Seminar W7 – 917

The mixed product
(the triple scalar product)

$\vec{a}, \vec{b}, \vec{c} \in \mathcal{U}$

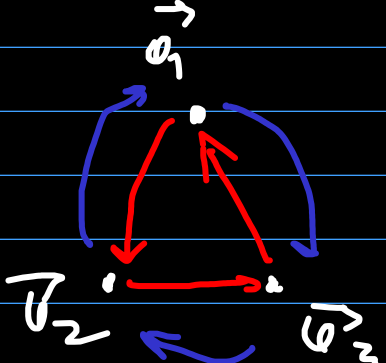
$$(\vec{a}, \vec{b}, \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

If the reference system is orthonormed and direct, then:

$\vec{u}_1(a_1, b_1, c_1)$, $\vec{u}_2(a_2, b_2, c_2)$

$\vec{u}_3(a_3, b_3, c_3)$

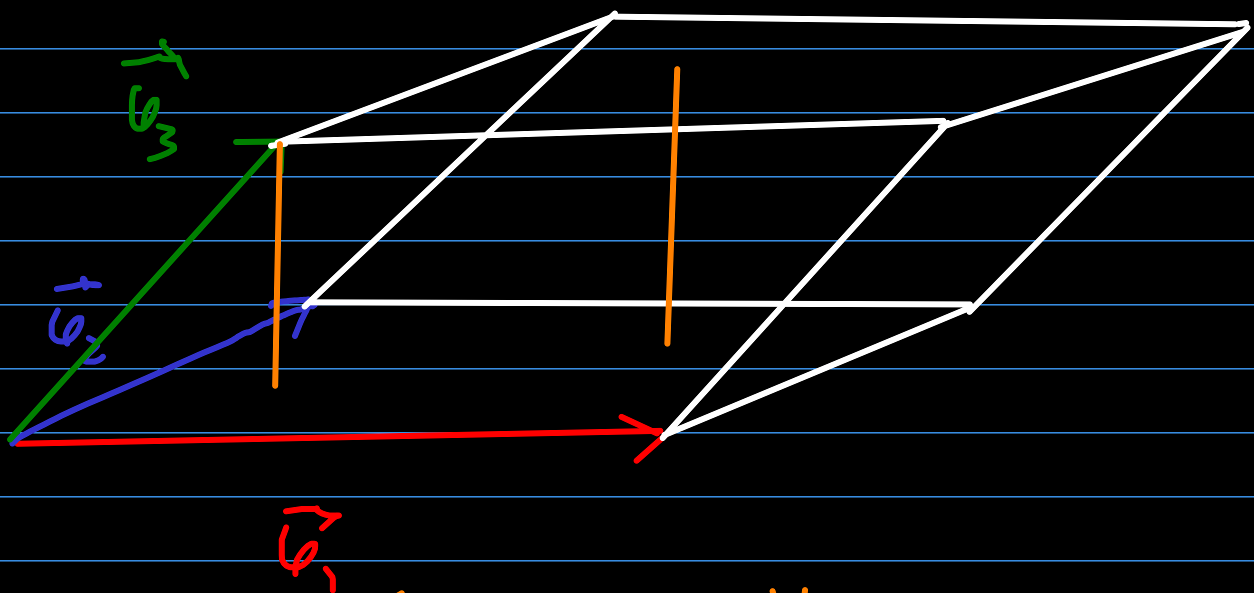
$$(\vec{u}_1, \vec{u}_2, \vec{u}_3) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$



$$(\vec{u}_1, \vec{u}_2, \vec{u}_3) = (\vec{u}_2, \vec{u}_3, \vec{u}_1) = (\vec{u}_3, \vec{u}_1, \vec{u}_2) =$$

$$= -(\vec{u}_1, \vec{u}_3, \vec{u}_2) = -(\vec{u}_2, \vec{u}_1, \vec{u}_3) = -(\vec{u}_3, \vec{u}_2, \vec{u}_1)$$

$|(\vec{u}_1, \vec{u}_2, \vec{u}_3)|$ = Volume of the parallelepiped given by $\vec{u}_1, \vec{u}_2, \vec{u}_3$



$$\text{height} = \frac{|(\vec{u}_1, \vec{u}_2, \vec{u}_3)|}{||\vec{u}_1 \times \vec{u}_2||}$$

The distance between two lines and the common perpendicular

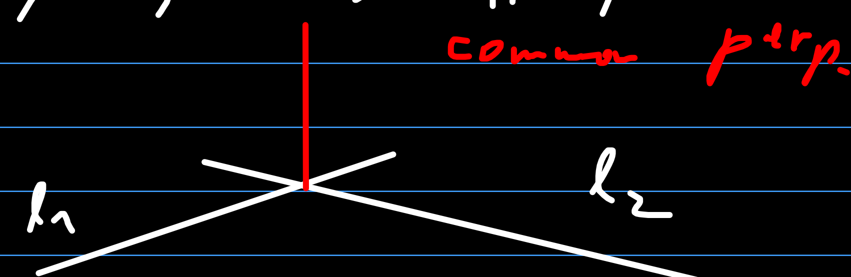
l_1, l_2 lines

common perp. = a line l that is perpendicular to l_1 and l_2 and intersects them

$$\mathrm{dist}(l_1, l_2) = \min_{\substack{M_1 \in l_1 \\ M_2 \in l_2}} (M_1 M_2)$$

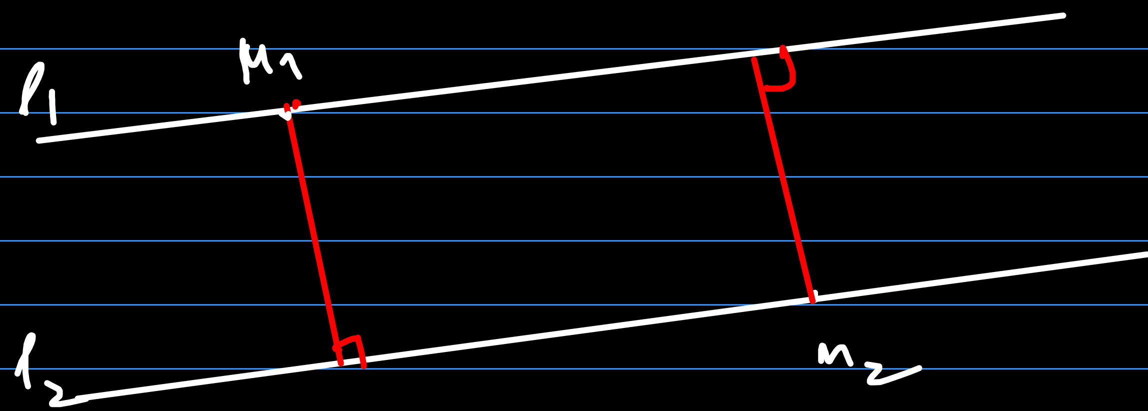
$\hookrightarrow$ the length of the common perpendicular

- $l_1 \cap l_2 \neq \emptyset$, $\mathrm{dist}(l_1, l_2) = 0$



- $l_1 \cap l_2 \neq \emptyset$, $l_1 \parallel l_2$

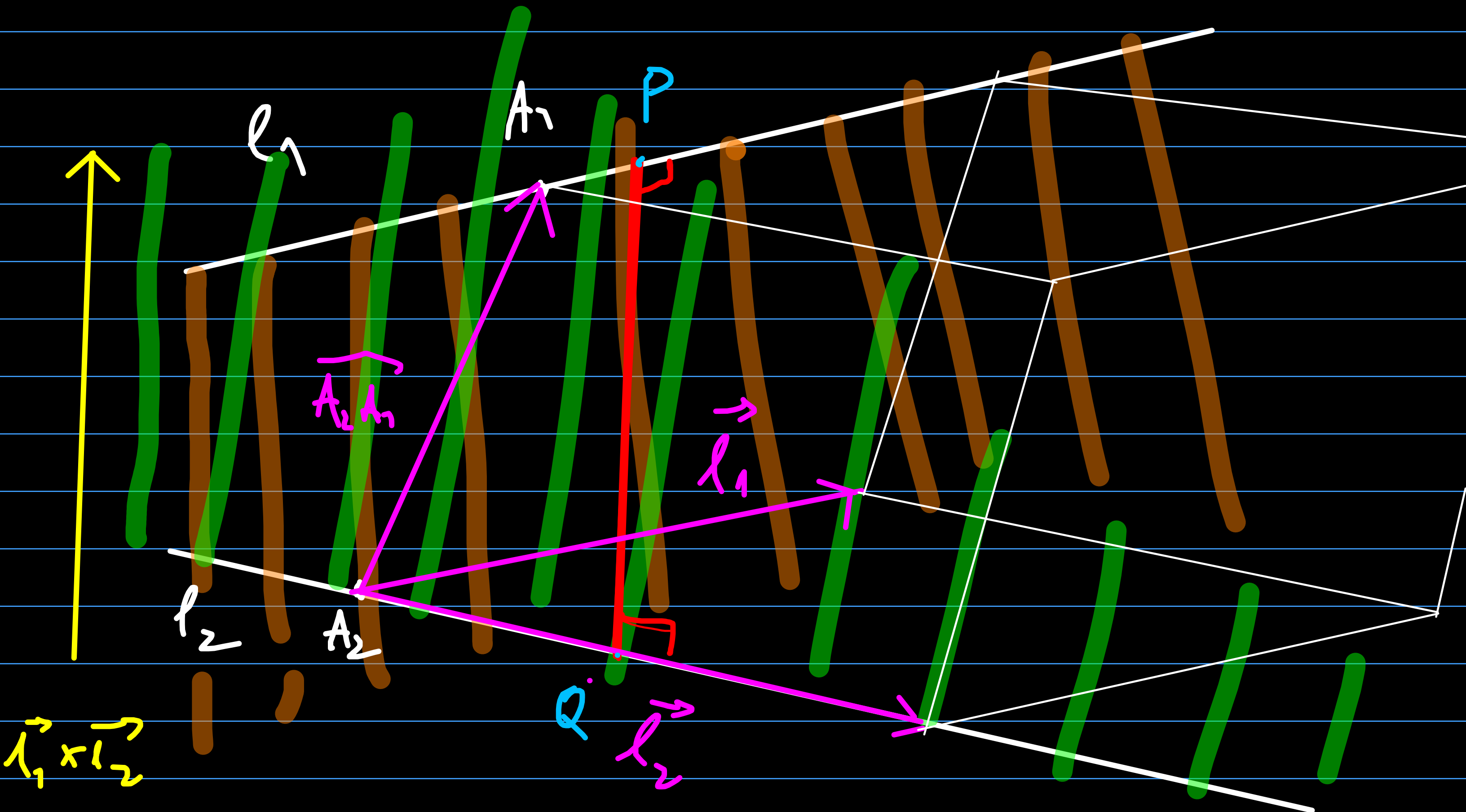
$dist(l_1, l_2) = dist(M_1, l_2) = dist(M_2, l_1)$
$\forall M_1 \in l_1$, $\forall M_2 \in l_2$



common perp. = any perpendicular from a point on one line onto the other.

- $l_1 \cap l_2 = \emptyset$, $l_1 \nparallel l_2$

$\rightarrow$ l_1 and l_2 are skew (or noncoplanar)



We pick random points

$A_1 \in \ell_1$, $A_2 \in \ell_2$

Π_1 = the plane that contains ℓ_1 and is parallel with $\vec{\ell_1} \times \vec{\ell_2}$

Π_2 = the plane that contains ℓ_2 and is parallel with $\vec{\ell_1} \times \vec{\ell_2}$

the common perpendicular $= \Pi_1 \cap \Pi_2$

$\rightarrow$ the reason why $\Pi_1 \nparallel \Pi_2$:

we would have $\vec{n}_{\Pi_1} \perp \vec{l}_1, \vec{l}_2, \vec{l}_1 \times \vec{l}_2$

(which is impossible)

$\text{dist}(l_1, l_2) = PQ =$ height in the parallelepiped $=$

$$= \frac{|(\overrightarrow{A_1A_2}, \vec{l}_1, \vec{l}_2)|}{||\vec{l}_1 \times \vec{l}_2||}.$$

7.8. Find the distance between the lines M_1M_2 and ℓ, where $M_1(-1,0,1)$, $M_2(-2,1,0)$ and

$$\ell : \begin{cases} x+y+z=1 \\ 2x-y-5z=0 \end{cases}$$

as well as the equations of the common perpendicular.

$$\ell_2 := M_1M_2 \quad , \quad \ell_1 := \ell$$

$$\ell_1 : \begin{cases} x+y+z=1 \\ 2x-y-5z=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = 2x-5z \\ x+2x-5z+z=1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = 2x-5z \\ 3x-4z-1=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = \dfrac{1+4z}{3} \\ y = 2 \cdot \dfrac{1+4z}{3} - 5z = \\ \quad = -\dfrac{7}{3} z + \dfrac{2}{3} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \dfrac{4}{3}\alpha + \dfrac{1}{3} \\ y = -\dfrac{7}{3}\alpha + \dfrac{2}{3} \\ z = \alpha \end{cases}$$

$$\Rightarrow \quad \vec{l_1}\,(4, -7, 3).$$

$$l_2 = M_1M_2 : \quad \frac{x+1}{-1} = \frac{y}{1} = \frac{z-1}{-1}$$

$$\Rightarrow \vec{l_2}\,(-1, 1, -1)$$

$$\vec{l_1} \parallel \vec{l_2} \Rightarrow l_1 \parallel l_2$$

We pick $A_1 \in l_1$ by plugging in $\alpha = 2$: $A_1(3, -4, 2)$

We pick $A_2 \in l_2$, $A_2 := M_2$

$$\overrightarrow{A_1A_2} = \vec{r}_{A_2} - \vec{r}_{A_1} = (-2, 1, 0) - (3, -4, 2)$$
$$= (-5, 5, -2)$$

$$dist(l_1, l_2) = \frac{|(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2})|}{\|\vec{l_1} \times \vec{l_2}\|}$$

$$(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}) = \begin{vmatrix} -5 & 5 & -2 \\ 4 & -7 & 3 \\ -1 & 1 & -1 \end{vmatrix} =$$

$$= -35 - 15 - 8 + 14 + 15 + 20 =$$
$$= -9$$

$$\vec{l_1} \times \vec{l_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -7 & 3 \\ -1 & 1 & -1 \end{vmatrix} =$$

$= 7\vec{i} - 3\vec{j} + 4\vec{k} - 7\vec{k} - 3\vec{i} + 4\vec{j} =$

$= 4\vec{i} + \vec{j} - 3\vec{k}$

$\Rightarrow \| \vec{l_1} \times \vec{l_2} \| = \sqrt{16 + 1 + 9} = \sqrt{26}$

$$dist(l_1, l_2) = \frac{9}{\sqrt{26}}.$$

Π_1 = plane that contains l_1 and is parallel to $\vec{l_1} \times \vec{l_2}$

$A_1(3, -4, 2)$, $\vec{l_1}(4, -7, 3)$

$\vec{l_1} \times \vec{l_2}(4, 1, -3)$

$$\Pi_1 : \begin{vmatrix} x-3 & y+4 & z-2 \\ 4 & -7 & 3 \\ 4 & 1 & -3 \end{vmatrix} = 0$$

$\Pi_1 : 18(x-3) + 24(y+4) + 28(z-2) = 0$

$\Pi_1: \; 9(x-3)+12(y+4)+14(z-1)=0$

$\Pi_1: \; 9x+12y+14z-9=0$

Π_2 = plane that contains l_2 and is parallel to $\vec{l_1}\times\vec{l_2}$

$A_2(-1,0,1)$, $\vec{l_2}(1,-1,1)$

$$\Pi_2: \begin{vmatrix} x+1 & y & z-1 \\ 1 & -1 & 1 \\ 4 & 1 & -3 \end{vmatrix} = 0$$

$\Rightarrow \Pi_2: \; 2(x+1)+7y+5(z-1)=0$

$\Rightarrow \Pi_2: \; 2x+7y+5z-3=0$

$$\Rightarrow l \doteq \Pi_1 \cap \Pi_2 : \begin{cases} 9x+12y+14z-9=0 \\ 2x+7y+5z-3=0 \end{cases}$$

The coplanarity condition

l_1, l_2 lines, $A_1 \in l_1$, $A_2 \in l_2$

l_1, l_2 coplanar $\iff \overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}$ linearly dependent

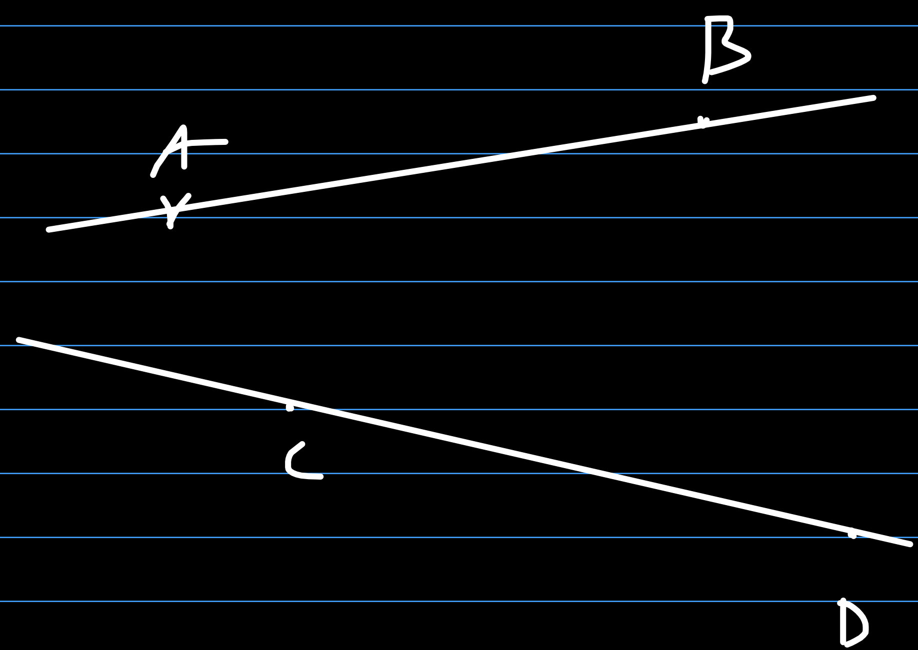
$\iff (\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}) = 0$

7-5. Find the value of the parameter α for which the pencil of planes through the line AB has a common plane with the pencil of planes through CD, where:

$A(1, 2\alpha, \alpha)$, $B(3, 2, 1)$, $C(-\alpha, 0, \alpha)$, $D(-1, 3, -1)$

$\overrightarrow{AB}(2, 2-2\alpha, 1-\alpha)$, $\overrightarrow{CD}(-1+\alpha, 3, -1-\alpha)$

$\overrightarrow{BD}(-4, 1, -2)$



We choose $B \in AB$, $D \in CD$

We write:

$$(\overrightarrow{BD}, \overrightarrow{AB}, \overrightarrow{CD}) = \begin{vmatrix} -4 & 1 & 2 \\ 2 & 2-2\alpha & 1-\alpha \\ \alpha-1 & 3 & -\alpha-1 \end{vmatrix} =$$

$$= (-4)\cdot \begin{vmatrix} 2-2\alpha & 1-\alpha \\ 3 & -\alpha-1 \end{vmatrix} -$$

$$- \begin{vmatrix} 2 & 1-\alpha \\ \alpha-1 & -\alpha-1 \end{vmatrix} + 2\cdot \begin{vmatrix} 2 & 2-2\alpha \\ \alpha-1 & 3 \end{vmatrix} =$$

$$= (-4)\cdot\left(-2(1-\alpha)(\alpha+1) - 3 + 3\alpha\right)$$

$$- \left(-2\alpha - 2 + \alpha^2 - 2\alpha + 1\right) +$$

$$+2\cdot(6+2\alpha^2-4\alpha+2) =$$

$$= -4(2\alpha^2-2-3+3\alpha) -$$

$$-(\alpha^2-4\alpha-1)+2(2\alpha^2-4\alpha+4)$$

$$= -5\alpha^2-16\alpha+37$$

$$\Rightarrow \alpha_{1,2} = \frac{16 \pm \sqrt{256+740}}{-10} =$$

$$= \frac{16 \pm \sqrt{1496}}{-10}$$

7.1. (a) $|(\vec{a},\vec{b},\vec{c})| \leq \|\vec{a}\|\cdot\|\vec{b}\|\|\vec{c}\|$

(b) $(\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}) =$

$$= 2(\vec{a},\vec{b},\vec{c})$$

(a) $|(\vec{a},\vec{b},\vec{c})| = \left|\|\vec{a}\|\cdot\|\vec{b}\times\vec{c}\|\cdot\cos(\vec{a},\vec{b}\times\vec{c})\right|$

$$= \left|\|\vec{a}\|\cdot\|\vec{b}\|\cdot\|\vec{c}\|\cdot\sin(\widehat{\vec{b},\vec{c}})\cdot\cos(\widehat{\vec{a},\vec{b}\times\vec{c}})\right|$$

$$= \|\vec{a}\|\cdot\|\vec{b}\|\|\vec{c}\|\cdot\underbrace{|\sin(\widehat{\vec{b},\vec{c}})|}_{\leq 1}\cdot\underbrace{|\cos(\vec{a},\vec{b}\times\vec{c})|}_{\leq 1}$$

$$\leq \|\vec{a}\|\cdot\|\vec{b}\|\|\vec{c}\|$$