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DSP: Quiz 1

Complex numbers 1

Rectangular → Polar

1.) $z = 4 + 4j$

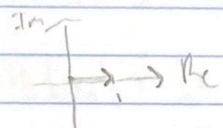
$$|z| = \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

$$\phi = 45° = \pi/4 \quad \text{graphically...}$$

So,

$$\boxed{z = (4\sqrt{2})\, e^{j(\pi/4)}}$$

2.) $z = 1$

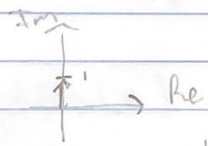


$|z| = 1$, $\phi = 0$

So,

$$\boxed{z = 1e^{j(0)} = 1}$$

$z = j$



$\phi = 90° = \frac{\pi}{2}$, $|z| = 1$

So, $\boxed{z = 1e^{j(\frac{\pi}{2})} = e^{j(\frac{\pi}{2})}}$

$z = -1$

$\phi = 180° = \pi$, $|z| = 1$

So, $\boxed{z = 1e^{j(\pi)} = e^{j(\pi)}}$

$z = -j$ $\quad$ $\phi = 270° = \frac{3\pi}{2}$, $|z| = 1$

So, $z = 1e^{j(\frac{3\pi}{2})} = e^{j(\frac{3\pi}{2})}$

the phase angle is moving 90° each time...

Polar → Rectangular

1.) $z = 2\exp(j\,\pi/4)$

$z = 2\cos(\frac{\pi}{4}) + j(2\sin(\frac{\pi}{4}))$

$\boxed{z = \frac{2}{\sqrt{2}} + j\frac{2}{\sqrt{2}}}$

2.) $z = 5\exp(j\pi)$

$z = 5\cos(\pi) + j(5\sin(\pi))$

$\boxed{z = -5 + j(0) = -5}$

Part 3

$$V(t) = a_1 \exp(j x_1) + a_2 \exp(j x_2)$$

$$x_i = 2\pi f t + \phi_i$$

1.) $|V|^2 = V^* V$

if $\phi_2 - \phi_1 = \pi \Rightarrow \phi_2 = \phi_1 + \pi$

$$\Rightarrow X_2 = 2\pi f t + \phi_2 = 2\pi f t + \phi_1 + \pi = X_1 + \pi$$

$$V = a_1 \cos(x_1) + j a_1 \sin(x_1) + a_2 \cos(X_2) + a_2 j \sin(x_2)$$

$$= a_1 \cos(x_1) + j a_1 \sin(x_1) + a_2 \cos(x_1 + \pi) + j a_2 \sin(x_1 + \pi)$$

note: $\cos(x+\pi) = -\cos(x)$
$\sin(x+\pi) = -\sin(x)$

$$V = a_1 \cos(x_1) + j a_1 \sin(x_1) - a_2 \cos(x_1) - j a_2 \sin(x_1)$$

$$V = (a_1 - a_2) \cos(x_1) + (a_1 - a_2) j \sin(x_1)$$

$$V = (a_1 - a_2) e^{j x_1}$$

$$\Rightarrow V^* = (a_1 - a_2) e^{-j x_1}$$

$$|V|^2 = V V^* = (a_1 - a_2) \cancel{e^{j x_1}} (a_1 - a_2) \cancel{e^{-j x_1}}$$

$$\boxed{|V|^2 = (a_1 - a_2)^2}$$

if $\Phi_2 - \Phi_1 = 0 \Rightarrow \Phi_1 - \Phi_2 = 0$

$$VV^* = a_1^2 + a_2^2 + a_1 a_2 \left(\underbrace{\exp(j(0))}_{1} + \underbrace{\exp(j(0))}_{1} \right)$$

$$= a_1^2 + a_2^2 + a_1 a_2$$

$$|V|^2 = VV^* = (a_1 + a_2)^2$$

2.) $\Phi_V = \tan^{-1}\left(\frac{Im\{V\}}{Re\{V\}}\right)$

$$V = a_1 \exp(jx_1) + a_2 \exp(jx_2)$$

$$Re\{V\} = a_1 \cos(x_1) + a_2 \cos(x_2)$$
$$Im\{V\} = a_1 \sin(x_1) + a_2 \sin(x_2)$$

$$x_i = 2\pi f t + \Phi_i$$

if $\Phi_2 - \Phi_1 = 0 \Rightarrow \Phi_1 - \Phi_2 = 0 \Rightarrow \Phi_1 = \Phi_2$

$$\Rightarrow x_1 = 2\pi f t + \Phi_1 = 2\pi f t + \Phi_2 = x_2$$

$$x_1 = x_2 = x$$

so, $Re\{V\} = (a_1 + a_2)\cos(x)$

$Im\{V\} = (a_1 + a_2)\sin(x)$

$$\frac{Im\{V\}}{Re\{V\}} = \left(\frac{\cancel{a_1 + a_2}}{\cancel{a_1 + a_2}}\right) \frac{\sin(x)}{\cos(x)} = \tan(x)$$

$$\Phi_V = \tan^{-1}(\tan(x)) = \underline{x}.$$

2.) If $\phi_2 - \phi_1 = \pi \Rightarrow \phi_2 = \pi + \phi_1$

$$x_2 = 2\pi f t + \phi_2 = 2\pi f t + \phi_1 + \pi$$
$$\Rightarrow x_2 = x_1 + \pi$$

$$\sin(x+\pi) = -\sin(x)$$
$$\cos(x+\pi) = -\cos(x)$$

So,
$$\sin(x_2) = \sin(x_1+\pi) = -\sin(x_1)$$
$$\cos(x_2) = \cos(x_1+\pi) = -\cos(x_1)$$

$$\mathrm{Im}\{v\} = a_1 \sin(x_1) + a_2 \sin(x_2)$$
$$= a_1 \sin(x_1) - a_2 \sin(x_1)$$
$$= (a_1 - a_2)\sin(x_1)$$

$$\mathrm{Re}\{v\} = a_1 \cos(x_1) + a_2 \cos(x_2)$$
$$= a_1 \cos(x_1) - a_2 \cos(x_1)$$
$$= (a_1 - a_2)\cos(x_1)$$

$$\frac{\mathrm{Im}\{v\}}{\mathrm{Re}\{v\}} = \frac{\cancel{(a_1-a_2)}\sin(x_1)}{\cancel{(a_1-a_2)}\cos(x_1)} = \tan(x_1)$$

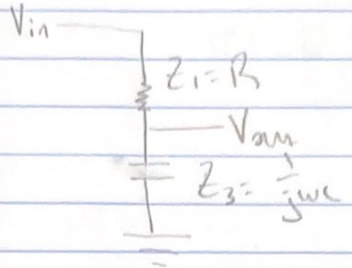
$$\underline{\phi_v = \tan^{-1}(\tan(x_1)) = x_1}$$

but also $\tan(x_1) = \tan(x_2 - \pi) = \tan(x_2)$

so $\underline{\phi_v = \tan^{-1}(\tan(x_2)) = x_2}$ also

These answers intuitively match the idea of constructive / destructive interference in the way that magnitudes add or subtract given phase differences and the way that resultant values have the same phase as the inputs.

RC Circuit



$$h(\omega) = \frac{V_{out}}{V_{in}}$$

$$V_{out} - Z_3 = 0$$
$$V_{in} - Z_1 - Z_3 = 0$$

$$\frac{V_{out}}{V_{in}} = \frac{Z_3}{Z_1 + Z_3}$$

$$h(\omega) = \frac{Z_3}{Z_1 + Z_3} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}$$

$$= \frac{1}{Rj\omega C + 1} \frac{(1 - Rj\omega C)}{(1 - Rj\omega C)}$$

$$= \frac{(1 - Rj\omega C)}{1^2 + (R\omega C)^2}$$

$$h(\omega) = \left(\frac{1}{1 + (R\omega C)^2}\right) - j\left(\frac{R\omega C}{1 + (R\omega C)^2}\right) \quad * \ \tau = RC$$

$$\boxed{h(\omega) = \left(\frac{1}{1 + (\omega\tau)^2}\right) - j\left(\frac{\omega\tau}{1 + (\omega\tau)^2}\right)}$$

$$|h(\omega)| = \left(\left(\frac{1}{1 + (\omega\tau)^2}\right)^2 + \left(\frac{\omega\tau}{1 + (\omega\tau)^2}\right)^2\right)^{\frac{1}{2}}$$

$\underline{w=0}$

$$|h(0)| = \left(\left(\frac{1}{1+0}\right)^2 + \left(\frac{0}{1+0}\right)^2\right)^{1/2} = 1$$

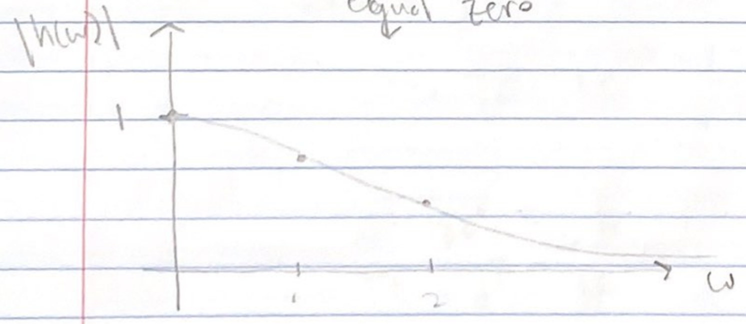
$\underline{w=1}$

$$|h(1)| = \left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \approx 0.7$$

$\underline{w=2}$

$$|h(2)| = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \sqrt{\frac{1}{25} + \frac{4}{25}} = \sqrt{\frac{5}{25}} = \frac{\sqrt{5}}{5} \approx 0.45$$

Numbers will continue to shrink as w grows since denominator contains w^2 ... will approach zero but cannot equal zero



So high frequencies have low amplitude in this filter!