NAILYN LOPEZ

Quiz 1: Digital Signal Processing

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1. For the following exercise, recall that the real part of a complex number is $\Re\{z\}$ and the imaginary part is $\Im\{z\}$. Suppose we have a voltage signal as a function of time: $v(t) = 2.5\cos(2\pi ft - \pi/4)$. The signal has an amplitude of 2.5 Volts, a frequency $f = 1$ kHz, and a phase shift of $\pi/4$ (45 degrees). Let $\phi = 2\pi ft - \pi/4$. (a) Show that

$$v(t) = \Re\{2.5e^{j\phi}\} \quad (1)$$

(b) Show that

$$v(t) = \Im\{2.5e^{j(\phi-\pi/2)}\} \quad (2)$$

(a) $v(t) = 2.5\cos(2\pi ft - \pi/4)$
$\phi = 2\pi ft - \frac{\pi}{4}$
$e^{j\phi} = \cos\phi + j\sin\phi$
$2.5e^{j\phi} = 2.5(\cos\phi + j\sin\phi)$
$\Re(2.5e^{j\phi}) = 2.5\cos\phi$
$v(t) = 2.5\cos(2\pi ft - \pi/4)$
$v(t) = \Re(2.5e^{j\phi})$

(b) $\phi = \frac{\pi}{2} = (2\pi ft - \pi/4) - \frac{\pi}{2}$
$= 2\pi ft - \frac{3\pi}{4}$
$e^{j(\phi-\pi/2)} = \cos(\phi = \frac{\pi}{2}) + j\sin(\phi \frac{\pi}{2})$
$\cos(\phi - \pi/2) = \sin\phi$, $\sin(\phi - \pi/2) = -\cos\phi$
$e^{j(\phi-\pi/2)} = \sin\phi - j\cos\phi$
$2.5e^{j(\phi-\pi/2)} = 2.5(\sin\phi - j\cos\phi) = -2.5\cos\phi$
$v(t) = \{2.5e^{j(\phi-\pi/2)}$

2. **Sampling a sine wave**: Let a set of sample times be 0, Δt, $2\Delta t$, ... $n\Delta t$. Let the frequency and period of a sinusoidal signal be f, and $T = 1/f$. (a) Show that kHz^{-1} is 1 millisecond. (b) If the period is 5 ns, what is the frequency? (c) Suppose we are sampling a sinusoidal signal with $f = 5$ kHz. If our sampling frequency, f_s is 50 kHz, how many samples per period? (d) If our $\Delta t = 1/f_s = 0.002$ ms, how many samples per period?

(a) $1\text{kHz}^{-1} = \frac{1}{10^3\text{Hz}}$
$= \frac{1}{1000}s = 1m$

(b) $f = \frac{1}{T} = \frac{1}{5\times10^9 s}$
$f = 2\times10^8\text{Hz} = 200\text{MHz}$

(c) $T = \frac{1}{f} = \frac{1}{5\times10^3} = 0.0002$
$s > 0.2ms$
$\Delta t = \frac{1}{f_s} = \frac{1}{50\times10^3} = 0.00002s$
$= 0.02ms$

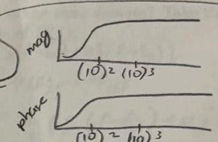
$\Delta t = \frac{1}{f_s} = \frac{1}{50\times10^3} = 0.00002s$
$= 0.02ms$
$\frac{T}{\Delta t} = \frac{0.2ms}{0.02ms} = 10$

(d) $\Delta t = 0.002ms$
$T = 0.2ms$
$\frac{T}{\Delta t} = \frac{0.2ms}{0.002ms}$
$= 100$

3. **Digitizing voltages**: Suppose we are dealing with an AC circuit that produces waveforms for audio systems. The output runs from 0 to 2.56 Volts. (a) If we can *digitize* the new voltage range into 256 steps, what is the voltage range between steps, ΔV? (b) What power of 2 gives 256? (c) If we double the number of **bits**, what is the new ΔV?

(a) $\Delta V = \frac{2.56V}{256} = 0.01V$
$\Delta V = 10mV$
(b) $(2)^8 = 256$
(c) $(2)(8) = 16\text{bits}$
$(2)^{16} = 65536$
$\Delta V = \frac{2.56V}{65536}$
$\Delta V = 3.91\times10^{-5}V$
$= 39.1\mu V$

4. Consider a signal with 2.5 V amplitude, and a DC offset of 2.5 V: $s(t) = 2.5\sin(2\pi ft) + 2.5$. (a) Write a short code in `octave` that produces and plots this signal, with $f = 10$ Hz, and $\Delta t = 1$ ms. (b) Use the `randn` function to create a noise vector of the same size as $s(t)$, but with a mean of 0 and a standard deviation of 1.0: `n = randn(size(t))`. (c) Plot the signal plus noise on the same graph: `plot(t,z)`, where $z = s + n$. (d) What is the signal-to-noise ratio (SNR) of the sine wave plus noise? (e) Use the `hist` command to create a histogram of the values of z.



5. The response of a simple high-pass RC filter is

$$R(f) = j\omega\tau/(1 + j\omega\tau) \quad (3)$$

(See memory bank). (a) Find the magnitude[1] of Eq. 3. (b) Find the phase angle of Eq. 3. (c) Graph the magnitude and phase angle versus frequency, by hand. (d) Suppose a signal has a an amplitude of A at a frequency f: $A(f)$. The filtered amplitude is $R(f)A(f)$. If $A = 1$ at $f = 0.5$ kHz, $R = 1$ kΩ, and $C = 1$ μF, what is the filtered amplitude $A(f)R(f)$?[2]

(a) $R(f) = \frac{j\omega\tau}{1+j\omega\tau}$
$|R(f)| = \left|\frac{j\omega\tau}{1+j\omega\tau}\right|$
$|z| = \sqrt{a^2+b^2}$
$|1+j\omega\tau| = \sqrt{1^2+(\omega\tau)^2}$
$= \sqrt{1+(\omega\tau)^2}$
$|j\omega\tau| = |\omega\tau| = \omega\tau$
$|R(f)| = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}$
$|R(f)| = \frac{2\pi fRC}{\sqrt{1+(2\pi fRC)^2}}$

(b) $\theta(f) = \tan^{-1}(\omega\tau)$
$\theta(f) = \frac{\pi}{2} - \tan^{-1}(\omega\tau)$
$\theta(f) = \tan^{-1}(2\pi RC) - \frac{\pi}{2}$

(c) $|R(f)| = \frac{0}{\sqrt{1}} = 0$
$|R(f)| = \frac{\omega\tau}{\omega\tau} = 1$

(d) $\tau = RC = (1000)(1.0^{-6}) = 10^{-3}s$
$\omega = 2\pi(500) = 1000\pi\ \text{rad/s}$
$|R(f)| = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}$
$|R(500)| = \frac{(1000\pi)(10^{-3})}{\sqrt{1-(1000\pi\times10^{-3})^2}}$
$= \frac{\pi}{\sqrt{1+\pi^2}}$

6. (a) If the sampling rate is 10 kHz, and the analog signal frequency is 2.5 kHz, what is the sampled frequency? (b) If the sampling rate is 10 kHz, and the analog signal frequency is 5 kHz, what is the sampled frequency? (c) If the sampling rate is 10 kHz, and the analog signal frequency

[1] Hint: multiply the top and bottom by the complex conjugate of the denominator.

[2] This filtered amplitude is a result of the *convolution theorem*, which we will encounter in a later chapter.

$= \frac{\pi}{\sqrt{1+\pi^2}} = \frac{3.14}{\sqrt{1+9.86}} = 0.95$

$A(f)R(f) = 1\times 0.95V$

is 15 kHz, what is the sampled frequency? (d) If the sampling rate is 10 kHz, and the analog signal frequency is 20 kHz, what is the sampled frequency?

(a) $f_a = 2.5\,kHz$

(b) $f_a = 5\,kHz$

(c) $f_a = |15 - 1(10)|$

$f_a = |15 - 10|$

$f_a = 5\,kHz$

(d) $f_a = |20 - 2(10)|$

$f_a = |20 - 20|$

$f_a = 0\,kHz$

7. Let a system S act on a signal $s(t)$ as follows: $S[s(t)] = s(t - T/2)$. (a) If $s(t) = 2\sin(2\pi f t)$, and $T = 1/f$, what is $S[s(t)]$? (b) Graph the input and output of S. (c) What is $s(t) + S[s(t)]$?

(a) $S[s(t)] = s(t - \frac{T}{2}) = 2\sin(2\pi f(t - \frac{1}{2f}))$

$\sin(A-B) = \sin A\cos B - \cos A \sin B$

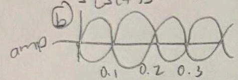
$A = 2\pi f t$ & $B = 2\pi f \frac{1}{2f} = \pi$

$S[s(t)] = 2\sin(\pi f t - \pi)$

$\sin(x - \pi) = -\sin x$

$S[s(t)] = -2\sin(2\pi f t)$

(b)



(c) $s(t) + S[s(t)]$

$= 2\sin(2\pi f t) + (-2\sin(2\pi f t))$

8. Suppose a signal component is the impulse $x[n] = [000200...]$, with 100 total samples. (a) If $y[n] = S(x[n]) = -x[n-1]$, what is $y[n]$? (b) If $y[n] = S(x[n]) = (x[n])^2$, what is $y[n]$? (c) Are the systems S in parts (a) and (b) linear or nonlinear?

(a) $y[n] = -x[n-1]$

$y[n] = [0, 0, -2, 00]$

(b) $y[n] = [0, 00, [2]^2, 00]$

$y[n] = [0,0,0,4,0,0...]$

(c) (a) linear (b) nonlinear

9. Determine if the following functions are even or odd:

- $\cos(2\pi f t)$: $\to \cos(2\pi f(-t)) = \cos(-2\pi f t)$ ✓ even
- $\exp(-(t/\sigma)^2)$: $\to (\frac{-t}{\sigma})^2 = (\frac{t}{\sigma})^2 = \exp(-(\frac{t}{\sigma})^2)$ even
- $\exp(-\alpha t)$: $\to \exp(-\alpha(-t)) = \exp(\alpha t) \neq \exp(-\alpha t)$ neither
- $at^2 + bt + c$: $\to a(-t)^2 + b(-t) + c = at^2 - bt + c$ odd

10. Using the properties of integrals and complex numbers, show that the Fourier transform operator is: (a) homogeneous, (b) additive, (c) and shift-invariant (up to a complex constant).

(a) $F(cf(t)) = \int_{-\infty}^{\infty} c f(t) e^{-i\omega t} dt$

$= c\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

$= cF(f(t))$

(b) $F(f_1(t) + f_2(t)) = \int_{-\infty}^{\infty} [f_1(t) + f_2(t)] e^{-i\omega t} dt$

$F(f_1(t) + f_2(t)) = \int_{-\infty}^{\infty} f_1(t) e^{-i\omega t} dt + \int_{-\infty}^{\infty} f_2(t) e^{-i\omega t} dt$

$= F(f_1(t)) + F(f_2(t))$

(c) $F(f(t - t_0)) = \int_{-\infty}^{\infty} f(t - t_0) e^{-i\omega t} dt$

$= \int_{-\infty}^{\infty} f(u) e^{-i\omega(u + t_0)} du$

$= e^{-i\omega t_0} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du = e^{-i\omega t_0} \hat{f}(\omega)$ 2

11. The **Dirac δ-function** is a distribution defined by the following property:

$$f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt \tag{4}$$

In words, the integral of a δ-function times a function f is the value of the function at t_0.(a) What is the Fourier transform of $a\delta(t - t_0)$? (b) What is the magnitude of the result? (c) What is the phase angle?

(a) $f(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

$F(\delta(t - t_0)) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-i\omega t} dt$

$= e^{-i\omega t_0} = e^{-i\omega t_0}$

(b) $|e^{-i\omega t_0}| = 1$

$|F(\delta(t - t_0))| = 1$

(c) $PA = -\omega t_0$

12. (a) Suppose we have a signal in the *frequency domain*: $F(f) = (a/2)(\delta(f - f_0) + a\delta(f + f_0))$. What is this signal in the *time domain*? Take the **inverse Fourier transform** of $F(f)$. (b) Suppose we have a signal in the *frequency domain*: $F(f) = (a/2j)(\delta(f - f_0) - a\delta(f + f_0))$. What is this signal in the *time domain*? Take the **inverse Fourier transform** of $F(f)$.

(a) $F(f) = \frac{a}{2}(\delta(f - f_0) + a\delta(f + f_0))$

$f(t) = \int_{-\infty}^{\infty} F(f) e^{i2\pi f t} df$

$f(t) = \int_{-\infty}^{\infty} \frac{a}{2}(\delta(f - f_0) + a\delta(f + f_0)) e^{i2\pi f t} df$

$f(t) = \frac{a}{2}[e^{i2\pi f_0 t} + a e^{-i2\pi f_0 t}]$

$f(t) = \frac{a}{2} e^{i2\pi f_0 t} + \frac{a^2}{2} e^{-i2\pi f_0 t}$

(b) $F(f) = \frac{a}{2i}(\delta(f - f_0) - a\delta(f + f_0))$

$f(t) = \int_{-\infty}^{\infty} F(f) e^{i2\pi f t} dt$

13. **Amplitude modulation.** (a) Express the following functions as complex exponentials: $A\cos(2\pi f_{LO} t)$ and $(m/A)\cos(2\pi f_A t)$. The frequencies f_{LO} and f_A are the local oscillator (carrier) and audio frequencies, respectively. (b) Multiply the two functions, and show that the result is a pair of sinusoids at two new frequencies. What are the new frequencies?

(a) $\cos(2\pi f t) = Re(e^{i2\pi f t})$

$A\cos(2\pi f_{LO} t) = A\,Re(e^{j2\pi f_{LO} t})$

$A\cos(2\pi f_{LO} t) = \frac{A}{2}(e^{j2\pi f_{LO} t} + e^{-j2\pi f_{LO} t})$

$\frac{m}{A}\cos(2\pi f_A t) = \frac{m}{A} Re(e^{j2\pi f_A t})$

$\frac{m}{A}\cos(2\pi f_A t) = \frac{m}{2A}(e^{j2\pi f_A t} + e^{-j2\pi f_A t})$

(b) $A\cos(2\pi f_{LO} t)(\frac{m}{A})(\cos(2\pi f_A t))$

$(\frac{A}{2}(e^{j2\pi f_{LO} t} + e^{-j2\pi f_{LO} t}))(\frac{m}{2A}(e^{j2\pi f_A t} + e^{-j2\pi f_A t}))$

$= \frac{m}{4}(e^{j2\pi f_{LO} t} + e^{-j2\pi f_{LO} t})(e^{j2\pi f_A t} + e^{-j2\pi f_A t})$

4.
(a) Define parameters and generate the signal

```
f = 10;          % Frequency in Hz
dt = 0.001;      % Time step (1 ms = 0.001 s)
t = 0:dt:1;      % Time vector from 0 to 1 second
s = 2.5 * sin(2 * pi * f * t) + 2.5;  % Signal with DC offset
```

(b) Generate noise

```
n = randn(size(t));  % Gaussian noise with mean 0 and standard deviation 1
```

(c) Compute noisy signal and plot both

```
z = s + n;
figure;
plot(t, s, 'b', 'LineWidth', 1.5); hold on; % Plot signal in blue
plot(t, z, 'r', 'LineWidth', 1.2);          % Plot noisy signal in red
xlabel('Time (s)');
ylabel('Amplitude');
title('Signal with Noise');
legend('Original Signal', 'Signal + Noise');
grid on;
```

(d) Compute Signal-to-Noise Ratio (SNR)

```
signal_power = mean(s.^2);  % Average power of signal
noise_power = mean(n.^2);   % Average power of noise
SNR = 10 * log10(signal_power / noise_power);
fprintf('Signal-to-Noise Ratio (SNR): %.2f dB\n', SNR);
```

(e) Create a histogram of noisy signal values

```
figure;
hist(z, 50);  % Histogram with 50 bins
xlabel('Amplitude');
ylabel('Frequency');
title('Histogram of Noisy Signal');
grid on;
```

1.
(a) - Delta Function Creation

```
fs = 20000; % Sampling frequency (20 kHz)
duration = 2; % Duration in seconds
n_samples = fs * duration; % Number of samples in the delta function
```

Create a delta function (impulse) for 2 seconds

```
delta_function = zeros(1, n_samples);
delta_function(1) = 1; % Impulse at the first sample
```

b) - Echo every 0.25 seconds
echo_interval = 0.25 * fs; % 0.25 seconds interval in samples
echo_signal = delta_function;

Add echo at 0.25s, 0.5s, 0.75s, etc. with decreasing amplitude
for i = echo_interval:echo_interval:n_samples
    echo_signal(i) = echo_signal(i) + 0.5^(i/echo_interval) * 1; % Decaying amplitude
end

(c) - Further modify the echo to decrease amplitude by half with each echo
he loop above already takes care of this behavior by multiplying the amplitude by half for each subsequent echo.

(d) - Create a sine tone
f_sine = 1000; % Frequency of the sine tone (1 kHz)
t_sine = 0:1/fs:1-1/fs; % Time vector for 1 second duration
sine_tone = sin(2 * pi * f_sine * t_sine); % Sine wave for 1 second

Extend the sine wave to a 2-second signal with zeros after 1 second
sine_wave = [sine_tone, zeros(1, n_samples - length(sine_tone))];

(e) - Convolve the echo response with the sine tone
output_signal = conv(sine_wave, echo_signal);

Normalize the output signal to avoid clipping
output_signal = output_signal / max(abs(output_signal));