Complex Analysis of Askaryan Radiation: Towards UHE-ν energy Reconstruction via the Hilbert Envelope of Observed Signals

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I. INTRODUCTION

The introduction.

II. UNITS, DEFINITIONS, AND CONVENTIONS

The units.

III. COLLECTION OF MAIN RESULTS

Here is a list of the basic results and ideas for this paper.

- Let the signal model $s(t)$ be

$$s(t) = -E_0 t e^{-\frac{1}{2}(t/\sigma_t)^2} \tag{1}$$

  This is the off-cone field equation from [1]. The parameter σ_{t} is the pulse width, and it depends two quantities: the longitudinal length of the UHE-ν-induced cascade, and the angle at which the cascade is observed relative to the Cherenkov angle. The parameter E_0 is the amplitude normalization, and it depends on two parameters: σ_{t}, and ω_0, the cutoff frequency from the cascade form factor.

- Let $\widehat{s}(t)$ represent the Hilbert transform of $s(t)$. The *analytic signal* of $s(t)$ is

$$s_{\mathrm{a}}(t) = s(t) + j\widehat{s}(t) \tag{2}$$

  The magnitude of the analytic signal, $|s_a(t)|$, is the *envelope* of the signal. The Hilbert transform $\widehat{s}(t)$ is equivalent to the convolution of $s(t)$ and the tempered distribution $h(t) = 1/(\pi t)$.

- Let $S(f)$ be the Fourier transform of $s(t)$. The Fourier transform of the analytic signal is

$$\mathcal{F}\{s_{\mathrm{a}}(t)\}_f = S_{\mathrm{a}}(f) = S(f)(1 + \operatorname{sgn} f) \tag{3}$$

  The sign function, sgn gives -1 if $f < 0$, 0 if $f = 0$, and 1 if $f > 1$.

- Taking the inverse Fourier transform of Eq. 3, the analytic signal may be written in terms of $S(f)$:

$$s_{\mathrm{a}}(t) = 2\int_0^\infty S(f) e^{2\pi j f t} df \tag{4}$$

- The Fourier transform of Eq. 1 is

$$S(f) = E_0 \sigma_t^3 (2\pi)^{3/2} j f e^{-2\pi^2 f^2 \sigma_t^2} \tag{5}$$

- Using the gaussian spectral width σ_{f} from [2], and the guassian width of $s(t)$ from [1], it was shown in [1] that the uncertainty principle holds for off-cone signals:

$$\sigma_{\mathrm{t}}\sigma_{\mathrm{f}} \geq \frac{1}{2\pi} \tag{6}$$

  The equality is reached in the limit the far-field parameter limits to zero: $\eta \to 0$. This makes the signal spectrum

$$S(f) = E_0 \sigma_t^3 (2\pi)^{3/2} j f e^{-\frac{1}{2}(f/\sigma_f)^2} \tag{7}$$

  Inserting $S(f)$ into Eq. 4, $s_{\mathrm{a}}(t)$ is

$$s_{\mathrm{a}}(t) = \frac{E_0\sigma_t^3(2\pi)^{3/2}}{\pi}\frac{d}{dt}\int_0^\infty e^{-\frac{1}{2}(f/\sigma_f)^2} e^{2\pi j f t} df \tag{8}$$

- Let $k^2/4 = \frac{1}{2}(f/\sigma_f)^2$, and $x = t/(\sqrt{2}\sigma_t)$. Equation 8 can be broken into real and imaginary parts:

$$s_{\mathrm{a}}(t) = \frac{E_0\sigma_{\mathrm{t}}}{\sqrt{2\pi}}\frac{dI}{dx} \tag{9}$$

$$\Re\{I\} = \int_0^\infty e^{-k^2/4}\cos(kx) dk \tag{10}$$

$$\Im\{I\} = \int_0^\infty e^{-k^2/4}\sin(kx) dk \tag{11}$$

  The real part of I is even, so it can be extended to $(-\infty, \infty)$ if it is multiplied by 1/2. The result is

$$\Re\{I\} = \sqrt{\pi} e^{-x^2} \tag{12}$$

  The imaginary part of I is proportional to *Dawson's integral*, $D(x)$*:*

$$\Im\{I\} = 2D(x) \tag{13}$$

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- The overall analytic signal, $s_a(t)$, is

$$s_a(t) = -E_0 \left(t e^{-\frac{1}{2}(t/\sigma_t)^2} - \frac{2j\sigma_t}{\sqrt{2\pi}} \frac{dD(x)}{dx} \right) \tag{14}$$

  The signal envelope is $|s_a(t)|$. It is important to note that, though $D(x)$ is not evaluated analytically, a high-precision algorithm for computing $D(x)$ was given in [3]. Note that $s_a(0) \neq 0$, since $dD(x)/dx = 1 - 2xD(x)$.

- Signal data in detectors designed to observe Askaryan pulses is equivalent to the convolution of the signal and detector response functions. Signal models are convolved with measured detector responses to create *signal templates*. Signal templates are cross-correlated with observed data to identify UHE-ν signals. The oscillations of signal templates and observed data can introduce various uncertainties when cross-correlated. This problem intensifies when the signal-to-noise ratio between Askaryan pulse data and thermal noise decreases. To reduce these uncertainties, the Hilbert envelope of observed signals is used in cross-correlations instead of the original signals. We seek an analytic equation for the Hilbert envelope of the data. That is, we seek the envelope of the convolution of the analytic signal model with a typical detector response. The RLC damped oscillator is a standard circuit model for the RF dipole antennas used in RNO-G and the proposed IceCube Gen2 [4–6].

- There are two paths to calculating the final result. The first option involves three steps. First, the detector response, $r(t)$ is convolved with $s(t)$. Second, the analytic signal of the result is found. Third, the magnitude of the analytic signal is computed, which can be compared to envelopes of observed signals. The second option involves computing the envelope of the convolution of $r(t)$ with $s(t)$ directly from $s_a(t)$ and $r_a(t)$.

- Let $x(t) * y(t)$ represent the convolution of two functions $x(t)$ and $y(t)$. Let the envelope of the convolution be $\mathcal{E}_{x*y}(t)$. $\mathcal{E}_{x*y}(t)$, $x_a(t)$, and $y_a(t)$ are related by

$$\mathcal{E}_{x*y}(t) = \frac{1}{2} |x_a(t) * y_a(t)| \tag{15}$$

  We should include the proof here (I have to track it down in my notebooks).

- The analytic signal of the Askaryan pulse is given by Eq. 14. The RLC damped oscillator response and corresponding analytic signal are

$$r(t) = R_0 e^{-\gamma t} \cos(2\pi f_0 t) \tag{16}$$
$$r_a(t) = R_0 e^{-\gamma t} e^{2\pi j f_0 t} \tag{17}$$

  The parameter γ is the *decay constant*, and the parameter f_0 is the resonance frequency. Note that the envelope of $r(t)$, $|r_a(t)|$, is simply $R_0 \exp(-\gamma t)$, as it should be. The proof of Eq. 17 is as follows:

$$r(t) = R_0 e^{-\gamma t} \cos(2\pi f_0 t) \tag{18}$$
$$R(f) = \frac{R_0}{2\pi j} \left(\frac{f - \frac{j\gamma}{2\pi}}{(f - z_+)(f - z_-)} \right) \tag{19}$$
$$z_+ = f_0 + \frac{j\gamma}{2\pi} \tag{20}$$
$$z_- = -f_0 + \frac{j\gamma}{2\pi} \tag{21}$$
$$\widehat{r}(t) = \mathcal{F}^{-1} \left\{ -j \operatorname{sgn}(f) R(f) \right\} \tag{22}$$
$$\widehat{r}(t) = \frac{R_0}{2j} \left(e^{2\pi j f_0 t} - e^{-2\pi j f_0 t} \right) e^{-\gamma t} \tag{23}$$
$$\widehat{r}(t) = R_0 \sin(2\pi f_0 t) e^{-\gamma t} \tag{24}$$
$$r_a(t) = R_0 \left(\cos(2\pi f_0 t) + j \sin(2\pi f_0 t) \right) e^{-\gamma t} \tag{25}$$
$$r_a(t) = R_0 e^{2\pi j f_0 t} e^{-\gamma t} \tag{26}$$

  In evaluating the inverse Fourier transform in Eq. 22, the poles at z_+ and z_- must be enclosed in separate contour integrals.

- Equation 15 will now be applied, using the definition of $r_a(t)$ and the Fourier transform of $s(t)$, $S(f)$. The Fourier transform of $r_a(t) * s_a(t)$ is $R_a(f) S_a(f)$. Using the definition of the analytic signal, we have

$$R_a(f) S_a(f) = R_a(f)(1 + \operatorname{sgn}(f)) S(f) \tag{27}$$

  The Fourier transform of $r_a(t)$ is

$$R_a(f) = \frac{R_0}{2\pi j} \frac{1}{f - z_0} \tag{28}$$
$$z_0 = f_0 + \frac{j\gamma}{2\pi} \tag{29}$$

  The Fourier transform $S(f)$ is given by Eq. 7. Inserting the spectra into Eq. 27 and taking the inverse Fourier transform should give $r_a(t) * s_a(t)$. Note the inverse transform integral is only over positive frequencies, due to the $1 + \operatorname{sgn}(f)$ factor. Introducing the time derivative removes a factor of $(2\pi j f)$ from the numerator of the integrand. Note also that it is prudent to consider the signal $s(t)$ shifted by a t_0, to ensure the entire signal is convolved with $r(t)$. For time-shifts, the Fourier transform responds like:

$$\mathcal{F} \left\{ s(t - t_0) \right\} = e^{-2\pi f t_0} S(f) \tag{30}$$

  The result is

$$r_a(t) * s_a(t) =$$
$$-j \sqrt{\frac{2}{\pi}} R_0 E_0 \sigma_t^3 \frac{d}{dt} \int_0^\infty \frac{e^{2\pi j f (t - t_0)} e^{-\frac{1}{2}\left(\frac{f}{\sigma_f}\right)^2}}{f - z} df \tag{31}$$

Completing the square in the exponent, and letting $x = (t - t_0)/(\sqrt{2}\sigma_t)$, $y = f/(\sqrt{2}\sigma_f)$, and $z_0 \to f_0/(\sqrt{2}\sigma_f) + j\gamma/(2\pi\sqrt{2}\sigma_f)$, we find

$$r_a(t) * s_a(t) = \\ -\frac{j}{\sqrt{\pi}} R_0 E_0 \sigma_t^2 \frac{d}{dx}\left(e^{-x^2}\int_0^\infty \frac{e^{(x+jy)^2}}{y - z_0} dy\right) \quad (32)$$

All that remains is to evaluate the integral

$$I(z, z_0) = \int_0^\infty \frac{e^{(x+jy)^2}}{y - z_0} dy \quad (33)$$

Let $z = x + jy$, $dy = -jdz$, and $z_1 = x + jz_0$. The integral becomes, as $R \to \infty$,

$$I(x, z_0) = \int_x^{x+jR} \frac{e^{z^2}}{z - z_1} dz \quad (34)$$

$I(x, z_0)$ can be evaluated via contour integration, with a rectangular contour that encloses the pole if $x > \Im\{z_0\}$. Let I_1 extend from $(x, 0)$ to (x, jR). Let I_2 extend from (x, jR) to $(0, jR)$. Let I_3 extend from $(0, jR)$ to $(0, 0)$. Finally, let I_4 extend from $(0, 0)$ to $(x, 0)$. Note that $I_1 = I(x, z_0)$. For the integral I_2, note that the exponential factor is proportional to $\exp(-R^2)$, while the denominator contains another factor of R. Since $dz = dx$, the integrand goes as $\exp(-R^2)/R$, vanishing as $R \to \infty$. For I_3, $x = 0$. Thus, it reduces to

$$I_3 = -\int_0^R \frac{e^{-y^2}}{y - z_0} dy \quad (35)$$

Let $y \to -y$, so that

$$I_3 = -\int_0^R \frac{e^{-y^2}}{y + z_0} dy = -G(z_0) \quad (36)$$

The function $G(z_0)$ is known as the Goodwin-Staton integral. $G(z_0)$ appears in radiation transport, neutron transport, and optical calculations. It is commonly used in problems involving diffusion approximations, particularly in media with semi-infinite boundary conditions. The final segment of the contour is

$$I_4 = \frac{j}{z_0}\int_0^x e^{x'^2} dx' \quad (37)$$

The sum of contour segments is $I_1 + I_2 + I_3 + I_4 = I_C$, where I_C is given by the Cauchy integral formula: $2\pi j e^{z_1^2}$. Solving for $I(x, z_0)$, we find

$$I(x, z_0) = 2\pi j e^{z_1^2} + G(z_0) - \frac{j}{z_0}\int_0^x e^{x'^2} dx' \quad (38)$$

If $x < \Im\{z_0\}$,

$$I(x, z_0) = G(z_0) - \frac{j}{z_0}\int_0^x e^{x'^2} dx' \quad (39)$$

Inserting this result into Eq. 32,

$$r_a(t) * s_a(t) = \\ -\frac{j}{\sqrt{\pi}} R_0 E_0 \sigma_t^2 \left(\frac{dA(x, z_0)}{dx} - \frac{j}{z_0}\frac{dD(x)}{dx}\right) \quad (40)$$

with

$$A(x, z_0) = 2\pi j e^{z_1^2} e^{-x^2} + G(z_0)e^{-x^2}, \quad x > \Im\{z_0\} \quad (41)$$
$$A(x, z_0) = G(z_0)e^{-x^2}, \quad x < \Im\{z_0\} \quad (42)$$

and

$$D(x) = e^{-x^2}\int_0^x e^{x'^2} dx' \quad (43)$$

What remains is to compute the magnitude of Eq. 40. Let the overall scale factor in Eq. 40 be $R_0 E_0 \sigma_t^2/\sqrt{\pi}$, let $z_0 = x_0 + jy_0$, and let $x > \Im\{z_0\}$. Upon taking the magnitude, the quantity in parentheses generates four terms. The first is

$$\frac{dA^*}{dx}\frac{dA}{dx} = \\ 16\pi^2|z_0|^2 e^{-4xy_0 - 2x_0^2 - 2y_0^2} \\ + 4G(z_0)x^2 e^{-2x^2} \\ + 16\pi G(z_0)x e^{-x^2} e^{-\gamma t} e^{-x_0^2 + y_0^2} \\ \times (x_0 \cos(2x_0(x - y_0)) - y_0 \sin(2x_0(x - y_0))) \quad (44)$$

The second is

$$\frac{1}{z_0^2}\left(\frac{dD(x)}{dx}\right)^2 = \frac{1}{z_0^2}\left(1 - 4xD(x) + 4x^2 D^2(x)\right) \quad (45)$$

The combination of the third and forth terms is

$$\frac{dD}{dx}\left\{\frac{j}{z_0^*}\frac{dA}{dx} - \frac{j}{z_0}\frac{dA^*}{dx}\right\} = \\ -\frac{2}{|z_0|^2}\frac{dD}{dx}\frac{d}{dx}(-2\pi y_0 e^{-2xy_0} e^{y_0^2 - x_0^2} \sin(2x_0(x - x_0)) \\ + y_0 G(z_0)e^{-x^2} \\ + 2\pi x_0 e^{-2xy_0} e^{y_0^2 - x_0^2} \cos(2x_0(x - x_0))) \quad (46)$$

IV. CONCLUSION

The conclusion.

Appendix A: Details

The details.

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