CALCULUS-BASED PHYSICS-1: MECHANICS (PHYS150-01): UNIT 2

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WEEK 3 SUMMARY

1. Displacement, velocity and acceleration vectors as functions of time, in 3D
2. General kinematic equations of constant acceleration in 2D
3. Simplifying for free-fall: projectile motion
    - **PhET Activity:** Range, trajectory, and air-resistance

VECTORS AS FUNCTIONS OF TIME

In general, the displacement of an object depends on time:

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \tag{1}$$

- $x(t)$ is the displacement in the x-direction
- $y(t)$ is the displacement in the y-direction
- $z(t)$ is the displacement in the z-direction

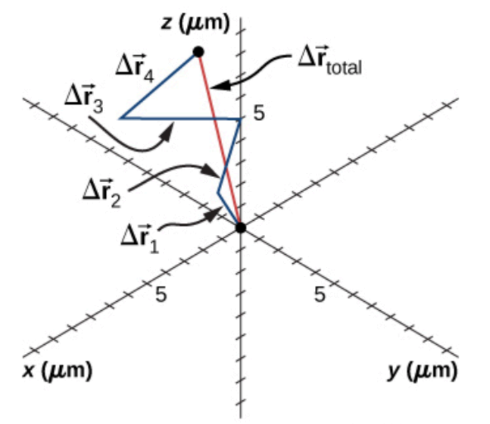

Figure 1: An example of a displacement vector at different moments in time.

The particle in Fig. 1 has four displacement vectors at four moments in time:

- $\vec{r}_1 = 2.0\hat{i} + 1.0\hat{j} + 3.0\hat{k} \quad (\mu m)$ at t_1
- $\vec{r}_2 = -1.0\hat{i} + 0.0\hat{j} + 3.0\hat{k} \quad (\mu m)$ at t_2
- $\vec{r}_3 = 4.0\hat{i} + -2.0\hat{j} + 1.0\hat{k} \quad (\mu m)$ at t_3
- $\vec{r}_4 = -3.0\hat{i} + 1.0\hat{j} + 2.0\hat{k} \quad (\mu m)$ at t_4

What is the total displacement of the particle from the origin?

We can think of this type of problem as an accounting problem, lining up columns (units: μm):

t_i	$\vec{r}_\mathrm{i}(t_\mathrm{i})$	$x(t_\mathrm{i})$	$y(t_\mathrm{i})$	$y(t_\mathrm{i})$
t_1	$\vec{r}_1(t_1)$	2.0	1.0	3.0
t_2	$\vec{r}_2(t_2)$	-1.0	0.0	3.0
t_3	$\vec{r}_3(t_3)$	4.0	-2.0	1.0
t_4	$\vec{r}_4(t_4)$	-3.0	1.0	2.0
t_total	$\vec{r}_\mathrm{total}(t_\mathrm{total})$	2.0	0.0	9.0

Figure 2: Accounting for the different displacement components, in units of μm.

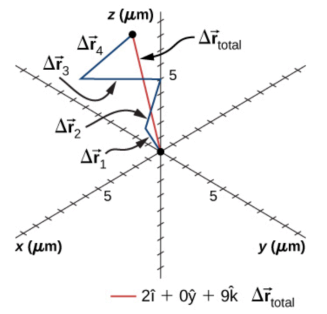

Figure 3: The total displacement of the particle is $\vec{r}_{\text{total}} = 2.0\hat{i} + 0.0\hat{k} + 9.0\hat{k} \quad (\mu m)$.

Professor: work some examples here with displacement and velocity.

The 18th hole at Pebble Beach Golf Course is a dogleg to the left of length 496.0 meters. The fairway off the tee is taken to be the x direction. A golfer hits his tee shot a distance of 300 meters, corresponding to a displacement of $\vec{r}_1 = 300.0\hat{i} \quad (m)$, and then hits a second shot 189.0 meters with $\vec{r}_2 = 172.0\hat{i} + 80.3\hat{j} \quad m$. What is the final displacement from the tee?

- A: $\vec{r}_{\mathrm{final}} = 172.0\hat{i} + 80.3\hat{j} \quad (m)$
- B: $\vec{r}_{\mathrm{final}} = 172.0\hat{i} + 380.3\hat{j} \quad (m)$
- C: $\vec{r}_{\mathrm{final}} = 472.0\hat{i} + 0.0\hat{j} \quad (m)$
- D: $\vec{r}_{\mathrm{final}} = 472.0\hat{i} + 80.3\hat{j} \quad (m)$

If the first shot takes 5.0 seconds, the second shot takes 4.0 seconds, and the walking time in between the shots is 60.0 seconds, what is the average velocity vector for the ball after the two shots?

- A: $\vec{r}_{\text{final}} = 1.7\hat{i} + 8.3\hat{j} \quad (m/s)$
- B: $\vec{v}_{\text{final}} = 172.0\hat{i} + 80.3\hat{j} \quad (m/s)$
- C: $\vec{v}_{\text{final}} = 6.8\hat{i} + 1.2\hat{j} \quad (m)$
- D: $\vec{v}_{\text{final}} = 6.8\hat{i} + 1.2\hat{j} \quad (m/s)$

The prior problem indicates something you may already have guessed:

$$\vec{v}_{\mathrm{avg}}(t) = v_{\mathrm{x}}(t)\hat{i} + v_{\mathrm{y}}(t)\hat{j} + v_{\mathrm{z}}(t)\hat{k} = \frac{\Delta\vec{r}}{\Delta t} \tag{2}$$

- $v_{\mathrm{x}}(t)$ is the avg. velocity in the x-direction
- $v_{\mathrm{y}}(t)$ is the avg. velocity in the y-direction
- $v_{\mathrm{z}}(t)$ is the avg. velocity in the z-direction

In other words, we divide each displacement component by the time, to get a vector where each component is the average velocity in that direction. $\Delta\vec{r} = \vec{r}_{\mathrm{f}} - \vec{r}_{\mathrm{i}}$.

Instantaneously, Eq. 2 is true, if we take the limit $\Delta t \to 0$:

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \tag{3}$$

- $\frac{dx}{dt}$ is the instantaneous velocity in the x-direction
- $\frac{dy}{dt}$ is the instantaneous velocity in the y-direction
- $\frac{dz}{dt}$ is the instantaneous velocity in the z-direction

The position of a particle is $\vec{r}(t) = 4.0t^2\hat{i} - 3.0\hat{j} + 2.0t^2\hat{k}$ (m). What is the velocity vector at $t = 2$ seconds? What is the average velocity between $t = 0$ and $t = 2$ seconds?

- A: $16\hat{i} + 8\hat{k}$ (m/s), $8\hat{i} + 4\hat{k}$ (m/s)
- B: $8\hat{i} + 4\hat{k}$ (m/s), $4\hat{i} + 2\hat{k}$ (m/s)
- C: $8\hat{i} + 8\hat{k}$ (m/s), $4\hat{i} + 4\hat{k}$ (m/s)
- D: $4\hat{i} + 2\hat{k}$ (m/s), $4\hat{i} + 2\hat{k}$ (m/s)

Instantaneously, from Eq. 3:

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{dv_{\mathrm{x}}}{dt}\hat{i} + \frac{dv_{\mathrm{y}}}{dt}\hat{j} + \frac{dv_{\mathrm{z}}}{dt}\hat{k} \tag{4}$$

- $\frac{dv_{\mathrm{x}}}{dt}$ is the instantaneous acceleration in the x-direction
- $\frac{dv_{\mathrm{y}}}{dt}$ is the instantaneous acceleration in the y-direction
- $\frac{dv_{\mathrm{z}}}{dt}$ is the instantaneous acceleration in the z-direction

The velocity of a particle is $\vec{v}(t) = 8.0t\hat{i} + 4.0t\hat{k}$ (m/s). What is the acceleration vector at $t = 2$ seconds? What is the average acceleration between $t = 0$ and $t = 2$ seconds?

- A: $4\hat{i} + 4\hat{k}$ (m/s^2), $2\hat{i} + 2\hat{k}$ (m/s^2)
- B: $8\hat{i} + 4\hat{k}$ (m/s^2), $8\hat{i} + 4\hat{k}$ (m/s^2)
- C: $8\hat{i} + 8\hat{k}$ (m/s^2), $4\hat{i} + 4\hat{k}$ (m/s^2)
- D: $4\hat{i} + 8\hat{k}$ (m/s^2), $2\hat{i} + 4\hat{k}$ (m/s^2)

The displacement of a particle is $\vec{x}(t) = (2t + 3)\hat{i} + (\frac{3}{2}t^2 + 2t + 3.0)\hat{j}$ (m). What is the horizontal velocity (the $\hat{i}$-component of the velocity) at $t = 4$ seconds? At $t = 10$ seconds?

- A: 4 m/s, 4 m/s
- B: 2 m/s, 4 m/s
- C: 2 m/s, 2 m/s
- D: 4 m/s, 2 m/s

The displacement of a particle is $\vec{x}(t) = (2t+3)\hat{i} + (\frac{3}{2}t^2 + 2t + 3.0)\hat{j}$ (m). What is the vertical velocity (the $\hat{j}$-component of the velocity) at $t = 4$ seconds? At $t = 10$ seconds?

- A: 14 m/s, 32 m/s
- B: 32 m/s, 14 m/s
- C: 12 m/s, 30 m/s
- D: 30 m/s, 12 m/s

Notice in the previous example, the x-velocity and y-velocity were not the same function.

In the kinematic description of motion, *we are able to treat the different components of motion separately.* In many cases, motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

Motions in displacement components are independent.

(Exception: non-conservative forces. More on this later.)

COMBINING FREE-FALL AND VECTOR COMPONENTS: PROJECTILE MOTION

World record basketball shot:
`https://youtu.be/gm2_6DX_0Bw`

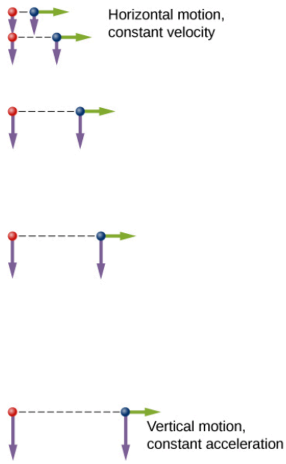

Figure 4: The red particle accelerates vertically, with no horizontal velocity. The blue particle acclerates vertically, with some horizontal velocity.

We now have learned that (a) motions in displacement components are *independent*, and (b) when acceleration is in one direction (vertical) only, the motion is *projectile motion*. Our usual equations of motion for no acceleration (horizontal), and constant acceleration (vertical) apply *independently*:

$$y(t) = y_0 + v_{0,\mathrm{y}} t - \frac{1}{2} g t^2 \quad (5)$$

$$v_{\mathrm{y}}(t) = -gt + v_{0,\mathrm{y}} \quad (6)$$

$$v_{\mathrm{y}}^2 = v_{\mathrm{y},0}^2 - 2g(y - y_0) \quad (7)$$

$$x(t) = x_0 + v_{0,\mathrm{x}} t \quad (8)$$

$$v_{\mathrm{x}}(t) = v_{0,\mathrm{x}} \quad (9)$$

Projectile motion is a good topic to introduce the concept of *boundary conditions*. The *physics* of projectile motion is the same for all situations, but the *individual cases and numbers* might not be the same.

Suppose we are given the initial velocity and angle of a object that undergoes projectile motion. To use Eqs. 5-9, we need $v_{0,\mathrm{x}}$ and $v_{0,\mathrm{y}}$, the initial horizontal and vertical velocity components, respectively.

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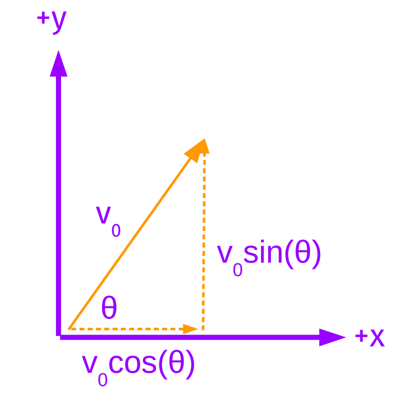

Figure 5: The initial velocity v_0 is broken into components.

During a fireworks display, a shell is shot into the air with an initial speed of 50 m/s, at an angle of 60° above horizontal. The fuse is timed to ignite the shell just as it reaches its highest point above the ground. Calculate the height at which the shell explodes.

- A: 190 m
- B: 100 m
- C: 110 m
- D: 250 m

How much time passes between the launch and the explosion?

- A: 3.9 seconds
- B: 4.3 seconds
- C: 5.1 seconds
- D: 10.0 seconds

What is the horizontal displacement of the shell when it explodes?

- A: 108 meters
- B: 98 meters
- C: 98 degrees
- D: 150 meters

PHET ACTIVITY

Let's try gaining visual intuition about projectile motion through the following program:

`https://phet.colorado.edu/en/simulation/projectile-motion`

1. Derive the *range equation* (Professor on board).
2. Plot the range versus initial velocity, for some fixed θ. Use Excel to derive the relationship by fitting a trend line to the data.
3. Plot the range versus θ, for some fixed initial velocity. Use Excel to derive the relationship by fitting a trend line to the data.
4. Now, turn on air resistance, and repeat the prior two exercises. What do you notice about the trend lines?
5. Play with the air resistance parameter by tuning it with the tools on the right side of the screen. What do you notice?

Projectile motion in two dimensions, with constant acceleration in one dimension, produces *quadratic curves*.

How do we obtain the trajectory, or $y(x)$ for these curves?

Looking at the x-direction:

$$x = v_0 \cos(\theta) t \tag{10}$$

$$t = \frac{x}{v_0 \cos(\theta)} \tag{11}$$

Substituting in Eq. 11 for t into the equation for vertical displacement gives:

$$y(t) - y_0 = -\frac{1}{2} g \frac{x^2}{v_0^2 \cos^2(\theta)} + \tan(\theta) x \tag{12}$$

$$y(t) - y_0 = -\left(\frac{g}{2 v_0^2 \cos^2(\theta)} \right) x^2 + \tan(\theta) x \tag{13}$$

$$y(x) - y_0 = -a x^2 + b x \tag{14}$$

In Eq. 14, we are simply saying that $y(x)$ is some quadratic. (It's still true that y and x are both functions of *time*, however, those functions of time are related).

Other useful equations are for the *time-of-flight*, and the *range*, concepts we've already seen in several examples:

$$T_{\mathrm{tof}} = \frac{2v_0 \sin\theta}{g} \tag{15}$$

$$R = \frac{v_0^2 \sin 2\theta}{g} \tag{16}$$

A space explorer is on a moon around another planet, and wants to measure g. She tosses a pebble from an initial height of 2 meter, at an angle of 45 degrees above horizontal, with an initial velocity of 2 m/s. When it lands, the horizontal displacement is 10 meters. What is the gravitational acceleration g?

- A: 0.125 m/s^2
- B: 0.25 m/s^2
- C: 0.5 m/s^2
- D: 1.0 m/s^2

Algebraic challenge: Show that the ratio of the range to the time is just the horizontal velocity, using the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$.

CONCLUSION

1. Displacement, velocity and acceleration vectors as functions of time, in 3D
2. General kinematic equations of constant acceleration in 2D
3. Simplifying for free-fall: projectile motion
    - **PhET Activity:** Range, trajectory, and air-resistance