Seminar W1 - 915

02 December 2020 12:14

$CC = \max\left(M, S + \frac{M}{2}\right)$

$FG = \max\left(\frac{4}{10} CC + \frac{6}{10} E, E\right)$

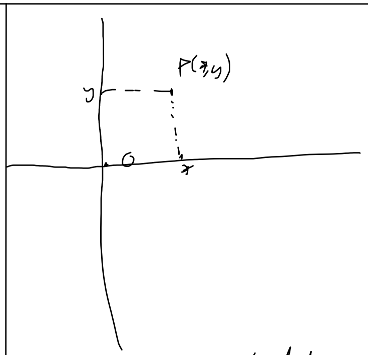
Minimal requirements:

↳ $CC \geq 4.5$, $E \geq 4.5$

$S = 2p/3p$

On Analytic Geometry

↳ "we represent points in the plane/the space as vectors of coordinates" → Descartes



Say E is the Euclidean plane (resp. space)

A Cartesian reference system is a pair $(O; b)$

O ↓ point "origin"

b ↳ basis of $\vec{E}$ ↗ the vectors on $\vec{E}$

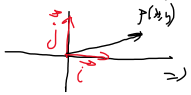
$P \in E$

$[P]_{\mathcal{R}} = [\vec{OP}]_b$

$\vec{OP} = \vec{r}_P$ ↓ the position vector of P with respect to $\mathcal{R}$.

In the most of the encountered case: $E = \mathbb{R}^2$, $\vec{E} = \mathbb{R}^2$

$\Rightarrow b = (\vec{i}, \vec{j})$



$[P]_{\mathcal{R}} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \vec{r}_P = x \cdot \vec{i} + y \cdot \vec{j}$

1.1. Consider a tetrahedron $ABCD$. Find:

(a) $\vec{AB} + \vec{BC} + \vec{CD}$

(b) $\vec{AD} + \vec{CB} + \vec{DC}$

(c) $\vec{AB} + \vec{BC} + \vec{DA} + \vec{CD}$

(a) $\vec{AB} + \vec{BC} = \vec{AC}$

$\vec{AC} + \vec{CD} = \vec{AD}$

$\Rightarrow \vec{AB} + \vec{BC} + \vec{CD} = \vec{AD}$

(b) $\vec{AD} + \vec{CB} + \vec{DC} = (\vec{AD} + \vec{DC}) + \vec{CB} =$

$= \vec{AC} + \vec{CB} = \vec{AB}$

(c) $\underbrace{\vec{AB} + \vec{BC}}_{\vec{AC}} + \vec{DA} + \vec{CD} = \vec{AC} + \vec{DA} + \vec{CD} =$

$= \vec{AD} + \vec{DA} = \vec{0}$

$\vec{AB} = \vec{AO} + \vec{OB} = \vec{r}_B - \vec{r}_A$

· O

M midpoint of $[AB]$

$\vec{r}_M = \frac{\vec{r}_A + \vec{r}_B}{2}$

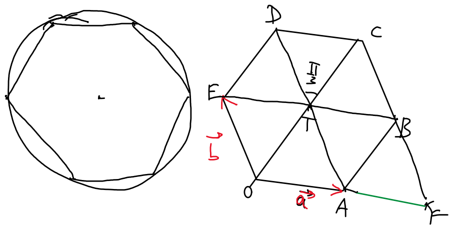
$\frac{AM}{MB} = k \in \mathbb{R}$

$\vec{r}_M = \frac{\vec{r}_A + k\vec{r}_B}{k+1}$

1.2. $OABCDE$ regular hexagon

$\vec{OA} = \vec{a}$, $\vec{OE} = \vec{b}$

Express the vectors $\vec{OB}, \vec{OC}, \vec{OD}$ in terms of $\vec{a}$ and $\vec{b}$. Show that $\vec{OC} + \vec{OD} + \vec{DE} = 3\vec{OC}$



Say T is the intersection point of the diagonals

$\Delta OAT \equiv \Delta TCD \Rightarrow OA \underline{\parallel} DC$

$\Rightarrow \vec{OA} = \vec{DC}$

Let $F \in OA$ s.t. A is the midpoint of OF $\Rightarrow$ TAFB parallelogram

OEBF parallelogram

$\Rightarrow OE \underline{\parallel} FB \Rightarrow \vec{FB} = \vec{b}$

$\vec{OC} = \vec{OF} + \vec{FC}$

DCFA parall. $\Rightarrow \vec{FC} = \vec{AD}$

$\vec{AT} = \vec{OE} = \frac{1}{2}\vec{AD} \Rightarrow \vec{AD} = 2\vec{OE} = 2\vec{b}$

$\Rightarrow \vec{FC} = 2\vec{b} \Rightarrow \vec{OC} = 2\vec{a} + 2\vec{b}$

$\vec{OB} = \vec{OA} + \vec{AF} + \vec{FB} = \vec{a} + \vec{a} + \vec{b} = 2\vec{a} + \vec{b}$

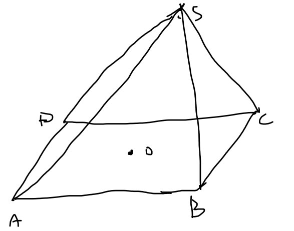
$\vec{OD} = \vec{OC} - \vec{DC} = 2\vec{a} + 2\vec{b} - \vec{a} = \vec{a} + 2\vec{b}$

$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} + \vec{OE} = \vec{a} + 2\vec{a} + \vec{b} + 2\vec{a} + 2\vec{b} + \vec{a} + 2\vec{b} + \vec{b} = 6\vec{a} + 6\vec{b} =$

$= 3 \cdot (2\vec{a} + 2\vec{b}) = 3 \cdot \vec{OC}$

1.3. Consider a pyramid with the vertex in S and the basis a parallelogram ABCD, whose diagonals are concurrent in a point O.

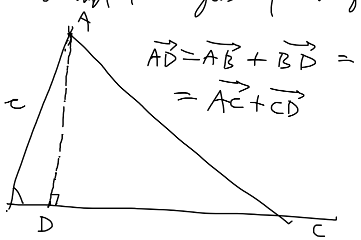
Show the equality $\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4\vec{SO}$



$\vec{SO} = \frac{\vec{SA} + \vec{SC}}{2} = \frac{\vec{SB} + \vec{SD}}{2}$

$\Rightarrow 2\vec{SO} = \frac{\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD}}{2}$

1.5. In a ΔABC we consider the height AD from the vertex A ($D \in BC$). Find the decomposition of the vector $\vec{AD}$ in terms of the vectors $\vec{c} = \vec{AB}$, $\vec{b} = \vec{AC}$ and the angles of the triangle



$\vec{AD} = \vec{AB} + \vec{BD} = \vec{AC} + \vec{CD}$

Let a, b, c be the sides BC, CA, AB

$2\vec{AD} = \vec{AB} + \vec{AC} + \vec{BD} + \vec{CD} =$

$= \vec{c} + \vec{b} + \vec{BD} + \vec{CD}$

In the right triangle ABD:

$BD = c \cos \hat{B}$

$\vec{BD} = \vec{BC} \cdot \frac{\|\vec{BD}\|}{\|\vec{BC}\|} = \vec{BC} \cdot \frac{c \cos \hat{B}}{a} =$

$= (\vec{BA} + \vec{AC}) \cdot \frac{c \cos \hat{B}}{a} =$

$= (\vec{b} - \vec{c}) \cdot \frac{c \cos \hat{B}}{a}$

$\Rightarrow \vec{AD} = \vec{c} + (\vec{b} - \vec{c}) \cdot \frac{c \cdot \cos \hat{B}}{a}$

$\Rightarrow b^2 = c^2 + a^2 - 2ac \cos \hat{B}$

$\Rightarrow \cos \hat{B} = \frac{c^2 + a^2 - b^2}{2ac}$

$\vec{AD} = \frac{c}{a} \cdot \frac{b^2 - a^2 - c^2}{2ac} \cdot \vec{b} + \left(1 - \frac{c}{a} \cdot \frac{b^2 - a^2 - c^2}{2ac}\right) \cdot \vec{c}$

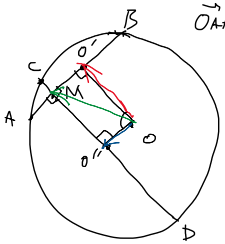
$\vec{AD} = \frac{b^2 - a^2 - c^2}{2a^2} \cdot \vec{b} + \frac{3a^2 + c^2 - b^2}{2a^2} \cdot \vec{c}$

1.7. (3p) Consider two perpendicular chords AB and CD of a given circle and

$\{M\} = AB \cap CD$

Show that $\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 2\vec{OM}$

(O is the centre of the circle)



$\vec{OA}+\vec{OB}+\vec{OC}+\vec{OD} \overset{?}{=} 2\vec{OM}$

The sum of the angles of a convex n-gon is $\pi(n-2)$

$\vec{OA}+\vec{OB}+\vec{OC}+\vec{OD} = (\vec{OA}+\vec{OB}) + (\vec{OC}+\vec{OD})$

Let O' be the midpoint of [AB] and O'' the midpoint of [CD]

$\left.\begin{array}{l}\Delta AOB \text{ isosceles} \Rightarrow OO' \perp AB \\ \Delta COD \text{ isosceles} \Rightarrow OO'' \perp CD\end{array}\right\} \Rightarrow m(\widehat{O'OO''}) = \frac{\pi}{2}$

$\Rightarrow MO'OO''$ rectangle $\Rightarrow \vec{OM} = \vec{OO'} + \vec{OO''}$

O' midpoint of [AB] $\Rightarrow \vec{OO'} = \frac{\vec{OA}+\vec{OB}}{2}$

O'' midpoint of [CD] $\Rightarrow \vec{OO''} = \frac{\vec{OC}+\vec{OD}}{2}$

$\Rightarrow 2\vec{OM} = \vec{OA}+\vec{OB}+\vec{OC}+\vec{OD}$