Complex analysis of Askaryan radiation: energy and angular reconstruction of ultra-high energy neutrinos

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Outline: from mathematical physics to UHE-ν observations

Background

- UHE-ν flux
- The Askaryan effect
- RF UHE-ν detectors

The Askaryan signal

- Analytic radiation model
- Signal *envelopes*
- Uncertainty principles

Event reconstruction

- Mathematics $\leftrightarrow$ signals
- **NuRadioMC**: MC software
- Initial results

Figure 1: Whittier College.

Background

UHE-ν detection via the Askaryan Effect

- $E_\nu \geq 10^{16}$ eV
- **Flux is small**
- Cascades $\leftrightarrow$ RF
- Ice is transparent
- Station arrays

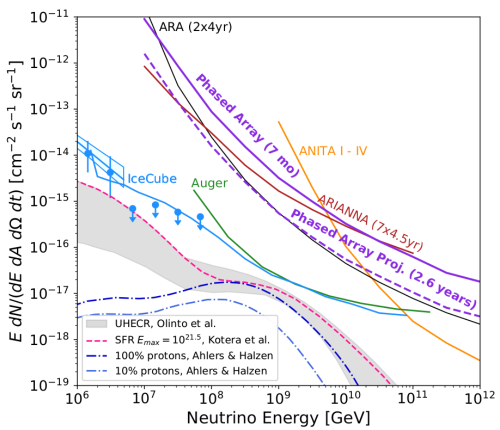

Figure 2: UHE-ν flux predictions and upper limits.

UHE-ν detection via the Askaryan Effect

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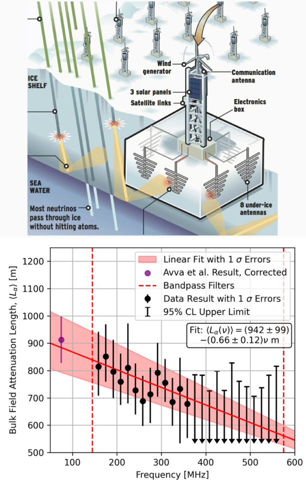

Figure 3: (Top) Detectors use the Askaryan effect. (Bottom) RF attenuation lengths in ice.

Background: in-ice UHE-ν observations

UHE-ν detection via the Askaryan Effect

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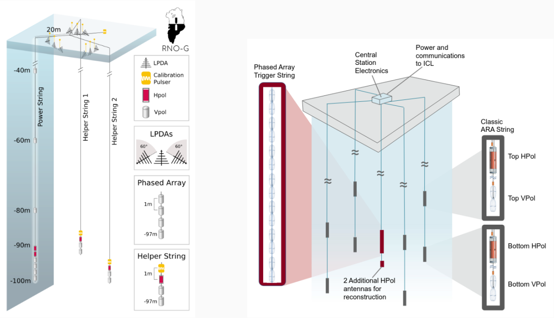

Figure 4: In-ice UHE-ν detectors. (Left) RNO-G (Greenland). (Right) ARA (Antarctica).

The Askaryan Signal

The Askaryan Signal: equation for $\vec{E}$-field

Askaryan electric field, $\vec{E}(r,t)$, [V m^{-1}]

$$r\vec{E}(t,\theta) = -\frac{E_0\omega_0\sin(\theta)}{8\pi p}t_r e^{-\frac{t_r^2}{4p}+p\omega_0^2}\operatorname{erfc}(\sqrt{p}\omega_0) \quad (1)$$

Geometric parameters

r	vertex distance [m]
E_0	E-field amplitude [V GHz^{-2}]
t_r	Retarded time [ns][a]
θ	Observation angle [rad]
p	$\sigma_t = \sqrt{2p}$ (Eq. 2) [ns^2]

[a] Definition: $t_\mathrm{r} = t_\mathrm{ref} - nR/c$.

Particle physics parameters

ω_0	Form-factor frequency [GHz]
θ_C	Cherenkov angle [rad]
p	$\sigma_t = \sqrt{2p}$ (Eq. 2) [ns^2]
a	Cascade length (Eq. 2) [m]

$$p = \frac{1}{2}\left(\frac{a}{c}\right)^2(\cos\theta - \cos\theta_C)^2 \quad (2)$$

$$\sqrt{2p} \approx (a/c)|\theta - \theta_\mathrm{C}|\sin\theta_\mathrm{C} \quad (3)$$

$$\sigma_t = \sqrt{2p} \propto a\Delta\theta \quad (4)$$

The Askaryan Signal: $\vec{E}$-field time-dependence, analytic signal

Askaryan electric field, $\vec{E}(r,t)$, [V m^{-1}], time-dependence

$$s(t) = \vec{E}(t,\theta) = -E_0 t e^{-\frac{1}{2}\left(\frac{t}{\sigma_t}\right)^2} \quad (5)$$

Askaryan electric field *analytic signal*

$$s_a(t) = -E_0 \left(t e^{-\frac{1}{2}(t/\sigma_t)^2} - \frac{2j\sigma_t}{\sqrt{2\pi}} \frac{dD(x)}{dx} \right) \quad (6)$$

The signal envelope

- $s_a(t) = s(t) + j\,\widehat{s}(t)$
- $\widehat{s}(t)$, Hilbert transform
- $|s_a(t)|$, signal *envelope*

Special functions and variables

- $D(x)$, Dawson function
- $x = t/(\sqrt{2}\sigma_t)$, normalized time

The Askaryan Signal: detected signals

Common response function for RF channels $r(t)$, [m ns^{-1}]

$$r(t) = R_0 \cos(2\pi f_0 t) e^{-\gamma t} \tag{7}$$

Common analytic signal for RF channels $r_a(t)$, [m ns^{-1}]

$$r_a(t) = R_0 e^{2\pi j f_0 t - \gamma t} \tag{8}$$

The signal envelope

- $r_a(t) = r(t) + j\,\widehat{r}(t)$
- $\widehat{r}(t)$, Hilbert transform
- $|r_a(t)|$, signal *envelope*

RF channels: RLC circuits

- $|r_a(t)| = R_0 \exp(-\gamma t)$
- γ, damping coefficient [GHz]
- f_0, resonance frequency [GHz]

The Askaryan Signal: detected signals

Detected signals, $r(t) * s(t)$, [V]

$$r(t) * s(t) = \int_{-\infty}^{\infty} r(\tau) s(t-\tau) d\tau \rightarrow \int_{0}^{\infty} r(\tau) s(t-\tau) d\tau \quad (9)$$

Theorem: the envelope of detected signal

Let $\mathcal{E}_{r*s}(t)$ represent the *envelope* of the convolution of $r(t)$ and $s(t)$. If $s_a(t)$ and $r_a(t)$ are the analytic signals of $s(t)$ and $r(t)$, respectively, then

$$\mathcal{E}_{r*s}(t) = \frac{1}{2} |r_a(t) * s_a(t)| \quad (10)$$

The Askaryan Signal: uncertainty principles

Uncertainty principles within Askaryan field

Let $\Delta\theta = \theta - \theta_{\mathrm{C}}$. Let a be the cascade length, c be the speed of light, σ_t be the pulse width, and σ_f be the width of the Fourier spectrum.

$$a\Delta\theta = \frac{c\sigma_t}{\sin\theta_{\mathrm{C}}} \tag{11}$$

Further, $s(t)$, and $S(f)$ (the Fourier transform), have widths that satisfy

$$\sigma_t\sigma_f = \frac{1}{2\pi}\left(1+\eta^2\right)^{1/2} \tag{12}$$

In the far-field, $\eta \to 0$.

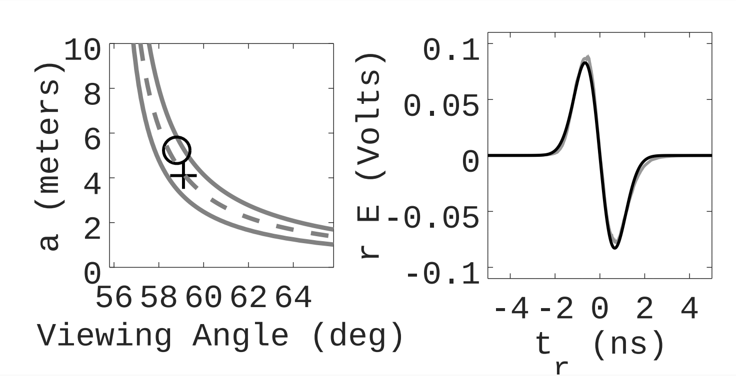

Figure 5: Fitting Askaryan field to NuRadioMC output. (Left) reconstruction of a and $\Delta\theta$, mathematical $\vec{E}$-field vs. MC $\vec{E}$-field. Circle: MC true. Cross: reconstruction.

Energy reconstruction potential: $a^2 \propto \ln(E_{\mathrm{C}}/E_{\mathrm{crit}})$

Event Reconstruction

Event Reconstruction: calculating $\mathcal{E}_{r*s}(t)$

Stages of the calculation:

- Calculate $r_a(t)$ and $s_a(t)$ (already shown).
- Convolve $r_a(t)$ and $\Re\{s_a(t)\}$ (time domain).
- Convolve $r_a(t)$ and $\Im\{s_a(t)\}$ (Fourier domain).
- Combine results and take the magnitude, multiply by (1/2).

Event Reconstruction: $r_a(t) * \Re\{s_a(t)\}$

Result for $r_a(t) * \Re\{s_a(t)\}$

$$r_a(t) * \Re\{s_a(t)\} = \sqrt{\frac{\pi}{2}} R_0 \sigma_t s(t) w(z) + R_0 E_0 \sigma_t^2 e^{-\frac{1}{2}(t/\sigma_t)^2} (1 + j\sqrt{\pi} z w(z)) \quad (13)$$

Complex poles:

- $z_1 = f_0/(\sqrt{2}\sigma_f) + j\gamma/(2\pi\sqrt{2}\sigma_f)$
- $z_1 = z + jx$

The Faddeeva function

- $w(z) = e^{-z^2} \operatorname{erfc}(-jz)$
- $w(z) = e^{y^2} \operatorname{erfc}(y),\ z \to jy$

Combine real and imaginary parts: $\mathcal{E}_{r*s}(t) = \frac{1}{2}|r_a(t) * s_a(t)|$

Event Reconstruction: $r_a(t) * \Im\{s_a(t)\}$

Result for $r_a(t) * \Im\{s_a(t)\}$

$$r_a(t) * \Im\left\{s_a(t)\right\} = \sqrt{\pi}E_0\sigma_t^2 z_1 e^{-z_1^2} r_a(t) + \sqrt{\frac{2}{\pi}} G(z_1) R_0 \sigma_t s(t) \quad (14)$$

Complex poles:

- $z_1 = f_0/(\sqrt{2}\sigma_f) + j\gamma/(2\pi\sqrt{2}\sigma_f)$
- $z_1 = z + jx$

The Goodwin-Staton integral

- $G(z) = \sqrt{\pi}F(z) - \frac{1}{2}e^{-z^2}\,\mathrm{Ei}(z^2)$
- $\mathrm{Ei}(z)$ is an exponential integral

Combine real and imaginary parts: $\mathcal{E}_{r*s}(t) = \frac{1}{2}|r_a(t) * s_a(t)|$

Event Reconstruction: graphs of $\mathcal{E}_{r*s}(t)$

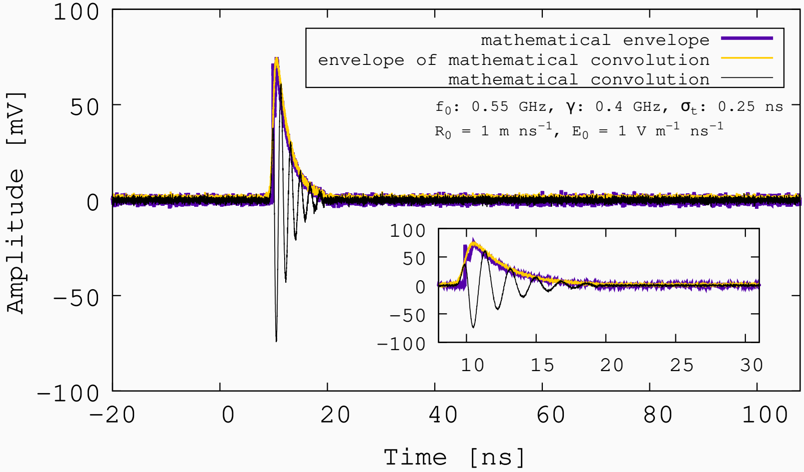

Figure 6: Black line: $s(t) * r(t)$. Gold line: the envelope of $s(t) * r(t)$ computed with the Python3 package scipy.special.hilbert. Purple line: $\mathcal{E}_{r*s}(t)$.

Event Reconstruction: graphs of $\mathcal{E}_{r*s}(t)$

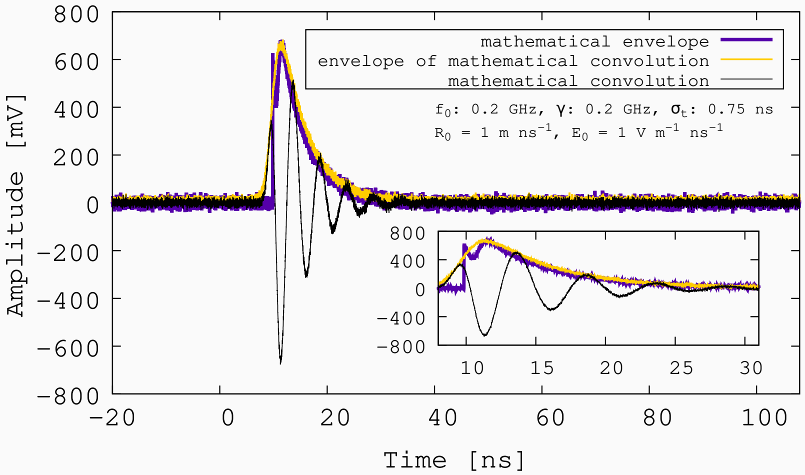

Figure 7: Same is Fig. 6, with different parameters.

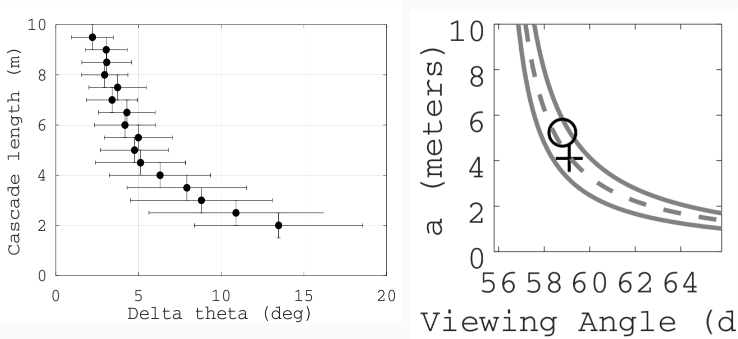

Figure 8: (Left) Reconstructed a and $\Delta\theta$, from NuRadioMC. (Right) Fig. 5 (right), for comparison.

Conclusion

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Figure 9: Whittier College.

References

References

1. P. Allison et al., Low-threshold ultrahigh-energy neutrino search with the Askaryan Radio Array, Phys Rev D 105, 122006 (2022).
2. J. A. Aguilar et al., In situ, broadband measurement of the radio frequency attenuation length at Summit Station, Greenland, J. Glaciol. 68, 1234 (2022).
3. S. Agarwal et al., Instrument design and performance of the first seven stations of RNO-G, arXiv (2024).
4. **J. C. Hanson and R. Hartig**, Complex analysis of Askaryan radiation: A fully analytic model in the time domain, Phys Rev D 105, 123019 (2022).
5. **J. C. Hanson and A. L. Connolly**, Complex analysis of Askaryan radiation: A fully analytic treatment including the LPM effect and Cascade Form Factor, Astroparticle Physics 91, 75 (2017).
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Bonus Slides

Calculate the signal envelope with SciPy

```
import numpy as np
import scipy.signal.hilbert as hilbert
env_out = np.abs(hilbert(sig_in))
```

1. The hilbert function computes the *analytic signal*
2. The absolute value of the analytic signal is the envelope

Calculate the envelope of the convolution of two functions

```
import numpy as np
import scipy.signal.hilbert as hilbert
env_out = 0.5*np.abs(np.conv(
    hilbert(sig_in_1),
    hilbert(sig_in_2),
    'same'))
```

This is equivalent to:

```
import numpy as np
import scipy.signal.hilbert as hilbert
env_out = np.abs(hilbert(
    np.conv(sig_in_1,sig_in_2,'same')))
```