CALCULUS-BASED PHYSICS-2: ELECTRICITY, MAGNETISM, AND THERMODYNAMICS (PHYS180-02): UNIT 1

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Welcome to Physics! Have a seat, right over here...

SUMMARY

Reading: Chapters 5-6

1. Charge, conductors and insulators
2. Coulomb’s law and electric Fields
3. Electric fields of charge distributions
4. Gauss’s Law

CHARGE, CONDUCTORS AND INSULATORS

Charge the following intrinsic properties:

1. Charge is conserved globally (charge cannot be created nor destroyed). Mass has the same property.
2. Charge is conserved locally (if we pull charge out of the system, charge will flow into the system).
3. Charge is quantized, with an electron (for example) having the fundamental negative unit, and a proton (for example) having the fundamental positive unit.
4. The laws of physics are the same for positive and negative charges.
5. The two kinds of charge emit fields that attract each other; fields emitted by charges of the same type repel such charges.

Benjamin Franklin and the Leyden Jar. (Good paper topic).

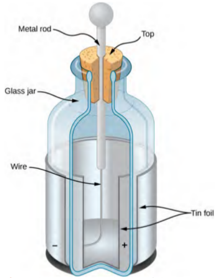

Figure 5.6 A Leyden jar (an early version of what is now called a capacitor) allowed experimenters to store large amounts of electric charge. Benjamin Franklin used such a jar to demonstrate that lightning behaved exactly like the electricity he got from the equipment in his laboratory.

Figure 1: A Leyden jar was an early version of a capacitor. Benjamin Franklin guessed that one type of charge moves and another remains stationary, explaining several behaviors of charged objects.

CHARGE, CONDUCTORS AND INSULATORS

The rest of the properties of charge are connected to the development of the structure of the atom, and we will return to this topic at the end of the semeter.

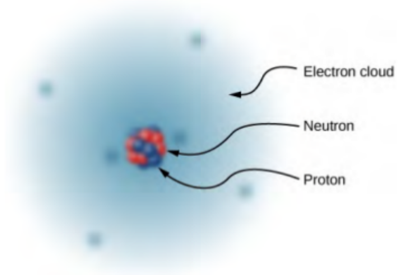

Figure 5.8 The nucleus of a carbon atom is composed of six protons and six neutrons. As in hydrogen, the surrounding six electrons do not have definite locations and so can be considered to be a sort of cloud surrounding the nucleus.

Figure 2: A sketch of our current atomic paradigm.

Suppose an ion is composed of six protons, eight neutrons, and five electrons. What is the net charge?

- A: +1
- B: 0
- C: -1
- D: -2

A rod with a positive charge is held next to a *conductor* (an object were charge can move around freely). Which of the following is true?

- A: The charges in the conductor all remain in place because charge is conserved.
- B: The negative charges in the conductor move toward the positive charges in the rod.
- C: The positive charges remain in place but the negative charges move away from the rod.
- D: The positive charges move toward the rod and the negative charges remain in place.

An *insulator* with a net positive charge is held next to an *insulator* with a net negative charge. Which of the following is true?

- A: The charges in the conductor all remain in place, and the force is attractive.
- B: The charges in the conductor all move around until the force is attractive.
- C: The charges in the conductor all remain in place, and the force is repellent.
- D: The charges in the conductor all move around until the force is repellent.

The boundary conditions of problems can vary depending on the materials involved:

Insulator: A material in which there are no free charges available to conduct electricity. Charges may be fixed in position within an insulator.

Conductor: A material in which there are free charges available to conduct electricity. Charges may not be fixed in position within a conductor.

Semi-conductor: A material in which there are free charges available to conduct electricity if certain requirements are met.

ACTIVITY: PHET CHARGES AND FIELDS

At your tables, go to the following URL:

`https://phet.colorado.edu/en/simulation/charges-and-fields`

Click on the java app to get it running. Notice the following:

1. This is a 2D coordinate space, and you can activate the grid lines at right, by clicking *grid*.
2. Clicking *values* gives you the measurement scale.
3. Click *electric field*, or make sure it is activated.
4. Verify the length scale with the **ruler tool**, shaped like a tape measure. It can be dragged from the box at right.

`https://phet.colorado.edu/en/simulation/charges-and-fields`

Click and drag a positive charge into the 2D coordinate system. This is analagous to charging an insulator.

1. Drag the yellow tool at the bottom into the space, and use it to measure the field strength. Notice the units are in V/m and m.
2. Write down in your notes the field strength versus distance. Use 25 cm distance increments, and record 15 data points in two columns.
3. Copy the data to Excel. Let r be the distance. In a third column, multiply the field strength by r^2.
4. In a fourth column, compute the base-10 logarithm of r^2 times the field strength. In a fifth column, compute the base-10 logarithm of the distance.

https://phet.colorado.edu/en/simulation/charges-and-fields

Click and drag a positive charge into the 2D coordinate system. This is analagous to charging an insulator.

1. Plot $\log_{10}(r^2E)$ vs. $\log_{10}(r)$ and estimate the slope. How close to zero do you think it is? What are some sources of error that contribute to the uncertainty in the slope?
2. Repeat this same exercise, but instead of measuring field strength versus *distance*, measure it in one location, versus *charge*. Take 15 data points in two columns and plot the results in Excel. What is the slope of the line? Notice the units of charge are nC.

COULOMB'S LAW AND ELECTRIC FIELDS

Coulomb's Law describes the force between charges.

Coulomb's Law

The electric force, or **Coulomb force**, between two electrically charged systems with charges q_1 and q_2 separated by a distance r is

$$\vec{F}_{\mathrm{C}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r} \tag{1}$$

In Eq. 1, $\hat{r} = \vec{r}/|\vec{r}|$, and $\epsilon_0 = 8.85418782 \times 10^{-12} N^{-1} m^{-2} C^2$, called the *perimittivity of free space.*

Coulomb Field

The electric field corresponding to Eq. 1, experienced by a charge q and generated by a charge Q is

$$\vec{E}_{\mathrm{C}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} \tag{2}$$

In Eq. 2, r remains the separation between q and Q.

Thus we have: $\vec{F}_{\mathrm{C}} = q\vec{E}_{\mathrm{C}}$.

The SI Unit of charge is the Coulomb, which is equal to the amount of charge in a "current" of 1 amp for 1 second (more on this later). **The charge of an electron is** 1.6×10^{-19} **Coulombs, or C.**

Suppose a charge $+q$ experiences the Coulomb field of another charge of $-2q$, separated by a distance r. Which of the following is true, if the charges are in free space?

- A: The charge $+q$ accelerates towards the other charge, and the charge $-2q$ remains stationary, because it is larger.
- B: The charge $-2q$ accelerates towards the other charge, and the charge $+q$ remains stationary, because it is the positive charge.
- C: The charge $-2q$ accelerates towards the other charge, and the charge $+q$ remains stationary, because it is smaller.
- D: The charges accelerate towards each other.

Newton's Third Law still applies, but in *materials* the positive charges are stationary.

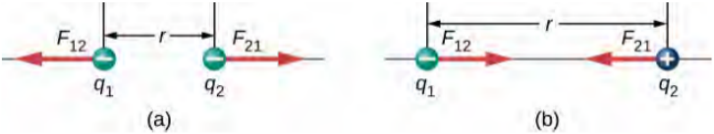

Figure 5.14 The electrostatic force $\vec{\mathbf{F}}$ between point charges q_1 and q_2 separated by a distance r is given by Coulomb's law. Note that Newton's third law (every force exerted creates an equal and opposite force) applies as usual—the force on q_1 is equal in magnitude and opposite in direction to the force it exerts on q_2. (a) Like charges; (b) unlike charges.

Figure 3: Newton's Third Law applies to the Coulomb force, as it does for all forces.

Suppose a charge $2q$ is at rest at the origin (0,0). Where should we place a charge $-2q$ such that the field is zero at (0,3)?

- A: (0,3)
- B: (0,6)
- C: (3,0)
- D: (6,0)

The Coulomb force equation gives a vector, and so does the corresponding electric field. Like a gravitational field, this effect has a vector at each point in space, so we refer to the Coulomb force and the Coulomb field as *vector fields*.

Vector field: An assignment of a vector to each point in a subset of space.

Which of the following is true of vectors $\vec{v}_\mathrm{i}$ in the lower left-hand corner of the figure at right?

- A: They are probably $\vec{v}_\mathrm{i} = -\hat{i} - \hat{j}$
- B: They are probably $\vec{v}_\mathrm{i} = \hat{i} + \hat{j}$
- C: They are probably $\vec{v}_\mathrm{i} = -\hat{i} + \hat{j}$
- D: They are probably $\vec{v}_\mathrm{i} = \hat{i} - \hat{j}$

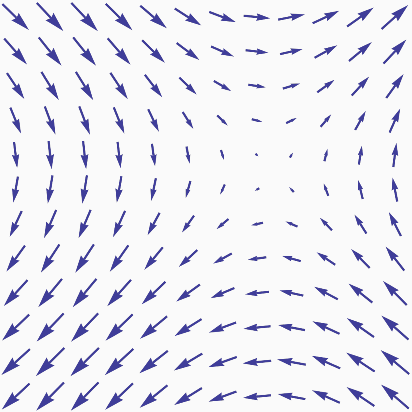

Figure 4: A vector field of vectors $\vec{v}_\mathrm{i}$. Let $\hat{j}$ represent up, and $\hat{i}$ represent right.

Which of the following is true of vectors $\vec{v}_{\mathrm{i}}$ in the upper left-hand corner of the figure at right?

- A: They are probably $\vec{v}_{\mathrm{i}} = -\hat{i} - \hat{j}$
- B: They are probably $\vec{v}_{\mathrm{i}} = \hat{i} + \hat{j}$
- C: They are probably $\vec{v}_{\mathrm{i}} = -\hat{i} + \hat{j}$
- D: They are probably $\vec{v}_{\mathrm{i}} = \hat{i} - \hat{j}$

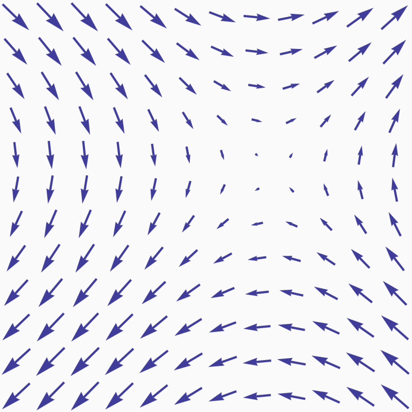

Figure 5: A vector field of vectors $\vec{v}_{\mathrm{i}}$. Let $\hat{j}$ represent up, and $\hat{i}$ represent right.

Group board exercise: What is the angle of the net electric field for the *test charge* at the point (1,1) in Fig. 6?

Group board exercise: What is the magnitude of the net electric field for the *test charge* at the point (1,1) in Fig. 6, if the distances have units of nanometers, and q is the charge of an electron, 1.6×10^{-19} C? (Let $\frac{1}{4\pi\epsilon_0} = 9 \times 10^9$ N m^2 C^{-2}).

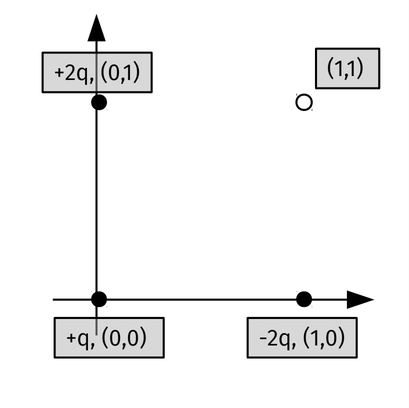

Figure 6: Three charges create a field for a hypothetical *test charge*.

What is the angle of the E-field at point (1,1) in Fig. 7 at right?

- A: 0 deg
- B: 45 deg
- C: 90 deg
- D: 135 deg

What is the fastest way to solve this problem?

- A: Blind luck
- B: Do the algebra
- C: Symmetry
- D: Numerical estimation

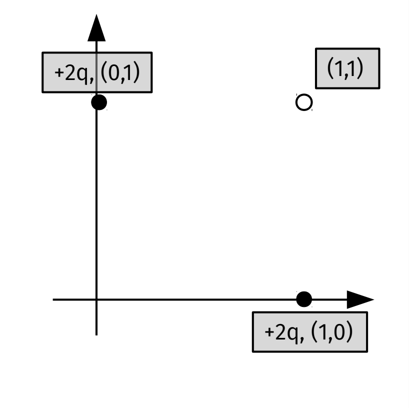

Figure 7: Two charges create a field for a hypothetical *test charge*.

THE SUPERPOSITION PRINCIPLE AND SYMMETRY

The forces of N fixed charges on a test charge Q create a net force, where the individual forces simply add like vectors. This is known as the **superposition principle**.

$$\vec{F}_{\mathrm{C,Net}} = \frac{1}{4\pi\epsilon_0} Q \sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{r}_i = Q\vec{E}_{\mathrm{C,Net}} \tag{3}$$

$$\vec{E}_{\mathrm{C,Net}} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{r}_i \tag{4}$$

For the expressions of fields built from the superposition principle, let's adopt a notation:

$$\vec{E}_{\mathrm{C,Net}}(P) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\hat{r}_i \tag{5}$$

Equation 5 represents the field at a *position* $P = P(x, y, z)$, relative to the positions $\vec{r}_i$ of the source charges.

Table exercise: Calculate $\vec{E}_{\mathrm{C,Net}}(P)$, if $P = (1, 1)$.

Table exercise: Calculate $\vec{E}_{\mathrm{C,Net}}(P)$, if $P = (-1, -1)$.

Group discussion: What does it mean if $P = (1, 0)$?

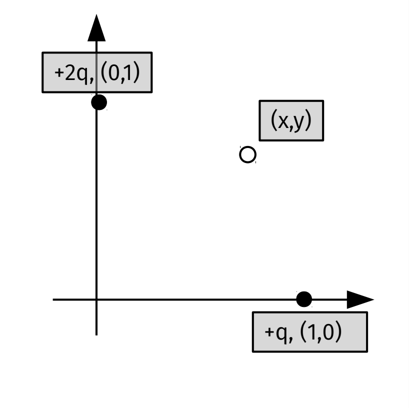

Figure 8: Two charges create a field for a hypothetical *test charge*.

The following problem is an example of solving for a field analytically, and *testing various limits*. Upon taking limits results are often simple and intuitive.

Two charges $+q$ are on the fixed in an insulator on the x-axis. Solve for the E-field at $P = (0, 0, z)$.

Show that the general solution is

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0} \frac{2qz}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} \hat{k} \quad (6)$$

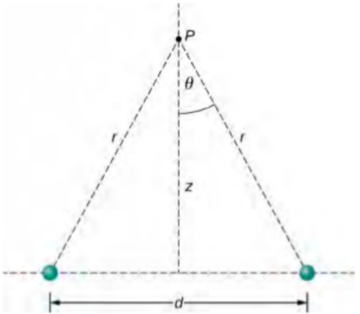

Figure 5.20 Finding the field of two identical source charges at the point P. Due to the symmetry, the net field at P is entirely vertical. (Notice that this is *not* true away from the midline between the charges.)

Figure 9: Solve for the E-field as a function of z, d, and q.

Show that the general solution is

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0} \frac{2qz}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} \hat{k} \quad (7)$$

Take the following two limits:
1) $z \gg d$ and 2) $z = 0$. What are the results?

Keep these results in mind, because we are about to start drawing **vector fields,** in order to visualize the algebra.

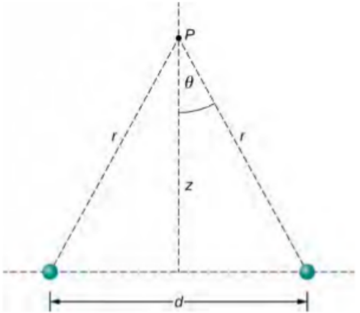

Figure 5.20 Finding the field of two identical source charges at the point P. Due to the symmetry, the net field at P is entirely vertical. (Notice that this is *not* true away from the midline between the charges.)

Figure 10: Solve for the E-field as a function of z, d, and q.

PhET Simulation of E-fields from Charges:

`https://phet.colorado.edu/en/simulation/charges-and-fields`

1. Create the situation in the prior problem, in Fig. 10.
2. Use the yellow sensor object to determine the local direction of the E-field at various points along the z-axis.
    - Do the results match the limit $z \gg d$?
    - Do the results match the limit $z = 0$, halfway between the charges?
    - Where is the field maximal?
3. Make sure you can see above and below the charges, and repeat steps 1 and 2 for negative z-values. What do you find?

PhET Simulation of E-fields from Charges:

Build E-fields with the following properties, by adding single charges. Let the *z-axis be upwards*, and let the *x-axis be to the right.*

1. Build an electric field that has **reflection symmetry** across the z-axis, with at least five charges.
2. Build an electric field that has *radial symmetry* about the origin, with at least six charges.
3. Build an electric field that would be the same if I rotated the picture by 90 degrees (**4-fold symmetry**) with at least four charges, some negative and some positive.
4. Build an electric field that would be the same if I rotated the picture by 45 degrees (**8-fold symmetry**) with at least eight charges, some negative and some positive.

PhET Simulation of E-fields from Charges:

The lesson is that the E-field has the *symmetry properties* of the *charge distribution*.

When we connect the vectors in a vector field, the results are figures like Fig. 11. Fields by convention originate from positive charges and terminate on negative ones.

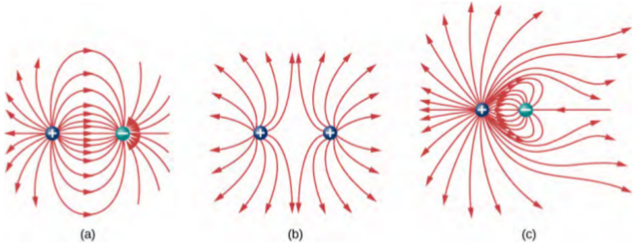

Figure 5.31 Three typical electric field diagrams. (a) A dipole. (b) Two identical charges. (c) Two charges with opposite signs and different magnitudes. Can you tell from the diagram which charge has the larger magnitude?

Figure 11: Field-line diagrams. The density of lines indicates electric field strength.

E-FIELDS OF CHARGE DISTRIBUTIONS

Welcome to calculus! Let $k = 1/(4\pi\epsilon_0)$.

$$\vec{E}(P) = k\sum_{i=1}^{N}\left(\frac{q_i}{r_i^2}\right)\hat{r} \tag{8}$$

$$\vec{E}(P) = k\int_{line}\left(\frac{\lambda dl}{r^2}\right)\hat{r} \tag{9}$$

$$\vec{E}(P) = k\int_{surface}\left(\frac{\sigma dA}{r^2}\right)\hat{r} \tag{10}$$

$$\vec{E}(P) = k\int_{volume}\left(\frac{\rho dV}{r^2}\right)\hat{r} \tag{11}$$

The functions λ, σ, and ρ are just charge densities. They decribe where charge is, and how much there is.

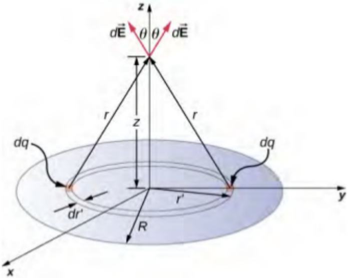

Figure 5.25 A uniformly charged disk. As in the line charge example, the field above the center of this disk can be calculated by taking advantage of the symmetry of the charge distribution.

Figure 12: We are going to work this example together, and other examples will be left to homework.

Observe on board.

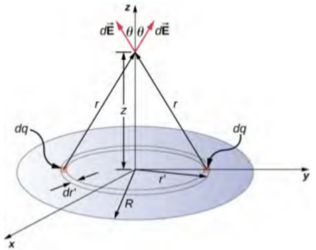

Figure 5.25 A uniformly charged disk. As in the line charge example, the field above the center of this disk can be calculated by taking advantage of the symmetry of the charge distribution.

Figure 13: We are going to work this example together, and other examples will be left to homework.

Result:

$$\boxed{\vec{E} = k\left(2\pi\sigma - \frac{2\pi\sigma z}{\sqrt{R^2+z^2}}\right)\hat{k}} \tag{12}$$

$$\boxed{\vec{E} = k\left(2\pi\sigma - \frac{2\pi\sigma z}{\sqrt{R^2 + z^2}}\right)\hat{k}} \tag{13}$$

Which of the following not true of Eq. 13?

- A: Taking the limit $R \to \infty$ yields a constant field.
- B: Taking the limit $z \to 0$ yields a constant field.
- C: The charge distribution has radial symmetry, so the field cannot have horizontal components.
- D: Taking the value $z = R$ represents a minimum in the field strength.

$$\boxed{\vec{E} = k\left(2\pi\sigma - \frac{2\pi\sigma z}{\sqrt{R^2 + z^2}}\right)\hat{k}} \tag{14}$$

What happens to Eq. 14, in the limit that $R \to \infty$?

- A: The field decreases to zero.
- B: The field is constant.
- C: The field grows increasingly positive.
- D: The field grows increasingly negative.

In the limit that $R \to \infty$,

$$\vec{E} = 2\pi\sigma k\hat{k} = \frac{\sigma}{2\epsilon_0}\hat{k} \tag{15}$$

Equation for the electric field of a uniform infinite disk.

Imagine two infinite disks with equal uniform charge distributions, some distance apart. One has positive charge, the other negative charge. What is the E-field between them?

- A: 0
- B: $\frac{\sigma}{2\epsilon_0}$
- C: $\frac{\sigma}{\epsilon_0}$
- D: $\frac{\sigma}{4\epsilon_0}$

Imagine two infinite disks with equal uniform charge distributions, some distance apart. Both have positive charge. What is the E-field between them?

- A: 0
- B: $\frac{\sigma}{2\epsilon_0}$
- C: $\frac{\sigma}{\epsilon_0}$
- D: $\frac{\sigma}{4\epsilon_0}$

Other interesting charge distributions:

- A line of charge with length L and total charge $Q = \lambda L$, where $P = (0, 0, z)$ above midpoint:

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0} \frac{\lambda L}{z\sqrt{z^2 + \frac{1}{4}L^2}} \hat{k} \tag{16}$$

- Equation 16, but with $L \to \infty$:

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{z} \hat{k} \tag{17}$$

Other interesting charge distributions:

- A ring of radius R and total charge $Q = 2\pi R\lambda$, where $P = (0, 0, z)$ above midpoint:

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{2\pi R\lambda z}{(z^2 + R^2)^{3/2}}\hat{k} \tag{18}$$

- Equation 18, but with $z \gg R$:

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{2\pi R\lambda}{z^2}\hat{k} \tag{19}$$

In Eq. 18, what does the quantity $2\pi R\lambda$ represent?

- A: The total charge density on the ring
- B: The circumference of the ring
- C: The magnitude of the electric field from the ring
- D: The total charge on the ring

Let $Q_{\mathrm{tot}} = 2\pi R\lambda$. That makes Eq. 19

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\mathrm{tot}}}{z^2}\hat{k} \tag{20}$$

This is identical to the electric field of what charge distribution? (Think back to the definition of the electric field).

- A: A plane with charge density Q_{tot}/A, where A is the area
- B: A line with total charge Q_{tot}
- C: A dipole of charge $\pm Q_{\mathrm{tot}}$
- D: A point charge Q_{tot}

As we shall see, the last few results follow from a notion known as Gauss's Law. First we need two concepts:

- The **area vector** of a surface
- The concept of flux

GAUSS'S LAW

Let $\vec{A}$ be a vector that:

- has a magnitude equal to the area of a surface
- has a direction that is orthogonal to the surface

If a surface has area A, then $\vec{A} = A\hat{n}$, where $\hat{n}$ is normal, and pointed outward orthogonally from the surface. What does *outward* mean?

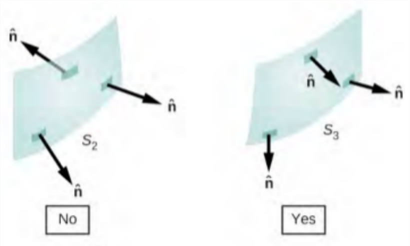

Figure 14: The convention for the area vector direction is outward, not inward.

Figure 15: We may think of a surface S as the sum of many infinitesimal square patches dS_i, each with an area vector $\vec{dA_i}$ equal in magnitude but not direction.

A rectangular surface has length a, width b, and it is located in the x-y plane. What is the area vector of the rectangular surface?

- A: $b^2\hat{k}$
- B: $a^2\hat{i}$
- C: $ab\hat{k}$
- D: $ab\hat{j}$

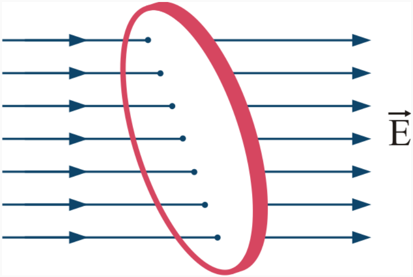

Figure 16: A circular patch, with an external electric field. How many electric field lines would pass through the circular patch if it was tilted to 90° from the field? How about 0°?

This behavior indicates a **dot-product** is working (zero and maximal over a span of 90 degrees). But a dot-product of which two quantities?

Electric flux:

$$\boxed{\Phi = \vec{E} \cdot \vec{A}} = EA\cos\theta \tag{21}$$

Assumptions:

- θ is the angle between the area vector and the field
- The electric field is uniform over this surface
- The surface is flat

These assumptions do not hold for any arbitrary configuration of charge and surfaces. However, if we zoom in closely enough to an individual patch, it does.

The units of electric flux are N C^{-1} m^2.

How can we obtain the flux of more complex electric fields through more complex surfaces? Simple: zoom in, get the flux from a patch, add it to the total:

$$\Phi = \sum_{i}^{N} \vec{E}_i \cdot d\vec{A}_i \tag{22}$$

In situations like this, the summation usually becomes an integral if we make dA small and N very large. That is, break the surface into many small patches. But now we are dealing with two vectors, $\vec{E}$ and $d\vec{A}$...

$$\boxed{\Phi = \int_{\mathrm{S}} \vec{E} \cdot d\vec{A}} \tag{23}$$

Equation 23 is known as a *surface integral*. We can do these! Let's try an easy one. Let $\vec{E} = E_0 x\hat{k}$, and S be a square of width x_0 and height y_0 in the x-y plane.

Group board exercise: Compute the electric flux Φ, and check the units to make sure they are correct.

$\Phi = \frac{1}{2} E_0 y_0 x_0^2$. (This has units of electric field times area). Which of the following would increase the flux?

- A: If x_0 or y_0 were to grow larger
- B: If E_0 were to grow larger
- C: If $\vec{E}$ were directed at an angle to the z-axis
- D: Both A and B

Imagine now that there is another, identical surface *above* the first surface, but it is *upside-down*. The electric field passes through both surfaces. What is the total flux, the sum of the fluxes through both surfaces?

- A: $\Phi = \frac{1}{2}E_0 y_0 x_0^2$
- B: $\Phi = E_0 y_0 x_0^2$
- C: 0
- D: $\Phi = -\frac{1}{2}E_0 y_0 x_0^2$

The answer is zero because, for the *upside-down* surface, the area vector is in the direction $-\hat{k}$. In fact, this result applies to any **closed surface**:

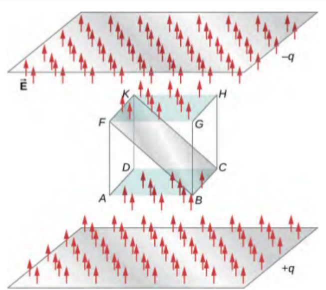

Figure 17: The flux through a closed surface due to an external field is zero.

We can encapsulate this idea in the following equation:

$$\boxed{\Phi = \oint_{\mathrm{S}} \vec{E}_{\mathrm{ext}} \cdot d\vec{A} = 0} \tag{24}$$

In general:

$$\boxed{\Phi = \oint_{\mathrm{S}} \vec{E} \cdot d\vec{A}} \tag{25}$$

However this implies something interesting about closed surfaces. If the flux is non-zero through a closed surface, the field cannot be an external field. It has to be an *internal field*.

How do you make an electric field *internal* to a closed surface? How do you make an electric field in general? **Charge.** Charge is the origin of any electric field.

$$\boxed{\Phi = \oint_{\mathrm{S}} \vec{E} \cdot d\vec{A} \propto Q} \tag{26}$$

If the charge Q is outside the surface, then by definition the field is external to the surface. If the surface *encloses the charge*, then the total flux is non-zero, and[1]

$$\boxed{\frac{Q_{\mathrm{enc}}}{\epsilon_0} = \oint_{\mathrm{S}} \vec{E} \cdot d\vec{A}} \tag{27}$$

[1] Formal proof relies on the *divergence theorem*, from Calculus III.

A charge Q is at the origin. Compute the electric field via Gauss's Law, at $P = (0, 0, R)$.

Group board exercise.

CONCLUSION

UNIT 3 SUMMARY

Reading: Chapters 5-7

1. Charge, Conductors and Insulators
2. Coulomb's Law and Electric Fields
3. E-fields of Charge Distributions
4. Gauss's Law

ANSWERS

- +1
- The positive charges remain in place but the negative charges move away from the rod.
- The charges in the conductor all remain in place, and the force is attractive.
- The charges accelerate towards each other.
- (0,6)
- They are probably $\vec{v}_{\mathrm{i}} = -\hat{i} - \hat{j}$
- $\vec{v}_{\mathrm{i}} = \hat{i} - \hat{j}$
- 45 deg
- Symmetry