Teaching Demonstration: Electronic Filters

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Abstract

This note contains a short teaching demonstration on electronic filters. The theory and design of citcuits that filter signals, and the algorithmic implementation of signal filters in computer code will be addressed.

1 An Essential Math Tool

The Fourier transform of a function $f(t)$ is defined as:

$$\mathcal{F}(f(t)) = \tilde{F}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \tag{1}$$

In Eq. 1, ω is the angular frequency, measured in radians per unit time. Let $f(t) = g'(t)$. Substituting into Eq. 1, and integrating by parts, we have

$$\tilde{F}(\omega) = g(t) e^{-j\omega t}|_{-\infty}^{\infty} + j\omega \int_{\infty}^{-\infty} g(t) e^{-j\omega t} dt \tag{2}$$

For physical signals that represent finite energy, $\lim_{|t|\to\infty} g(t) = 0$. This requirement simplifies Eq. 2 by making the first term on the right-hand side vanish. We have

$$\tilde{F}(\omega) = j\omega \int_{\infty}^{-\infty} g(t) e^{-j\omega t} dt = j\omega \mathcal{F}(g(t)) \tag{3}$$

The result may be summarized:

$$\boxed{\mathcal{F}(g'(t)) = j\omega \mathcal{F}(g(t))} \tag{4}$$

One utility of this result is that differential equations in the time-domain may be converted to algebraic equations in the Fourier domain, making them easier to apply.

2 Three Simple Circuits

A voltage divider lowers the output voltage ($v_{out}(t)$) relative to the input voltage ($v_{in}(t)$) by some ratio, given by the resistances R_1 and R_2 in the following circuit:

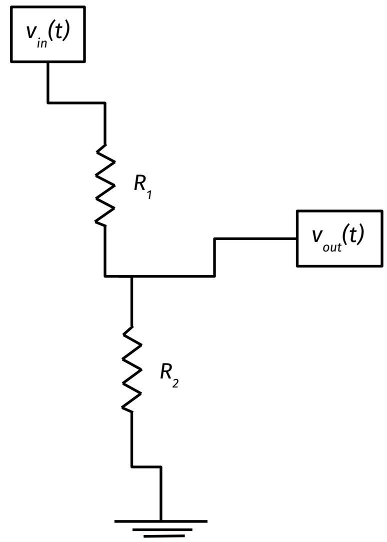

Figure 1: A two-resistor voltage divider.

The input and output voltages with respect to ground must both follow Ohm's law, where $i(t)$ is the current:

$$v_{in}(t) = (R_1 + R_2)i(t) \tag{5}$$

$$v_{out}(t) = (R_2)i(t) \tag{6}$$

The resistances R_1 and R_2 do not depend on frequency. Taking the Fourier transform of both sides of Eqs. 5 and 6, and dividing Eq. 6 by Eq. 5, we have

$$\boxed{\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = \frac{R_2}{R_1 + R_2}\frac{i(\omega)}{i(\omega)} = \frac{R_2}{R_1 + R_2}} \tag{7}$$

The ratio of output voltage to input voltage is sometimes named the *transfer function.* In this case, the transfer function only affects the overall amplitude of the output signal, while not affecting any particular frequency, nor shifting any phases. A circuit in which the resistor labelled R_2 has been replaced with a capacitor labelled C is shown in Fig. 2.

The capacitance C is a constant, defined as the ratio of the charge stored on the capacitor, q, to the voltage required to place that charge on the capacitor, V:

$$C = \frac{q}{V} \tag{8}$$

$$VC = q \tag{9}$$

$$C\frac{dV}{dt} = \frac{dq}{dt} = i(t) \tag{10}$$

$$\tilde{i}(\omega) = j\omega C\tilde{V}(\omega) \tag{11}$$

$$\frac{\tilde{V}(\omega)}{\tilde{i}(\omega)} = \frac{1}{j\omega C} \tag{12}$$

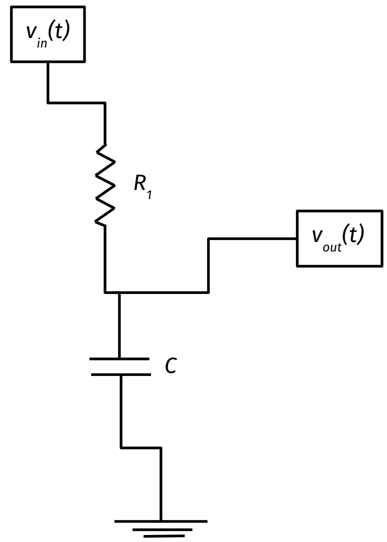

Figure 2: A frequency-dependent voltage divider, also known as a single-pole RC low-pass filter.

Thus, Ohm's law says that the frequency-dependent resistance, or *impedance*, of a capacitor is

$$\boxed{Z_C = \frac{1}{j\omega C}} \tag{13}$$

The transfer function of the RC circuit in Fig. 2 is the same as Eq. 7, with R_2 replaced with Z_C (and $R_1 = R$):

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1} \tag{14}$$

Let the *time-constant* be defined as $\tau = RC$, and $\omega_0 = 1/\tau$. Equation 14 may be written:

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = -\frac{j\omega_0}{\omega - j\omega_0} \tag{15}$$

The magnitude and phase of Eq. 15 are

$$\boxed{M_{LP}(\omega) = \sqrt{\frac{\omega_0^2}{\omega^2 + \omega_0^2}}} \tag{16}$$

$$\boxed{\phi_{LP}(\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)} \tag{17}$$

From Eq. 16, we can see that the low-pass transfer function $M_{LP}(\omega)$ attenuates frequencies much larger than ω_0. From Eq. 17, the circuit introduces a phase-shift that is approximately linear ($-\omega/\omega_0$) for frequencies that are not attenuated. A final interesting quantity to consider is the *group delay*, defined as the negative derivative of the phase:

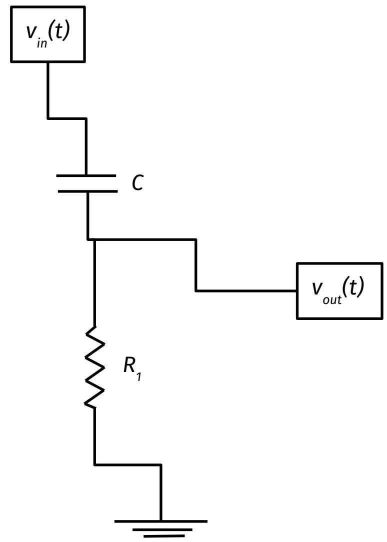

Figure 3: A frequency-dependent voltage divider, also known as a single-pole RC high-pass filter.

$$\boxed{-\frac{d\phi}{d\omega} = \tau_{LP}(\omega) = \frac{\omega_0}{\omega^2 + \omega_0^2}} \tag{18}$$

Equation 18 shows that frequency modes below ω_0 are delayed in time by a factor of $\approx \omega_0^{-1}$. A final simple circuit to consider is the same as Fig. 2, with the capacitor and resistor exchanged (Fig. 3). Following the same arguments as the low-pass case, the complex transfer function is

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = \frac{\omega}{\omega - j\omega_0} \tag{19}$$

The magnitude and phase of Eq. 19 are

$$\boxed{M_{HP}(\omega) = \sqrt{\frac{\omega^2}{\omega^2 + \omega_0^2}}} \tag{20}$$

$$\boxed{\phi_{HP}(\omega) = \tan^{-1}\left(\frac{\omega_0}{\omega}\right)} \tag{21}$$

The group-delay turns out to be the same as Eq. 18

$$\boxed{-\frac{d\phi}{d\omega} = \tau_{HP}(\omega) = \frac{\omega_0}{\omega^2 + \omega_0^2}} \tag{22}$$

Unlike the voltage divider in Fig. 1, which has zero group delay, the circuits in Figs. 2 and 3 have capacitors. The group delay in these cases is driven by how quickly these capacitors can be charged and discharged, regardless of where they are in the circuit.

2.1 Poles and Zeros

Notice that Eqs. 14 and 19 become infinite if the frequency is treated as a complex number, with the value $\omega = j\omega_0$. Treating the frequency as a complex number is useful for further analysis using *contour integration*, allowing us to perform integrals with singularities. A complex frequency that makes the transfer functions infinite is called a *pole*, and a frequency that makes the transfer functions zero is called a *zero*. Both the low-pass and high-pass filter examples above have just one pole each. The low-pass filter has a zero whenever $|\omega| \to \infty$, and the high-pass filter has a zero for $\omega = 0$. This is why these circuits are named *single-pole* low and high-pass filters.

3 Summary of Simple Circuits

The key results so far may be summarized:

- The *transfer function* of a circuit may be defined as the ratio of the output voltage to the input voltage in the Fourier domain.
- The transfer function of a *voltage divider* is given by Eq. 7, a frequency-independent ratio with no phase shift.
- The transfer function of a *single-pole low-pass filter* is given by Eq. 14. Magnitude: Eq. 16 Phase: Eq. 17
- The transfer function of a *single-pole high-pass filter* is given by Eq. 19. Magnitude: Eq. 20 Phase: Eq. 21

4 Building a Passive Differentiator

Consider a single-pole high pass filter, with transfer function given by Eq. 19. Choose a value for ω_0 much larger than any frequency in the expected input signal: $\omega_0 \gg \omega$. The transfer function is approximately:

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} \approx \frac{\omega}{-j\omega_0} = j\omega\tau = j\omega RC \tag{23}$$

Rearranging Eq. 23, and switching back to the time-domain:

$$\tilde{v}_{out}(\omega) \approx j\omega\tau\tilde{v}_{in}(\omega) \tag{24}$$

$$v_{out}(t) \approx \tau\frac{dv_{in}}{dt} \tag{25}$$

Equation 25 shows that with the correct choice of resistance and capacitance, the circuit output is the derivative of the input, with a *gain* equal to $\tau = RC$. This circuit is known as a *passive differentiator*.

5 Building a Passive Integrator

Consider a single-pole low-pass filter, with transfer function given by Eq. 14. Choose a value for ω_0 much smaller than any frequency in the expected input signal: $\omega_0 \ll \omega$. The transfer function is approximately:

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} \approx \frac{-j\omega_0}{\omega} \tag{26}$$

Rearranging Eq. 26, switching back to the time-domain, and integrating both sides:

$$j\omega\tilde{v}_{out}(\omega) \approx \omega_0\tilde{v}_{in}(\omega) \tag{27}$$

$$\frac{dv_{out}}{dt} = \omega_0 v_{in}(t) \tag{28}$$

$$v_{out}(t) = \frac{1}{RC}\int_{t_1}^{t_2} v_{in}(t)dt \tag{29}$$

Equation 29 shows that with the correct choice of resistance and capacitance, the circuit output is the integral of the input between two set times, with a *gain* equal to $1/RC$. This circuit is known as a *passive integrator*.

6 Summary of Passive Differentiator and Integrator

- By choosing a large value of RC, relative to input frequencies, the output of a single-pole high-pass filter is proportional to the derivative of the input, with gain RC.
- By choosing a small value of RC, relative to input frequencies, the output of a single-pole low-pass filter is proportional to the integral of the input, with gain $1/RC$.

7 Bonus: Circuits with Inductors

An *inductor* is a circuit-component that stores energy in an internal magnetic field. The definition of *inductance* is the amount of internal magnetic flux, ϕ, generated due to a change in the current, $i(t)$:

$$L = \frac{d\phi}{di} \tag{30}$$

Applying the chain rule to the right-hand side, and applying Faraday's Law of Induction ($d\phi/dt = v$):

$$L = \frac{d\phi}{dt}\frac{dt}{di} = v\frac{dt}{di} = v/i(t) \tag{31}$$

Faraday's Law of Induction, in this case, states that a voltage v appears in circuits that causes the ensuing current to preserve the internal magnetic field. Solving for the voltage, and taking the Fourier transform of both sides:

$$\tilde{v}(\omega) = j\omega L\tilde{i}(\omega) \tag{32}$$

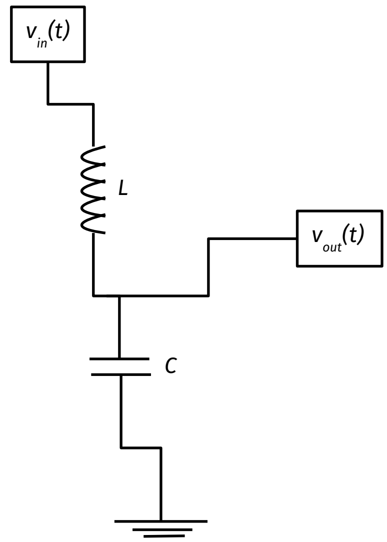

Figure 4: A simple LC-resonator (two-pole).

Thus, the impedance of an inductor is

$$\boxed{Z_L = j\omega L} \tag{33}$$

An example of a circuit comprised of an inductor and a capacitor is shown in Fig. 4. Following the voltage-divider example, but with $Z_L = j\omega L$ and $Z_C = 1/j\omega C$, we find the transfer function is real:

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = \frac{1}{1 - \omega^2 LC} \tag{34}$$

Define a *resonance frequency* $\omega_R^{-2} = LC$, so that

$$\boxed{\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = \frac{\omega_R^2}{\omega_R^2 - \omega^2} = \frac{\omega_R^2}{(\omega_R - \omega)(\omega_R + \omega)}} \tag{35}$$

The DC value at $\omega = 0$ of the transfer function is 1, and the transfer function approaches zero in the limit of $\omega \gg \omega_R$. Thus, it is tempting to think of this circuit as a two-pole low-pass filter. However, for $\omega \approx \omega_R$, the magnitude (absolute value) of the transfer function also approaches infinity. Thus, we can think of this circuit as more of a *resonator*, rather than a pure filter. Finally, there is a 180-degree phase-shift if the frequency is greater than the resonance frequency.

8 Bonus: Summary of RL Circuit

The key results of the prior section are as follows:

- The transfer function of a *two-pole RL resonator* is given by Eq. 35.
- The magnitude of Eq. 35 approaches infinity for $\omega \approx \omega_R$

- The output phase flips by π relative to the input phase, depending on whether $\omega > \omega_R$ or $\omega < \omega_R$.