# Seminar W13 — 9.12

**Projective plane** $\underline{\underline{\mathbb{R}\mathbb{P}^2}} = \mathbb{P}^2(\mathbb{R}) = \mathbb{P}(\mathbb{R}^2) = \mathbb{P}\mathbb{R}^2$

$$\mathbb{R}\mathbb{P}^2 = \left\{ [x : y : z] \;\middle|\; \begin{array}{c} x, y, z \in \mathbb{R} \\ (x, y, z) \neq (0, 0, 0) \end{array} \right\}$$

homogeneous (projective) vector

$$[x : y : z] = [\lambda x : \lambda y : \lambda z]$$

$$\forall \lambda \in \mathbb{R} \setminus \{0\}$$

$$\mathbb{R}\mathbb{P}^2 = \frac{\mathbb{R}^3 \setminus \{(0,0,0)\}}{\sim}$$

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \overset{\text{def}}{\iff} \exists \lambda \in \mathbb{R} \setminus \{0\} :$$

$$(x_2, y_2, z_2) = \lambda \cdot (x_1, y_1, z_1)$$

$\mathbb{RP}^2$ = the lines in $\mathbb{R}^3$ that contain the origin $(0,0,0)$

$$\underbrace{\mathbb{RP}^2}_{\text{projective plane}} = \underbrace{\mathbb{RA}^2}_{\text{affine plane } (\cong \mathbb{R}^2)} \cup \underbrace{\mathbb{R}_\infty}_{\text{line at infinity}}$$

$$\mathbb{RA}^2 = \{ [x:y:z] \in \mathbb{RP}^2 \mid z \neq 0 \} =$$

$$= \{ [\underbrace{\tfrac{x}{z}}_{X} : \underbrace{\tfrac{y}{z}}_{Y} : 1] \in \mathbb{RP}^2 \mid z \neq 0 \} =$$

$$= \{ [X : Y : 1] \mid X, Y \in \mathbb{R} \}$$

$\mathbb{RA}^2 \longrightarrow \mathbb{A}^2$ bijective

$$[x:y:z] \longmapsto \left( \frac{x}{z}, \frac{y}{z} \right)$$

This is how we embed $\mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$

$$\mathbb{R}_\infty = \{ [x:y:z] \in \mathbb{RP}^2 \mid z = 0 \} =$$

$$= \left\{ [x:y:0] \in \mathbb{RP}^2 \;\middle|\; \begin{array}{l} x,y \in \mathbb{R} \\ (x,y) \neq (0,0) \end{array} \right\}$$

Every homogeneous vector $[x:y:0]$ corresponds to all the parallel lines in the (affine) plane that have the direction vector $(x,y)$.

<u>Why we care:</u>

$\varphi_1, \varphi_2$ affine transformations

$$\varphi_1 \begin{pmatrix} x \\ y \end{pmatrix} = M_1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_1$$

$$\varphi_2 \begin{pmatrix} x \\ y \end{pmatrix} = M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_2$$

$$(\varphi_1 \circ \varphi_2) \begin{pmatrix} x \\ y \end{pmatrix} = \varphi_1 \left( M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_2 \right) =$$

$$= M_1 \cdot \left( M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_2 \right) + v_1 =$$

$$= M_1 M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + M_1 u_2 + u_1$$

Instead of defining an affine transformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ as $\varphi\begin{pmatrix} x \\ y \end{pmatrix} = M \cdot \begin{pmatrix} x \\ y \end{pmatrix} + u_0 =$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} ax + by + x_0 \\ cx + dy + y_0 \end{pmatrix}$$

we define it as a projective transformation $\varphi : \mathbb{RP}^2 \to \mathbb{RP}^2$

$$\varphi \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \left( \begin{array}{cc|c} \multicolumn{2}{c|}{M} & u_0 \\ \hline 0 & 0 & 1 \end{array} \right) \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + x_0 \\ cx + dy + y_0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \varphi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by + x_0 \\ cx + dy + y_0 \end{pmatrix}$$

$\Psi$ is a <u>projective transformation</u>:

$$\Psi : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2.$$

$$\Psi\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix}$$

$\Psi$ is an affine transformation if

$$a_{31} = 0 \qquad a_{32} = 0 \qquad a_{33} \neq 0$$

(usually $a_{33} = 1$)

13.1. Find the concatenation (product) of an anticlockwise rotation about the origin through an angle $\frac{3\pi}{2}$, followed by a scaling by a factor of 3 units in the $x$-direction and 2 units in the $y$-direction.

$$S(3,2) \circ R_{\frac{3\pi}{2}}$$

$$[R_{\frac{3\pi}{2}}] = \begin{pmatrix} \cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2} & 0 \\ \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[S(3,2)] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

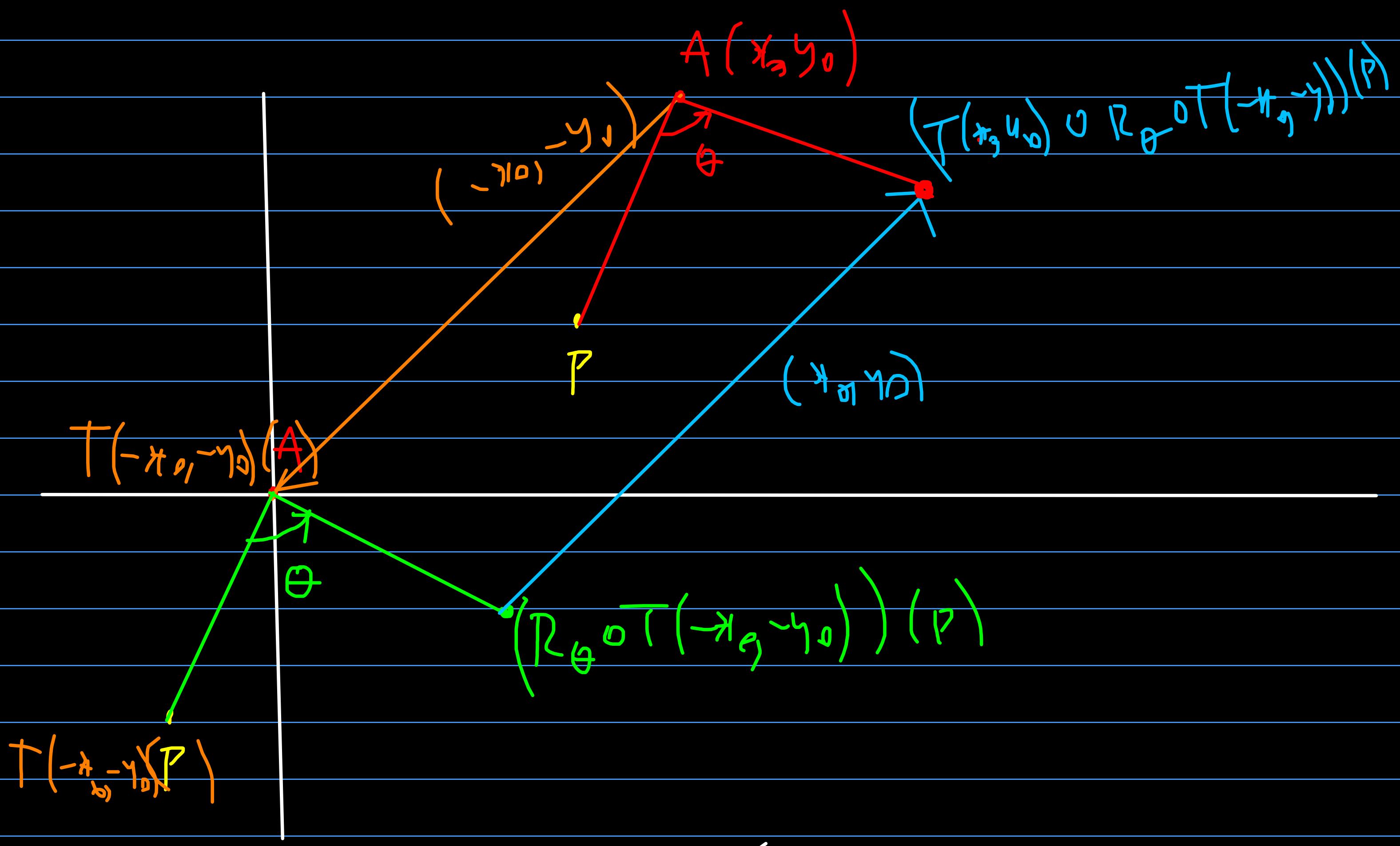
$$\left[S(3,2)\circ R_{\frac{3\pi}{2}}\right] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(S(3,2)\circ R_{\frac{3\pi}{2}}\right)\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 3y \\ -2x \\ 1 \end{bmatrix}$$

$$\left(S(3,2)\circ R_{\frac{3\pi}{2}}\right)\begin{pmatrix} x \\ y \\ \end{pmatrix} = \begin{pmatrix} 3y \\ -2x \end{pmatrix}$$

13.3. $R_\theta(x_0, y_0) = T(x_0, y_0) \circ R_\theta \circ T(-x_0, -y_0)$




$$[R_\theta(x_0, y_0)] = \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_0 = -x_0 \cos\theta + y_0 \sin\theta + x_0$$

$$\beta_0 = -x_0 \sin\theta - y_0 \cos\theta + y_0$$

13.4. $P(x_0, y_0)$, $Q(x_1, y_1)$, $P \neq Q$

Show that $R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0)$ is a translation.

$$[R_{-\theta}(x_1, y_1)] = \begin{pmatrix} \cos\theta & \sin\theta & \alpha_1 \\ -\sin\theta & \cos\theta & \beta_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[R_\theta(x_0, y_0)] = \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[R_{-\theta}(x_1, y_1) \circ R_{\theta}(x_0, y_0)] =$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & \alpha_1 \\ -\sin\theta & \cos\theta & \beta_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$=: M$$

$$m_{1,1} = \cos^2\theta + \sin^2\theta = 1$$

$$m_{1,2} = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$$

$$m_{1,3} = \alpha_0 \cdot \cos\theta + \beta_0 \cdot \sin\theta + \alpha_1$$

$$m_{2,1} = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$$

$$m_{2,2} = \cos^2\theta + \sin^2\theta = 1$$

$$m_{2,3} = -\alpha_0 \sin\theta + \beta_0 \cos\theta + \beta_1$$

$$m_{3,1} = 0, \quad m_{3,2} = 0, \quad m_{3,3} = 1$$

$$\Rightarrow [R_{-\theta}(x_1, y_1) \circ R_{\theta}(x_0, y_0)] =$$

$$= \left(\begin{array}{cc|c} 1 & 0 & m_{1,3} \\ 0 & 1 & m_{2,3} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \quad R_{-\theta}(x_1, y_1) \circ R_{\theta}(x_0, y_0) =$$

$$= T(m_{1,3}, m_{2,3})$$

$$m_{1,3} = \alpha_0 \cdot \cos\theta + \beta_0 \cdot \sin\theta + \alpha_1$$

$$m_{2,3} = -\alpha_0 \sin\theta + \beta_0 \cos\theta + \beta_1$$

$$\alpha_0 = -x_0 \cos\theta + y_0 \sin\theta + x_0$$

$$\beta_0 = -x_0 \sin\theta - y_0 \cos\theta + y_0$$

$$\alpha_1 = -x_1 \cos\theta - y_1 \sin\theta + x_1$$

$$\beta_1 = x_1 \sin\theta - y_1 \cos\theta + y_1$$

$$m_{1,3} = \cos\theta\left(-x_0 \cos\theta + y_0 \sin\theta + x_0\right) +$$

$$+ \sin\theta\left(-x_0 \sin\theta - y_0 \cos\theta + y_0\right) +$$

$$+ \left(-x_1 \cos\theta - y_1 \sin\theta + x_1\right) =$$

$$= \underline{-x_0 \cos^2\theta} + \underline{y_0 \sin\theta \cos\theta} + x_0 \cos\theta$$

$$\underline{- x_0 \sin^2\theta} \underline{- y_0 \sin\theta \cos\theta} + y_0 \sin\theta -$$

$-x_1 \cos\theta - y_1 \sin\theta + \underline{x_1} =$

$= (x_1 - x_0) + \cos\theta\,(x_0 - x_1) +$

$+ \sin\theta\,(y_0 - y_1) =$

$= (x_1 - x_0)\underbrace{(1-\cos\theta)}_{2\sin^2\frac{\theta}{2}} - (y_1 - y_0)\cdot\underbrace{\sin\theta}_{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$m_{2,3} = \cdots$

13.*. $l_1, l_2$ parallel lines. Show that

$r_{l_1} \circ r_{l_2}$ is a translation:

$l_1: ax + by + c_1 = 0$

$l_2: ax + by + c_2 = 0$

$\alpha := a^2 - b^2, \quad \beta = -2ab, \quad \gamma = a^2 + b^2$

$$[r_{\ell_1}] = \begin{pmatrix} \frac{-\alpha}{\gamma} & \frac{\beta}{\gamma} & -\frac{2ac_1}{\gamma} \\ \frac{\beta}{\gamma} & \frac{\alpha}{\gamma} & -\frac{2bc_1}{\gamma} \\ 0 & 0 & 1 \end{pmatrix}$$

$$[r_{\ell_1}] = \begin{pmatrix} -\alpha & \beta & -2ac_1 \\ \beta & \alpha & -2bc_1 \\ 0 & 0 & \gamma \end{pmatrix}$$

$$[r_{\ell_2}] = \begin{pmatrix} -\alpha & \beta & -2ac_2 \\ \beta & \alpha & -2bc_2 \\ 0 & 0 & \gamma \end{pmatrix}$$

$$[r_{\ell_1} \circ r_{\ell_2}] = \begin{pmatrix} -\alpha & \beta & -2ac_1 \\ \beta & \alpha & -2bc_1 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} -\alpha & \beta & -2ac_2 \\ \beta & \alpha & -2bc_2 \\ 0 & 0 & \gamma \end{pmatrix}$$

$$=: M$$

$$m_{1,1} = \alpha^2 + \beta^2 = (a^2+b^2)^2$$

$$m_{1,2} = -\alpha\beta + \alpha\beta = 0$$

$$m_{1,3} = 2ac_2 \cdot \alpha - 2bc_2 \cdot \beta - 2ac_1 \cdot \gamma$$

$$m_{2,1} = -\beta\alpha + \alpha\beta = 0$$

$$m_{2,2} = \alpha^2 + \beta^2$$

$$m_{2,3} = -2ac_2\beta - 2bc_2\alpha - 2bc_1\gamma$$

$$m_{3,1} = 0 \quad , \quad m_{3,2} = 0 \quad , \quad m_{3,3} = \gamma^2$$

$$[r_{l_1} \circ r_{l_2}] = \begin{pmatrix} (a^2+b^2)^2 & 0 & m_{1,3} \\ 0 & (a^2+b^2)^2 & m_{2,3} \\ 0 & 0 & (a^2+b^2)^2 \end{pmatrix}$$

$$(r_{l_1} \circ r_{l_2}) = \begin{pmatrix} 1 & 0 & \frac{m_{1,3}}{(a^2+b^2)^2} \\ 0 & 1 & \frac{m_{2,3}}{(a^2+b^2)^2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow r_{l_1} \circ r_{l_2} = T\left( \frac{m_{1,3}}{(a^2+b^2)^2} , \frac{m_{2,3}}{(a^2+b^2)^2} \right)$$