Seminar W 12 – 9.1.7

Affine transformations (plane)

$y = mx + n$ affine function

$f : \mathbb{R} \to \mathbb{R}$

$x \mapsto mx + n$

$f(x_1 + x_2) \neq f(x_1) + f(x_2)$

$y = mx$ linear function

$\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ affine transformation if:

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = M \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \vec{v}_0 =$$

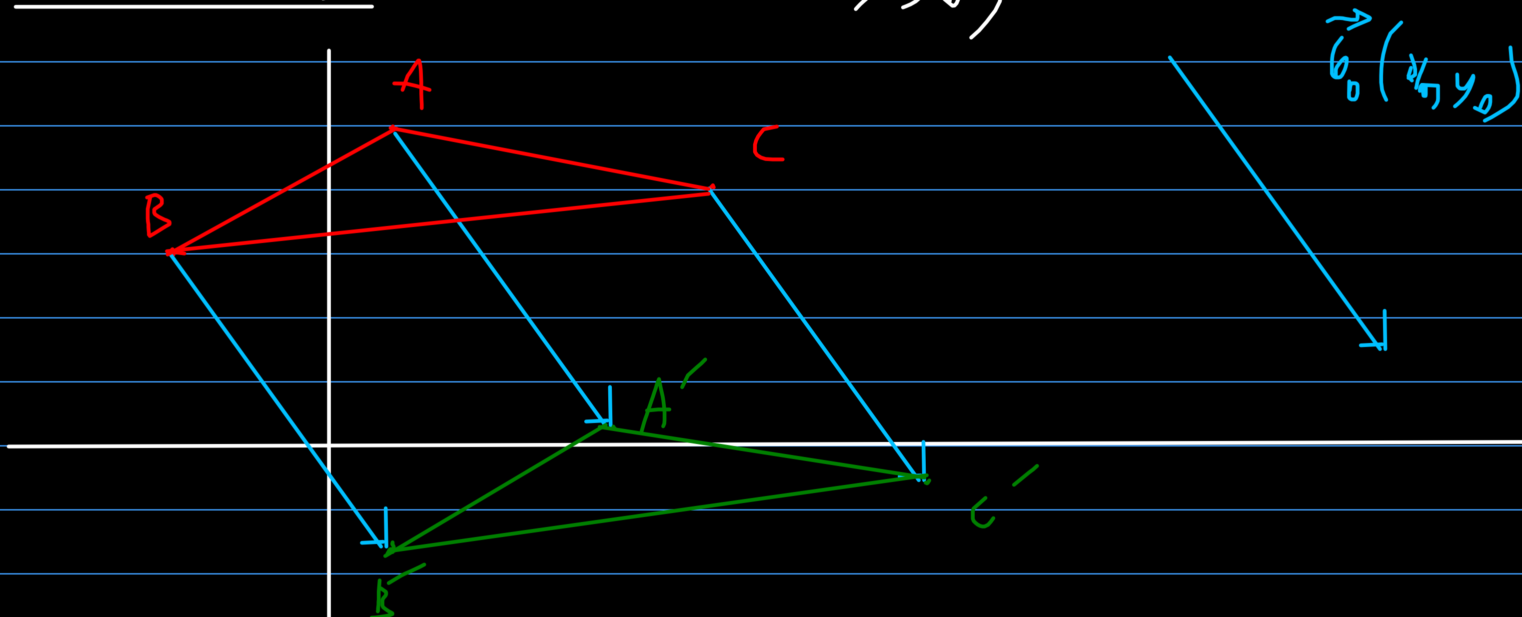
$M \in M_{2\times 2}(\mathbb{R})$, $\quad \vec{v}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x + a_{12}y + x_0 \\ a_{21}x + a_{22}y + y_0 \end{pmatrix}$$

→ they preserve lines and parallelism.
(but not distances and angles)

<u>Translations</u>: $T(x_0, y_0)$



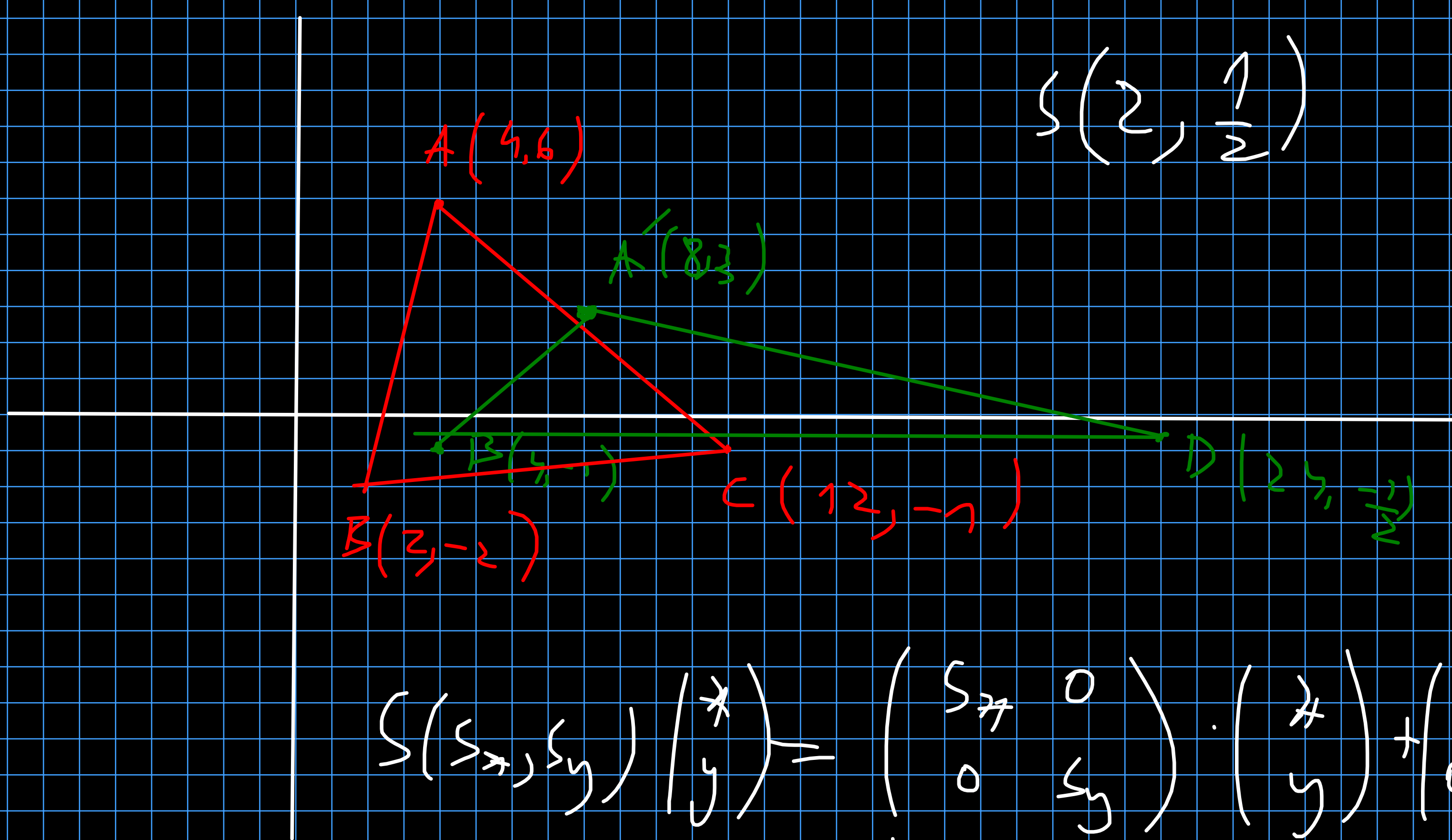
$$T(x_0, y_0)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x + x_0 \\ y + y_0 \end{pmatrix}$$

If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ affine transformation, how do we check if it is a translation. If so, by which vector?

- Pick a point A, let $A' := \varphi(A)$
- If φ is a translation, then $\varphi = T(\overrightarrow{AA'})$
- Check for all the other points B with $\varphi(B) = B'$ that $\overrightarrow{BB'} = \overrightarrow{AA'}$

<u>Scalings</u>: $S(s_x, s_y)$

$S(2, \frac{1}{2})$



$$S(s_x, s_y)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} s_x \cdot x \\ s_y \cdot y \end{pmatrix}$$

if $s_x = s_y \Rightarrow S(s_x, s_y)$ "homothety"

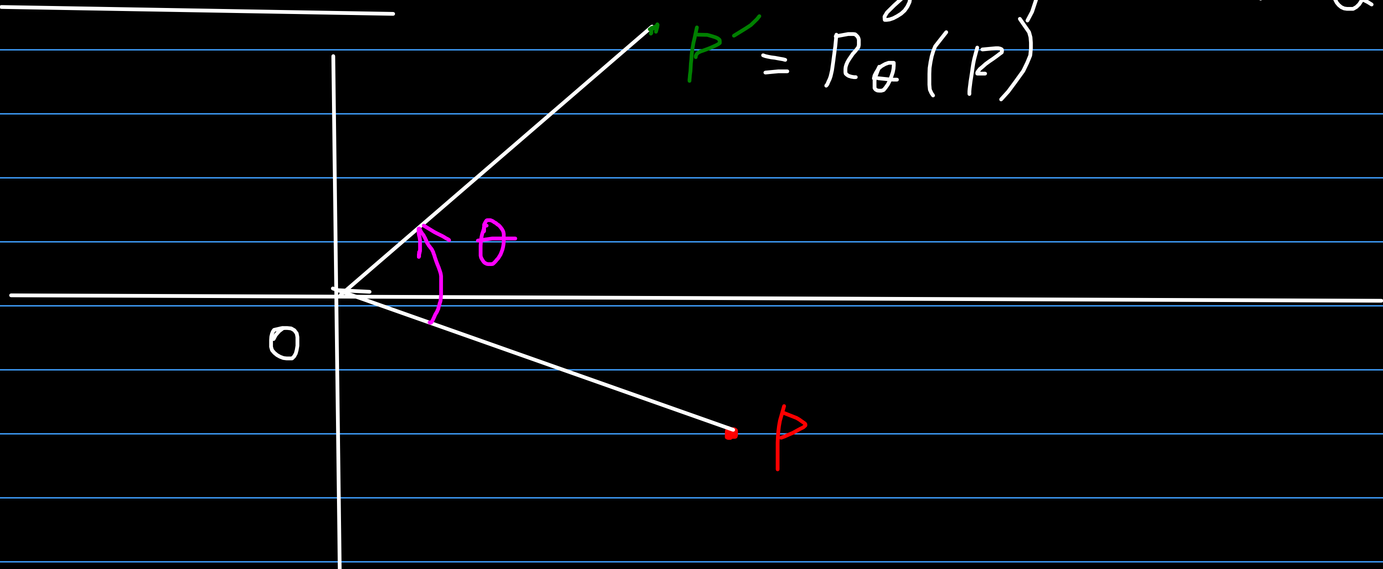
If $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ affine transformation, how do we check if it is a scaling. If so, by which factors?

- Pick a point A, $A' := \varphi(A)$
- If φ is indeed a scaling, then

$$s_x = \frac{x_{A'}}{x_A}, \quad s_y = \frac{y_{A'}}{y_A}$$

- Check this against all the other points

Rotations: (around the origin) $\quad R_\theta$



$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ affine transformation, how do we check if it is a rotation. If so, by which angle? (Bonus question: Around which point)

in general $P_0 = 0$

- Check if $Fix(\varphi) = \{P_0\}$

$\left[f : A \to B, \ Fix(f) = \{x \in A \mid f(x) = x\} \right]$

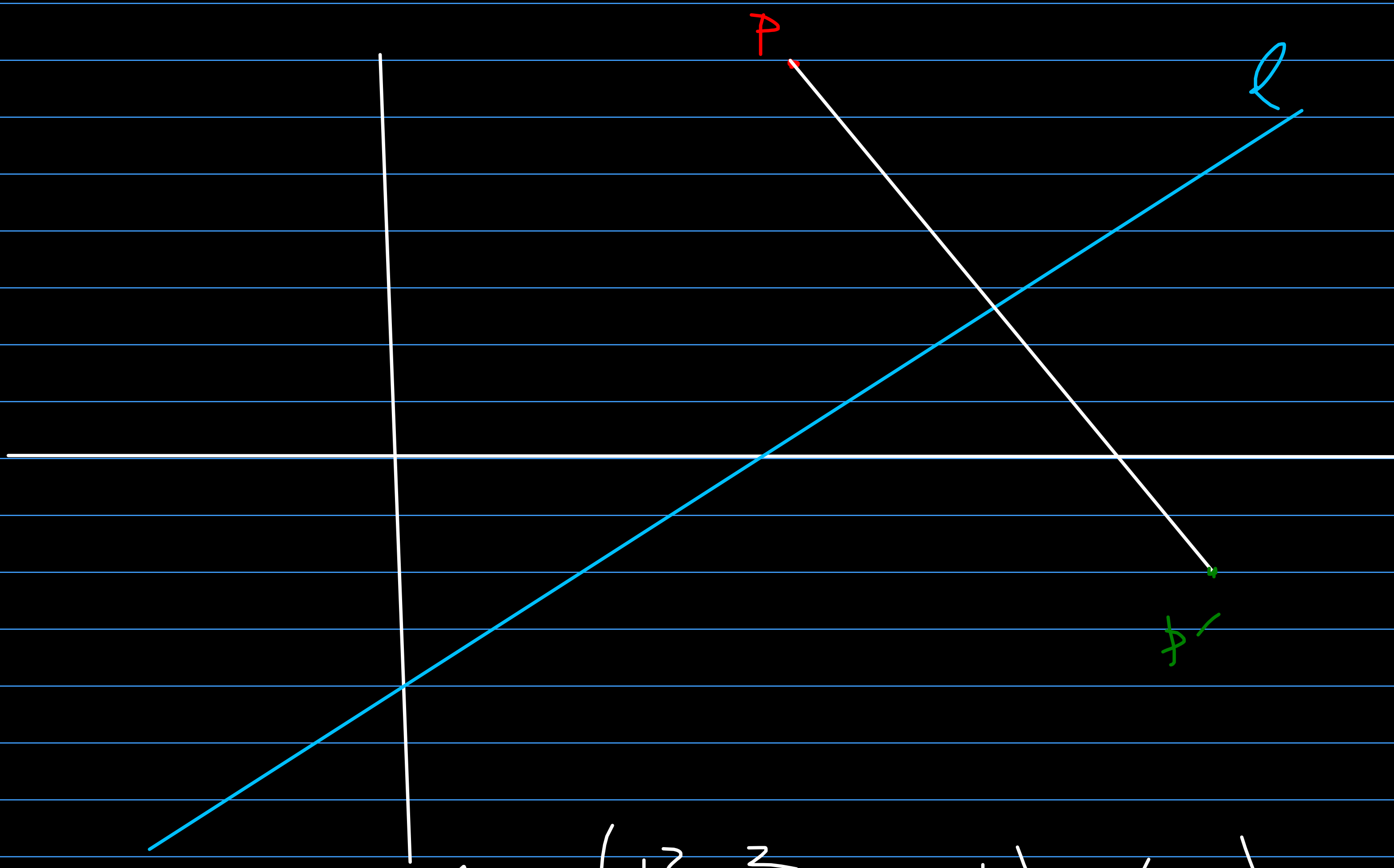
- If so, then check if $\forall P \in \mathbb{R}^2$, $P' = \varphi(P)$ we have $P_0 P = P_0 P'$

- If so, then check if $\forall P \in \mathbb{R}^2$, $P' = \varphi(P)$ we have $m(\widehat{P P_0 P'})$ is the same

- If so, then this angle is θ.

Reflection (orthogonal, with respect to a line ℓ):

r_ℓ , $\ell : ax + by + c = 0$



$$r_\ell\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{a^2+b^2}\cdot\begin{pmatrix} b^2-a^2 & -2ab\\ -2ab & a^2-b^2\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} +$$

$$+ \frac{-2c}{a^2+b^2}\cdot\begin{pmatrix}a\\b\end{pmatrix}$$

If $c=0$ (i.e. if $\ell \ni 0$), then

r_l is a linear transformation.

$r_x := r_{Ox} : (x,y) \longmapsto (x,-y)$

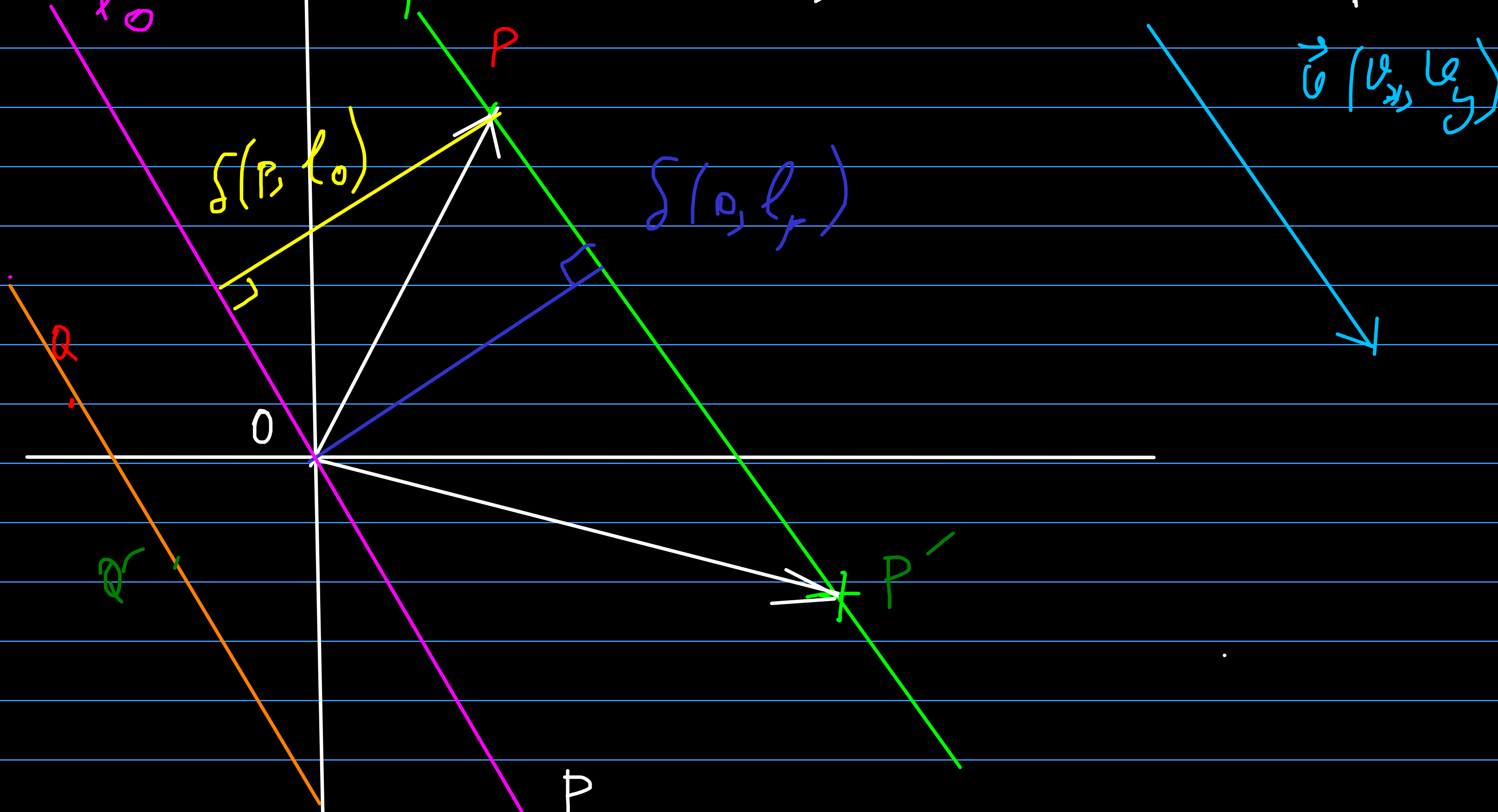
$r_y := r_{Oy} : (x,y) \longmapsto (-x,y)$

If $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ affine transformation, how do we check if it is a reflection? If so, with respect to which axis?

- Verify if $Fix(\varphi) = l$, l line
- If so, then φ is a reflection or a shear.
- Check if for any A, $A' = \varphi(A)$ l is the perpendicular bisector of the segment AA'.

Shears: $sh(\vec{u}, r)$

$\vec{u} \in \mathbb{R}^2, \ \|\vec{u}\| = 1, \ r \in \mathbb{R}$



$$sh(\vec{u}, r)\overset{P}{\overset{\shortparallel}{\begin{pmatrix} x \\ y \end{pmatrix}}} =$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} + r \cdot \delta(O, l_P) \cdot \vec{u}$$

$l: ax + by + c = 0$

$P(x_P, y_P) \Rightarrow \delta(P, l) = \dfrac{ax_P + by_P + c}{\sqrt{a^2 + b^2}}$

$$sh(\vec{u}, r)\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} - r \cdot \delta(P, \ell_0) \cdot \vec{u}$$

$$sh(\vec{u}, r)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - r u_x u_y & r u_x^2 \\ -r u_y^2 & 1 + r u_x u_y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

12.1. Find the image of ΔABC through the reflection with respect to the line

$$d: \; x - y = 2$$

$A(-1, 2)$, $B(-2, -1)$, $C(3, 3)$

$$d: \; x - y - 2 = 0$$

$$a = 1, \; b = -1, \; c = -2$$

$$r_d\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \cdot \begin{pmatrix} b^2 - a^2 & -2ab \\ -2ab & a^2 - b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{-2c}{a^2+b^2} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$r_d\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{2}\cdot\begin{pmatrix}0 & 2\\2 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + 2\cdot\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}2\\-2\end{pmatrix} =$$

$$= \begin{pmatrix}y+2\\x-2\end{pmatrix}$$

$$r_d(A) = r_d(-1,2) = (4,-3)$$

$$r_d(B) = r_d(-2,-1) =$$

$$= \left(\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\end{pmatrix} + \begin{pmatrix}2\\-2\end{pmatrix}\right)^t =$$

$$= \left(\begin{pmatrix}-1\\-2\end{pmatrix} + \begin{pmatrix}2\\-2\end{pmatrix}\right)^t = \begin{pmatrix}1\\-4\end{pmatrix}^t = (1,-4)$$

$$r_d(C) = r_d(3,3) = (5,1)$$

12.2. Find the image of ΔABC through the clockwise rotation of angle 30^0 when $A(6,4)$, $B(6,2)$, $C(10,6)$

$$[R_{-\frac{\pi}{6}}] = \begin{pmatrix} \cos(-\frac{\pi}{6}) & -\sin(-\frac{\pi}{6}) \\ \sin(-\frac{\pi}{6}) & \cos(-\frac{\pi}{6}) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$R_{-\frac{\pi}{6}}(6,4) = (3\sqrt{3}+2, -3+2\sqrt{3})$$

$$R_{-\frac{\pi}{6}}(B) = R_{-\frac{\pi}{6}}(6,2) = (3\sqrt{3}+1, -3+\sqrt{3})$$

$$R_{-\frac{\pi}{6}}(C) = R_{-\frac{\pi}{6}}(10,6) = (5\sqrt{3}+3, -5+3\sqrt{3})$$

12.3. ABCD quadrilateral

$A(1,1)$, $B(3,1)$, $C(2,2)$, $D(\frac{3}{2},3)$

Find the images of this quadrilateral through the transformations:

(a) $T(1,2)$, r_x, $R_{-\frac{\pi}{2}}$

(b) $S(2,\frac{5}{2})$, r_y, $R_{\frac{\pi}{2}}$

(c) $Sh((\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}), \frac{3}{2})$

(a) $T(1,2)(A) = T(1,2)(1,1) = (2,3)$

$T(1,2)(B) = T(1,2)(3,1) = (4,3)$

$r_x(C) = r_x(2,2) = (2,-2)$

$$[R_{-\frac{\pi}{2}}] = \begin{pmatrix} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

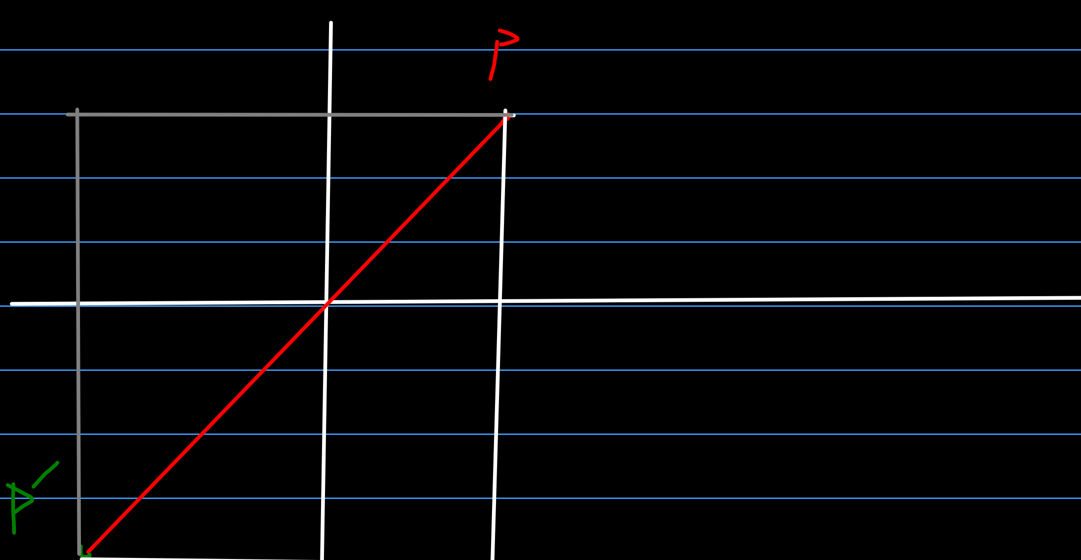
$$R_{-\frac{\pi}{2}}(D) = R_{-\frac{\pi}{2}}\left(\frac{3}{2}, 3\right) = \left(3, -\frac{3}{2}\right)$$

$$R_{-\frac{\pi}{2}}(A) = R_{-\frac{\pi}{2}}(1,1) = (1,-1)$$

(5) $$S\left(2, \frac{5}{2}\right)(B) = S\left(2, \frac{5}{2}\right)(3,1) = \left(6, \frac{5}{2}\right)$$

$$S\left(2, \frac{5}{2}\right)(C) = S\left(2, \frac{5}{2}\right)(2,2) = (4,5)$$

$$r_y(A) = r_y(1,1) = (-1,1)$$



$$t\left(R_{\frac{\pi}{2}}\right) = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$R_{\frac{\pi}{2}}(c) = R_{\frac{\pi}{2}}(2,2) = (-2,2)$$

$$[sh(\vec{u}, r)] = \begin{pmatrix} 1 - r u_x u_y & r u_x^2 \\ -r u_y^2 & 1 + r u_x u_y \end{pmatrix}$$

$$\Rightarrow \left[sh\left(\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \frac{3}{2}\right)\right] =$$

$$= \begin{pmatrix} 1 - \frac{3}{2} \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} & \frac{3}{2} \cdot \frac{4}{5} \\ -\frac{3}{2} \cdot \frac{1}{5} & 1 + \frac{3}{2} \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{5} & \frac{6}{5} \\ -\frac{3}{10} & \frac{8}{5} \end{pmatrix}$$

$$sh\left(\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \frac{3}{2}\right)(A) = \left(\frac{8}{5}, \frac{13}{10}\right)$$

$$sh\left(\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \frac{3}{2}\right)(B) = \left(\frac{12}{5}, \frac{7}{10}\right)$$