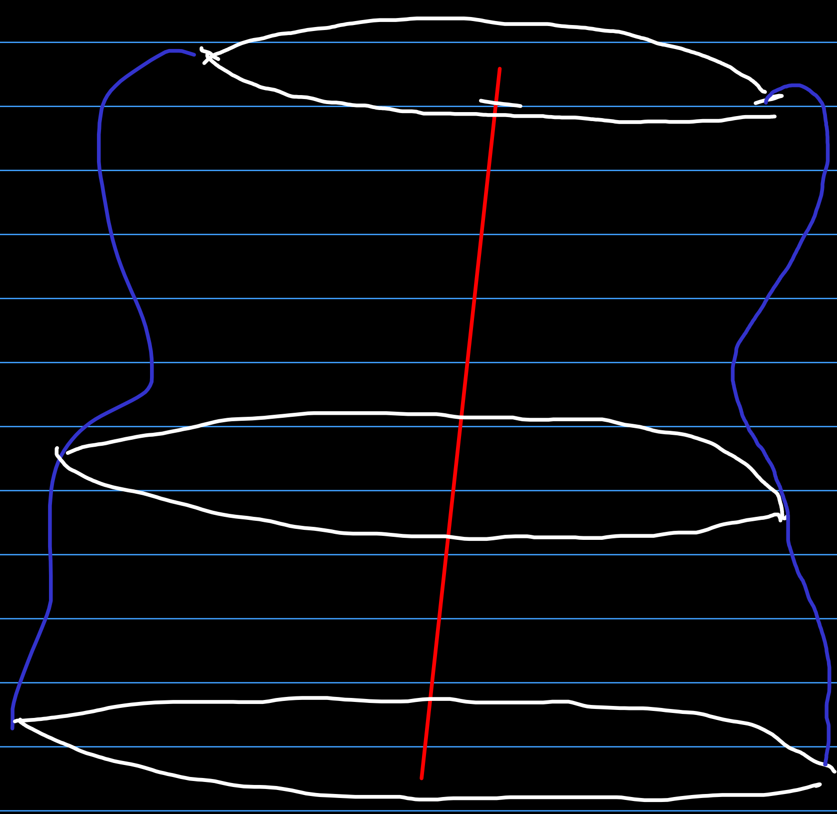
$$d_{\lambda,\mu} : \begin{cases} Ax + By + Cz + D = \lambda \\ \Pi_1 = \mu \, \Pi_2 \end{cases}$$

$$\left(x(A_1 - \mu A_2) + y(B_1 - \mu B_2) + z \cdot (C_1 - \mu C_2) + D_1 - \mu D_2 = 0 \right)$$

Revolution surfaces

$\mathcal{C}$ curve, ℓ line (axis of rotation)

We rotate $\mathcal{C}$ around ℓ

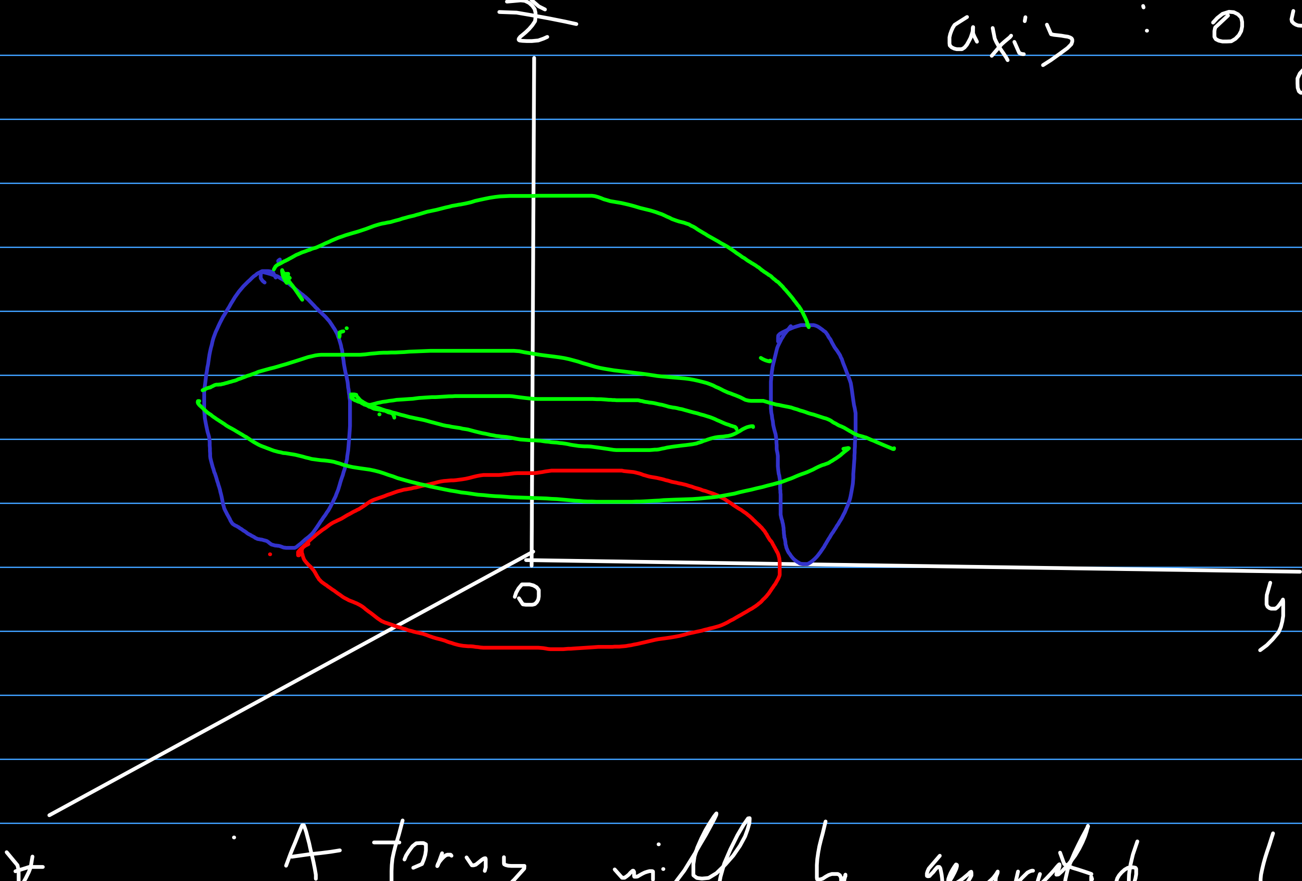


The difference from ruled surfaces is that we don't have generatrices (generating lines, $d_{\lambda,\mu}$) but generating circles

11.5. $\mathscr{C}$ circle in (xOz)

$$\mathscr{C}: \begin{cases} x^2+y^2+z^2=1 \\ z=0 \end{cases}$$

axis : Oy



A torus will be generated by circles centered on Oz

$$\mathscr{C}_{\lambda,\mu}: \begin{cases} (x-x_0)^2+(y-y_0)^2+z^2=\lambda \\ z=\mu \end{cases}$$

x_0, y_0 fixed on the axis