Homework 3, Unit 0: Foundations and Fundamentals

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1 Memory Bank

- **Homogeneous system:** Let k be a constant, and let $s_{\text{in}}(t)$ and $s_{\text{out}}(t)$ be the input and output signals to a system S, respectively. S is *homogeneous* if:

$$s_{\text{out}}(t) = S[s_{\text{in}}(t)] \tag{1}$$
$$ks_{\text{out}}(t) = S[ks_{\text{in}}(t)] \tag{2}$$

- **Additive system:** Let $s_1(t)$ and $s_2(t)$ be two input signals to a system S, with outputs $s_1'(t)$ and $s_2'(t)$. S is *additive* if:

$$s_1'(t) = S[s_1(t)] \tag{3}$$
$$s_2'(t) = S[s_2(t)] \tag{4}$$
$$s_1'(t) + s_2'(t) = S[s_1(t) + s_2(t)] \tag{5}$$

- **Shift-invariant system:** Let $s_{\text{in}}(t)$ and $s_{\text{out}}(t)$ be input and output signals to a system S, and let t_0 be a constant. S is *shift invariant* if:

$$s_{\text{out}}(t) = S[s_{\text{in}}(t)] \tag{6}$$
$$s_{\text{out}}(t - t_0) = S[s_{\text{in}}(t - t_0)] \tag{7}$$

- $F(f) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-2\pi jft}dt$ ... The Fourier Transform.
- $\mathcal{F}^{-1}\{F(f)\} = \int_{-\infty}^{\infty} F(f)e^{2\pi jft}df$ ... The Inverse Fourier Transform.
- The **Dirac δ-function** is a distribution defined by the following property:

$$f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt \tag{8}$$

In words, the integral of a δ-function times a function f is the value of the function at t_0.

- **Convolution**: this is an operation that characterizes the response $h[n]$ of a linear system.

$$y[i] = h[n] * x[n] = \sum_{j=0}^{M-1} h[j]x[i-j] \tag{9}$$

In words, the output at sample i is equal to the produce of the system response h and the input signal x, summed over the proceeding M samples (from $j = 0$ to $j = M - 1$).

2 Linear Systems

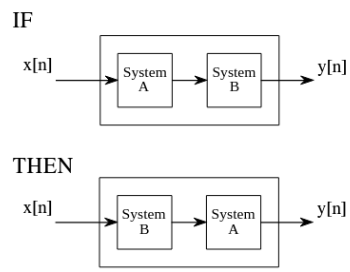


Figure 1: Linear systems **commute**.

1. Consider Fig. 1, which depicts two linear systems A and B. Symbolically, systems A and B **commute** if $A\{B\{x[n]\}\} = B\{A\{x[n]\}\}$. (a) Let $A\{x[n]\} = 2x[n] - 1$, and $B\{x[n]\} = 0.5x[n]$. Which system, A or B, is a linear system? For the system that is not linear, which linear property does it break? (b) Modify the non-linear system to make it linear, and show that A and B commute.

a) For A, Check $A\{x_1[n] + x_2[n]\} = A\{x_1[n]\} + A\{x_2[n]\}$

$2x_1[n] + 2x_2[n] - 1 \neq 2x_1[n] + 2x_2[n] - 2$

CHECK: $A\{c \cdot x[n]\} = c \cdot A\{x[n]\}$

$2c \cdot x[n] - 1 \neq 2c \cdot x[n] - c$

For B, Check $B\{x_1[n] + x_2[n]\} = B\{x_1[n]\} + B\{x_2[n]\}$

$0.5x_1[n] + 0.5x_2[n] = 0.5x_1[n] + 0.5x_2[n]$ ✓

CHECK: $B\{c \cdot x[n]\} = c \cdot B\{x[n]\}$

$c \cdot 0.5x[n] = c \cdot 0.5x[n]$ ✓

$\boxed{\text{B is linear, A is not and breaks additive and Homogeneous property}}$

b) $A\{x[n]\} = 2x[n]$ is linear

then, $A\{B\{x[n]\}\} = B\{A\{x[n]\}\}$

$2 \cdot (0.5x[n]) = 0.5(2x[n])$

$x[n] = x[n]$ ✓ $\boxed{\text{So now A and B commute}}$

2. Consider Eq. 8 in the Memory Bank. Let $f(t) = a_1\cos(2\pi f_1 t) + a_2\cos(2\pi f_2 t)$, with $T_1 = 1/f_1$, $T_2 = 1/f_2$, and $f_2 = 2f_1$. Evaluate the following:

- $\int_{-\infty}^{\infty} f(t)\delta(t - T_1)dt = a_1 + a_2$
- $\int_{-\infty}^{\infty} f(t)\delta(t - T_2)dt = -a_1 + a_2$

$\int_{-\infty}^{\infty} f(t)\delta(t - T_1)dt$

$f(T_1) = a_1\cos(2\pi f_1 T_1) + a_2\cos(2\pi f_2 T_1)$

$= a_1\cos(2\pi f_1 \cdot \frac{1}{f_1}) + a_2\cos(2\pi \cdot 2f_1 \cdot \frac{1}{f_1})$

$= a_1\cos(2\pi) + a_2\cos(4\pi)$

$= a_1(1) + a_2(1)$

$\int_{-\infty}^{\infty} f(t)\delta(t - T_2)dt$

$f(T_2) = a_1\cos(2\pi f_1 T_2) + a_2\cos(2\pi f_2 T_2)$

$= a_1\cos(2\pi \cdot f_1 \cdot \frac{1}{2f_1}) + a_2\cos(2\pi \cdot 2f_1 \cdot \frac{1}{2f_1})$

$= a_1\cos(\pi) + a_2\cos(2\pi)$

$= a_1(-1) + a_2(1)$

3. Let $f(t) = a\delta(t - t_0)$. (a) Show that the magnitude of the **Fourier transform** of this impulse is a. (b) Show that the phase angle, ϕ, is $-2\pi f t_0$. (c) Show that the group delay, $\tau_g = -d\phi/d\omega$ is t_0.

a) $F(w) = \int_{-\infty}^{\infty} a\,\delta(t - t_0)\, e^{-jwt}\, dt$

$F(w) = a e^{-jwt_0}$

$|F(w)| = |a e^{-jwt_0}| \rightarrow \boxed{|F(w)| = |a|}$

b) $\phi(w) = \arg(F(w)) = \arg(a e^{-jwt_0})$

$\phi(w) = -wt_0$

$\boxed{\phi(f) = -2\pi f t_0}$

c) $T_g = -\frac{d\phi(w)}{dw}$, since $\phi(w) = -wt_0$, $\frac{d\phi(w)}{dw} = -t_0$

$\boxed{T_g = -(t_0) = t_0}$

4. Let $\delta[n]$ represent a digital impulse: $[1000\ 0000]$[1]. (a) If $y[n] = S[x[n]] = 0.5x[n-2]$, what is $S[\delta[n]]$? (b) $y[n]$ is the *impulse response* of S. What is the *step response*, if the step input is $s[n] = [01111111]$?

a) $S[\delta[n]] = 0.5\,\delta[n-2]$

$S[\delta[n]] = [0010\ 0000]$

$\boxed{S[\delta[n]] = 0.5\,\delta[n-2]}$

b) $y[n] = 0.5\,\delta[n-2] = [0\ 0\ .5\ .5\ .5\ .5\ .5\ .5]$

3 Fourier Transforms and Basic Filters

1. Suppose we pass a signal $s(t)$ into a low-pass filter. The signal as a function of frequency is $S(f)$, the Fourier transform of $s(t)$. The output of the low-pass filter will be $S(f)$ times $1/(1 + j\omega\tau)$, where $\omega = 2\pi f$, and $\tau = RC$. That is, the output will be $S(f)/(1 + j\omega\tau)$. (a) Calculate the Fourier transform $S(f)$, if $s(t) = a\delta(t - t_0)$ (as we did in class). (b) Suppose we pass our impulse $s(t)$ into a low-pass filter. What is the magnitude of the output, as a function of frequency? (c) Repeat this exercise, but with a high-pass filter response: $j\omega\tau/(1 + j\omega\tau)$.

a) $S(f) = \int_{-\infty}^{\infty} a\,\delta(t - t_0)\, e^{-j2\pi f t}\, dt$

$S(f) = a e^{-j2\pi f t_0}$

b) $H(f)_{LPF} = \frac{1}{1+jw\tau} = \frac{1}{1+j2\pi f\tau}$

$y(f)_{LPF} = \frac{a e^{-j2\pi f t_0}}{1+j2\pi f\tau}$

$|y(f)| = \left|\frac{a e^{-j2\pi f t_0}}{1+j2\pi f\tau}\right|$

$= \frac{|a|}{\sqrt{1+(2\pi f\tau)^2}}$

c) $H(f)_{HPF} = \frac{jw\tau}{1+jw\tau} = \frac{j2\pi f\tau}{1+j2\pi f\tau}$

$Y(f) = \frac{a e^{-j2\pi f t_0} \cdot j2\pi f\tau}{1+j2\pi f\tau}$

$|Y(f)| = \left|\frac{a e^{-j2\pi f t_0} \cdot j2\pi f\tau}{1+j2\pi f\tau}\right|$

$= \frac{|a| \cdot 2\pi f\tau}{\sqrt{1+(2\pi f\tau)^2}}$

[1]Let the index for data in this list of numbers start with $n = 0$.

2. For the output spectra of the previous exercise, low-pass and high-pass, calculate the group delays.[2]

$T_g(f)_{LPF} = -\frac{d}{dw}\left[-\tan^{-1}(w\tau)\right]$

$= \frac{d}{dw}\tan^{-1}(w\tau) = \frac{\tau}{1+w^2\tau^2}$

$T_g = \frac{\tau}{1+(2\pi f\tau)^2}$

$T_g(f)_{HPF} = -\frac{d}{dw}\left(\frac{\pi}{2} - \tan^{-1}(w\tau)\right)$

$= -\left(-\frac{\tau}{1+w^2\tau^2}\right) = \frac{\tau}{1+(2\pi f\tau)^2}$

3. (a) Show that the inverse Fourier transform of $S(f) = (a/2)(\delta(f - f_0) + \delta(f + f_0))$ is a cosine function. (b) Show that the inverse Fourier transform of $S(f) = (a/2j)(\delta(f - f_0) - \delta(f + f_0))$ is a sine function.

a) $s(t) = \int_{-\infty}^{\infty} S(f)\, e^{j2\pi f t}\, df$

$s(t) = \int_{-\infty}^{\infty} \frac{a}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] e^{j2\pi f t}\, df$

$s(t) = \frac{a}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]$

$s(t) = \frac{a}{2}\left[2\cos(2\pi f_0 t)\right]$

$s(t) = a\cos(2\pi f_0 t)$

b) $s(t) = \int_{-\infty}^{\infty} \frac{a}{2j}\left[\delta(f - f_0) - \delta(f + f_0)\right] e^{j2\pi f t}\, df$

$s(t) = \frac{a}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right]$

$s(t) = \frac{a}{2j}\left[2j\sin(2\pi f_0 t)\right]$

$s(t) = a\sin(2\pi f_0 t)$

4 Convolution and Octave Code

1. For the following exercises, use Eq. 9. Let the digital impulse be $\delta[n]$ which is 1 for $n = 0$, and 0 if $n \neq 0$. For example, $\delta[n - 5]$ is 1 when $n = 5$. (a) Show that if $x[n] = \delta[n]$, $y[n] = h[n] * x[n] = h[n]$. That is, if the input data is an impulse, the output is the system response. (b) Show that if the input impulse is shifted ($x[n] = \delta[n - n_0]$), the output is $h[n]$, shifted by the same amount.

a) $x[n] = \delta[n]$

$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

$y[i] = \sum_{j=0}^{m-1} h[j]\,\delta[i - j] = h[i]$

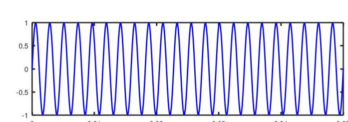
$y[i] = h[i]$

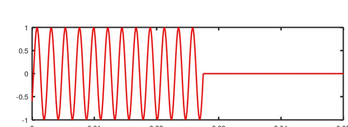
b) $x[n] = \delta[n - n_0]$

$y[i] = \sum_{j=0}^{m-1} h[j]\,\delta[i - j - n_0]$

$y[i] = h[i - n_0]$

2. In `octave`, use the `conv` function to convolve a 440 Hz sine wave with a $\delta[n - n_0]$ impulse. Shift the phase of the sine output by varying n_0.





[2]Hint: multiply the numerator a
complex conjugate of the denominato
expression into real and imaginary pa