Seminar V/9 – 914

Conics

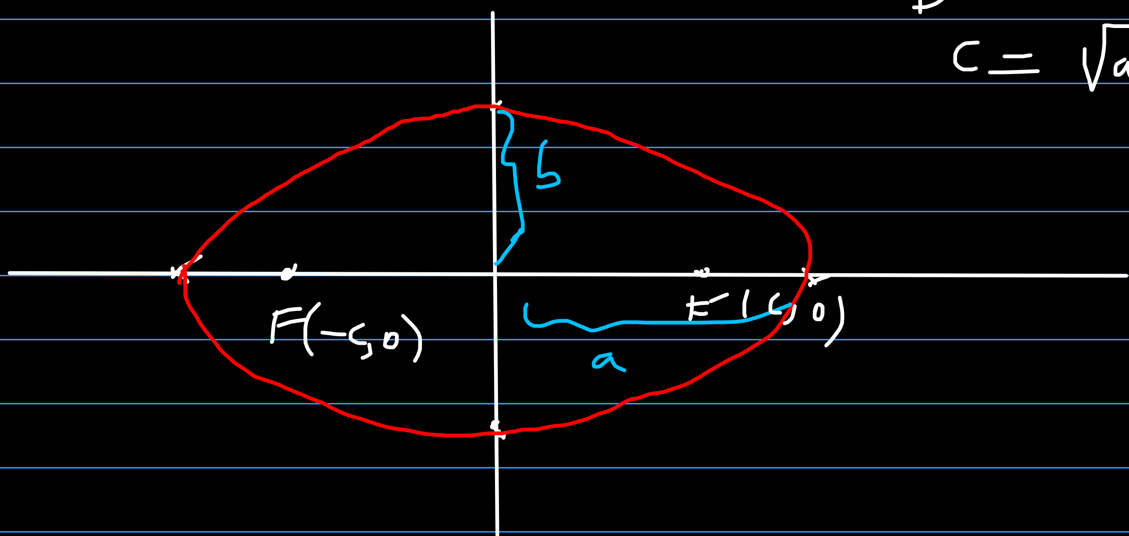
$\ell : ax + by + c = 0$

$\mathcal{C} : a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$

→ conic: ellipse, hyperbola, parabola

Ellipse: $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$c = \sqrt{a^2 - b^2}$



→ locus of points M s.t. $MF + MF' = 2a$

$$T_{\mathcal{E}}(x_0,y_0): \quad \frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1$$

$$f: [0, 2\pi) \to \mathbb{R}^2$$

$$t \mapsto (a\cos t,\ b\sin t)$$

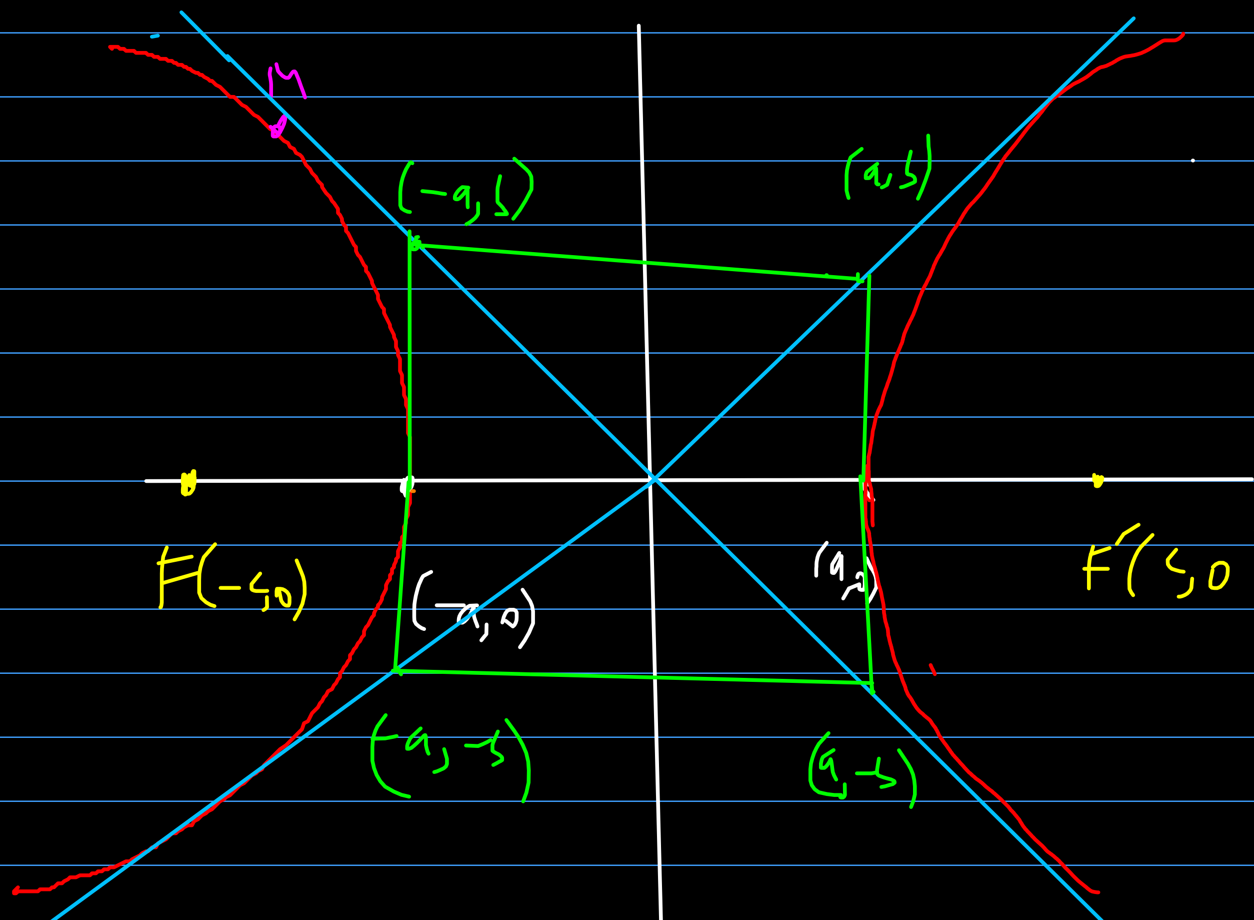
is a parametrization for $\mathcal{E}$.

$$\mathcal{E}: \begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, \quad t \in [0, 2\pi)$$

a parametric equation for the ellipse

"the good"

Hyperbola: $\mathcal{H}: \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



$c = \sqrt{a^2+b^2}$

→ locus of points M so that

$$|MF - MF'| = 2a$$

$$T_{\mathcal{H}}(x_0, y_0): \quad \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$$

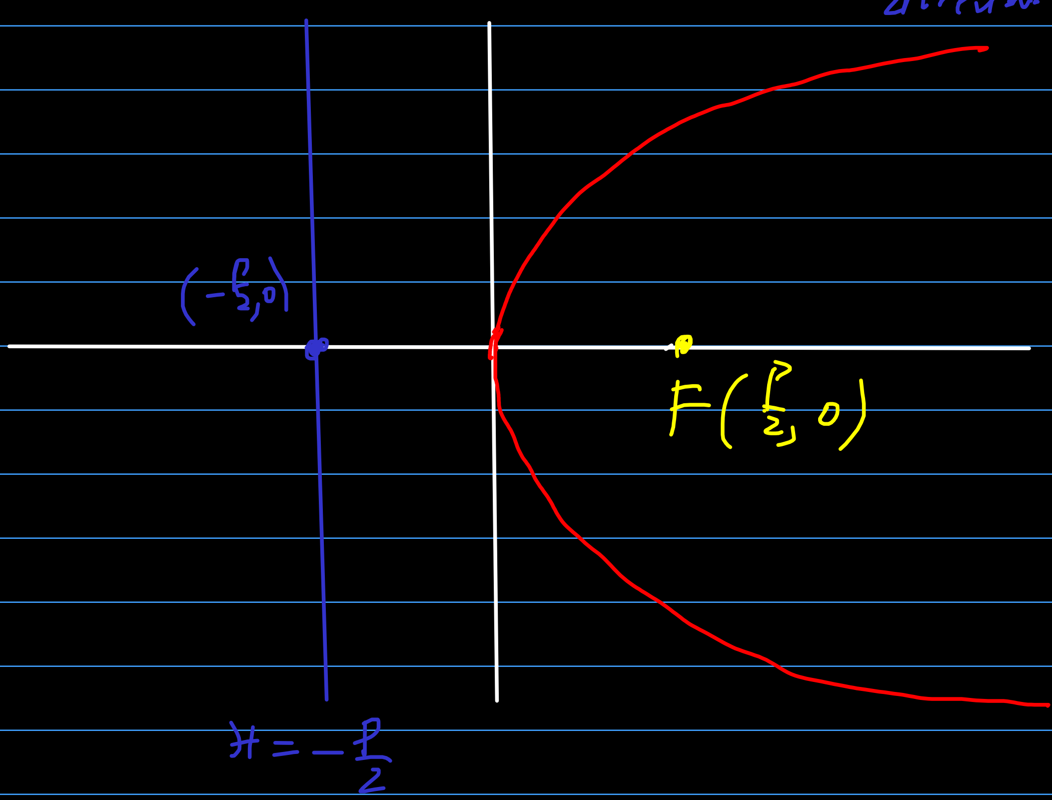
oblical asymptotes: $y = \pm \frac{b}{a} x$

$\mathcal{H}: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \iff y = \pm b\sqrt{\frac{x^2}{a^2} - 1}$

"the bad"

<u>Parabola</u>: $\mathcal{P}: y^2 = 2px$

→ locus of points equidistant to a point P (called the focus) and a line g (called the directrix or directrix line)



$$T_P(x_0, y_0): \quad y y_0 = p(x + x_0)$$

"the ugly"

9.2. Find the equations of the tangent lines to the ellipse

$$\mathcal{E}: x^2 + 4y^2 - 20 = 0$$

which are orthogonal to the line

$$\ell: 2x - 2y - 13 = 0$$

Proof: Let d be a tangent line that satisfies the condition

$$m_\ell = 1 \Rightarrow m_d = -1 \Rightarrow d: y = -x + c$$

$$d \cap \mathcal{E}: \begin{cases} y = -x + c \\ x^2 + 4y^2 - 20 = 0 \end{cases} \Leftrightarrow \begin{cases} y = -x + c \\ x^2 + 4(-x+c)^2 - 20 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = -x+c \\ x^2+4x^2-8xc+4c^2-20=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = -x+c \\ 5x^2-8xc+4c^2-20=0 \end{cases}$$

d tangent to $\mathcal{C} \Leftrightarrow |d \cap \mathcal{C}| = 1 \Leftrightarrow$

$\Leftrightarrow$ $5x^2-8xc+4c^2-20=0$

has a unique solution $\Leftrightarrow$

$\Leftrightarrow$ $\Delta = 0$

$$\Delta = 64c^2 - 20\cdot(4c^2-20) =$$

$$= 64c^2 - 80c^2+400 =$$

$$= 400 - 16c^2$$

$\Rightarrow d: y=-x+c$ tangent iff $c^2 = \frac{400}{16}$

so iff $c = \pm 5$

$\Rightarrow$ The tangents to the curve that are orthogonal to ℓ are $y = -x+5$ and $y = -x-5$

1.8. Find the equation of the tangent line to the parabola $\mathcal{P}: y^2 - 8x = 0$, that is parallel to $d: 2x+2y-3 = 0$.

We write the tangent to $\mathcal{P}$ in the point (x_0, y_0):

$$T_{\mathcal{P}}(x_0, y_0): \; y y_0 = 4(x+x_0)$$

If $y_0 \neq 0$:

$$\left.\begin{array}{l} m_{T_{\mathcal{P}}(x_0,y_0)} = \dfrac{4}{y_0} \\ \\ m_d = -1 \end{array}\right\} \begin{array}{l} T_{\mathcal{P}}(x_0,y_0) \parallel d \\ \Rightarrow \quad y_0 = -4 \end{array}$$

Because $(x_0, y_0) \in P$;

$$y_0^2 = 8x_0$$

$$\Rightarrow 16 = 8x_0 \Rightarrow x_0 = 2$$

$$\Rightarrow T_P(2, -4) : -4y = 4(x+2) \Rightarrow$$

$$\Leftrightarrow x + y + 2 = 0$$

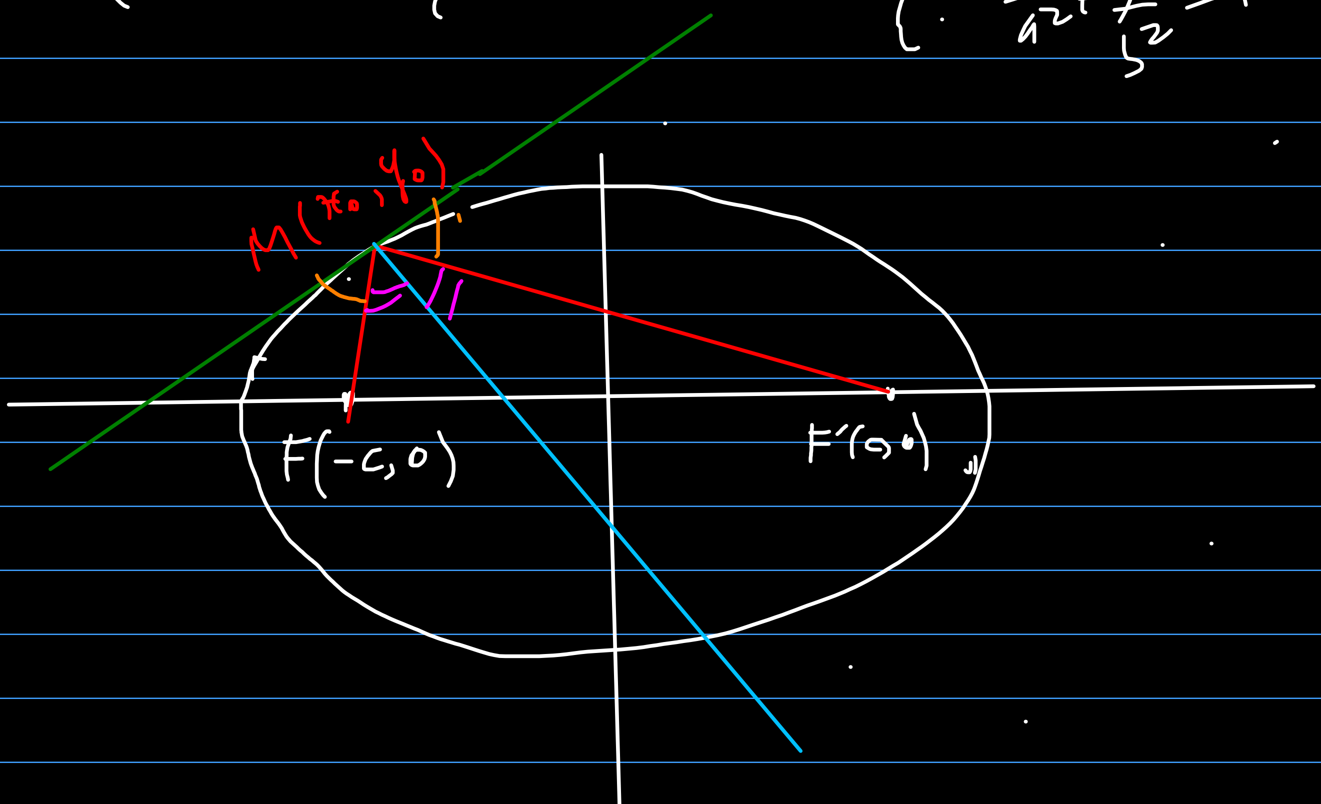
2/ $y_0 = 0 \Rightarrow x_0 = 0$

$$\Rightarrow T_P(0,0) : x = 0$$

$$m_{T_P(0,0)} = \infty \Rightarrow T_P(0,0) \not\parallel d$$

9.12. Show that a ray of light through a focus of an ellipse reflects to a ray that passes through the other focus

$\mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



What we need to show is that for every $M(x_0, y_0) \in \mathcal{E}$:

$N_{\mathcal{E}}(x_0, y_0)$ is the bisector of the angle $\widehat{FMF'}$

$$N_{\mathcal{E}}(x_0,y_0):\quad \frac{x-x_0}{f'_x(x_0,y_0)}=\frac{y-y_0}{f'_y(x_0,y_0)}.$$

$$f(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$$

$$f'_x(x_0,y_0)=\frac{2x_0}{a^2}$$

$$f'_y(x_0,y_0)=\frac{2y_0}{b^2}$$

$$N_{\mathcal{E}}(x_0,y_0):\quad \begin{cases} x=x_0+\lambda\cdot\frac{2x_0}{a^2}\\ y=y_0+\lambda\cdot\frac{2y_0}{b^2}\end{cases}$$

$$\forall\lambda\in\mathbb{R}$$

$M(x_0,y_0)$, $F(-c,0)$

$$MF:\quad \frac{x+c}{x_0+c}=\frac{y}{y_0}\iff$$

$$\iff xy_0-y(x_0+c)+cy_0=0$$

Let $T(x,y) \in N_{\ell}(x_0, y_0)$

$$\left[\begin{array}{l} \ell : ax+by+c=0, \quad M(x_0, y_0) \\ \mathrm{dist}(M, \ell) = \dfrac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}} \end{array} \right]$$

$$\mathrm{dist}(T, MF) = \frac{|x y_0 - y(x_0+c) + c y_0|}{\sqrt{y_0^2 + (x_0+c)^2}} =$$

$$= \frac{\left| \left(x_0 + \lambda \cdot \frac{2x_0}{a^2}\right) y_0 - (x_0+c) \cdot \left(y_0 + \lambda \cdot \frac{2y_0}{b^2}\right) + c y_0 \right|}{\sqrt{y_0^2 + (x_0+c)^2}}$$

$$= \frac{\left| x_0 y_0 + \lambda \cdot \frac{2x_0 y_0}{a^2} - x_0 y_0 - \lambda \frac{2x_0 y_0}{b^2} - c y_0 - \frac{2\lambda c y_0}{b^2} + c y_0 \right|}{\sqrt{y_0^2 + (x_0+c)^2}}$$

$$= \frac{|2\lambda| \cdot \left| \frac{x_0 y_0}{a^2} - \frac{x_0 y_0}{b^2} - \frac{c y_0}{b^2} \right|}{\sqrt{y_0^2 + (x_0 + c)^2}} =$$

$$= \frac{|2\lambda| \cdot \left| x_0 y_0 \cdot \frac{\overbrace{b^2 - a^2}^{-c^2}}{a^2 b^2} - \frac{c y_0}{b^2} \right|}{\sqrt{y_0^2 + (x_0 + c)^2}} =$$

$$= \frac{|2\lambda| \cdot |c| \cdot \left| \frac{x_0 y_0 c}{a^2 b^2} - \frac{y_0}{b^2} \right|}{} =$$

...

$$= \frac{|2\lambda| \cdot |c| \cdot \left| \frac{y_0}{b^2} \right| \cdot \left| \frac{x_0 c}{a^2} - 1 \right|}{\sqrt{y_0^2 + (x_0 + c)^2}}$$

$$\text{dist}(T, MF') = \frac{|2\lambda| \cdot |c| \cdot \left|\frac{y_0}{b^2}\right| \left|\frac{-x_0 c}{a^2} - 1\right|}{\sqrt{y_0^2 + (x_0 - c)^2}}$$

We still need to prove:

$$\frac{\left(\frac{x_0 c}{a^2} - 1\right)^2}{y_0^2 + (x_0 + c)^2} = \frac{\left(\frac{x_0 c}{a^2} + 1\right)^2}{y_0^2 + (x_0 - c)^2}$$

we will use:

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1 \quad \text{and} \quad c = \sqrt{a^2 - b^2}$$