Digital Signal Processing: COSC360

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January 12, 2022

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Previous lectures covered:

- Complex numbers 1: Arithmetic and some calculus (continuous and discete) ... see Chapter 30 of text

This lecture will cover:

- Complex numbers 2: The Fourier series and Fourier transform (continuous and discrete)

Next lecture will cover:

- *Time-permitting*: The Laplace transform (continuous and discrete)

Complex numbers 2: theory and examples

Complex numbers 2: theory and examples

Review: Let's work the following examples.

1. Let $z_1 = x_1 + jy_1$, and $z_2 = x_2 + jy_2$. Simplify $z = \frac{z_1^* z_2}{|z_1|^2 + |z_2|^2}$ into real and imaginary parts.

2. Express z in polar form and plot it for $x_1 = y_1 = 1.0$, and $x_2 = y_2 = -1.0$.

3. Express the function $v(t) = v_0 \cos(\omega t + \phi_0)$ as a phasor, and plot it.

The **Fourier series** representation of a function $f(x)$ is written:

$$S(x) = \frac{A_0}{2} + \sum_{i=1}^{\infty} \left(A_n \cos(nx) + B_n \sin(nx) \right) \tag{1}$$

with

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx \tag{2}$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx \tag{3}$$

Complex numbers 2: theory and examples

Let's obtain the **Fourier series** coefficients A_n and B_n for a square-wave signal:

$$f(x) = 1, \quad 0 \leq x \leq \pi, \quad 0, \pi < x \leq 2\pi \qquad (4)$$

(Observe on board). The result: $A_0 = 1.0$, all other $A_n = 0$, odd B_n values follow $2/(n\pi)$, even $B_n = 0$ as well.

Create octave code that plots this (see Moodle for example). Initially, plot a solution that has the first 5 coefficients.

Plot a solution with the first 20 coefficients. *Hint: can you find how to use the for-loop in octave?*

Let's obtain the **Fourier series** coefficients A_n and B_n for a sawtooth-wave signal:

$$f(x) = x, \quad 0 < x < 2\pi \tag{5}$$

Notice that (for technical reasons) we are always dealing with *repeating signals.* First obtain the coefficients A_n and B_n (we can use complex numbers to help with the integral).

Copy your previous octave code and modify it to plot this solution for the first 20 coefficients.

```
for i=[1:n]
    ...
endfor
```

Complex numbers 2: theory and examples

We now have a way to describe a *continuous signal* **digitally.** Ponder this for a moment. How is this possible? Aren't we representing an infinite quantity of information with a finite quantity?
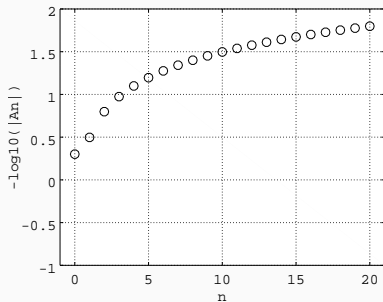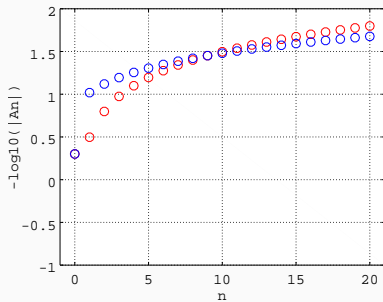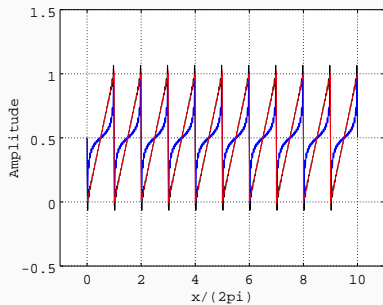

Figure 1: The negative base-10 logarithm of the Fourier coefficients A_n describing a saw-tooth wave.

Complex numbers 2: theory and examples

The octave **play** and **audioplayer** command:

```
fs = 44000.0;
dt = 1/fs;
t = [0:dt:2.0];
v = 0.5*sin(2.0*pi*3000.0.*t);
player = audioplayer(v,fs,8);
play(player)
```

These should define a digital waveform to play, and play it over the speakers.

Activity:

1. Create a sawtooth-wave and play it with octave.
2. Create a distorted sawtooth-wave and play it with octave.
3. Compare the sounds. They are different, correct?
4. Plot the spectral coefficients and discuss how the sound is affected by the distortion.
5. Bonus: write code that plays a short tune.[1] How can you get multiple notes to play at once? What is an **octave**, in the sense of musical notes and frequencies?

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[1] This is more of a programming/engineering exercise than mathematics.

Notice that in order to *plot* the sampled signal using a finite number of Fourier coefficients, we had to define *x* as

```
x = x_start:dx:x_end
```

What happens in our code if we decrease the overall number of samples by increasing *dx* while keeping the other two variables constant?

Try this with either your sound-playing code or code that plots sawtooth waves. *This is known as the sampling problem, and we will return to it in a moment.*

Finally, let's bring the concepts of *sampling* and *complex numbers* together, for the **Fourier transform**.

First, let's split our representation of the function $f(t)$ with the Fourier series into two parts, absorb the constant into the first term, and re-scale by $1/\pi$:

$$f(t) = \frac{1}{\pi} \sum_{i=1}^{N} A_n \cos(nt) + \frac{1}{\pi} \sum_{i=1}^{N} B_n \sin(nt) \qquad (6)$$

The first term is the "even" term, and the second term is the "odd" term, because of the even-ness and odd-ness of the cosines and sines (hopefully this started to dawn on us in in the sawtooth and square-wave exercises).

Complex numbers 2: theory and examples

Now our coefficient equations read (because of the missing π):

$$A_n = \int f(t)\cos(nt)dt \tag{7}$$

$$B_n = \int f(t)\sin(nt)dt \tag{8}$$

Consider the following function:

$$F(n) = A_n - jB_n = \int f(t)\cos(nt)dt - j\int f(t)\sin(nt)dt \tag{9}$$

Integrals are *linear*, in that they distribute over the same function:

$$F(n) = \int f(t)(\cos(nt) - j\sin(nt))dt = \int f(t)\exp(-jnt)dt \tag{10}$$

Now we have:
$$F(n) = \int f(t) \exp(-jnt) dt \tag{11}$$

Important question: what are the units of t? Don't they have to be unit-less? Otherwise we cannot take the exponential of integers times seconds. Let $t \rightarrow t/T$, where T is the *period*:

$$F(n) = \frac{1}{T} \int f(t) \exp(-jnt/T) dt \tag{12}$$

Imagine a wave oscillating in the lowest *harmonic*[2]. The *frequency* of the lowest harmonic is $f = 1/T$. Suppose the integer n *selects* the harmonic (units: Hz):

$$f = \frac{n}{T} \tag{13}$$

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[2]Professor: draw a picture of this.

Now let's have the integer select the harmonic, but in angular frequency:

$$\omega = 2\pi f = \frac{n}{T} \tag{14}$$

This in turn causes the function $F(n)$ to be[3]

$$F(n) = \frac{1}{T} \int_{-T}^{T} f(t) \exp(-2j\pi ft)dt = \frac{1}{T} \int_{-T}^{T} f(t) \exp(-j\omega t)dt \tag{15}$$

We can choose any high-frequency we wish by taking $n \to \infty$. Currently, though, the lowest frequency we can get is for $n = 1$, or $\omega = T^{-1}$. If we let $T \to \infty$, then we can get any frequency, but that makes the limits of our integral infinite:

$$F(n) = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} f(t) \exp(-2j\pi ft)dt \tag{16}$$

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[3]Remember this integral is over a finite interval. Let's call it $[-T, T]$.

For a special class of functions[4] this integral is simply:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t) dt \tag{17}$$

It turns out the inverse operation looks like:

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(j\omega t d\omega) \tag{18}$$

Let's do some *easy* examples together. Let's obtain $F(\omega)$ for the following functions:

1. $f(t) = 1.0 \ |t| \leq T/2$

2. $f(t) = \exp(-t/\tau)$ for $t \geq 0$, zero everywhere else.

[4]Mumble mumble ... wait until you take quantum mechanics.

1. $T\left(\frac{\sin(x)}{x}\right)$, where $x = \frac{\omega T}{2}$
2. $\tau/(1 + j\omega\tau)$

1. Use octave to plot each result.
2. Modify the script so that the *time-domain* signal and the *frequency-domain* signal are plotted one above the other.

To make multiple plots:

```
figure(1);
subplot(2,1,1)
plot(x,y);
subplot(2,1,2);
plot(x2,y2);
```

Now let's try a strange example. Consider the *delta-function*, or delta-distribution, defined by

$$f(t_0) = \int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt \qquad (19)$$

In other words, the δ-function is 1.0 at the point t_0, but zero everywhere else[5]. What is the Fourier transform of the function $a\delta(t - t_0)$?

$$F(\omega) = \int_{-\infty}^{\infty} a\delta(t - t_0) \exp(-j\omega t) dt \qquad (20)$$

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[5]...sort of, but don't ask questions.

Sines and cosines in the *frequnecy domain?* Why not!

$$F(\omega) = a\cos(\omega t_0) - aj\sin(\omega t_0) \tag{21}$$

Now let's work out the following:

1. $|F(\omega)|^2$
2. Phase-angle of $F(\omega)$: $\phi(\omega)$, and the **group-delay**: $-\frac{d\phi}{d\omega}$.

Suppose we have the "modulus" of a time-domain function, like this:

$$|f(t)|^2 = \int_{-\infty}^{\infty} f^*(t)f(t)dt \tag{22}$$

(This does not really work for the δ-function). What should the power be for $f(t) = a\delta(t - t_0)$, if power is proportional to the square of f?

Did you notice something interesting?

$$|F(\omega)|^2 = |f(t)|^2 = a^2 \tag{23}$$

It turns out that all the functions for which the Fourier-transform is well-behaved have this property, called *Parseval's theorem.* For the time-domain:

$$|f(t)|^2 = \int_{-\infty}^{\infty} f^*(t)f(t)dt \tag{24}$$

This is a useful check of energy/power conservation when dealing with multiple signals in the time and frequency domains.

We'll return to this in a moment, when we think about the discrete cases.

Complex numbers 2: theory and examples

Let's review the useful properties of the Fourier transform. Let the Fourier transform operation on a function $f(t)$ be $F\{f(t)\}$. What happens for each of the following? (Observe on board).

- $f(t) = ag(t) + bh(t)$. $F\{f(t)\} =$ (linearity).
- $f(t) = g(t - t_0)$. $F\{f(t)\} =$ (time-shifting).
- $f(t) = \exp(j\omega t_0)g(t)$. $F\{f(t)\} =$ (frequency-shifting).
- $f(t) = g(at)$. $F\{f(t)\} =$ (time-scaling).

These properties open up a set of tasks we can now perform. For example, we know the Fourier transform of the square-wave $f(t)$[6] was

$$F\{f(t)\} = T\left(\frac{\sin(x)}{x}\right) \tag{25}$$

with $x = \omega T/2$. We can calculate the Fourier-transform $F\{g(t)\}$ if

$$g(t) = 2f(t) + f(t - \pi/2) \tag{26}$$

Shouldn't this just be a combination of the $sin(x)/x$ functions from before?

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[6]This is often called a sync-function, and be ready to recall it later when we encounter sampling in detail.

Other properties of the Fourier transform:

- $f(t) = g^*(t)$. $F\{f(t)\} =$ (complex conjugation).
- $F\{f(t)\}_{\omega=0} =$ (DC-component, or 0th-component).

For example, suppose

$$f(t) = \exp(-t/\tau) \ \ if \ t \geq 0, \ \ 0 \ \ if \ t < 0 \tag{27}$$

What's the DC-component of the Fourier spectrum? You can even think of this one in terms of just units. (Even though we calculated this one earlier), can we sketch the Fourier spectrum of $f(t)$ using the DC-component and some intuition about the limit?

Finally, there is a useful property when derivatives are involved.

- $f(t) = g'(t)$. $F\{f(t)\} =$ (Fourier transform of a derivative).

This one involves *integration by parts:*

$$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du \tag{28}$$

In summary:

$$\boxed{F\{f'(t)\} = -j\omega F\{f(t)\}} \tag{29}$$

What is $F\{f''(t)\}$? What is it for many derivatives? Well, how about differential equations? Consider this one:

$$af''(t) - bf'(t) + c = 0 \tag{30}$$

Taking the Fourier-transform of both sides, you can solve for $F\{f(t)\}$! Then you just have to take the inverse Fourier transform.

Solve this (familiar) differential equation, using the Fourier transform ideas:

$$f''(t) + k^2 f(t) = 0 \qquad (31)$$

For anyone who has taken the physics of waves and oscillations, this equation looks familiar because it produces simple harmonic motion.

Complex numbers 2: theory and examples

There is so much more here, especially in electrical engineering!
Remember the AC-circuit elements from before? We can now understand the origin of the impedances! Recall that Ohm's law is $v = iR$ in *both the time-domain and the frequency domain.*

- (The resistor is trivial). if R is a constant, then the Fourier transform of a constant is a constant. Thus, $Z_R = R$.

- Definition of capacitance: $Q = CV$. Take the derivative of both sides and then take the Fourier transform of both sides, compare to Ohm's law.

- Definition of inductance: $L = \phi_B/i$, the ratio of magnetic flux to current. Solve for the flux, then take the negative derivative of both sides. Use Lenz's law $v = -d\phi_B/dt$. Now take the Fourier transform of both sides and rearrange. Compare to Ohm's law.

Coding with Octave, and Application: Low and high pass RC filters

$$\mathcal{F}(f(t)) = \tilde{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad (32)$$

In Eq. 31, ω is the angular frequency, measured in radians per unit time. Let $f(t) = g'(t)$. Substituting into Eq. 31, and integrating by parts, we have

$$\tilde{F}(\omega) = g(t)e^{-j\omega t}|_{-\infty}^{\infty} + j\omega \int_{\infty}^{-\infty} g(t)e^{-j\omega t}dt \qquad (33)$$

For physical signals that represent finite energy, $\lim_{|t|\to\infty} g(t) = 0$. This requirement simplifies Eq. 32 by making the first term on the right-hand side vanish.

$$\tilde{F}(\omega) = j\omega \int_{\infty}^{-\infty} g(t)e^{-j\omega t}dt = j\omega \mathcal{F}(g(t)) \tag{34}$$

The result may be summarized:

$$\boxed{\mathcal{F}(g'(t)) = j\omega \mathcal{F}(g(t))} \tag{35}$$

A voltage divider lowers the output voltage ($v_{out}(t)$) relative to the input voltage ($v_{in}(t)$) by some ratio, given by the resistances R_1 and R_2 in the following circuit:
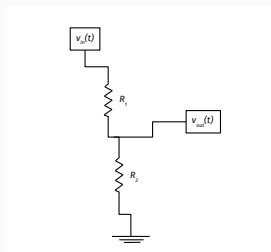

Figure 3: A two-resistor voltage divider.

The input and output voltages with respect to ground must both follow Ohm's law, where $i(t)$ is the current:

$$v_{in}(t) = (R_1 + R_2)i(t) \tag{36}$$

$$v_{out}(t) = (R_2)i(t) \tag{37}$$

The resistances R_1 and R_2 do not depend on frequency. Taking the Fourier transform of both sides of Eqs. 35 and 36, and dividing Eq. 36 by Eq. 35, we have

$$\boxed{\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = \frac{R_2}{R_1 + R_2}\frac{i(\omega)}{i(\omega)} = \frac{R_2}{R_1 + R_2}} \tag{38}$$

The ratio of output voltage to input voltage is sometimes named the *transfer function.*

Transfer function

A mathematical function relating the output or response of a system to the input or stimulus.

In Eq. 37, the transfer function only affects the overall amplitude of the output signal, while not affecting any particular frequency, nor shifting any phases.

Activity: Write octave code defining a *maximum frequency ω_{max}*, and a *frequency resolution df.* For a given pair of resistances, plot the transfer function of the voltage divider versus frequency between 0 and ω_{max} (it should be a flat line). Label the axes appropriately.

A circuit in which the resistor labelled R_2 has been replaced with a capacitor labelled C is shown in Fig. 4.

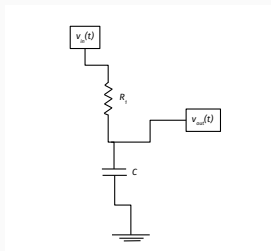Figure 4: A frequency-dependent voltage divider, also known as a single-pole RC low-pass filter.

The capacitance *C* is a constant, defined as the ratio of the charge stored on the capacitor, *q*, to the voltage required to place that charge on the capacitor, *V*:

$$C = \frac{q}{V} \tag{39}$$

$$VC = q \tag{40}$$

$$C\frac{dV}{dt} = \frac{dq}{dt} = i(t) \tag{41}$$

$$\tilde{i}(\omega) = j\omega C \tilde{V}(\omega) \tag{42}$$

$$\frac{\tilde{V}(\omega)}{\tilde{i}(\omega)} = \frac{1}{j\omega C} \tag{43}$$

Thus, Ohm's law says that the frequency-dependent resistance, or *impedance*, of a capacitor is

$$Z_C = \frac{1}{j\omega C} \tag{44}$$

The transfer function of the RC circuit in Fig. 4 is the same as Eq. 37, with R_2 replaced with Z_C (and $R_1 = R$):

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1} \tag{45}$$

Activity: Write octave code definining a *maximum frequency ω_{max}*, and a *frequency resolution df*. For a given resistance and capacitance, plot the transfer function magnitude versus frequency between 0 and ω_{max}, as well as for the phase. Label the axes appropriately, indicating that the curves are for a *low-pass filter*.

Octave application: Low and high-pass RC filters

Let the *time-constant* be defined as $\tau = RC$, and $\omega_0 = 1/\tau$. Equation 44 may be written:

$$\frac{\tilde{v}_{out}(\omega)}{\tilde{v}_{in}(\omega)} = -\frac{j\omega_0}{\omega - j\omega_0} \tag{46}$$

The magnitude and phase of Eq. 45 are

$$\boxed{M_{LP}(\omega) = \sqrt{\frac{\omega_0^2}{\omega^2 + \omega_0^2}}} \tag{47}$$

$$\boxed{\phi_{LP}(\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)} \tag{48}$$

From Eq. 46, we can see that the low-pass transfer function $M_{LP}(\omega)$ attenuates frequencies much larger than ω_0. From Eq. 47, the circuit introduces a phase-shift that is approximately linear $(-\omega/\omega_0)$ for frequencies that are not attenuated. A final interesting quantity to consider is the *group delay*, defined as the negative derivative of the phase:

$$-\frac{d\phi}{d\omega} = \tau_{LP}(\omega) = \frac{\omega_0}{\omega^2 + \omega_0^2} \qquad (49)$$

Equation 48 shows that frequency modes below ω_0 are delayed in time by a factor of $\approx \omega_0^{-1}$.

A final simple circuit to consider is the same as Fig. 4, with the capacitor and resistor exchanged (Fig. 5). Following the same arguments as the low-pass case, the complex transfer function is

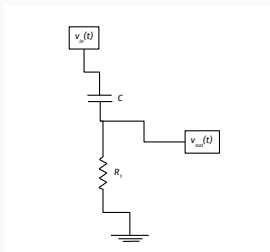$$\frac{\tilde{V}_{out}(\omega)}{\tilde{V}_{in}(\omega)} = \frac{\omega}{\omega - j\omega_0} \tag{50}$$

Figure 5: A frequency-dependent voltage divider, also known as a single-pole RC high-pass filter.

Activity: Write octave code definining a *maximum frequency ω_{max}*, and a *frequency resolution df.* For a given resistance and capacitance, plot the transfer function magnitude versus frequency between 0 and ω_{max}, as well as for the phase. Label the axes appropriately, indicating that the curves are for a *high-pass filter.*

Octave application: Low and high-pass RC filters

The magnitude and phase of Eq. 49 are

$$M_{HP}(\omega) = \sqrt{\frac{\omega^2}{\omega^2 + \omega_0^2}} \tag{51}$$

$$\phi_{HP}(\omega) = \tan^{-1}\left(\frac{\omega_0}{\omega}\right) \tag{52}$$

The group-delay turns out to be the same as Eq. 48

$$-\frac{d\phi}{d\omega} = \tau_{HP}(\omega) = \frac{\omega_0}{\omega^2 + \omega_0^2} \tag{53}$$

Unlike the voltage divider in Fig. 3, which has zero group delay, the circuits in Figs. 4 and 5 have capacitors. The group delay in these cases is driven by how quickly these capacitors can be charged and discharged, regardless of where they are in the circuit.

Notice that Eqs. 44 and 49 become infinite if the frequency is treated as a complex number, with the value $\omega = j\omega_0$. Treating the frequency as a complex number is useful for further analysis using *contour integration*, allowing us to perform integrals with singularities. A complex frequency that makes the transfer functions infinite is called a *pole*, and a frequency that makes the transfer functions zero is called a *zero*. Both the low-pass and high-pass filter examples above have just one pole each. The low-pass filter has a zero whenever $|\omega| \to \infty$, and the high-pass filter has a zero for $\omega = 0$. This is why these circuits are named *single-pole* low and high-pass filters.

The key results so far may be summarized:

- The *transfer function* of a circuit may be defined as the ratio of the output voltage to the input voltage in the Fourier domain.

- The transfer function of a *voltage divider* is given by Eq. 37, a frequency-independent ratio with no phase shift.

- The transfer function of a *single-pole low-pass filter* is given by Eq. 44. Magnitude: Eq. 46 Phase: Eq. 47

- The transfer function of a *single-pole high-pass filter* is given by Eq. 49. Magnitude: Eq. 50 Phase: Eq. 51

Notice we had to derive the transfer function first, in each case, and then use octave to create the plots. How do we use tools like octave to obtain the transfer function? Or, what if we *cannot* derive the transfer function, but have to simply *measure it*? (This is usually the case).

To understand the origin of the transfer function for arbitrarily complex systems, we'll begin the topic of linear, time-invariant (LTI) systems.

Conclusion

Previous lectures covered:

- Complex numbers 1: Arithmetic and some calculus (continuous and discete) … see Chapter 30 of text

This lecture will cover:

- Complex numbers 2: The Fourier series and Fourier transform (continuous and discrete)

Next lecture will cover:

- *Time-permitting*: The Laplace transform (continuous and discrete)