Friday warm-up: Kinematics III, and the Cross-Product

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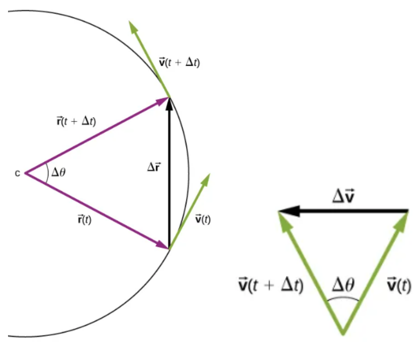

Figure 1: Geometric picture of centripetal acceleration.

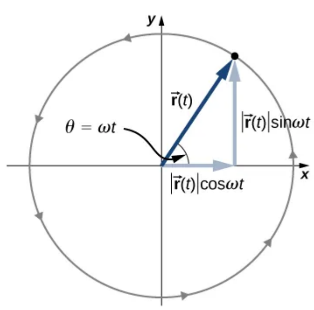

Figure 2: Algebraic picture of centripetal acceleration.

1 Memory Bank

1. $s = r\theta$ ... Arc length s, radius r, and the radian, θ.
2. $\vec{r}(t) = r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j}$
3. $\omega = \Delta\theta/\Delta t$ ... Average *angular* velocity
4. $\vec{v} = \vec{\omega} \times \vec{r}$ ... Relationship between *tangential* velocity, radius, and *angular* velocity
5. The **cross-product**:
   - $\hat{i} \times \hat{j} = \hat{k}$
   - $\hat{k} \times \hat{i} = \hat{j}$
   - $\hat{j} \times \hat{k} = \hat{i}$
   - Non-alphabetical order: multiply by -1. For example, $\hat{k} \times \hat{i} = -\hat{j}$
   - $\hat{i} \times \hat{i} = 0$, $\hat{j} \times \hat{j} = 0$, $\hat{k} \times \hat{k} = 0$

2 The Cross Product of Vectors

1. Multiply these vectors using the *cross-product*: (a) $\vec{v}_1 = 2\hat{i}$, and $\vec{v}_2 = 3\hat{j}$. (b) $\vec{v}_1 = 2\hat{i} + 2\hat{j}$, and $\vec{v}_2 = \hat{i} - \hat{j}$.

3 Kinematics III

1. In Fig. 1, the angle $\Delta\theta$ is the same in both triangles. (a) Why does $\Delta v/v = \Delta r/r$? (b) Multiply both sides by v, to obtain $\Delta v = v/r\Delta r$. Now divide both sides by Δt, and take the limit that $\Delta t \to 0$. What is the result? (c) Let $s(t) = r\theta(t)$. Take the derivative of both sides, and let $d\theta(t)/dt = \omega$. What do you find? (d) Now eliminate v in favor of ω in the expression you derived for the acceleration. (d) Suppose a warrior is using a *sling* to hurl a stone at an enemy. The stone circles above his head with radius $r = 1$ m. If the stone makes 1 revolution every 0.25 seconds, what is its speed and acceleration?

2. Consider Fig. 1. The angle $\theta(t) = \omega t$ describes the position of a system circling the origin at constant speed. Prove the following

$$\vec{r}(t) = r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j} \quad (1)$$

$$\vec{a} = -\omega^2\vec{r} \quad (2)$$

3. (a) Draw a diagram of a circle in the xy-plane, centered at the origin, and draw the *axis of rotation* along the z-axis. (b) Show that if $\vec{v}$ is tangent to the circle, and $\vec{r}$ is the displacement from the origin, $\vec{\omega}$ constantly points in the z-direction, if $\vec{v} = \vec{\omega} \times \vec{r}$.