Homework 4, Unit 0: Foundations and Fundamentals

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March 8, 2025

1 Memory Bank

- **Convolution**: this is an operation that characterizes the response $h[n]$ of a linear system.

$$y[i] = h[n] * x[n] = \sum_{j=0}^{M-1} h[j]x[i-j] \tag{1}$$

In words, the output at sample i is equal to the produce of the system response h and the input signal x, summed over the proceeding M samples (from $j = 0$ to $j = M - 1$).

- **Discrete Delta Function**, $\delta[n]$: A standard impulse response that contains one non-zero sample. It has the following property:

$$x[n] = \delta[n] * x[n] \tag{2}$$

- **Discrete Fourier Transform**, for a sampled, digitized signal x_n:

$$X_{\mathrm{k}} = \sum_{n=0}^{N-1} x_n e^{-2\pi j(k/N)n} \tag{3}$$

- In DFT analysis, we often need to know the Δt, time duration for samples, and the sampling rate, f_{s}. Note that $1/f_{\mathrm{s}} = \Delta t$.
- For a sinusoid of frequency f (Hz), the period is $T = 1/f$ (seconds).
- **Inverse Discrete Fourier Transform**, for a sampled, digitized signal X_k in the frequency domain:

$$x_{\mathrm{n}} = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{2\pi j(k/N)n} \tag{4}$$

2 Impulse Response

1. **Impulse response of audio echo system.** Let the sampling frequency be 20 kHz. (a) Start with a 2-second $\delta[n]$. How many samples should it contain? (b) Modify the $\delta[n]$ to create an echo every 0.2 seconds, and give the locations of the non-zero samples. (c) Modify the response function to make each echo half the amplitude as the previous echo. (d) Test your DSP echo on a sine-tone that is 0.1 seconds long.

2. **Impulse response of a band-pass filter.** Let $l[n]$ and $h[n]$ be the impulse responses of single-pole low and high pass filters with the same cutoff frequency, f_{c}, respectively. (a) Show that, when an input signal $s[n]$ is split into two copies and sent to $l[n]$ and $h[n]$ *in parallel*, the sum of the outputs is still $s[n]$. (b) Show that the result in (a) implies that $h[n] = \delta[n] - l[n]$. (c) Now assume the cutoff frequencies are different for $h[n]$ and $l[n]$. If the filters act *in series*, the result is a *band pass* filter, if (choose A, B, C, or D):
   - A: the f_{c} of $l[n]$ is lower than that of $h[n]$.
   - B: the f_{c} of $h[n]$ is lower than that of $l[n]$.
   - C: the f_{c} of $l[n]$ is equal to that of $h[n]$.
   - D: the f_{c} of $l[n]$ and $h[n]$ are equal to one-half the sampling frequency.

   A bandpass filter filters data below one cutoff frequency, and above another cutoff frequency, leaving a “pass band” in the spectrum.

3 Discrete Fourier Transform, Filtering, and Noise

1. **Discrete Fourier Transform properties.** (a) Knowing that the DFT is a complex sum (see Eq. 3), prove that the DFT as a DSP operator is additive and homogeneous. (b) Let $X_k = \delta[k]$ be a frequency-domain signal equal to a constant at the frequency corresponding to $k = k_0$ in Eq. 4, and zero otherwise. Show that the *inverse* DFT (see Eq.4) of $\delta[k]$ is a complex sinusoid with frequency k_0. This is one way to demonstrate *sinusoidal fidelity,* that the frequency represented in the time-domain is the same one represented in the frequency domain.

2. **Spectrum of a Square Pulse.** Download the Code Lab 8 (`compare_spectra.m`) from the course Moodle page. (a) Run the code, and explain in your own words why the magnitude of the Fourier spectrum *widens* as the pulse width *narrows.* In the figure generated by the code, the Fourier spectra are shown in the left column, while the time-domain signals are shown in the right column. (b) Measure the width of the time-domain signals and the Fourier spectra in a consistent fashion, and show that the product of the time-domain width and Fourier domain width is a constant. *This is known as the uncertainty principle, that the width of the signal in one domain is inversely proportional to the width in the other domain.*