Complex Analysis of Askaryan Radiation: Towards UHE-ν energy Reconstruction via the Hilbert Envelope of Observed Signals

Jordan C. Hanson* and Raymond Hartig
Department of Physics and Astronomy, Whittier College
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I. INTRODUCTION

The introduction.

II. UNITS, DEFINITIONS, AND CONVENTIONS

The units.

III. COLLECTION OF MAIN RESULTS

Here is a list of the basic results and ideas for this paper.

- Let the signal model $s(t)$ be

$$s(t) = -E_0 t e^{-\frac{1}{2}(t/\sigma_t)^2} \tag{1}$$

  This is the off-cone field equation from [1]. The parameter σ_{t} is the pulse width, and it depends two quantities: the longitudinal length of the UHE-ν-induced cascade, and the angle at which the cascade is observed relative to the Cherenkov angle. The parameter E_0 is the amplitude normalization, and it depends on two parameters: σ_{t}, and ω_0, the cutoff frequency from the cascade form factor.

- Let $\widehat{s}(t)$ represent the Hilbert transform of $s(t)$. The *analytic signal* of $s(t)$ is

$$s_{\mathrm{a}}(t) = s(t) + j\widehat{s}(t) \tag{2}$$

  The magnitude of the analytic signal, $|s_a(t)|$, is the *envelope* of the signal. The Hilbert transform $\widehat{s}(t)$ is equivalent to the convolution of $s(t)$ and the tempered distribution $h(t) = 1/(\pi t)$.

- Let $S(f)$ be the Fourier transform of $s(t)$. The Fourier transform of the analytic signal is

$$\mathcal{F}\{s_{\mathrm{a}}(t)\}_f = S_{\mathrm{a}}(f) = S(f)(1 + \operatorname{sgn} f) \tag{3}$$

  The sign function, sgn gives -1 if $f < 0$, 0 if $f = 0$, and 1 if $f > 1$.

- Taking the inverse Fourier transform of Eq. 3, the analytic signal may be written in terms of $S(f)$:

$$s_{\mathrm{a}}(t) = 2\int_0^\infty S(f) e^{2\pi j f t} df \tag{4}$$

- The Fourier transform of Eq. 1 is

$$S(f) = E_0 \sigma_t^3 (2\pi)^{3/2} j f e^{-2\pi^2 f^2 \sigma_t^2} \tag{5}$$

- Using the gaussian spectral width σ_{f} from [2], and the guassian width of $s(t)$ from [1], it was shown in [1] that the uncertainty principle holds for off-cone signals:

$$\sigma_{\mathrm{t}}\sigma_{\mathrm{f}} \geq \frac{1}{2\pi} \tag{6}$$

  The equality is reached in the limit the far-field parameter limits to zero: $\eta \to 0$. This makes the signal spectrum

$$S(f) = E_0 \sigma_t^3 (2\pi)^{3/2} j f e^{-\frac{1}{2}(f/\sigma_f)^2} \tag{7}$$

  Inserting $S(f)$ into Eq. 4, $s_{\mathrm{a}}(t)$ is

$$s_{\mathrm{a}}(t) = \frac{E_0 \sigma_t^3 (2\pi)^{3/2}}{\pi} \frac{d}{dt} \int_0^\infty e^{-\frac{1}{2}(f/\sigma_f)^2} e^{2\pi j f t} df \tag{8}$$

- Let $k^2/4 = \frac{1}{2}\left(f/\sigma_f\right)^2$, and $x = t/(\sqrt{2}\sigma_t)$. Equation 8 can be broken into real and imaginary parts:

$$s_{\mathrm{a}}(t) = \frac{E_0 \sigma_{\mathrm{t}}}{\sqrt{2\pi}} \frac{dI}{dx} \tag{9}$$

$$\Re\{I\} = \int_0^\infty e^{-k^2/4} \cos(kx) dk \tag{10}$$

$$\Im\{I\} = \int_0^\infty e^{-k^2/4} \sin(kx) dk \tag{11}$$

  The real part of I is even, so it can be extended to $(-\infty, \infty)$ if it is multiplied by 1/2. The result is

$$\Re\{I\} = \sqrt{\pi} e^{-x^2} \tag{12}$$

  The imaginary part of I is proportional to *Dawson's integral, $D(x)$:*

$$\Im\{I\} = 2D(x) \tag{13}$$

*Electronic address: jhanson2@whittier.edu

- The overall analytic signal, $s_a(t)$, is

$$s_a(t) = -E_0 \left(t e^{-\frac{1}{2}(t/\sigma_t)^2} - \frac{2j\sigma_t}{\sqrt{2\pi}} \frac{dD(x)}{dx} \right) \tag{14}$$

The signal envelope is $|s_a(t)|$. It is important to note that, though $D(x)$ is not evaluated analytically, a high-precision algorithm for computing $D(x)$ was given in [3]. Note that $s_a(0) \neq 0$, since $dD(x)/dx = 1 - 2xD(x)$.

- Signal data in detectors designed to observe Askaryan pulses is equivalent to the convolution of the signal and detector response functions. Signal models are convolved with measured detector responses to create *signal templates*. Signal templates are cross-correlated with observed data to identify UHE-ν signals. The oscillations of signal templates and observed data can introduce various uncertainties when cross-correlated. This problem intensifies when the signal-to-noise ratio between Askaryan pulse data and thermal noise decreases. To reduce these uncertainties, the Hilbert envelope of observed signals is used in cross-correlations instead of the original signals. We seek an analytic equation for the Hilbert envelope of the data. That is, we seek the envelope of the convolution of the analytic signal model with a typical detector response. The RLC damped oscillator is a standard circuit model for the RF dipole antennas used in RNO-G and the proposed IceCube Gen2 [4–6].

- There are two paths to calculating the final result. The first option involves three steps. First, the detector response, $r(t)$ is convolved with $s(t)$. Second, the analytic signal of the result is found. Third, the magnitude of the analytic signal is computed, which can be compared to envelopes of observed signals. The second option involves computing the envelope of the convolution of $r(t)$ with $s(t)$ directly from $s_a(t)$ and $r_a(t)$.

- Let $x(t)*y(t)$ represent the convolution of two functions $x(t)$ and $y(t)$. Let the envelope of the convolution be $\mathcal{E}_{x*y}(t)$. $\mathcal{E}_{x*y}(t)$, $x_a(t)$, and $y_a(t)$ are related by

$$\mathcal{E}_{x*y}(t) = \frac{1}{2}|x_a(t) * y_a(t)| \tag{15}$$

The analytic signal of the Askaryan pulse is given by Eq. 14. The RLC damped oscillator response and corresponding analytic signal are

$$r(t) = R_0 e^{-\gamma t} \cos(2\pi j f_0 t) \tag{16}$$

$$r_a(t) = R_0 e^{-\gamma t} e^{2\pi j f_0 t} \tag{17}$$

The parameter γ is the *decay constant*, and the parameter f_0 is the resonance frequency. Note that the envelope of $r(t)$, $|r_a(t)|$, is simply $R_0 \exp(-\gamma t)$, as it should be.

- The analytic signal and response convolution inside the magnitude on the right hand side of Eq. 15 can be broken into real and imaginary parts of $s_a(t)$:

$$r_a(t) * s_a(t) = \tag{18}$$

$$r_a(t) * \Re\{s_a(t)\} + \tag{19}$$

$$j r_a(t) * \Im\{s_a(t)\} \tag{20}$$

Let $\gamma' = \sqrt{2}\sigma_t\gamma$, $f_0' = \sqrt{2}\sigma_t f_0$, and $z_0 = \gamma' - 2\pi j f_0'$. The result for $r_a(t) * \Re\{s_a(t)\}$ is

$$r_a(t) * \Re\{s_a(t)\} = -\sqrt{\pi} E_0 R_0 \sigma_t^2 e^{-\gamma t} e^{2\pi j f_0 t} \frac{dQ}{dz_0} \tag{21}$$

$$Q(z_0) = e^{z_0^2/4} \operatorname{erfc}(z_0/2) \tag{22}$$

- The result for $r_a(t) * \Im\{s_a(t)\}$ is

$$r_a(t) * \Im\{s_a(t)\} = -\sqrt{\frac{\pi}{2}} R_0 E_0 \sigma_t t e^{-\frac{1}{2}(t/\sigma_t)^2} Q(z_0) \tag{23}$$

Note that the complementary error function and Gaussian functions are *entire functions*, so dQ/dz_0 is defined for all z_0.

- The results in Eqs. 21-23 may be expressed in terms of $s(t)$ and $\mathrm{r_a}(t)$:

$$r_a(t) * \Re\{s_a(t)\} = -\sqrt{\pi} E_0 \sigma_t^2 \mathrm{r_a}(t) \frac{dQ}{dz_0} \tag{24}$$

$$r_a(t) * \Im\{s_a(t)\} = \sqrt{\frac{\pi}{2}} R_0 \sigma_t s(t) Q(z_0) \tag{25}$$

- The convolution $r_a(t) * s_a(t)$ is therefore

$$r_a(t) * s_a(t) = -\sqrt{\pi} E_0 \sigma_t^2 \mathrm{r_a}(t) \frac{dQ}{dz_0} + j\sqrt{\frac{\pi}{2}} R_0 \sigma_t s(t) Q(z_0) \tag{26}$$

- The envelope of the convolution of $s(t)$ and $r(t)$ is therefore

$$\mathcal{E}_{r*s}(t) = \frac{1}{2}|\mathrm{r_a}(t) * \mathrm{s_a}(t)| \tag{27}$$

This result can be further simplified. There are several intermediate steps that aid in the simplification. First, note that $Q(z_0)$ is related to the *Voigt functions* $U(x,t)$ and $V(x,t)$:

$$U(x,t) = jV(x,t) = \frac{\pi}{4t} e^{z^2} \operatorname{erfc}(z) \tag{28}$$

$$z = \frac{1 - jx}{2\sqrt{t}} \tag{29}$$

Note that t and x are real parameters, and that the Voigt functions are also real. Let $z = \frac{1}{2} z_0$. The parameters are connected to z_0 by the following:

$$\Re\{z_0\} = \frac{1}{\sqrt{t}} \quad (30)$$

$$\gamma' = \frac{1}{\sqrt{t}} \quad (31)$$

$$\Im\{z_0\} = -\frac{x}{\sqrt{t}} \quad (32)$$

$$x = \frac{2\pi f_0'}{\gamma'} = \frac{2\pi f_0}{\gamma} \quad (33)$$

It is significant that x is the ratio of the detector parameters. Thus, $Q(z_0)$ may be written in terms of U and V:

$$Q(z_0) = \frac{2}{\gamma'\sqrt{\pi}}\left(U + jV\right) \quad (34)$$

Equation 25 simplifies:

$$\boxed{r_a(t) * \Im\{s_a(t)\} = \frac{R_0 s(t)}{\gamma}\left(U + jV\right)} \quad (35)$$

To simplify Eq. 24, we need the derivative of $Q(z_0)$ with respect to z_0:

$$\frac{dQ}{dz_0} = \frac{1}{2} z_0 Q(z_0) - \frac{1}{\sqrt{\pi}} \quad (36)$$

Combining Eqs. 24, 29, and 36, we find

$$r_a(t) * \Re\{s_a(t)\} = E_0\sigma_t^2 r_a\left(1 - \frac{z_0}{\gamma'}\left(U + jV\right)\right) \quad (37)$$

Using the definitions of z_0, γ', and Eq. 33, we find

$$\boxed{r_a(t) * \Re\{s_a(t)\} = E_0\sigma_t^2 r_a\left(1 - (1 + jx)\left(U + jV\right)\right)} \quad (38)$$

There are useful limits for Voigt functions:

$$\lim_{t\to 0} U(x,t) = \frac{1}{1+x^2} \quad (39)$$

$$\lim_{t\to 0} V(x,t) = \frac{x}{1+x^2} \quad (40)$$

Taking this limit for Eqs. 35 and 38, we find, to first-order in $1/\gamma'$,

$$r_a(t) * \Im\{s_a(t)\} \approx \frac{R_0 s(t)}{\gamma} \quad (41)$$

$$r_a(t) * \Re\{s_a(t)\} \approx 0 \quad (42)$$

- Using Eq. 15, we find

$$\boxed{\mathcal{E}_{r*s}(t) = \frac{R_0|s(t)|}{2\gamma}} \quad (43)$$

This result appears to have (1) the right units, (2) the right limits, and (3) the right symmetry. However, the *width* of this theoretical envelope is limited by the width of $s(t)$. The limit that the Voigt parameter $t \to 0$ implies that $\gamma' \gg 1$. This is not necessarily a good limit for realistic detectors.

IV. CONCLUSION

The conclusion.

Appendix A: Details

The details.

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