Seminar W13 - 9.16

Projective plane

$\mathbb{RP}^2 = \mathbb{PR}^2 = \mathbb{P}^2(\mathbb{R})$

$$\mathbb{RP}^2 = \left\{ [x : y : z] \;\middle|\; \begin{array}{l} x, y, z \in \mathbb{R} \\ (x, y, z) \neq (0,0,0) \end{array} \right\}$$

homogeneous coordinates / vectors

$$[x : y : z] = [\lambda x : \lambda y : \lambda z]$$

$$\forall \lambda \in \mathbb{R} \setminus \{0\}$$

$$\mathbb{RP}^2 = \frac{\mathbb{R}^3 \setminus \{0\}}{\sim}$$

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \iff \exists \lambda \in \mathbb{R} \setminus \{0\} : (x_2, y_2, z_2) = \lambda (x_1, y_1, z_1)$$

$\mathbb{RP}^2$ = the set of lines in $\mathbb{R}^3$ that contain the origin $(0,0,0)$

$$\mathbb{RP}^2 = \mathbb{RA}^2 \cup \mathbb{R}\infty$$

projective plane — affine plane (the one that we usually work with) — line at infinity

$$\mathbb{RA}^2 = \{ [x:y:z] \in \mathbb{RP}^2 \mid z \neq 0 \} =$$

$$= \{ [\underbrace{\tfrac{x}{z}}_{= X} : \underbrace{\tfrac{y}{z}}_{= Y} : 1] \in \mathbb{RP}^2 \mid z \neq 0 \} =$$

$$= \{ [X : Y : 1] \in \mathbb{RP}^2 \mid X, Y \in \mathbb{R} \}$$

$\mathbb{RA}^2 \to \mathbb{R}^2$ bijective map

$$[x:y:z] \mapsto \left(\frac{x}{z}, \frac{y}{z} \right)$$

This is how we embed

$$\mathbb{R}^2 \hookrightarrow \mathbb{RP}^2$$

$$\mathbb{R}\infty = \{ [x:y:z] \mid z = 0 \} =$$

$= \{ [x : y : 0] \mid x, y \in \mathbb{R}, (x,y) \neq (0,0) \}$

Every projective point $[x : y : 0]$ corresponds to all the parallel lines in $\mathbb{R}^2$ whose direction vector is (x,y).

<u>Why we care:</u>

φ_1, φ_2 affine transformations

$$\varphi_1 \begin{pmatrix} x \\ y \end{pmatrix} = M_1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \omega_1$$

$$\varphi_2 \begin{pmatrix} x \\ y \end{pmatrix} = M_2 \begin{pmatrix} x \\ y \end{pmatrix} + \omega_2$$

$$(\varphi_1 \circ \varphi_2) \begin{pmatrix} x \\ y \end{pmatrix} = \varphi_1 \left(\varphi_2 \begin{pmatrix} x \\ y \end{pmatrix} \right) =$$

$$= \varphi_1 \left(M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \omega_2 \right) =$$

$$= M_1 \cdot \left(M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \omega_2 \right) + \omega_1 =$$

$$= M_1 M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + M_1 \omega_2 + \omega_1$$

Instead of defining $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ (which is affine) as

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix} + v_0$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} ax + by + x_0 \\ cx + dy + y_0 \end{pmatrix}$$

we see it (temporarily) as a projective transformation $\varphi : \mathbb{RP}^2 \to \mathbb{RP}^2$

$$\Rightarrow \varphi\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \left(\begin{array}{cc|c} & M & v_0 \\ \hline 0 & 0 & 1 \end{array}\right) \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} ax + by + x_0 \\ cx + dy + y_0 \\ 1 \end{bmatrix}$$

From here we just deduce that

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by + x_0 \\ cx + dy + y_0 \end{pmatrix}$$

A projective transformation is a function

$$\Psi : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$$

$$\Psi \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11} x + a_{12} y + a_{13} z \\ a_{21} x + a_{22} y + a_{23} z \\ a_{31} x + a_{32} y + a_{33} z \end{bmatrix}$$

A projective transformation Ψ is affine if $a_{31} = a_{32} = 0$ and $a_{33} \neq 0$

13.1. Find the concatenation (product, composition) of an anticlockwise rotation about the origin through an angle of $\frac{3\pi}{2}$ followed by a scaling by a factor of 3 units in the x direction and 2 units in the y direction

$$S(3,2) \circ R_{\frac{3\pi}{2}}.$$

$$[R_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[R_{\frac{3\pi}{2}}] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

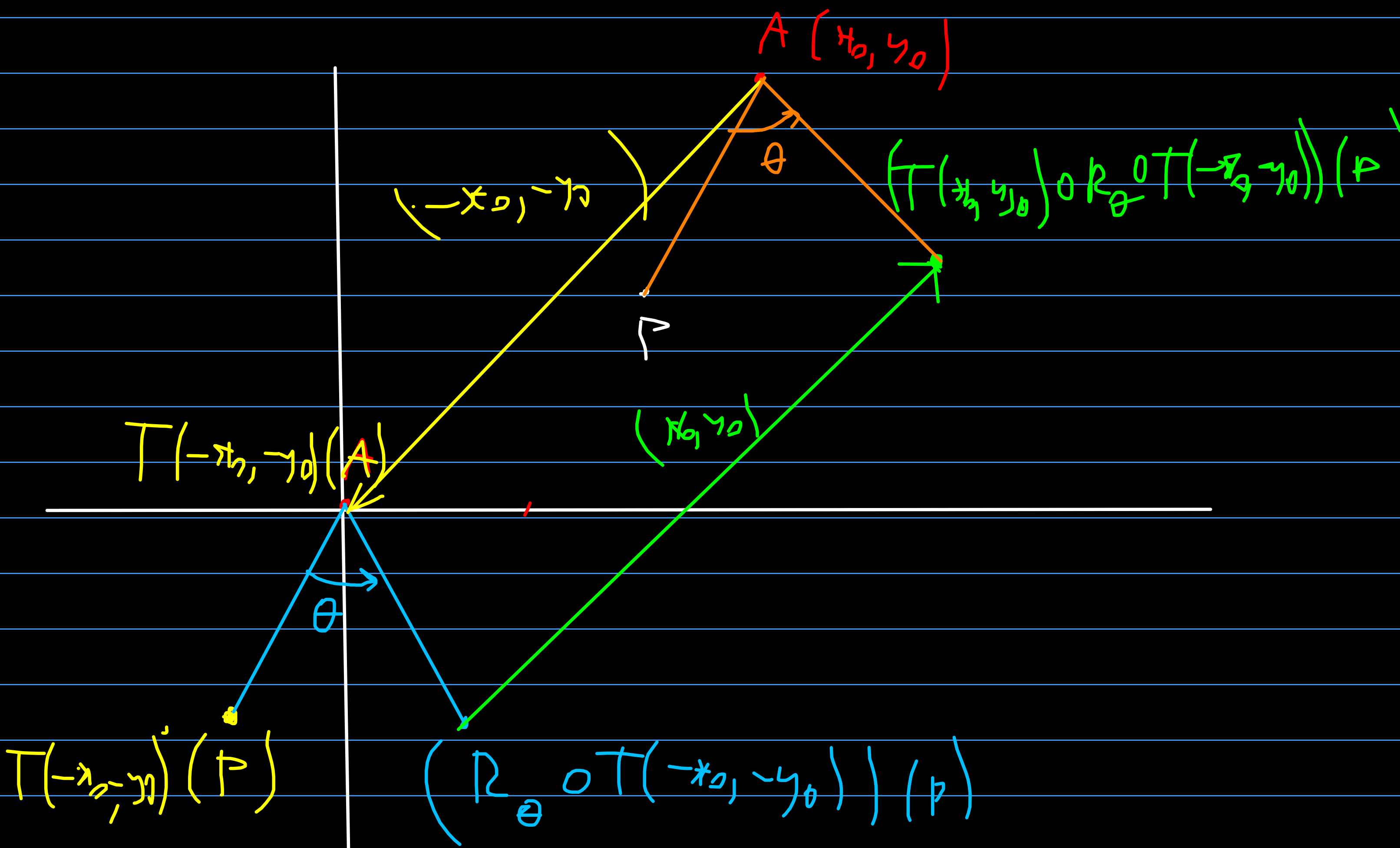
$$[S(3,2)] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[S(3,2) \circ R_{\frac{3\pi}{2}}] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \left(S(3,2) \circ R_{\frac{3\pi}{2}}\right)\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 3y \\ -2x \\ 1 \end{bmatrix}$$

$$\Rightarrow \left(S(3,2) \circ R_{\frac{3\pi}{2}}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ -2x \end{pmatrix}$$

13.3. $R_\theta(x_0, y_0) = T(x_0, y_0) \circ R_\theta \circ T(-x_0, -y_0)$



$$[R_\theta(x_0, y_0)] = \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_0 = -x_0 \cos\theta + y_0 \sin\theta + x_0$$

$$\beta_0 = -x_0 \sin\theta - y_0 \cos\theta + y_0$$

13-4. $P(x_0, y_0)$, $Q(x_1, y_1)$, $Q \neq P$

show that $R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0)$

is a translation.

$$[R_{-\theta}(x_1, y_1)] = \begin{pmatrix} \cos\theta & \sin\theta & \alpha_1 \\ -\sin\theta & \cos\theta & \beta_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[R_\theta(x_0, y_0)] = \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left[R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0) \right] =$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & \alpha_1 \\ -\sin\theta & \cos\theta & \beta_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=: M$$

$$m_{11} = \cos^2\theta + \sin^2\theta = 1$$

$$m_{12} = \sin\theta\cos\theta - \sin\theta\cos\theta = 0$$

$$m_{13} = \alpha_0 \cdot \cos\theta + \beta_0 \cdot \sin\theta + \alpha_1$$

$$m_{21} = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$$

$$m_{22} = \sin^2\theta + \cos^2\theta = 1$$

$$m_{23} = -\alpha_0 \sin\theta + \beta_0 \cos\theta + \beta_1$$

$$m_{31} = 0, \quad m_{32} = 0, \quad m_{33} = 1$$

$$\Rightarrow \left[R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0) \right] = \begin{pmatrix} 1 & 0 & m_{13} \\ 0 & 1 & m_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0) = T(m_{13}, m_{23})$$

$$m_{13} = \alpha_0 \cdot \cos\theta + \beta_0 \cdot \sin\theta + \alpha_1$$

$$m_{23} = -\alpha_0 \sin\theta + \beta_0 \cos\theta + \beta_1$$

$$\alpha_0 = -x_0 \cos\theta + y_0 \sin\theta + x_0$$

$$\beta_0 = -x_0 \sin\theta - y_0 \cos\theta + y_0$$

$$\alpha_1 = -x_1 \cos\theta - y_1 \sin\theta + x_1$$

$$\beta_1 = x_1 \sin\theta - y_1 \cos\theta + y_1$$

$$m_{13} = \cos\theta \left(-x_0 \cos\theta + y_0 \sin\theta + x_0\right) +$$

$$+ \sin\theta \cdot \left(-x_0 \sin\theta - y_0 \cos\theta + y_0\right) -$$

$$- x_1 \cos\theta - y_1 \sin\theta + x_1 =$$

$$= -x_0 \cos^2\theta + y_0 \sin\theta \cos\theta + x_0 \cos\theta -$$

$$- x_0 \sin^2\theta - y_0 \sin\theta \cos\theta + y_0 \sin\theta -$$

$$- x_1 \cos\theta - y_1 \sin\theta + x_1 =$$

$$= (x_1 - x_0) + \cos\theta \left(x_0 - x_1\right) + \sin\theta \left(y_0 - y_1\right)$$

$$= (x_0 - x_1)(\cos\theta - 1) + (y_0 - y_1) \cdot \sin\theta$$

$$m_{23} = -\sin\theta(-x_0\cos\theta + y_0\sin\theta + x_0)$$
$$+ \cos\theta \cdot (-x_0\sin\theta - y_0\cos\theta + y_0) +$$
$$+ x_1\sin\theta - y_1\cos\theta + y_1 =$$

$$= (y_1 - y_0) + \sin\theta(x_1 - x_0) +$$
$$+ \cos\theta \cdot (y_0 - y_1) =$$

$$= -(x_0 - x_1)\sin\theta + (y_0 - y_1)(\cos\theta - 1)$$

13. *. Consider two parallel lines l_1 and l_2. Show that $r_{l_1} \circ r_{l_2}$ is a translation.

l_1: $ax + by + c_1 = 0$

l_2: $ax + by + c_2 = 0$

$$[r_{l_1}] = \begin{pmatrix} b^2-a^2 & -2ab & -2ac_1 \\ -2ab & a^2-b^2 & -2bc_1 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

$$[r_{l_2}] = \begin{pmatrix} b^2-a^2 & -2ab & -2ac_2 \\ -2ab & a^2-b^2 & -2bc_2 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

$$[r_{l_1} \circ r_{l_2}] = \begin{pmatrix} b^2-a^2 & -2ab & -2ac_1 \\ -2ab & a^2-b^2 & -2bc_1 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \cdot$$

$$\begin{pmatrix} b^2-a^2 & -2ab & -2ac_2 \\ -2ab & a^2-b^2 & -2bc_2 \\ 0 & 0 & a^2+b^2 \end{pmatrix} =: M$$

$$m_{11} = (b^2-a^2)^2 + 4a^2b^2 = b^4 + a^4 - 2a^2b^2 + 4a^2b^2 =$$
$$= (a^2+b^2)^2$$

$$m_{12} = -(b^2-a^2)\cdot 2ab - 2ab(a^2-b^2) = 0$$

$$m_{13} = (b^2-a^2)(-2ac_2) - 2ab(-2bc_2) -$$
$$- 2ac_1(a^2+b^2)$$

$$m_{21} = -2ab(b^2-a^2) + (a^2-b^2)\cdot(-2ab) =$$
$$= 0$$

$$m_{22} = 4a^2b^2 + (a^2-b^2)^2 = (a^2+b^2)^2$$

$$m_{23} = -2ab\cdot(-2ac_2) + (a^2-b^2)\cdot(-2bc_2) +$$
$$+ (-2bc_1)\cdot(a^2+b^2)$$

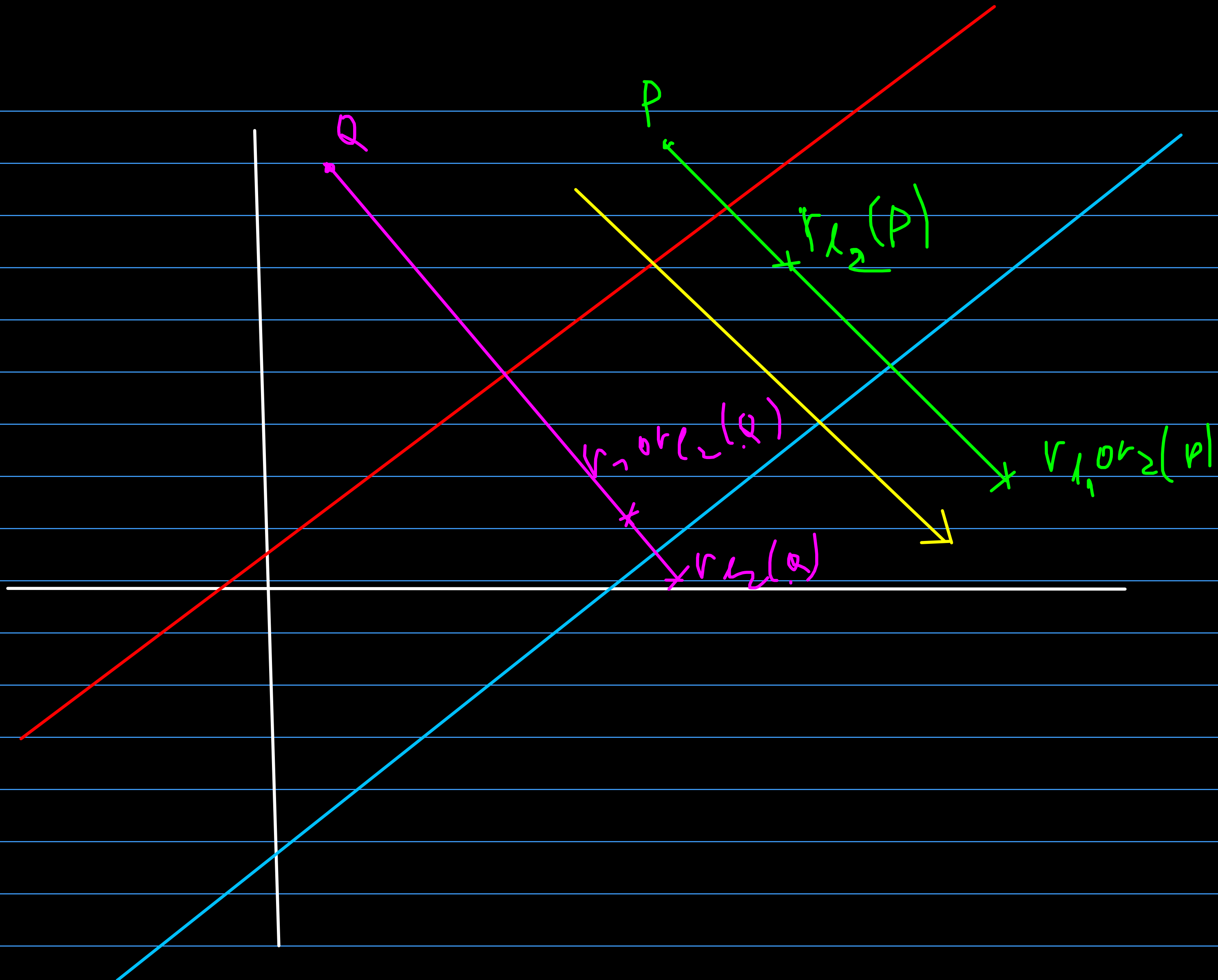
$$m_{31} = 0 \qquad m_{32} = 0 \qquad m_{33} = (a^2+b^2)^2$$

$$\Rightarrow [r_{\ell_1} \circ r_{\ell_2}] = \begin{pmatrix} (a^2+b^2)^2 & 0 & m_{13} \\ 0 & (a^2+b^2)^2 & m_{23} \\ 0 & 0 & (a^2+b^2)^2 \end{pmatrix}$$

$$[r_{l_1} \circ r_{l_2}] = \begin{pmatrix} 1 & 0 & \frac{m_{13}}{(a^2+b^2)^2} \\ 0 & 1 & \frac{m_{23}}{(a^2+b^2)^2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$m_{13} = (b^2-a^2)\cdot(-2ac_2) - 2ab(-2bc_2) -$$
$$- 2ac_1(a^2+b^2) =$$
$$= -2ab^2c_2 + 2a^3c_2 + 4ab^2c_2 -$$
$$- 2a^3c_1 - 2ab^2c_1 =$$
$$= 2ab^2(c_2-c_1) + 2a^3(c_2-c_1) =$$
$$= 2a(c_2-c_1)\cdot(a^2+b^2)$$

$$m_{23} = -2ab\cdot(-2ac_2) + (a^2-b^2)\cdot(-2bc_2) +$$
$$+ (-2bc_1)\cdot(a^2+b^2) =$$
$$= 4a^2bc_2 - 2a^2bc_2 + 2b^3c_2 - 2a^2bc_1 -$$
$$- 2b^3c_1 = 2a^2b(c_2-c_1) + 2b^3(c_2-c_1)$$
$$= 2b(c_2-c_1)\cdot(a^2+b^2)$$



Q
P
$r_{l_2}(P)$
r_1 or $r_2(Q)$
r_1 or $r_2(P)$
$r_{l_2}(Q)$