Calculus-Based Physics-1: Mechanics (PHYS150-01): Unit 0

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Opening Remarks - Welcome!

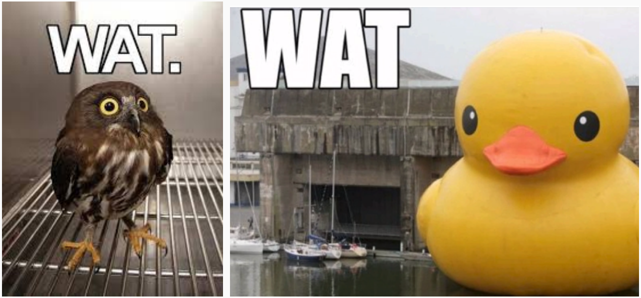


Figure 1: Taking physics for the first time.

Summary

Week 1 Summary

Physics - $\phi\upsilon\sigma\iota\kappa\acute{\eta}$ - "phusiké": *knowledge of nature*
from $\phi\acute{\upsilon}\sigma\iota\varsigma$ - "phúsis": *nature*

1. Estimation/Unit Analysis - Chapters 1.1 - 1.4
    - Estimating the correct order of magnitude
    - Unit analysis - dealing with the units of numbers
2. Coordinates and vectors - Chapters 2.1 - 2.4
    - Scalars and vectors
    - Cartesian (rectangular) coordinates, displacement
    - Vector addition, subtraction, and multiplication
3. Review of Calculus Techniques
    - The derivative, derivatives of elementary functions
    - Function approximation
    - Anti-derivatives and integration

Estimation/Unit Analysis - Chapters 1.1 - 1.4

Estimation/Unit Analysis

In science and engineering, estimation is to obtain a quantity in the absence of precision, informed by rational constraints.

1. **Define relevant unit scales**: (mg, g, or kg), (m/s or km/hr)
2. **Obtain complex quantities** from simple ones
    - Obtain *areas* and *volumes* from *lengths*
    - Obtain *rates* from *numerators* and *denominators*
3. **Taking advantage of scaling problems**
    - Knowing *relationship* between variables
    - Using that *relationship* to obtain new information
4. **Constrain the unknown with upper and lower limits**

Estimation/Unit Analysis

Units: Which of the following represents a *volume*?

- A: 10 gm
- B: 10 cm^2
- C: 1 cm^3
- D: 1 cm s^{-1}

Units: If a grain of sand within a fluid sinks 15 cm in 5 seconds, what is the speed of the grain?

- A: 3 cm
- B: 3 s
- C: 3 s/cm
- D: 3 cm/s

Estimation/Unit Analysis

Unit conversion: If a person weights 120 lbs, what is their weight in kilograms?[1]

- A: 54.5 kg
- B: 264 kg
- C: 54.5 lbs
- D: 264 lbs

[1] One kilogram is 2.2 lbs.

Unit conversion: A density is a mass divided by a volume. For example, water has a density of 1 gm cm^{-3}. What is the density of water in kg m^{-3}?

- A: 1 kg m^{-3}
- B: 10 kg m^{-3}
- C: 100 kg m^{-3}
- D: 1000 kg m^{-3}

Group exercise on complex units: A *vitrolero* is a classic container for serving *agua fresca*. It has a diameter of 20 cm, and a height of 30 cm. How many cups can we serve from the vitrolero if we put 0.5 liters of agua fresca in each cup?

- *Hint:* 1 liter is 1000 mL
- *Hint:* 1 mL is 1 cm^3
- **Volume:** The volume of a cylinder is π times the radius of the base, squared, , times the height: $\pi r^2 h$.

Unit scale: A generation is about one-third of a lifetime. Determine how many generations have passed since the year 0 AD[2].

- A: 10
- B: 20
- C: 60
- D: 100

[2] What is the appropriate scale here?

Unit scale: (a) What fraction of Earth's diameter[3] is the greatest ocean depth (11 km below sea level)? (b) The greatest mountain height (8.8 km above sea level)?

- A: 8.6×10^{-2}, 6.9×10^{-2}
- B: 8.6×10^{-3}, 6.9×10^{-3}
- C: 8.6×10^{-4}, 6.9×10^{-4}
- D: 8.6×10^{-5}, 6.9×10^{-3}

[3] The diameter of the Earth is 12,800 km.

Complex quantities: If a Whittier College athlete ran the 5k race at a track meet in 35 minutes, what was her average speed?

- A: 0.3 meters per second
- B: 3 meters per second
- C: 30 meters per second
- D: 300 meters per second

Complex quantities: Suppose you won the lottery and received \$1 billion USD. Because your life is dope, you stack that paper over the Whittier College soccer field. Each stack contains 100 bills, and each bill is worth \$100. If you evenly cover the field, how high is the money level?

- A: 0.5 inch
- B: 1 inch
- C: 2 inches
- D: 1 foot

Scaling problem: Supposed you have an ideal gas in a cylinder of fixed volume. If the pressure begins as 100 kPa, and you *double* the temperature of the gas, what is the new pressure?

- A: 100 kPa
- B: 50 kPa
- C: 10 kPa
- D: 200 kPa

Scaling problem: Supposed you have an ideal gas in a cylinder of fixed volume. If the pressure begins as 100 kPa, and you *halve* the temperature of the gas, what is the new pressure?

- A: 100 kPa
- B: 50 kPa
- C: 10 kPa
- D: 200 kPa

Upper/lower limits: How many undergraduate students are there at Whittier College[4]?

- A: 5,000
- B: 1,000
- C: 1,250
- D: 500

[4] What is the absolute lower limit, and what is the upper limit?

Upper/lower limits: What is the average yearly college tuition in the United States (before subtracting grants and scholarships)?

- A: \$5,000
- B: \$10,000
- C: \$25,000
- D: \$40,000

What information affects the upper and lower limits here?

Coordinates and Vectors - Chapters 2.1 - 2.4

Activity Link

Who understands coordinates and vectors better than anyone else?

`https://youtu.be/0B7WL7nhIF4?si=_dl4t_GwL98aXWFa`

Physics requires mathematical objects to build equations that capture the behavior of nature. Two examples of such objects are scalar and vector quantities. Each type of object obeys similar but different rules.

1. Scalar quantities
    - mass: $m_1 + (m_2 + m_3) = (m_1 + m_2) + m_3$
    - speed: $v_1(v_2 + v_3) = v_1 v_2 + v_1 v_3$
    - charge: $q_1 \left(\frac{1}{q_1}\right) = 1$, $q_1(0) = 0$
2. Vector quantities
    - velocity: $\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = (\vec{v}_1 + \vec{v}_2) + \vec{v}_3$
    - tension: $\vec{t}_1 \cdot (\vec{t}_2 + \vec{t}_3) = \vec{t}_1 \cdot \vec{t}_2 + \vec{t}_1 \cdot \vec{t}_3$

A vector may be expressed as *a list of scalars*: $\vec{v} = (4, 2)$ (a vector with two *components*), $\vec{u} = (3, 4, 5)$ (three *components*). Now, we know how to add and subtract scalars. How do we add and subtract vectors?

What is
$(1, 3, 8)+$
$(0, 2, 1)$?
Answer: $(1, 5, 9)$

In other words, when adding vectors, we add them component by component.

How do we subtract vectors? In the same fashion:

What is
$(1, 3, 8)-$
$(0, 2, 1)$?
Answer: $(1, 1, 7)$

In other words, when subtracting vectors, we subtract them component by component.

A HTML-based demonstration for adding vectors:

`https://phet.colorado.edu/en/simulations/vector-addition`

Notice several things:

- Produce vectors that *cancel* each other.
- What happens when vectors are parallel and orthogonal?

How do we multiply vectors? In the same fashion, *for one kind of multiplication*:

What is
$(1, 3, 8) \cdot (0, 2, 1)$?
Answer: $1 \cdot 0 + 3 \cdot 2 + 8 \cdot 1 = 14$

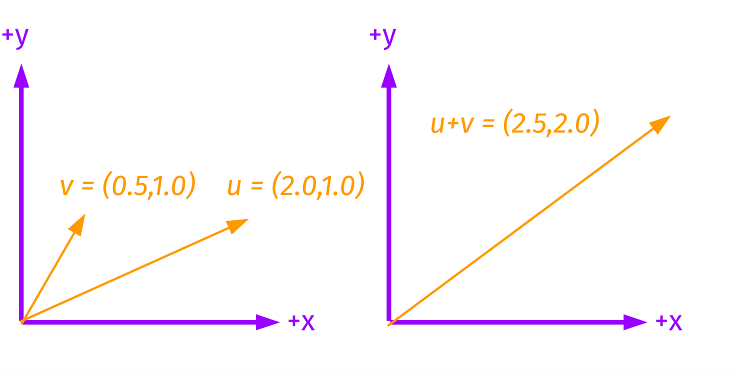
This kind of multiplication is known as the dot-product. There is also the *cross-product*, which we will save for later.

The components of a vector may describe quantities in a coordinate system, such as *Cartesian coordinates* - after René Descartes. Vectors in the 3D Cartesian coordinate system (x,y,z) may be written in the following notation:

$$\boxed{\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}}$$

- a: The amount in the +x-direction, $\hat{i}$: a vector of length 1, in the +x-direction
- b: The amount in the +y-direction, $\hat{j}$: a vector of length 1, in the +y-direction
- c: The amount in the +z-direction, $\hat{k}$: a vector of length 1, in the +z-direction

Coordinates and Vectors - Vectors (Chapters 2.1-2.3)



(a) (b)

Figure 2: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) What is $\vec{u} + \vec{v}$? Adding components: $\vec{u} + \vec{v} = 2.5\hat{i} + 2.0\hat{j}$.

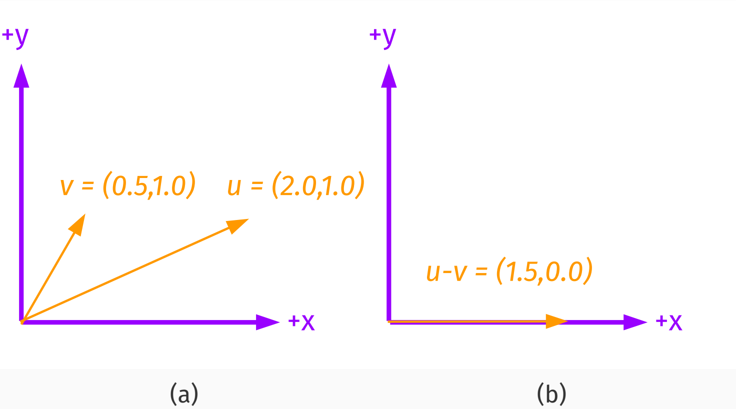
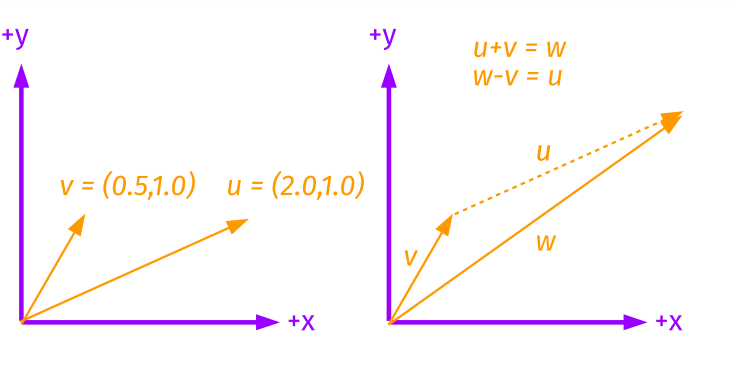


Figure 3: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) What is $\vec{u} - \vec{v}$? Subtracting components: $\vec{u} - \vec{v} = 1.5\hat{i} + 0.0\hat{j}$.



(a) (b)

Figure 4: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) To compute $\vec{w} - \vec{v}$, arrange the vectors to get a sense of the result, $\vec{u}$.

$\vec{p} = 4\hat{i} + 2\hat{j}$. $\vec{q} = -4\hat{i} + 2\hat{j}$.
Compute $\vec{p} \cdot \vec{q}$.

- A: 12
- B: -12
- C: 4
- D: 8

$\vec{p} = -1\hat{i} + 6\hat{j}$. $\vec{q} = 3\hat{i} + 0.5\hat{j}$.
Compute $\vec{p} \cdot \vec{q}$.

- A: -1
- B: 1
- C: 0
- D: 3

Why was the last answer zero? Look at it graphically:

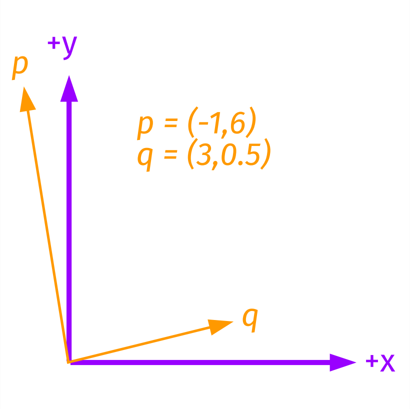


Figure 5: Two vectors $\vec{p}$ and $\vec{q}$ are *orthogonal* if $\vec{p} \cdot \vec{q} = 0$.

What if the vectors are parallel? Look at it graphically:

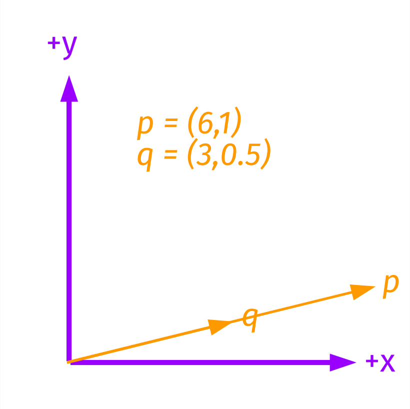


Figure 6: Two vectors $\vec{p}$ and $\vec{q}$ are *parallel* if $\vec{p} \cdot \vec{q}$ is maximal.

The *length* or *norm* of a vector $\vec{v} = a\hat{i} + b\hat{j}$ is $|\vec{v}| = \sqrt{a^2 + b^2}$.

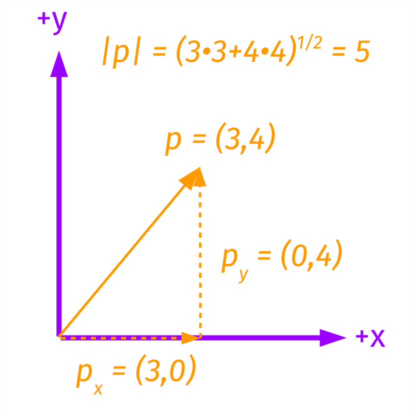


Figure 7: Computing the norm of a vector $\vec{p}$.

Notice that $\sqrt{\vec{p} \cdot \vec{p}} = |\vec{p}|$.
Let θ_p be the angle between $\vec{p}$ and the x-axis.
$p_x = \vec{p} \cdot \hat{i} = |\vec{p}| \cos(\theta_p)$.
$p_y = \vec{p} \cdot \hat{j} = |\vec{p}| \sin(\theta_p)$.

Theorem: The dot product of two vectors $\vec{p}$ and $\vec{q}$ is $|u||v| \cos(\theta)$, if θ is the angle between them.

Proof: $\vec{p} \cdot \vec{q} = p_x q_x + p_y q_y = |p||q| \cos\theta_p \cos\theta_q + |p||q| \sin\theta_q \sin\theta_q$
$= |p||q|(\cos\theta_p \cos\theta_q + \sin\theta_p \sin\theta_q) = |p||q| \cos(\theta_p - \theta_q)$
$= |p||q| \cos\theta$.

$\boxed{\vec{p} \cdot \vec{q} = |p||q| \cos\theta}$

Coordinates and Vectors - Dot Product (Chapters 2.1-2.3)

An object moves at 2 m/s at $\theta = 60°$ with respect to the x-axis. What is the velocity of the object?

- A: $(1\hat{i} + 1\hat{j})$ m/s
- B: $(\sqrt{3}\hat{i} + 1\hat{j})$ m/s
- C: $(\sqrt{3}\hat{i} + \sqrt{3}\hat{j})$ m/s
- D: $(1\hat{i} + \sqrt{3}\hat{j})$ m/s

What is the dot product of this velocity with another velocity: 5 m/s along the x-axis?

- A: 1 $(m/s)^2$
- B: 5 $(m/s)^2$
- C: 10 $(m/s)^2$
- D: 5 (m/s)

Is it possible to multiply vectors and scalars? Of course: $a_1\vec{p} = a_1 p_x\hat{i} + a_1 p_y\hat{j}$.

Also, multiplication properties still hold. For example: $(a_1 + a_2)\vec{p} = a_1\vec{p} + a_2\vec{p}$.

A spacecraft moves at 400 m/s, at an angle of 30 degrees with respect to the x-axis. If it fires two thrusters that boost the x-component and y-component of the velocity by 25% and 50%, respectively, what is the final velocity?

- A: $(433\hat{i} + 300\hat{j})$ m/s
- B: $(300\hat{i} + 433\hat{j})$ m/s
- C: 400 m/s
- D: $(400\hat{i} + 433\hat{j})$ m/s

We define the *position* of an object as a vector locating it in a given coordinate system. The scalar *distance* is the norm of the position vector, that is, the distance to to the origin.

Now we can introduce the concept of dislacement: a vector describing a movement of an object.

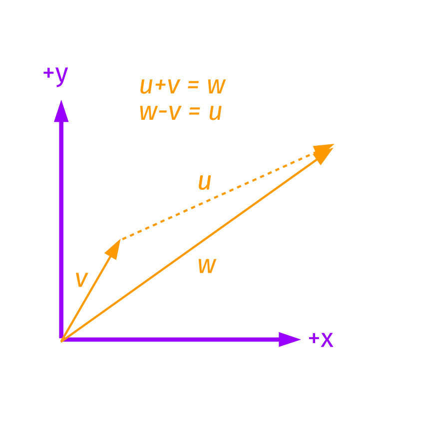


Figure 8: Suppose an object moves from position $\vec{v}$ to $\vec{w}$. In this case, the displacement is $\vec{u}$. **Thus, the final position is the initial position, plus the displacement.**

It follows that the *displacement* is zero if the initial and final positions are the same, but the *distance travelled* is not.

Suppose a jet fighter travelling at 800 km per hour banks such that it flies in a circle of radius 0.5 km. How long does it take to complete the circle? What is the distance traveled, and what is the displacement?

- A: 2π km, 28 seconds, 2π km
- B: π km, 14 seconds, π km
- C: π km, 28 seconds, π km
- D: π km, 14 seconds, 0 km

Average velocity is the ratio of the displacement to the elapsed time.

$$\boxed{\vec{v}_{\mathrm{avg}} = \frac{\Delta\vec{x}}{\Delta t}} \tag{1}$$

The *average speed* is the norm of the average velocity:

$$\boxed{v_{\mathrm{avg}} = \frac{|\Delta\vec{x}|}{\Delta t}} \tag{2}$$

If the motion is in one dimension, then the average speed is

$$v_{\mathrm{avg}} = \frac{x_{\mathrm{f}} - x_{\mathrm{i}}}{t_{\mathrm{f}} - t_{\mathrm{i}}} \tag{3}$$

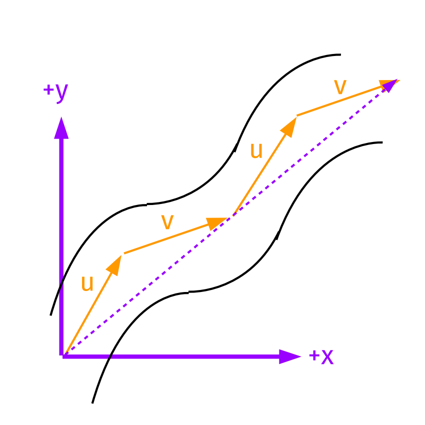


Figure 9: A Formula-1 driver keeps his car on the track by following a path approximated by the position vectors *u*, *v*, *u*, and *v*. The dashed arrow represents the total displacement.

If $\vec{u} = (20\hat{i} + 30\hat{j})$ m, and $\vec{v} = (30\hat{i} + 20\hat{j})$ m, what is the total displacement? If the elapsed time is 10 seconds, what is the average velocity?

- A: $(50\hat{i} + 50\hat{j})$ m, 14 m/s
- B: $(80\hat{i} + 100\hat{j})$ m, 10 m/s
- C: $(100\hat{i} + 100\hat{j})$ m, 14 m/s
- D: $(50\hat{i} + 150\hat{j})$ m, 10 m/s

Review of Calculus Techniques

Review of Calculus Techniques

1. Computing limits
    - Examples in mathematics
    - Examples in physics
2. Differentiation
    - Definition of the derivative
    - Examples of derivatives
3. Integration
    - Definition of the integral
    - Examples of integrals

Consider the function $f(x)$, defined below:

$$f(x) = \frac{1}{1 + x^2} \tag{4}$$

Evaluate the function at the following points:

- $x = 0$
- $x = 10$
- $x = 100$
- $x = 1000$

What is the *limiting value* of the function as $x \to \infty$? What is the *limiting value* of the function as $x \to -\infty$?

Consider the function $f(x)$, defined below:

$$f(x) = \exp(x) = e^x \tag{5}$$

Evaluate the function at the following points:

- $x = 0$
- $x = 10$
- $x = 100$
- $x = 1000$

What is the *limiting value* of the function as $x \to \infty$? What is the *limiting value* of the function as $x \to -\infty$?

Review of Calculus Techniques - Computing Limits

Consider the function $f(x)$, defined below:

$$f(x) = \sin(x) \tag{6}$$

Evaluate the function at the following points:

- $x = \pi$
- $x = -\pi$
- $x = 0.1$
- $x = 0.01$

What is the *limiting value* of the function as $x \to 0$? The procedure is straightforward if the function is *continuous* and *differentiable*.

Review of Calculus Techniques - Differentiation

Derivative of a Function

Let $f(t)$ be a continuous function on an interval $[a, b]$, and $a < t < b$. The derivative of $f(t)$ is

$$f'(t) = \frac{df}{dt} = lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{7}$$

List of Common Derivatives

Here is a link to lists of common derivatives: `https://en.wikipedia.org/wiki/Derivative` `https://en.wikipedia.org/wiki/Differentiation_rules`

Professor: work some examples.

Review of Calculus Techniques - Integration

Integral of a Function

Let $f(t)$ be a continuous function on an interval $[a, b]$, and $a < t_i < b$, where t_i are regular points within the integral. The Riemann integral is

$$I = \sum_i f(t_i)\Delta t_i \rightarrow \int_a^b f(t)dt \tag{8}$$

List of Common Derivatives

Here is a link to a list of common integrals: `https://en.wikipedia.org/wiki/Lists_of_integrals`

Professor: work some examples.

Conclusion

Week 1 Summary

1. Methods of approximation
   - Estimating the correct order of magnitude
   - Function approximation
   - Unit analysis
2. Coordinates and vectors
   - Scalars and vectors
   - Cartesian (rectangular) coordinates, displacement
   - Vector addition, subtraction, and multiplication
3. Review of Calculus Techniques
   - Limits
   - Differentiation
   - Integration