Calculus-Based Physics-1: Mechanics (PHYS150-01): Unit 0

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Opening Remarks - Welcome!

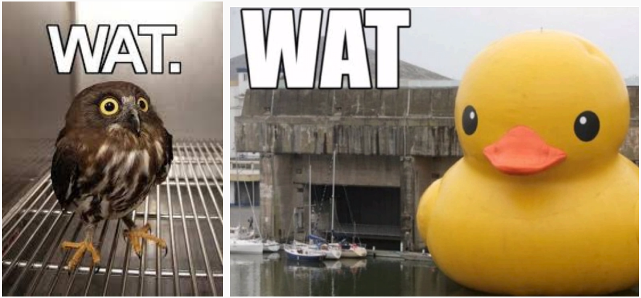


Figure 1: Taking physics for the first time.

Summary

Week 1 Summary

Physics - $\phi\upsilon\sigma\iota\kappa\acute{\eta}$ - "phusiké": *knowledge of nature*
from $\phi\acute{\upsilon}\sigma\iota\varsigma$ - "phúsis": *nature*

1. Estimation/Unit Analysis - Chapters 1.1 - 1.4
   - Estimating the correct order of magnitude
   - Unit analysis - dealing with the units of numbers
2. Coordinates and vectors - Chapters 2.1 - 2.4
   - Scalars and vectors
   - Cartesian (rectangular) coordinates, displacement
   - Vector addition, subtraction, and multiplication
3. Review of Calculus Techniques
   - The derivative, derivatives of elementary functions
   - Function approximation
   - Anti-derivatives and integration

Estimation/Unit Analysis - Chapters 1.1 - 1.4

Estimation/Unit Analysis

In science and engineering, estimation is to obtain a quantity in the absence of precision, informed by rational constraints.

1. **Define relevant unit scales**: (mg, g, or kg), (m/s or km/hr)
2. **Obtain complex quantities** from simple ones
    - Obtain *areas* and *volumes* from *lengths*
    - Obtain *rates* from *numerators* and *denominators*
3. **Taking advantage of scaling problems**
    - Knowing *relationship* between variables
    - Using that *relationship* to obtain new information
4. **Constrain the unknown with upper and lower limits**

Estimation/Unit Analysis

Units: Which of the following represents a *volume*?

- A: 10 gm
- B: 10 cm^2
- C: 1 cm^3
- D: 1 cm s^{-1}

Units: If a grain of sand within a fluid sinks 15 cm in 5 seconds, what is the speed of the grain?

- A: 3 cm
- B: 3 s
- C: 3 s/cm
- D: 3 cm/s

Unit conversion: If a person weights 120 lbs, what is their weight in kilograms?[1]

- A: 54.5 kg
- B: 264 kg
- C: 54.5 lbs
- D: 264 lbs

[1]One kilogram is 2.2 lbs.

Unit conversion: A density is a mass divided by a volume. For example, water has a density of 1 gm cm^{-3}. What is the density of water in kg m^{-3}?

- A: 1 kg m^{-3}
- B: 10 kg m^{-3}
- C: 100 kg m^{-3}
- D: 1000 kg m^{-3}

Group exercise on complex units: A *vitrolero* is a classic container for serving *agua fresca*. It has a diameter of 20 cm, and a height of 30 cm. How many cups can we serve from the vitrolero if we put 0.5 liters of agua fresca in each cup?

- *Hint:* 1 liter is 1000 mL
- *Hint:* 1 mL is 1 cm^3
- **Volume:** The volume of a cylinder is π times the radius of the base, squared, , times the height: $\pi r^2 h$.

Unit scale: A generation is about one-third of a lifetime. Determine how many generations have passed since the year 0 AD[2].

- A: 10
- B: 20
- C: 60
- D: 100

[2] What is the appropriate scale here?

Unit scale: (a) What fraction of Earth's diameter[3] is the greatest ocean depth (11 km below sea level)? (b) The greatest mountain height (8.8 km above sea level)?

- A: 8.6×10^{-2}, 6.9×10^{-2}
- B: 8.6×10^{-3}, 6.9×10^{-3}
- C: 8.6×10^{-4}, 6.9×10^{-4}
- D: 8.6×10^{-5}, 6.9×10^{-3}

[3]The diameter of the Earth is 12,800 km.

Complex quantities: If a Whittier College athlete ran the 5k race at a track meet in 35 minutes, what was her average speed?

- A: 0.3 meters per second
- B: 3 meters per second
- C: 30 meters per second
- D: 300 meters per second

Complex quantities: Suppose you won the lottery and received $1 billion USD. Because your life is dope, you stack that paper over the Whittier College soccer field. Each stack contains 100 bills, and each bill is worth $100. If you evenly cover the field, how high is the money level?

- A: 0.5 inch
- B: 1 inch
- C: 2 inches
- D: 1 foot

Scaling problem: Supposed you have an ideal gas in a cylinder of fixed volume. If the pressure begins as 100 kPa, and you *double* the temperature of the gas, what is the new pressure?

- A: 100 kPa
- B: 50 kPa
- C: 10 kPa
- D: 200 kPa

Scaling problem: Supposed you have an ideal gas in a cylinder of fixed volume. If the pressure begins as 100 kPa, and you *halve* the temperature of the gas, what is the new pressure?

- A: 100 kPa
- B: 50 kPa
- C: 10 kPa
- D: 200 kPa

Upper/lower limits: How many undergraduate students are there at Whittier College[4]?

- A: 5,000
- B: 1,000
- C: 1,250
- D: 500

[4] What is the absolute lower limit, and what is the upper limit?

Upper/lower limits: What is the average yearly college tuition in the United States (before subtracting grants and scholarships)?

- A: \$5,000
- B: \$10,000
- C: \$25,000
- D: \$40,000

What information affects the upper and lower limits here?

Coordinates and Vectors - Chapters 2.1 - 2.4

Coordinates and Vectors - Applications: displacement

Activity Link

Who understands coordinates and vectors better than anyone else?

`https://youtu.be/0B7WL7nhIF4?si=_dl4t_GwL98aXWFa`

Physics requires mathematical objects to build equations that capture the behavior of nature. Two examples of such objects are scalar and vector quantities. Each type of object obeys similar but different rules.

1. Scalar quantities
   - mass: $m_1 + (m_2 + m_3) = (m_1 + m_2) + m_3$
   - speed: $v_1(v_2 + v_3) = v_1 v_2 + v_1 v_3$
   - charge: $q_1 \left(\frac{1}{q_1}\right) = 1$, $q_1(0) = 0$
2. Vector quantities
   - velocity: $\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = (\vec{v}_1 + \vec{v}_2) + \vec{v}_3$
   - tension: $\vec{t}_1 \cdot (\vec{t}_2 + \vec{t}_3) = \vec{t}_1 \cdot \vec{t}_2 + \vec{t}_1 \cdot \vec{t}_3$

A vector may be expressed as *a list of scalars*: $\vec{v} = (4, 2)$ (a vector with two *components*), $\vec{u} = (3, 4, 5)$ (three *components*). Now, we know how to add and subtract scalars. How do we add and subtract vectors?

What is
$(1, 3, 8)+$
$(0, 2, 1)$?
Answer: $(1, 5, 9)$

In other words, when adding vectors, we add them component by component.

How do we subtract vectors? In the same fashion:

What is
$(1, 3, 8)-$
$(0, 2, 1)$?
Answer: $(1, 1, 7)$

In other words, when subtracting vectors, we subtract them component by component.

A HTML-based demonstration for adding vectors:

`https://phet.colorado.edu/en/simulations/vector-addition`

Notice several things:

- Produce vectors that *cancel* each other.
- What happens when vectors are parallel and orthogonal?

How do we multiply vectors? In the same fashion, *for one kind of multiplication*:

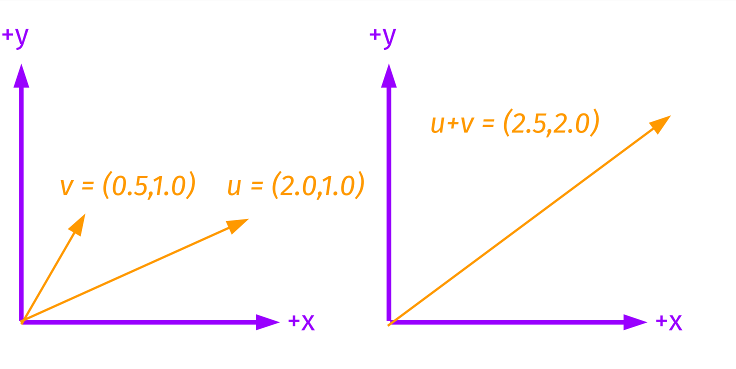
What is
$(1, 3, 8) \cdot (0, 2, 1)$?
Answer: $1 \cdot 0 + 3 \cdot 2 + 8 \cdot 1 = 14$

This kind of multiplication is known as the dot-product. There is also the *cross-product*, which we will save for later.

The components of a vector may describe quantities in a coordinate system, such as *Cartesian coordinates* - after René Descartes. Vectors in the 3D Cartesian coordinate system (x,y,z) may be written in the following notation:

$$\boxed{\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}}$$

- a: The amount in the +x-direction, $\hat{i}$: a vector of length 1, in the +x-direction
- b: The amount in the +y-direction, $\hat{j}$: a vector of length 1, in the +y-direction
- c: The amount in the +z-direction, $\hat{k}$: a vector of length 1, in the +z-direction



(a) (b)

Figure 2: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) What is $\vec{u} + \vec{v}$? Adding components: $\vec{u} + \vec{v} = 2.5\hat{i} + 2.0\hat{j}$.

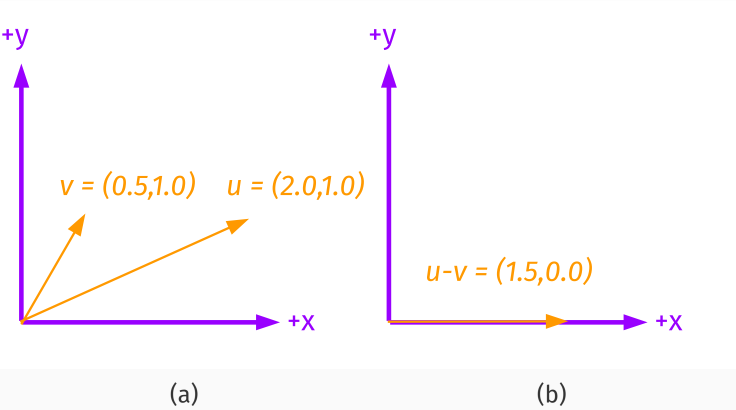
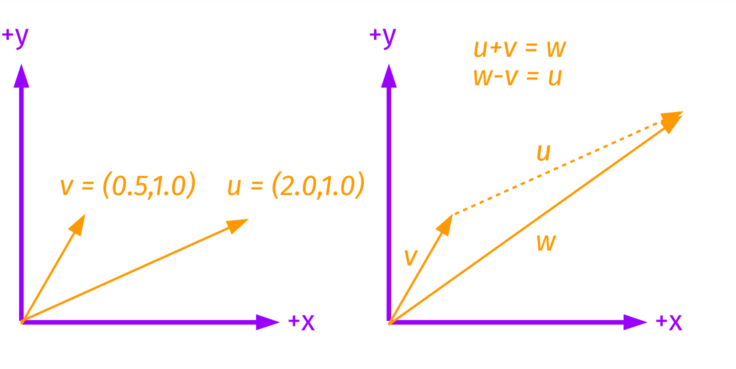


Figure 3: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) What is $\vec{u} - \vec{v}$? Subtracting components: $\vec{u} - \vec{v} = 1.5\hat{i} + 0.0\hat{j}$.



(a) (b)

Figure 4: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) To compute $\vec{w} - \vec{v}$, arrange the vectors to get a sense of the result, $\vec{u}$.

$\vec{p} = 4\hat{i} + 2\hat{j}$. $\vec{q} = -4\hat{i} + 2\hat{j}$.
Compute $\vec{p} \cdot \vec{q}$.

- A: 12
- B: -12
- C: 4
- D: 8

$\vec{p} = -1\hat{i} + 6\hat{j}$. $\vec{q} = 3\hat{i} + 0.5\hat{j}$.
Compute $\vec{p} \cdot \vec{q}$.

- A: -1
- B: 1
- C: 0
- D: 3

Why was the last answer zero? Look at it graphically:

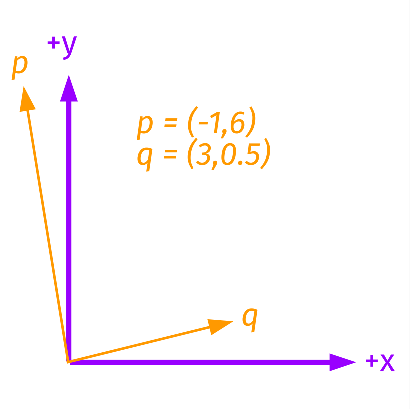


Figure 5: Two vectors $\vec{p}$ and $\vec{q}$ are *orthogonal* if $\vec{p} \cdot \vec{q} = 0$.

What if the vectors are parallel? Look at it graphically:

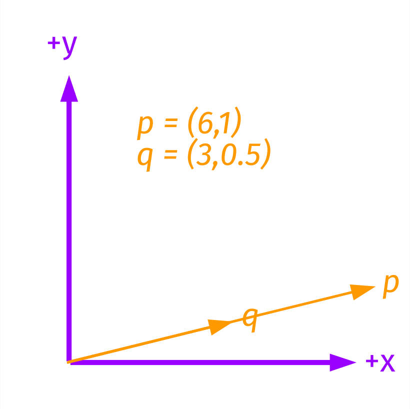


Figure 6: Two vectors $\vec{p}$ and $\vec{q}$ are *parallel* if $\vec{p} \cdot \vec{q}$ is maximal.

The *length* or *norm* of a vector $\vec{v} = a\hat{i} + b\hat{j}$ is $|\vec{v}| = \sqrt{a^2 + b^2}$.

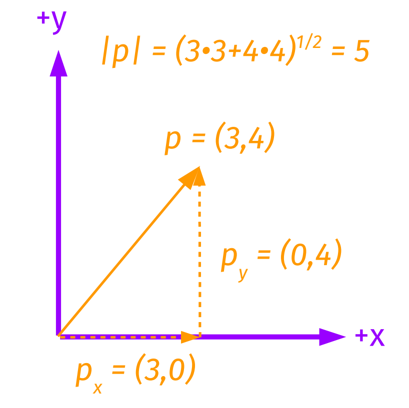


Figure 7: Computing the norm of a vector $\vec{p}$.

Notice that $\sqrt{\vec{p} \cdot \vec{p}} = |\vec{p}|$.
Let θ_p be the angle between $\vec{p}$ and the x-axis.
$p_x = \vec{p} \cdot \hat{i} = |\vec{p}| \cos(\theta_p)$.
$p_y = \vec{p} \cdot \hat{j} = |\vec{p}| \sin(\theta_p)$.

Theorem: The dot product of two vectors $\vec{p}$ and $\vec{q}$ is $|u||v| \cos(\theta)$, if θ is the angle between them.

Proof: $\vec{p} \cdot \vec{q} = p_x q_x + p_y q_y = |p||q| \cos\theta_p \cos\theta_q + |p||q| \sin\theta_q \sin\theta_q$
$= |p||q|(\cos\theta_p \cos\theta_q + \sin\theta_p \sin\theta_q) = |p||q| \cos(\theta_p - \theta_q)$
$= |p||q| \cos\theta$.
$\boxed{\vec{p} \cdot \vec{q} = |p||q| \cos\theta}$

An object moves at 2 m/s at $\theta = 60°$ with respect to the x-axis. What is the velocity of the object?

- A: $(1\hat{i} + 1\hat{j})$ m/s
- B: $(\sqrt{3}\hat{i} + 1\hat{j})$ m/s
- C: $(\sqrt{3}\hat{i} + \sqrt{3}\hat{j})$ m/s
- D: $(1\hat{i} + \sqrt{3}\hat{j})$ m/s

What is the dot product of this velocity with another velocity: 5 m/s along the x-axis?

- A: 1 $(m/s)^2$
- B: 5 $(m/s)^2$
- C: 10 $(m/s)^2$
- D: 5 (m/s)

Is it possible to multiply vectors and scalars? Of course: $a_1\vec{p} = a_1 p_x \hat{i} + a_1 p_y \hat{j}$.

Also, multiplication properties still hold. For example: $(a_1 + a_2)\vec{p} = a_1\vec{p} + a_2\vec{p}$.

A spacecraft moves at 400 m/s, at an angle of 30 degrees with respect to the x-axis. If it fires two thrusters that boost the x-component and y-component of the velocity by 25% and 50%, respectively, what is the final velocity?

- A: $(433\hat{i} + 300\hat{j})$ m/s
- B: $(300\hat{i} + 433\hat{j})$ m/s
- C: 400 m/s
- D: $(400\hat{i} + 433\hat{j})$ m/s

We define the *position* of an object as a vector locating it in a given coordinate system. The scalar *distance* is the norm of the position vector, that is, the distance to to the origin.

Now we can introduce the concept of dislacement: a vector describing a movement of an object.

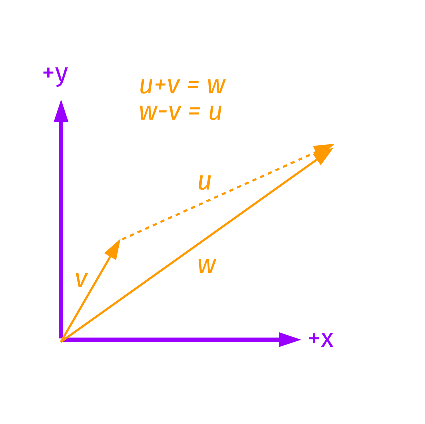


Figure 8: Suppose an object moves from position $\vec{v}$ to $\vec{w}$. In this case, the displacement is $\vec{u}$. **Thus, the final position is the initial position, plus the displacement.**

It follows that the *displacement* is zero if the initial and final positions are the same, but the *distance travelled* is not.

Suppose a jet fighter travelling at 800 km per hour banks such that it flies in a circle of radius 0.5 km. How long does it take to complete the circle? What is the distance traveled, and what is the displacement?

- A: 2π km, 28 seconds, 2π km
- B: π km, 14 seconds, π km
- C: π km, 28 seconds, π km
- D: π km, 14 seconds, 0 km

Coordinates and Vectors - Average Velocity (Chapter 3.1)

Average velocity is the ratio of the displacement to the elapsed time.

$$\boxed{\vec{v}_{\mathrm{avg}} = \frac{\Delta\vec{x}}{\Delta t}} \tag{1}$$

The *average speed* is the norm of the average velocity:

$$\boxed{v_{\mathrm{avg}} = \frac{|\Delta\vec{x}|}{\Delta t}} \tag{2}$$

If the motion is in one dimension, then the average speed is

$$v_{\mathrm{avg}} = \frac{x_{\mathrm{f}} - x_{\mathrm{i}}}{t_{\mathrm{f}} - t_{\mathrm{i}}} \tag{3}$$

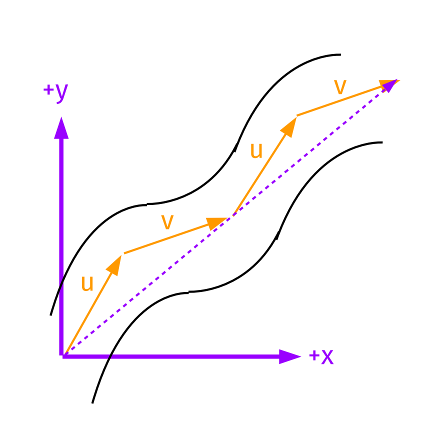


Figure 9: A Formula-1 driver keeps his car on the track by following a path approximated by the position vectors *u*, *v*, *u*, and *v*. The dashed arrow represents the total displacement.

If $\vec{u} = (20\hat{i} + 30\hat{j})$ m, and $\vec{v} = (30\hat{i} + 20\hat{j})$ m, what is the total displacement? If the elapsed time is 10 seconds, what is the average velocity?

- A: $(50\hat{i} + 50\hat{j})$ m, 14 m/s
- B: $(80\hat{i} + 100\hat{j})$ m, 10 m/s
- C: $(100\hat{i} + 100\hat{j})$ m, 14 m/s
- D: $(50\hat{i} + 150\hat{j})$ m, 10 m/s

Review of Calculus Techniques

In science and engineering, function approximation or expanding a function is a technique in which a simple function is used obtain the value of a more complicated function near a point where they are approximately equal.

1. Memorizing special cases
    - $\sin(x) \approx x$, when $|x| < 1$
    - $\tan(x) \approx x$, when $|x| < 1$
    - $(1+x)^{1/2} \approx 1 + \frac{1}{2}x$, when $|x| < 1$
    - $\exp(x) \approx 1 + x$, when $|x| < 1$
2. Utilizing the **Taylor Series** (more on this later)
    - $f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$

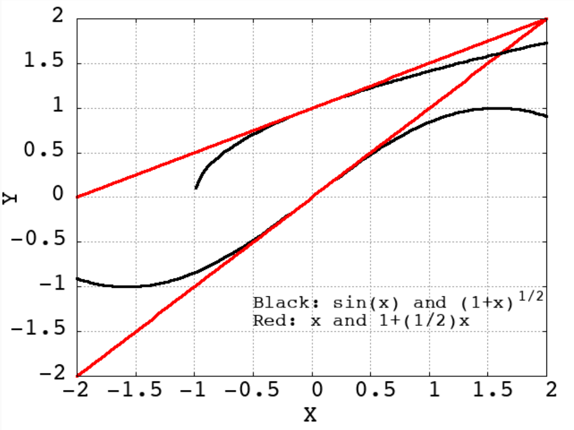


Figure 10: Certain functions may be approximated by simpler ones. In this case, $\sin(x)$ is approximated by x near $x = 0$, and $(1 + x)^{1/2}$ is approximated by $1 + \frac{1}{2}x$ near $x = 0$.

Methods of approximation - Function approximation

The height in meters of a surfer above some average height as he bobs in the waves is described by $h(t) = sin(\pi t/2)$. What is his height at 0.2 seconds? What is his height at -0.2 seconds?

- A: 10 meters, -10 meters
- B: $\pi/10$ meters, -$\pi/10$ meters
- C: -0.1 meters, 0.1 meters
- D: -π meters, π meters

The value of an investment in dollars, v, versus time in years, t, follows the form $v(t) = P\exp(rt)$, where P is the value at $t = 0$, and $r = 1/3$. What is $v(1)$, the value after one year?

- A: $\approx 1/3P$
- B: $\approx 2/3P$
- C: $\approx 3/2P$
- D: $\approx 4/3P$

Review of Calculus - Skill 1: Taking Limits

Taking a limit has a variety of uses in physics. Consider the following example:

The weight of an object sitting on the Earth's surface is given by the following equation, which we will encounter later in physics:

$$w = G\frac{mM}{r^2} \tag{4}$$

- w: weight
- G: a constant of nature
- m: the object's mass
- M: the mass of the Earth
- r: the distance between the center of the Earth and the object

The weight of an object sitting on the Earth's surface is given by the following equation, which we will encounter later in physics:

$$w = G\frac{mM}{r^2} \tag{5}$$

Let R be the radius of the Earth, r_0 be the object's height above the Earth's surface, and $\epsilon = r_0/R \ll 1$. Also, let $g = GM/R^2$. Rearranging the equation for weight:

$$w = mg(1+\epsilon)^{-2} \tag{6}$$

Since $\epsilon \ll 1$, take the limit:

$$\lim_{\epsilon \to 0} w = \lim_{\epsilon \to 0} mg(1+\epsilon)^{-2} = mg \tag{7}$$

Thus, for practical calculations where the object is not far from the Earth's surface, the weight is $w = mg$, or the mass times some measurable constant, g.

Often in other branches of physics, we often encounter the *sinc function*:

$$s(t) = \frac{sin(t)}{t} \tag{8}$$

What is $\lim_{t \to 0} s(t)$? It looks like $0/0$, but that is not definited mathematically.

We can use L'Hôpital's Rule: take the *derivative* of the numerator and the denominator, then take the limit:

$$\lim_{t\to 0} s(t) = \lim_{t\to 0} \left| \frac{\frac{d}{dt}\sin(t)}{\frac{d}{dt}t} \right| = \lim_{t\to 0} \cos(t) = 1 \tag{9}$$

This approach is valid for cases like $0/0$ or ∞/∞. But what is the derivative? How do we know that the derivative of $\sin(t)$ is $\cos(t)$, and the derivative of t is one? The definition of the derivative:

$$f'(t) = \dot{f} = \frac{df}{dt} \equiv \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \tag{10}$$

"The change in y over the change in x at h."

Review of Calculus - Skill 2: Taking Derivatives

1. $f(t) = t^n$
2. $f(t) = \exp(t)$
3. $f(t) = \ln(t)$
4. $f(t) = \sin(t)$
5. $f(t) = \cos(t)$
6. $f(t) = \tan(t)$
7. $f(t) = a$

1. $f' = nt^{n-1}$
2. $f' = \exp(t)$
3. $f' = 1/t$
4. $f' = \cos(t)$
5. $f' = -\sin(t)$
6. $f' = \sec^2(x)$
7. $f' = 0$

1. $g(t) = af(t)$
2. $g(t) = a(t) + b(t)$
3. $g(t) = a(t)b(t)$
4. $g(t) = a(t)/b(t)$
5. $g(t) = a(b(t))$

1. $g' = af'$
2. $g' = a' + b'$
3. $g' = ab' + a'b$
4. $g' = \frac{ba' - ab'}{b^2}$
5. $g' = a'(b)b'$

An *anti-derivative* just reverses the action of the derivative. Also called an indefinite integral. For example:

$$x(t) = at^n \tag{11}$$

$$x' = nat^{n-1} \tag{12}$$

$$\int nat^{n-1}dt = at^n + C \tag{13}$$

In Eq. 13, the $\int$ symbol represents integration (a big S for "summation"). There is a constant C because, technically, if we take the derivative of Eq. 13, the result is Eq. 12 (derivative of a constant C is zero).

An integral is the difference in the value of the anti-derivative evaluated at two points:

$$\int_c^d nat^{n-1}dt = a(d)^n - a(c)^n \tag{14}$$

"Raise the power by one, and divide the whole thing by that number." - Just like derivatives, there are many simple cases to memorize.

$$\int_a^b \cos(t)dt = \sin(b) - \sin(a) \tag{15}$$

1. $f(t) = t^n$
2. $f(t) = \exp(t)$
3. $f(t) = 1/t$
4. $f(t) = \sin(t)$
5. $f(t) = \cos(t)$
6. $f(t) = a$
7. ...

1. $\int f(t)dt = (n+1)^{-1}t^{n+1} + C$
2. $\int f(t)dt = \exp(t) + C$
3. $\int f(t)dt = \ln(t) + C$
4. $\int \sin(t)dt = -\cos(t) + C$
5. $\int \cos(t)dt = \sin(t) + C$
6. $\int f(t) = at + C$
7. ...

A Few Calculus Problems

Let $x(t) = 5t^2 - 2t + 7$ in the $\hat{x}$-direction, where x is in meters and t is in seconds. What is the velocity (take the derivative) at $t = 1$ second?

- A: 8 m/s in the $\hat{y}$-direction
- B: 6 m/s in the $\hat{x}$-direction
- C: 8 m/s in the $\hat{x}$-direction
- D: 6 m/s in the $\hat{y}$-direction

Let $v(t) = 2t + 2$ in the $\hat{x}$-direction. What is the position versus time (take the integral)?

- A: $t^2 + 2t + C$
- B: $t^2 + 2t$
- C: $t + 2 + C$
- D: $t^3 2t^2 + C$

Conclusion

Week 1 Summary

1. Methods of approximation
   - Estimating the correct order of magnitude
   - Function approximation
   - Unit analysis
2. Coordinates and vectors
   - Scalars and vectors
   - Cartesian (rectangular) coordinates, displacement
   - Vector addition, subtraction, and multiplication
3. Review of Calculus Techniques
   - Limits
   - Differentiation
   - Integration