CALCULUS-BASED PHYSICS-2: ELECTRICITY AND MAGNETISM (PHYS180-02): UNIT 2

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SUMMARY

Reading: chapters 9-10

1. Chapter 9: Current and resistance
2. Chapter 10: Ohm's law and Circuit Analysis

PREVIEW OF WHAT'S COMING

(Preview of Unit 3). Suppose we have a system that obeys

$$v(t) = R\frac{dQ}{dt} = i(t)R \tag{1}$$

This is called **Ohm's law.**

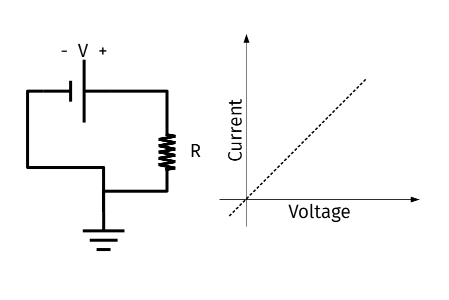

Figure 1: A simple circuit with a resistor element, some voltage, and *ground*. This just means that 0V is at the negative terminal.

(Preview of Unit 3). What if we add a capacitor?

$$V_0 = R\frac{dQ}{dt} + \frac{Q}{C} \tag{2}$$

This is called **Ohm's law.**

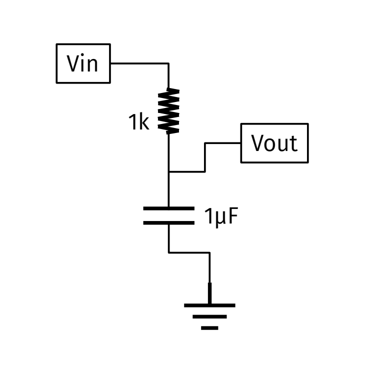

Figure 2: A simple circuit with a resistor and capacitor elements.

(Preview of Unit 3). What if we add a capacitor? We can show that

$$V_{out}(t) = V_0 \exp(-t/\tau) \tag{3}$$

with $\tau = RC$. Thus, if we send a signal V_{in} down a "very long" coaxial cable, with some capacitance per unit length, it will not exit the cable.

For example: `https://www.pasternack.com/images/ProductPDF/LMR-400.pdf`

What is the attenuation per 100 m of this cable at 150 MHz?

CURRENT

Notions of current:

- $I = \frac{\Delta Q}{\Delta t}$ - The derivative of charge
- The *movement* of electrons
- The *flow* of charge
- Number of Coulombs per second (1 Amp = C/s)

There is an interesting problem with the notion of current as movement of charges.

Speed of typical electronic signals: $\approx 10^8$ m/s

Typical speed of actual charges passing through a conductor under voltage: $\approx 10^{-4}$ m/s

Since there is a 12 order of magnitude range, it's probably a good idea to ponder...

Are the electrons colliding/interacting to form electrical signals? Or just moving all together?

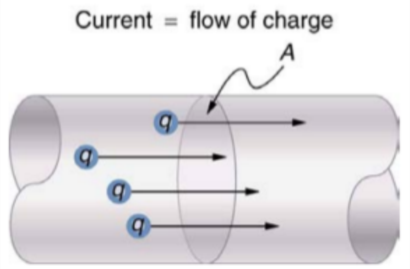

Figure 3: The *drift velocity* is the average velocity of an electron, and current is derived from this average velocity.

So we see how electrical signals can move near the speed of light, but we measure the movements of electrons in circuits to be slow. Can we make a calculation to understand the speed of the electrons?

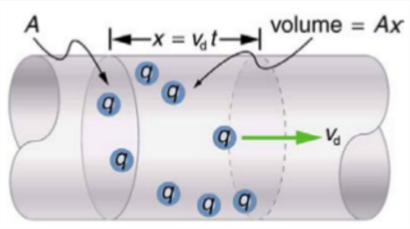

Figure 4: Consider the volume V of conductor with cross-sectional area A and length Δx, having n free electrons per unit volume.

An **amp** is one *Coulomb* per *second*. The definition of current is

$$I = \frac{\Delta Q}{\Delta t} = \frac{qnA\Delta x}{\Delta t} = qnAv_{\mathrm{d}} \tag{4}$$

Solving for drift velocity:

$$v_{\mathrm{d}} = \frac{I}{qnA} \tag{5}$$

Suppose our conductor is a wire with radius r and $A = \pi r^2$. Substituting,

$$v_{\mathrm{d}} = \frac{I}{\pi qnr^2} \tag{6}$$

Remember that $q = 1.6 \times 10^{-19}$ C, and n is the number of free electrons *per atom per unit volume*. How do we get this number?

Number density: The total number of objects in a system is equal to the *number density* times the volume of the system.

$$N = nV \tag{7}$$

- N: Total number
- n: *number density*
- V: Volume

Example: Number of Stars in the Milky Way. How many stars are in our galaxy? Assume the galaxy is a disk of height h and radius r. We observe n stars per unit volume.

- $r = 50 \times 10^3$ *light-years*
- $h = 2 \times 10^3$ *light-years*
- $n = 10^{-2}$ *light-year* $^{-3}$

1. Compute the volume in *light-years*^{3}
2. Multiply the volume by the number density to obtain the total number.
3. Compare the result with others' results.

How many **conduction** electrons are there in a cube of copper that is 1 micron (1 μm = 10^{-6} m) on a side?

- Copper has a density of 8.8 grams per cubic centimeter.
- Copper has an atomic weight of 63.54 grams per mole. (*Do you remember what a mole is?*).
- There are $N_A = 6.02 \times 10^{23}$ atoms per mole.
- Only one electron per atom of copper is a conduction electron.

---

1. Divide the density by the atomic weight. What are the units?
2. Multiply by N_A (Avogadro's number). What are the units?
3. Convert the units from cm^{-3} to μm^{-3}.

Number density: Let's examine copper, a common wire material with one free electron per atom. Copper has a density of 8800 kg/m^3, and 0.06354 kg/mol. There are 6.02×10^{23} atoms/mol. How many free electrons per m^3 of copper? (Remember that there is only one conduction electron per copper atom).

- A: 10^{26} free electrons per m^3
- B: 10^{27} free electrons per m^3
- C: 10^{28} free electrons per m^3
- D: 10^{29} free electrons per m^3

Consider a copper wire with radius $r = 2.053$ mm that is carrying 20.0 A of current. Using $q = 1.6 \times 10^{19}$, and $n = 8.34 \times 10^{28}$ electrons/m^3, and $v_{\mathrm{d}} = I/(\pi q n r^2)$, compute the drift velocity of charge in the wire. *This is a common situation in household wiring.*

- A: 10^{-1} m/s
- B: 10^{-2} m/s
- C: 10^{-3} m/s
- D: 10^{-4} m/s

Drift speed vs. signal speed. Given that the electrons move at 1 mm/s, how is it that electric signals move at 10^8 m/s?

Electrical signals are waves of charge:
`https://phet.colorado.edu/en/simulation/legacy/wave-on-a-string`

Current as the derivative of charge: $I = dQ/dt$. Suppose a bolt of lightning lasts for 0.5 ms, and carries a current of 30 kA (kilo-amps). How many Coulombs of charge does it deliver?

- A: 1.5 C
- B: 15 C
- C: 150 C
- D: 1500 C

Current as the derivative of charge: $I = dQ/dt$. Suppose a bolt of lightning lasts for 5 ms, and the current is described by $I(t) = \alpha t$, with $\alpha = 2 \times 10^3$ C/s^2. How many Coulombs of charge does it deliver?

- A: 2.5 C
- B: 0.25 C
- C: 0.025 C
- D: 0.0025 C

RESISTIVITY AND RESISTANCE

Resistivity ρ and **conductivity** σ are intrinsic properties of materials, and they are reciprocals of each other:

$$\rho = 1/\sigma \tag{8}$$

Let $\vec{J}$ be the *current density*. The most general form of Ohm's law is

$$\boxed{\vec{J} = \sigma \vec{E}} \tag{9}$$

Let's assume that current is flowing down a conductor of length L and cross-sectional area $\vec{A}$, parallel to $\vec{E}$. Integrate both sides:

$$I = \sigma \int \vec{E} \cdot d\vec{A} = \sigma E A = \frac{\sigma V}{L} A \tag{10}$$

$$V = \frac{L\rho}{A} I = IR \tag{11}$$

Thus, the *resistance* of an object is

$$R = \frac{L\rho}{A} \tag{12}$$

The resistivity has units of Ohm meters. Which of the following is true of a conductor?

- A: Lengthening it increases the resistance, and widening it increases the resistance.
- B: Lengthening it decreases the resistance, and widening it increases the resistance.
- C: Lengthening it increases the resistance, and widening it decreases the resistance.
- D: Lengthening it decreases the resistance, and widening it decreases the resistance.

Thus, the *resistance* of an object is

$$R = \frac{L\rho}{A} \tag{13}$$

Suppose the length of a conductor is doubled, and the area is quadrupled. By how much does the resistance change?

- A: It decreases by a factor of two.
- B: It decreases by a factor of four.
- C: It increases by a factor of two.
- D: It increases by a factor of four.

What is the resistance of a 1 meter-long wire with area 3 mm^2 that made from a metal that has $\rho = 10^{-7}\ \Omega$ m?

- A: 0.01 Ohms
- B: 0.1 Ohms
- C: 1.0 Ohms
- D: 10.0 Ohms

Material	Conductivity, σ $(\Omega \cdot \mathrm{m})^{-1}$	Resistivity, ρ $(\Omega \cdot \mathrm{m})$	Temperature Coefficient, α $(°\mathrm{C})^{-1}$
Conductors			
Silver	6.29×10^{7}	1.59×10^{-8}	0.0038
Copper	5.95×10^{7}	1.68×10^{-8}	0.0039
Gold	4.10×10^{7}	2.44×10^{-8}	0.0034
Aluminum	3.77×10^{7}	2.65×10^{-8}	0.0039
Tungsten	1.79×10^{7}	5.60×10^{-8}	0.0045
Iron	1.03×10^{7}	9.71×10^{-8}	0.0065
Platinum	0.94×10^{7}	10.60×10^{-8}	0.0039
Steel	0.50×10^{7}	20.00×10^{-8}	
Lead	0.45×10^{7}	22.00×10^{-8}	
Manganin (Cu, Mn, Ni alloy)	0.21×10^{7}	48.20×10^{-8}	0.000002
Constantan (Cu, Ni alloy)	0.20×10^{7}	49.00×10^{-8}	0.00003
Mercury	0.10×10^{7}	98.00×10^{-8}	0.0009
Nichrome (Ni, Fe, Cr alloy)	0.10×10^{7}	100.00×10^{-8}	0.0004

Figure 5: Resistivities of various metals.

Like the expansion of metals with increasing temperature, *resistivity has a temperature dependence.* Let $\Delta T = T - T_0$, with $T_0 = 20.0$ C.

$$\rho = \rho_0 \exp\left(\alpha \Delta T\right) \tag{14}$$

Using a Taylor series around $\Delta T = 0$, we find

$$\rho(T) \approx \rho_0 \left(1 + \alpha \Delta T\right) \tag{15}$$

Material	Conductivity, σ $(\Omega\cdot\mathrm{m})^{-1}$	Resistivity, ρ $(\Omega\cdot\mathrm{m})$	Temperature Coefficient, α $(^{\circ}\mathrm{C})^{-1}$
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Mercury	0.10×10^{7}	98.00×10^{-8}	0.0009
Nichrome (Ni, Fe, Cr alloy)	0.10×10^{7}	100.00×10^{-8}	0.0004

Figure 6: Resistivities of various metals. Notice the temperature coefficients are small, so the Taylor series is justified.

Consider a copper wire that has a resistance of 0.1 Ohms at 20 degrees C. What will be the resistance of the wire at 40 degrees C, if the temperature coefficient of copper resistivity is 0.004?

- A: 0.104 Ohms
- B: 0.108 Ohms
- C: 0.112 Ohms
- D: 0.116 Ohms

Consider a copper wire that has a resistance of 5 Ohms at 20 degrees C. What will be the resistance of the wire at 50 degrees C, if the temperature coefficient of copper resistivity is 0.004?

- A: 4.8 Ohms
- B: 5.0 Ohms
- C: 5.6 Ohms
- D: 6.0 Ohms

Given that we found a 12% increase in the resistance of the wire in the prior example, what would happen to the voltage delivered to a device at the other end of that wire?

- A: It would be unaffected.
- B: It would decrease by 12 percent.
- C: It would increase by 12 percent.
- D: It would decrease by 6 percent.

So you can see why we don't do DC power transmission over long distances.

POWER

POWER

Notice that if $P = dU/dt$ and $U = qV$, then $P = Vdq/dt = IV$, at a constant voltage.

$$P = IV = \frac{I^2}{R} \tag{16}$$

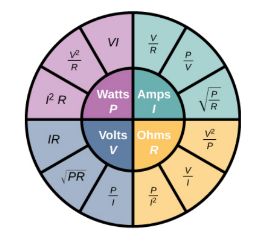

Figure 7: The many relationships between electrical quantities in DC circuits.

Practical example: home energy consumption. (Example 9.10).

BATTERIES AND INTERNAL RESISTANCE

1. A battery is a series of electrochemical *cells*.
2. The **cathode** gives electrons to the **anode**.
3. Molecular acid-base reactions “pump” charge to a higher potential at the cost of some internal resistance.

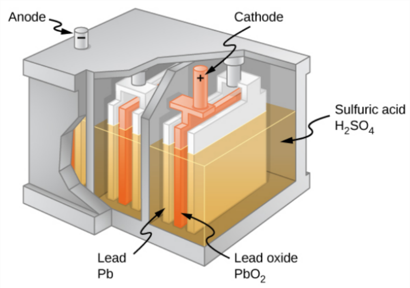

Figure 8: A simplified view of a liquid Pb-acid battery.

1. A battery is a series of electrochemical *cells*.
2. The **cathode** gives electrons to the **anode**.
3. Molecular acid-base reactions “pump” charge to a higher potential at the cost of some internal resistance.

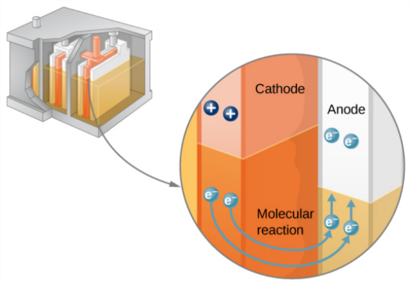

Figure 9: A microscopic picture of Fig. 9.

1. The *internal resistance* of a battery obeys the following model.
2. $V_{term} = \epsilon - Ir_{int}$

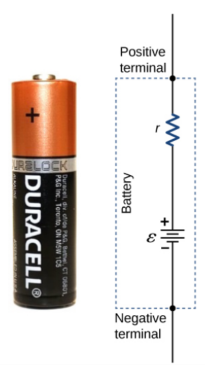

Figure 10: A good model of battery internal resistance.

1. The *internal resistance* of a battery obeys the following model.
2. $V_{term} = \epsilon - Ir_{int}$. A battery may be *rated* at ϵ, but not produce ϵ.
3. To most accurately predict current flow, both r and R must be included (note the labeling of the circuit).

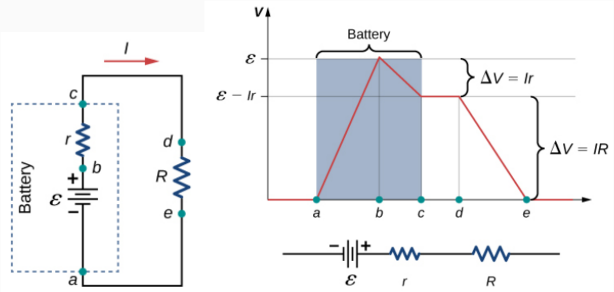

Figure 11: A good model of battery internal resistance.

A 5V battery powers a motor with effective resistance of 45 Ω. If the current draw is 100 mA, what is the internal battery resistance? (*Hint:* draw a diagram and go around the loop).

- A: 3 Ω
- B: 30 Ω
- C: 5 Ω
- D: 50 Ω

OHM'S LAW, KIRCHHOFF'S RULES AND SIMPLE CIRCUITS

Voltage and capacitance can be combined with another concept, resistance, to build an understanding of circuits. Ohm's law is the relationship between *current* and *voltage*:

$$V = IR = \frac{dQ}{dt}R \tag{17}$$

Current is the *flow of charge.* The unit of resistance is called the Ohm, and the unit of current is called the amp, for Ampère.

A 9 V battery powers a smoke detector, and the current is 0.009 amps. What is the resistance of the circuit powering the smoke detector?

- A: 1000 Ohms
- B: 100 Ohms
- C: 10 Ohms
- D: 1 Ohm

A 3.3 V battery powers a christmas light, and the current is 0.33 amps. What is the resistance of the circuit powering the smoke detector?

- A: 1000 Ohms
- B: 100 Ohms
- C: 10 Ohms
- D: 1 Ohm

Resistors, capacitors, voltages, and current can be combined conceptually to form **DC Circuits** (direct-current). The following rules govern the interaction of resistors with each other, and capacitors with each other:

- For resistors *in series*: $R = R_1 + R_2 + R_3 + ...$
- For resistors *in parallel*: $R^{-1} = R_1^{-1} + R_2^{-1} + R_3^{-1} + ...$
- For capacitors *in parallel*: $C = C_1 + C_2 + C_3 + ...$
- For capacitors *in series*: $C^{-1} = C_1^{-1} + C_2^{-1} + C_3^{-1} + ...$

Observe on board: difference between in series, and in parallel? (Hint, different voltage for in series, same voltage for in parallel).

Kirchhoff's rules:

1. Summing the voltage in a loop must equal zero ($\vec{E}$-fields are conservative, energy is conserved).
2. Current through a node is conserved (charge is conserved).

Observe on board.

Two resistors have 1 kΩ each (1000 Ohms). What is the effective resistance if they are added *in parallel*?

- A: 1000 Ohms
- B: 500 Ohms
- C: 2000 Ohms
- D: 200 Ohms

Two capacitors have 1 nF each. What is the effective capacitance if they are added *in series*?

- A: 0.5 nF
- B: 5 nF
- C: 1 nF
- D: 2 nF

Two capacitors have 1 nF each. What is the effective capacitance if they are added *in parallel*?

- A: 0.5 nF
- B: 5 nF
- C: 1 nF
- D: 2 nF

1. Summing the voltage in a loop must equal zero ($\vec{E}$-fields are conservative, energy is conserved).
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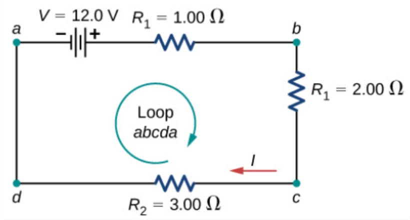

Figure 12: Using a loop to solve for the current, I. The resistor rule for series resistors comes out of the calculation (observe on board).

1. Summing the voltage in a loop must equal zero ($\vec{E}$-fields are conservative, energy is conserved).
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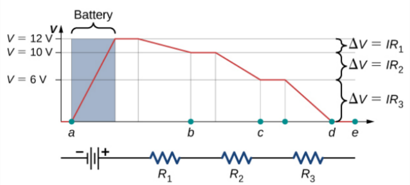

Figure 13: The result is a complete understanding of voltage and current in the DC circuit.

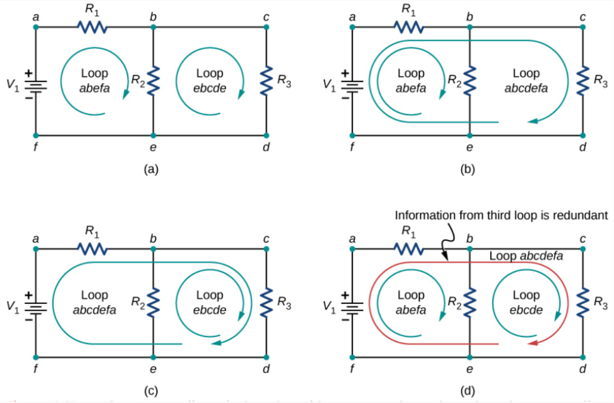

Figure 14: Loops and nodes produce a **system of equations**. The number of equations must be equal to the number of unknowns. *Pay attention to negative/positive signs for voltage and current!*

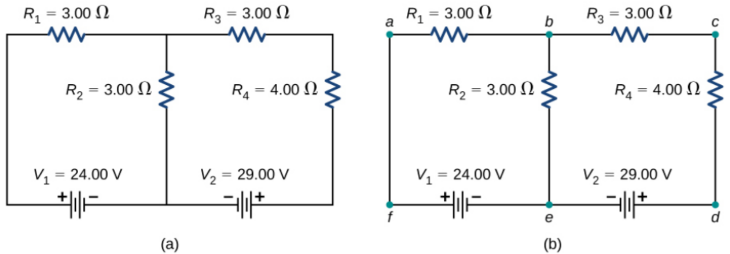

Figure 10.26 (a) A multi-loop circuit. (b) Label the circuit to help with orientation.

Figure 15: (1) Label the circuit. (2) Identify nodes with current labels, and loops. (3) Apply Kirchhoff's Rules.

Work together in groups at tables. (We'll take this in steps. First identify two loops and a node...)

PHET: DC CIRCUIT MODELING

Use DC circuit PhET simulation to model Fig. 16, checking the currents I_1, I_2, and I_3.

`https://phet.colorado.edu/en/simulation/circuit-construction-kit-dc-virtual-lab`

Do you obtain the same results as the example? What is the power consumption of this circuit?

GRAPHICAL ANALYSIS OF SIMPLE CIRCUITS

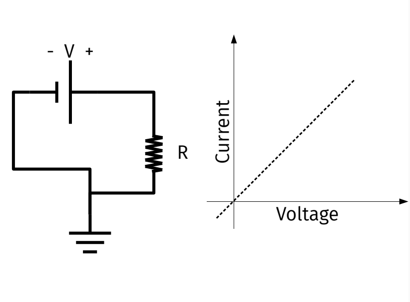

Figure 16: Circuits components are represented graphically by iV curves.

If the resistance R is increased, what will happen?

- A: The slope on the graph will increase
- B: The slope on the graph will decrease
- C: The slope will stay the same
- D: Cannot determine what will happen

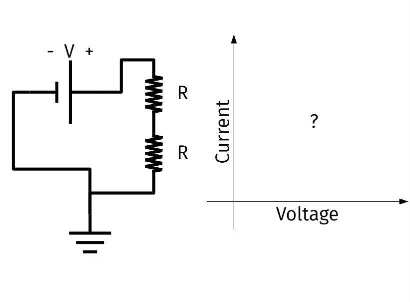

Figure 17: Circuits components are represented graphically by iV curves.

Should the slope now be greater than, less than, or equal to the that of Fig. 17?

- A: Greater than Fig. 17
- B: Less than Fig. 17
- C: Equal to Fig. 17
- D: Cannot determine.

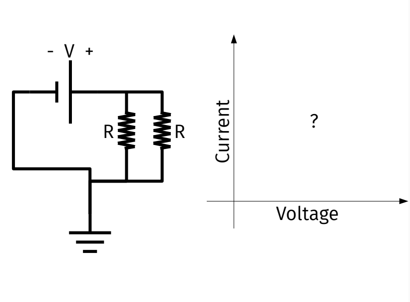

Figure 18: Circuits components are represented graphically by iV curves.

Should the slope now be greater than, less than, or equal to the that of Fig. 17?

- A: Greater than Fig. 17
- B: Less than Fig. 17
- C: Equal to Fig. 17
- D: Cannot determine.

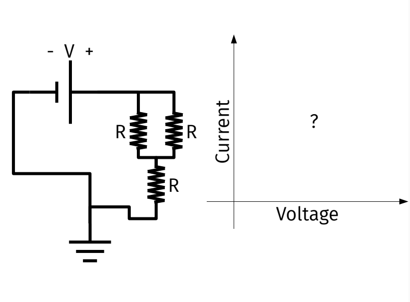

Figure 19: Circuits components are represented graphically by iV curves.

Should the slope now be greater than, less than, or equal to the that of Fig. 17?

- A: Greater than Fig. 17
- B: Less than Fig. 17
- C: Equal to Fig. 17
- D: Cannot determine.

Group board exercise: Solve for the current.

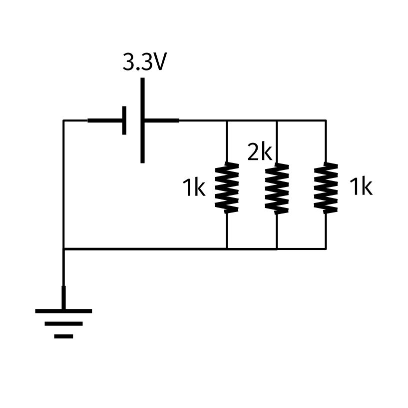

Figure 20: Three resistors in parallel.

Group board exercise: Solve for the current.

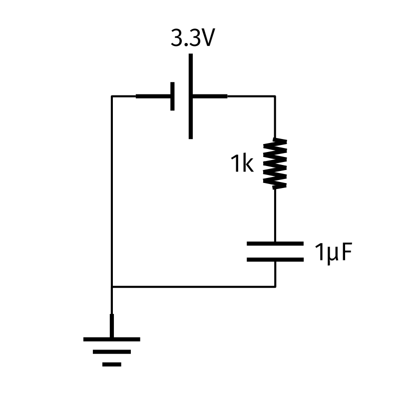

Figure 21: A resistor in series with a capacitor...

Using Kirchhoff's loop rule:

$$V - iR - QC^{-1} = 0 \tag{18}$$

$$Q_0 = CV \tag{19}$$

$$\tau = RC \tag{20}$$

$$\tau\dot{Q} + Q = Q_0 \tag{21}$$

Group board exercise: Show that the solution for the charge on the capacitor is

$$Q(t) = Q_0 \left(1 - \exp(-t/\tau)\right) \tag{22}$$

Take the derivative of both sides to find the current versus time. Graph the current.

So we see that the current is an exponential function:

$$i(t) = i_0 e^{-t/\tau} \tag{23}$$

($i_0 = V/R$, the initial current). What is the current in a circuit with a 1k resistor and a 1 nF capacitor after 3 μs, if the voltage is 5 V?

- A: 0.25 mA
- B: 0.5 mA
- C: 1.0 mA
- D: 2.0 mA

So we see that the current is an exponential function:

$$i(t) = i_0 e^{-t/\tau} \tag{24}$$

($i_0 = V/R$, the initial current). Thinking of the same circuit: when is the current 0.0 mA?

- A: 3 microseconds
- B: 30 microseconds
- C: 300 microseconds
- D: never

Batteries obey Kirchhoff's Rules as well.

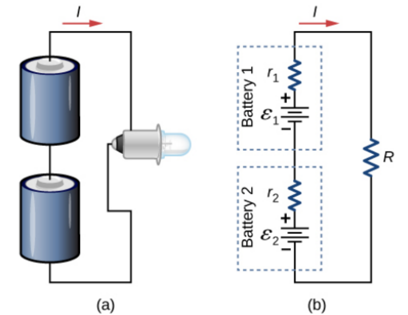

Figure 10.32 Two batteries connect in series to an LED bulb, as found in a flashlight.

Figure 22: Two batteries connected in series. The net voltage is going to be affected by the internal resistances.

Batteries obey Kirchhoff's Rules as well.

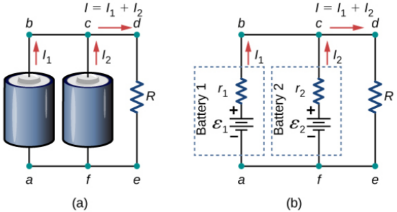

Figure 10.33 (a) Two batteries connect in parallel to a load resistor. (b) The circuit diagram shows the shows battery as an emf source and an internal resistor. The two emf sources have identical emfs (each labeled by ε) connected in parallel that produce the same emf.

Figure 23: Two batteries connected in parallel. The net voltage is going to be affected by the internal resistances.

Two 1.5 V batteries with internal resistances of 1Ω each are connected in series, along with a 10 Ω resistor. What is the current draw?

- A: 0.3 A
- B: 0.25 A
- C: 0.1 A
- D: 0.15 A

CONCLUSION

Reading: chapters 9-10

1. Chapter 9: Current and resistance
    - Skip Section 9.3 for time constraints...resistivity
    - Briefly cover power (examples and summary of formulae)
2. Chapter 10: Ohm's law and Circuit Analysis
    - Batteries and internal resistance
    - Kirchhoff's Rules
    - RC Circuits
    - Batteries in series and in parallel