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Homework #01

1.- Prove $z^* = \dfrac{x_1 - j y_1}{x_2 - j y_2}$ when $z = \dfrac{x_1 + j y_1}{x_2 + j y_2}$

So we start with:

$$z = \frac{x_1 + j y_1}{x_2 + j y_2} \cdot \overbrace{\left(\frac{x_2 - j y_2}{x_2 - j y_2} \right)}$$

This is "1" because the numerator and denominator are equal.
We also multiply this since because we have the conjugation in the denominators

$$z = \frac{(x_1 + j y_1)(x_2 - j y_2)}{x_2^2 + y_2^2}$$

$j^2 = -1$

$$z = \frac{x_1 x_2 - x_1 j y_2 + j y_1 x_2 - y_1 y_2 \overbrace{j^2}}{x_2^2 + y_2^2}$$

$$z = \frac{x_1 x_2 + y_1 y_2 - x_1 y_2 j + y_1 x_2 j}{x_2^2 + y_2^2}$$

$$z = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right) j$$

This is like: $z = a + b j$
then: $z^* = a - b j$

So: $$z^* = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} - \left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right) j$$

$$z^* = \frac{x_1 x_2 + y_1 y_2 - x_2 y_1 j + x_1 y_2 j}{x_2^2 + y_2^2}$$

copying everything but the numerator in a different order:

$$z^* = \frac{x_1 x_2 + x_1 y_2 j - x_2 y_1 j + y_1 y_2}{x_2^2 + y_2^2}$$

But: $y_1 y_2 = - y_1 y_2 j^2$

because: $j^2 = -1$

So:

$$z^* = \frac{x_1 x_2 + x_1 y_2 j - x_2 y_1 j - y_1 y_2 j^2}{x_2^2 + y_2^2}$$

$$z^* = \frac{x_1 (x_2 + y_2 j) - y_1 j (x_2 + y_2 j)}{x_2^2 + y_2^2}$$

$$z^* = \frac{(x_2 + y_2 j)(x_1 - y_1 j)}{x_2^2 + y_2^2}$$

But:

$$x_2^2 + y_2^2 = (x_2 + j y_2)(x_2 - j y_2)$$

So: $$z^* = \frac{\cancel{(x_2 + j y_2)}(x_1 - j y_1)}{\cancel{(x_2 + j y_2)}(x_2 - j y_2)}$$

$$z^* = \frac{x_1 - j y_1}{x_2 - j y_2}$$

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2.- Prove: $e^{j\phi} = \cos\phi + j\sin\phi$

Using Taylor series: $e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{j\phi} = 1 + \frac{j\phi}{1!} + \frac{(j\phi)^2}{2!} + \frac{(j\phi)^3}{3!} + \frac{(j\phi)^4}{4!} + \dots$$

$$e^{j\phi} = 1 + j\frac{\phi}{1!} - \frac{\phi^2}{2!} - i\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + \dots$$

$$e^{j\phi} = \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \dots\right) + j\left(\frac{\phi}{1!} - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \dots\right)$$

$$e^{j\phi} = \sum_{n=0}^{\infty} (-1)^n \frac{\phi^{2n}}{(2n)!} + j\sum_{n=0}^{\infty} \frac{\phi^{2n+1}}{(2n+1)!} \cdot (-1)^n$$

But: Using the Taylor series expansion:

$$\sin(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{(2K+1)!} \cdot x^{2K+1}$$

$$\cos(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{(2K)!} x^{2K}$$

In conclusion:

$$\boxed{e^{j\phi} = \cos\phi + j\sin\phi}$$

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3.- Prove $\cos(x) = \frac{1}{2}(e^{jx} + e^{-jx})$

As we prove in #2:

$$e^{jx} = \cos(x) + j\sin(x) \quad \ldots (\mathrm{I})$$

For the same reasion:

$$e^{-jx} = \cos(x) - j\sin(x) \quad \ldots (\mathrm{II})$$

$(\mathrm{I}) + (\mathrm{II})$:

$$e^{jx} + e^{-jx} = \cos(x) + \cancel{j\sin(x)} + \cos(x) - \cancel{j\sin(x)}$$

$$\Rightarrow e^{jx} + e^{-jx} = 2\cos(x)$$

$$2\cos(x) = e^{jx} + e^{-jx}$$

$$\cos(x) = \frac{1}{2}(e^{jx} + e^{-jx}) \quad \blacksquare$$