RC Circuits Lab: Electronic Filters

March 15, 2019

1 An Essential Math Tool

The Fourier transform of a function $f(t)$ is defined as:

$$\mathcal{F}(f(t)) = \tilde{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \tag{1}$$

In Eq. 1, ω is the angular frequency, measured in radians per unit time. Let $f(t) = g'(t)$. Substituting into Eq. 1, and integrating by parts, we have

$$\tilde{F}(\omega) = g(t)e^{-j\omega t}|_{-\infty}^{\infty} + j\omega \int_{\infty}^{-\infty} g(t)e^{-j\omega t}dt \tag{2}$$

For physical signals that represent finite energy, $\lim_{|t|\to\infty} g(t) = 0$. This requirement simplifies Eq. 2 by making the first term on the right-hand side vanish. We have

$$\tilde{F}(\omega) = j\omega \int_{\infty}^{-\infty} g(t)e^{-j\omega t}dt = j\omega\mathcal{F}(g(t)) \tag{3}$$

The result may be summarized:

$$\boxed{\mathcal{F}(g'(t)) = j\omega\mathcal{F}(g(t))} \tag{4}$$

One utility of this result is that differential equations in the time-domain may be converted to algebraic equations in the Fourier domain, making them easier to apply.

2 Two Simple Circuits

A simple RC circuit is shown in Fig. 1. The resistance R is given, and the capacitance C is the ratio of the charge stored on the capacitor, q, to the voltage required to place that charge on the capacitor, V:

$$q = CV \tag{5}$$

$$i(t) = C\frac{dV}{dt} \tag{6}$$

$$\tilde{i}(\omega) = j\omega C\tilde{V}(\omega) \tag{7}$$

$$\frac{\tilde{V}(\omega)}{\tilde{i}(\omega)} = \frac{1}{j\omega C} \tag{8}$$

Thus, Ohm's law says that the frequency-dependent resistance, or *impedance*, of a capacitor is

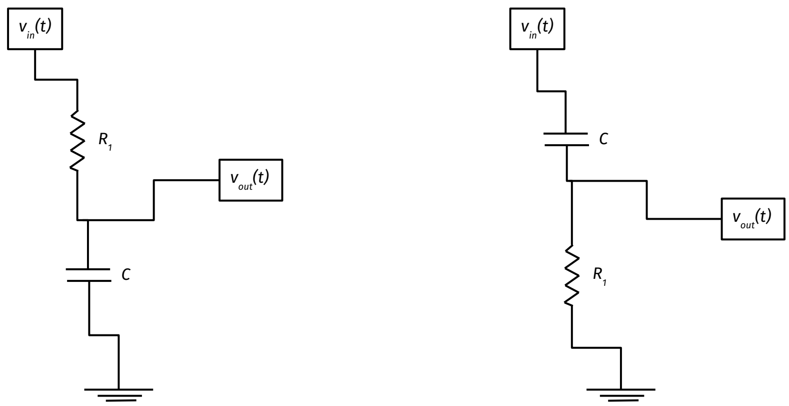

Figure 1: (Left) A single-pole RC low-pass filter. (Right) A single-pole RC high-pass filter.

$$\boxed{Z_C = \frac{1}{j\omega C}} \tag{9}$$

The **transfer function** of the RC circuit in Fig. 1 is the ratio of the output voltage to the input voltage, as with a voltage divider. However, the derivation of this ratio produces

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1} \tag{10}$$

Let the *time-constant* be defined as $\tau = RC$, and $\omega_0 = 1/\tau$. Equation 10 may be written:

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = -\frac{j\omega_0}{\omega - j\omega_0} \tag{11}$$

The magnitude and phase of Eq. 11 are

$$M_{LP}(\omega) = \left(1 + \left(\frac{\omega}{\omega_0}\right)^2\right)^{-1/2} \tag{12}$$

$$\phi_{LP}(\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right) \tag{13}$$

The low-pass transfer function $M_{LP}(\omega)$ attenuates frequencies much larger than ω_0, and the $\phi_{LP}(\omega)$ function shows that there is a frequency-dependent phase-shift. The high-pass filter in Fig. 1 (right) is similar to the low-pass filter in Fig. 1 (left). Following the same arguments as the low-pass case, the complex transfer function is

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = \frac{\omega}{\omega - j\omega_0} \tag{14}$$

The magnitude and phase of Eq. 14 are

$$M_{HP}(\omega) = \left(1 + \left(\frac{\omega_0}{\omega}\right)^2\right)^{-1/2} \tag{15}$$

$$\phi_{HP}(\omega) = \tan^{-1}\left(\frac{\omega_0}{\omega}\right) \tag{16}$$

Unlike the voltage divider, the circuits in Fig. 1 have capacitors. The filtering in these cases is driven by how quickly these capacitors can be charged and discharged, regardless of where they are in the circuit.

3 Building a Passive Differentiator

Consider a single-pole high pass filter, with transfer function given by Eq. 14. Choose a value for ω_0 much larger than any frequency in the expected input signal: $\omega_0 \gg \omega$. The transfer function is approximately:

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} \approx \frac{\omega}{-j\omega_0} = j\omega\tau = j\omega RC \tag{17}$$

Rearranging Eq. 17, and switching back to the time-domain:

$$\tilde{v}_{out}(\omega) \approx j\omega\tau\tilde{v}_{in}(\omega) \tag{18}$$

$$v_{out}(t) \approx \tau\frac{dv_{in}}{dt} \tag{19}$$

Equation 19 shows that with the correct choice of resistance and capacitance, the circuit output is the derivative of the input, with a *gain* equal to $\tau = RC$. This circuit is known as a *passive differentiator*.

4 Building a Passive Integrator

Consider a single-pole low-pass filter, with transfer function given by Eq. 10. Choose a value for ω_0 much smaller than any frequency in the expected input signal: $\omega_0 \ll \omega$. The transfer function is approximately:

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} \approx \frac{-j\omega_0}{\omega} \tag{20}$$

Rearranging Eq. 20, switching back to the time-domain, and integrating both sides:

$$j\omega\tilde{v}_{out}(\omega) \approx \omega_0\tilde{v}_{in}(\omega) \tag{21}$$

$$\frac{dv_{out}}{dt} = \omega_0 v_{in}(t) \tag{22}$$

$$v_{out}(t) = \frac{1}{RC}\int_{t_1}^{t_2} v_{in}(t)dt \tag{23}$$

Equation 23 shows that with the correct choice of resistance and capacitance, the circuit output is the integral of the input between two set times, with a *gain* equal to $1/RC$. This circuit is known as a *passive integrator*.

5 Summary of Passive Differentiator and Integrator

- By choosing a large value of RC, relative to input frequencies, the output of a single-pole high-pass filter is proportional to the derivative of the input, with gain RC.
- By choosing a small value of RC, relative to input frequencies, the output of a single-pole low-pass filter is proportional to the integral of the input, with gain $1/RC$.