Calculus-Based Physics-1, Mechanics (PHYS150): Unit 3

Jordan Hanson

October 7, 2024

Whittier College Department of Physics and Astronomy

Unit 3 Summary

Week 3 Summary

1. Work has a scientifically precise definition
    - Units
    - As a product of force and displacement vectors
2. Kinetic Energy and the Work-Energy Theorem
3. Gravitational potential energy
    - Potential energy
    - *Simplifying otherwise complex calculations*
    - Potential energy near Earth's surface
    - ...in space
4. Definition of a **conservative force**
    - Relationship between conservative forces and potential energy
    - Conservation of energy for conservative forces

Definitions of Work

Physical Definition of Work

Let $\vec{F}$ be a force exerted on a system, which is displaced by a displacement $\vec{x}$. The **work** done on the system is $W = \vec{F} \cdot \vec{x}$

The units of work are N m = kg m/s^2, or *Joules*.

Definitions of Work

Let θ be the angle between the force and the displacement. Then this equation

$$W = \vec{F} \cdot \vec{x} \tag{1}$$

becomes

$$W = Fx \cos\theta \tag{2}$$

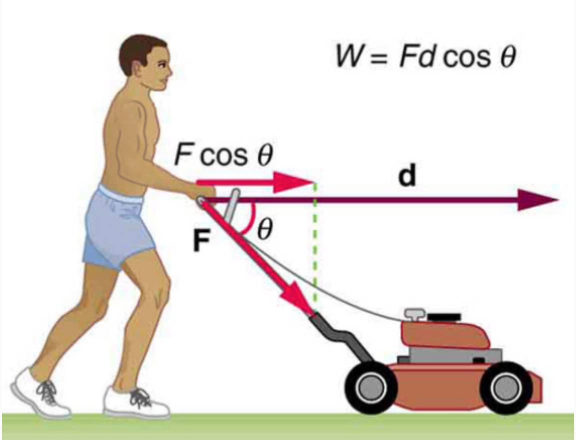


Figure 1: A case where $\theta \neq 0$.

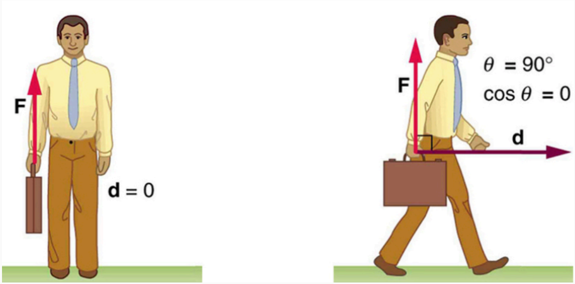


Figure 2: (Left): A case where $x = 0$, so $W = 0$. (Right): A case where $\theta = 90°$, so $W = 0$.

Just because an action requires *energy* doesn't mean we are performing *work*. It requires muscular energy to hold up a heavy briefcase but this is not what we mean by work. Work is about moving objects.

Definitions of Work

What about Newton's 3rd Law? If one system A exerts a force F_{AB} on a system B, then Newton's 3rd law states that system B exerts a force $-F_{\mathrm{AB}}$ on system A.

If the work done by A on B is $W = (F_{\mathrm{AB}})x\cos\theta$, then the work done by B on A is $W = -(F_{\mathrm{AB}})x\cos\theta$.

In Fig. 1, the work done by the man on the mower is positive, but the work done by the mower on the man is negative.

Definitions of Work

More units of energy:

Unit Name	Definition	Value
electron-volt (eV)	energy of 1 e^- through 1 V	1.60×10^{-19} J
1 Rydberg (Rd)	ionize 1 hydrogen atom	21.8×10^{-19} J
Joule	**1 N·m**	**1.0 J**
foot-pound	1 ft·lb	1.36 J
calorie	Raise 1 gram of water 1° C	4.184 J
British Thermal Unit	Raise 1 lb of ice to boil (°F)	1054.3 J
food calorie (kcal)	1000 calories	4184 J
kilowatt hours	1 kilowatt system for 1 hr	3.6×10^6 J
gasoline galon equiv.	burning a galon of gas	$\approx 120 \times 10^6$ J
$E = mc^2$, 1 mole of H^+	Rest mass (fusion/fission)	9×10^{13} J

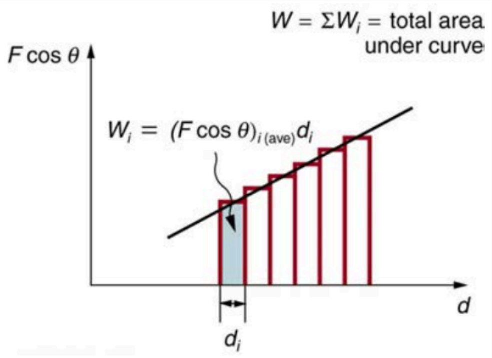


Figure 3: Breaking the displacement $\vec{x}$ into pieces, and summing them.

This interpretation naturally leads to the subject of *integration* in calculus.

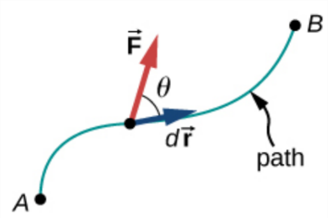


Figure 4: Summary of the concept of the work integral.

$$W = \int_{AB} \vec{F} \cdot d\vec{r} \tag{3}$$

Definitions of Work

Consider the case where the force doing the work on the system of mass m is friction:

$$W = \int_{AB} \vec{F} \cdot d\vec{r} = -\int_{AB} \mu_{\mathrm{k}} N dx = -\mu_{\mathrm{k}} mg \int_{AB} dx \tag{4}$$

- Friction acts in opposite direction, so the dot product gives a minus sign
- Friction acts along path AB (whatever direction of motion is)

The driver of a 900 kg car slams on the breaks, and the tires slide on the pavement with $\mu_{\mathrm{k}} = 0.2$. The initial speed is 25 m/s. Assuming $g = 10$ m/s^2, how far does the car travel before coming to a stop?

- A: 312.5 m
- B: 625 m
- C: 31.25 m
- D: 62.5 m

What is the work done on the car?

- A: 280 J
- B: -280 J
- C: 280 kJ
- D: -280 kJ

Take your *algebraic* answer for the breaking distance (displacement from two slides ago), and substitute it into the expression for the work done on the car (force of friction times displacement). What do you get? (*Check your units*).

- A: mv_{i}^2
- B: $mg\frac{1}{2}mv_{\mathrm{i}}^2$
- C: $-\frac{1}{2}mv_{\mathrm{i}}^2$
- D: $\mu_{\mathrm{k}}mv_{\mathrm{i}}^2$

Keep this result in mind...

Suppose we raise the 900 kg car by a displacement of 10 meters. What is the work done on the car?

- A: 9 J
- B: 90 J
- C: 900 kJ
- D: 90 kJ

What if we drop the car from a height of 10 m? What is the final velocity of the car?

- A: $10\sqrt{2}$ m/s
- B: 20 m/s
- C: 10 m/s
- D: $10\sqrt{10}$ m/s

Stand up, and show in your groups that the work done on the falling car is equal to $\frac{1}{2}mv_{\mathrm{f}}^2$, both numerically and algebraically.

Applications of the Concept of Work

Work done on an oscillator. Let $\vec{x}$ be the displacement of a mass m attached to an oscillator (a spring), with spring constant k. The spring applies a force $\vec{F}$ to the mass:

$$F = -kx \tag{5}$$

$$W = \int \vec{F} \cdot d\vec{x} \tag{6}$$

$$W = -k \int_{AB} x dx = -\frac{1}{2} k x^2 \tag{7}$$

Equation 7 the work done *on the mass by the spring*. The work performed *by the system applying the force is*:

$$W_{\mathrm{spring}} = \frac{1}{2} k x^2 \tag{8}$$

Work done on an oscillator:

$$W_{\mathrm{spring}} = \frac{1}{2}kx^2 \tag{9}$$

For two points A and B that are neither at equilibrium point of oscillator:

$$W_{\mathrm{spring,AB}} = \frac{1}{2}kx_{\mathrm{B}}^2 - \frac{1}{2}kx_{\mathrm{A}}^2 = \frac{1}{2}k(x_{\mathrm{B}}^2 - x_{\mathrm{A}}^2) \tag{10}$$

What about gravity as the force?

$$W_{\mathrm{g}} = \int_A^B -mg\hat{j} \cdot dy\hat{j} = -mg(y_{\mathrm{B}} - y_{\mathrm{A}}) = -mg\Delta y \qquad (11)$$

Why the minus sign? Gravity does negative work if we raise an object Δy. We contribute $W_{\mathrm{g}} = mg\Delta y$ (J) of energy to raise an object Δy against gravity.

Applications of the Concept of Work

List of work applications:

- Mass m through distance l against friction: $W_{\mathrm{f}} = \mu_{\mathrm{k}} mgl$
- Displacing an oscillator (spring) from x_{A} to x_{B}: $W_{\mathrm{s}} = \frac{1}{2}k(x_{\mathrm{B}}^2 - x_{\mathrm{A}}^2)$
- Displacing an object Δy with mass m: $W_{\mathrm{g}} = mg\Delta y$

Cases corresponding to $W = 0$:

- The normal force ($\vec{F} \cdot d\vec{x} = 0$)
- Centripetal force ($\vec{F} \cdot d\vec{x} = 0$)
- *Static friction* ($d\vec{x} = 0$)

Kinetic Energy and the Work-Energy Theorem

The formal proof involves combining **Newton's Second Law**, and the **Definition of Work**:

$$W = \int_{\mathrm{AB}} \vec{F} \cdot d\vec{x} \tag{12}$$

$$W = \int_{\mathrm{AB}} (m\vec{a}) \cdot d\vec{x} \tag{13}$$

$$W = m \int_{\mathrm{AB}} \left(\frac{d\vec{v}}{dt}\right) \cdot d\vec{x} \tag{14}$$

$$W = m \int_{\mathrm{AB}} \vec{v} \cdot d\vec{v} \tag{15}$$

$$W = \frac{1}{2} m v_{\mathrm{B}}^2 - \frac{1}{2} m v_{\mathrm{A}}^2 = \Delta KE \tag{16}$$

$$W = \Delta KE \tag{17}$$

Kinetic Energy and the Work-Energy Theorem

The Work-Energy Theorem

If W is the work done on a system, and KE is the kinetic energy of the system, then $W = \Delta KE$.

A firework of mass 1 kg is launched straight upwards. The gunpowder releases 500 J of energy. What is the velocity of the shell as it leaves the launcher? How high does it fly straight upwards?

- A: 31.6 m/s, 50 m
- B: 22.4 m/s, 5 m
- C: 31.6 m/s, 20 m
- D: 22.4 m/s, 50 m

A slingshot is like a spring with a spring constant of 2000 N/m. If a projectile is placed in the slingshot pouch, how much work is required to draw it back 10 cm?

- A: 1 J
- B: 5 J
- C: 10 J
- D: 50 J

If the pouch is released, what is the final velocity of the projectile if all of the work is converted into kinetic energy? (The mass m is 1 kg).

- A: 1 m/s
- B: $\sqrt{20}$ m/s
- C: 100 m/s
- D: $\sqrt{30}$ m/s

Recall the problem from the midterm about measuring coefficient of *kinetic* friction. Suppose we compress a spring, and use the spring to slide a 1 kg steel block across a wet surface with unknown kinetic friction coefficient μ_{k}. If the spring has $k = 1000$ N/m, we compress it by 10 cm, and the plate slides 10 m, what is μ_{k}?

- A: 0.001
- B: 1
- C: 0.05
- D: 0.2

Gravitational Potential Energy

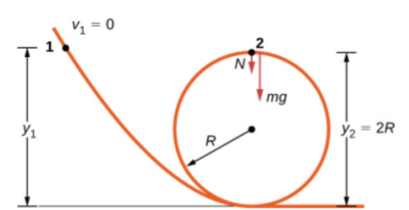
Gravitational Potential Energy

Suppose we raise an object against gravity, and then, it remains motionless. *Does it have energy*? Well, it **cannot have any kinetic energy**, because it has no velocity. Work has been performed on it, however, so we say that it has **potential energy**.

Potential energy $\leftrightarrow$ Kinetic energy

The famous roller coaster problem: Suppose a roller coaster has a circular loop with radius R, and the cars begin on a hill of height y_1. Since work has been performed on the cars to get them that high, they have energy $W = mgy_1$ (they have mass m). If the cars are released, the kinetic energy propels them through the loop. What is the ratio of y_1 to R?

- A: $y_1 = \frac{1}{2}R$
- B: $y_1 = R$
- C: $y_1 = \frac{3}{2}R$
- D: $y_1 = \frac{5}{2}R$



Lab Activity: Gravitational Potential Energy and Work-Energy Theorem

Lab Activity - See Lab Handout

(See laboratory activity handout).

Lab Activity: Roller Coaster

Lab Activity for Gravitational Potential Energy

Materials (5 min):

- Pipe liner
- A ball bearing
- Scissors
- Tape
- Ruler

Build (10 min):

- Cut pipe liner in half, lengthwise
- Shape the pipe liner into the shape on prior slide
- Use tape to fix in place

Measure (15 minutes):

- Measure the initial height required to traverse loop
- Compare to loop radius
- Report the radius and ten initial height measurements

Bonus: *Two loops?*

Interactive simulation:
`https://openstaxcollege.org/l/21PhETSimRamp`

- Push an object up the ramp (which includes friction)
- Copy the energy plot to your lab notebook, and explain each feature of it.
    1. Where is the energy constant?
    2. Where is the energy changing?

Power

Power

Power is the derivative of **work**:

$$P = \frac{dW}{dt} \tag{18}$$

$$W = \int P dt \tag{19}$$

The unit of power is the Watt, or 1 J/s = 1 N m/s.

Power

A common scale for home power usage is the kW hr, or kilowatt hours, which is a unit of energy. If a home consumes 1000 kW hr in one month, what is the average power consumed at a given time in this home? (Hint: divide by the time of one month).

- A: 0.1 kW
- B: 1.4 kW
- C: 10 kW
- D: 1.2 MW

According to Google, central AC uses about 3500 W for a home. Think about this the next time you get your electric bill...

Power

Since $P = \frac{dW}{dt}$ we may show that $P = \vec{F} \cdot \vec{v}$. (Example).

Definition of Conservative Force

Definition of Conservative Force

Definition of Conservative Force

Let $\vec{F}(x, y)$ describe some *force field*. The field is **conservative** if the line integral around a closed path is zero:

$$\oint_p \vec{F} \cdot d\vec{r} = 0 \tag{20}$$

Returning the system to the same point implies no net work.

Corollary 1

The relationship between derivatives of the components is

$$\frac{dF_y}{dx} = \frac{dF_x}{dy} \tag{21}$$

Definition of Conservative Force

Consider the case of gravity: $\vec{F}(x, y) = 0\hat{i} - mg\hat{j}$. So it depends neither on x nor y. Let's check that it is conservative:

$$\frac{dF_y}{dx} = 0 \tag{22}$$

$$\frac{dF_x}{dy} = 0 \tag{23}$$

$$\frac{dF_y}{dx} = \frac{dF_x}{dy} \tag{24}$$

Definition of Conservative Force

Check if the following force is conservative:

$$\vec{F} = x^2\hat{i} + y^2\hat{j} \tag{25}$$

Hint: when you picture a conservative force, picture symmetry.

Check if the following force is conservative:

$$\vec{F} = y^2\hat{i} + x^2\hat{j} \tag{26}$$

How would you draw these forces?

Corollary 2

A conservative force is a gradient of a scalar function, the potential energy.

$$\vec{F} = -\nabla U \tag{27}$$

Conclusion

Unit 6 Summary

1. Work has a scientifically precise definition
   - Units
   - As a product of force and displacement vectors
2. Kinetic Energy and the Work-Energy Theorem
3. Gravitational potential energy
   - Potential energy
   - *Simplifying otherwise complex calculations*
   - Potential energy near Earth's surface
   - …in space
4. Definition of a **conservative force**
   - Relationship between conservative forces and potential energy
   - Conservation of energy for conservative forces