CALCULUS-BASED PHYSICS-1: MECHANICS (PHYS150-01): UNIT 1

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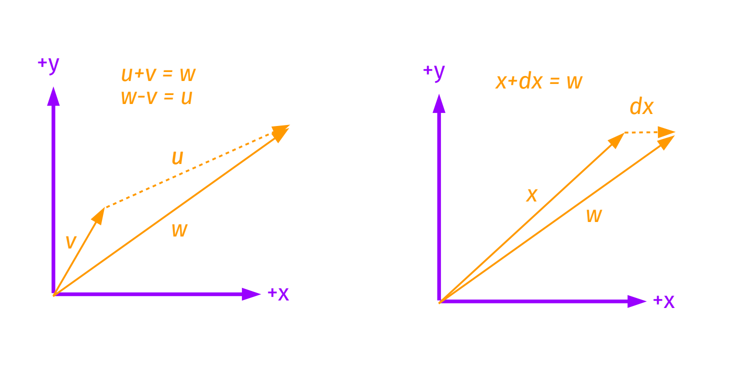
UNIT 0 REVIEW

1. Methods of approximation
    - Estimating the correct order of magnitude
    - Function approximation
    - Unit analysis
2. Coordinates and vectors
    - Scalars and vectors
    - Cartesian (rectangular) coordinates, displacement
    - Vector addition, subtraction, and multiplication
3. Review of Calculus Techniques
    - Limits
    - Differentiation
    - Integration

UNIT 1 SUMMARY

1. Displacement, and instantaneous velocity and acceleration
    - *Mathematics review*: taking derivatives
    - Average velocity and average acceleration
2. The case of constant acceleration
    - An *an equation of motion* for constant acceleration
    - Derivation of common equations of motion
    - Average quantities and exercises
3. **Lab Activity: Measuring acceleration of gravity: g**
4. Exercises with vectors, graphs, and equations of motion

DISPLACEMENT, AND INSTANTANEOUS VELOCITY AND ACCELERATION



(a) (b)

Figure 1: (Left): The displacement vector is $\vec{u}$. (Right) Treat displacement for a small change in time, dt, and call it $d\vec{x}$.

Let $f(t) = A\sin(Bt) + Ct^2$.
Compute f'.

- A: $f'(t) = AB\sin(Bt) + 2Ct$
- B: $f'(t) = AB\cos(Bt) + 2C$
- C: $f'(t) = AB\sin(Bt) + 2Ct$
- D: $f'(t) = AB\cos(Bt) + 2Ct$

Let $f(t) = (4t - 1)/(3t + 2)$.
Compute f'.

- A: $f'(t) = \frac{4}{3t+2}$
- B: $f'(t) = \frac{4}{(3t+2)^2} + \frac{12t-3}{(3t+2)^2}$
- C: $f'(t) = \frac{4}{3t+2} + \frac{12t-3}{(3t+2)^2}$
- D: $f'(t) = \frac{12t-3}{(3t+2)^2}$

Definition of instantaneous velocity vector:

$$\boxed{v(t) = \frac{d\vec{x}}{dt}} \tag{1}$$

Simple example: Let the vector position of an object be

$$\vec{x}(t) = (2t\hat{i} - 3t^2\hat{j}) \quad m \tag{2}$$

Then

$$\vec{v}(t) = (2\hat{i} - 6t\hat{j}) \quad m/s \tag{3}$$

Definition of instantaneous *acceleration* vector:

$$\boxed{a(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\frac{d\vec{x}}{dt}} \tag{4}$$

Simple example: Let the vector position of an object be

$$\vec{x}(t) = (2t\hat{i} - 3t^2\hat{j}) \quad m \tag{5}$$

Then

$$\vec{v}(t) = (-6\hat{j}) \quad m/s^2 \tag{6}$$

Interesting... If the motion of an object is *quadratic* in time, then the acceleration is a constant.

Let the displacement versus time of an object be

$$\vec{y}(t) = (-\frac{1}{2}gt^2 + v_i t + y_0)\hat{j} \quad (m) \tag{7}$$

If Eq. 7 gives the displacement in the $\hat{j}$ direction, then what are the velocity and acceleration?

THE CASE OF CONSTANT ACCELERATION

Using the definitions of instantaneous velocity and acceleration:

$$\frac{d\vec{y}}{dt} = (-gt + v_i)\hat{j} \quad (m/s) \tag{8}$$

$$\frac{d}{dt}\frac{d\vec{y}}{dt} = (-g)\hat{j} \quad (m/s^2) \tag{9}$$

The acceleration is just some constant, g, in the $-\hat{j}$ direction. This leads to a *linear* equation for the velocity, and a *quadratic* equation for the displacement.

So we have the following three equations for a system experiencing constant acceleration:

$$\vec{y}(t) = (-\frac{1}{2}gt^2 + v_{\mathrm{i}}t + y_0)\hat{j} \quad (m) \tag{10}$$

$$\vec{v}(t) = (-gt + v_{\mathrm{i}})\hat{j} \quad (m/s) \tag{11}$$

$$\vec{a}(t) = (-g)\hat{j} \quad (m/s^2) \tag{12}$$

What if we solve for time in Eq. 11, after taking the magnitude of the vector?

$$\frac{v - v_{\mathrm{i}}}{-g} = t \tag{13}$$

Now substitute Eq. 13 into Eq. 10:

$$y = -\frac{1}{2}g\left(\frac{v - v_{\mathrm{i}}}{-g}\right)^2 + v_{\mathrm{i}}\left(\frac{v - v_{\mathrm{i}}}{-g}\right) + y_0 \tag{14}$$

$$-2g(y - y_0) = (v - v_{\mathrm{i}})^2 + 2v_{\mathrm{i}}(v - v_{\mathrm{i}}) \tag{15}$$

$$-2g(y - y_0) = v^2 - v_{\mathrm{i}}^2 \tag{16}$$

$$-2g(y - y_0) + v_{\mathrm{i}}^2 = v^2 \tag{17}$$

Equation 17 provides a way to obtain the velocity of an accelerating system at some displacement without knowing the time.

A particle moves along the x-axis according to $x(t) = (10t - 2t^2)\hat{i}$ m. What is the instantaneous velocity at $t = 2$ seconds and $t = 3$ seconds? What is the average of these two numbers?

- A: 2 m/s, -2 m/s, 2 m/s
- B: 2 m/s, 4 m/s, 3 m/s
- C: 10 m/s, 8 m/s, 9 m/s
- D: 2 m/s, -2 m/s, 0 m/s

Let $x(t) = (10t - 2t^2)\hat{i}$ m, from prior exercise. What is the displacement between $t = 2$ seconds and $t = 3$ seconds?

- A: 0 m
- B: 10 m
- C: -4 m
- D: 3 m

Notice in the previous two problems: the *instantaneous velocity* is not the *average velocity*. The average velocity between two and three seconds was 0 m/s, but the instantaneous velocity was not zero at either point. However, the *displacement* was 0 m in this time interval. The *average velocity* must be

$$\boxed{\bar{v} = \frac{x_\mathrm{f} - x_\mathrm{i}}{t_\mathrm{f} - t_\mathrm{i}}} \tag{18}$$

On February 15, 2013, a meteor entered Earth's atmosphere over Chelyabinsk, Russia, and exploded at an altitude of 23.5 km. Eyewitnesses could feel the intense heat from the fireball, and the blast wave from the explosion blew out windows in buildings. The blast wave took approximately 2 minutes 30 seconds to reach ground level. What was the average velocity of the blast wave? Compare this with the speed of sound, which is 343 m/s at sea level.

- A: 35 m/s (10% speed of sound)
- B: 100 m/s (30% speed of sound)
- C: 150 m/s (40% speed of sound)
- D: 350 m/s (100% speed of sound)

Notice that if we take the limit $t_\mathrm{f} \to t_\mathrm{i}$, or $\Delta t = t_\mathrm{f} - t_\mathrm{i} \to 0$,

$$\lim_{\Delta t \to 0} \bar{v} = \lim_{\Delta t \to 0} \frac{x_\mathrm{f} - x_\mathrm{i}}{t_\mathrm{f} - t_\mathrm{i}} \tag{19}$$

$$\lim_{\Delta t \to 0} \frac{x_\mathrm{f} - x_\mathrm{i}}{\Delta t} = \frac{dx}{dt} = v(t) \tag{20}$$

The limit of the average velocity as the time interval approaches zero is the instantaneous velocity. What about acceleration?

A particle moves along the x-axis according to $x(t) = (10t - 2t^2)\hat{i}$ m. What is the instantaneous acceleration at $t = 2$ seconds and $t = 3$ seconds? What is the average of these two numbers?

- A: 2 $\mathrm{m/s^2}$, 2 $\mathrm{m/s^2}$, 2 $\mathrm{m/s^2}$
- B: -4 m/s, -4 m/s, -4 m/s
- C: -4 $\mathrm{m/s^2}$, -4 $\mathrm{m/s^2}$, -4 $\mathrm{m/s^2}$
- D: 0 $\mathrm{m/s^2}$, 0 $\mathrm{m/s^2}$, 0 $\mathrm{m/s^2}$

Notice that the *average acceleration* and the *instantaneous acceleration* are equal. This implies that the acceleration is constant. Similar to the definition of *average velocity*, we have the *average acceleration*:

$$\bar{a} = \boxed{\frac{v_{\mathrm{f}} - v_{\mathrm{i}}}{t_{\mathrm{f}} - t_{\mathrm{i}}}} \tag{21}$$

A cheetah can accelerate from rest to a speed of 35.0 m/s in 7.00 s. What is its average acceleration, if it's headed in the $-\hat{i}$ direction?

- A: -5 m/s^2
- B: 2 m/s^2
- C: 10 m/s^2
- D: 5 m/s^2

LAB ACTIVITY: FREE-FALL

This is a simple lab exercise to see if we understand the kinematic equations well enough to predict where an object will land. Let's assume we are rolling a marble off the edge of the lab table at some known initial horizontal speed $v_{x,i}$. When it leaves the table, it will begin to accelerate:

$$v_y(t) = v_{i,y} - gt \tag{22}$$

We know that $v_{i,y} = 0$ m/s. Thus, we know the velocity as a function of time because we know $v_{x,i}$ will not change and $v_y(t)$ is known.

Suppose the height of the table is h. Show that

$$h = -\frac{1}{2}gt^2 \tag{23}$$

Use this equation to solve for t. (Remember that the object has $v_{x,i}$, but acceleration is in the y-direction.

The result should be

$$t = \sqrt{\frac{2h}{g}} \tag{24}$$

1. Measure *h* with the ruler, and predict *t*.
2. Drop the marble from the height and confirm the prediction using a stopwatch.
3. If the marble was moving at some constant *horizontal* velocity, $v_{x,i}$, where would it land?
4. Roll the marble next to a ruler that is laid on the table such that you can time it and obtain $v_{x,i}$ as the marble leaves the table. Make sure the ruler is perpendicular to the edge. Mark where the marble lands and measure the horizontal distance from the table (call it Δx).

Using $\Delta x = v_{x,i}t$, find the predicted horizontal distance of landing, Δx. Does it agree with your measurement? Why or why not?

LAB ACTIVITY: MEASURING ACCELERATION OF GRAVITY

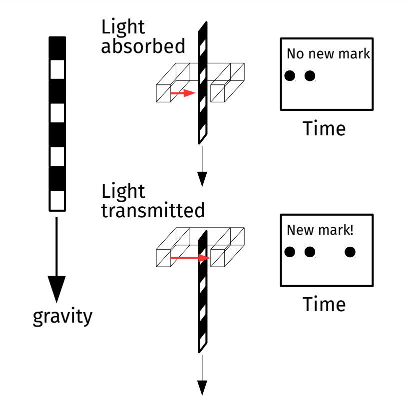


Figure 2: (Left) A *picket* is marked at regular intervals with black strips. (Right) Upon dropping the picket through a *photo-gate*, the strips will block the photo-gate and we will record when this happens on a clock.

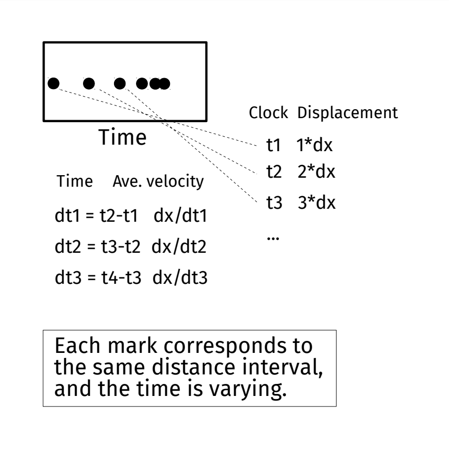


Figure 3: We can measure the velocity versus time of the picket by taking the ratio of *displacements* to *times*.

Acceleration is the change in velocity, so once we have the velocities (dx/dt_{i}), we can take more ratios:

$$t1' = (t2 + t1)/2 \quad dx/dt1 \tag{25}$$
$$t2' = (t3 + t2)/2 \quad dx/dt2 \tag{26}$$
$$t3' = (t4 + t3)/2 \quad dx/dt3 \tag{27}$$
$$\ldots \tag{28}$$

$$t1' \quad \frac{dx/dt2 - dx/dt1}{t2' - t1'} \tag{29}$$
$$t2' \quad \frac{dx/dt3 - dx/dt2}{t3' - t2'} \tag{30}$$
$$t3' \quad \frac{dx/dt4 - dx/dt3}{t4' - t3'} \tag{31}$$
$$\ldots \tag{32}$$

Once we have the acclerations $\frac{dx/dt2-dx/dt1}{t2'-t1'}$, ..., we can compare them with each other and compute the *average* and *standard deviation*.

$$\bar{a}_{\mathrm{meas}} = N^{-1} \sum_{\mathrm{i}}^{N} a_{\mathrm{meas,i}} \tag{33}$$

$$\sigma_{\mathrm{meas}}^2 = N^{-1} \sum_{\mathrm{i}}^{N} (a_{\mathrm{meas,i}} - \bar{a}_{\mathrm{meas}})^2 \tag{34}$$

Quote the result like this: $\bar{a}_{\mathrm{meas}} \pm \sigma_{\mathrm{meas}}$. *The mean plus or minus one standard deviation.* The result of this experiment is $g = \bar{a}_{\mathrm{meas}}$, the acceleration due to gravity near the Earth's surface.

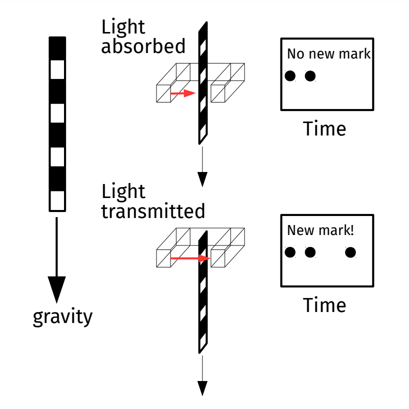


Figure 4: Does adding more mass to the picket change the answers?

EXERCISES WITH VECTORS, GRAPHS, AND EQUATIONS OF MOTION

We have a system of equations describing motion of classical particles undergoing constant acceleration:

$$x = x_0 + \bar{v}t \tag{35}$$

$$\bar{v} = (v + v_0)/2 \tag{36}$$

$$v = v_0 + at \tag{37}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \tag{38}$$

$$v^2 = v_0^2 + 2a(x - x_0) \tag{39}$$

A particle moves in a straight line with an initial velocity of 30 m/s and a constant acceleration of 30 m/s^2. What is the displacement at $t = 5$ seconds? What is the velocity at $t = 5$ seconds?

- A: 900 m, 180 m
- B: 180 m, 525 m/s
- C: 525 m, 180 m/s
- D: 700 m, 200 m/s

A particle is moving at 5 m/s, 60 degrees with respect to the x-axis. At $t = 0$ seconds, it begins to accelerate at 1 m/s^2. What is the speed after 3 seconds?

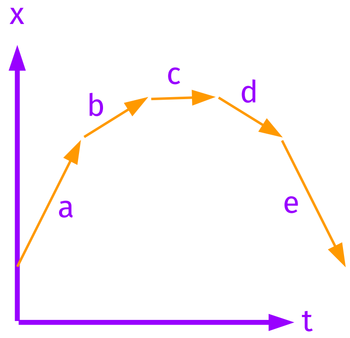
- A: 8 m/s
- B: 4 m/s
- C: -4 m/s
- D: -8 m/s

If the particle is at (0,0) at $t = 0$, where is the particle at $t = 3$ seconds?

- A: (7.5,13) m
- B: (16.9,9.75) m
- C: (9.75,16.9) m
- D: (13,7.5) m

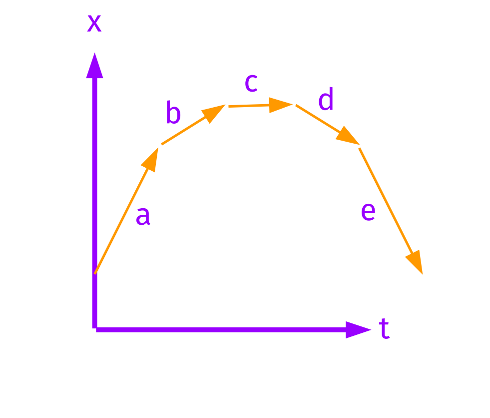
Which segment(s) of the motion described by the plot at right has $v \approx 0$ (m/s)?

- A
- B and D
- C
- E



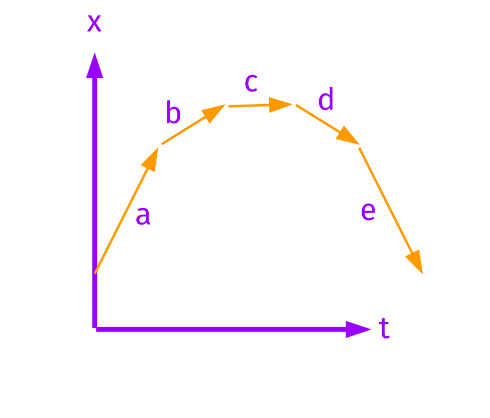
Which segment(s) of the motion described by the plot at right has the largest velocity?

- A
- B
- D
- E



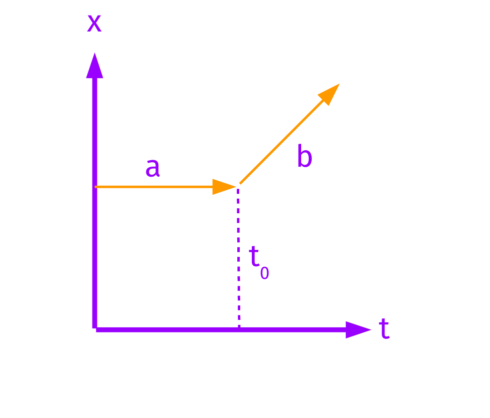
Does the motion described by the plot correspond to negative or positive acceleration?

- Negative
- Positive



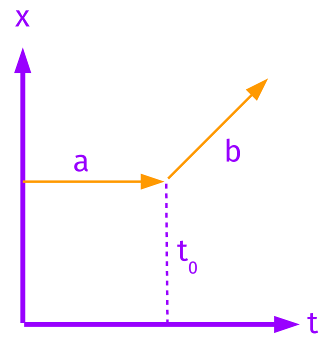
In which region(s) is the acceleration zero m/s^2?

- A only
- B only
- Both A and B
- None



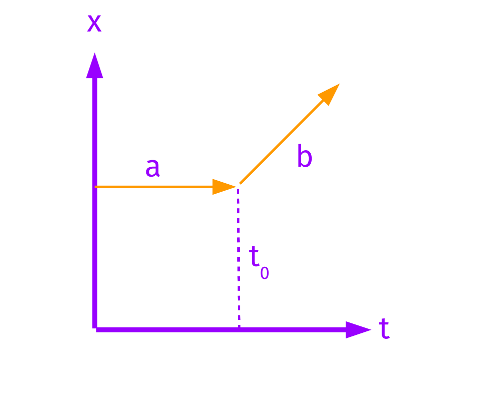
At $t = t_0$, what is the acceleration?

- Negative and large
- Positive and large
- Positive, but small
- Negative, but small



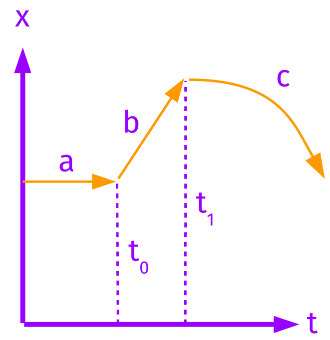
What is the average velocity?

- Less than the slope of region B
- Greater than the slope of region B
- Zero



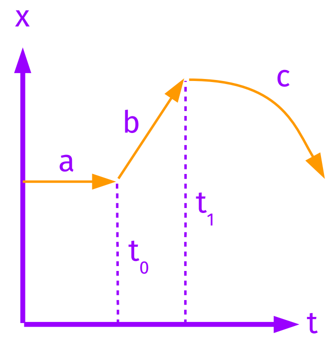
What is the sign of the acceleration at $t = t_1$? What is the sign of the acceleration at $t = t_0$?

- Negative, positive
- Positive, negative
- Positive, positive
- Negative, negative



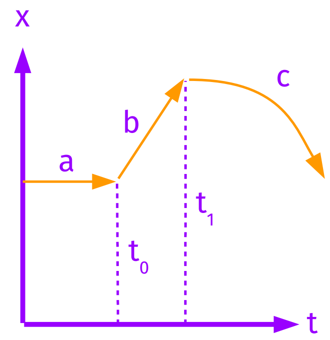
What is most likely the total displacement?

- Positive and large
- Negative and large
- Zero
- Cannot discern from graph



What is most likely the acceleration during segment C?

- Positive and increasing
- Negative and increasing
- Negative and decreasing
- Negative and constant



A particle is moving along the y-axis, with an initial speed of 10 m/s, and an acceleration of -10 m/s^2. What is the displacement when the speed has decreased to 1 m/s?

- A: 3 m
- B: 4 m
- C: 5 m
- D: 6 m

A particle is moving along the y-axis, with an initial speed of 10 m/s, and an acceleration of -10 m/s^2. How much time has elapsed when the particle has a speed of 1 m/s?

- A: 0.9 seconds
- B: 10 seconds
- C: 0.5 seconds
- D: 1.5 seconds

We have a gap in our abilities to solve problems with constant acceleration. We can know how to use *differentiation* to obtain other quantities/equations, including the approximate quantities of *average velocity* and *average acceleration*. What about integration?

$$\int_{t_1}^{t_2} v(t)dt = x(t) \tag{40}$$

Equation 40 follows from the *fundamental theorem of calculus* (more on this later).

A particle is moving along the z-axis, according to $\vec{v}(t) = (v_0 + g_{\mathrm{m}} t)\hat{k}$ (m/s), where $v_0 = 2$ (m/s) at $t = 0$, and $g_{\mathrm{m}} = g/6$, the acceleration due to gravity on the moon (9.8/6 m/s^2). What is the displacement after 2 seconds have elapsed?

- A: 4/3 m
- B: 10/3 m
- C: 1 m
- D: 2/3 m

What about integration?

$$\int_{t_1}^{t_2} a(t)dt = v(t) \tag{41}$$

Equation 41 follows from the *fundamental theorem of calculus* (more on this later).

At $t = 0$, a particle has an acceleration $\vec{a}(t) = -(3\hat{i} + 4\hat{j})$ m/s^2. What is the velocity after 3 seconds?

- A: $(-3\hat{i} - 4\hat{j})$ m/s
- B: $(3\hat{i} + 4\hat{j})$ m/s
- C: $(-9\hat{i} - 12\hat{j})$ m/s
- D: $(-12\hat{i} - 9\hat{j})$ m/s

CONCLUSION

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