Calculus-Based Physics-1: Mechanics (PHYS150-02): Unit 0

Jordan Hanson

September 2, 2022

Whittier College Department of Physics and Astronomy

Course Introduction

1. Professor Jordan Hanson
2. Contact: jhanson2@whittier.edu, SLC 212
3. Syllabus: Moodle
4. Office hours: Discord 918particle#5083
5. Course pre-requisites: Calculus 1 (concurrently or previously, high-school equivalent).
6. Text: University Physics Volume 1 (see syllabus)
7. Homework: OpenStax Tutor (see syllabus)

Opening Remarks - Welcome!

Figure 1: The usual look from a student taking physics for the first time.

Summary

Week 1 Summary

Physics - $\phi\upsilon\sigma\iota\kappa\acute{\eta}$ - "phusiké": *knowledge of nature*
from $\phi\acute{\upsilon}\sigma\iota\varsigma$ - "phúsis": *nature*

1. Methods of approximation
    - Estimating the correct order of magnitude
    - Function approximation
    - Unit analysis
2. Coordinates and vectors
    - Scalars and vectors
    - Cartesian (rectangular) coordinates, displacement
    - Vector addition, subtraction, and multiplication
3. Review of Calculus Techniques
    - Differentiation of power series.
    - Integration of power series.

Methods of approximation

Methods of approximation - Estimation: Chapter 1

In science and engineering, estimation is to obtain a quantity in the absence of precision, informed by rational constraints.

1. Define relevant scales
    - 1 *AU* for the solar system (distance from Sun to Earth)
    - 1 *angstrom* (10^{-10} meters) for cell membranes
2. Obtain complex quantities from simple ones
    - Obtain *areas* and *volumes* from *lengths*
    - Obtain *rates* from *numerators* and *denominators*
3. Constrain the unknown with upper and lower limits
    - The solar system is *less than one light-year across*
    - An insect is *at least one millimeter* long

Professor: work several examples.

Estimate the mass of ants in an ant colony. Assume that the colony is a species known to have 10^5 ants (roughly) per colony.

- A: 0.01 kg
- B: 0.1 kg
- C: 1 kg
- D: 10 kg

An adult blue whale is about 30 meters long. What is the mass of a blue whale calf? (1 tonne = 1000 kg).

- A: 300 kg
- B: 0.3 tonnes
- C: 3 tonnes
- D: 50 tonnes

How long does it take an airliner to fly across the Atlantic ocean? Assume the velocity is 500 mph, and the radius of the Earth is 7000 km.

- A: 10 hours
- B: 15 hours
- C: 2 hours
- D: 4 hours

A person has 150 kg of mass and is about 1.5 meters tall. What is his density (mass divided by volume)?

- A: 100 kg/m^3
- B: 400 kg/m^3
- C: 1000 kg/m^3
- D: 10^4 kg/m^3

A jar of coffee beans sits on a counter at a cafe. If the jar is about 10 cm tall, and has a radius of about 5 cm, estimate the number of beans inside.

- A: 300 beans
- B: 3000 beans
- C: 30,000 beans
- D: 30 beans

What is the volume of a basketball?

- A: 700 cm^3
- B: 7000 cm^3
- C: 700 cm^2
- D: 7000 cm^2

In science and engineering, function approximation or expanding a function is a technique in which a simple function is used obtain the value of a more complicated function near a point where they are approximately equal.

1. Memorizing special cases
    - $\sin(x) \approx x$, when $|x| < 1$
    - $\tan(x) \approx x$, when $|x| < 1$
    - $(1 + x)^{1/2} \approx 1 + \frac{1}{2}x$, when $|x| < 1$
    - $\exp(x) \approx 1 + x$, when $|x| < 1$
2. Utilizing the **Taylor Series** (more on this later)
    - $f(x) \approx f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots$

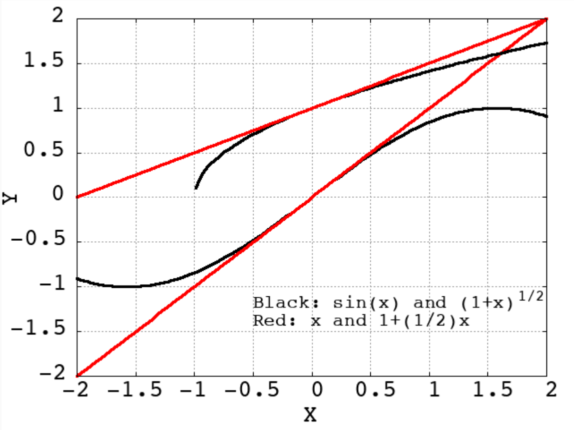


Figure 2: Certain functions may be approximated by simpler ones. In this case, $\sin(x)$ is approximated by x near $x = 0$, and $(1 + x)^{1/2}$ is approximated by $1 + \frac{1}{2}x$ near $x = 0$.

The height in meters of a surfer above some average height as he bobs in the waves is described by $h(t) = sin(\pi t/2)$. What is his height at 0.2 seconds? What is his height at -0.2 seconds?

- A: 10 meters, -10 meters
- B: $\pi/10$ meters, -$\pi/10$ meters
- C: -0.1 meters, 0.1 meters
- D: -π meters, π meters

The value of an investment in dollars, v, versus time in years, t, follows the form $v(t) = P\exp(rt)$, where P is the value at $t = 0$, and $r = 1/3$. What is $v(1)$, the value after one year?

- A: $\approx 1/3P$
- B: $\approx 2/3P$
- C: $\approx 3/2P$
- D: $\approx 4/3P$

Physics requires units to relate ideas to the real world, and unit analysis is a powerful tool to eliminate incorrect results and to facilitate estimation.

1. SI units, and kilogram-meter-second unit set
    - mass: kilogram (gram = 10^{-3} kg, milligram = 10^{-6} kg)
    - length: meter (millimeter = 10^{-3} m, kilometer = 10^{3} m)
    - time: second (1 year $\approx \pi \times 10^{7}$ sec, 1 hour = 3600 sec)
2. Unit analysis
    - If we are calculating a density, the units should work out to be kg/m^3
    - Identifying the fundamental unit in a complex calculation often simplifies it (when done properly, this reveals the beauty of physics)

Professor: work several examples.

A millenium is 1000 years. If a glacier retreats at a pace of 10 cm per year, what is this rate in meters per millenium?

- A: 0.1 meter per millenium
- B: 1 meter per millenium
- C: 10 meters per millenium
- D: 100 meters per millenium

Ice has a density of 0.917 grams per centimeter cubed. What is this density in kilograms per meter cubed?

- A: 91.7 kg/m^3
- B: 917 kg/m^3
- C: 9170 kg/m^3
- D: 9.17 kg/m^3

Sometimes, the beauty of physics arrises from choosing the right unit.

`http://joshworth.com/dev/pixelspace/pixelspace_solarsystem.html`

The Sun in this ruler is at 0 km, and Jupiter is at about 780,000,000 km (good luck finding it). Clearly, the kilometer is the wrong unit to choose for interplanetary distances. What if we defined a new unit, the astronomical unit, as the distance between the Earth and the Sun?

Planetary orbital radii in AU (geometric means):

Mercury	0.379
Venus	0.722
Earth	1.00
Mars	1.52
Jupiter	5.20
Saturn	9.54
Uranus	19.2
Neptune	30.1

Figure 3: Why such simple numbers? There is a set of simple relationships between the *orbital period* and the *orbital radius* of planets called Kepler's Laws, which led to the discovery of Newton's Law of Gravity.

Coordinates and Vectors

Introduction of the problem, and group activity:
Navigation in the film The Hunt for Red October.
`https://youtu.be/4unk6siO-tI`

Physics requires mathematical objects to build equations that capture the behavior of nature. Two examples of such objects are scalar and vector quantities. Each type of object obeys similar but different rules.

1. Scalar quantities
   - mass: $m_1 + (m_2 + m_3) = (m_1 + m_2) + m_3$
   - speed: $v_1(v_2 + v_3) = v_1 v_2 + v_1 v_3$
   - charge: $q_1 \left(\frac{1}{q_1}\right) = 1$, $q_1(0) = 0$
2. Vector quantities
   - velocity: $\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = (\vec{v}_1 + \vec{v}_2) + \vec{v}_3$
   - tension: $\vec{t}_1 \cdot (\vec{t}_2 + \vec{t}_3) = \vec{t}_1 \cdot \vec{t}_2 + \vec{t}_1 \cdot \vec{t}_3$

A vector may be expressed as *a list of scalars*: $\vec{v} = (4, 2)$ (a vector with two *components*), $\vec{u} = (3, 4, 5)$ (three *components*). Now, we know how to add and subtract scalars. How do we add and subtract vectors?

What is
$(1, 3, 8)+$
$(0, 2, 1)$?
Answer: $(1, 5, 9)$

In other words, when adding vectors, we add them component by component.

How do we subtract vectors? In the same fashion:

What is
$(1, 3, 8)-$
$(0, 2, 1)$?
Answer: $(1, 1, 7)$

In other words, when subtracting vectors, we subtract them component by component.

A HTML-based demonstration for adding vectors:

`https://phet.colorado.edu/en/simulations/vector-addition`

Notice several things:

- Produce vectors that *cancel* each other.
- What happens when vectors are parallel and orthogonal?

How do we multiply vectors? In the same fashion, *for one kind of multiplication*:

What is
$(1, 3, 8) \cdot (0, 2, 1)$?
Answer: $1 \cdot 0 + 3 \cdot 2 + 8 \cdot 1 = 14$

This kind of multiplication is known as the dot-product. There is also the *cross-product*, which we will save for later.

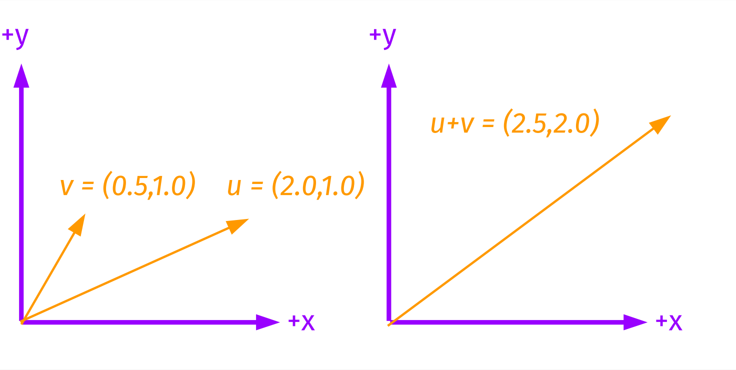
Coordinates and Vectors - Coordinates (Chapters 2.1-2.3)

The components of a vector may describe quantities in a coordinate system, such as *Cartesian coordinates* - after René Descartes. Vectors in the 3D Cartesian coordinate system (x,y,z) may be written in the following notation:

$$\boxed{\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}}$$

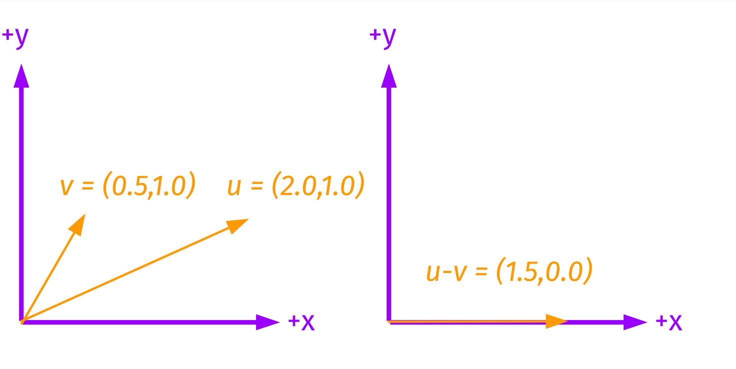
- a: The amount in the +x-direction, $\hat{i}$: a vector of length 1, in the +x-direction
- b: The amount in the +y-direction, $\hat{j}$: a vector of length 1, in the +y-direction
- c: The amount in the +z-direction, $\hat{k}$: a vector of length 1, in the +z-direction

Coordinates and Vectors - Vectors (Chapters 2.1-2.3)



(a) (b)

Figure 4: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) What is $\vec{u} + \vec{v}$? Adding components: $\vec{u} + \vec{v} = 2.5\hat{i} + 2.0\hat{j}$.



(a) (b)

Figure 5: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) What is $\vec{u} - \vec{v}$? Subtracting components: $\vec{u} - \vec{v} = 1.5\hat{i} + 0.0\hat{j}$.

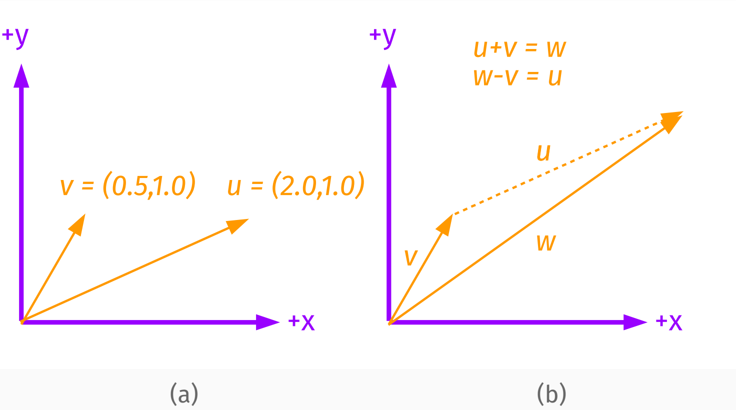


Figure 6: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) To compute $\vec{w} - \vec{v}$, arrange the vectors to get a sense of the result, $\vec{u}$.

$\vec{p} = 4\hat{i} + 2\hat{j}$. $\vec{q} = -4\hat{i} + 2\hat{j}$.
Compute $\vec{p} \cdot \vec{q}$.

- A: 12
- B: -12
- C: 4
- D: 8

$\vec{p} = -1\hat{i} + 6\hat{j}$. $\vec{q} = 3\hat{i} + 0.5\hat{j}$.
Compute $\vec{p} \cdot \vec{q}$.

- A: -1
- B: 1
- C: 0
- D: 3

Why was the last answer zero? Look at it graphically:

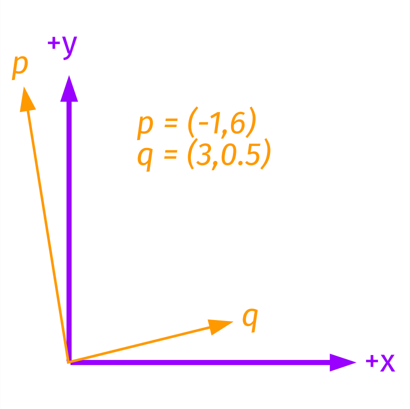


Figure 7: Two vectors $\vec{p}$ and $\vec{q}$ are *orthogonal* if $\vec{p} \cdot \vec{q} = 0$.

What if the vectors are parallel? Look at it graphically:

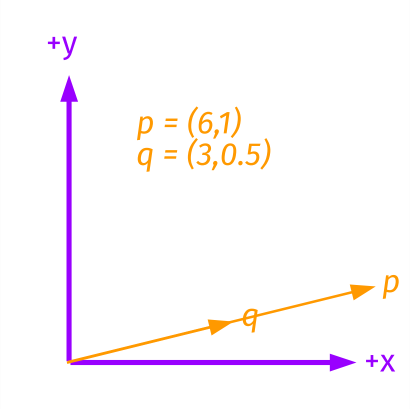


Figure 8: Two vectors $\vec{p}$ and $\vec{q}$ are *parallel* if $\vec{p} \cdot \vec{q}$ is maximal.

The *length* or *norm* of a vector $\vec{v} = a\hat{i} + b\hat{j}$ is $|\vec{v}| = \sqrt{a^2 + b^2}$.

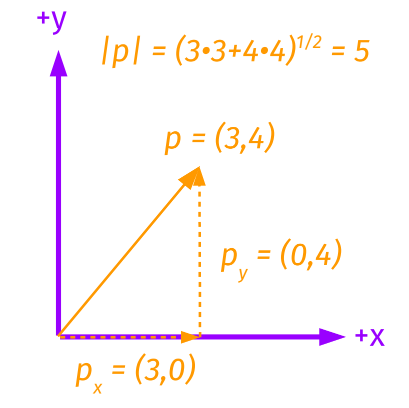


Figure 9: Computing the norm of a vector $\vec{p}$.

Notice that $\sqrt{\vec{p}\cdot\vec{p}} = |\vec{p}|$.
Let θ_p be the angle between $\vec{p}$ and the x-axis.
$p_x = \vec{p}\cdot\hat{i} = |\vec{p}|\cos(\theta_p)$.
$p_y = \vec{p}\cdot\hat{j} = |\vec{p}|\sin(\theta_p)$.

Theorem: The dot product of two vectors $\vec{p}$ and $\vec{q}$ is $|u||v|\cos(\theta)$, if θ is the angle between them.

Proof: $\vec{p}\cdot\vec{q} = p_x q_x + p_y q_y = |p||q|\cos\theta_p\cos\theta_q + |p||q|\sin\theta_q\sin\theta_q$
$= |p||q|(\cos\theta_p\cos\theta_q + \sin\theta_p\sin\theta_q) = |p||q|\cos(\theta_p - \theta_q)$
$= |p||q|\cos\theta$.
$\boxed{\vec{p}\cdot\vec{q} = |p||q|\cos\theta}$

An object moves at 2 m/s at $\theta = 60°$ with respect to the x-axis. What is the velocity of the object?

- A: $(1\hat{i} + 1\hat{j})$ m/s
- B: $(\sqrt{3}\hat{i} + 1\hat{j})$ m/s
- C: $(\sqrt{3}\hat{i} + \sqrt{3}\hat{j})$ m/s
- D: $(1\hat{i} + \sqrt{3}\hat{j})$ m/s

What is the dot product of this velocity with another velocity: 5 m/s along the x-axis?

- A: 1 $(m/s)^2$
- B: 5 $(m/s)^2$
- C: 10 $(m/s)^2$
- D: 5 (m/s)

Is it possible to multiply vectors and scalars? Of course: $a_1\vec{p} = a_1 p_x\hat{i} + a_1 p_y\hat{j}$.

Also, multiplication properties still hold. For example: $(a_1 + a_2)\vec{p} = a_1\vec{p} + a_2\vec{p}$.

A spacecraft moves at 400 m/s, at an angle of 30 degrees with respect to the x-axis. If it fires two thrusters that boost the x-component and y-component of the velocity by 25% and 50%, respectively, what is the final velocity?

- A: $(433\hat{i} + 300\hat{j})$ m/s
- B: $(300\hat{i} + 433\hat{j})$ m/s
- C: 400 m/s
- D: $(400\hat{i} + 433\hat{j})$ m/s

We define the *position* of an object as a vector locating it in a given coordinate system. The scalar *distance* is the norm of the position vector, that is, the distance to to the origin.

Now we can introduce the concept of dislacement: a vector describing a movement of an object.

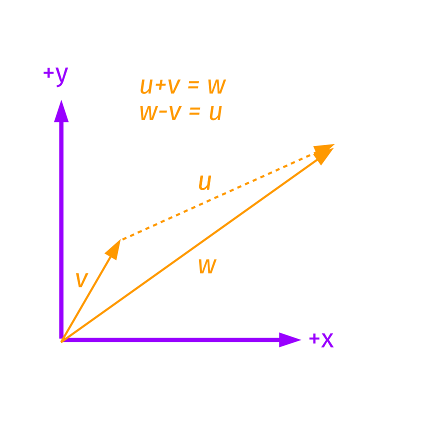


Figure 10: Suppose an object moves from position $\vec{v}$ to $\vec{w}$. In this case, the displacement is $\vec{u}$. **Thus, the final position is the initial position, plus the displacement.**

It follows that the *displacement* is zero if the initial and final positions are the same, but the *distance travelled* is not.

Suppose a jet fighter travelling at 800 km per hour banks such that it flies in a circle of radius 0.5 km. How long does it take to complete the circle? What is the distance traveled, and what is the displacement?

- A: 2π km, 28 seconds, 2π km
- B: π km, 14 seconds, π km
- C: π km, 28 seconds, π km
- D: π km, 14 seconds, 0 km

Conclusion of Chapter 2

For a list of helpful equations and mathematical definitions, see Key Equations section of Chapter 2.

Average velocity is the ratio of the displacement to the elapsed time.

$$\boxed{\vec{v}_{\mathrm{avg}} = \frac{\Delta\vec{x}}{\Delta t}} \tag{1}$$

The *average speed* is the norm of the average velocity:

$$\boxed{v_{\mathrm{avg}} = \frac{|\Delta\vec{x}|}{\Delta t}} \tag{2}$$

If the motion is in one dimension, then the average speed is

$$v_{\mathrm{avg}} = \frac{x_{\mathrm{f}} - x_{\mathrm{i}}}{t_{\mathrm{f}} - t_{\mathrm{i}}} \tag{3}$$

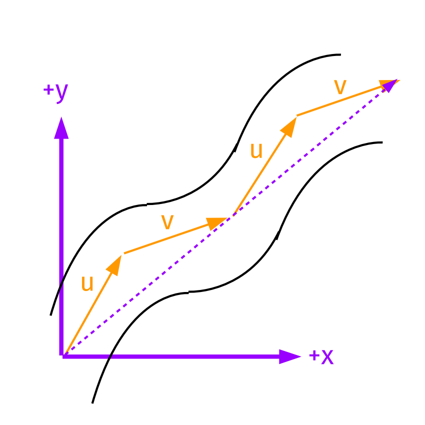


Figure 11: A Formula-1 driver keeps his car on the track by following a path approximated by the position vectors *u*, *v*, *u*, and *v*. The dashed arrow represents the total displacement.

If $\vec{u} = (20\hat{i} + 30\hat{j})$ m, and $\vec{v} = (30\hat{i} + 20\hat{j})$ m, what is the total displacement? If the elapsed time is 10 seconds, what is the average velocity?

- A: $(50\hat{i} + 50\hat{j})$ m, 14 m/s
- B: $(80\hat{i} + 100\hat{j})$ m, 10 m/s
- C: $(100\hat{i} + 100\hat{j})$ m, 14 m/s
- D: $(50\hat{i} + 150\hat{j})$ m, 10 m/s

Review of Calculus Techniques

Taking a limit has a variety of uses in physics. Consider the following example:

The weight of an object sitting on the Earth's surface is given by the following equation, which we will encounter later in physics:

$$w = G\frac{mM}{r^2} \tag{4}$$

- w: weight
- G: a constant of nature
- m: the object's mass
- M: the mass of the Earth
- r: the distance between the center of the Earth and the object

The weight of an object sitting on the Earth's surface is given by the following equation, which we will encounter later in physics:

$$w = G\frac{mM}{r^2} \tag{5}$$

Let R be the radius of the Earth, r_0 be the object's height above the Earth's surface, and $\epsilon = r_0/R \ll 1$. Also, let $g = GM/R^2$. Rearranging the equation for weight:

$$w = mg(1+\epsilon)^{-2} \tag{6}$$

Since $\epsilon \ll 1$, take the limit:

$$\lim_{\epsilon \to 0} w = \lim_{\epsilon \to 0} mg(1+\epsilon)^{-2} = mg \tag{7}$$

Thus, for practical calculations where the object is not far from the Earth's surface, the weight is $w = mg$, or the mass times some measurable constant, g.

Often in other branches of physics, we often encounter the *sinc function*:

$$s(t) = \frac{sin(t)}{t} \tag{8}$$

What is $\lim_{t\to 0} s(t)$? It looks like $0/0$, but that is not definited mathematically.

Review of Calculus - Skill 1: Taking Limits

We can use L'Hôpital's Rule: take the *derivative* of the numerator and the denominator, then take the limit:

$$\lim_{t \to 0} s(t) = \lim_{t \to 0} \left| \frac{\frac{d}{dt}\sin(t)}{\frac{d}{dt}t} \right| = \lim_{t \to 0} \cos(t) = 1 \tag{9}$$

This approach is valid for cases like $0/0$ or ∞/∞. But what is the derivative? How do we know that the derivative of $\sin(t)$ is $\cos(t)$, and the derivative of t is one? The definition of the derivative:

$$f'(t) = \dot{f} = \frac{df}{dt} \equiv \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \tag{10}$$

"The change in y over the change in x at h."

1. $f(t) = t^n$
2. $f(t) = \exp(t)$
3. $f(t) = \ln(t)$
4. $f(t) = \sin(t)$
5. $f(t) = \cos(t)$
6. $f(t) = \tan(t)$
7. $f(t) = a$

1. $f' = nt^{n-1}$
2. $f' = \exp(t)$
3. $f' = 1/t$
4. $f' = \cos(t)$
5. $f' = -\sin(t)$
6. $f' = \sec^2(x)$
7. $f' = 0$

Review of Calculus - Skill 2: Derivative Properties

1. $g(t) = af(t)$
2. $g(t) = a(t) + b(t)$
3. $g(t) = a(t)b(t)$
4. $g(t) = a(t)/b(t)$
5. $g(t) = a(b(t))$

1. $g' = af'$
2. $g' = a' + b'$
3. $g' = ab' + a'b$
4. $g' = \frac{ba' - ab'}{b^2}$
5. $g' = a'(b)b'$

An *anti-derivative* just reverses the action of the derivative. Also called an indefinite integral. For example:

$$x(t) = at^n \tag{11}$$

$$x' = nat^{n-1} \tag{12}$$

$$\int nat^{n-1}dt = at^n + C \tag{13}$$

In Eq. 13, the $\int$ symbol represents integration (a big S for "summation"). There is a constant C because, technically, if we take the derivative of Eq. 13, the result is Eq. 12 (derivative of a constant C is zero).

An integral is the difference in the value of the anti-derivative evaluated at two points:

$$\int_c^d nat^{n-1}dt = a(d)^n - a(c)^n \tag{14}$$

"Raise the power by one, and divide the whole thing by that number." - Just like derivatives, there are many simple cases to memorize.

$$\int_a^b \cos(t)dt = \sin(b) - \sin(a) \tag{15}$$

Review of Calculus - Skill 3: Anti-derivatives and integrals

1. $f(t) = t^n$
2. $f(t) = \exp(t)$
3. $f(t) = 1/t$
4. $f(t) = \sin(t)$
5. $f(t) = \cos(t)$
6. $f(t) = a$
7. ...

1. $\int f(t)dt = (n+1)^{-1}t^{n+1} + C$
2. $\int f(t)dt = \exp(t) + C$
3. $\int f(t)dt = \ln(t) + C$
4. $\int \sin(t)dt = -\cos(t) + C$
5. $\int \cos(t)dt = \sin(t) + C$
6. $\int f(t) = at + C$
7. ...

A Few Calculus Problems

Let $x(t) = 5t^2 - 2t + 7$ in the $\hat{x}$-direction, where x is in meters and t is in seconds. What is the velocity (take the derivative) at $t = 1$ second?

- A: 8 m/s in the $\hat{y}$-direction
- B: 6 m/s in the $\hat{x}$-direction
- C: 8 m/s in the $\hat{x}$-direction
- D: 6 m/s in the $\hat{y}$-direction

Let $v(t) = 2t + 2$ in the $\hat{x}$-direction. What is the position versus time (take the integral)?

- A: $t^2 + 2t + C$
- B: $t^2 + 2t$
- C: $t + 2 + C$
- D: $t^3 2t^2 + C$

Conclusion

Week 1 Summary

1. Methods of approximation
    - Estimating the correct order of magnitude
    - Function approximation
    - Unit analysis
2. Coordinates and vectors
    - Scalars and vectors
    - Cartesian (rectangular) coordinates, displacement
    - Vector addition, subtraction, and multiplication
3. Review of Calculus Techniques
    - Limits
    - Differentiation
    - Integration