Seminar W1 - 913

02 December 2020 12:14

$cc = \max(M, S + \frac{M}{2})$

S = points from the seminar

$FG = \max(\frac{4}{10}cc + \frac{6}{10}E, E)$

Min. conditions:

$cc \geq 4.5$, $E \geq 4.5$

S = exercises (2, 3)

$S \leq 12$

$cc \leq 12$

Analytic Geometry

→ Cartesian Geometry

→ Descartes

Let E^2 (or E^3) be the intuitive Euclidean plane (space)

A reference system for E^2/E^3 is $\mathcal{R} = (O; b)$

O → origin; b → basis of $\vec{E}$

Say we take $E^2 = \mathbb{R}^2$

$b = (v_1, v_2)$

(v_1, v_2) basis $\Leftrightarrow$ v_1, v_2 lin. indep. $\Leftrightarrow$

$\Leftrightarrow \nexists \alpha \in \mathbb{R} : v_2 = \alpha v_1$

$\Leftrightarrow$ v_1 and v_2 are not parallel

Once we fix a reference system (i.e. origin + axis vectors) we can talk about coordinates of points

Let P be a point in E^2

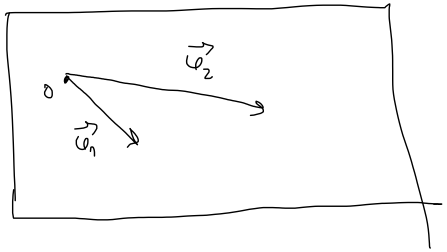
$b = (v_1, v_2)$

$[P]_{\mathcal{R}} = [\underbrace{\vec{OP}}_{=\vec{r}_P}]_b = \begin{pmatrix} x \\ y \end{pmatrix}$

$\vec{r}_P$ → the position vector of P w.r.t. to $\mathcal{R}$

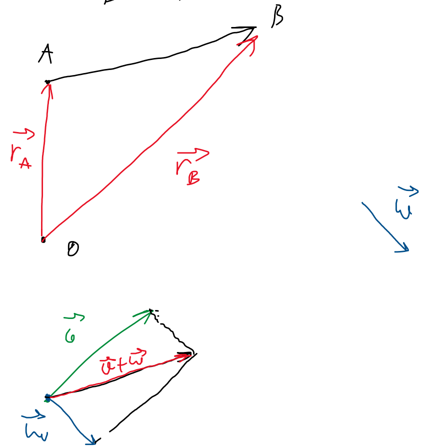
$\Rightarrow P(x, y)$

$\vec{OP} = x \cdot \vec{v_1} + y \cdot \vec{v_2}$

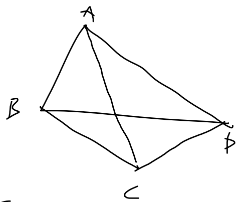


Let $A, B \in E^2$, $\mathcal{R}$ reference system

$\vec{AB} = \vec{r_B} - \vec{r_A}$



1.1. Consider a tetrahedron ABCD



Find the following sums of vectors:

(a) $\vec{AB} + \vec{BC} + \vec{CD}$

(b) $\vec{AD} + \vec{CB} + \vec{DC}$

(c) $\vec{AB} + \vec{BC} + \vec{DA} + \vec{CD}$

(a) $\left.\begin{array}{l} \vec{AB} + \vec{BC} = \vec{AC} \\ \vec{AC} + \vec{CD} = \vec{AD} \end{array}\right| \Rightarrow \vec{AB} + \vec{BC} + \vec{CD} = \vec{AD}$

(b) $\vec{AD} + \vec{CB} + \vec{DC} =$

$\vec{AD} + \vec{DC} = \vec{AC}$

$\vec{AC} + \vec{CB} = \vec{AB}$

$\Rightarrow \vec{AD} + \vec{CB} + \vec{DC} = \vec{AB}$

(c) $\vec{AB} + \vec{BC} + \vec{DA} + \vec{CD}$

$\left.\begin{array}{l} \vec{AB} + \vec{BC} = \vec{AC} \\ \vec{AC} + \vec{DA} = \vec{DC} \\ \vec{DC} + \vec{CD} = \vec{0} \end{array}\right\} \Rightarrow \vec{AB} + \vec{BC} + \vec{DA} + \vec{CD} = \vec{0}$

A, B points, M midpoint of [AB]

Let O be a point.

$\vec{OM} = \frac{\vec{OA} + \vec{OB}}{2}$

If we fix a reference system.

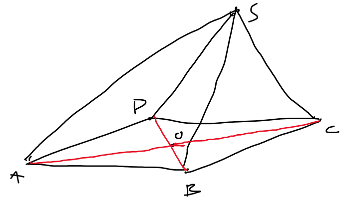
$\vec{r_M} = \frac{\vec{r_A} + \vec{r_B}}{2}$

$\frac{AM}{MB} = k \Rightarrow \vec{r_M} = \frac{\vec{r_A} + k\vec{r_B}}{1+k}$

1.3. Consider a pyramid with the vertex in S and the basis a parallelogram ABCD, whose diagonals are concurrent in O.

Show the equality:

$\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4\vec{SO}$



$\vec{SO} = \vec{SA} + \vec{AO} = \vec{SB} + \vec{BO} = \vec{SC} + \vec{CO} =$

$= \vec{SD} + \vec{DO}$

$\Rightarrow 4\vec{SO} = \vec{SA} + \vec{AO} + \vec{SB} + \vec{BO} + \vec{SC} + \vec{CO} + \vec{SD} + \vec{DO} =$

O is the midpoint of AC

$\Rightarrow \vec{AO} = \vec{OC} \Rightarrow \vec{AO} + \vec{CO} = \vec{0}$

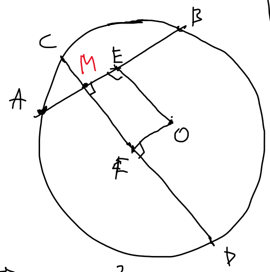
O is the midpoint of BD

$\Rightarrow \vec{BO} = \vec{OD} \Rightarrow \vec{BO} + \vec{DO} = \vec{0}$

$\Rightarrow 4\vec{SO} = \vec{SA} + \vec{SB} + \vec{SC} + \vec{SD}$

1.7. Consider two perpendicular chords AB and CD of a given circle and $\{M\} = AB \cap CD$. Let O be the center of the circle. Show that:

$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 2\vec{OM}$



The sum of all angles of a convex n-gon is $\pi(n-2)$

$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} \overset{?}{=} 2\vec{OM}$

$\frac{\vec{OA} + \vec{OB}}{2} + \frac{\vec{OC} + \vec{OD}}{2} \overset{!}{=} \vec{OM}$

$\left.\begin{array}{l} \Delta OAB \text{ isosceles} \\ \text{Let } E \text{ be the midpoint of } [AB] \end{array}\right\} \Rightarrow OE \perp AB$

$\Rightarrow \frac{\vec{OA} + \vec{OB}}{2} = \vec{OE}$

$\left(\left.\begin{array}{l} OE \perp AB \\ CD \perp AB \end{array}\right| \Rightarrow OE \parallel CD\right)$ Not necessary

$\left.\begin{array}{l} \text{Let } F \text{ be the midpoint of } [CD] \\ \Delta OCD \text{ isosceles} \end{array}\right| \Rightarrow$

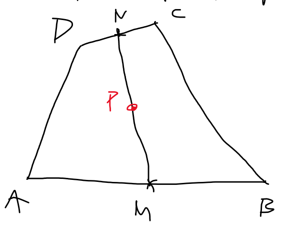
$\Rightarrow OF \perp CD$

$\left.\begin{array}{l} OF \perp CD \\ OE \perp AB \\ AB \perp CD \end{array}\right\} \Rightarrow OEMF$ rectangle

OM diagonal in the parallelogram OEMF

$\Rightarrow \vec{OM} = \vec{OE} + \vec{OF} = \frac{\vec{OA} + \vec{OB}}{2} + \frac{\vec{OC} + \vec{OD}}{2}$

1.6. M, N are the midpoints of two opposite edges of a given quadrilateral ABCD and P is the midpoint of [MN]. Show that: $\vec{PA}+\vec{PB}+\vec{PC}+\vec{PD}=\vec{0}$



$\vec{PA}=\vec{PM}+\vec{MA}$, $\vec{PB}=\vec{PM}+\vec{MB}$

$\vec{PC}=\vec{PN}+\vec{NC}$, $\vec{PD}=\vec{PN}+\vec{ND}$

N midpoint of [CD] $\Rightarrow \vec{DN}=\vec{NC} \Rightarrow \vec{NC}+\vec{ND}=\vec{0}$

M midpoint of [AB] $\Rightarrow \vec{AM}=\vec{MB} \Rightarrow$

$\Rightarrow \vec{MA}+\vec{MB}=\vec{0}$

$\Rightarrow \vec{PA}+\vec{PB}+\vec{PC}+\vec{PD} = \vec{PM}+\vec{MA}+\vec{PM}+\vec{MB}+$

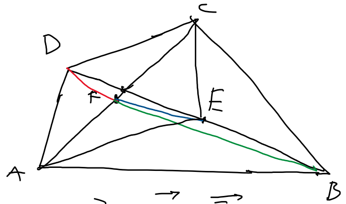
$+\vec{PN}+\vec{NC}+\vec{PN}+\vec{ND} =$

$= 2\vec{PM}+2\vec{PN}+\underbrace{(\vec{MA}+\vec{MB})}_{=\vec{0}}+\underbrace{(\vec{NC}+\vec{ND})}_{=\vec{0}}$

$=\vec{0}$, because $\vec{PM}=-\vec{PN}$, due to P being the midpoint of [MN]

1.4. E, F midpoints of the diagonals of a quadrilateral ABCD. Show that:

$\vec{EF}=\frac{1}{2}\cdot(\vec{AB}-\vec{CD})=\frac{1}{2}(\vec{AD}+\vec{CB})$



$\vec{EF}=\frac{\vec{EA}+\vec{EC}}{2}$

$\vec{FE}=\frac{\vec{FD}+\vec{FB}}{2}$

$\Rightarrow \vec{DF}=\frac{\vec{DA}+\vec{DC}}{2}$; $\vec{FD}=\frac{\vec{AD}+\vec{CD}}{2}$

$\vec{FB}=\frac{\vec{AB}+\vec{CB}}{2}$

$\vec{FA}=\frac{\vec{DA}+\vec{BA}}{2}$

$\vec{EC}=\frac{\vec{DC}+\vec{BC}}{2}$

$\vec{EF}=\frac{\vec{EA}+\vec{EC}}{2}=\frac{\vec{DA}+\vec{BA}+\vec{DC}+\vec{BC}}{4}$

$\vec{FE}=\frac{\vec{FB}+\vec{FB}}{2}=\frac{\vec{AB}+\vec{CB}+\vec{AB}+\vec{CD}}{4}$

$\vec{EF}=\frac{2\vec{r_A}+2\vec{r_C}-2\vec{r_B}-2\vec{r_D}}{4}=$

$=\frac{2(\vec{r_A}-\vec{r_B})+2(\vec{r_C}-\vec{r_D})}{4}=\frac{1}{2}(\vec{BA}+\vec{DC})$