CALCULUS-BASED PHYSICS-1: MECHANICS (PHYS150-01): WEEK 3

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WEEK 2 REVIEW

1. Displacement, and instantaneous velocity and acceleration
    - *Mathematics review*: taking derivatives
    - Average velocity and average acceleration
2. The case of constant acceleration
    - An *an equation of motion* for constant acceleration
    - Derivation of common equations of motion
    - Average quantities and exercises
3. **Lab Activity: Measuring acceleration of gravity: g**
4. Exercises with vectors, graphs, and equations of motion

WEEK 2 REVIEW PROBLEMS

If a subway train is moving to the left (has a negative velocity) and then comes to a stop, what is the direction of its acceleration? Is the acceleration positive or negative?

- A: To the right, positive
- B: To the right, negative
- C: To the left, positive
- D: To the left, negative

An object that is thrown straight up falls back to Earth. When is its velocity zero? Does its velocity change direction? Does the acceleration change sign?

- A: During flight, yes, no
- B: At the peak height, yes, yes
- C: At the peak height, yes, no
- D: During flight, no, no

WEEK 3 SUMMARY

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1. Displacement, velocity and acceleration vectors as functions of time
    - Breaking into components
    - Derivatives of components
2. Combining free-fall and vector components: projectile motion
    - The independence of velocity components
    - **Lab-activity: testing component independence**
3. Relative motion and reference frames
    - Relative motion in one-dimension
    - Relative motion in two-dimensions

VECTORS AS FUNCTIONS OF TIME

In general, the displacement of an object depends on time:

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \tag{1}$$

- $x(t)$ is the displacement in the x-direction
- $y(t)$ is the displacement in the y-direction
- $z(t)$ is the displacement in the z-direction

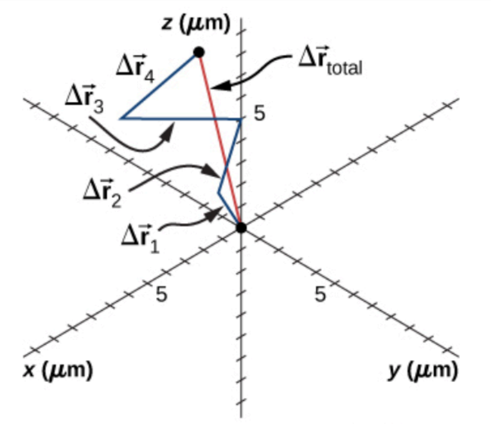

Figure 1: An example of a displacement vector at different moments in time.

The particle in Fig. 1 has four displacement vectors at four moments in time:

- $\vec{r}_1 = 2.0\hat{i} + 1.0\hat{j} + 3.0\hat{k} \quad (\mu m)$ at t_1
- $\vec{r}_2 = -1.0\hat{i} + 0.0\hat{j} + 3.0\hat{k} \quad (\mu m)$ at t_2
- $\vec{r}_3 = 4.0\hat{i} + -2.0\hat{j} + 1.0\hat{k} \quad (\mu m)$ at t_3
- $\vec{r}_4 = -3.0\hat{i} + 1.0\hat{j} + 2.0\hat{k} \quad (\mu m)$ at t_4

What is the total displacement of the particle from the origin?

We can think of this type of problem as an accounting problem, lining up columns (units: μm):

t_{i}	$\vec{r}_{\mathrm{i}}(t_{\mathrm{i}})$	$x(t_{\mathrm{i}})$	$y(t_{\mathrm{i}})$	$y(t_{\mathrm{i}})$
t_1	$\vec{r}_1(t_1)$	2.0	1.0	3.0
t_2	$\vec{r}_2(t_2)$	-1.0	0.0	3.0
t_3	$\vec{r}_3(t_3)$	4.0	-2.0	1.0
t_4	$\vec{r}_4(t_4)$	-3.0	1.0	2.0
t_{total}	$\vec{r}_{\mathrm{total}}(t_{\mathrm{total}})$	2.0	0.0	9.0

Figure 2: Accounting for the different displacement components, in units of μm.

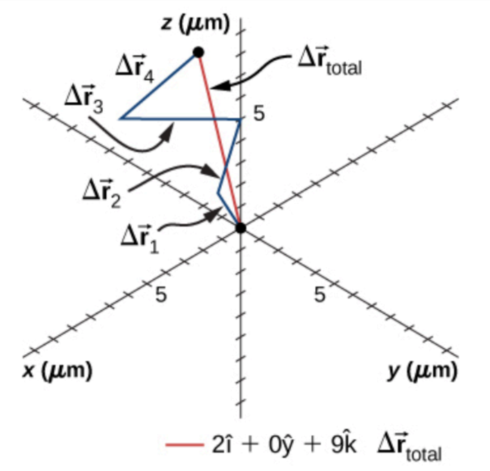

Figure 3: The total displacement of the particle is $\vec{r}_{\text{total}} = 2.0\hat{i} + 0.0\hat{k} + 9.0\hat{k} \quad (\mu m)$.

The 18th hole at Pebble Beach Golf Course is a dogleg to the left of length 496.0 meters. The fairway off the tee is taken to be the x direction. A golfer hits his tee shot a distance of 300 meters, corresponding to a displacement of $\vec{r}_1 = 300.0\hat{i} \quad (m)$, and then hits a second shot 189.0 meters with $\vec{r}_2 = 172.0\hat{i} + 80.3\hat{j} \quad m$. What is the final displacement from the tee?

- A: $\vec{r}_{\text{final}} = 172.0\hat{i} + 80.3\hat{j} \quad (m)$
- B: $\vec{r}_{\text{final}} = 172.0\hat{i} + 380.3\hat{j} \quad (m)$
- C: $\vec{r}_{\text{final}} = 472.0\hat{i} + 0.0\hat{j} \quad (m)$
- D: $\vec{r}_{\text{final}} = 472.0\hat{i} + 80.3\hat{j} \quad (m)$

If the first shot takes 5.0 seconds, the second shot takes 4.0 seconds, and the walking time in between the shots is 60.0 seconds, what is the average velocity vector for the ball after the two shots?

- A: $\vec{r}_{\text{final}} = 1.7\hat{i} + 8.3\hat{j} \quad (m/s)$
- B: $\vec{v}_{\text{final}} = 172.0\hat{i} + 80.3\hat{j} \quad (m/s)$
- C: $\vec{v}_{\text{final}} = 6.8\hat{i} + 1.2\hat{j} \quad (m)$
- D: $\vec{v}_{\text{final}} = 6.8\hat{i} + 1.2\hat{j} \quad (m/s)$

The prior problem indicates something you may already have guessed:

$$\vec{v}_{\mathrm{avg}}(t) = v_{\mathrm{x}}(t)\hat{i} + v_{\mathrm{y}}(t)\hat{j} + v_{\mathrm{z}}(t)\hat{k} = \frac{\Delta\vec{r}}{\Delta t} \qquad (2)$$

- $v_{\mathrm{x}}(t)$ is the avg. velocity in the x-direction
- $v_{\mathrm{y}}(t)$ is the avg. velocity in the y-direction
- $v_{\mathrm{z}}(t)$ is the avg. velocity in the z-direction

In other words, we divide each displacement component by the time, to get a vector where each component is the average velocity in that direction. $\Delta\vec{r} = \vec{r}_{\mathrm{f}} - \vec{r}_{\mathrm{i}}$.

Instantaneously, Eq. 2 is true, if we take the limit $\Delta t \rightarrow 0$:

$$\vec{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \tag{3}$$

- $\frac{dx}{dt}$ is the instantaneous velocity in the x-direction
- $\frac{dy}{dt}$ is the instantaneous velocity in the y-direction
- $\frac{dz}{dt}$ is the instantaneous velocity in the z-direction

The position of a particle is $\vec{r}(t) = 4.0t^2\hat{i} - 3.0\hat{j} + 2.0t^2\hat{k}$ (m). What is the velocity vector at $t = 2$ seconds? What is the average velocity between $t = 0$ and $t = 2$ seconds?

- A: $16\hat{x} + 8\hat{z}$ (m/s), $8\hat{x} + 4\hat{z}$ (m/s)
- B: $8\hat{x} + 4\hat{z}$ (m/s), $4\hat{x} + 2\hat{z}$ (m/s)
- C: $8\hat{x} + 8\hat{z}$ (m/s), $4\hat{x} + 4\hat{z}$ (m/s)
- D: $4\hat{x} + 2\hat{z}$ (m/s), $4\hat{x} + 2\hat{z}$ (m/s)

Instantaneously, from Eq. 3:

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{dv_{\mathrm{x}}}{dt}\hat{i} + \frac{dv_{\mathrm{y}}}{dt}\hat{j} + \frac{dv_{\mathrm{z}}}{dt}\hat{k} \tag{4}$$

- $\frac{dv_{\mathrm{x}}}{dt}$ is the instantaneous acceleration in the x-direction
- $\frac{dv_{\mathrm{y}}}{dt}$ is the instantaneous acceleration in the y-direction
- $\frac{dv_{\mathrm{z}}}{dt}$ is the instantaneous acceleration in the z-direction

The velocity of a particle is $\vec{v}(t) = 8.0t\hat{i} + 4.0t\hat{k}$ (m/s). What is the acceleration vector at $t = 2$ seconds? What is the average acceleration between $t = 0$ and $t = 2$ seconds?

- A: $4\hat{i} + 4\hat{k}$ (m/s^2), $2\hat{i} + 2\hat{k}$ (m/s^2)
- B: $8\hat{i} + 4\hat{k}$ (m/s^2), $8\hat{i} + 4\hat{k}$ (m/s^2)
- C: $8\hat{i} + 8\hat{k}$ (m/s^2), $4\hat{i} + 4\hat{k}$ (m/s^2)
- D: $4\hat{i} + 8\hat{k}$ (m/s^2), $2\hat{i} + 4\hat{k}$ (m/s^2)

The displacement of a particle is $\vec{x}(t) = (2t+3)\hat{i} + (\frac{3}{2}t^2 + 2t + 3.0)\hat{j}$ (m). What is the horizontal velocity (the $\hat{i}$-component of the velocity) at $t = 4$ seconds? At $t = 10$ seconds?

- A: 4 m/s, 4 m/s
- B: 2 m/s, 4 m/s
- C: 2 m/s, 2 m/s
- D: 4 m/s, 2 m/s

The displacement of a particle is $\vec{x}(t) = (2t+3)\hat{i} + (\frac{3}{2}t^2 + 2t + 3.0)\hat{j}$ (m). What is the vertical velocity (the $\hat{j}$-component of the velocity) at $t = 4$ seconds? At $t = 10$ seconds?

- A: 14 m/s, 32 m/s
- B: 32 m/s, 14 m/s
- C: 12 m/s, 30 m/s
- D: 30 m/s, 12 m/s

Notice in the previous example, the x-velocity and y-velocity were not the same function.

In the kinematic description of motion, *we are able to treat the different components of motion separately.* In many cases, motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

Motions in displacement components are independent.

(Exception: non-conservative forces. More on this later.)

COMBINING FREE-FALL AND VECTOR COMPONENTS: PROJECTILE MOTION

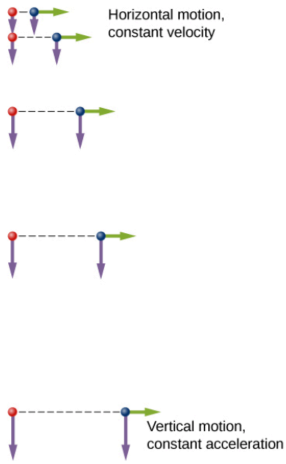

Figure 4: The red particle accelerates vertically, with no horizontal velocity. The blue particle acclerates vertically, with some horizontal velocity.

LAB ACTIVITY

Is this true? Figure 4 is testable by experiment.

Procedure:

1. Obtain two marbles, a meter stick, and a stopwatch.
2. Measure the height of the lab bench, Δx.
3. We are going to drop a marble from this height (Δx) and record the time. Show first algebraically that the predicted time for the marble to fall is $t = \sqrt{2\Delta x/g}$.
4. Measure t for several trials. Does it match the expected result $\sqrt{2\Delta x/g}$? What are sources of error?
5. Repeat the measurement, but **roll the marble off of the table instead of dropping it** from Δx. Does the average result for t change?

We now have learned that (a) motions in displacement components are *independent*, and (b) when acceleration is in one direction (vertical) only, the motion is *projectile motion*. Our usual equations of motion for no acceleration (horizontal), and constant acceleration (vertical) apply *independently*:

$$y(t) = y_0 + v_{0,\mathrm{y}}t - \frac{1}{2}gt^2 \quad (5)$$

$$v_{\mathrm{y}}(t) = -gt + v_{0,\mathrm{y}} \quad (6)$$

$$v_{\mathrm{y}}^2 = v_{\mathrm{y},0}^2 - 2g(y - y_0) \quad (7)$$

$$x(t) = x_0 + v_{0,\mathrm{x}}t \quad (8)$$

$$v_{\mathrm{x}}(t) = v_{0,\mathrm{x}} \quad (9)$$

Projectile motion is a good topic to introduce the concept of *boundary conditions*. The *physics* of projectile motion is the same for all situations, but the *individual cases and numbers* might not be the same.

Suppose we are given the initial velocity and angle of a object that undergoes projectile motion. To use Eqs. 5-9, we need $v_{0,\mathrm{x}}$ and $v_{0,\mathrm{y}}$, the initial horizontal and vertical velocity components, respectively.

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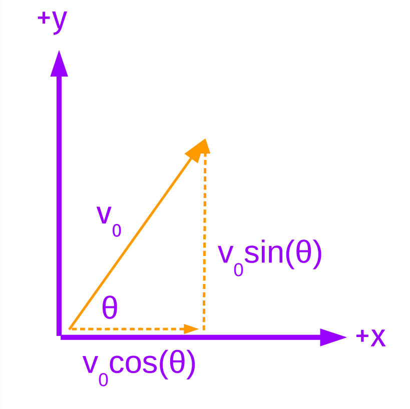

Figure 5: The initial velocity v_0 is broken into components.

During a fireworks display, a shell is shot into the air with an initial speed of 50 m/s, at an angle of 60° above horizontal. The fuse is timed to ignite the shell just as it reaches its highest point above the ground. Calculate the height at which the shell explodes.

- A: 190 m
- B: 100 m
- C: 110 m
- D: 250 m

How much time passes between the launch and the explosion?

- A: 3.9 seconds
- B: 4.3 seconds
- C: 5.1 seconds
- D: 10.0 seconds

What is the horizontal displacement of the shell when it explodes?

- A: 108 meters
- B: 98 meters
- C: 98 degrees
- D: 150 meters

Let's try gaining visual intuition about projectile motion through the following program:

`http://galileoandeinstein.physics.virginia.edu/more_stuff/Applets/Projectile/projectile.html`

1. First, set air resistance to zero, at bottom right.
2. Make ten measurements of g by creating some projectile trajectories, and taking the ratio $g = v_{0,y}^2/(2\Delta y)$. What value do you obtain, on average?
3. Now, set air resistance to $b/m \approx 0.02$, and repeat the ten measurements. What value do you obtain?
4. Explain why this value is smaller, larger, or equal to the first set of measurements.

Projectile motion in two dimensions, with constant acceleration in one dimension, produces *quadratic curves*.

How do we obtain the trajectory, or $y(x)$ for these curves?

Looking at the x-direction:

$$x = v_0 \cos(\theta) t \tag{10}$$

$$t = \frac{x}{v_0 \cos(\theta)} \tag{11}$$

Substituting in Eq. 11 for t into the equation for vertical displacement gives:

$$y(t) - y_0 = -\frac{1}{2} g \frac{x^2}{v_0^2 \cos^2(\theta)} + \tan(\theta) x \tag{12}$$

$$y(t) - y_0 = -\left(\frac{g}{2v_0^2 \cos^2(\theta)}\right) x^2 + \tan(\theta) x \tag{13}$$

$$y(x) - y_0 = -\left(\frac{g}{2v_0^2 \cos^2(\theta)}\right) x^2 + \tan(\theta) x \tag{14}$$

In Eq. 14, we are simply saying that $y(x)$ is some quadratic. (It's still true that y and x are both functions of *time*, however, those functions of time are related).

A space explorer is on a moon around another planet, and wants to measure g. She tosses a pebble from an initial height of 2 meter, at an angle of 45 degrees above horizontal, with an initial velocity of 2 m/s. When it lands, the horizontal displacement is 10 meters. What is the gravitational acceleration g?

- A: 0.125 m/s^2
- B: 0.25 m/s^2
- C: 0.5 m/s^2
- D: 1.0 m/s^2

Other useful equations are for the *time-of-flight*, and the *range*, concepts we've already seen in several examples:

$$T_{\mathrm{tof}} = \frac{2v_0 \sin\theta}{g} \tag{15}$$

$$R = \frac{v_0^2 \sin 2\theta}{g} \tag{16}$$

Algebraic challenge: Show that the ratio of the range to the time is just the horizontal velocity, using the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$.

RELATIVE MOTION AND REFERENCE FRAMES

Thus far, we have been discussing *kinematics* with respect to a *fixed frame of reference*. Usually, we think of this frame of reference as the Earth. If we kick a soccer ball, we know how to use equations to describe the motion. What if we are moving, and the ball is moving with us, when we kick it?

Michael is running at 5 m/s, with a soccer ball rolling with him at the same speed. He shoots the ball, such that he judges the speed to be 10 m/s. What is the speed of the ball for someone who is observing, and standing still?

- A: 5 m/s
- B: 10 m/s
- C: 15 m/s
- D: Cannot determine.

So we see that relative motion is about adding and subtracting vectors. Let's introduce a notation so that we always add vectors from reference frames correctly. If two frames of reference (think of them as moving coordinate systems) S and S' are both moving with respect to each other. The position of a particle P in frame S is

$$\vec{r}_{\mathrm{PS}} = \vec{r}_{\mathrm{PS'}} + \vec{r}_{\mathrm{S'S}} \tag{17}$$

$$\vec{r}_{\mathrm{PS}} = \vec{r}_{\mathrm{PS'}} + \vec{r}_{\mathrm{S'S}} \tag{18}$$

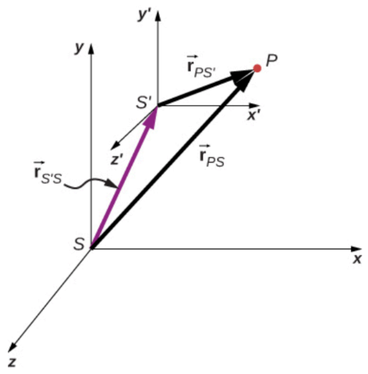

Figure 6: “To get where you are going, first you must know where you are.”

For positions:

$$\vec{r}_{\mathrm{PS}} = \vec{r}_{\mathrm{PS'}} + \vec{r}_{\mathrm{S'S}} \tag{19}$$

Since velocities are just the time-derivatives of positions:

$$\vec{v}_{\mathrm{PS}} = \vec{v}_{\mathrm{PS'}} + \vec{v}_{\mathrm{S'S}} \tag{20}$$

...and accelerations are just the time-derivatives of velocities:

$$\vec{a}_{\mathrm{PS}} = \vec{a}_{\mathrm{PS'}} + \vec{a}_{\mathrm{S'S}} \tag{21}$$

The old train-robber problem: a robber is riding a horse alongside a train moving in the $\hat{j}$ direction, at 5 m/s initially. The robber angles his run at 30 degrees with respect to the $\hat{j}$ direction, and she is 10 m to the side of the train. Her horse runs at a speed of 6 m/s. What is the horse and robber's velocity, relative to the train?

- A: $3\hat{i}$ m/s
- B: $\sqrt{2}\hat{i} + \sqrt{2}\hat{j}$ m/s
- C: $(3\sqrt{3} - 5)\hat{i} + 3\hat{j}$ m/s
- D: $3\hat{i} + (3\sqrt{3} - 5)\hat{j}$ m/s

The old train-robber problem: The conductor realizes the robber is there! When the robber is 6 m away from the train, the conductor accelerates by 1 m/s^2. What is the train's speed along the track *relative to the robber* when the robber reaches the train? (Check: is this faster or slower than what it was initially?)

- A: $7 - 3\sqrt{3}$ m/s
- B: $3\sqrt{3} - 5$ m/s
- C: $3\sqrt{3}$ m/s
- D: $\sqrt{3}$ m/s

The old train-robber problem: The conductor sees that the horse no longer has a rider, and assumes the robber made it onto the train. He holds on tight, and pulls the emergency brake. The ensuing deceleration is -3 m/s^2 from an initial train speed of 10 m/s. At this moment the robber is dropping down from the hatch to the car carrying the gold bars. The drop takes 0.5 seconds. What is the horizontal velocity of the bars relative to the robber just before he lands?

- A: -7 m/s
- B: -1.5 m/s
- C: 7 m/s
- D: 10 m/s

The old train-robber problem: The robber tumbles forward, but turns back and pounces on the gold. The conductor and robber both notice a long bridge over a ravine approaching, 0.5 km away at $t = 0$. The robber begins to load gold bars into the pack, each taking about 5 seconds. The conductor releases the brake and begins to accelerate at 3 m/s^2 from an initial speed of 10 m/s. How many gold bars can the robber load before the robber reaches the ravine?

- A: ≈ 1
- B: ≈ 2
- C: ≈ 3
- D: ≈ 4

The old train-robber problem: If the robber reaches the edge of the ravine in 15 seconds, at what speed is the train moving relative to the ground? Would any human be able to jump fast enough in the other direction to counteract the train moving the robber into air over the ravine?

- A: ≈ 10 m/s, yes
- B: ≈ 15 m/s, yes
- C: ≈ 55 m/s, no
- D: ≈ 55 m/s, yes

The old train-robber problem: The robber realizes she cannot jump fast enough up-track to avoid being pulled into the ravine. The conductor opens the gold car door with a pistol aimed, and shouts "Hands up!" The robber concludes which of the following?

- A: Crime doesn't pay
- B: Studying physics helps robbers commit crimes more efficiently
- C: She should have gone into a career in science
- D: Gold is heavy

Relative motion is also a good model for the difference between "air-speed" and "ground-speed." Suppose a plane can fly at 100 kph in still air, but experiences a tailwind of 20 kph. What is the "ground-speed" of the aircraft?

- A: 80 kph
- B: 120 kph
- C: 100 kph
- D: 20 kph

Suppose the wind changes. The plane is traveling due East ($\hat{i}$ direction), and the wind moves towards Northeast (45 degrees with respect to East), and has a new speed of 40 kph. What is the ground speed of the aircraft?

- A: $100\hat{i} + 40/\sqrt{2}\hat{j}$ kph
- B: $(100 + 40/\sqrt{2})\hat{i} + 40/\sqrt{2}\hat{j}$ kph
- C: $40/\sqrt{2}\hat{i} + 40/\sqrt{2}\hat{j}$ kph
- D: $(100 + 40/\sqrt{2})\hat{i}$ kph

CONCLUSION

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2. Combining free-fall and vector components: projectile motion
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    - **Lab-activity: testing component independence**
3. Relative motion and reference frames
    - Relative motion in one-dimension
    - Relative motion in two-dimensions

ANSWERS

- To the right, positive
- At the peak height, yes, yes
- $\vec{r}_{\text{final}} = 472.0\hat{i} + 80.3\hat{j} \quad (m)$
- $\vec{v}_{\text{final}} = 6.8\hat{i} + 1.2\hat{j} \quad (m/s)$
- $16\hat{x} + 8\hat{z}$ (m/s), $8\hat{x} + 4\hat{z}$ (m/s)
- $8\hat{i} + 4\hat{k}$ (m/s^2), $8\hat{i} + 4\hat{k}$ (m/s^2)
- 2 m/s, 2 m/s
- 14 m/s, 32 m/s
- 190 m
- 4.3 seconds
- 108 meters
- 0.5 m/s^2
- $R/T = v_0 \cos\theta = v_{x,0}$
- $3\hat{i} + (3\sqrt{3} - 5)\hat{j}$ m/s
- $7 - 3\sqrt{3}$ m/s
- -1.5 m/s
- ≈ 3
- ≈ 55 m/s, no
- 120 kph
- $(100 + 40/\sqrt{2})\hat{i} + 40/\sqrt{2}\hat{j}$ kph