Seminar W12 - 9.1.6

Affine transformations (plane)

$y = mx + n$ affine function

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto mx + n$$

$$f(x_1 + x_2) \neq f(x_1) + f(x_2)$$

$y = mx$ linear function

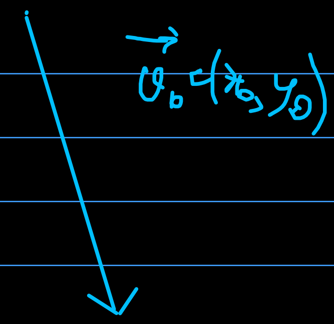
$\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transformation if

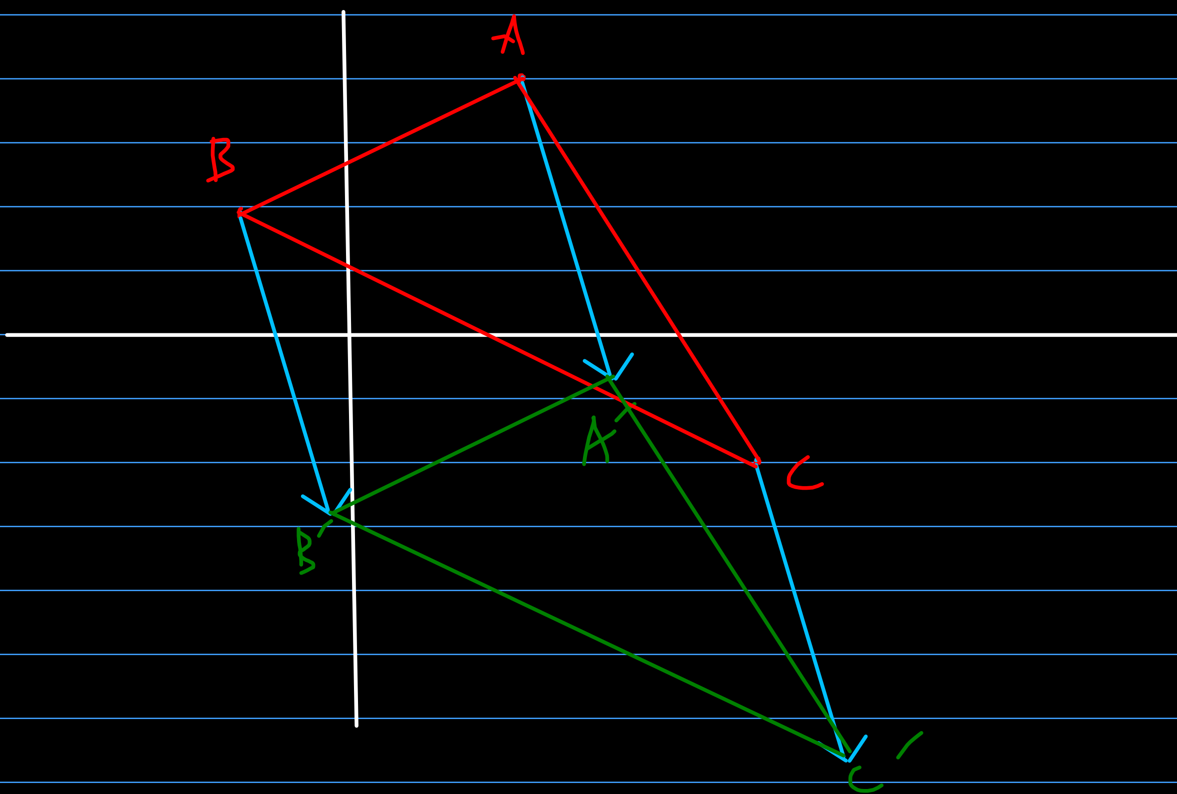
$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{M}_{\in M_{2,2}(\mathbb{R})} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}_{= \vec{v_0}}$$

$\to$ they preserve lines and parallelism

(but not necessarily distances and angles)

Translations : $T(x_0, y_0)$





$$T(x_0, y_0)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x + x_0 \\ y + y_0 \end{pmatrix}$$

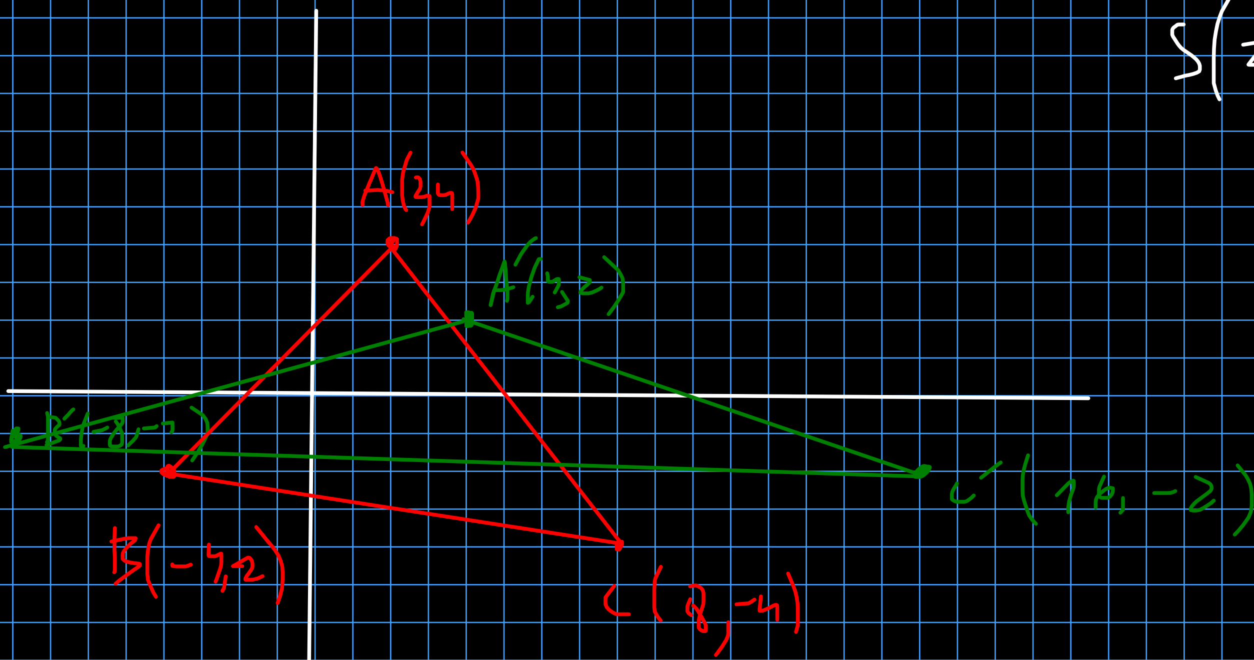
If we have $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ an affine transformation, how do we tell if φ is a translation?

- Pick a point A, let $A' = \varphi(A)$
- If φ is a translation, then $\varphi = T(\overrightarrow{AA'})$
- Check this against all other points B.

Scalings:

$$S(s_x, s_y)$$

$$S\left(2, \frac{1}{2}\right)$$



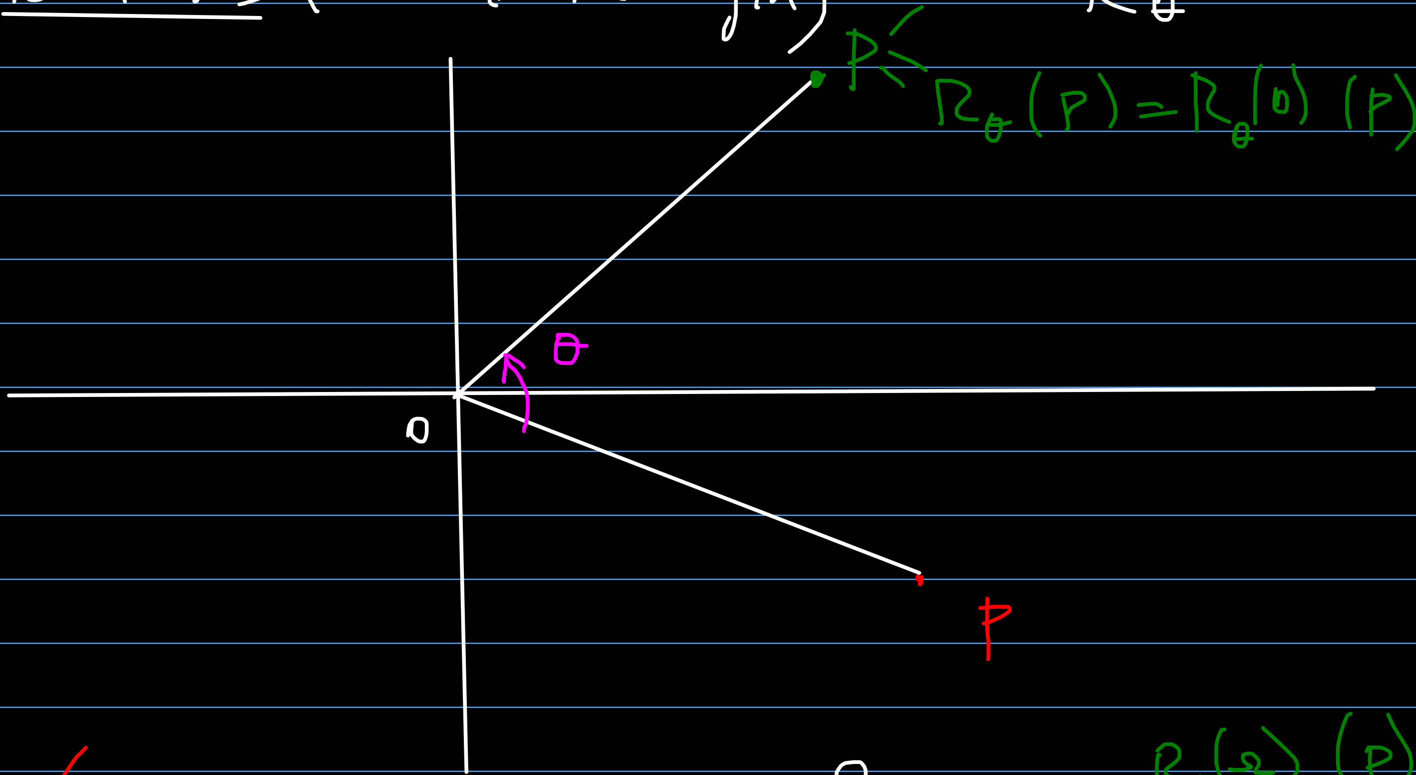
$$S(s_x,s_y)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}s_x & 0\\ 0 & s_y\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}s_x\\ s_y\end{pmatrix}$$

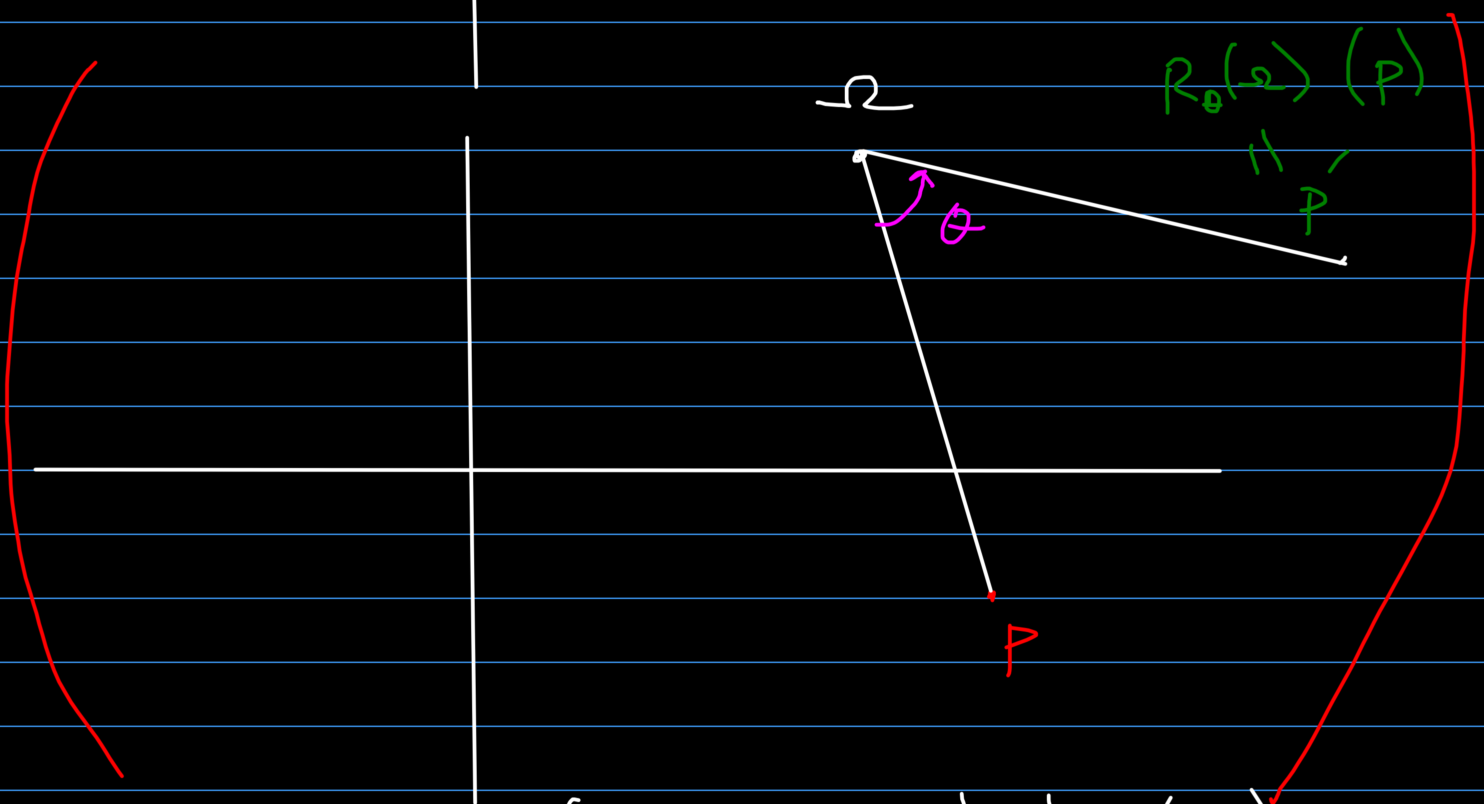
If $s_x = s_y \Rightarrow S(s_x, s_y)$ "homothety"

If we have $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ an affine transformation, how do we tell if φ is a scaling?

- Pick a point A, $A' = \varphi(A)$
- If φ is a scaling $\Rightarrow s_x = \frac{x_{A'}}{x_A}$ $s_y = \frac{y_{A'}}{y_A}$
- Check this against all other points.

Rotations (around the origin) $\quad R_\theta$





$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$f: A \to B$, $\mathrm{Fix}(f) = \{x \in A \mid f(x) = x\}$

If we have $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ an affine transformation, how do we tell if φ is a rotation?

- Check if $\mathrm{Fix}(\varphi) = \{P_0\}$
- If so, then $\varphi = R_\theta(P_0)$, we don't know θ

(- Check if $\forall P$: $P_0P = P_0\varphi(P)$)

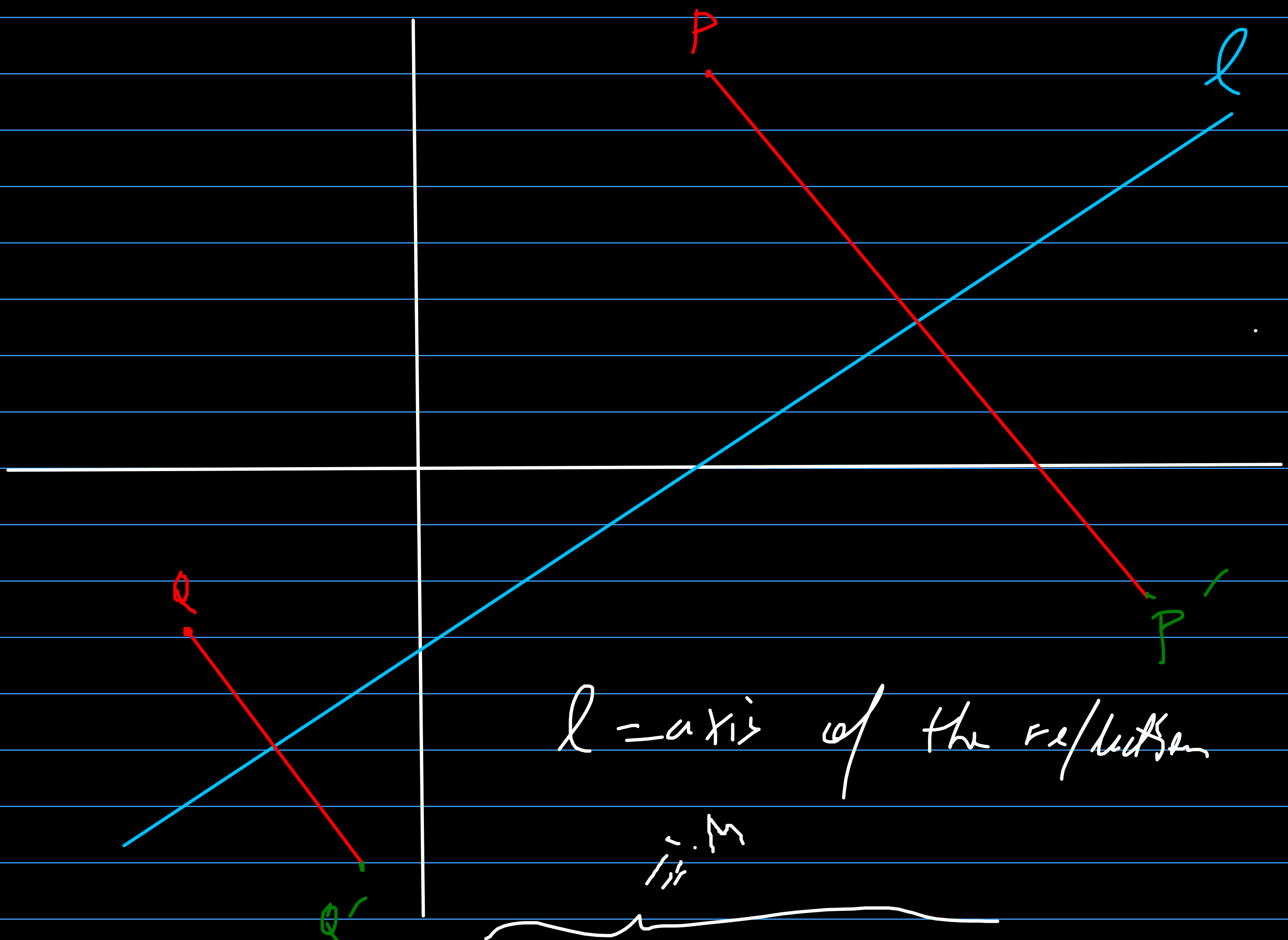
- Check if $\forall P$: $m(\widehat{PP_0P'})$ is the same

If so, $\theta = m(\widehat{PP_0P'})$

Reflections (orthogonal): r_ℓ

(w.r. to a line)

$\ell: \ ax+by+c=0$



ℓ = axis of the reflection

$$r_\ell \begin{pmatrix} x \\ y \end{pmatrix} = \overbrace{\begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} \\ \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{pmatrix}}^{M} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + $$

$$+ \begin{pmatrix} \frac{-2ac}{a^2+b^2} \\ -\frac{2bc}{a^2+b^2} \end{pmatrix} \longrightarrow \vec{b_0}$$

If $c=0$ (i.e. if $\ell \ni 0$), then

$$r_\ell \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} \\ \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

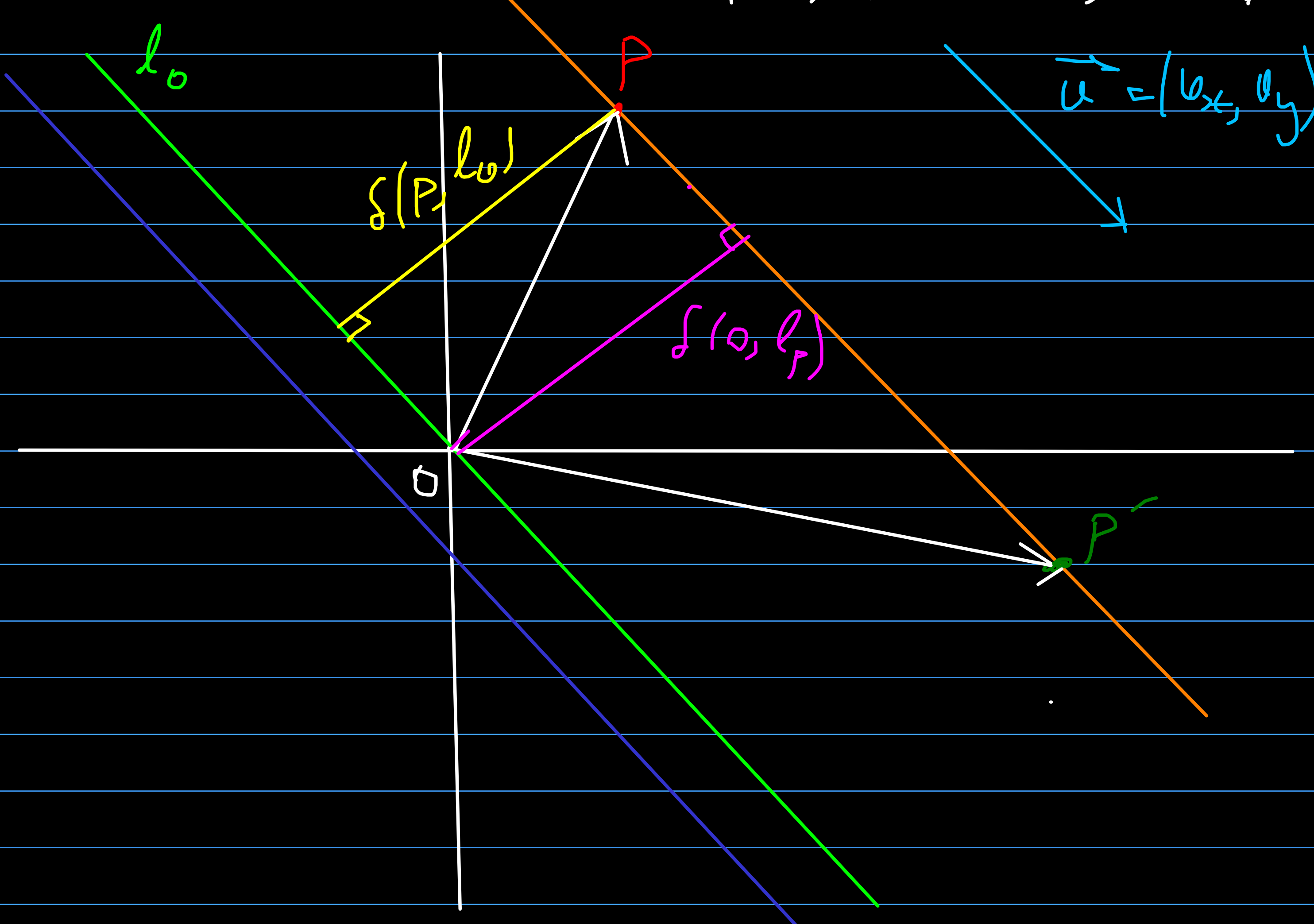
If we have $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ an affine transformation, how do we tell if φ is a reflection?

- Check that $Fix(\varphi) = \ell$, ℓ line
- Check that $\forall A$, $A' = \varphi(A)$, then ℓ is the perpendicular bisector of the segment AA'

Shearing

$$sh(\vec{u}, r)$$

$\vec{u} \in \mathbb{R}^2$, $\|\vec{u}\| = 1$, $r \in \mathbb{R}$



$$sh(\vec{u}, r)(P) = sh(\vec{u}, r)\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} + r \cdot \delta(O, l_M) \cdot \vec{u} = P'$$

l_M = line through M with direction $\vec{u}$

$\ell: ax+by+c=0$

$P(x_P, y_P)$

$$\delta(P,\ell) = \frac{ax_P + by_P + c}{\sqrt{a^2+b^2}}$$

$\hookrightarrow$ oriented distance!

$$sh(\vec{u},r)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - r \cdot \delta(P,\ell_0) \cdot \vec{u}$$

$$sh(\vec{u},r)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - r u_x u_y & r u_x^2 \\ -r u_y^2 & 1 + r u_x u_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

If we have $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ an affine transformation, how do we tell if φ is a shear?

- Check if $Fix(\varphi) = \ell$

Check if φ is a reflection

Check if $\forall A$, $A' = \varphi(A)$ $\overrightarrow{AA'} \parallel \ell$

- If we know that φ is a shearing with axis ℓ, we choose $\vec{u} = \frac{1}{\|\vec{\ell}\|} \cdot \vec{\ell}$
- Find r using the definition

11.1. Find the image of the triangle ABC through r_d.

$d: x - y = 2$

$A(-1, 2)$, $B(-2, -1)$, $C(3, 3)$

$d: x - y - 2 = 0$

$$r_d\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \cdot \begin{pmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$+ \frac{-2c}{a^2+b^2} \begin{pmatrix} a \\ b \end{pmatrix}$$

Here $a = 1$, $b = -1$, $c = -2$.

$$r_d\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \frac{4}{2} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$r_d\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}2\\-2\end{pmatrix} =$$

$$= \begin{pmatrix}y+2\\x-2\end{pmatrix}$$

$$r_d(A) = r_d(-1,2) = (4, -3)$$

$$r_d(B) = r_d(-3,-1) =$$

$$= \left(\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\end{pmatrix} + \begin{pmatrix}2\\-2\end{pmatrix}\right)^t =$$

$$= \left(\begin{pmatrix}-1\\-2\end{pmatrix} + \begin{pmatrix}3\\-2\end{pmatrix}\right)^t = \begin{pmatrix}-3\\-4\end{pmatrix}^t = (-3, -4)$$

12.2. Find the image of the ΔABC through the clockwise rotation of angle $\frac{\pi}{6}$, where $A(6,4)$, $B(6,2)$, $C(10,6)$

$$[R_{-\frac{\pi}{6}}] = \begin{pmatrix} \cos\left(-\frac{\pi}{6}\right) & -\sin\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$R_{-\frac{\pi}{6}}(A) = R_{-\frac{\pi}{6}}(6,4) = \left(3\sqrt{3}+2, -3+2\sqrt{3}\right)$$

$$R_{-\frac{\pi}{6}}(B) = R_{-\frac{\pi}{6}}(6,2) = \left(3\sqrt{3}+1, -3+\sqrt{3}\right)$$

$$R_{-\frac{\pi}{6}}(C) = R_{-\frac{\pi}{6}}(10,6) = \left(5\sqrt{3}+3, -5+3\sqrt{3}\right)$$

12.3 ABCD quadrilateral

$A(1,1)$, $B(3,1)$, $C(2,2)$, $D(\frac{3}{2},3)$

Find the images of ABCD through the following transformations:

(a) $T(1,2)$, r_x, $R_{-\frac{\pi}{2}}$

(b) $S(2,\frac{5}{2})$, r_y, $R_{\frac{\pi}{2}}$

(c) $Sh\left(\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right),\frac{3}{2}\right)$

(a) $T(1,2)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+1\\y+2\end{pmatrix}$

$T(1,2)(A)=(2,3)$

$T(1,2)(B)=(4,2)$

$T(1,2)(C)=(3,4)$

$T(1,2)(D)=(\frac{5}{2},5)$

$$r_x \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$r_x(B) = r_x(3,1) = (3,-1)$$

$$R_{-\frac{\pi}{2}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

$$R_{-\frac{\pi}{2}}(C) = R_{-\frac{\pi}{2}}(2,2) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

(5) $$S(2, 2.5) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{5}{2}y \end{pmatrix}$$

$$S(2, \tfrac{5}{2})(D) = S(2, \tfrac{5}{2})(\tfrac{3}{2}, 3) = (3, \tfrac{15}{2})$$

$$r_y \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

$$r_y(A) = r_y(1,1) = (-1, 1)$$

$$R_{\frac{\pi}{2}}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} =$$

$$= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-y\\ x\end{pmatrix}$$

$$R_{\frac{\pi}{2}}(D) = R_{\frac{\pi}{2}}\left(3, \frac{3}{2}\right) = \left(-3, \frac{3}{2}\right)$$

(c)

$$Sh(\vec{u}, r)\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1 - r u_x u_y & r u_x^2\\ -r u_y^2 & 1 + r u_x u_y\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$u_x = \frac{2}{\sqrt{5}} \qquad u_y = \frac{1}{\sqrt{5}} \qquad r = \frac{3}{2}$$

$$\Rightarrow [Sh(u, \vec{r})] = \begin{pmatrix}1 - \frac{3}{2}\cdot\frac{2}{5} & \frac{3}{2}\cdot\frac{4}{5}\\ -\frac{3}{2}\cdot\frac{1}{5} & 1 + \frac{3}{2}\cdot\frac{2}{5}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{2}{5} & \frac{6}{5}\\ -\frac{3}{10} & \frac{8}{5}\end{pmatrix}$$

$$Sh(\vec{u}, r)(C) = \left(\begin{pmatrix} 2/5 & 6/5 \\ -3/10 & 8/5 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right)^t$$

$$= \begin{pmatrix} \frac{16}{5} \\ \frac{13}{5} \end{pmatrix}^t = \left(\frac{16}{5}, \frac{13}{5} \right)$$