

Calculus-Based Physics-1: Mechanics (PHYS150-01): Unit 0

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Opening Remarks - Welcome!

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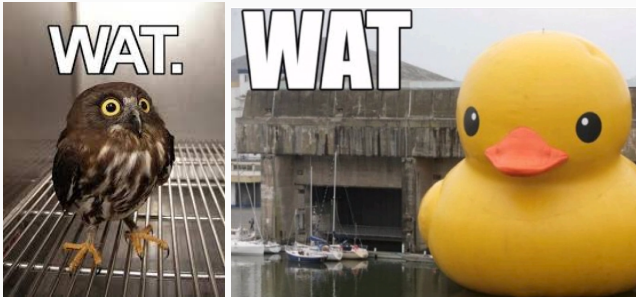


Figure 1: Taking physics for the first time.

Summary

Week 1 Summary

Physics - φυσική - "phusiké": *knowledge of nature*
from φύσις - "phúsis": *nature*

1. Estimation/Unit Analysis - Chapters 1.1 - 1.4
 - **Estimating** the correct order of magnitude
 - **Unit analysis** - dealing with the units of numbers
2. Coordinates and vectors - Chapters 2.1 - 2.4
 - **Scalars** and **vectors**
 - **Cartesian** (rectangular) coordinates, displacement
 - **Vector** addition, subtraction, and multiplication
3. Review of Calculus Techniques
 - The derivative, derivatives of elementary functions
 - **Function** approximation
 - Anti-derivatives and integration

Estimation/Unit Analysis - Chapters 1.1 - 1.4

Estimation/Unit Analysis

In science and engineering, **estimation** is to obtain a quantity in the absence of precision, informed by rational constraints.

1. Define relevant **unit scales**: (mg, g, or kg), (m/s or km/hr)
2. Obtain **complex quantities** from simple ones
 - Obtain *areas* and *volumes* from *lengths*
 - Obtain *rates* from *numerators* and *denominators*
3. Taking advantage of **scaling problems**
 - Knowing *relationship* between variables
 - Using that *relationship* to obtain new information
4. Constrain the unknown with **upper** and **lower** limits

Units: Which of the following represents a *volume*?

- A: 10 gm
- B: 10 cm²
- C: 1 cm³
- D: 1 cm s⁻¹

Estimation/Unit Analysis

Units: If a grain of sand within a fluid sinks 15 cm in 5 seconds, what is the speed of the grain?

- A: 3 cm
- B: 3 s
- C: 3 s/cm
- D: 3 cm/s

Unit conversion: If a person weights 120 lbs, what is their weight in kilograms?¹

- A: 54.5 kg
- B: 264 kg
- C: 54.5 lbs
- D: 264 lbs

¹One kilogram is 2.2 lbs.

Estimation/Unit Analysis

Unit conversion: A **density** is a mass divided by a volume. For example, water has a density of 1 gm cm^{-3} . What is the density of water in kg m^{-3} ?

- A: 1 kg m^{-3}
- B: 10 kg m^{-3}
- C: 100 kg m^{-3}
- D: 1000 kg m^{-3}

Group exercise on complex units: A *vitrolero* is a classic container for serving *agua fresca*. It has a diameter of 20 cm, and a height of 30 cm. How many cups can we serve from the vitrolero if we put 0.5 liters of *agua fresca* in each cup?

- *Hint:* 1 liter is 1000 mL
- *Hint:* 1 mL is 1 cm³
- **Volume:** The volume of a cylinder is π times the radius of the base, squared, , times the height: $\pi r^2 h$.

Estimation/Unit Analysis

Unit scale: A generation is about one-third of a lifetime.
Determine how many generations have passed since the year 0 AD².

- A: 10
- B: 20
- C: 60
- D: 100

²What is the appropriate scale here?

Estimation/Unit Analysis

Unit scale: (a) What fraction of Earth's diameter³ is the greatest ocean depth (11 km below sea level)? (b) The greatest mountain height (8.8 km above sea level)?

- A: 8.6×10^{-2} , 6.9×10^{-2}
- B: 8.6×10^{-3} , 6.9×10^{-3}
- C: 8.6×10^{-4} , 6.9×10^{-4}
- D: 8.6×10^{-5} , 6.9×10^{-3}

³The diameter of the Earth is 12,800 km.

Complex quantities: If a Whittier College athlete ran the 5k race at a track meet in 35 minutes, what was her average speed?

- A: 0.3 meters per second
- B: 3 meters per second
- C: 30 meters per second
- D: 300 meters per second

Complex quantities: Suppose you won the lottery and received \$1 billion USD. Because your life is dope, you stack that paper over the Whittier College soccer field. Each stack contains 100 bills, and each bill is worth \$100. If you evenly cover the field, how high is the money level?

- A: 0.5 inch
- B: 1 inch
- C: 2 inches
- D: 1 foot

Scaling problem: Supposed you have an ideal gas in a cylinder of fixed volume. If the pressure begins as 100 kPa, and you *double* the temperature of the gas, what is the new pressure?

- A: 100 kPa
- B: 50 kPa
- C: 10 kPa
- D: 200 kPa

Scaling problem: Supposed you have an ideal gas in a cylinder of fixed volume. If the pressure begins as 100 kPa, and you *halve* the temperature of the gas, what is the new pressure?

- A: 100 kPa
- B: 50 kPa
- C: 10 kPa
- D: 200 kPa

Upper/lower limits: How many undergraduate students are there at Whittier College⁴?

- A: 5,000
- B: 1,000
- C: 1,250
- D: 500

⁴What is the absolute lower limit, and what is the upper limit?

Estimation/Unit Analysis

Upper/lower limits: What is the average yearly college tuition in the United States (before subtracting grants and scholarships)?

- A: \$5,000
- B: \$10,000
- C: \$25,000
- D: \$40,000

What information affects the **upper** and **lower** limits here?

Coordinates and Vectors - Chapters 2.1 - 2.4

Activity Link

Who understands coordinates and vectors better than anyone else?

https://youtu.be/0B7WL7nhIF4?si=_dl4t_GwL98aXWFa

Coordinates and Vectors - Scalars, Vectors (Chapters 2.1-2.3)

Physics requires **mathematical objects** to build equations that capture the behavior of nature. Two examples of such objects are **scalar** and **vector** quantities. Each type of object obeys similar but different rules.

1. Scalar quantities

- mass: $m_1 + (m_2 + m_3) = (m_1 + m_2) + m_3$
- speed: $v_1(v_2 + v_3) = v_1v_2 + v_1v_3$
- charge: $q_1\left(\frac{1}{q_1}\right) = 1, q_1(0) = 0$

2. Vector quantities

- velocity: $\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = (\vec{v}_1 + \vec{v}_2) + \vec{v}_3$
- tension: $\vec{t}_1 \cdot (\vec{t}_2 + \vec{t}_3) = \vec{t}_1 \cdot \vec{t}_2 + \vec{t}_1 \cdot \vec{t}_3$

Coordinates and Vectors - Scalars, Vectors (Chapters 2.1-2.3)

A vector may be expressed as *a list of scalars*: $\vec{v} = (4, 2)$ (a vector with two *components*), $\vec{u} = (3, 4, 5)$ (three *components*). Now, we know how to add and subtract scalars. How do we add and subtract vectors?

What is

$(1, 3, 8) +$

$(0, 2, 1)$?

Answer: $(1, 5, 9)$

In other words, when adding vectors, we add them component by component.

Coordinates and Vectors - Scalars, Vectors (Chapters 2.1-2.3)

How do we subtract vectors? In the same fashion:

What is

$(1, 3, 8) -$

$(0, 2, 1)$?

Answer: $(1, 1, 7)$

In other words, when subtracting vectors, we subtract them component by component.

Coordinates and Vectors - Scalars, Vectors (Chapters 2.1-2.3)

A HTML-based demonstration for adding vectors:

`https://phet.colorado.edu/en/simulations/
vector-addition`

Notice several things:

- Produce vectors that *cancel* each other.
- What happens when vectors are parallel and orthogonal?

Coordinates and Vectors - Scalars, Vectors (Chapters 2.1-2.3)

How do we multiply vectors? In the same fashion, *for one kind of multiplication*:

What is

$$(1, 3, 8) \cdot (0, 2, 1)?$$

$$\text{Answer: } 1 \cdot 0 + 3 \cdot 2 + 8 \cdot 1 = 14$$

This kind of multiplication is known as the dot-product. There is also the *cross-product*, which we will save for later.

Coordinates and Vectors - Coordinates (Chapters 2.1-2.3)

The components of a vector may describe quantities in a **coordinate system**, such as *Cartesian coordinates* - after René Descartes.

Vectors in the 3D Cartesian coordinate system (x,y,z) may be written in the following notation:

$$\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$$

- a: The amount in the +x-direction, \hat{i} : a vector of length 1, in the +x-direction
- b: The amount in the +y-direction, \hat{j} : a vector of length 1, in the +y-direction
- c: The amount in the +z-direction, \hat{k} : a vector of length 1, in the +z-direction

Coordinates and Vectors - Vectors (Chapters 2.1-2.3)

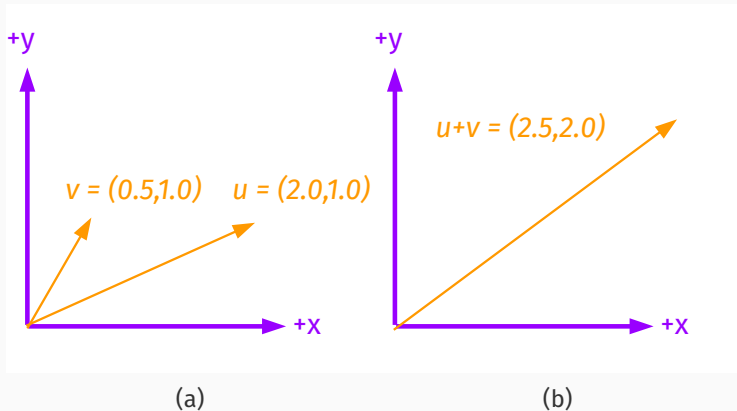


Figure 2: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) What is $\vec{u} + \vec{v}$? Adding components: $\vec{u} + \vec{v} = 2.5\hat{i} + 2.0\hat{j}$.

Coordinates and Vectors - Vectors (Chapters 2.1-2.3)

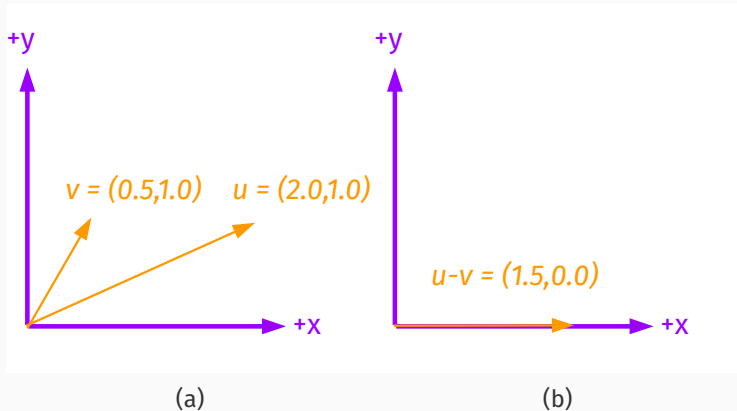


Figure 3: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) What is $\vec{u} - \vec{v}$? Subtracting components: $\vec{u} - \vec{v} = 1.5\hat{i} + 0.0\hat{j}$.

Coordinates and Vectors - Vectors (Chapters 2.1-2.3)

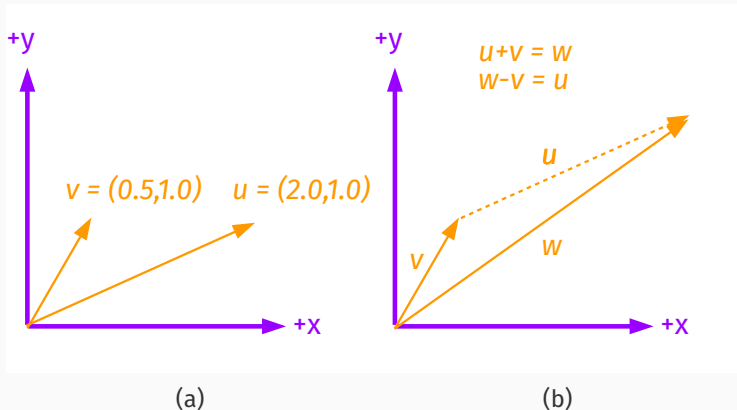


Figure 4: (a) Two vectors in a two-dimensional Cartesian coordinate system: $\vec{u} = 0.5\hat{i} + 1.0\hat{j}$ and $\vec{v} = 2.0\hat{i} + 1.0\hat{j}$. (b) To compute $\vec{w} - \vec{v}$, arrange the vectors to get a sense of the result, \vec{u} .

Coordinates and Vectors - Vectors (Chapters 2.1-2.3)

$$\vec{p} = 4\hat{i} + 2\hat{j}. \quad \vec{q} = -4\hat{i} + 2\hat{j}.$$

Compute $\vec{p} \cdot \vec{q}$.

- A: 12
- B: -12
- C: 4
- D: 8

$$\vec{p} = -1\hat{i} + 6\hat{j}. \quad \vec{q} = 3\hat{i} + 0.5\hat{j}.$$

Compute $\vec{p} \cdot \vec{q}$.

- A: -1
- B: 1
- C: 0
- D: 3

Coordinates and Vectors - Vectors (Chapters 2.1-2.3)

Why was the last answer zero? Look at it graphically:

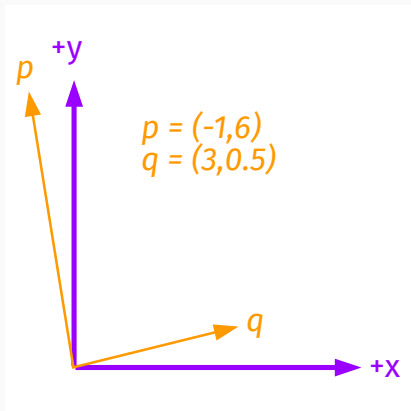


Figure 5: Two vectors \vec{p} and \vec{q} are *orthogonal* if $\vec{p} \cdot \vec{q} = 0$.

Coordinates and Vectors - Vectors (Chapters 2.1-2.3)

What if the vectors are parallel? Look at it graphically:

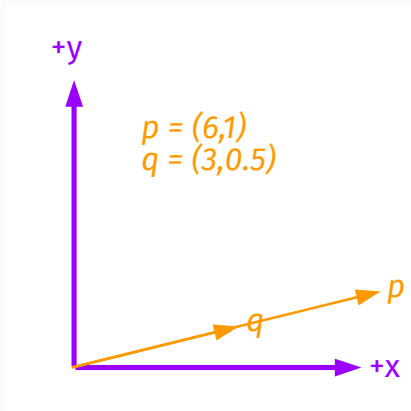


Figure 6: Two vectors \vec{p} and \vec{q} are *parallel* if $\vec{p} \cdot \vec{q}$ is maximal.

Coordinates and Vectors - Dot Product (Chapters 2.1-2.3)

The *length* or *norm* of a vector $\vec{v} = a\hat{i} + b\hat{j}$ is $|\vec{v}| = \sqrt{a^2 + b^2}$.

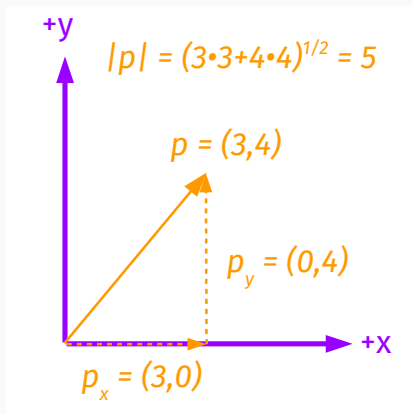


Figure 7: Computing the norm of a vector \vec{p} .

Coordinates and Vectors - Dot Product (Chapters 2.1-2.3)

Notice that $\sqrt{\vec{p} \cdot \vec{p}} = |\vec{p}|$.

Let θ_p be the angle between \vec{p} and the x-axis.

$$p_x = \vec{p} \cdot \hat{i} = |\vec{p}| \cos(\theta_p).$$

$$p_y = \vec{p} \cdot \hat{j} = |\vec{p}| \sin(\theta_p).$$

Theorem: The dot product of two vectors \vec{p} and \vec{q} is $|\vec{p}||\vec{q}| \cos(\theta)$, if θ is the angle between them.

$$\begin{aligned} \text{Proof: } \vec{p} \cdot \vec{q} &= p_x q_x + p_y q_y = |p||q| \cos \theta_p \cos \theta_q + |p||q| \sin \theta_p \sin \theta_q \\ &= |p||q| (\cos \theta_p \cos \theta_q + \sin \theta_p \sin \theta_q) = |p||q| \cos(\theta_p - \theta_q) \\ &= |p||q| \cos \theta. \end{aligned}$$

$$\boxed{\vec{p} \cdot \vec{q} = |p||q| \cos \theta}$$

Coordinates and Vectors - Dot Product (Chapters 2.1-2.3)

An object moves at 2 m/s at $\theta = 60^\circ$ with respect to the x-axis. What is the velocity of the object?

- A: $(1\hat{i} + 1\hat{j})$ m/s
- B: $(\sqrt{3}\hat{i} + 1\hat{j})$ m/s
- C: $(\sqrt{3}\hat{i} + \sqrt{3}\hat{j})$ m/s
- D: $(1\hat{i} + \sqrt{3}\hat{j})$ m/s

What is the dot product of this velocity with another velocity: 5 m/s along the x-axis?

- A: 1 (m/s)^2
- B: 5 (m/s)^2
- C: 10 (m/s)^2
- D: 5 (m/s)

Coordinates and Vectors - Scalars, Vectors (Chapters 2.1-2.3)

Is it possible to multiply vectors and scalars? Of course:

$$a_1\vec{p} = a_1p_x\hat{i} + a_1p_y\hat{j}.$$

Also, multiplication properties still hold. For example:

$$(a_1 + a_2)\vec{p} = a_1\vec{p} + a_2\vec{p}.$$

A spacecraft moves at 400 m/s, at an angle of 30 degrees with respect to the x-axis. If it fires two thrusters that boost the x-component and y-component of the velocity by 25% and 50%, respectively, what is the final velocity?

- A: $(433\hat{i} + 300\hat{j})$ m/s
- B: $(300\hat{i} + 433\hat{j})$ m/s
- C: 400 m/s
- D: $(400\hat{i} + 433\hat{j})$ m/s

Coordinates and Vectors - Displacement (Chapters 2.1-2.3)

We define the *position* of an object as a vector locating it in a given coordinate system. The scalar *distance* is the norm of the position vector, that is, the distance to to the origin.

Now we can introduce the concept of **displacement**: a vector describing a movement of an object.

Coordinates and Vectors - Displacement (Chapters 2.1-2.3)

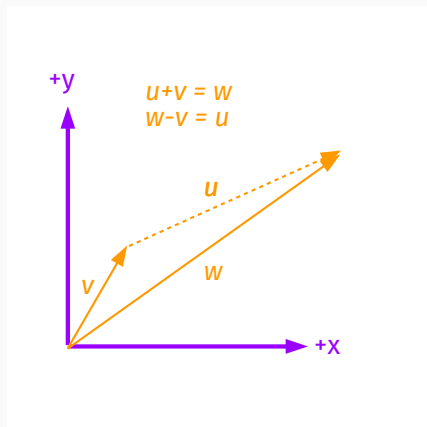


Figure 8: Suppose an object moves from position \vec{v} to \vec{w} . In this case, the displacement is \vec{u} . Thus, the final position is the initial position, plus the displacement.

Coordinates and Vectors - Displacement (Chapters 2.1-2.3)

It follows that the *displacement* is zero if the initial and final positions are the same, but the *distance travelled* is not.

Suppose a jet fighter travelling at 800 km per hour banks such that it flies in a circle of radius 0.5 km. How long does it take to complete the circle? What is the distance traveled, and what is the displacement?

- A: 2π km, 28 seconds, 2π km
- B: π km, 14 seconds, π km
- C: π km, 28 seconds, π km
- D: π km, 14 seconds, 0 km

Coordinates and Vectors - Average Velocity (Chapter 3.1)

Average velocity is the ratio of the displacement to the elapsed time.

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{x}}{\Delta t} \quad (1)$$

The *average speed* is the norm of the average velocity:

$$v_{\text{avg}} = \frac{|\Delta \vec{x}|}{\Delta t} \quad (2)$$

If the motion is in one dimension, then the average speed is

$$v_{\text{avg}} = \frac{x_f - x_i}{t_f - t_i} \quad (3)$$

Coordinates and Vectors - Average Velocity (Chapter 3.1)

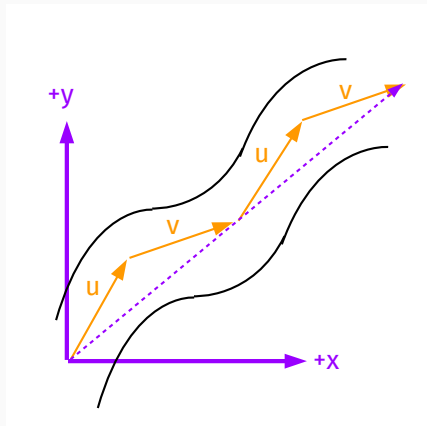


Figure 9: A Formula-1 driver keeps his car on the track by following a path approximated by the position vectors u , v , u , and v . The dashed arrow represents the total displacement.

Coordinates and Vectors - Average Velocity (Chapter 3.1)

If $\vec{u} = (20\hat{i} + 30\hat{j})$ m, and $\vec{v} = (30\hat{i} + 20\hat{j})$ m, what is the total displacement? If the elapsed time is 10 seconds, what is the average velocity?

- A: $(50\hat{i} + 50\hat{j})$ m, 14 m/s
- B: $(80\hat{i} + 100\hat{j})$ m, 10 m/s
- C: $(100\hat{i} + 100\hat{j})$ m, 14 m/s
- D: $(50\hat{i} + 150\hat{j})$ m, 10 m/s

Review of Calculus Techniques

Review of Calculus Techniques

1. Computing limits

- Examples in mathematics
- Examples in physics

2. Differentiation

- Definition of the derivative
- Examples of derivatives

3. Integration

- Definition of the integral
- Examples of integrals

Review of Calculus Techniques - Computing Limits

Consider the function $f(x)$, defined below:

$$f(x) = \frac{1}{1+x^2} \quad (4)$$

Evaluate the function at the following points:

- $x = 0$
- $x = 10$
- $x = 100$
- $x = 1000$

What is the *limiting value* of the function as $x \rightarrow \infty$? What is the *limiting value* of the function as $x \rightarrow -\infty$?

Review of Calculus Techniques - Computing Limits

Consider the function $f(x)$, defined below:

$$f(x) = \exp(x) = e^x \quad (5)$$

Evaluate the function at the following points:

- $x = 0$
- $x = 10$
- $x = 100$
- $x = 1000$

What is the *limiting value* of the function as $x \rightarrow \infty$? What is the *limiting value* of the function as $x \rightarrow -\infty$?

Review of Calculus Techniques - Computing Limits

Consider the function $f(x)$, defined below:

$$f(x) = \sin(x) \tag{6}$$

Evaluate the function at the following points:

- $x = \pi$
- $x = -\pi$
- $x = 0.1$
- $x = 0.01$

What is the *limiting value* of the function as $x \rightarrow 0$? The procedure is straightforward if the function is *continuous* and *differentiable*.

Review of Calculus Techniques - Differentiation

Derivative of a Function

Let $f(t)$ be a continuous function on an interval $[a, b]$, and $a < t < b$. The derivative of $f(t)$ is

$$f'(t) = \frac{df}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \quad (7)$$

List of Common Derivatives

Here is a link to lists of common derivatives: <https://en.wikipedia.org/wiki/Derivative>
https://en.wikipedia.org/wiki/Differentiation_rules

Professor: work some examples.

Review of Calculus Techniques - Integration

Integral of a Function

Let $f(t)$ be a continuous function on an interval $[a, b]$, and $a < t_i < b$, where t_i are regular points within the interval. The Riemann integral is

$$I = \sum_i f(t_i) \Delta t_i \rightarrow \int_a^b f(t) dt \quad (8)$$

List of Common Derivatives

Here is a link to a list of common integrals: https://en.wikipedia.org/wiki/Lists_of_integrals

Professor: work some examples.

Conclusion

Week 1 Summary

1. Methods of approximation
 - **Estimating** the correct order of magnitude
 - **Function** approximation
 - **Unit analysis**
2. Coordinates and vectors
 - **Scalars** and **vectors**
 - **Cartesian** (rectangular) coordinates, displacement
 - **Vector** addition, subtraction, and multiplication
3. Review of Calculus Techniques
 - Limits
 - Differentiation
 - Integration