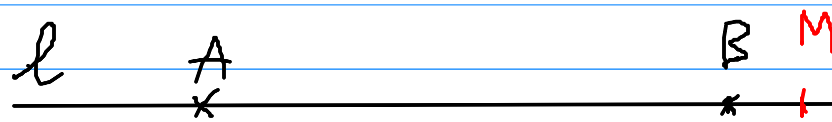
Seminar W2 - 916



l line in the Euclidean plane

We fix a reference system.

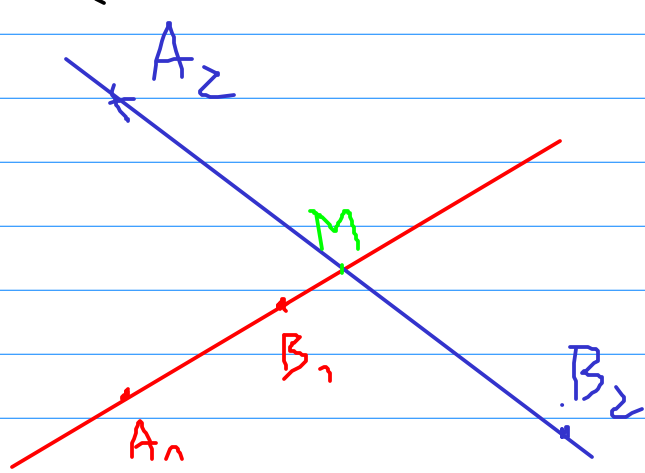
Then $\forall M \in l \ \exists!\ \lambda \in \mathbb{R}$ s.t.:

$$\vec{r}_M = \lambda \vec{r}_A + (1-\lambda)\vec{r}_B$$

$$\boxed{\text{If } M \in [AB] \text{ and } \frac{AM}{MB} = \alpha \text{ then } \vec{r}_M = \frac{\alpha}{\alpha+1}\vec{r}_B + \frac{1}{\alpha+1}\vec{r}_A}$$

Let $\{M\} = l_1 \cap l_2$, $A_1, B_1 \in l_1$

$A_2, B_2 \in l_2$



Template for proofs

Step 1: We write the fact that $M \in \ell_1$ and $M \in \ell_2$ by using the vector equation

$$\exists \lambda, \mu \in \mathbb{R}: \vec{r_M} = \lambda \vec{r_{A_1}} + (1-\lambda) \cdot \vec{r_{B_1}} \quad (1)$$
$$\vec{r_M} = \mu \vec{r_{B_1}} + (1-\mu) \vec{r_{B_2}} \quad (2)$$

Step 2: We find two vectors $\vec{u}, \vec{w}$ that are always linearly independent

Step 3: We write $\vec{r_{A_1}}, \vec{r_{B_1}}, \vec{r_{A_2}}, \vec{r_{B_2}}$ in terms of $\vec{u}$ and $\vec{w}$.

Step 4: You have obtained from (1) and (2) that:

$$\alpha(\lambda,\mu)\cdot\vec{u} + \beta(\lambda,\mu)\cdot\vec{w} = \vec{0}$$

<u>Step 5</u>: $\vec{u},\vec{w}$ lin. indep. $\Rightarrow \begin{cases}\alpha(\lambda,\mu)=0\\ \beta(\lambda,\mu)=0\end{cases}$

Solve the system to get λ (and μ)

<u>Step 6</u>: Replace λ (or μ) in (1) (or (2))

<u>Step 7</u>: Rejoice! For you have found $\vec{r_M}$ in terms of $\vec{u}$ and $\vec{w}$

1.1. ΔABC, G centroid, H orthocenter, I incenter, O circumcenter

$\downarrow$

inters. point of

perpendicular bisectors

We fix a reference system. Show that:

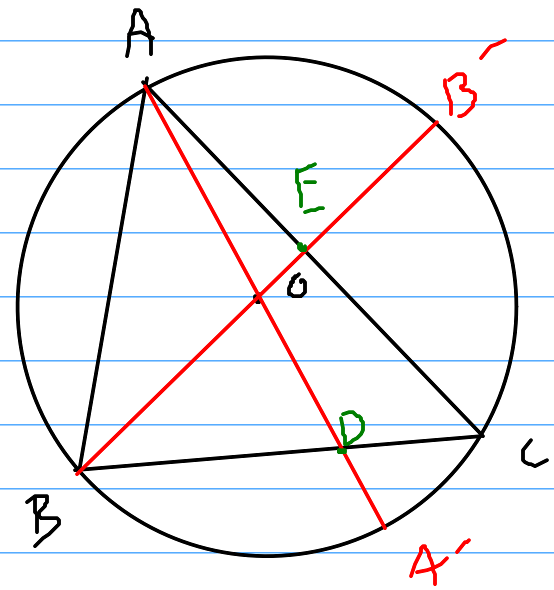
(a) $$\vec{r_G} = \frac{\vec{r_A} + \vec{r_B} + \vec{r_C}}{3}$$

(b) $$\vec{r_I} = \frac{a\vec{r_A} + b\vec{r_B} + c\vec{r_C}}{a+b+c}$$

$a = BC$, $b = CA$, $c = AB$

(c) $$\vec{r_H} = \frac{\tan A \cdot \vec{r_A} + \tan B \cdot \vec{r_B} + \tan C \cdot \vec{r_C}}{\tan A + \tan B + \tan C}$$

(d) $$\vec{r_O} = \frac{\sin 2A \cdot \vec{r_A} + \sin 2B \cdot \vec{r_B} + \sin 2C \cdot \vec{r_C}}{\sin 2A + \sin 2B + \sin 2C}$$

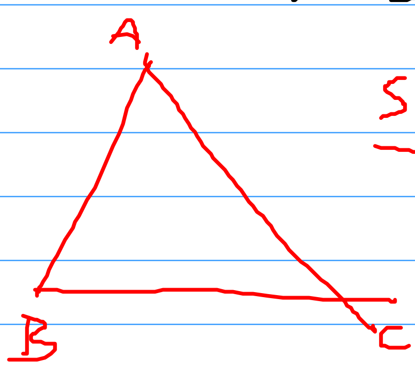


We draw the diameters AA', BB'

$AA' \cap BC = \{D\}$, $BB' \cap AC = \{E\}$

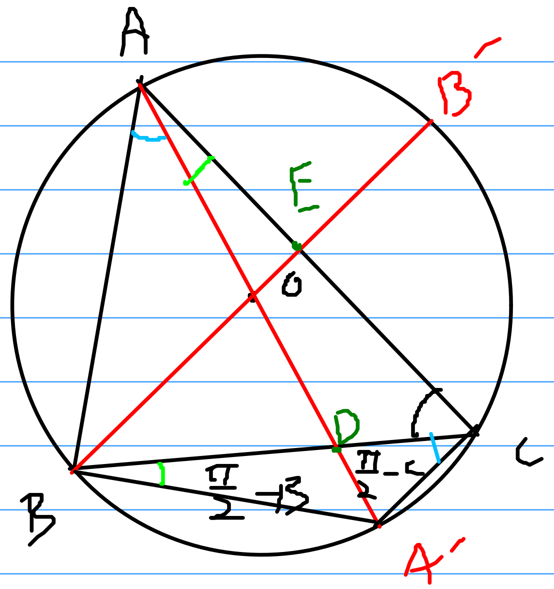
Show that $\dfrac{BD}{DC} = \dfrac{\sin 2C}{\sin 2B}$

$\left(\dfrac{AE}{EC} = \dfrac{\sin 2C}{\sin 2A} \right)$



Sine thm.: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

R = radius of the circumscribed circle



Using the sine thm.

in $\triangle ABD$:

$$\frac{BD}{\sin \widehat{BAD}} = \frac{AD}{\sin \hat{B}} = \frac{c}{\sin(\widehat{ADB})}$$

Using the sine thm in

$\triangle ADC$: $$\frac{CD}{\sin \widehat{CAD}} = \frac{b}{\sin(\widehat{ADC})} = \frac{AD}{\sin \hat{C}}$$

$$\Rightarrow \frac{BD}{CD} \cdot \frac{\sin \widehat{CAD}}{\sin(\widehat{BAD})} = \frac{\dfrac{AD}{\sin \hat{B}}}{\dfrac{AD}{\sin \hat{C}}} = \frac{\sin C}{\sin B}$$

$\widehat{BAA'} \equiv \widehat{BCA'}$ (because they both correspond to $\widehat{BA'}$)

$\widehat{CAA'} \equiv \widehat{CBA'}$ (because they both correspond to $\widehat{CA'}$)

$$m(\widehat{ACA'}) = \frac{\pi}{2} \Rightarrow m(\widehat{BCA'}) = \frac{\pi}{2} - m(\hat{C})$$

$m(\widehat{ABA'}) = \frac{\pi}{2} \Rightarrow m(\widehat{CBA'}) = \frac{\pi}{2} - m(\hat{B})$

$\Rightarrow \sin(\widehat{CAD}) = \sin(\frac{\pi}{2} - B) = \cos(B)$

$\sin(\widehat{BAD}) = \sin(\frac{\pi}{2} - C) = \cos C$

$$\Rightarrow \frac{BD}{CD} \cdot \frac{\sin(\widehat{CAD})}{\sin(\widehat{BAD})} = \frac{\sin C}{\sin B}$$

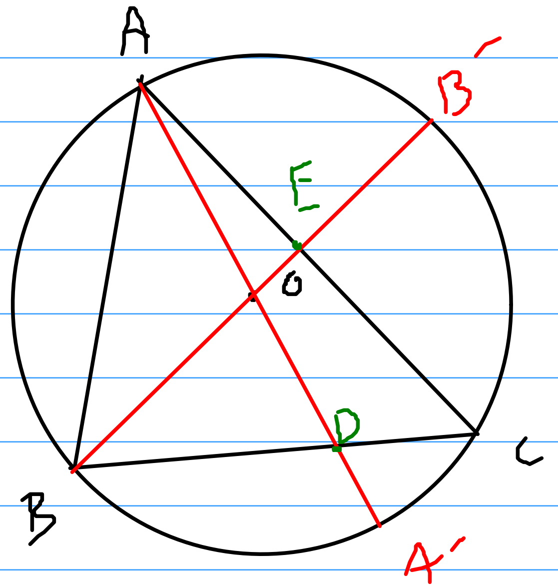
$$\Rightarrow \frac{BD}{CD} = \frac{\sin C}{\sin B} \cdot \frac{\cos C}{\cos B} = \frac{\sin 2C}{\sin 2B}$$

$$\Rightarrow \frac{BD}{CD} = \frac{\sin 2C}{\sin 2B} \qquad \frac{AE}{EC} = \frac{\sin 2C}{\sin 2A}$$

We will solve 1.1.(d)

11. d $\qquad \vec{r_O} = \dfrac{\sin 2A \cdot \vec{r_A} + \sin 2B \cdot \vec{r_B} + \sin 2C \cdot \vec{r_C}}{\sin 2A + \sin 2B + \sin 2C}$

We know: $\dfrac{BD}{DC} = \dfrac{\sin 2C}{\sin 2B} \qquad \dfrac{AE}{EC} = \dfrac{\sin 2C}{\sin 2A}$



$\sin 2A =: \alpha$

$\sin 2B =: \beta$

$\sin 2C =: \gamma$

$$\vec{r_O} = (1-\lambda)\vec{r_A} + \lambda \vec{r_D}$$

$$= (1-\mu)\vec{r_B} + \mu \vec{r_E} \qquad \lambda, \mu \in \mathbb{R}$$

$$\frac{BD}{DC} = \frac{\gamma}{\beta} \Rightarrow \vec{r_D} = \frac{\vec{r_B} + \frac{\gamma}{\beta}\vec{r_C}}{1+\frac{\gamma}{\beta}} = \frac{\beta \vec{r_B} + \gamma \vec{r_C}}{\beta + \gamma}$$

$$\frac{AE}{EC} = \frac{\gamma}{\alpha} \Rightarrow \vec{r_E} = \frac{\alpha \vec{r_A} + \gamma \vec{r_C}}{\alpha + \gamma}$$

$$\Rightarrow (1-\lambda)\vec{r_A} + \frac{\lambda\beta}{\beta+\gamma}\vec{r_B} + \frac{\lambda\gamma}{\beta+\gamma}\vec{r_C} =$$

$$= (1-\mu)\vec{r_B} + \frac{\alpha\mu}{\alpha+\gamma}\vec{r_A} + \frac{\gamma\mu}{\alpha+\gamma}\vec{r_C}$$

$$\Rightarrow \left(1-\lambda-\frac{\alpha\mu}{\alpha+\gamma}\right)\vec{r_A} + \left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)\vec{r_B} +$$

$$+\left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\gamma\mu}{\alpha+\gamma}\right)\vec{r_C} = \vec{0} \qquad (*)$$

$\vec{AB}$ and $\vec{AC}$ are linearly independent, because, if not, then the triangle would be degenerate..

We choose $\vec{v} = \vec{AB}$, $\vec{w} = \vec{AC}$

$\vec{r_B} = \vec{r_A} + \vec{v}$, $\vec{r_C} = \vec{r_A} + \vec{w}$

We replace them in $(*)$

$$\left(1-\lambda-\frac{\alpha\mu}{\alpha+\gamma}\right)\vec{r_A} + \left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)\cdot(\vec{r_A}+\vec{v}) +$$

$$+\left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)\cdot(\vec{r_A}+\vec{w}) = \vec{0}$$

$$\left(1-\lambda-\frac{\alpha\mu}{\alpha+\gamma}+\frac{\lambda\beta}{\beta+\gamma}-1+\mu+\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)\cdot\vec{r_A}+\left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)\vec{u}+\left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)\cdot\vec{w}=\vec{0}$$

$$\Rightarrow \left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)\cdot\vec{u}+\left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)\cdot\vec{w}=\vec{0}$$

$\vec{u},\vec{w}$ lin. indep.

$$\Rightarrow \begin{cases} \dfrac{\lambda\beta}{\beta+\gamma}-1+\mu=0 \\[2ex] \dfrac{\lambda\gamma}{\beta+\gamma}-\dfrac{\mu\gamma}{\alpha+\gamma}=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu=-\dfrac{\lambda\beta}{\beta+\gamma}+1 \\[2ex] \dfrac{\lambda\gamma}{\beta+\gamma}+\dfrac{\beta\gamma\cdot\lambda}{(\alpha+\gamma)(\beta+\gamma)}-\dfrac{\gamma}{\alpha+\gamma}=0 \end{cases}$$

$$\cancel{\Rightarrow \lambda=\frac{\frac{\gamma}{\alpha+\gamma}}{\frac{\gamma}{\beta+\gamma}}}$$

$(\alpha+\gamma)(\beta+\gamma)$ $\quad\gamma$

$$\lambda = \frac{\frac{\gamma}{\alpha+\gamma}}{\frac{\gamma}{\beta+\gamma} + \frac{\beta\gamma}{(\alpha+\gamma)(\beta+\gamma)}} =$$

$$= \frac{\beta+\gamma}{(\alpha+\gamma)+\beta} = \frac{\beta+\gamma}{\alpha+\beta+\gamma}$$

$$\Rightarrow \vec{r_0} = (1-\lambda)\vec{r_A} + \frac{\lambda\beta}{\beta+\gamma}\vec{r_B} + \frac{\lambda\gamma}{\beta+\gamma}\vec{r_C}$$

$$\Rightarrow \vec{r_0} = \frac{\alpha}{\alpha+\beta+\gamma}\vec{r_A} + \frac{\beta}{\alpha+\beta+\gamma}\vec{r_B} + \frac{\gamma}{\alpha+\beta+\gamma}\vec{r_C}$$

$$\Rightarrow \vec{r_0} = \frac{\sin 2A}{\sin 2A + \sin 2B + \sin 2C}\vec{r_A} + \dots$$

1.2. Consider the nonzero angle $\widehat{BOB'}$ and the points $A \in [OB]$, $A' \in [OB']$.

$$\{M\} = AB' \cap A'B$$

$$\{N\} = AA' \cap BB'$$

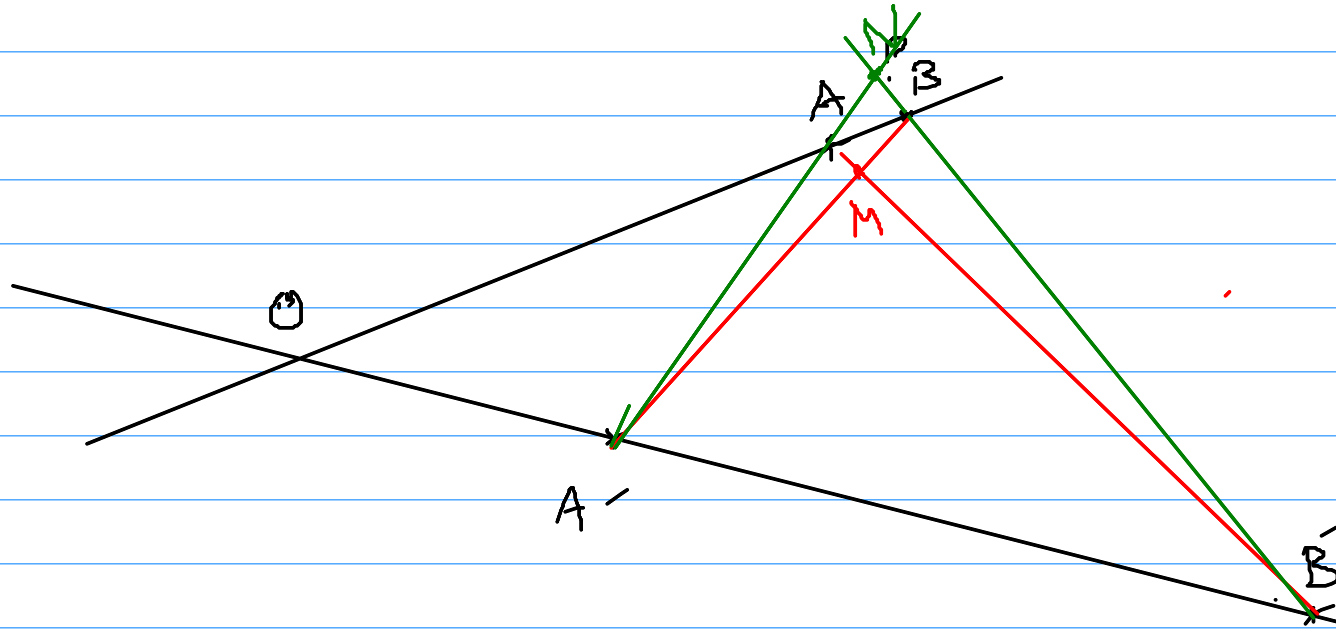
$$\vec{u} := \overrightarrow{OA}, \quad \vec{v} = \overrightarrow{OA'}$$

$$\overrightarrow{OB} = m \cdot \overrightarrow{OA}, \quad \overrightarrow{OB'} = n \cdot \overrightarrow{OA'}$$

Show that:

$$\overrightarrow{OM} = m \cdot \frac{1-n}{1-mn} \cdot \overrightarrow{OA} + n \cdot \frac{1-m}{1-mn} \cdot \overrightarrow{OA'}$$

$$\overrightarrow{ON} = m \cdot \frac{n-1}{n-m} \cdot \overrightarrow{OA} + n \cdot \frac{m-1}{m-n} \overrightarrow{OA'}$$



$$\vec{OM} = \lambda\cdot\vec{OA} + (1-\lambda)\,\vec{OB'} = \lambda\cdot\vec{u} + (1-\lambda)\cdot n\cdot\vec{v}$$

$$= \mu\cdot\vec{OB} + (1-\mu)\cdot\vec{OA'} = \mu\cdot m\cdot\vec{u} + (1-\mu)\cdot\vec{v}$$

$$\Rightarrow (\lambda-\mu m)\cdot\vec{u} + ((1-\lambda)n - 1 + \mu)\cdot\vec{v} = \vec{0}$$

$\vec{u}, \vec{v}$ linearly independent

$$\Rightarrow \begin{cases} \lambda - \mu m = 0 \\ (1-\lambda)n - 1 + \mu = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \lambda = \mu m \\ n - n\mu m - 1 + \mu = 0 \end{cases}$$

$$\Rightarrow \mu = \frac{n-1}{1-nm} \Rightarrow 1-\mu = \frac{-n(m+1)}{1-mn}$$

$$\Rightarrow \vec{OM} = \vec{r}_M = \frac{(n-1)m}{1-n.m}\cdot\vec{u} + \frac{-n(m+1)}{1-mn}\vec{v}$$

Same thing for $\vec{ON}$.