Complex Analysis of Askaryan Radiation: Towards UHE-ν energy Reconstruction via the Hilbert Envelope of Observed Signals

Jordan C. Hanson* and Raymond Hartig
Department of Physics and Astronomy, Whittier College
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I. INTRODUCTION

The introduction.

II. UNITS, DEFINITIONS, AND CONVENTIONS

The units.

III. COLLECTION OF MAIN RESULTS

Here is a list of the basic results and ideas for this paper.

- Let the signal model $s(t)$ be

$$s(t) = -E_0 t e^{-\frac{1}{2}(t/\sigma_t)^2} \tag{1}$$

This is the off-cone field equation from [1]. The parameter σ_{t} is the pulse width, and it depends two quantities: the longitudinal length of the UHE-ν-induced cascade, and the angle at which the cascade is observed relative to the Cherenkov angle. The parameter E_0 is the amplitude normalization, and it depends on two parameters: σ_{t}, and ω_0, the cutoff frequency from the cascade form factor.

- Let $\widehat{s}(t)$ represent the Hilbert transform of $s(t)$. The *analytic signal* of $s(t)$ is

$$s_{\mathrm{a}}(t) = s(t) + j\widehat{s}(t) \tag{2}$$

The magnitude of the analytic signal, $|s_a(t)|$, is the *envelope* of the signal. The Hilbert transform $\widehat{s}(t)$ is equivalent to the convolution of $s(t)$ and the tempered distribution $h(t) = 1/(\pi t)$.

- Let $S(f)$ be the Fourier transform of $s(t)$. The Fourier transform of the analytic signal is

$$\mathcal{F}\{s_{\mathrm{a}}(t)\}_f = S_{\mathrm{a}}(f) = S(f)(1 + \operatorname{sgn} f) \tag{3}$$

The sign function, sgn gives -1 if $f < 0$, 0 if $f = 0$, and 1 if $f > 1$.

- Taking the inverse Fourier transform of Eq. 3, the analytic signal may be written in terms of $S(f)$:

$$s_{\mathrm{a}}(t) = 2\int_0^\infty S(f) e^{2\pi j f t} df \tag{4}$$

- The Fourier transform of Eq. 1 is

$$S(f) = E_0 \sigma_t^3 (2\pi)^{3/2} j f e^{-2\pi^2 f^2 \sigma_t^2} \tag{5}$$

- Using the gaussian spectral width σ_{f} from [2], and the guassian width of $s(t)$ from [1], it was shown in [1] that the uncertainty principle holds for off-cone signals:

$$\sigma_t \sigma_f \geq \frac{1}{2\pi} \tag{6}$$

The equality is reached in the limit the far-field parameter limits to zero: $\eta \to 0$. This makes the signal spectrum

$$S(f) = E_0 \sigma_t^3 (2\pi)^{3/2} j f e^{-\frac{1}{2}(f/\sigma_f)^2} \tag{7}$$

Inserting $S(f)$ into Eq. 4, $s_{\mathrm{a}}(t)$ is

$$s_{\mathrm{a}}(t) = \frac{E_0 \sigma_t^3 (2\pi)^{3/2}}{\pi} \frac{d}{dt} \int_0^\infty e^{-\frac{1}{2}(f/\sigma_f)^2} e^{2\pi j f t} df \tag{8}$$

- Let $k^2/4 = \frac{1}{2}\left(f/\sigma_f\right)^2$, and $x = t/(\sqrt{2}\sigma_t)$. Equation 8 can be broken into real and imaginary parts:

$$s_{\mathrm{a}}(t) = \frac{E_0 \sigma_{\mathrm{t}}}{\sqrt{2\pi}} \frac{dI}{dx} \tag{9}$$

$$\Re\{I\} = \int_0^\infty e^{-k^2/4} \cos(kx) dk \tag{10}$$

$$\Im\{I\} = \int_0^\infty e^{-k^2/4} \sin(kx) dk \tag{11}$$

The real part of I is even, so it can be extended to $(-\infty, \infty)$ if it is multiplied by 1/2. The result is

$$\Re\{I\} = \sqrt{\pi} e^{-x^2} \tag{12}$$

The imaginary part of I is proportional to *Dawson's integral, $D(x)$:*

$$\Im\{I\} = 2D(x) \tag{13}$$

*Electronic address: `jhanson2@whittier.edu`

- The overall analytic signal, $s_a(t)$, is

$$s_a(t) = -E_0 \left(t e^{-\frac{1}{2}(t/\sigma_t)^2} - \frac{2j\sigma_t}{\sqrt{2\pi}} \frac{dD(x)}{dx} \right) \tag{14}$$

The signal envelope is $|s_a(t)|$. It is important to note that, though $D(x)$ is not evaluated analytically, a high-precision algorithm for computing $D(x)$ was given in [3]. Note that $s_a(0) \neq 0$, since $dD(x)/dx = 1 - 2xD(x)$.

- Signal data in detectors designed to observe Askaryan pulses is equivalent to the convolution of the signal and detector response functions. Signal models are convolved with measured detector responses to create *signal templates*. Signal templates are cross-correlated with observed data to identify UHE-ν signals. The oscillations of signal templates and observed data can introduce various uncertainties when cross-correlated. This problem intensifies when the signal-to-noise ratio between Askaryan pulse data and thermal noise decreases. To reduce these uncertainties, the Hilbert envelope of observed signals is used in cross-correlations instead of the original signals. We seek an analytic equation for the Hilbert envelope of the data. That is, we seek the envelope of the convolution of the analytic signal model with a typical detector response. The RLC damped oscillator is a standard circuit model for the RF dipole antennas used in RNO-G and the proposed IceCube Gen2 [4–6].

- There are two paths to calculating the final result. The first option involves three steps. First, the detector response, $r(t)$ is convolved with $s(t)$. Second, the analytic signal of the result is found. Third, the magnitude of the analytic signal is computed, which can be compared to envelopes of observed signals. The second option involves computing the envelope of the convolution of $r(t)$ with $s(t)$ directly from $s_a(t)$ and $r_a(t)$.

- Let $s(t) * r(t)$ represent the convolution of $s(t)$ and $r(t)$. Let the envelope of the convolution be $\mathcal{E}_{s*r}(t)$. $\mathcal{E}_{s*r}(t)$, $s_a(t)$, and $r_a(t)$ are related by

$$\mathcal{E}_{s*r}(t) = \frac{1}{2} |s_a(t) * r_a(t)| \tag{15}$$

The proof of Eq. 15 is based on two ideas. First, the Hilbert transform of a function $s(t)$ is equivalent to convolving it with the "tempered distribution" $h(t) = 1/(\pi t)$. Second, computing the Hilbert transform twice yields the original function, multiplied by -1: $h * h * s = -s$. Given the definitions of the analytic signal and the Hilbert transform,

$$(s * r)_a(t) = s * r + j\ \widehat{s * r} \tag{16}$$
$$\mathcal{E}_{s*r}(t) = |s * r + js * r * h| \tag{17}$$

However,

$$r_a * s_a = (r + j\hat{r}) * (s + j\hat{s}) \tag{18}$$
$$r_a * s_a = r * s + jr * \hat{s} + j\hat{r} * s - \hat{r} * \hat{s} \tag{19}$$
$$r_a * s_a = r * s - r * h * s * h + 2jh * r * s \tag{20}$$
$$r_a * s_a = r * s - h * h * r * s + 2jh * r * s \tag{21}$$
$$r_a * s_a = 2r * s + 2jh * r * s \tag{22}$$

Multiplying both sides 1/2 and taking the magnitude completes the proof:

$$\frac{1}{2} |r_a * s_a| = |r * s + jh * r * s| = \mathcal{E}_{s*r}(t) \tag{23}$$

- The RLC damped oscillator impulse response and corresponding analytic signal are

$$r(t) = R_0 e^{-2\pi\gamma_f t} \cos(2\pi f_0 t) \tag{24}$$
$$r_a(t) = R_0 e^{-2\pi\gamma_f t} e^{2\pi j f_0 t} \tag{25}$$

The parameters γ_f and f_0 are the *decay constant* that corresponds to the *fall time* of the output signal, and the resonance frequency. Note that the envelope of $r(t)$, $|r_a(t)|$, is $R_0 \exp(-2\pi\gamma_f t)$, as expected. To prove Eq. 25, first compute the Fourier transform of $r(t)$:

$$R(f) = \frac{R_0}{4\pi j} \left(\frac{1}{f - z_+} + \frac{1}{1 - z_-} \right) \tag{26}$$
$$z_+ = f_0 + j\gamma_f \tag{27}$$
$$z_- = -f_0 + j\gamma_f \tag{28}$$

Given Eq. 4, the procedure to find $r_a(t)$ is to multiply the *negative* frequency components by 0 and the *positive* frequency components by 2, and take the inverse Fourier transform. The inverse Fourier transform may be completed by extension to the complex plane using the upper infinite semi-circle as a contour, and applying Jordan's lemma. The residue from the pole at z_+ drives the final result.

- The goal is now to apply Eq. 15 by convolving $s_a(t)$ with $r_a(t)$. The real part of $s_a(t)$ can be convolved with $r_a(t)$ in the time domain. As in Eq. 14, real part of $s_a(t)$ is $s(t)$. Since $r_a(t < 0) = 0$, the convolution is written

$$r_a(t) * \Re\{s_a(t)\} = \int_0^\infty r_a(\tau) s(t - \tau) d\tau \tag{29}$$

Let $x = t/(\sqrt{2}\sigma_t)$, $y = \tau/(\sqrt{2}\sigma_t)$, and $z_0 = \sqrt{2}\sigma_t(\gamma - 2\pi j f_0)$. The convolution may be organized as follows

$$r_a(t) * \Re\{s_a(t)\} = R_0 s(t) I + R_0 E_0 \sigma_t^2 e^{-\frac{1}{2}(t/\sigma_t^2)} \frac{dI}{dt} \tag{30}$$

with

$$I(x, z_0) = \sqrt{2}\sigma_t \int_0^\infty e^{-y^2 + (2x - z_0)y} dy \tag{31}$$

Let $b = x - z_0/2$. Completing the square in the exponent produces

$$I(x, z_0) = \sqrt{2}\sigma_t e^{b^2} \int_0^\infty e^{-(y-b)^2} dy \tag{32}$$

The integral may be cast as a complementary error function by substituting $u = y - b$ and letting $b = jz$:

$$I(x, z_0) = \sqrt{\frac{\pi}{2}} \sigma_t e^{-z^2} \operatorname{erfc}(-jz) \tag{33}$$

Thus, the integral is proportional to the *Faddeeva function*, $w(z)$:

$$I(x, z_0) = \sqrt{\frac{\pi}{2}} \sigma_t w(z) \tag{34}$$

Substituting Eq. 34 into Eq. 30, and simplifying, produces

$$\begin{aligned} r_a(t) * \Re\{s_a(t)\} &= \sqrt{\frac{\pi}{2}} R_0 \sigma_t s(t) w(z) \\ &- j\frac{\sqrt{\pi}}{2} R_0 E_0 \sigma_t^2 e^{-\frac{1}{2}(t/\sigma_t)^2} \frac{dw(z)}{dz} \end{aligned} \tag{35}$$

- Turning to the convolution of $r_a(t)$ with $\Im(s_a)$, note that the imaginary part of Eq. 14 contains $dD(x)/dx$, the derivative of the Dawson function. To complete the convolution, the inverse Fourier transform of the product of the Fourier transforms of $r_a(t)$ and $\Im\{s_a(t)\}$ is calculated. To complete this step, the Fourier transform of $dD(x)/dx$ is needed:

$$x = t/(\sqrt{2}\sigma_t) \tag{36}$$

$$f' = \sqrt{2}\sigma_t f \tag{37}$$

$$\mathcal{F}\left\{\frac{dD(x)}{dx}\right\} = \int_{-\infty}^\infty \frac{dD(x)}{dx} e^{-2\pi jft} dt \tag{38}$$

$$\mathcal{F}\left\{\frac{dD(x)}{dx}\right\} = \sqrt{2}\sigma_t (2\pi jf') \int_{-\infty}^\infty D(x) e^{-2\pi jf'x} dx \tag{39}$$

$$\mathcal{F}\left\{\frac{dD(x)}{dx}\right\} = 2\pi^2 \sigma_t^2 f e^{-\frac{1}{2}(f/\sigma_f)^2} \operatorname{sgn}(f) \tag{40}$$

Let $z_0 = (\gamma - 2\pi jf_0)$, so that the Fourier transform of r_a is

$$\mathcal{F}\{r_a(t)\} = R_0 \int_0^\infty e^{-z_0 t} e^{-2\pi jft} dt \tag{41}$$

$$\mathcal{F}\{r_a(t)\} = \frac{R_0}{z_0 + 2\pi jf} \tag{42}$$

Let $z_1 = f_0 + j\gamma/(2\pi)$. Multiplying the Fourier transforms, and simplifying, yields

$$\begin{aligned} &\mathcal{F}\{r_a(t) * \Im\{s_a(t)\}\} = \\ &\quad - j\sqrt{2\pi} R_0 E_0 \sigma^3 \frac{f \operatorname{sgn}(f) e^{-\frac{1}{2}(f/\sigma_f)^2}}{f - z_1} \end{aligned} \tag{43}$$

The inverse Fourier transform now gives $r_a(t) * \Im\{s_a(t)\}$:

$$\begin{aligned} &r_a(t) * \Im\{s_a(t)\} = \\ &- j\sqrt{2\pi} R_0 E_0 \sigma_t^3 \int_{-\infty}^\infty \frac{f \operatorname{sgn}(f) e^{-\frac{1}{2}(f/\sigma_f)^2} e^{2\pi jft}}{f - z_1} df \end{aligned} \tag{44}$$

Let $y = f/(\sqrt{2}\sigma_f)$, $x = t/(\sqrt{2}\sigma_t)$, $\operatorname{sgn}(f) = \operatorname{sgn}(y)$, and $z_1 \to f_0/(\sqrt{2}\sigma_f) + j\gamma/(2\pi\sqrt{2}\sigma_f)$. The inverse Fourier transform can be organized as:

$$\begin{aligned} &r_a(t) * \Im\{s_a(t)\} = \\ &\quad - \frac{R_0 E_0 \sigma_t^3}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^\infty \frac{\operatorname{sgn}(y) e^{-y^2 + 2jxy}}{y - z_1} dy \end{aligned} \tag{45}$$

Completing the square in the exponent, factoring a unit of j from the exponent, and changing the derivative from d/dt to d/dx gives

$$\begin{aligned} &r_a(t) * \Im\{s_a(t)\} = \\ &\quad - \frac{R_0 E_0 \sigma_t^2}{2\sqrt{\pi}} \frac{d}{dx} e^{-x^2} \int_{-\infty}^\infty \frac{\operatorname{sgn}(y) e^{(x+jy)^2}}{y - z_1} dy \end{aligned} \tag{46}$$

Let $z = x + jy$, with $y = \Im(z)$. In this substitution, x is treated as a constant, so $dz = jy$. Let $z_2 = x + jz_1$, while $z_1 = f_0/(\sqrt{2}\sigma_f) + j\gamma/(2\pi\sqrt{2}\sigma_f)$. The result is

$$\begin{aligned} &r_a(t) * \Im\{s_a(t)\} = \\ &\quad - \frac{R_0 E_0 \sigma_t^2}{2\sqrt{\pi}} \frac{d}{dx} e^{-x^2} \int_{x-j\infty}^{x+j\infty} \frac{\operatorname{sgn}(\Im(z)) e^{z^2}}{y - z_2} dz \end{aligned} \tag{47}$$

The integral may be finished by extending z to the complex plane and using contour integration. The location of the pole is in the first quadrant, unless $t = (\gamma\sigma_t)\sigma_t$. In that case, $\Re(z_2) = 0$, and the pole is located on the imaginary axis. For typical values of σ_t and γ, $t \approx 0.1$ ns. Since $r(t) = 0$ if $t < 0$, the location of z_2 is usually in the first quadrant. Let the line integrals of the rectangular contour be: I_1, $(x, -jR) \to (x, jR)$, I_2, $(x, jR) \to (0, jR)$, I_3, $(0, jR) \to (0, -jR)$, and I_4, $(0, -jR) \to (x, -jR)$. I_2 and I_4 are both proportional to $\exp(-R^2)$, so $I_2 \to 0$ and $I_4 \to 0$ when $R \to \infty$. I_1 is the integral in Eq. 47. Let $w = -y$, so that I_3 is

$$I_3 = -\int_0^R \frac{e^{-y^2}}{y - z_1} dy + \int_0^R \frac{e^{-w^2}}{w + z_1} dw \tag{48}$$

Integrals of the type in I_3 are known as *Goodwin-Staton* functions, and are related to the other error functions. The Goodwin-Staton function, $G(z_1)$, is odd, so $-G(-z_1) = G(z_1)$. The result for I_3 is

$$I_3 = -G(-z_1) + G(z_1) = 2G(z_1) \tag{49}$$

The residue of the enclosed contour $I_C = I_1 + I_2 + I_3 + I_4$ is $\exp(z_2^2)$. With I_2, I_3, and I_4 completed, the solution for I_1 is

$$I_1 = 2\pi j e^{z_2^2} - 2G(z_1) \tag{50}$$

Inserting this result into Eq. 47, and simplifying, gives

$$r_a(t) * \Im\left\{s_a(t)\right\} = \\ -\frac{R_0 E_0 \sigma_t^2}{2\sqrt{\pi}}\frac{d}{dx}\left(2\pi j e^{2jxz_1 - z_1^2} - 2G(z_1)e^{-x^2}\right) \tag{51}$$

Notice that $2\pi jxz_1 = 2\pi j f_0 t - \gamma t$. Evaluating the derivative and simplifying gives the final result:

$$r_a(t) * \Im\left\{s_a(t)\right\} = \\ \sqrt{\pi}E_0\sigma_t^2 z_1 e^{-z_1^2} r_a(t) + \sqrt{\frac{2}{\pi}}G(z_1)R_0\sigma_t s(t) \tag{52}$$

- Combining Eq. 35 and Eq. 52 gives $r_a(t) * s_a(t)$:

$$r_a(t) * s_a(t) = \\ \sqrt{\frac{\pi}{2}}R_0\sigma_t s(t)w(z) - j\frac{\sqrt{\pi}}{2}R_0E_0\sigma_t^2 e^{-\frac{1}{2}(t/\sigma_t)^2}\frac{dw(z)}{dz} + \\ j\sqrt{\pi}E_0\sigma_t^2 z_1 e^{-z_1^2} r_a(t) + j\sqrt{\frac{2}{\pi}}G(z_1)R_0\sigma_t s(t) \tag{53}$$

The units of convolution should be $R_0E_0\sigma_t^2$, and each term in Eq. 53 has these units. The first term is proportional to $R_0\sigma_t s(t)$, and $s(t)$ has units of $E_0\sigma_t$, so the unit product is $R_0E_0\sigma_t^2$. The units of the second term are $R_0E_0\sigma_t^2$. The third term has units of $E_0\sigma_t^2 r_a(t)$, and $r_a(t)$ has units of R_0, so the unit product is $R_0E_0\sigma_t^2$. The fourth term has units of $R_0\sigma_t s(t)$, like the first term. Thus, all units check. To check the limits, the parameters within z and z_1 must be recalled. Combining prior definitions, the results are

$$z = \sqrt{2}\sigma_t\pi f_0 + j\left(\frac{\sigma_t\gamma}{\sqrt{2}} - x\right) \tag{54}$$

$$z_1 = f_0/(\sqrt{2}\sigma_f) + j\gamma/(2\pi\sqrt{2}\sigma_f) \tag{55}$$

Recall that $\sigma_t = 1/(2\pi\sigma_f)$, and $x = t/(\sqrt{2}\sigma_t)$. Substituting this into the definition of z reveals that

$$z_1 = z + jx \tag{56}$$

Thus, when $t = 0$, the two poles are equal, since $t = 0$ means $x = 0$. Taking the magnitude of Eq. 53, and multiplying by 1/2, gives the final result:

$$\mathcal{E}_{r*s}(t) = \frac{1}{2}|r_a(t) * s_a(t)| \tag{57}$$

- Two examples of comparisons of Eq. 53 to the convolution of $s(t)$ and $r(t)$ are given in Fig. 1 below.

- It is important to note that the convolution of $s(t)$ and $r(t)$ may be done analytically in the time-domain. Starting with the definitions of $s(t)$, $r(t)$, and the convolution integral:

$$s * r = \int_0^\infty r(\tau)s(t-\tau)d\tau \tag{58}$$

$$s * r = -E_0\int_0^\infty r(\tau)(t-\tau)e^{-\frac{1}{2}\left(\frac{t-\tau}{\sigma_t}\right)^2}d\tau \tag{59}$$

$$s * r = \left(s(t) + E_0\sigma_t^2 e^{-\frac{1}{2}\left(\frac{t}{\sigma_t}\right)^2}\frac{d}{dt}\right)I(t,\sigma_t) \tag{60}$$

$$I(t,\sigma_t) = \int_0^\infty r(\tau)e^{(t\tau)/\sigma_t^2}e^{-\frac{1}{2}\left(\frac{\tau}{\sigma_t}\right)^2}d\tau \tag{61}$$

To compute $I(t,\sigma_t)$, let

$$z = \gamma - 2\pi j f_0 \tag{62}$$

$$z_0 = \sqrt{2}\sigma_t z \tag{63}$$

$$x = t/(\sqrt{2}\sigma_t) \tag{64}$$

$$y = \tau/(\sqrt{2}\sigma_t) \tag{65}$$

$$r(\tau) = \Re\left\{R_0 e^{-z\tau}\right\} \tag{66}$$

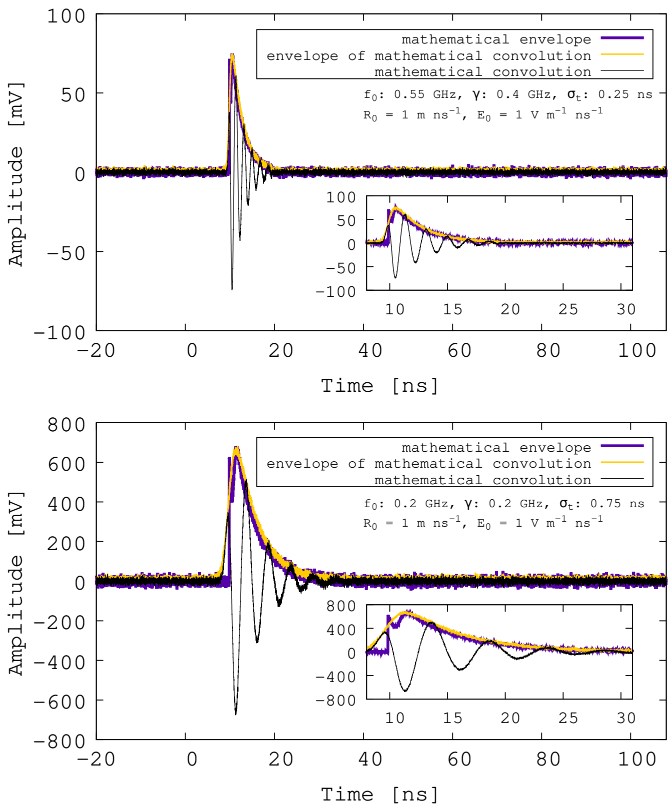

FIG. 1: (Top) The thin black line represents $s(t) * r(t)$. The dark gray envelope represents the envelope of $s(t) * r(t)$ computed with the Python3 package scipy.special.hilbert. The light gray envelope represents Eq. 53. (Bottom) Same as top, for different parameter values.

These substitutions simply $I(t,\sigma_t)$ to

$$I(t,\sigma_t) = \sqrt{2}\sigma_t R_0 \Re\left\{\int_0^\infty e^{(2y-z_0)x} e^{-x^2} dx\right\} \tag{67}$$

Let $b = y - z_0/2$, Completing the square in the exponent gives

$$I(t,\sigma_t) = \sqrt{\frac{\pi}{2}}\sigma_t R_0 \Re\left\{e^{b^2}\,\mathrm{erfc}(-b)\right\} \tag{68}$$

Let $b = jc$, so that

$$I(t,\sigma_t) = \sqrt{\frac{\pi}{2}}\sigma_t R_0 \Re\{e^{-c^2}\,\mathrm{erfc}(-jc)\} \tag{69}$$

$$I(t,\sigma_t) = \sqrt{\frac{\pi}{2}}\sigma_t R_0 \Re\{w(c)\} \tag{70}$$

The Faddeeva function $w(z)$ may be broken into real and imaginary parts using the Voigt functions:

$$U(p,q) + jV(p,q) = \sqrt{\frac{\pi}{4q}} w(z) \tag{71}$$

The relationship between p, q, and z is $z = (1 - jp)/(2\sqrt{q})$. Using this relationship, we find

$$p = \left(\frac{t}{\sigma_t}\right)\left(\frac{\sigma_f}{f_0}\right) - \frac{\gamma}{2\pi f_0} \tag{72}$$

$$q = \frac{1}{2}\left(\frac{\sigma_f}{f_0}\right)^2 \tag{73}$$

Thus, the integral $I(t,\sigma_t)$ becomes

$$I(t,\sigma_t) = \sigma_t R_0 \left(\frac{\sigma_f}{f_0}\right) U(p,q) \tag{74}$$

Substituting the result for $I(t,\sigma_t)$ in Eq. 60 gives

$$s * r = R_0\sigma_t\left(\frac{\sigma_f}{f_0}\right)\left(s(t) + E_0\sigma_t^2 e^{-\frac{1}{2}\left(\frac{t}{\sigma_t}\right)^2}\frac{d}{dt}\right)U(p,q) \tag{75}$$

Equation 75 can be simplified in two ways. First, note that $\sigma_t\sigma_f = 1/(2\pi)$, so $R_0\sigma_t(\sigma_f/f_0) = R_0/(2\pi f_0)$. Second, the derivative of the Voigt function may be simplified using the chain rule:

$$\frac{dU(p,q)}{dt} = \frac{dU(p,q)}{dp}\frac{dp}{dt} = \frac{dU(p,q)}{dp}\frac{\sigma_f}{\sigma_t f_0} \tag{76}$$

Thus, Eq. 75 becomes

$$s * r = \frac{R_0}{2\pi f_0}\left(s(t) + E_0\sigma_t\left(\frac{\sigma_f}{f_0}\right)e^{-\frac{1}{2}\left(\frac{t}{\sigma_t}\right)^2}\frac{d}{dp}\right)U(p,q) \tag{77}$$

The envelope of Eq. 77 can be computed numerically, which represents an alternative approach to that represented by Eq. 15.

IV. CONCLUSION

The conclusion.

Appendix A: Details

The details.

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