

CALCULUS-BASED PHYSICS-1: MECHANICS (PHYS150-01): WEEK 3

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WEEK 2 REVIEW

1. Displacement, and instantaneous velocity and acceleration
 - *Mathematics review*: taking derivatives
 - Average velocity and average acceleration
2. The case of constant acceleration
 - An *equation of motion* for constant acceleration
 - Derivation of **common equations of motion**
 - Average quantities and exercises
3. **Lab Activity: Measuring acceleration of gravity: g**
4. Exercises with vectors, graphs, and equations of motion

WEEK 2 REVIEW PROBLEMS

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If a subway train is moving to the left (has a negative velocity) and then comes to a stop, what is the direction of its acceleration? Is the acceleration positive or negative?

- A: To the right, positive
- B: To the right, negative
- C: To the left, positive
- D: To the left, negative

An object that is thrown straight up falls back to Earth. When is its velocity zero? Does its velocity change direction? Does the acceleration change sign?

- A: During flight, yes, no
- B: At the peak height, yes, yes
- C: At the peak height, yes, no
- D: During flight, no, no

WEEK 3 SUMMARY

1. Displacement, velocity and acceleration vectors as functions of time
 - Breaking into components
 - Derivatives of components
2. Combining free-fall and vector components: projectile motion
 - The independence of velocity components
 - **Lab-activity: testing component independence**
3. Relative motion and reference frames
 - Relative motion in one-dimension
 - Relative motion in two-dimensions

VECTORS AS FUNCTIONS OF TIME

In general, the displacement of an object depends on time:

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \quad (1)$$

- $x(t)$ is the displacement in the x-direction
- $y(t)$ is the displacement in the y-direction
- $z(t)$ is the displacement in the z-direction

VECTORS AS FUNCTIONS OF TIME

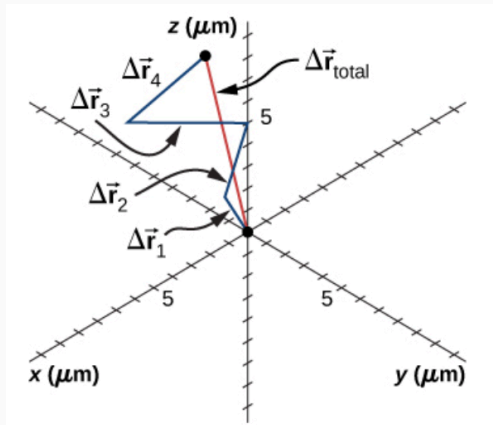


Figure 1: An example of a displacement vector at different moments in time.

The particle in Fig. 1 has four displacement vectors at four moments in time:

- $\vec{r}_1 = 2.0\hat{i} + 1.0\hat{j} + 3.0\hat{k} \quad (\mu m) \text{ at } t_1$
- $\vec{r}_2 = -1.0\hat{i} + 0.0\hat{j} + 3.0\hat{k} \quad (\mu m) \text{ at } t_2$
- $\vec{r}_3 = 4.0\hat{i} + -2.0\hat{j} + 1.0\hat{k} \quad (\mu m) \text{ at } t_3$
- $\vec{r}_4 = -3.0\hat{i} + 1.0\hat{j} + 2.0\hat{k} \quad (\mu m) \text{ at } t_4$

What is the total displacement of the particle from the origin?

We can think of this type of problem as an accounting problem, lining up columns (units: μm):

t_i	$\vec{r}_i(t_i)$	$x(t_i)$	$y(t_i)$	$y(t_i)$
t_1	$\vec{r}_1(t_1)$	2.0	1.0	3.0
t_2	$\vec{r}_2(t_2)$	-1.0	0.0	3.0
t_3	$\vec{r}_3(t_3)$	4.0	-2.0	1.0
t_4	$\vec{r}_4(t_4)$	-3.0	1.0	2.0
t_{total}	$\vec{r}_{\text{total}}(t_{\text{total}})$	2.0	0.0	9.0

Figure 2: Accounting for the different displacement components, in units of μm .

VECTORS AS FUNCTIONS OF TIME

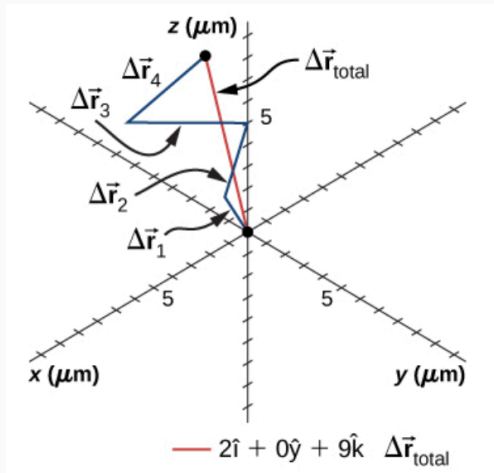


Figure 3: The total displacement of the particle is $\vec{r}_{\text{total}} = 2.0\hat{i} + 0.0\hat{k} + 9.0\hat{k}$ (μm).

The 18th hole at Pebble Beach Golf Course is a dogleg to the left of length 496.0 meters. The fairway off the tee is taken to be the x direction. A golfer hits his tee shot a distance of 300 meters, corresponding to a displacement of $\vec{r}_1 = 300.0\hat{i} \text{ (m)}$, and then hits a second shot 189.0 meters with $\vec{r}_2 = 172.0\hat{i} + 80.3\hat{j} \text{ m}$. What is the final displacement from the tee?

- A: $\vec{r}_{\text{final}} = 172.0\hat{i} + 80.3\hat{j} \text{ (m)}$
- B: $\vec{r}_{\text{final}} = 172.0\hat{i} + 380.3\hat{j} \text{ (m)}$
- C: $\vec{r}_{\text{final}} = 472.0\hat{i} + 0.0\hat{j} \text{ (m)}$
- D: $\vec{r}_{\text{final}} = 472.0\hat{i} + 80.3\hat{j} \text{ (m)}$

If the first shot takes 5.0 seconds, the second shot takes 4.0 seconds, and the walking time in between the shots is 60.0 seconds, what is the average velocity vector for the ball after the two shots?

- A: $\vec{r}_{\text{final}} = 1.7\hat{i} + 8.3\hat{j} \quad (m/s)$
- B: $\vec{v}_{\text{final}} = 172.0\hat{i} + 80.3\hat{j} \quad (m/s)$
- C: $\vec{v}_{\text{final}} = 6.8\hat{i} + 1.2\hat{j} \quad (m)$
- D: $\vec{v}_{\text{final}} = 6.8\hat{i} + 1.2\hat{j} \quad (m/s)$

The prior problem indicates something you may already have guessed:

$$\vec{v}_{\text{avg}}(t) = v_x(t)\hat{i} + v_y(t)\hat{j} + v_z(t)\hat{k} = \frac{\Delta\vec{r}}{\Delta t} \quad (2)$$

- $v_x(t)$ is the avg. velocity in the x-direction
- $v_y(t)$ is the avg. velocity in the y-direction
- $v_z(t)$ is the avg. velocity in the z-direction

In other words, we divide each displacement component by the time, to get a vector where each component is the average velocity in that direction. $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$.

Instantaneously, Eq. 2 is true, if we take the limit $\Delta t \rightarrow 0$:

$$\vec{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k} \quad (3)$$

- $\frac{dx}{dt}$ is the instantaneous velocity in the x-direction
- $\frac{dy}{dt}$ is the instantaneous velocity in the y-direction
- $\frac{dz}{dt}$ is the instantaneous velocity in the z-direction

The position of a particle is $\vec{r}(t) = 4.0t^2\hat{i} - 3.0\hat{j} + 2.0t^2\hat{k}$ (m).

What is the velocity vector at $t = 2$ seconds? What is the average velocity between $t = 0$ and $t = 2$ seconds?

- A: $16\hat{x} + 8\hat{z}$ (m/s), $8\hat{x} + 4\hat{z}$ (m/s)
- B: $8\hat{x} + 4\hat{z}$ (m/s), $4\hat{x} + 2\hat{z}$ (m/s)
- C: $8\hat{x} + 8\hat{z}$ (m/s), $4\hat{x} + 4\hat{z}$ (m/s)
- D: $4\hat{x} + 2\hat{z}$ (m/s), $4\hat{x} + 2\hat{z}$ (m/s)

Instantaneously, from Eq. 3:

$$\vec{a}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} + \frac{dv_z}{dt} \hat{k} \quad (4)$$

- $\frac{dv_x}{dt}$ is the instantaneous acceleration in the x-direction
- $\frac{dv_y}{dt}$ is the instantaneous acceleration in the y-direction
- $\frac{dv_z}{dt}$ is the instantaneous acceleration in the z-direction

The velocity of a particle is $\vec{v}(t) = 8.0t\hat{i} + 4.0t\hat{k}$ (m/s). What is the acceleration vector at $t = 2$ seconds? What is the average acceleration between $t = 0$ and $t = 2$ seconds?

- A: $4\hat{i} + 4\hat{k}$ (m/s²), $2\hat{i} + 2\hat{k}$ (m/s²)
- B: $8\hat{i} + 4\hat{k}$ (m/s²), $8\hat{i} + 4\hat{k}$ (m/s²)
- C: $8\hat{i} + 8\hat{k}$ (m/s²), $4\hat{i} + 4\hat{k}$ (m/s²)
- D: $4\hat{i} + 8\hat{k}$ (m/s²), $2\hat{i} + 4\hat{k}$ (m/s²)

The displacement of a particle is $\vec{x}(t) = (2t + 3)\hat{i} + (\frac{3}{2}t^2 + 2t + 3.0)\hat{j}$ (m). What is the horizontal velocity (the \hat{i} -component of the velocity) at $t = 4$ seconds? At $t = 10$ seconds?

- A: 4 m/s, 4 m/s
- B: 2 m/s, 4 m/s
- C: 2 m/s, 2 m/s
- D: 4 m/s, 2 m/s

The displacement of a particle is $\vec{x}(t) = (2t + 3)\hat{i} + (\frac{3}{2}t^2 + 2t + 3.0)\hat{j}$ (m). What is the vertical velocity (the \hat{j} -component of the velocity) at $t = 4$ seconds? At $t = 10$ seconds?

- A: 14 m/s, 32 m/s
- B: 32 m/s, 14 m/s
- C: 12 m/s, 30 m/s
- D: 30 m/s, 12 m/s

Notice in the previous example, the x-velocity and y-velocity were not the same function.

In the kinematic description of motion, *we are able to treat the different components of motion separately*. In many cases, motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

Motions in displacement components are independent.

(Exception: non-conservative forces. More on this later.)

COMBINING FREE-FALL AND VECTOR COMPONENTS: PROJECTILE MOTION

PROJECTILE MOTION

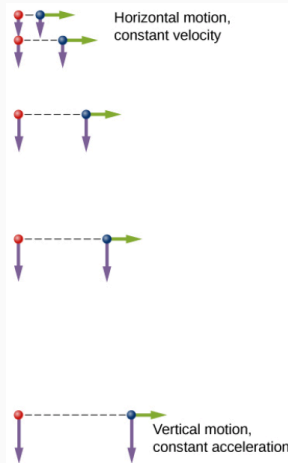


Figure 4: The red particle accelerates vertically, with no horizontal velocity. The blue particle accelerates vertically, with some horizontal velocity.

LAB ACTIVITY

Is this true? Figure 4 is testable by experiment.

Procedure:

1. Obtain two marbles, a meter stick, and a stopwatch.
2. Measure the height of the lab bench, Δx .
3. We are going to drop a marble from this height (Δx) and record the time. Show first algebraically that the predicted time for the marble to fall is $t = \sqrt{2\Delta x/g}$.
4. Measure t for several trials. Does it match the expected result $\sqrt{2\Delta x/g}$? What are sources of error?
5. Repeat the measurement, but **roll the marble off of the table instead of dropping it** from Δx . Does the average result for t change?

PROJECTILE MOTION

We now have learned that (a) motions in displacement components are *independent*, and (b) when acceleration is in **one direction** (vertical) only, the motion is *projectile motion*. Our usual equations of motion for no acceleration (horizontal), and constant acceleration (vertical) apply *independently*:

$$y(t) = y_0 + v_{0,y}t - \frac{1}{2}gt^2 \quad (5)$$

$$v_y(t) = -gt + v_{0,y} \quad (6)$$

$$v_y^2 = v_{y,0}^2 - 2g(y - y_0) \quad (7)$$

$$x(t) = x_0 + v_{0,x}t \quad (8)$$

$$v_x(t) = v_{0,x} \quad (9)$$

Projectile motion is a good topic to introduce the concept of *boundary conditions*. The *physics* of projectile motion is the same for all situations, but the *individual cases and numbers* might not be the same.

Suppose we are given the initial velocity and angle of a object that undergoes projectile motion. To use Eqs. 5-9, we need $v_{0,x}$ and $v_{0,y}$, the initial horizontal and vertical velocity components, respectively.

PROJECTILE MOTION

Suppose we are given the initial velocity and angle of a object that undergoes projectile motion. To use Eqs. 5-9, we need $v_{0,x}$ and $v_{0,y}$, the initial horizontal and vertical velocity components, respectively.

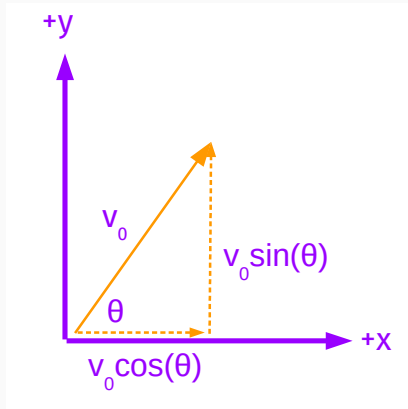


Figure 5: The initial velocity v_0 is broken into components.

During a fireworks display, a shell is shot into the air with an initial speed of 50 m/s , at an angle of 60° above horizontal. The fuse is timed to ignite the shell just as it reaches its highest point above the ground. Calculate the height at which the shell explodes.

- A: 190 m
- B: 100 m
- C: 110 m
- D: 250 m

How much time passes between the launch and the explosion?

- A: 3.9 seconds
- B: 4.3 seconds
- C: 5.1 seconds
- D: 10.0 seconds

What is the horizontal displacement of the shell when it explodes?

- A: 108 meters
- B: 98 meters
- C: 98 degrees
- D: 150 meters

PROJECTILE MOTION

Let's try gaining visual intuition about projectile motion through the following program:

http://galileoandeinstein.physics.virginia.edu/more_stuff/Applets/Projectile/projectile.html

1. First, set air resistance to zero, at bottom right.
2. Make ten measurements of g by creating some projectile trajectories, and taking the ratio $g = v_{0,y}^2 / (2\Delta y)$. What value do you obtain, on average?
3. Now, set air resistance to $b/m \approx 0.02$, and repeat the ten measurements. What value do you obtain?
4. Explain why this value is smaller, larger, or equal to the first set of measurements.

Projectile motion in two dimensions, with constant acceleration in one dimension, produces *quadratic curves*. How do we obtain the **trajectory**, or $y(x)$ for these curves? Looking at the x-direction:

$$x = v_0 \cos(\theta)t \quad (10)$$

$$t = \frac{x}{v_0 \cos(\theta)} \quad (11)$$

PROJECTILE MOTION

Substituting in Eq. 11 for t into the equation for vertical displacement gives:

$$y(t) - y_0 = -\frac{1}{2}g \frac{x^2}{v_0^2 \cos^2(\theta)} + \tan(\theta)x \quad (12)$$

$$y(t) - y_0 = -\left(\frac{g}{2v_0^2 \cos^2(\theta)}\right)x^2 + \tan(\theta)x \quad (13)$$

$$y(x) - y_0 = -\left(\frac{g}{2v_0^2 \cos^2(\theta)}\right)x^2 + \tan(\theta)x \quad (14)$$

In Eq. 14, we are simply saying that $y(x)$ is some quadratic. (It's still true that y and x are both functions of *time*, however, those functions of time are related).

A space explorer is on a moon around another planet, and wants to measure g . She tosses a pebble from an initial height of 2 meter, at an angle of 45 degrees above horizontal, with an initial velocity of 2 m/s. When it lands, the horizontal displacement is 10 meters. What is the gravitational acceleration g ?

- A: 0.125 m/s^2
- B: 0.25 m/s^2
- C: 0.5 m/s^2
- D: 1.0 m/s^2

Other useful equations are for the *time-of-flight*, and the *range*, concepts we've already seen in several examples:

$$T_{\text{tof}} = \frac{2v_0 \sin \theta}{g} \quad (15)$$

$$R = \frac{v_0^2 \sin 2\theta}{g} \quad (16)$$

Algebraic challenge: Show that the ratio of the range to the time is just the horizontal velocity, using the trigonometric identity $\sin(2\theta) = 2 \sin \theta \cos \theta$.

RELATIVE MOTION AND REFERENCE FRAMES

Thus far, we have been discussing *kinematics* with respect to a *fixed frame of reference*. Usually, we think of this frame of reference as the Earth. If we kick a soccer ball, we know how to use equations to describe the motion. What if we are moving, and the ball is moving with us, when we kick it?

Michael is running at 5 m/s, with a soccer ball rolling with him at the same speed. He shoots the ball, such that he judges the speed to be 10 m/s. What is the speed of the ball for someone who is observing, and standing still?

- A: 5 m/s
- B: 10 m/s
- C: 15 m/s
- D: Cannot determine.

So we see that relative motion is about adding and subtracting vectors. Let's introduce a notation so that we always add vectors from reference frames correctly. If two frames of reference (think of them as moving coordinate systems) S and S' are both moving with respect to each other. The position of a particle P in frame S is

$$\vec{r}_{PS} = \vec{r}_{PS'} + \vec{r}_{S'S} \quad (17)$$

$$\vec{r}_{PS} = \vec{r}_{PS'} + \vec{r}_{S'S} \quad (18)$$

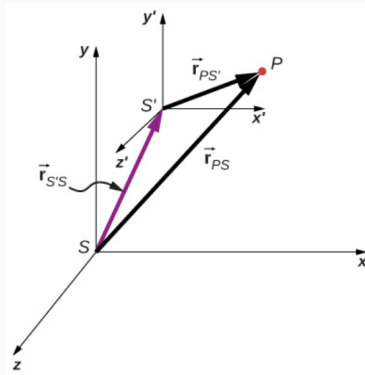


Figure 6: “To get where you are going, first you must know where you are.”

For positions:

$$\vec{r}_{PS} = \vec{r}_{PS'} + \vec{r}_{S'S} \quad (19)$$

Since velocities are just the time-derivatives of positions:

$$\vec{v}_{PS} = \vec{v}_{PS'} + \vec{v}_{S'S} \quad (20)$$

...and accelerations are just the time-derivatives of velocities:

$$\vec{a}_{PS} = \vec{a}_{PS'} + \vec{a}_{S'S} \quad (21)$$

ANSWERS

ANSWERS

- To the right, positive
- At the peak height, yes, yes
- $\vec{r}_{\text{final}} = 472.0\hat{i} + 80.3\hat{j} \quad (m)$
- $\vec{v}_{\text{final}} = 6.8\hat{i} + 1.2\hat{j} \quad (m/s)$
- $16\hat{x} + 8\hat{z} \quad (m/s), 8\hat{x} + 4\hat{z} \quad (m/s)$
- $8\hat{i} + 4\hat{k} \quad (m/s^2), 8\hat{i} + 4\hat{k} \quad (m/s^2)$
- 2 m/s, 2 m/s
- 14 m/s, 32 m/s
- 190 m
- 4.3 seconds
- 108 meters
- 0.5 m/s^2
- $R/T = v_0 \cos \theta = v_{x,0}$