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Homework 4, Unit 0: Foundations and Fundamentals

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1 Memory Bank

- **Convolution**: this is an operation that characterizes the response $h[n]$ of a linear system.

$$y[i] = h[n] * x[n] = \sum_{j=0}^{M-1} h[j]x[i-j] \tag{1}$$

In words, the output at sample i is equal to the produce of the system response h and the input signal x, summed over the proceeding M samples (from $j = 0$ to $j = M - 1$).

- **Discrete Delta Function**, $\delta[n]$: A standard impulse response that contains one non-zero sample. It has the following property:

$$x[n] = \delta[n] * x[n] \tag{2}$$

- **Discrete Fourier Transform**, for a sampled, digitized signal x_n:

$$X_{\mathrm{k}} = \sum_{n=0}^{N-1} x_n e^{-2\pi j(k/N)n} \tag{3}$$

- In DFT analysis, we often need to know the Δt, time duration for samples, and the sampling rate, f_{s}. Note that $1/f_{\mathrm{s}} = \Delta t$.
- For a sinusoid of frequency f (Hz), the period is $T = 1/f$ (seconds).
- **Inverse Discrete Fourier Transform**, for a sampled, digitized signal X_k in the frequency domain:

$$x_{\mathrm{n}} = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{2\pi j(k/N)n} \tag{4}$$

2 Impulse Response

1. **Impulse response of audio echo system.** Let the sampling frequency be 20 kHz. (a) Start with a 2-second $\delta[n]$. How many samples should it contain? (b) Modify the $\delta[n]$ to create an echo every 0.2 seconds, and give the locations of the non-zero samples. (c) Modify the response function to make each echo half the amplitude as the previous echo. (d) Test your DSP echo on a sine-tone that is 0.1 seconds long.

2. **Impulse response of a band-pass filter.** Let $l[n]$ and $h[n]$ be the impulse responses of single-pole low and high pass filters with the same cutoff frequency, f_{c}, respectively. (a) Show that, when an input signal $s[n]$ is split into two copies and sent to $l[n]$ and $h[n]$ *in parallel*, the sum of the outputs is still $s[n]$. (b) Show that the result in (a) implies that $h[n] = \delta[n] - l[n]$. (c) Now assume the cutoff frequencies are different for $h[n]$ and $l[n]$. If the filters act *in series*, the result is a *band pass* filter, if (choose A, B, C, or D):
   - A: the f_{c} of $l[n]$ is lower than that of $h[n]$.
   - B: the f_{c} of $h[n]$ is lower than that of $l[n]$.
   - C: the f_{c} of $l[n]$ is equal to that of $h[n]$.
   - D: the f_{c} of $l[n]$ and $h[n]$ are equal to one-half the sampling frequency.

   A bandpass filter filters data below one cutoff frequency, and above another cutoff frequency, leaving a "pass band" in the spectrum.

3 Discrete Fourier Transform, Filtering, and Noise

1. **Discrete Fourier Transform properties.** (a) Knowing that the DFT is a complex sum (see Eq. 3), prove that the DFT as a DSP operator is additive and homogeneous. (b) Let $X_k = \delta[k]$ be a frequency-domain signal equal to a constant at the frequency corresponding to $k = k_0$ in Eq. 4, and zero otherwise. Show that the *inverse* DFT (see Eq.4) of $\delta[k]$ is a complex sinusoid with frequency k_0. This is one way to demonstrate *sinusoidal fidelity,* that the frequency represented in the time-domain is the same one represented in the frequency domain.

2. **Spectrum of a Square Pulse.** Download the Code Lab 8 (`compare_spectra.m`) from the course Moodle page. (a) Run the code, and explain in your own words why the magnitude of the Fourier spectrum *widens* as the pulse width *narrows.* In the figure generated by the code, the Fourier spectra are shown in the left column, while the time-domain signals are shown in the right column. (b) Measure the width of the time-domain signals and the Fourier spectra in a consistent fashion, and show that the product of the time-domain width and Fourier domain width is a constant. *This is known as the uncertainty principle, that the width of the signal in one domain is inversely proportional to the width in the other domain.*

2 Impulse Response

1. **Impulse response of audio echo system.** Let the sampling frequency be 20 kHz. (a) Start with a 2-second $\delta[n]$. How many samples should it contain? (b) Modify the $\delta[n]$ to create an echo every 0.2 seconds, and give the locations of the non-zero samples. (c) Modify the response function to make each echo half the amplitude as the previous echo. (d) Test your DSP echo on a sine-tone that is 0.1 seconds long.

a). $f_s = 20 \times 10^3\ Hz = 20 \times 10^3$ samples/sec

$T = 2\,sec$

$n = 20 \times 10^3$ samples/sec $\cdot\, 2\,sec = 4 \times 10^4$ samples

$\delta[n] = [0] \rightarrow n = 0$

b). $t_e = 0.2\,sec$

$2\,sec \times \frac{1 \text{ sample point}}{0.2\,sec} = 10$ pts.

$$\frac{4 \times 10^4}{10} = 4000$$

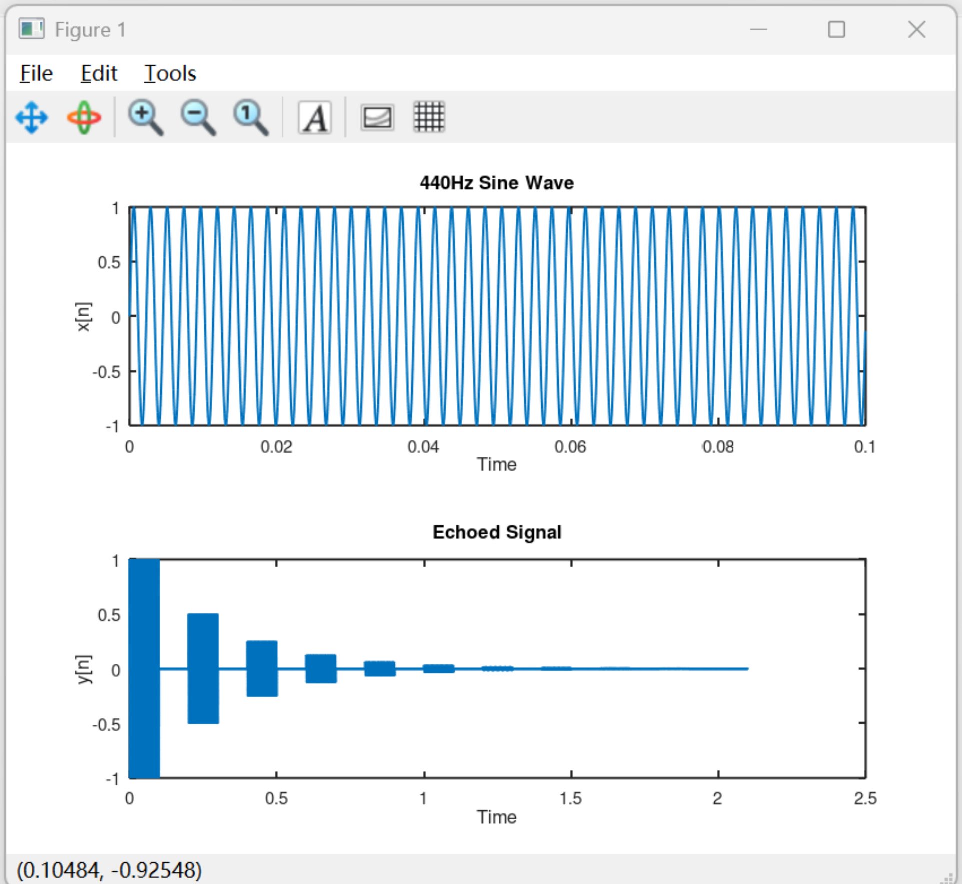
$\delta[n] = [0, 4000, 8000, 12000, 16000, 20000, 24000,$
$28000, 32000, 36000]$

c). $h[n] = \delta[n] + \frac{1}{2}[n-4000] + \frac{1}{2^2}[n-8000] \cdots + \frac{1}{2^9}[n-36000]$

$= [1, 0.5, \ldots\ldots, \frac{1}{2^9}]$

d).

```
clear;
close;
home;

fs = 20000;
T = 0.1;
N = fs * T;

t = (0:N - 1) / fs;
x = sin(2 * pi * 440 * t); %440Hz sin wave
T_impulse = 2;
N_impulse = fs * T_impulse;
h = zeros(1, N_impulse);
num_echoes = 10;
N_e = fs * 0.2;

for k = 0:num_echoes-1
    h(k * N_e + 1) = (1/2)^k;
end

y = conv(x, h);

subplot(2,1,1);
plot(t, x);
title('440Hz Sine Wave');
xlabel('Time');
ylabel('x[n]');

t_y = (0:length(y)-1) / fs;
subplot(2,1,2);
plot(t_y, y);
title('Echoed Signal');
xlabel('Time');
ylabel('y[n]');
```



2. **Impulse response of a band-pass filter.** Let $l[n]$ and $h[n]$ be the impulse responses of single-pole low and high pass filters with the same cutoff frequency, f_c, respectively. (a) Show that, when an input signal $s[n]$ is split into two copies and sent to $l[n]$ and $h[n]$ *in parallel*, the sum of the outputs is still $s[n]$. (b) Show that the result in (a) implies that $h[n] = \delta[n] - l[n]$.

a). $y_L[n] = L[n] \cdot x[n]$

we know $H(f) + L(f) = 1$

Then $L[n] + h[n] = \delta[n]$

$y_h[n] = h[n] \cdot x[n]$

since $y[n] = y_L[n] + y_h[n]$

and. $L[n] + h[n] = \delta[n]$

Then $y[n] = L[n] \cdot x[n] + h[n] \cdot x[n]$

$= (L[n] + h[n]) \cdot x[n]$

$= \delta[n]\, x[n]$

$= x[n] =$ input signal $S[n]$

b). since $h[n] + L[n] = \delta[n]$

$\Downarrow$

$h[n] = \delta[n] - L[n]$

(c) Now assume the cutoff frequencies are different for $h[n]$ and $l[n]$. If the filters act *in series*, the result is a *band pass* filter, if (choose A, B, C, or D):

- A: the f_c of $l[n]$ is lower than that of $h[n]$. $f_L < f_c < f_h$ ✓
- B: the f_c of $h[n]$ is lower than that of $l[n]$. $f_h < f_c < f_L$ ✗
- C: the f_c of $l[n]$ is equal to that of $h[n]$. $f_L = f_c = f_h$ ✗
- D: the f_c of $l[n]$ and $h[n]$ are equal to one-half the sampling frequency.

A bandpass filter filters data below one cutoff frequency, and above another cutoff frequency, leaving a "pass band" in the spectrum.

D:

$$\left.\begin{aligned} H_{LPF} &= \begin{cases} 1 & f_c < \frac{f_s}{2} \\ 0 & f_c > \frac{f_s}{2} \end{cases} \\ H_{HPF} &= \begin{cases} 0 & f_c < \frac{f_s}{2} \\ 1 & f_c > \frac{f_s}{2} \end{cases} \end{aligned}\right\} f_c \longrightarrow 0$$

✗

3 Discrete Fourier Transform, Filtering, and Noise

1. **Discrete Fourier Transform properties.** (a) Knowing that the DFT is a complex sum (see Eq. 3), prove that the DFT as a DSP operator is additive and homogeneous. (b) Let $X_k = \delta[k]$ be a frequency-domain signal equal to a constant at the frequency corresponding to $k = k_0$ in Eq. 4, and zero otherwise. Show that the *inverse* DFT (see Eq.4) of $\delta[k]$ is a complex sinusoid with frequency k_0. This is one way to demonstrate *sinusoidal fidelity*, that the frequency represented in the time-domain is the same one represented in the frequency domain.

a). ① additivity : WTS $x_n = a_n + b_n \longrightarrow X_k = A_k + B_k$

We know $X_k = \sum_{n=0}^{N-1} x_n e^{-2\pi j\left(\frac{k}{N}\right)n}$

$$= \sum_{n=0}^{N-1} (a_n + b_n) e^{-2\pi j\left(\frac{k}{N}\right)n}$$

$$= \sum_{n=0}^{N-1} a_n e^{-2\pi j\left(\frac{k}{N}\right)n} + \sum_{n=0}^{N-1} b_n e^{-2\pi j\left(\frac{k}{N}\right)n}$$

$$= A_k + B_k$$

② Homogenety : WTS $y_n = c x_n \longrightarrow Y_k = c X_k$

$$Y_k = \sum_{n=0}^{N-1} (c x_n) e^{-2\pi j\left(\frac{k}{N}\right)n}$$

$$= c \sum_{n=0}^{N-1} x_n e^{-2\pi j\left(\frac{k}{N}\right)n}$$

$$= c X_k$$

b). $X_k = \delta[k-k_0] \Rightarrow X_k = \begin{cases} 0 & k \neq k_0 \\ 1 & k = k_0 \end{cases}$

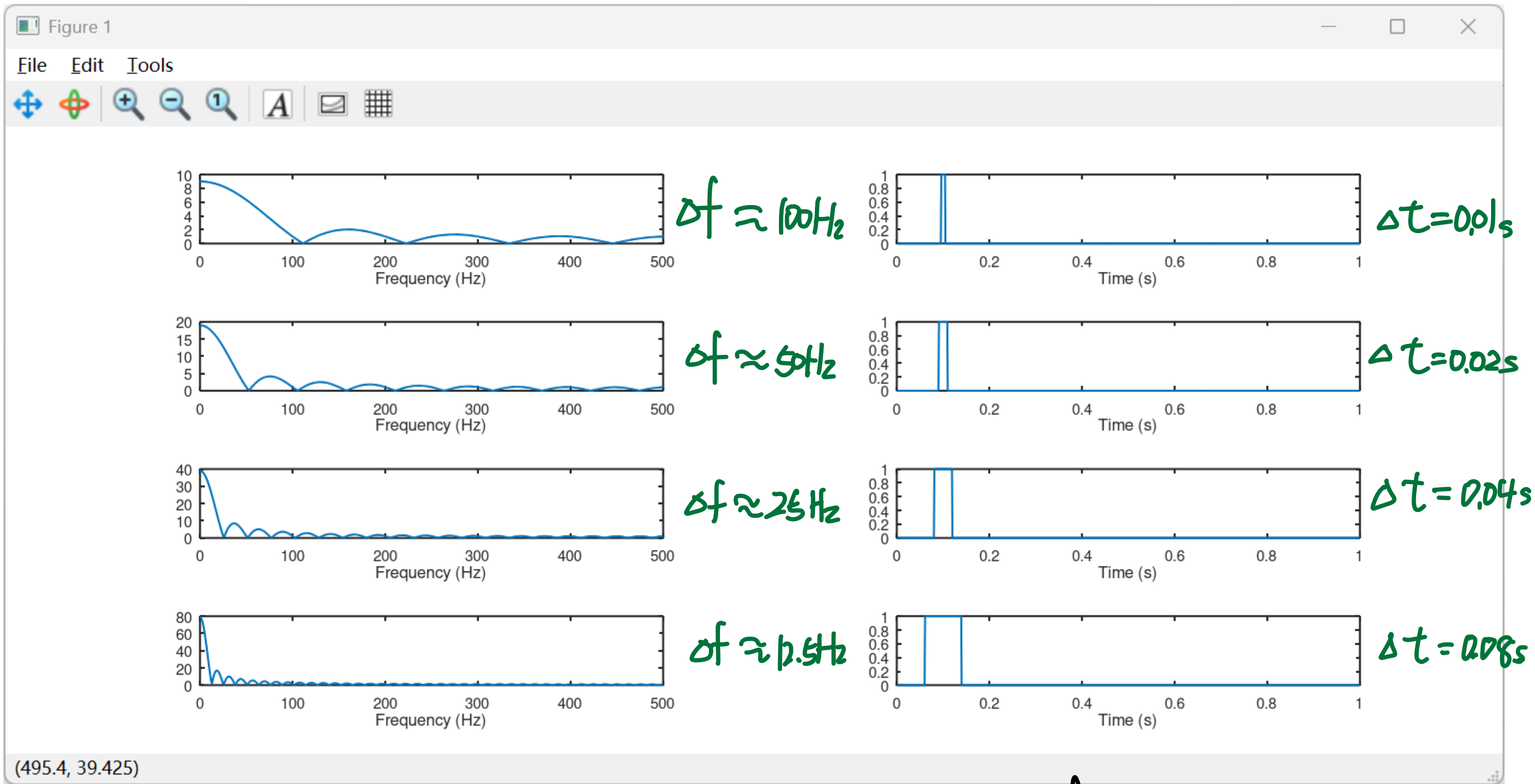
IDFT : $x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{-2\pi j (\frac{k}{N}) n}$

$= \frac{1}{N} \sum_{k=0}^{N-1} \delta[k-k_0] e^{-2\pi j (\frac{k}{N}) n}$

$= \frac{1}{N} \cdot 1 \cdot e^{-2\pi j (\frac{k_0}{N}) n}$

$= \frac{1}{N} e^{-2\pi j (\frac{k_0}{N}) n} \longrightarrow$ sin wave

2. **Spectrum of a Square Pulse.** Download the Code Lab 8 (`compare_spectra.m`) from the course Moodle page. (a) Run the code, and explain in your own words why the magnitude of the Fourier spectrum *widens* as the pulse width *narrows*. In the figure generated by the code, the Fourier spectra are shown in the left column, while the time-domain signals are shown in the right column. (b) Measure the width of the time-domain signals and the Fourier spectra in a consistent fashion, and show that the product of the time-domain width and Fourier domain width is a constant. *This is known as the uncertainty principle, that the width of the signal in one domain is inversely proportional to the width in the other domain.*



a) The graph shows that a shorter pulse requires more high frequency components to be constructed, which cause a wider spectrum;

where as a longer paule is mainly composed of low frequency, having narrower spectrum

The reason is that in DFT

$\frac{1}{f_s} = \Delta t$ where f_s is inverse relation with Δt

and we know that $\Delta f = \frac{f_s}{N}$, where f_s is proportion to Δf

Thus. Δf and Δt is inverse relation

b). from the graph we can measure

Δf	Δt	$\Delta f \cdot \Delta t$
100Hz	0.01 sec	1
50Hz	0.02sec	1
25Hz	0.04sec	1
12.5Hz	0.08Hz	1