

Complex Analysis of Askaryan Radiation: A Fully Analytic Model in the Time-Domain

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The detection of ultra-high energy (UHE, ≥ 10 PeV) neutrinos via detectors designed to utilize the Askaryan effect has been a long-time goal of the astroparticle physics community. The Askaryan effect describes radio-frequency (RF) radiation from high-energy cascades. When a UHE neutrino initiates a cascade, cascade properties are imprinted on the radiation. Thus, observed radiation properties must be used to reconstruct the UHE neutrino event. Analytic Askaryan models have three advantages when used for UHE neutrino reconstruction. First, cascade properties may be derived from the match between analytic function and observed data. Second, analytic models minimize computational intensity in simulation packages. Third, analytic models can be embedded in firmware to enhance the real-time sensitivity of detectors. We present a fully analytic Askaryan model in the time-domain for UHE neutrino-induced cascades in dense media that builds upon prior models in the genre. We then show that our model matches semi-analytic parameterizations used in Monte Carlo simulations for the design of IceCube-Gen2. We find correlation coefficients greater than 0.95 and fractional power differences $< 5\%$ between the the fully analytic and semi-analytic approaches.

I. INTRODUCTION

The extrasolar flux of neutrinos with energies between $[0.01-1]$ PeV has been measured by the IceCube collaboration [1]. Previous analyses have shown that the discovery of UHE neutrinos (UHE- ν) will require an expansion in detector volume because the flux is expected to decrease with energy [2–6]. The UHE- ν flux could potentially explain the origin of UHE cosmic rays (UHECR), and provides the opportunity to study electroweak interactions at record-breaking energies [7, 8]. Utilizing the Askaryan effect expands the effective volume of UHE- ν detector designs, because this effect offers a way to detect UHE- ν with radio pulses that travel more than 1 km in sufficiently RF-transparent media such as Antarctic and Greenlandic ice [9–11].

The Askaryan effect occurs within a dense medium with an index of refraction n . A relativistic particle with $v > c/n$ initiates a high-energy cascade with negative total charge. The charge radiates energy in the RF bandwidth, and the radiation may be detected if the medium does not significantly attenuate the signal [12, 13]. The IceCube EHE analysis has constrained the UHE- ν flux to be $E_\nu^2 \phi_\nu \leq 2 \times 10^{-8} \text{ GeV cm}^{-2} \text{ s}^{-1} \text{ sr}^{-1}$ between $[5 \times 10^{15} - 2 \times 10^{19}] \text{ eV}$ [4]. Arrays of $\mathcal{O}(100)$ *in situ* detectors encompassing effective areas of $\approx 10^4 \text{ m}^2 \text{ sr}$ per station, spaced by $\mathcal{O}(1)$ RF attenuation length could discover a UHE- ν flux beyond the EHE limits. The most suitable ice formations exist in Antarctica and Greenland, and a group of prototype Askaryan-class detectors has been deployed. These detectors seek to probe unexplored UHE- ν flux parameter-space from astrophysical and cosmogenic sources [5, 6, 14, 15].

Askaryan radiation was first measured in the laboratory in silica sand, and later ice [16–18]. Cascade properties affect the amplitude and phase of the radiation. At RF wavelengths, cascade particles radiate coherently, and the radiation amplitude scales with the total track length of the excess negative charge. The RF pulse shape is influenced by the *longitudinal length* of the cascade, and the pulse is strongest when the viewing angle is close to the Cherenkov angle, θ_C . The *excess charge profile* describes the excessive negative charge versus longitudinal position on the cascade axis. Radiation wavelengths shorter than the *lateral width* of the cascade, perpendicular to the cascade axis, are attenuated. At energies far above 10 PeV in ice, however, excess charge profiles generated by *electromagnetic* cascades experience the LPM effect and can have multiple peaks [19, 20]. This theoretical foundation has been constructed from a variety of experimental and simulation results.

The field of Askaryan-class detectors requires this foundation for at least two reasons. First, the theoretical form of the Askaryan RF pulse is used to optimize RF detector designs. Askaryan models are incorporated into simulations [21–23] in order to calculate expected signals and aid in detector design. For example, reconstruction tools for the radio component of IceCube-Gen2 combine machine learning and insights from Askaryan radiation physics [24–26]. Second, Askaryan models are used as templates to search large data sets for signal candidates [5, 27]. The signal-to-noise ratios (SNRs) at RF channels are expected to be small ($\text{SNR} \approx 3$). The amplitude of the radiated field decreases with the vertex distance ($1/r$), and the signal is attenuated by the ice [9, 28, 29]. Low SNR signals require correspondingly low RF trigger thresholds, but signals must be sampled for a bandwidth of $[0.1-1] \text{ GHz}$. Thus, RF channels are triggered at high rates by thermal noise. UHE- ν signals will be hidden within millions of thermal triggers. Template-waveform

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matching between models and data is a powerful technique for isolating potential UHE- ν candidates [27, 30].

Askaryan models fall into three categories: full Monte Carlo (MC), semi-analytic, and fully analytic. The original work by E. Zas, F. Halzen, and T. Stanev (ZHS) [13] was a full MC model. The properties of cascades with total energy ≤ 1 PeV were examined. A parameterization for the Askaryan field below 1 GHz was offered, attenuating modes above 1 GHz via a frequency-dependent form factor tied to the lateral cascade width. The semi-analytic approach was introduced by J. Alvarez-Muñiz *et al* (ARVZ) [31]. This approach accounts for fluctuations in the charge excess profile, and provides an analytic vector potential observed at the Cherenkov angle. The vector potential at the Cherenkov angle is labeled the form factor, and observed fields are derived from the derivative of the vector potential once convolved with a charge excess profile from MC. Recent work also accounts for differences in fit parameters from electromagnetic and hadronic cascades, and other interaction channels, while matching full MC simulations [32].

Finally, fully analytic models of Askaryan radiation from first principles have been introduced. J. Ralston and R. Buny (RB) gave a fully analytic model valid for observations of cascades in the near and far-field, with the transition encapsulated by a parameter η [33]. The result was complex frequency-domain model. Recently, a model and software implementation was given by J. C. Hanson and A. Connolly (JCH+AC) that built upon RB by providing an analytic form factor derived from GEANT4 simulations, and accounted for LPM elongation [34]. This work connected the location of poles in the complex frequency plane to η and the form factor. The poles combine to form a low-pass filter for the Askaryan radiation. The JCH+AC results match the ZHS results while demonstrating the physical origins of model parameters. The RB and JCH+AC results are given in the Fourier domain, but most UHE- ν searches (like template matching) have taken place in the time-domain. *The goal of this work is to present a fully analytic time-domain model valid for all viewing angles θ and $\eta < 1$, and to demonstrate that it matches semi-analytic models.*

In Section II, the cascade geometry, units, and vocabulary are defined. In Section III, we describe how the JCH+AC form factor fits into the current model [34]. In Section IV, the analytic Askaryan field, observed at $\theta = \theta_C$ (*on-cone*), is presented. In Section V, the analytic Askaryan field observed for $\theta \neq \theta_C$ (*off-cone*) is presented. In Section VI, fully analytic fields are matched to semi-analytic fields generated with NuRadioMC [23] at 10 PeV (electromagnetic cascades) and 100 PeV (hadronic cascades). Though the LPM effect is activated in NuRadioMC, it has a negligible influence on the waveform comparison at these energies. In Section VII, the results are summarized and potential applications of the fully analytic time-domain model are described.

II. UNITS, DEFINITIONS, AND CONVENTIONS

The coordinate system of the Askaryan radiation from a vector current density \vec{J} is shown in Fig. 1 (a)-(b). Primed cylindrical coordinates refer \vec{J} , and the unprimed spherical coordinates refer to the observer. The zenith or *viewing angle* is measured with respect to the *longitudinal axis* (z'). The observer displacement is $r = |\vec{x} - \vec{x}'|$, in the \hat{r} direction. The origin is located where the cascade has the highest instantaneous charge density (ICD). The ICD is treated with cylindrical symmetry and therefore no ϕ' -dependence. This assumption is based on the large number of cascade particles and momentum conservation. The lateral extent of the ICD is along the *lateral axis* (ρ'). The viewing angle is θ in spherical coordinates, and the Cherenkov angle occurs when θ satisfies $\cos(\theta_C) = 1/n_{\text{ice}}$ with $n_{\text{ice}} = 1.78 \pm 0.003$ [35].

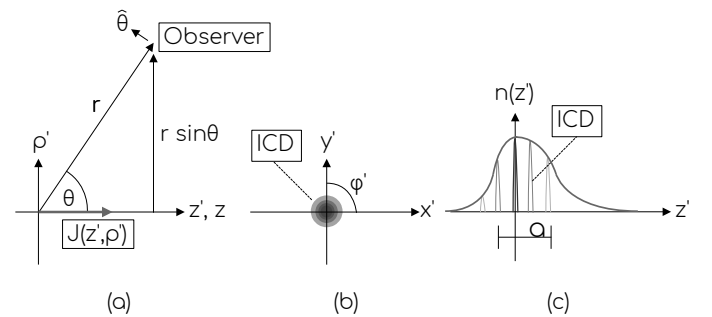


FIG. 1: (a) Side view of the coordinate systems used in the analysis. Spherical unprimed coordinates refer to the observer. Primed cylindrical coordinates refer to $\vec{J}(\rho', z')$. (b) Front view of the coordinate system. The instantaneous charge density (ICD) is assumed to have no ϕ' -dependence. (c) The function $n(z')$ describes the total cascade excess charge, and it has a characteristic width a . The ICD has an instantaneous width much smaller than a [34].

In Fig. 1 (c), an archetypal excess charge profile $n(z')$ is shown with characteristic longitudinal length a . The individual ICDs represent the excess charge density for small windows of time, and $n(z')$ refers to the total excess charge as a function of z' . Approximating the central portion of $n(z')$ as a Gaussian distribution $N(\mu, \sigma)$ corresponds to setting $a = 2\sigma$. Askaryan radiation occurs because $n(z')$ represents excess negative charge [13, 34, 36]. Cascades may be characterized as *electromagnetic*, initiated by charged outgoing leptons from UHE- ν interactions, or *hadronic*, initiated by the nuclei with which the UHE- ν interact. Electromagnetic cascades follow the Greisen distribution and hadronic cascades follow the Gaisser-Hillas distribution. An example of such an implementation via the ARVZ semi-analytic parameterization is AraSim [11].

The units of the electromagnetic field in the Fourier domain are V/m/Hz, often converted in the literature to

V/m/MHz. To make the distance-dependence explicit, both sides of field equations are multiplied by r , as in $r\vec{E} = \dots$, making the units V/Hz. Throughout this work, an overall field normalization constant E_0 is used *in the time domain*, and it is directly proportional to the cascade energy E_C . Thus, E_0 may be linearly scaled with energy, as in other Askaryan models. We show that the on-cone field amplitude is proportional to E_0 times a characteristic frequency-squared, so the units of E_0 are V/Hz². For off-cone results, we show that the field amplitude is proportional to E_0 times a characteristic frequency divided by a characteristic pulse width, and the units of E_0 remain V/Hz².

In Section III B, we review briefly the energy-dependence of the longitudinal length a in both the electromagnetic and hadronic cases. For the Greisen distribution with critical energy E_{crit} , it can be shown that if $n_{\text{max}} = n(z_{\text{max}})$, where $z_{\text{max}} = \ln(E_C/E_{\text{crit}})$, then $n_{\text{max}}a \sim E_C/E_{\text{crit}}$. Thus, the area under the curve $n(z')$ scales with E_C . RB show that the Askaryan radiation amplitude is proportional to $n_{\text{max}}a$ and therefore E_C . The cascade develops over a length $\approx a$, but the radiation is coherent over a length $\Delta z'_{\text{coh}}$ for which the displacement is constant to first order relative to a wavelength. The η parameter is the square of the ratio of a to $\Delta z'_{\text{coh}}$:

$$\eta = \left(\frac{a}{\Delta z'_{\text{coh}}} \right)^2 = \frac{k}{r} (a \sin \theta)^2 \quad (1)$$

In far-field, $\eta < 1$. In the first JCH+AC model, a limiting frequency ω_C (Equation 2) was shown to filter the Askaryan radiation [34]:

$$\eta = \frac{\omega}{\omega_C} \quad (2)$$

The effect of ω_C is described in Section IV. The Askaryan radiation is primarily polarized in the $\hat{\theta}$ -direction, with a small amount along \hat{r} [31, 34]. The wavevector is $k = (2\pi)/(n\lambda)$, where n is the index of refraction. A 3D wavevector is defined by RB, equivalent to $\vec{q} = nk(1, \vec{\rho}/R)$. The vector current density is treated by RB as a charge density times the velocity of the ICD: $\vec{J}(t, \vec{x}') = \rho(z' - vt, \rho')\vec{v}$. Further, the charge density is factored into $n(z')$ times ICD: $\rho(z' - vt, \rho') = n(z')f(z' - vt, \rho')$. The form factor \tilde{F} is the three-dimensional spatial Fourier transform of the ICD [33].

The result for \tilde{F} was derived analytically by JCH+AC [34], and that derivation is briefly described in Section III A. JCH+AC define a parameter σ , and \tilde{F} is a function of σ : $\tilde{F}(\sigma)$. The variable σ is the ratio of angular frequency to the low-pass cutoff frequency ω_{CF} of \tilde{F} :

$$\sigma = \frac{\omega}{\omega_{\text{CF}}} \quad (3)$$

Armed with \tilde{F} , the longitudinal length a and the corresponding energy-dependence on E_0 , the RB field equations $\vec{\mathcal{E}}$, and the displacement r , the Askaryan electromagnetic field may be assembled according to the following form [33]:

$$r\vec{E}(\omega, \theta) = E_0 \left(\frac{\omega}{2\pi} \right) \psi \vec{\mathcal{E}}(\omega, \theta) \tilde{F}(\omega, \theta) \quad (4)$$

The factor E_0 is proportional to cascade energy. The factor ω is the angular frequency. The variable ψ is $\psi = -i \exp(ikr) \sin \theta$. The function $\vec{\mathcal{E}}(\omega, \theta)$ contains the vector and complex pole structure of the field (see [33] and [34]). The model represented by Equation 4 is an *all- θ , all- ω* model. Equation 4 is valid at all frequencies not attenuated by $\tilde{F}(\omega, \theta)$, and all viewing angles not attenuated by $\vec{\mathcal{E}}(\omega, \theta)$. The first goal of this work is to build an *all- θ , all- t* model in the time-domain, derived from Equation 4, and the second goal is to compare it to semi-analytic parameterizations.

III. THE FORM FACTOR AND LONGITUDINAL LENGTH PARAMETER

To arrive at the main electromagnetic field in the time-domain, the individual pieces of Equation 4 must first be assembled. The first piece will be the form factor \tilde{F} that accounts for the 3D ICD, followed by some remarks about the energy-dependence of the longitudinal length parameter a .

A. The Form Factor

As presented in Section II the form factor is the 3D Fourier transform of the ICD $f(z', \vec{\rho}')$, with $\vec{q} = nk(1, \vec{\rho}/R)$:

$$F(\vec{q}) = \int d^3x' f(z', \rho') e^{-i\vec{q} \cdot \vec{x}'} \quad (5)$$

The goal is to evaluate \tilde{F} in the Fourier domain for an ICD definition informed by cascade simulations. Simulations of the cascade induced by UHE- ν indicate a thin wave of charge in z' spread uniformly in ϕ' , that decreases exponentially in ρ' . Using these observations JCH+AC complete the derivation in [34]. The final result was a simple analytic formula:

$$\tilde{F} = \frac{1}{(1 + \sigma^2)^{3/2}} \quad (6)$$

The form factor acts as a low-pass filter with the cutoff-frequency ω_{CF} introduced in Section II: $\sigma = \omega/\omega_{\text{CF}}$:

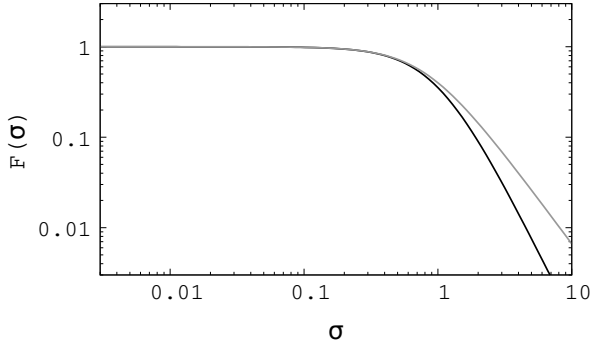


FIG. 2: (Black) Equation 6, graphed versus $\sigma = \omega/\omega_{\text{CF}}$. (Gray) The two-pole approximation.

$$\tilde{F} = \frac{1}{(1 + \sigma^2)^{3/2}} \approx \frac{1}{1 + \frac{3}{2}\sigma^2} \quad (7)$$

$$\tilde{F} \approx \frac{1}{(1 + i\sqrt{3/2}\sigma)(1 - i\sqrt{3/2}\sigma)} \quad (8)$$

$$\tilde{F} \approx \frac{\omega_0^2}{(\omega + i\omega_0)(\omega - i\omega_0)} \quad (9)$$

The definition $\omega_0 = \sqrt{2/3} \omega_{\text{CF}}$ has been used in the final step. Equation 9 matches the original ZHS parameterization (see Equation 20 of [13]).

1. A Note about the Molière Radius

In Section VIB, the decay constant l of the lateral component of the ICD is inferred from best-fit values of ω_0 . The connection between the l -parameter and ω_0 was described by JCH+AC [34]. Put simply, the ICD decays by a factor of $1/e$ a lateral distance l from the cascade axis. Note, however, that the l -parameter is not the Molière radius. The Molière radius is the lateral radius which forms a cylinder containing 90% of the energy deposition of the cascade. For ice with a density of 0.917 g cm^{-3} , one can estimate $R_M \approx 9.2 \text{ cm}$ using standard formulas. Although it is tempting to compare l to R_M , these parameters have different definitions. Knowing that l is related to ω_0 , l may be estimated as $\lambda/2$ in ice at the cutoff-frequency. At 3 GHz in ice, $\lambda/2 \approx 2.8 \text{ cm}$, and at 1 GHz in ice, $\lambda/2 \approx 8.4 \text{ cm}$. Although the results are at the same order of magnitude as R_M , there are three effects limiting the high-frequency spectrum of the radiation: ω_0 , ω_C , and the viewing angle. Thus, $l < R_M$ is possible for a radiation spectrum limited to $\lesssim 1 \text{ GHz}$.

B. The Longitudinal Length Parameter

The next piece required in the assembly of the main electromagnetic field is the energy-dependence of the

overall amplitude, and the energy dependence of the longitudinal length parameter, a , which is a part of $\tilde{\mathcal{E}}$ in Equation 4 [33]. What follows are two separate discussions, one for electromagnetic cascades, and one for hadronic cascades. Though we share these calculations for convenience, note that a variety of theoretical and experimental results on this topic are available [16] [37] [38].

1. Electromagnetic Case

The number of charged particles versus distance in radiation lengths $n(z')$ in an electromagnetic cascade taking place in a dense medium with initial cascade energy E_C , critical energy E_{crit} , normalization parameter n_0 , and age s is [34]

$$n(z') = \frac{n_0}{\sqrt{\ln(E_C/E_{\text{crit}})}} \exp \left\{ z' \left(1 - \frac{3}{2} \ln(s) \right) \right\} \quad (10)$$

To find the energy-dependent width of the Greisen distribution, four steps are necessary: (1) normalization of $n(z')$ as a fraction of the maximum excess charge, (2) conversion of $n(z')$ to $n(s)$, (3) determination of the width of $n(s)$ by approximating the central portion as a Gaussian distribution, and (4) conversion of the width from s units to radiation lengths z' , and then converting those results to a distance. Define the ratio $R = n(a/2)/n_{\text{max}}$. The final result in radiation lengths is

$$a = \sqrt{\ln(E_C/E_{\text{crit}})} \sqrt{-6 \ln(R)} \quad (11)$$

Since $R < 1$, $\ln(R) < 0$ and a is real-valued. Equation 11 is in radiation lengths. In solid ice the density is $\rho_{\text{ice}} = 0.917 \text{ g cm}^{-3}$, and the electromagnetic radiation length is $z_0 = 36.08 \text{ g cm}^{-2}$ [34]. Converting to distance gives

$$a = \frac{z_0}{\rho_{\text{ice}}} \sqrt{\ln(E_C/E_{\text{crit}})} \sqrt{-6 \ln(R)} \quad (12)$$

Note that $a \propto \sqrt{\ln(E_C)}$, as shown by RB and others. The product $n_{\text{max}}a$ is proportional to the energy E_C/E_{crit} . For this reason RB took $n_{\text{max}}a$ as the field normalization rather than E_C [33]. As an example, let $R = 0.4$, and $E_{\text{crit}} \approx 10^8 \text{ eV}$, gives $a \approx 4 \text{ meters}$ for $E_C = 10^{16} \text{ eV}$. We show in Section VI that our fitted a -values are close to 4 meters when matched to semi-analytic parameterizations.

2. Hadronic Case

The Gaisser-Hillas distribution describes hadronic cosmic-ray air showers, but has also been applied to

hadronic cascades in dense media [22]. The original function reads

$$n(z') = n_{max} \left(\frac{z' - z_0}{z_{max} - z_0} \right)^{(z_{max} - z_0)/\lambda} e^{\frac{z_{max} - z'}{\lambda}} \quad (13)$$

The variables are defined as follows: n_{max} is the instantaneous maximum number of particles in the cascade, z' is the longitudinal distance in radiation lengths, z_0 is the initial starting point, λ is the interaction length, and z'_{max} is the location of n_{max} . Using the same steps as the electromagnetic case, we find

$$a = \sqrt{\lambda z'_{max}} \sqrt{-8 \ln(R)} \quad (14)$$

The a parameter again goes as $\sqrt{z_{max}} \propto \sqrt{\ln(E_C)}$ which produces similar lengths as the electromagnetic case when scaled by the appropriate interaction length and ice density.

IV. ON-CONE FIELD EQUATIONS

The $\hat{\theta}$ -component of the electromagnetic field at $\theta = \theta_C$ will now be built in the time-domain from Equation 4. Setting $\theta = \theta_C$ in the general RB field equations (Appendix A), with Equation 6 for \hat{F} , $\sigma = \omega/\omega_{CF}$ and $\eta = \omega/\omega_{CF}$, and letting E_0 be proportional to cascade energy E_C produces Equation 45 from JCH+AC [34]:

$$r\tilde{E}(\omega, \theta_C) = \frac{(-i\omega)E_0 \sin(\theta_C) e^{i\omega r/c}}{(1 - i\omega/\omega_C)^{1/2} (1 + (\omega/\omega_{CF})^2)^{3/2}} \quad (15)$$

More detail is provided in Appendix A. Let the retarded time be $t_r = t - r/c$, and let $\omega_0 = \sqrt{\frac{2}{3}}\omega_{CF}$ and $\hat{E}_0 = E_0 \sin \theta_C$. The inverse Fourier transform of Equation 15 is

$$rE(t, \theta_C) = \frac{\hat{E}_0 i \omega_C \omega_0^2}{\pi} \frac{d}{dt_r} \int_{-\infty}^{\infty} \frac{e^{-i\omega t_r}}{(2i\omega_C + \omega)(\omega + i\omega_0)(\omega - i\omega_0)} d\omega \quad (16)$$

In Equation 16, the derivative with respect to the retarded time d/dt_r is introduced to remove a factor of $(-i\omega)$ from the numerator. Further, the three simple poles arise from approximation in the denominator. The poles are located on the imaginary axis, above and below

the origin. Thus, region of convergence of Equation 16 depends on the sign of the retarded time. The details of each case are provided in Appendix A. The results, including the effect of each pole and each sign of t_r , become

$$rE(t, \theta_C) = \frac{1}{3} \hat{E}_0 \omega_{CF}^2 \begin{cases} (1 - \frac{1}{2}\epsilon) e^{\omega_0 t_r} & t_r < 0 \\ (2e^{-2\omega_C t_r} - (1 + \frac{1}{2}\epsilon) e^{-\omega_0 t_r}) & t_r > 0 \end{cases} \quad (17)$$

Equation 17 represents the time-domain solution for the on-cone $\hat{\theta}$ -component of the Askaryan electric field. Table I summarizes the definitions of the parameters in Equation 17. Fit results for the parameters of Table I are shown in Section VI.

Notice that the overall field amplitude scales with energy ($E_0 \propto E_C \sim n_{max} a$) and with the second power of cutoff-frequency. The units of E_0 are therefore V Hz². The amplitude is asymmetric, and the parameter ϵ influences the asymmetry. The ϵ parameter was studied in JCH+AC in detail. For example, Fig. 10 of [34] shows that $\epsilon \approx [0.1 - 1]$ for $\sqrt{2\pi}\rho_0 \approx 20 \text{ m}^{-1}$ and $a \approx 4 \text{ m}$. These values are typical of 10 PeV cascades (see Sec-

tion. VI). The expression for ϵ is the product of the ratio of the lateral to longitudinal length, and the longitudinal length to the vertex distance, making it a physical parameter connecting the event geometry to the cascade shape. Figure 3 displays normalized examples of Equation 17 for different values of ω_0 , ω_C , and ϵ . The values shown are motivated by comparisons to semi-analytic parameterizations in Sec. VI.

Parameter	Definition
\hat{E}_0	$E_0 \sin(\theta_C)$
E_0	$\approx n_{\max} a$
ω_0	$\sqrt{\frac{2}{3}} \omega_{CF}$
ω_{CF}	$(c\sqrt{2\pi\rho_0})/(n \sin \theta)$
ω_C	$(rc)/(na^2 \sin^2 \theta)$
$\epsilon = \omega_0/\omega_C$	$\sqrt{2/3}(\sqrt{2\pi\rho_0}a)(a/r)$ (see Equation 46 of [34])
t_r	$t - r/c$

TABLE I: The parameters used to build Equation 17. Fitted values in comparison to semi-analytic parameterizations are shown in Section VI.

A. Verification of the Uncertainty Principle

Among the results of JCH+AC is the Gaussian width of the radiation in the Fourier domain: σ_ν , where ν represents the frequency. Generally speaking, the product of the width in the Fourier domain with the width in the time domain must be greater than or equal to some constant. Gaussian functions minimize this *uncertainty principle*. The following procedure is used to compute the width σ_t of the on-cone field. First, the branches corresponding to $t_r < 0$ and $t_r > 0$ are each treated as probability distribution functions, and normalized. Next the average positive and negative retarded times, $\bar{t}_{r,+}$ and $\bar{t}_{r,-}$, are computed:

$$\bar{t}_{r,+} = \frac{\omega_0}{\frac{1}{2}\epsilon - 1} \int_0^\infty 2t_r e^{-2\omega_C t_r} - \left(1 + \frac{1}{2}\epsilon\right) t_r e^{-\omega_0 t_r} dt_r \quad (18)$$

$$\bar{t}_{r,-} = \omega_0 \int_{-\infty}^0 t_r e^{\omega_0 t_r} dt_r \quad (19)$$

Subtracting the two averages yields σ_t , which simplifies to

$$\sigma_t = \bar{t}_{r,+} - \bar{t}_{r,-} = \frac{\epsilon + 2}{\omega_0} = \frac{1}{\omega_C} + \frac{2}{\omega_0} \quad (20)$$

The result has the correct units and the limiting cases are sensible. Suppose $\epsilon \rightarrow 0$ ($\omega_C \gg \omega_0$), then $\sigma_t \rightarrow 2/\omega_0$, which is expected from observing Equation 17 if the ω_C exponential disappears. If $\epsilon = 1$ ($\omega_C = \omega_0$), then $\sigma_t = 3/\omega_0$. That is, the pulse is wider if there is more than one relevant cutoff frequency. The result also reveals that as the pulse asymmetry grows with ϵ , so does the pulse width.

Equation 36 of JCH+AC [34] provides σ_ν :

$$\sigma_\nu = \frac{c}{2\pi a \Delta \cos \theta} (1 + \eta^2)^{1/2} \quad (21)$$

Expanding to first order in $\Delta \cos(\theta) = \cos(\theta) - \cos(\theta_C)$,

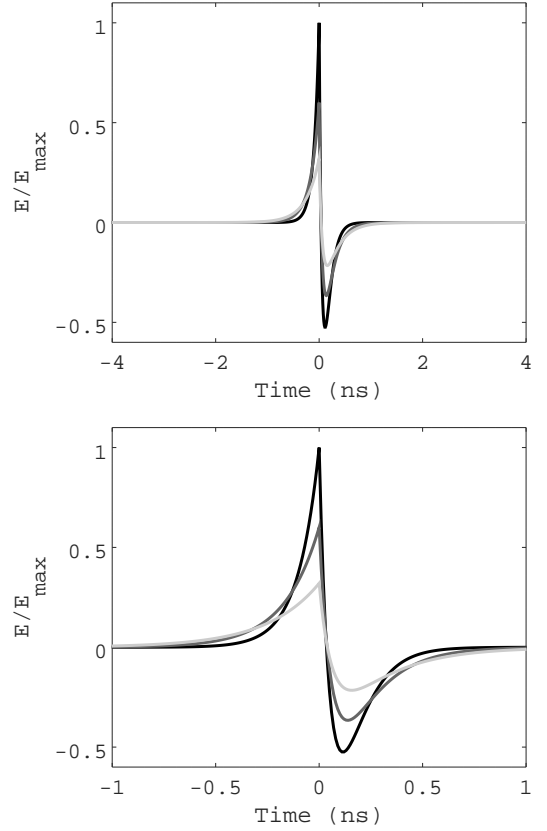


FIG. 3: (Top) Equation 17 from $[-4, 4]$ ns, with (black) $\omega_C = 2\pi(1.25)$ GHz, $\omega_0 = 2\pi(1.56)$ GHz, $\epsilon = 1.25$, (gray) $\omega_C = 2\pi(1.25)$ GHz, $\omega_0 = 2\pi(0.94)$ GHz, $\epsilon = 0.75$, (light gray) $\omega_C = 2\pi(1.25)$ GHz, $\omega_0 = 2\pi(0.625)$ GHz, $\epsilon = 0.5$. The amplitudes of all curves are normalized to the peak of the $\epsilon = 1.25$ (black) data. (Bottom) Same as top panel, plotted between $[-1, 1]$ ns.

$$\sigma_\nu \approx \frac{c}{2\pi a \sin(\theta_C) \Delta \theta} (1 + \eta^2)^{1/2} \quad (22)$$

From previous definitions, let $\omega_C^{-1} = a^2 \sin^2(\theta_C)/(rc)$, and $\omega_0^{-1} = l \sin(\theta_C)/c$, where $l = \sqrt{3/2}/(\sqrt{2\pi\rho_0})$. Noting that $\sigma_t = \omega_C^{-1} + 2\omega_0^{-1}$, the uncertainty product is

$$\sigma_\nu \sigma_t = \frac{1}{2\pi} \left(\left(\frac{a}{r}\right) \frac{\sin(\theta_C)}{\Delta \theta} + \sqrt{6} \left(\frac{l}{a}\right) \frac{1}{\Delta \theta} \right) (1 + \eta^2)^{1/2} \quad (23)$$

The limits $r \gg a$ and $ka \ll 1$ cause $\eta \rightarrow 0$. Assume these limits to be true, along with $\Delta \theta \rightarrow 0$. The result is

$$\sigma_\nu \sigma_t = \frac{1}{2\pi} \left(\left(\frac{a}{r}\right) \frac{\sin(\theta_C)}{\Delta \theta} + \sqrt{6} \left(\frac{l}{a}\right) \frac{1}{\Delta \theta} \right) \quad (24)$$

Therefore, in order to satisfy $\sigma_\nu \sigma_t > 1/(2\pi)$,

$$\left(\frac{a}{r}\right) \sin(\theta_C) + \sqrt{6} \left(\frac{l}{a}\right) > \Delta \theta \quad (25)$$

Variable	Definition
u	$1 - i\eta$
x	$\cos(\theta)$
x_C	$\cos(\theta_C)$
$\sin^2(\theta)$	$1 - x^2$
q	$(xx_C - x_C^2)/(1 - x^2)$
y	$(\frac{1}{2})(ka)^2(\cos\theta - \cos\theta_C)^2$
p	$\frac{1}{2}(\frac{a}{c})^2(\cos\theta - \cos\theta_C)^2$

TABLE II: Useful variables for the derivation of the off-cone Askaryan electromagnetic field.

Although $a/r \ll 1$ and $l/a \ll 1$, as long as these expressions do not approach zero as fast as $\Delta\theta \rightarrow 0$ in Equation 25, the uncertainty principle holds. Yet these are exactly the conditions of the problem. The observer distance r is assumed to be in the far-field and much larger than the longitudinal length, and at cascade energies at or above 10 PeV, the longitudinal length is computed to be much larger than the lateral ICD width. Thus, the result is that $\sigma_\nu \sigma_t > 1/(2\pi)$.

V. OFF-CONE FIELD EQUATIONS

Turning to the case for which $\theta \neq \theta_C$, the $\hat{\theta}$ -component of the electromagnetic field will now be built in the time-domain. The RB field equations for the $\hat{\theta}$ and \hat{r} components are summarized in both RB and JCH+AC [33, 34]. Recall the general form of the electromagnetic field, given in Equation 4:

$$r\vec{E}(\omega, \theta) = E_0 \left(\frac{\omega}{2\pi} \right) \psi \vec{\mathcal{E}}(\omega, \theta) \tilde{F}(\omega, \theta) \quad (26)$$

The first task is to simplify $\vec{\mathcal{E}}(\omega, \theta)$ before taking the inverse Fourier transform. The full definition of $\vec{\mathcal{E}}$ is given in Appendix A. The simplification involves expanding $\vec{\mathcal{E}}(\omega, \theta)$ in a Taylor series such that $u = 1 - i\eta \approx 1$, restricting $\eta < 1$. The restriction $\eta < 1$ may be satisfied by either $a \sin\theta/r \ll 1$, or $ka \sin\theta \ll 1$, or both. Once $\vec{\mathcal{E}}(\omega, \theta)$ is simplified, the inverse Fourier transform of Equation 26 may be evaluated to produce the result. Table II contains useful variable definitions, Table III contains useful function definitions, and Table IV contains special cases of the functions in Table III.

The original form of $\vec{\mathcal{E}}(\eta, \theta)$ is shown in Appendix A. Changing variables to u and x (Tab. II) and using the function definitions and values in Tabs. III-IV, $\vec{\mathcal{E}}(u, x) \cdot \hat{\theta} = \mathcal{E}(u, x)$ becomes

$$\mathcal{E}(u, x) = f(u, x)g(u, x)(1 - h(u, x)) \quad (27)$$

Expanding $\mathcal{E}(u, x)$ near $u = 1$ gives

Function	Definition
$f(u, x)$	$\left(u + 3 \frac{(1-u)^2}{u} \frac{x^2 - xx_C}{1-x^2} \right)^{-1/2}$
$g(u, x)$	$\exp\left(-\frac{1}{2}(ka)^2(x - x_C)^2 u^{-1}\right)$
$h(u, x)$	$\left(\frac{1-u}{u}\right) q$
$\vec{\mathcal{E}}(u, x) \cdot \hat{\theta}$	$f(u, x)g(u, x)(1 - h(u, x))$

TABLE III: Useful functions for the derivation of the off-cone Askaryan electromagnetic field. The last row contains the vector structure of the $\hat{\theta}$ -component of the field.

Function ($u = 1$)	Result
$f(x, 1)$	1
$\dot{f} _{u=1}$	$-\frac{1}{2}$
$g(x, 1)$	$\exp(-y)$
$\dot{g} _{u=1}$	$y \exp(-y)$
$h(x, 1)$	0
$\dot{h} _{u=1}$	$-q$

TABLE IV: Special cases of the functions defined in Table III, when $u = 1$.

$$\mathcal{E}(u, x) = \mathcal{E}(x, 1) + (u - 1)\dot{\mathcal{E}}(x, 1) + \mathcal{O}(u - 1)^2 \quad (28)$$

The details of the expansion are shown in Appendix B. The result is

$$\mathcal{E}(x, u) = e^{-y} \left(1 - \frac{1}{2}j\eta(2y + 2q - 1) \right) \quad (29)$$

The inverse Fourier transform of the $\hat{\theta}$ -component gives the time-domain results, after including the expanded $\mathcal{E}(u, x)$:

$$rE(t, \theta) = \mathcal{F}^{-1} \left\{ E_0 \left(\frac{\omega}{2\pi} \right) \tilde{F} \psi \mathcal{E} \right\} \quad (30)$$

Intriguingly, the result is proportional to the *line-broadening function*, H (DLMF 7.19, [39]) common to spectroscopy applications. There are three terms in Equation 29. Two terms ultimately vanish, being integrals over odd integrands (see Appendix B). The integral that remains contains H , with $\omega_1 = t_r/(2p)$:

$$I_0 = 2\pi i \left(\frac{\omega_C}{\omega_0} \right) e^{-\frac{t_r^2}{4p}} H(\sqrt{p}\omega_0, i\omega_1\sqrt{p}) \quad (31)$$

Though the line-broadening function cannot be expressed analytically, there are examples of polynomial expansions [40]. Note that, for situations relevant to the current problem, $\omega > \omega_1$. Requiring that $\omega > \omega_1$ amounts to a restriction between $\Delta\theta$ and $|t_r|$:

$$|t_r| < |2p\omega| \quad (32)$$

It is shown in the next section that $\sqrt{2p}$ is the pulse width σ_t , so $|2p\omega|$ has units of time. Using the results of Sec. V A below, the restriction on the retarded time may be written $|t_r|/\sigma_t < \omega\sigma_t = 2\pi(\sigma_t/T)$. That is, the accuracy of the waveform should be trusted within a number of pulse widths that is less than 2π times the ratio of the pulse width to the period of the lowest frequency. This is not a strong requirement, since the field quickly approaches zero after several pulse widths. Hereafter, this step will be called the *symmetric approximation*, because the result for $r\vec{E}(t_r, \theta)$ in Equation 33 is symmetric. The restriction on $\Delta\theta$ is formalized in Sec. V B. Solving I_0 using the symmetric approximation clears the way for the final result (see Appendix B):

$$rE(t, \theta) = -\frac{E_0\omega_0 \sin(\theta)}{8\pi p} t_r e^{-\frac{t_r^2}{4p} + p\omega_0^2} \operatorname{erfc}(\sqrt{p}\omega_0) \quad (33)$$

Equation 33 represents the time-domain solution for the off-cone $\hat{\theta}$ -component of the Askaryan electric field. Equation 33 is graphed in Figs. 4 and 5. In Fig. 4 (top), $E(t, \theta)$ is shown normalized to the maximum value for the angular range displayed, $[\theta_C + 1.5^\circ, \theta_C + 5.5^\circ]$, from $t = [-5, 5]$ ns. Pulses with viewing angles closer to θ_C have larger relative amplitudes and shorter pulse widths. Figure 4 (bottom) contains the same results, but for $t = [-1.5, 1.5]$ ns. The pulses are symmetric and all zero crossings are at $t_r = 0$ ns. Figure 5 contains contours of the same results as in Fig. 4. Table I summarizes the definitions of the parameters in Equation 33. Fit results for the parameters of Table I are shown in Section VI.

As in the on-cone result, the overall field amplitude scales with energy ($n_{\max}a$). However, the amplitude scales with the first power of cutoff-frequency. The units of E_0 remain V Hz^2 because of the factors of t_r and p , which have units of time and time-squared, respectively. The argument of the complementary error function, $\sqrt{p}\omega_0$, is unitless. This factor is strictly positive, so the range of the complementary error function is $(0, 1)$. The factor $\sqrt{p}\omega_0$ cannot be zero without setting $\theta = \theta_C$, or setting $\omega_{\text{CF}} = 0$. Both cases are not allowed. Equation 33 represents the *off-cone* ($\theta \neq \theta_C$) solution, so $p \neq 0$. Setting $\omega_{\text{CF}} = 0$ would require cascade particles to have no lateral momentum, which is not physical. Another possibility is that $p = 0$ if $a = 0$, but this implies $E_0 = 0$ (see Section III B). Therefore, $0 < \operatorname{erfc}(\sqrt{p}\omega_0) < 1$.

A. Verification of the Uncertainty Principle

As in Section IV A, the uncertainty principle should be checked. Equation 33 is an anti-symmetric Gaussian function with pulse width $\sigma_t = \sqrt{2p}$. Let $\Delta \cos \theta =$

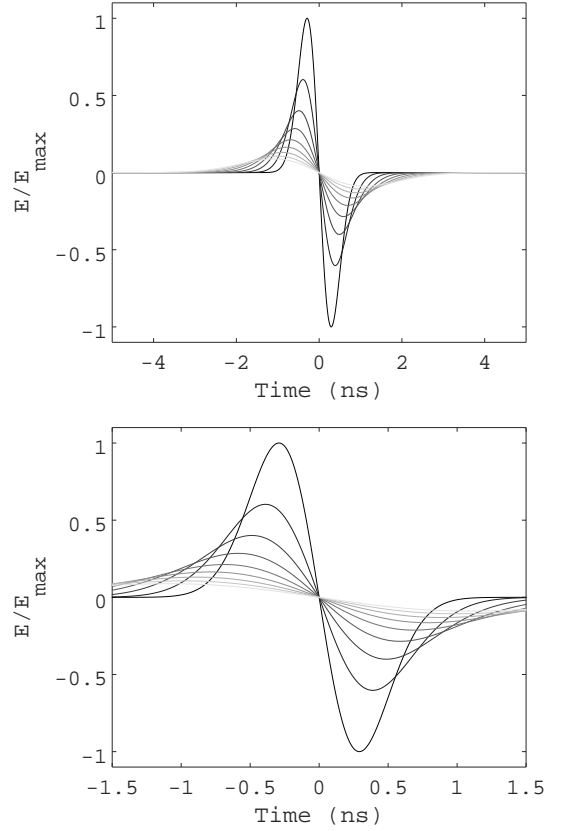


FIG. 4: $E(t, \theta)$ vs. t_r (Equation 33), normalized. The viewing angle θ is varied from $\theta_C + 1.5^\circ$ to $\theta_C + 5.5^\circ$ in steps of 0.5° . Top: $\omega_0/(2\pi) = 1.0$ GHz. Bottom: Same as top, zoomed in on central region.

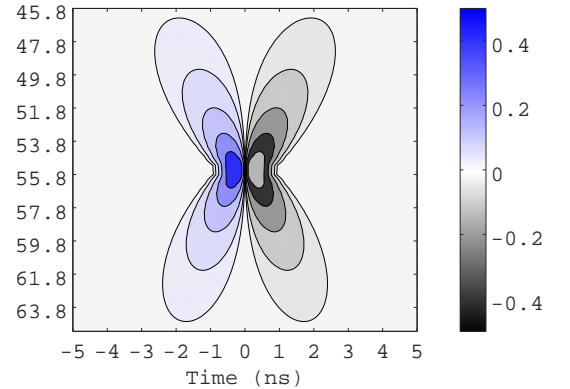


FIG. 5: Contours of $E(t, \theta)$ vs. θ vs. t_r (Equation 33), normalized. The normalization is the same as Fig. 4. Although the contour lines extend into the region near θ_C , Equation 5 is only being evaluated at $\Delta\theta > 1.5^\circ$ (see text for details).

$(\cos \theta - \cos \theta_C)$. Using Table II, the expression $\sqrt{2p}$ evaluates to

$$\sigma_t = \sqrt{2p} = \left(\frac{a}{c}\right) (\Delta \cos \theta) \quad (34)$$

Recall that σ_ν is given by

$$\sigma_\nu = \frac{c}{2\pi a \Delta \cos \theta} (1 + \eta^2)^{1/2} \quad (35)$$

The uncertainty product is

$$\sigma_t \sigma_\nu = \frac{1}{2\pi} (1 + \eta^2)^{1/2} \quad (36)$$

Expanding near $u \approx 1$ implies that $\eta \rightarrow 0$. This condition corresponds to observations in the far-field. Thus, the model obeys the uncertainty principle. For the special case $\eta = 0$, the model *minimizes* the uncertainty product.

B. Usage of the On-Cone versus Off-Cone Fields

The form of Equation 33, and the restriction between $\Delta\theta$ and $|t_r|$ from the symmetric approximation suggests the limit $\Delta\theta = \theta - \theta_C$ must be examined carefully. Since $p \propto (\cos \theta - \cos \theta_C)^2$, probing the model near $\theta = \theta_C$ is equivalent to taking the limit that $p \rightarrow 0$. Intriguingly, the p^{-1} -dependence in the field does not lead to a divergence. As the field grows in amplitude from p^{-1} as $p \rightarrow 0$, the field *width*, $\sqrt{2p}$, approaches zero. Thus, the power in the field does not diverge.

Equations 20 and 34 contain the pulse widths of the on-cone and off-cone fields, respectively. Power in the off-cone case is limited by the pulse width $\sqrt{2p}$, and the observed power should decrease in coherence as $\Delta\theta$ and the pulse width both increase. Thus, a reasonable constraint on $\Delta\theta_{\min}$ is given by setting the minimum off-cone pulse width equal to the on-cone pulse width. Setting Eqs. 20 and 34 equal:

$$\frac{1}{\omega_C} + \frac{2}{\omega_0} = \sqrt{2p} \quad (37)$$

Expanding the expression for p near $\theta = \theta_C$, and evaluating the square root leads to

$$\frac{1}{\omega_C} + \frac{2}{\omega_0} = \frac{a}{c} \sin \theta_C \Delta\theta_{\min} \quad (38)$$

Using $\epsilon = \omega_0/\omega_C$, and letting $k_0 = \omega_0/c$, the formula may be rearranged:

$$\epsilon + 2 = a k_0 \sin \theta_0 \Delta\theta_{\min} \quad (39)$$

Squaring both sides, and then dividing both sides by r yields

$$\frac{(\epsilon + 2)^2}{r} = k_0 \left(\frac{k_0 (a \sin \theta_C)^2}{r} \right) \Delta\theta_{\min}^2 \quad (40)$$

The quantity in parentheses is η , with $\omega = \omega_0$. Since $\eta = \omega/\omega_C$, setting $\omega = \omega_0$ means $\eta = \epsilon$. Solving for $\Delta\theta_{\min}$ gives

$$\Delta\theta_{\min} = \frac{\epsilon + 2}{\sqrt{\epsilon k_0 r}} \quad (41)$$

Assuming $\epsilon \approx 1$, $f_0 \approx 1$ GHz, $n = 1.78$ for solid ice, and $c = 0.3$ m ns $^{-1}$ (see Sec. VI A), $k_0 \approx 35$ m $^{-1}$. Taking $r = 1000$ m, $\Delta\theta_{\min} \approx 1^\circ$. Simple rules to remember for the application of the off-cone field are:

$$\Delta\theta_{\min} \geq 1^\circ \quad (42)$$

$$\Delta\theta_{\min} \propto \frac{1}{\sqrt{kr}} \quad (43)$$

VI. COMPARISON TO SEMI-ANALYTIC PARAMETERIZATIONS

The fully analytic model will now be compared to the ARZ semi-analytic parameterization [32] used in NuRadioMC [23] to predict signals in IceCube-Gen2 Radio. To provide concrete comparisons, a small set of waveforms were generated with NuRadioMC, for both electromagnetic and hadronic cascades. The electromagnetic cascades have an energy of $E_C = 10^{16}$ eV, while the hadronic cascades have an energy of $E_C = 10^{17}$ eV. The reason the EM energy is lower is to avoid the LPM effect, which changes the shape of $n(z')$, and requires more careful modeling [34]. To be clear, the comparison is between the fully analytic model of Equations 17 and 33 and the NuRadioMC implementation of the ARZ semi-analytic parameterization shown in [32].

The comparison involves three stages. First, waveforms and a -values are generated for each cascade type, energy, and angle: $\theta = \theta_C + 3.0^\circ$, and $\theta = \theta_C$. Second, Equations 17 and 33 are tuned to match the waveforms. In each fit, the Pearson correlation coefficient (ρ) between MC waveforms and Equations 17 or 33 is maximized, and the sum-squared of amplitude differences $((\Delta E)^2)$ is minimized. Finally, best-fit parameters are tabulated and compared with MC values from NuRadioMC.

Two remarks are important regarding the fit criteria. First, the Pearson correlation coefficient is not sensitive to differences in amplitude. The coefficient is normalized by the standard deviation of each signal, and the signals have no DC components:

$$\rho = \frac{\text{cov}(f_{\text{data}}, f_{\text{model}})}{\sigma_{\text{data}} \sigma_{\text{model}}} \quad (44)$$

Thus, the parameters that control correlation are those that scale t_r . Second, the parameters that control $(\Delta E)^2$ are those that scale the waveform amplitude. Note that

$$(\Delta E)^2 = \sum_{i=1}^N (E_{i,data} - E_{i,model})^2 \quad (45)$$

The $(\Delta E)^2$ minimization resembles χ^2 minimization, but lacks weighting in the denominator from statistical error in the fully analytic model. No statistical error has been associated with Equations 17 and 33. The utility of minimizing $(\Delta E)^2$ is that the final difference in power between model and data is minimized.

A. Waveform Comparison: $\theta = \theta_C$

Electromagnetic case. Six different electromagnetic cascades and the corresponding Askaryan fields were generated using the ARZ2019 model from NuRadioMC [23] [32] for comparison to Equation 17. The cascades have $E_C = 10$ PeV, and $r = 1000$ meters. The units of $\vec{E}(t_r, \theta_C)$ are mV/m versus nanoseconds, so the units of $r\vec{E}$ are Volts. The sampling rate of the digitized semi-analytic parameterizations was 100 GHz, with $N = 2048$ samples. Let $f_C = \omega_C/(2\pi)$ and $f_0 = \omega_0/(2\pi)$. The frequencies f_C and f_0 were varied from [0.6 - 6.0] GHz. The parameter E_0 was varied from [0.05 - 5.0] V GHz⁻². In a simple 2-level for-loop, the Pearson correlation coefficient ρ was maximized by varying f_0 and f_C . Next, the sum of the squared amplitude differences $(\Delta E)^2$ was minimized by varying E_0 , while holding f_0 and f_C fixed. Several other schemes were studied, including a 3-level for-loop, but the two-stage process produced the best results. The results are shown in Fig. 6.

Typically, maximizing ρ corresponds to minimizing $(\Delta E)^2$, but the waveforms fluctuate in shape and amplitude. In Fig. 7, $(\Delta E)^2$ is graphed versus ρ for one event. Typical best-fit ρ -values are ≈ 0.97 for this set, corresponding to best-fit $(\Delta E)^2$ values of $\approx 7\%$. Contours of $\rho > 0.95$ for f_0 versus f_C are shown in Fig. 6 (Left column). The crosses represent the best-fit location. The dashed gray line at $y = x$ corresponds to $f_0/f_C = \epsilon = 1$. Although Equation 17 contains an expansion to first order in ϵ , the motivation for the expansion is to cause the formula to resemble the derivative of the vector potential presented by ARZ [32]. The only real restriction is that $\epsilon \neq 2$, due to the singularity. Thus, the best-fit results avoid the solid black lines ($\epsilon = 2$) in Fig. 6, but find ϵ -values large enough to account for pulse symmetry. The best-fit waveforms are shown in Fig. 6 (Right column), where the gray curves correspond to ARZ from NuRadioMC and the black curves represent Equation 17.

Table V contains the best-fit results for the Equation 17 parameters, along with best-fit ρ -values and $(\Delta E)^2$ -values. The typical fractional power difference is $\approx 7\%$, and correlation coefficients are $\approx \rho = 0.97$. For every

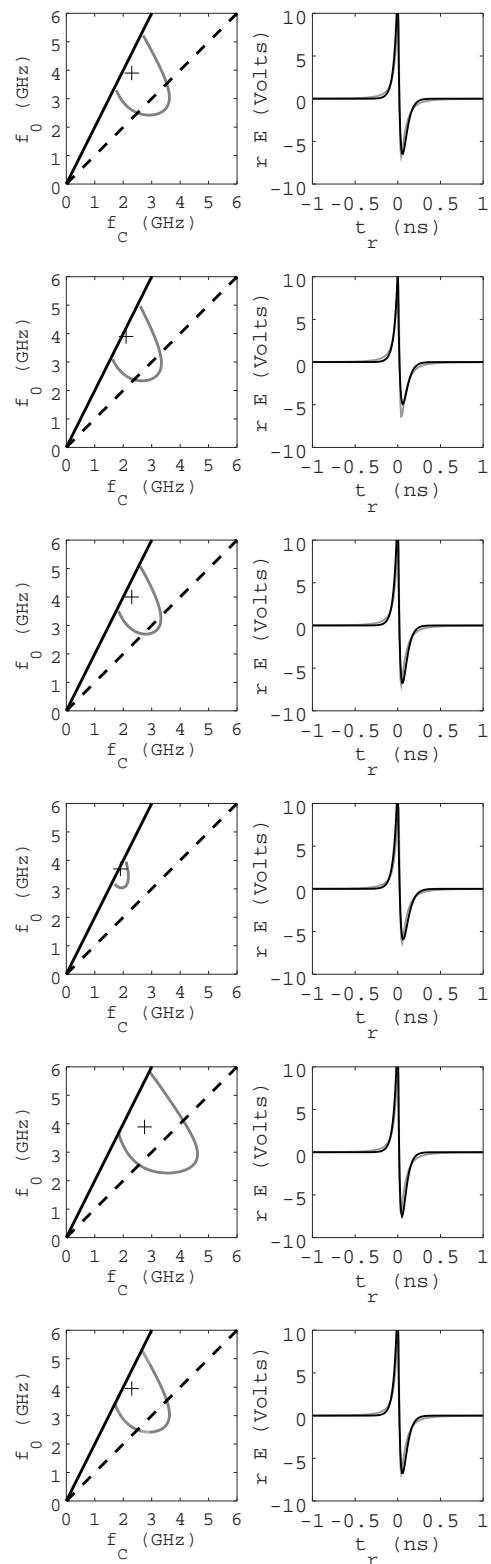


FIG. 6: **Fit results: electromagnetic case, $\theta = \theta_C$, $E_C = 10$ PeV.** The six rows (from top to bottom) correspond to NuRadioMC waveforms 1-6, 10 PeV electromagnetic cascades. (Left column) The best-fits for f_0 and f_C . Dashed line: $\epsilon = 1$. Solid line: $\epsilon = 2$. Gray contour: $\rho > 0.95$. Black cross: best-fit. (Right column) The best-fit waveforms. Gray: semi-analytic parameterizations from [23]. Black: Equation 17.

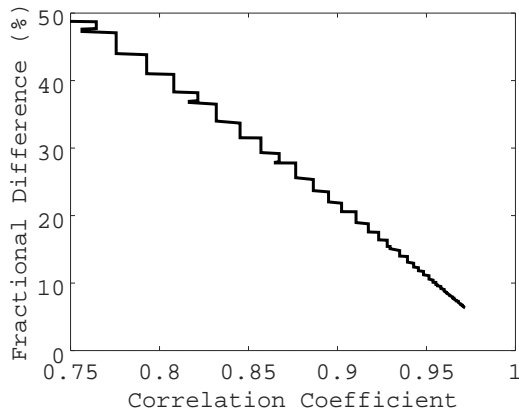


FIG. 7: The fractional difference in the sum of amplitude differences squared $((\Delta E)^2)$ versus correlation coefficient (ρ) for waveform 1 at $E_C = 10$ PeV, electromagnetic case.

waveform, $f_0 > f_C$, implying $\epsilon > 1$. This result will shift to $\epsilon < 1$ below for the hadronic events. To estimate the acceptable range for f_0 and f_C , the horizontal and vertical distances from the crosses to the $\rho > 0.95$ contour is assigned to the errors in Tab. V. Though this error definition leads to large fractional errors in the frequencies and a , the a -errors encompass the a -values from NuRadioMC. Though outside the scope of this work, future work would include a full-scale NuRadioMC run with thousands of waveform comparisons. The region in $[f_0, f_C]$ space for which UHE- ν signals are expected for IceCube-Gen2 Radio would be established, and man-made backgrounds that mimic UHE- ν could be rejected based on the best-fit location in the space. Finally, the connection between ϵ and cascade type (hadronic or electromagnetic) would be probed.

Hadronic case. Using the same procedure as the electromagnetic case, NuRadioMC was used to generate six hadronic cascades at 100 PeV for comparison to Equation 17. The energy was increased to show that the waveforms and model scale with energy. The hadronic cascades are not morphed by the LPM effect, so Equation 17 remains valid. The same fit procedure as the electromagnetic case was used, with two exceptions. First, the range for E_0 was increased by a factor of 10, because the cascade energy is $E_C = 100$ PeV. Second, the correlation contours represent $\rho = 0.985$. In a simple 2-level for-loop, the Pearson correlation coefficient ρ was maximized, fixing the complex poles. Next, the sum of the squared amplitude differences $(\Delta E)^2$ was minimized with the best-fit pole frequencies. The main results are shown in Figure 8.

The results shown in Figure 8 demonstrate that modeling hadronic cascades at $\theta = \theta_C$ is similar to the electromagnetic case, except that the contours indicate best-fits with $\epsilon < 1$. The amplitudes of the best-fit analytic functions in Figure 6 are smaller by a factor of 10 due to the smaller E_C . Recall that the amplitude of Equation

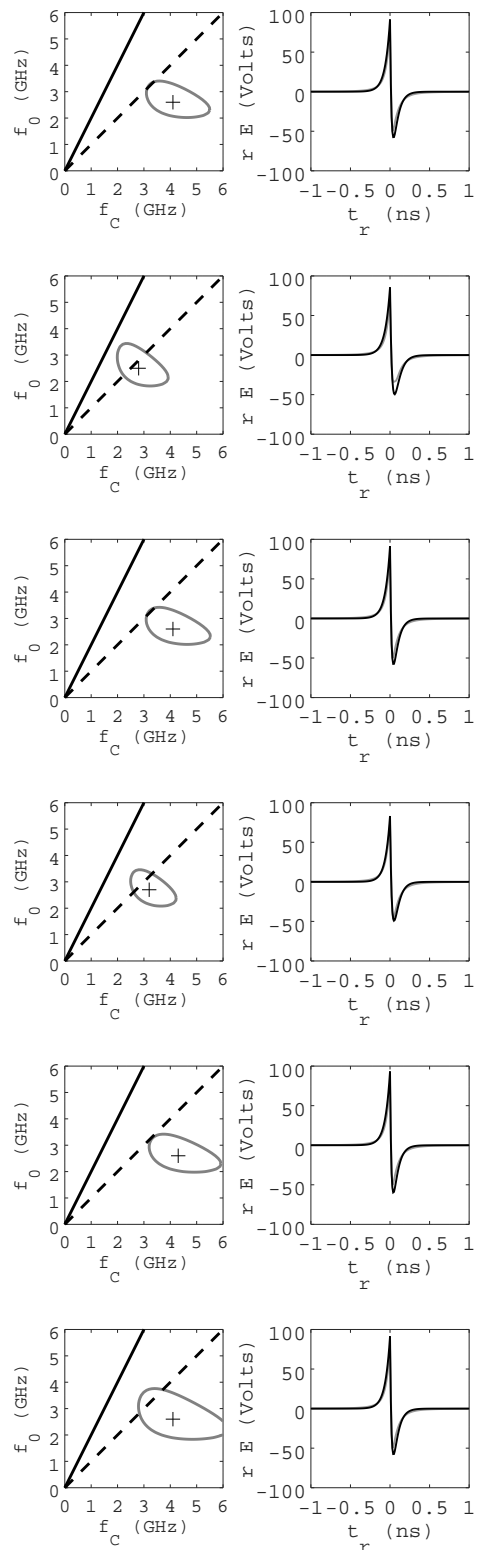


FIG. 8: **Fit results: hadronic case**, $\theta = \theta_C$, $E_C = 100$ PeV. The six rows (from top to bottom) correspond to NuRadioMC waveforms 1-6, 100 PeV hadronic cascades. (Left column) The best-fits for f_0 and f_C . Dashed line: $\epsilon = 1$. Solid line: $\epsilon = 2$. Gray contour: $\rho > 0.9$. Black cross: best-fit. (Right column) The best-fit waveforms. Gray: semi-analytic parameterizations from [23]. Black: Equation 17.

#	f_0 (GHz)	f_C (GHz)	E_0 (V GHz ⁻²)	a_{wave} (m), a_{MC} (m)	ρ	$(\Delta E)^2$ (%)
1	$3.9^{+0.2}_{-1.9}$	$2.3^{+1.3}_{-0.3}$	0.3	$4.1^{+1.2}_{-0.3}$, 4.85	0.97	6.5
2	$3.9^{+0.3}_{-1.5}$	$2.1^{+0.9}_{-0.1}$	0.5	$4.3^{+1.8}_{-0.2}$, 6.35	0.97	10.9
3	$4.0^{+1.2}_{-1.0}$	$2.3^{+0.8}_{-0.4}$	0.35	$4.1^{+0.7}_{-0.4}$, 4.48	0.96	7.5
4	$3.7^{+0.1}_{-0.5}$	$1.9^{+0.5}_{-0.1}$	1.85	$4.5^{+1.1}_{-0.3}$, 5.6	0.955	8.9
5	$3.9^{+1.4}_{-0.9}$	$2.7^{+1.4}_{-0.8}$	0.18	$4.0^{+2.0}_{-1.2}$, 4.48	0.97	5.7
6	$3.9^{+1.3}_{-1.9}$	$2.3^{+1.3}_{-0.3}$	0.31	$4.1^{+2.0}_{-0.5}$, 4.85	0.97	6.4
Ave.	3.88	2.3	0.6	4.18	0.966	7.7
Err.	3.08	0.1	0.3	0.07	0.003	0.8

TABLE V: **Fit results: electromagnetic case**, $\theta = \theta_C$, $E_C = 10$ PeV. The six rows (from top to bottom) correspond to NuRadioMC waveforms 1-6, 10 PeV electromagnetic cascades. From left to right, the form-factor cutoff-frequency, coherence cuoff-frequency, energy-scaling normalization, longitudinal length parameter, the best-fit correlation coefficient, and the relative power difference between NuRadioMC semi-analytic parameterization and the fully analytic model. The parameter means and errors in the mean are quoted in the bottom two rows.

17 goes as $E_0\omega_0^2$, so those parameters will combine to produce the factor of 10 difference. The gray contours in Fig. 8 correspond to $\rho = 0.985$, because the fits are slightly better compared to the electromagnetic case.

Table VI contains the best-fit parameters corresponding to Figure 8, along with results for ρ and $(\Delta E)^2$. The typical power difference $(\Delta E)^2$ has decreased significantly with respect to the electromagnetic case, despite using identical fit procedures. The ρ -values all exceed 0.985, and the $(\Delta E)^2$ results are typically below 2 percent. Intriguingly, $\epsilon < 1$ means higher f_C values, which in turn yields systematically low a -values, despite the increased energy. Reconstructed a -values are still within a factor of 2 of those derived from NuRadioMC. The ϵ parameter controls pulse asymmetry, and the hadronic pulses with $\epsilon < 1$ appear to be more asymmetric than the electromagnetic. Despite systematic offsets, the best-fit a and the NuRadioMC a value are tightly correlated (see Fig. 11 below).

B. Waveform Comparison: $\theta \neq \theta_C$

Electromagnetic case. Six different electromagnetic cascades and the corresponding Askaryan fields were generated with NuRadioMC for comparison to Equation 33. The events had $\theta = \theta_C + 3.0^\circ$, but r and E_C remained unchanged relative to Sec. VIA. The fit procedure is similar to Section VIA. One difference is that ω_0 only changes the waveform amplitude, along with E_0 . The pulse width $\sigma_t = \sqrt{2p}$ connects the longitudinal length a and the viewing angle with respect to the Cherenkov

#	f_0 (GHz)	f_C (GHz)	E_0 (V GHz ⁻²)	a_{wave} (m), a_{MC} (m)	ρ	$(\Delta E)^2$ (%)
1	$2.6^{+0.6}_{-0.6}$	$4.1^{+1.1}_{-1.0}$	1.0	$3.1^{+0.8}_{-0.8}$, 5.23	0.99	1.86
2	$2.5^{+0.7}_{-0.6}$	$2.8^{+0.9}_{-0.8}$	1.25	$3.75^{+1.2}_{-1.1}$, 6.35	0.99	1.83
3	$2.6^{+0.7}_{-0.6}$	$4.1^{+1.2}_{-0.9}$	1.0	$3.1^{+0.9}_{-0.7}$, 5.23	0.99	1.83
4	$2.7^{+0.6}_{-0.5}$	$3.2^{+0.8}_{-0.6}$	1.0	$3.5^{+0.9}_{-0.7}$, 6.35	0.99	2.5
5	$2.6^{+0.7}_{-0.6}$	$4.3^{+1.4}_{-1.1}$	1.0	$3.0^{+1.0}_{-0.75}$, 4.85	0.99	1.755
6	$2.6^{+1.4}_{-0.7}$	$4.1^{+1.9}_{-1.2}$	1.0	$3.1^{+1.4}_{-0.9}$, 5.23	0.99	1.86
Ave.	2.60	3.75	1.04	3.3	0.99	1.9
Err.	0.03	0.25	0.04	0.1	0.0	0.1

TABLE VI: **Fit results: hadronic case**, $\theta = \theta_C$, $E_C = 100$ PeV. The six rows (from top to bottom) correspond to NuRadioMC waveforms 1-6, 100 PeV hadronic cascades. From left to right, the form-factor cutoff-frequency, coherence cuoff-frequency, energy-scaling normalization, longitudinal length parameter, the best-fit correlation coefficient, and the relative power difference between NuRadioMC semi-analytic parameterization and the fully analytic model. The parameter means and errors in the mean are quoted in the bottom two rows.

angle $|\Delta \cos \theta| \approx \sin \theta_C (\theta - \theta_C)$.

The fit procedure was done in two stages. First, θ -values and a -values were scanned from $[\theta_C + 1.5^\circ, \theta_C + 10.0^\circ]$ and $[0.1, 10]$ meters, respectively, to determine the best-fit ρ . Once the best-fit values for a and θ are determined, $(\Delta E)^2$ is minimized by varying $f_0 = \omega_0/(2\pi)$ and E_0 from $[0.3, 3.0]$ GHz and $[0.1, 2.0]$ V GHz⁻², respectively. First, θ -values and a -values were scanned in a 2-level for loop. Once the best-fit values for a and θ were determined, $(\Delta E)^2$ was minimized by varying f_0 and E_0 in a second 2-level for loop. The main results are shown in Figure 9.

In Figure 9 (left column), the best-fit a -values and θ -values are marked with a cross. The circles represent the MC values from NuRadioMC. Circles and crosses lie on the dashed lines, because an uncertainty principle connects a -values to θ -values (see Section VA). Specifically, Equation 34 may be used to show, to first-order in $\Delta\theta = \theta - \theta_C$:

$$a\Delta\theta = \frac{c\sqrt{2p}}{\sin \theta_C} = \text{const} \quad (46)$$

Treating p as a constant derived from the waveform related to pulse width ($\sigma_t = \sqrt{2p}$) means that the product of a and $\Delta\theta$ is a constant. The parameters are therefore inversely proportional: $a = \text{const}/\Delta\theta$. Further, the $\rho > 0.95$ contour is centered along the dashed lines, indicating all good fits respect the uncertainty principle. Thus, measurement of the true pulse width (beyond that introduced by RF detection components) represents a strong constraint on the event geometry. The best-fit results happen to fall above and below the true values

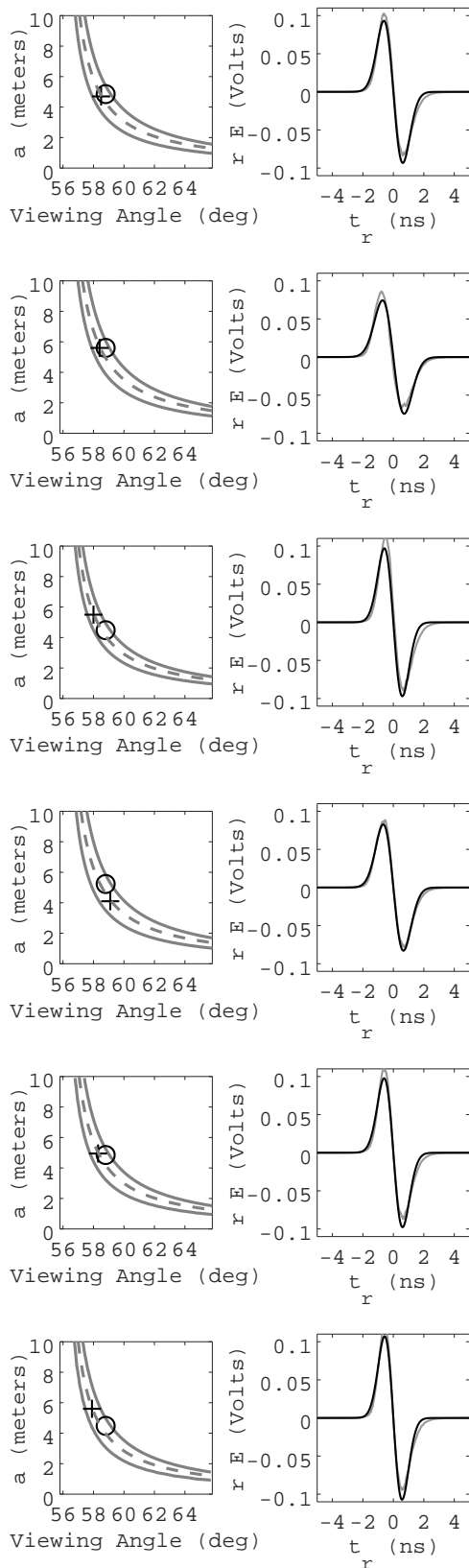


FIG. 9: **Fit results: electromagnetic case, $\theta \neq \theta_C$, $E_C = 10$ PeV.** The six rows (from top to bottom) correspond to NuRadioMC waveforms 1-6, 10 PeV electromagnetic cascades. (Left column) Best-fit θ and a -values. Crosses: best-fits. Circles: MC true values. Gray contour: $\rho > 0.95$. Dashed line: a versus θ from Equation 34 (uncertainty principle). (Right column) The best-fit waveforms. Gray: semi-analytic parameterizations from [23]. Black: Equation 33.

#	θ_{wave} (deg), θ_{MC} (deg)	a_{wave} (m), a_{MC} (m)	f_0 (GHz)	E_0 (V GHz ⁻²)	l (cm)	ρ	$(\Delta E)^2$ (%)
1	$58.5^{+0.7}_{-0.6}$, 58.8	$4.7^{+1.3}_{-1.0}$, 4.85	0.75	1.2	$3.4^{+0.9}_{-0.7}$	0.99	1.93
2	$58.4^{+0.6}_{-0.5}$, 58.8	$5.6^{+1.4}_{-1.1}$, 5.60	1.0	1.2	$2.6^{+0.4}_{-0.3}$	0.99	2.61
3	$58.0^{+0.5}_{-0.4}$, 58.8	$5.5^{+1.3}_{-1.0}$, 4.48	1.0	1.1	$2.6^{+0.3}_{-0.2}$	0.98	4.47
4	$59.1^{+0.9}_{-0.7}$, 58.8	$4.1^{+1.2}_{-0.9}$, 5.23	0.75	1.2	$3.4^{+0.5}_{-0.5}$	0.995	0.80
5	$58.3^{+0.7}_{-0.5}$, 58.8	$4.95^{+1.4}_{-1.1}$, 4.85	0.75	1.2	$3.4^{+0.4}_{-0.3}$	0.99	1.8
6	$57.9^{+0.6}_{-0.4}$, 58.8	$5.6^{+1.5}_{-1.2}$, 4.48	0.75	1.2	$3.5^{+0.5}_{-0.4}$	0.99	1.83
Ave.	58.4	5.1	0.83	1.18	3.2	0.989	2.2
Err.	0.2	0.2	0.05	0.02	0.2	0.002	0.5

TABLE VII: **Fit results: electromagnetic case, $\theta \neq \theta_C$, $E_C = 10$ PeV.** The six rows (from top to bottom) correspond to NuRadioMC waveforms 1-6, 10 PeV electromagnetic cascades. From left to right, the viewing angle, longitudinal length parameter, form-factor cutoff frequency, the energy-scaling normalization, the lateral width of the cascade $l = 1/(\sqrt{2\pi\rho_0})$, the best-fit correlation coefficient, and the relative power difference between NuRadioMC semi-analytic parameterization and the fully analytic model. The parameter means and errors in the mean are quoted in the bottom two rows.

along the dashed line, indicating no systematic bias. The best-fit waveforms are shown in Figure 9 (right column). Typical correlation coefficients exceed $\rho = 0.98$ and typical fractional power differences are less than 3%. Table VII contains the results from the fit procedure.

The results in Table VII demonstrate that Equation 33 is in excellent agreement with the semi-analytic parameterization. On average, the fits encompass the MC values of θ and a , and the results for f_0 and E_0 values match expectations. The results for l , the lateral width parameter, are derived from the f_0 results. Cascades with l -values of a few cm should have cutoff-frequencies in the ≈ 1 GHz range. The ρ -values are comparable to the hadronic on-cone data ($\rho \approx 0.98$), indicating that the *symmetric approximation* made in Section V does not incur much penalty. The fractional power differences $(\Delta E)^2$ are $\approx 2\%$, most likely due to the amplitude symmetry of the fully analytic model.

Hadronic case. Six different hadronic events and the corresponding Askaryan fields were generated for comparison to Equation 33. The fit procedure is the same as the electromagnetic case, except that the range for E_0 is expanded to $[1.0, 20.0]$ V GHz⁻², because the cascade energy is $E_C = 100$ PeV. The main results are shown in Figure 10.

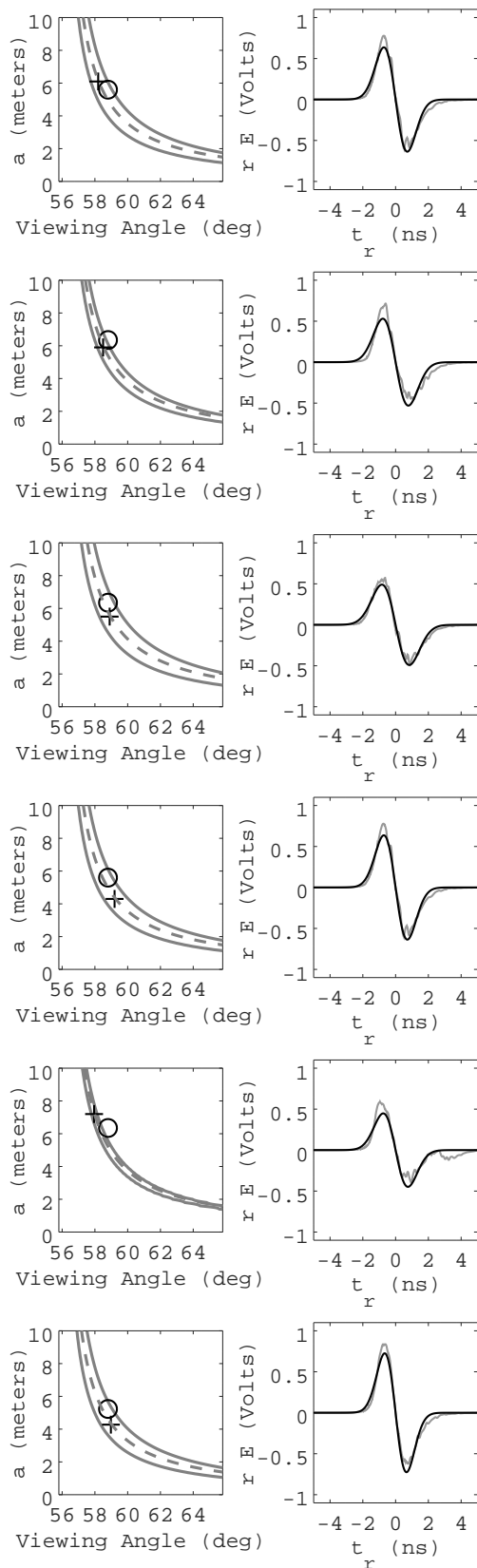


FIG. 10: **Fit results: hadronic case, $\theta \neq \theta_C$, $E_C = 100$ PeV.** The six rows (from top to bottom) correspond to NuRadioMC waveforms 1-6, 100 PeV hadronic cascades. (Left column) Best-fit θ and a -values. Crosses: best-fits. Circles: MC true values. Gray contour: $\rho > 0.95$. Dashed line: a versus θ from Equation 34 (uncertainty principle). (Right column) The best-fit waveforms. Gray: semi-analytic parameterizations from [23]. Black: Equation 33.

#	θ_{wave} (deg), θ_{MC} (deg)	a_{wave} (m), a_{MC} (m)	f_0 (GHz)	E_0 (V GHz ⁻²)	l (cm)	ρ	$(\Delta E)^2$ (%)
1	$58.2^{+0.6}_{-0.4}$, 58.8	$6.1^{+1.5}_{-1.2}$, 5.6	0.8	10.6	$3.2^{+0.5}_{-0.5}$	0.98	3.55
2	$58.5^{+0.4}_{-0.3}$, 58.8	$5.9^{+0.9}_{-0.8}$, 6.35	0.85	10.3	$3.0^{+0.3}_{-0.2}$	0.96	7.1
3	$58.9^{+0.8}_{-0.6}$, 58.8	$5.5^{+1.4}_{-1.1}$, 6.35	0.9	10.8	$2.8^{+0.5}_{-0.5}$	0.98	2.64
4	$59.2^{+0.8}_{-0.7}$, 58.8	$4.3^{+1.1}_{-0.8}$, 5.6	0.85	10.5	$3.0^{+0.5}_{-0.5}$	0.98	3.10
5	$58.0^{+0.2}_{-0.2}$, 58.8	$7.2^{+0.6}_{-0.6}$, 6.35	0.9	8.2	$2.9^{+0.3}_{-0.3}$	0.955	8.76
6	$59.0^{+0.8}_{-0.6}$, 58.8	$4.3^{+1.1}_{-0.9}$, 5.23	0.85	10.4	$3.0^{+0.5}_{-0.5}$	0.985	3.00
Ave.	58.6	5.5	0.86	10.1	3.2	0.973	5
Err.	0.2	0.5	0.015	0.4	0.2	0.005	1

TABLE VIII: **Fit results: hadronic case, $\theta \neq \theta_C$, $E_C = 100$ PeV.** The six rows (from top to bottom) correspond to NuRadioMC waveforms 1-6, 10 PeV hadronic cascades. From left to right, the viewing angle, longitudinal length parameter, form-factor cutoff frequency, the energy-scaling normalization, the lateral width of the cascade $l = 1/(\sqrt{2\pi\rho_0})$, the best-fit correlation coefficient, and the relative power difference between NuRadioMC semi-analytic parameterization and the fully analytic model. The parameter means and errors in the mean are quoted in the bottom two rows.

As with the electromagnetic case, ρ is maximized and $(\Delta E)^2$ is minimized. Table VIII contains the best-fit parameters, along with ρ and $(\Delta E)^2$. Solutions with $\rho \approx 0.98$ and $(\Delta E)^2 \approx 5\%$ were found. Similar to the results shown in Table VII, the results in Table VIII are in agreement with the MC values from NuRadioMC. Note that the f_0 and l values are statistically equivalent between Tables VII and VIII. The similarity leaves the amplitude scaling to the E_0 parameter. The E_0 -values match expectations for 100 PeV cascades, because they are a factor of 10 higher than those of the 10 PeV electromagnetic case.

One interesting question is whether the analytic model has potential to *reconstruct* cascade parameters from the waveform shape. What follows is a simple reconstruction exercise. For the first exercise, $\theta = \theta_C + 3.0^\circ$ is assumed fixed by another part of the analysis. For example, θ could be determined by measuring the cutoff-frequency in the Fourier domain below 1 GHz [34]. Fixing θ means a is the only parameter varied. Scanning Equation 33 over all NuRadioMC waveforms at fixed $\theta = \theta_C + 3.0^\circ$ yields Figure 11, in which the best-fit a -value is graphed versus the MC a -value. The a -errors from the digitized MC cascades profiles, and the a -errors from waveforms, are taken to be ± 10 cm (\pm two Δa step-sizes). A non-linear least-squares (NLLS) algorithm was applied to fit

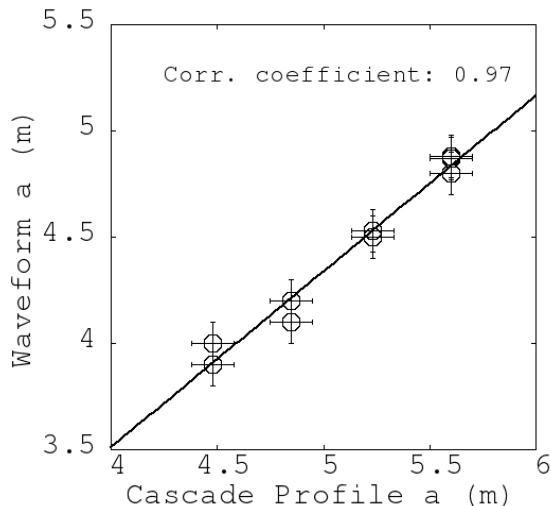


FIG. 11: The longitudinal length parameter a derived from the Equation 33 best-fit versus the a -value derived from the cascade profile in NuRadioMC. A linear fit and correlation coefficient are shown (slope: 0.83 ± 0.05 , intercept: 0.2 ± 0.2 (m), correlation coefficient = 0.97).

a function to the data, and a linear correlation appeared. The linear function is a good fit, and the correlation between waveform a -values and MC a -values is 0.97.

The results in Figure 11 imply a reconstruction technique for $\ln(E_C)$ using the formulas found in Section IIIB. Consider the relationship between a and $\ln(x)$, where $x = E_C/E_{\text{crit}}$: $a = c_1 \sqrt{\ln(x)}$. The fractional error in $\ln(x)$ is proportional to the fractional error in a :

$$\frac{\sigma_{\ln(x)}}{\ln(x)} = 2c_1 \left(\frac{\sigma_a}{a} \right) \quad (47)$$

VII. CONCLUSION

To advance the field of UHE- ν detection, a fully analytic Askaryan model in the time-domain has been presented, that accounts for the UHE- ν cascade energy, and lateral and longitudinal geometry. The fully analytic model was compared to published results generated with a semi-analytic parameterization used commonly in NuRadioMC, a simulation being used to design IceCube-Gen2. Pearson correlation coefficients between the fully analytic and semi-analytic parameterizations were found to be greater than 0.95, and typical fractional differences in total power were found to be $\approx 5\%$. What follows is a brief summary of new results, and a list of potential uses for a fully analytic model in UHE- ν detection.

A. Summary of New Results

The main results are summarized in Table IX. Beginning with the analytic form factor, it was shown that the

Result	Location
$\tilde{F}(\omega, \theta)$, analytic form factor	Eq. 6, Sec. III A
$a \propto \sqrt{\ln(E_C)}$, longitudinal parameter	Eqs. 11 and 14, Sec. III B
$r\vec{E}(t_r, \theta_C)$, on-cone field ($\hat{\theta}$)	Eq. 17, Sec. IV
$\sigma_t \sigma_\nu$, on-cone uncertainty principle	Eq. 24, Sec. IV A
$r\vec{E}(t_r, \theta)$, off-cone field ($\hat{\theta}$)	Eq. 33, Sec. V
$\sigma_t \sigma_\nu$, off-cone uncertainty principle	Eq. 36, Sec. V A
On-cone EM comparison to NuRadioMC	Fig. 6, Tab. V
On-cone HAD comparison to NuRadioMC	Fig. 8, Tab. VI
Off-cone EM comparison to NuRadioMC	Fig. 9, Tab. VII
Off-cone HAD comparison to NuRadioMC	Fig. 10, Tab. VIII

TABLE IX: A summary of results in this work.

lateral component of the cascade ICD leads to the effect of a two-pole, low-pass filter on the radiated Askaryan field. The analytic form factor was first presented in [34], however this work has introduced the first complete study of the effect of its complex poles on the Askaryan radiation. The longitudinal factor a was introduced by Ralston and Buny [33], who described the connection to cascade energy. In this work, a technique for cascade energy reconstruction was presented, based on constraining a -values by fitting analytic functions to observed waveforms. The connection was made for both electromagnetic cascades and hadronic cascades via common parameterizations of each cascade type.

The on-cone field equations were derived, similar to the final section of [34]. This work represents the first time the two pole frequencies f_0 and f_C and the asymmetry parameter ϵ have been used to characterize the time-domain field equations such that they match semi-analytic parameterizations. The ϵ parameter reveals a potential cascade type identification scheme. Finally, the on-cone field equations were shown to satisfy the uncertainty principle for Fourier transform pairs. The off-cone field equations were derived as well, and were shown to be in excellent agreement with semi-analytic parameterizations. Cascade parameters like a and the viewing angle θ were fit directly from the waveforms. Finally, matching off-cone field equations to data shows promise as an energy reconstruction.

To obtain the off-cone field, a Taylor series to first order in $(u - 1)$ was used, with $u = 1 - i\eta$. Further, an approximation involving $\eta = \omega/\omega_C$ was made in \tilde{F} . Both steps imply $\eta < 1$. The restriction $\eta < 1$ means that Eqs. 17 and 33 must be applied to the far-field. Given that a and θ_C are fixed by cascade physics and ice density, and that the relevant Askaryan bandwidth for ice is $[0.1 - 1]$ GHz, the parameter most easily varied within η is the observer distance r . Taking $\nu = 0.5$ GHz, $n = 1.78$, $c = 0.3$ m GHz, $\theta = \theta_C$, and $a = 5$ m, requiring that $\eta = 1$ gives $r \geq 0.4$ km. Scaling to $\nu = 0.25$ GHz gives $r \geq 0.2$ km. According to NuRadioMC [23] (Fig. 13), the r corresponding to UHE- ν at 10^{18} eV ranges from 0.7-3.2 km,

and 0.2 km is rare.

The “acceleration argument” invoked by RB in [33] states that if $r(t)$ points to the ICD, $r(t)$ must be constant enough to ensure that $\Delta r < \lambda$. Using the law of cosines, with two sides being r and $r + \Delta r$, and a third being a , the criteria that $(a/r)^2 \ll 1$ leads to $|\Delta r| \approx a/n$ which is $\mathcal{O}(2)$ m. When in doubt about usage and event geometry, determining if $(a/r)^2 \ll 1$ is a good check. If the UHE- ν event is a charged-current interaction with an electromagnetic cascade above the LPM energy, a grows faster than $\sqrt{\ln(E_C/E_{\text{crit}})}$ (Sec. III B) [20]. Since future work with this fully analytic treatment will focus on developing probability distributions for parameters like a as part of an energy reconstruction, it will be straightforward to study in parallel when the far-field requirement breaks down by comparing to ARZ models in NuRadioMC [23].

B. Utility of the Analytic Model

There are at least four main advantages of fully analytic models that accurately model the central Askaryan radiation peak. First, when analytic models are matched to observed data, cascade properties may be derived directly from the waveforms. Second, in large scale simulations required for the design of IceCube-Gen2 radio, evaluating a fully analytic model technically provides a speed advantage when compared to the semi-analytic parameterizations. Third, fully analytic models, combined with RF antenna response, can be embedded in firmware to form a *matched filter* that enhances detection probability. Fourth, parameters in analytic models may be *scaled* to produce results that apply to media of different density than ice. This application is useful for understanding potential signals in the Antarctic *firn*, or the upper layer of snow and ice that is of lower density than the solid ice beneath it.

The ability to fit cascade properties from waveforms will be a useful tool for IceCube-Gen2 radio. Examples of current reconstruction techniques include the forward-folding method [25] and information field theory (IFT) [26]. In particular, the longitudinal length parameter a leads to a reconstruction of $\ln(E_C)$, given knowledge of $\Delta\theta$ (Fig. 11 and Equation 47). Further, all designs for detector stations in IceCube-Gen2 radio include many distinct RF channels and one phased-array of channels. Treating each waveform like a separate trial would increase cascade parameter precision. Even pulse width measurements without *a priori* knowledge of a or θ contain information about (θ, a) via the uncertainty principle (see gray contours of Figures 4 and 5).

Much thought has been devoted to improving the speed of NuRadioMC and other codes that rely on Askaryan emissions models. In one example, AraSim, convolution with pre-simulated cascade profiles was dropped in favor of on-the-fly convolution with the Greisen (EM) and Gaisser-Hillas (hadronic) cascade profiles [22]. While this reduced the number of steps required to produce a sim-

ulated data set, it required the code to make a determination of the optimal time-window before convolution. To boost speed in NuRadioMC, a shower library is implemented that pulls realistic cascade profiles from memory and performs the convolutions event-by-event. This strategy provides accuracy regarding randomly forming sub-showers due to the LPM effect, but it still requires a convolution for each event [23]. A fully analytic model provides fast accuracy in the energy range of 10-100 PeV without the need for a shower library.

The most intriguing usage for a fully analytic Askaryan model would be to embed the model as a *matched filter* in detector firmware. Matched filters are commonly used in radar systems that maximize detected SNR by correlating incoming data with a pre-defined function. Because the cascade properties would not be known in advance, an array of matched filters could be implemented to form a *matched filter bank*. One example of this approach was the TARA experiment [41], which was designed to detect low-SNR cosmic ray radar echoes. This is similar to the challenge faced by IceCube-Gen2 radio: pushing the limit of low-SNR RF pulse detection in a remote setting. For example, a matched filter bank could be formed with an array of off-cone field formulas with fixed a -value and varying θ -values, which would then be convolved with the RF channel impulse response (see Section 6 of [27]).

Finally, a fully analytic model enhances the ability of IceCube-Gen2 radio to identify signals that originate in the firn. At the South Pole, the RF index of refraction begins around 1.35 and does not reach the solid ice value of 1.78 until 150-200 meters [28]. There are at least two signals that could originate in the firn: UHE- ν events that create Askaryan radiation, and UHE cosmic ray cascades partially inside or fully inside the firn. The altitude of the South Pole makes the latter possible. The Askaryan radiation of the firn UHE- ν events could be modeled via appropriate density-scaling of the longitudinal length a and the lateral width $1/(\sqrt{2\pi}\rho_0)$. Being lower in density, the firn would contain cascades of different size and shape relative to solid ice, which would alter the radiation. The Askaryan radiation of UHE cosmic ray cascades within the firn could be similarly modeled, with the caveat that the cosmic ray cascades would also generate geosynchrotron radiation. In the atmosphere, the geosynchrotron radiation dominates. In the firn, the UHE cosmic ray cascade would only be meters in length instead of kilometers, and the Askaryan radiation is expected to dominate.

VIII. ACKNOWLEDGEMENTS

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Appendix A: Details of the On-Cone Field Equation Derivation

The original equations for the $\hat{\theta}$ -component of $\vec{\mathcal{E}}$ are:

$$\mathcal{W}(\eta, \theta) = \frac{\exp\left(-\frac{1}{2}(ka)^2 \frac{(\cos\theta - \cos\theta_C)^2}{1-i\eta}\right)}{\left(1-i\eta\left(1-3i\eta \frac{\cos\theta}{\sin^2\theta} \frac{\cos\theta - \cos\theta_C}{1-i\eta}\right)\right)^{1/2}} \quad (\text{A1})$$

$$\vec{\mathcal{E}}(\eta, \theta) \cdot \hat{\theta} = \mathcal{W}(\eta, \theta) \left(1-i\eta \frac{\cos\theta_C}{\sin^2\theta} \frac{\cos\theta - \cos\theta_C}{1-i\eta}\right) \quad (\text{A2})$$

Letting $\theta = \theta_C$ yields

$$\vec{\mathcal{E}}(\eta, \theta) \cdot \hat{\theta} = \frac{1}{\sqrt{1-i\eta}} \quad (\text{A3})$$

The complete field from the original RB model [33], including the form factor \tilde{F} , $\psi = -i \exp(ikr) \sin\theta$, and $\vec{\mathcal{E}}$ is

$$r\vec{E}(\omega, \theta) = E_0 \left(\frac{\omega}{2\pi}\right) \psi \vec{\mathcal{E}}(\eta, \theta) \tilde{F} \quad (\text{A4})$$

Let Equation 6 for the form factor, with $\sigma = \omega/\omega_{\text{CF}}$ and $\eta = \omega/\omega_{\text{CF}}$, and letting E_0 be proportional to cascade energy E_C :

$$r\tilde{E}(\omega, \theta_C) = \frac{(-i\omega)E_0 \sin(\theta_C) e^{i\omega r/c}}{(1-i\omega/\omega_C)^{1/2} (1+(\omega/\omega_{\text{CF}})^2)^{3/2}} \quad (\text{A5})$$

Suppose $\omega < \omega_C$, and $\omega < \omega_{\text{CF}}$, such that the following approximations of the factors in the denominator are valid:

$$(1-i\omega/\omega_C)^{1/2} \approx 1 - \frac{i}{2} \frac{\omega}{\omega_C} \quad (\text{A6})$$

$$(1+(\omega/\omega_{\text{CF}})^2)^{3/2} \approx 1 + \frac{3}{2} \left(\frac{\omega}{\omega_{\text{CF}}}\right)^2 \quad (\text{A7})$$

Using the approximations introduces simple poles into the complex formula for the frequency-dependent electric field. Inserting the approximations in the denominator of Equation A5, we have

$$r\tilde{E}(\omega, \theta_C) = \frac{(-i\omega)E_0 \sin(\theta_C) e^{i\omega R/c}}{(1-\frac{i}{2}\omega/\omega_C) (1+\frac{3}{2}(\omega/\omega_{\text{CF}})^2)} \quad (\text{A8})$$

The denominator can be rearranged by factoring the ω coefficients, and defining $\omega_0 = \sqrt{\frac{2}{3}}\omega_{\text{CF}}$.

$$r\tilde{E}(\omega, \theta_C) = \frac{2i\omega_C\omega_0^2(-i\omega)E_0 \sin(\theta_C) e^{i\omega r/c}}{(2i\omega_C + \omega)(\omega + i\omega_0)(\omega - i\omega_0)} \quad (\text{A9})$$

Let $\hat{E}_0 = E_0 \sin(\theta_C)$, and let the retarded time be $t_r = t - r/c$. Taking the *inverse* Fourier transform, using the same sign convention as RB [33] ($f(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{F}(\omega) e^{-i\omega t} d\omega$), converts the field to the time-domain:

$$rE(t, \theta_C) = \frac{\hat{E}_0 i\omega_C \omega_0^2}{\pi} \frac{d}{dt_r} \int_{-\infty}^{\infty} \frac{e^{-i\omega t_r}}{(2i\omega_C + \omega)(\omega + i\omega_0)(\omega - i\omega_0)} d\omega \quad (\text{A10})$$

1. If $t_r > 0$: Consider the contour comprised of the real axis and the clockwise-oriented negative infinite semi-circle. On the contour, the exponential phase factor in Equation A10 goes as

$$\exp(-i\omega t_r) = \exp(-i(R \cos\phi + iR \sin\phi)t_r) \quad (\text{A11})$$

For the semi-circle, $\phi \in [\pi, 2\pi]$, so $\sin\phi < 0$ and $t_r > 0$. Exponential decay occurs and the integrand vanishes on the semi-circle for $|\omega| = R \rightarrow \infty$.

2. If $t_r < 0$: Consider the contour comprised of the

real axis and the counter-clockwise-oriented positive infinite semi-circle. On the contour, the exponential phase factor in Equation A10 goes again as

$$\exp(-i\omega t_r) = \exp(-i(R \cos\phi + iR \sin\phi)t_r) \quad (\text{A12})$$

For the semi-circle, $\phi \in [0, \pi]$, so $\sin\phi > 0$ and $t_r < 0$. Exponential decay occurs and the integrand vanishes on the semi-circle for $|\omega| = R \rightarrow \infty$.

Using cases 1 and 2, Equation A10 can be solved using

the Cauchy integral formula. Beginning with $t_r > 0$, two poles are enclosed in the semi-circle: one that originated from the coherence cutoff frequency, and the other that

originated from the form factor. The Cauchy integral formula yields

$$rE(t, \theta_C) = 2\hat{E}_0\omega_C\omega_0^2 \frac{d}{dt_r} \left(\frac{e^{-2\omega_C t_r}}{i^2(-2\omega_C + \omega_0)(-2\omega_C - \omega_0)} + \frac{e^{-\omega_0 t_r}}{i^2(-\omega_0 + 2\omega_C)(-2\omega_0)} \right) \quad (\text{A13})$$

Define the ratio of the cutoff frequencies: $\epsilon = \omega_0/\omega_C$. After evaluating the time derivatives, Equation A13 becomes

$$rE(t, \theta_C) = \hat{E}_0\omega_0^2 \left(\frac{e^{-2\omega_C t_r}}{(1 - \frac{\epsilon}{2})(1 + \frac{\epsilon}{2})} - \frac{e^{-\omega_0 t_r}}{(2)(1 - \frac{\epsilon}{2})} \right) \quad (\text{A14})$$

Expanding to linear order in ϵ , assuming $\epsilon < 1$, and recalling that $\omega_0^2 = \frac{2}{3}\omega_{CF}^2$:

$$rE(t, \theta_C) \approx \frac{1}{3}\hat{E}_0\omega_{CF}^2 \left(2e^{-2\omega_C t_r} - \left(1 + \frac{\epsilon}{2}\right) e^{-\omega_0 t_r} \right) \quad (\text{A15})$$

Turning to the case of $t_r < 0$, consider integrating Equation A10 along the contour comprised of the real axis and the counter-clockwise-oriented positive infinite semi-circle. The contour encloses one pole, and the exponent ensures convergence:

$$rE(t, \theta_C) = (2\pi i)\hat{E}_0(\pi)^{-1}i\omega_C\omega_0^2 \frac{d}{dt_r} \left(\frac{e^{\omega_0 t_r}}{(2i\omega_C + i\omega_0)(2i\omega_0)} \right) \quad (\text{A16})$$

After evaluating the derivative, the expression simplifies with $\epsilon = \omega_0/\omega_C$:

$$rE(t, \theta_C) = \frac{1}{2}\hat{E}_0\omega_0^2 \left(\frac{e^{\omega_0 t_r}}{1 + \frac{1}{2}\epsilon} \right) \quad (\text{A17})$$

Finally, using the same first-order approximation in ϵ as the $t_r > 0$ case:

$$rE(t, \theta_C) \approx \frac{1}{3}\hat{E}_0\omega_{CF}^2 \left(1 - \frac{1}{2}\epsilon \right) e^{\omega_0 t_r} \quad (\text{A18})$$

Collecting the $t_r > 0$ and $t_r < 0$ results together:

$$rE(t, \theta_C) = \frac{1}{3}\hat{E}_0\omega_{CF}^2 \begin{cases} \left(1 - \frac{1}{2}\epsilon\right) e^{\omega_0 t_r} & t_r < 0 \\ \left(2e^{-2\omega_C t_r} - \left(1 + \frac{1}{2}\epsilon\right) e^{-\omega_0 t_r}\right) & t_r > 0 \end{cases} \quad (\text{A19})$$

Appendix B: Details of the Off-Cone Field Equation Derivation

Using Tabs. II-IV, Equation A2 reduces to

$$\mathcal{E}(u, x) = f(u, x)g(u, x)(1 - h(u, x)) \quad (\text{B1})$$

Expanding to first-order with respect to u near ($u = 1$) gives

$$\mathcal{E}(u, x) = \mathcal{E}(x, 1) + (u - 1)\dot{\mathcal{E}}(x, 1) + \mathcal{O}(u - 1)^2 \quad (\text{B2})$$

The first term is $fg(1-h)$ evaluated at $u = 1$: $\exp(-y)$ (Table IV). The second term requires the first derivative of $\mathcal{E}(u, x)$ with respect to u , evaluated at $u = 1$.

$$\dot{\mathcal{E}}(u, x) = f\dot{g} + \dot{f}g - (f\dot{g}h + f\dot{g}h + \dot{f}gh) \quad (\text{B3})$$

$$\dot{\mathcal{E}}(1, x) = \left(f\dot{g} + \dot{f}g - (f\dot{g}h + f\dot{g}h + \dot{f}gh) \right) |_{u=1} \quad (\text{B4})$$

The first-derivatives of f , g , and h , evaluated at $u = 1$, are given in Tab. IV. Because $h(x, 1) = 0$, terms proportional to h will vanish. The result is

$$\dot{\mathcal{E}}(1, x) = \frac{1}{2}e^{-y} (2y + 2q - 1) \quad (\text{B5})$$

Inserting Equation B5 into Equation B2,

$$\mathcal{E}(u, x) = e^{-y} \left(1 + \frac{1}{2}(u - 1)(2y + 2q - 1) \right) \quad (\text{B6})$$

Using the definition of u (Table II), the result may be written

$$\mathcal{E}(u, x) = e^{-y} \left(1 - \frac{1}{2} j\eta (2y + 2q - 1) \right) \quad (\text{B7})$$

Proceeding with the inverse Fourier transform of the $\hat{\theta}$ -component:

$$rE(t, \theta) = \mathcal{F}^{-1} \left\{ E_0 \left(\frac{\omega}{2\pi} \right) \tilde{F} \psi \mathcal{E} \right\} \quad (\text{B8})$$

Let $\eta = \omega/\omega_C$, $y = p\omega^2$ (Table II). Inserting the Taylor series for \mathcal{E} , the form factor \tilde{F} , and $\psi = -i \exp(ikr) \sin \theta$ (Sec. II), and following the same steps as the on-cone case produces

$$2\pi r E(t, \theta) = \frac{E_0 \omega_0^2 \sin(\theta)}{4\pi i \omega_C} \frac{d}{dt_r} \int_{-\infty}^{\infty} \frac{e^{-i\omega t_r - p\omega^2} (2i\omega_C + 2p\omega^3 + (2q-1)\omega)}{\omega^2 + \omega_0^2} d\omega \quad (\text{B9})$$

Unlike the on-cone case, Equation B9 cannot be integrated with infinite semi-circle contours, because the exponential term diverges along the imaginary axis far from the origin. Let I_0 represent the constant term with respect to ω in the numerator:

$$I_0 = \int_{-\infty}^{\infty} \frac{e^{-i\omega t_r - p\omega^2} (2i\omega_C)}{\omega^2 + \omega_0^2} d\omega \quad (\text{B10})$$

Further, let I_1 and I_3 represent the linear and cubic terms, respectively. Completing the square in the exponent of I_0 , with $\omega_1 = t_r/(2p)$, yields

$$I_0 = 2i\omega_C e^{-\frac{t_r^2}{4p}} \int_{-\infty}^{\infty} \frac{e^{-p(\omega+i\omega_1)^2}}{\omega^2 + \omega_0^2} d\omega \quad (\text{B11})$$

Equation B11 may be re-cast as the *line-broadening function*, H (DLMF 7.19, [39]) common to spectroscopy applications:

$$I_0 = 2\pi i \left(\frac{\omega_C}{\omega_0} \right) e^{-\frac{t_r^2}{4p}} H(\sqrt{p}\omega_0, i\sqrt{p}\omega_1) \quad (\text{B12})$$

Assume that $\omega > \omega_1$. This approximating step will be called the *symmetric approximation*.

$$I_0 \approx 2i\omega_C e^{-\frac{t_r^2}{2p}} \int_{-\infty}^{\infty} \frac{e^{-p\omega^2}}{\omega^2 + \omega_0^2} d\omega \quad (\text{B13})$$

The result for I_0 involves the complementary error function (DLMF 7.7.1, [39]):

$$I_0 = 2i\omega_C e^{-\frac{t_r^2}{2p}} \pi \omega_0^{-1} e^{p\omega_0^2} \operatorname{erfc}(\sqrt{p}\omega_0) \quad (\text{B14})$$

The integrals I_1 and I_3 are zero by symmetry, with odd integrands over $(-\infty, \infty)$. Inserting the result for I_0 into Equation B9 and evaluating the derivative finishes the problem (see Sec. V).

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