Homework 1, Unit 0: Foundations and Fundamentals

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1 Memory Bank

- $\sqrt{-1} = j$ ... The fundamental imaginary unit.
- $z = x + jy$ ... A complex number.
- $\Re\{z\} = x$, $\Im\{z\} = y$ ... Real and imaginary parts.
- $z^* = x - jy$ ... The complex conjugate of z.
- $|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$ ... The magnitude of z.
- $\tan\phi = y/x$ ... The phase angle of z.
- $|z| = r$, so $x = r\cos\phi$, and $y = r\sin\phi$.
- **Taylor Series:** Let $f(t)$ be a continuous, differentiable function. Let $f^n(t)$ be the n-th derivative of $f(t)$, with $f^0(t) = f(t)$. The Taylor series is an infinite series, equivalent to $f(t)$, given by

$$f(t) = \sum_{n=0}^{\infty} \frac{f^n(t_0)}{n!}(t - t_0)^n \tag{1}$$

- **Euler's Identity:** $e^{j\phi} = \cos\phi + j\sin\phi$

2 Complex Numbers and Signals

1. Let $z_1 = 3 + 4j$, and $z_2 = -3 + 4j$. Evaluate:
   (a) Graph z_1 and z_2 in the complex plane.
   (b) $z_1 + z_2$
   (c) $z_1 - z_2$
   (d) $z_1 * z_2$
   (e) z_1/z_2
   (f) $|z_1|$
   (g) $|z_2|$
   (h) ϕ_1
   (i) ϕ_2
   (j) Write z_1 and z_2 in polar form.

2. Use Euler's Identity to show that

$$\cos(2\pi f t) = \frac{e^{2\pi j f t} + e^{-2\pi j f t}}{2} \tag{2}$$

$$\sin(2\pi f t) = \frac{e^{2\pi j f t} - e^{-2\pi j f t}}{2j} \tag{3}$$

3. Let $v_1(t) = 4\cos(2\pi f_1 t)$, $v_2(t) = 4\cos(2\pi f_2 t - \phi)$. Use the results of the previous exercise in the following questions. (a) Show that $P = v_1(t)v_2(t)$ is a pair of sinusoids with frequencies $f_+ = f_1 + f_2$ and $f_- = f_1 - f_2$, offset by a total phase shift of 2ϕ. (b) Show that $P_{\max} = 16$, if $\phi = 0$ and $f_1 = f_2$. Why is 16 the correct number?[1].

4. Suppose that

$$v_1(t) = \Im\left\{\exp(j(2\pi f t - \phi))\right\} \tag{4}$$

$$v_2(t) = \Im\left\{\exp(2\pi j f t)\right\} \tag{5}$$

Drop the portion of the complex phase containing the frequency f, and represent the signals with just $\exp(-j\phi)$ and 1. (a) Graph these signals by treating the 1 and $\exp(-j\phi)$ as complex numbers in polar form. (b) Add the complex numbers, and obtain formulas for the new magnitude and phase angle. (c) Test your formulas for $\phi = 45$ degrees. When you add two signals of the same frequency offset by a phase, you should obtain a new

[1] The product of two mixed signal voltages, divided by the resistance, is the power (in Watts). The formula is $P = v^2/R$.

signal at the same frequency with a new phase and amplitude. What happens when the signals are in phase ($\phi = 0$ degrees) and out of phase ($\phi = 180$ degrees)?

3 Probability and Statistics, Noise

1. Consider the following octave code:

```
clear;
close;
home;

x = randn(10000,1);
figure(1)
hist(x,30);
figure(2);
plot(x)
axis([-1 10001 -10 10]);
```

The octave workspace is cleared, and a vector of data `x` is created. This vector contains pseudo-random numbers drawn from *the Gaussian distribution*, with mean μ and standard deviation σ:

$$p(x)dx = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx \tag{6}$$

(a) Graph Eq. 6, and compare to Figure 1 created by the code. This figure contains a *histogram*, that counts how often the pseudo-random numbers fall into each of 30 bins. Does the histogram resemble Eq. 6? (b) Examine Figure 2 created by the code. If the numbers represent digitized, sampled noise voltages, they appear to be pure noise. (c) Write code that adds gaussian noise to a sine wave. (d) Notice that, as the amplitude of the sine wave decreases, the signal appears to be lost in the noise. The ratio of sine wave amplitude divided by σ in Eq. 6 is called the signal-to-noise ratio (SNR).

2. The octave function `rand` gives pseudo-random numbers drawn from a *uniform distribution*:

$$p(x)dx = \frac{dx}{b-a}, \quad a \leq x \leq b \tag{7}$$

This PDF is flat between a and b, where any number between these is equally likely to occur. The `rand` function has default settings of $b = 1$ and $a = 0$. Write an octave code that demonstrates that the sum of a large set of numbers drawn from `rand` is distributed according to Eq. 6. That is, we get gaussian noise from the repeated addition of many uniform random numbers.

4 ADC and DAC

1. Create an octave code that graphs a sine wave of frequency `f` and sampling frequency `fs` (see Code Lab 1 on Moodle for examples). Now tune the sampling frequency to with a factor of 2 of the signal frequency. Qualitatively, what happens to the signal graph?