2.

```
fs = 44100; % Sampling frequency (Hz)
f = 440;    % Frequency of sine wave (Hz)
T = 1/fs;   % Sampling period
t = 0:T:0.01; % Time vector (10 ms duration)
n0 = 20;    % Shift amount (change this to see phase shift)

% Generate the sine wave
x = sin(2 * pi * f * t);

% Create the shifted impulse δ[n - n0]
delta = zeros(1, length(x));
delta(n0 + 1) = 1; % Since Octave indices start at 1

% Convolve the sine wave with the impulse
y = conv(x, delta, 'same'); % Keep the output size the same as input

% Plot the original and shifted sine waves
figure;
plot(t, x, 'b', 'LineWidth', 1.5); hold on;
plot(t, y, 'r', 'LineWidth', 1.5);
legend('Original Sine Wave', 'Phase-Shifted Sine Wave');
xlabel('Time (s)');
ylabel('Amplitude');
title(['Sine Wave Shifted by n_0 = ', num2str(n0)]);
grid on;
```

NAILYN LOPEZ

(3)(b)
$s(t) = \int_{-\infty}^{\infty} \frac{a}{2j}(\delta(f-f_0) - \delta(f+f_0))e^{j2\pi ft}df$
$s(t) = \frac{a}{2j}e^{j2\pi f_0 t} - \frac{a}{2j}e^{-j2\pi f_0 t}$
$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$
$s(t) = a\sin(2\pi f_0 t)$

Homework 3, Unit 0: Foundations and Fundamentals

Prof. Jordan C. Hanson

February 19, 2025

1 Memory Bank

- **Homogeneous system:** Let k be a constant, and let $s_{\text{in}}(t)$ and $s_{\text{out}}(t)$ be the input and output signals to a system S, respectively. S is *homogeneous* if:

$$s_{\text{out}}(t) = S[s_{\text{in}}(t)] \quad (1)$$
$$ks_{\text{out}}(t) = S[ks_{\text{in}}(t)] \quad (2)$$

- **Additive system:** Let $s_1(t)$ and $s_2(t)$ be two input signals to a system S, with outputs $s_1'(t)$ and $s_2'(t)$. S is *additive* if:

$$s_1'(t) = S[s_1(t)] \quad (3)$$
$$s_2'(t) = S[s_2(t)] \quad (4)$$
$$s_1'(t) + s_2'(t) = S[s_1(t) + s_2(t)] \quad (5)$$

- **Shift-invariant system:** Let $s_{\text{in}}(t)$ and $s_{\text{out}}(t)$ be input and output signals to a system S, and let t_0 be a constant. S is *shift invariant* if:

$$s_{\text{out}}(t) = S[s_{\text{in}}(t)] \quad (6)$$
$$s_{\text{out}}(t - t_0) = S[s_{\text{in}}(t - t_0)] \quad (7)$$

- $F(f) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-2\pi jft}dt$ ... The Fourier Transform.
- $\mathcal{F}^{-1}\{F(f)\} = \int_{-\infty}^{\infty} F(f)e^{2\pi jft}df$ ... The Inverse Fourier Transform.
- The **Dirac δ-function** is a distribution defined by the following property:

$$f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt \quad (8)$$

In words, the integral of a δ-function times a function f is the value of the function at t_0.

- **Convolution**: this is an operation that characterizes the response $h[n]$ of a linear system.

$$y[i] = h[n] * x[n] = \sum_{j=0}^{M-1} h[j]x[i-j] \quad (9)$$

In words, the output at sample i is equal to the produce of the system response h and the input signal x, summed over the proceeding M samples (from $j = 0$ to $j = M - 1$).

$\int_{-\infty}^{\infty} f(t)\delta((t) - (T_2))dt$
$f(T_2) = a_1\cos(2\pi f_2 T_2) + a_2\cos(2\pi f_2 T_2)$
$f(T_2) = a_1\cos(2\pi f_1(\frac{1}{2f_1})) + a_2\cos(2\pi(2f_1)(\frac{1}{2f_1}))$
$f(T_2) = a_1\cos(\pi) + a_2\cos(2\pi)$
$f(T_2) = a_2(-1) + a_2(1)$
$f(T_2) = -a_1 + a_2$

2 Linear Systems

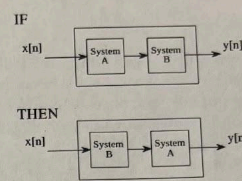

Figure 1: Linear systems **commute**.

1. Consider Fig. 1, which depicts two linear systems A and B. Symbolically, systems A and B **commute** if $A\{B\{x[n]\}\} = B\{A\{x[n]\}\}$. (a) Let $A\{x[n]\} = 2x[n] - 1$, and $B\{x[n]\} = 0.5x[n]$. Which system, A or B, is a linear system? For the system that is not linear, which linear property does it break? (b) Modify the non-linear system to make it linear, and show that A and B commute.

(a) A: non-linear
B: linear
(b) $A\{x[n]\} = 2x[n]$
$A\{B\{x[n]\}\} = B\{A\{x[n]\}\}$
$A\{B\{x[n]\}\} = A\{0.5x[n]\}$
$2(0.5x[n]) = x[n]$
$B\{A\{x[n]\}\} = B\{2x[n]\}$
$0.5(2x[n]) = x[n]$

2. Consider Eq. 8 in the Memory Bank. Let $f(t) = a_1\cos(2\pi f_1 t) + a_2\cos(2\pi f_2 t)$, with $T_1 = 1/f_1$, $T_2 = 1/f_2$, and $f_2 = 2f_1$. Evaluate the following:
   - $\int_{-\infty}^{\infty} f(t)\delta(t - T_1)dt$
   - $\int_{-\infty}^{\infty} f(t)\delta(t - T_2)dt$

$f(t) = a_1\cos(2\pi f_1 t) + a_2\cos(2\pi f_2 t)$
$f_2 = 2f_2$
$T_1 = \frac{1}{f_1}$ & $T_2 = \frac{1}{f_2} = \frac{1}{2f_1}$
$f(T_1) = a_1\cos(2\pi f_1 T_1) + a_2\cos(2\pi f_2 T_1)$
$f(T_1) = a_1\cos(2\pi f_1(\frac{1}{f_1})) + a_2\cos(2\pi(2f_1)(\frac{1}{f_1}))$
$= a\cos(2\pi) + a_2\cos(4\pi)$
$f(T_1) = a_1(1) + a_2(1) = a_1 + a_2$
$f(T_1) = a_1 + a_2$

3. Let $f(t) = a\delta(t - t_0)$. (a) Show that the magnitude of the **Fourier transform** of this impulse is a. (b) Show that the phase angle, ϕ, is $-2\pi f t_0$. (c) Show that the group delay, $\tau_g = -d\phi/d\omega$ is t_0.

(a) $f(t) = a\delta(t-t_0)$

$F(f) = \int_{-\infty}^{\infty} a\delta(t-t_0)e^{-j2\pi f t}\,dt$

$F(f) = ae^{-j2\pi f t_0}$

$|F(f)| = |ae^{-j2\pi f t_0}|$

$|F(f)| = |a| = a$

(b) $\Phi(f) = \arg(ae^{-j2\pi f t_0})$

$\Phi(f) = -2\pi f t_0$

(c) $\tau_g = \frac{d\phi(f)}{d\omega}$

$\phi(\omega) = -\omega t_0$

$\frac{d\phi(\omega)}{d\omega} = -t_0$

$\tau_g = -(-t_0) = t_0$

$\tau_g = t_0$

4. Let $\delta[n]$ represent a digital impulse: [1000 0000][1]. (a) If $y[n] = S[x[n]] = 0.5x[n-2]$, what is $S[\delta[n]]$? (b) $y[n]$ is the *impulse response* of S. What is the *step response*, if the step input is $s[n] = [01111111]$?

(a) $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{other} \end{cases}$

$S[x[n]] = 0.5x[n-2]$

$S[\delta[n]] = 0.5\delta[n-2]$

$S[\delta[n]] = 0.5\delta[n-2]$

$h[n] = 0.5\delta[n-2]$

(b) $s[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$y[n] = s[n] \times h[n]$

$= \sum_{k=-\infty}^{\infty} s[k]h[n-k]$

$y[n] = \sum_{n=-\infty}^{\infty} s_k \times [0.5\delta[n-k-2]]$

$y[n] = 0.5s[n-2]$

$y[n] = \begin{cases} 0, & n < 2 \\ 0.5, & n \geq 2 \end{cases}$

$y[n] = 0.5u[n-2]$

3 Fourier Transforms and Basic Filters

1. Suppose we pass a signal $s(t)$ into a low-pass filter. The signal as a function of frequency is $S(f)$, the Fourier transform of $s(t)$. The output of the low-pass filter will be $S(f)$ times $1/(1 + j\omega\tau)$, where $\omega = 2\pi f$, and $\tau = RC$. That is, the output will be $S(f)/(1 + j\omega\tau)$. (a) Calculate the Fourier transform $S(f)$, if $s(t) = a\delta(t - t_0)$ (as we did in class). (b) Suppose we pass our impulse $s(t)$ into a low-pass filter. What is the magnitude of the output, as a function of frequency? (c) Repeat this exercise, but with a high-pass filter response: $j\omega\tau/(1 + j\omega\tau)$.

(a) $S(f) = \int_{-\infty}^{\infty} a\delta(t-t_0)e^{-j2\pi f t}\,dt$

$S(f) = ae^{-2\pi f t_0}$

(b) $H = \frac{1}{1+j\omega t}$, $\omega = 2\pi f$

$y(f) = S(f)H(f) = \frac{ae^{-j2\pi f t_0}}{1+j2\pi f\tau}$

$|y(f)| = \left|\frac{ae^{-j2\pi f t_0}}{1+j2\pi f t}\right|$

$|y(f)| = \frac{|a|}{\sqrt{1+(2\pi f t)^2}}$

(c) $H(f) = \frac{j\omega t}{1+j\omega t} = \frac{j2\pi f t}{1+j2\pi f t}$

$Y(f) = S(f)H(f) = ae^{-j2\pi f t_0}\left(\frac{j2\pi f t}{1+j2\pi f t}\right)$

$|y(f)| = |ae^{-j2\pi f t_0}| \times \left|\frac{j2\pi f t}{1+j2\pi f t}\right|$

$|y(f)| = |a| \times \left(\frac{2\pi f\tau}{\sqrt{1+(2\pi f t)^2}}\right)$

2. For the output spectra of the previous exercise, low-pass and high-pass, calculate the group delays.[2]

(a) $H(f) = \frac{1}{1+j\omega t}$, $\omega = 2\pi f$

$\Phi(f) = \arg\left(\frac{1}{1+j\omega t}\right)$

$1+j\omega t = \sqrt{1+(\omega t)^2}\,e^{j\theta}$

$\theta = \tan^{-1}(\omega t)$

$\Phi(f) = -\tan^{-1}(\omega \tau)$

$\tau = -\frac{d}{d\omega}[-\tan^{-1}(\omega t)]$

$\frac{d}{d\omega}\tan^{-1}(\omega t) = \frac{\tau}{1+(\omega\tau)^2}$

$\tau = \frac{\tau}{1+(\omega\tau)^2}$

(b) $H(f) = \frac{j\omega\tau}{1+j\omega\tau}$

$\Phi(f) = \arg\left(\frac{j\omega t}{1+j\omega t}\right)$

$H(f) = \frac{\omega\tau e^{j\pi/2}}{\sqrt{1+(\omega t)^2}\,e^{j\theta}}$

$\Phi(f) = \frac{\pi}{2} - \tan^{-1}(\omega t)$

$\tau = -\frac{d}{d\omega}\left(\frac{\pi}{2} - \tan^{-1}(\omega t)\right)$

$\tau = \frac{\tau}{1+(\omega\tau)^2}$

3. (a) Show that the inverse Fourier transform of $S(f) = (a/2)(\delta(f - f_0) + \delta(f + f_0))$ is a cosine function. (b) Show that the inverse Fourier transform of $S(f) = (a/2j)(\delta(f - f_0) - \delta(f + f_0))$ is a sine function.

(a) $S(f) = \frac{a}{2}(\delta(f-f_0) + \delta(f+f_0))$

(b)

$s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi f t}\,df$

$s(t) = \int_{-\infty}^{\infty} \frac{a}{2}(\delta(f-f_0) + \delta(f+f_0))e^{j2\pi f t}\,dt$

$\int_{-\infty}^{\infty} \delta(f-f_0)g(f)\,df = g(f_0)$

$s(f) = \frac{a}{2}e^{j2\pi f_0 t} + \frac{a}{2}e^{-j2\pi f_0 t}$

$e^{j\theta} + e^{-j\theta} = 2\cos\theta$

$s(t) = a\cos(2\pi f_0 t)$

4 Convolution and Octave Code

1. For the following exercises, use Eq. 9. Let the digital impulse be $\delta[n]$ which is 1 for $n = 0$, and 0 if $n \neq 0$. For example, $\delta[n - 5]$ is 1 when $n = 5$. (a) Show that if $x[n] = \delta[n]$, $y[n] = h[n] * x[n] = h[n]$. That is, if the input data is an impulse, the output is the system response. (b) Show that if the input impulse is shifted ($x[n] = \delta[n - n_0]$), the output is $h[n]$, shifted by the same amount.

(a) $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

$y[n] = h[n] \times x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

$y[n] = \sum_{k=-\infty}^{\infty} h[k]\delta[n-k]$

$\sum_{k=-\infty}^{\infty} h[k]\delta[n-k] = h[n]$

$y[n] = h[n]$

(b) $x[n] = \delta[n-n_0]$

$y[n] = h[n] \times \delta[n-n_0]$

$y[n] = \sum_{k=-\infty}^{\infty} h[k]\delta[n-n_0-k] = y[n] = h[n-n_0]$

2. In **octave**, use the **conv** function to convolve a 440 Hz sine wave with a $\delta[n - n_0]$ impulse. Shift the phase of the sine output by varying n_0.

[1] Let the index for data in this list of numbers start with $n = 0$.

[2] Hint: multiply the numerator and denominator of ratios by the complex conjugate of the denominator, to aid in splitting the complex expression into real and imaginary parts.