HW3

Aman Kumpawat

February 28, 2025

2. Linear Systems

Problem 1.

(a) I define

$$A\{x[n]\} = 2x[n] - 1, \quad B\{x[n]\} = 0.5x[n].$$

I check if A is linear:

Homogeneity:

$$A\{kx[n]\} = 2(kx[n]) - 1 = 2kx[n] - 1 \neq k(2x[n] - 1) = 2kx[n] - k.$$

So A is not linear.

I check if B is linear:

Homogeneity:

$$B\{kx[n]\} = 0.5(kx[n]) = k\,0.5x[n] = k\,B\{x[n]\}.$$

Additivity:

$$B\{x_1[n]+x_2[n]\} = 0.5(x_1[n]+x_2[n]) = 0.5x_1[n]+0.5x_2[n] = B\{x_1[n]\}+B\{x_2[n]\}.$$

Therefore B is linear.

(b) I modify A by removing the constant term:

$$\tilde{A}\{x[n]\} = 2x[n].$$

Now I check commutation with B:

$$\tilde{A}\{B\{x[n]\}\} = \tilde{A}\{0.5x[n]\} = 2(0.5x[n]) = x[n],$$
$$B\{\tilde{A}\{x[n]\}\} = B\{2x[n]\} = 0.5(2x[n]) = x[n].$$

Thus $\tilde{A}\{B\{x[n]\}\} = B\{\tilde{A}\{x[n]\}\}$, so they commute.

Problem 2. I let

$$f(t) = a_1 \cos(2\pi f_1 t) + a_2 \cos(2\pi f_2 t), \quad T_1 = \frac{1}{f_1}, \quad T_2 = \frac{1}{f_2}, \quad f_2 = 2f_1.$$

Using $\int_{-\infty}^{\infty} f(t)\,\delta(t-T)\,dt = f(T)$, I do the following:

1. $\int f(t)\,\delta(t-T_1)\,dt = f(T_1)$. Then

$$f(T_1) = a_1 \cos(2\pi f_1 T_1) + a_2 \cos(2\pi f_2 T_1).$$

   Since $f_1 T_1 = 1, \quad f_2 T_1 = 2$, I get $\cos(2\pi \cdot 1) = 1, \quad \cos(2\pi \cdot 2) = 1$. So $f(T_1) = a_1 + a_2$.

2. $\int f(t)\,\delta(t-T_2)\,dt = f(T_2)$. Since $T_2 = \frac{1}{2f_1} = \frac{T_1}{2}$, I have $f_1 T_2 = \frac{1}{2}, \quad f_2 T_2 = 1$. So $f(T_2) = a_1 \cos(\pi) + a_2 \cos(2\pi) = -a_1 + a_2$.

Problem 3. I let $f(t) = a\,\delta(t-t_0)$.

Fourier transform:

$$F(f) = \int_{-\infty}^{\infty} a\,\delta(t-t_0)\,e^{-2\pi j f t}\,dt = a\,e^{-2\pi j f t_0}.$$

(a) Magnitude:

$$|F(f)| = |a\,e^{-2\pi j f t_0}| = |a|.$$

(b) Phase:

$$\arg\{F(f)\} = -\,2\pi f t_0.$$

(c) Group delay $\tau_g = -\frac{d}{d\omega}\big(-\omega t_0\big) = t_0$, with $\omega = 2\pi f$.

Problem 4. I let $\delta[n]$ be a digital impulse with $\delta[0] = 1$, and $\delta[n] = 0$ for $n \neq 0$. Suppose

$$y[n] = 0.5\,x[n-2].$$

1. If $x[n] = \delta[n]$, then $y[n] = 0.5\,\delta[n-2]$.

2. The impulse response is $0.5\,\delta[n-2]$. For a step $s[n] = [0, 1, 1, 1, \ldots]$, $y[n] = 0.5\,s[n-2]$. If $s[n] = u[n-1]$, then $s[n-2] = u[n-3]$, so $y[n] = 0.5\,u[n-3]$.

3. Fourier Transforms and Basic Filters

Problem 1. I let $s(t) = a\,\delta(t - t_0)$. A low-pass filter has $H_{\mathrm{LP}}(f) = \frac{1}{1+j\omega\tau}$, with $\omega = 2\pi f$.

(a) The Fourier transform of $s(t) = a\,\delta(t - t_0)$ is

$$S(f) = a\,e^{-2\pi j f t_0}.$$

(b) The output of the low-pass filter is $S_{\mathrm{out}}(f) = S(f)\,H_{\mathrm{LP}}(f)$, so

$$S_{\mathrm{out}}(f) = a\,e^{-2\pi j f t_0}\,\frac{1}{1 + j\omega\tau}.$$

Magnitude:

$$|S_{\mathrm{out}}(f)| = \frac{|a|}{\sqrt{1 + (\omega\tau)^2}} = \frac{a}{\sqrt{1 + (2\pi f\tau)^2}}.$$

(c) For a high-pass filter $H_{\mathrm{HP}}(f) = \frac{j\omega\tau}{1+j\omega\tau}$, I get

$$S'_{\mathrm{out}}(f) = a\,e^{-2\pi j f t_0}\,\frac{j\omega\tau}{1 + j\omega\tau}.$$

Magnitude:

$$|S'_{\mathrm{out}}(f)| = \frac{a\,|j\omega\tau|}{\sqrt{1 + (\omega\tau)^2}} = \frac{a\,(\omega\tau)}{\sqrt{1 + (\omega\tau)^2}} = \frac{a\,(2\pi f\tau)}{\sqrt{1 + (2\pi f\tau)^2}}.$$

Problem 2. For $S_{\mathrm{out}}(f) = \frac{a\,e^{-j\omega t_0}}{1+j\omega\tau}$, the phase is

$$\phi_{\mathrm{LP}}(\omega) = -\,\omega t_0 - \arg(1 + j\omega\tau).$$

Since $\arg(1+j\omega\tau) = \tan^{-1}(\omega\tau)$,

$$\phi_{\mathrm{LP}}(\omega) = -\,\omega t_0 - \tan^{-1}(\omega\tau).$$

The group delay is

$$\tau_{g,\mathrm{LP}}(\omega) = -\,\frac{d}{d\omega}\big(\phi_{\mathrm{LP}}(\omega)\big) = t_0 + \frac{\tau}{1+(\omega\tau)^2}.$$

For $S'_{\mathrm{out}}(f) = a\,e^{-j\omega t_0}\,\frac{j\omega\tau}{1+j\omega\tau}$, the phase includes $-\,\omega t_0 + \frac{\pi}{2} - \tan^{-1}(\omega\tau)$. Differentiating gives the same group delay:

$$\tau_{g,\mathrm{HP}}(\omega) = t_0 + \frac{\tau}{1+(\omega\tau)^2}.$$

Problem 3.

(a) I let $S(f) = \frac{a}{2}\big[\delta(f-f_0) + \delta(f+f_0)\big]$. Then

$$s(t) = \int_{-\infty}^{\infty} S(f)\,e^{2\pi jft}\,df = \frac{a}{2}\int \big[\delta(f-f_0)+\delta(f+f_0)\big]e^{2\pi jft}\,df.$$

By the sifting property,

$$s(t) = \frac{a}{2}\big[e^{2\pi jf_0t} + e^{-2\pi jf_0t}\big] = a\cos(2\pi f_0 t).$$

(b) I let $S(f) = \frac{a}{2j}\big[\delta(f-f_0) - \delta(f+f_0)\big]$. Then

$$s(t) = \int_{-\infty}^{\infty} \frac{a}{2j}\big[\delta(f-f_0)-\delta(f+f_0)\big]e^{2\pi jft}\,df = \frac{a}{2j}\big[e^{2\pi jf_0t} - e^{-2\pi jf_0t}\big].$$

Since $e^{j\theta} - e^{-j\theta} = 2j\,\sin(\theta)$, I have

$$s(t) = a\,\sin(2\pi f_0 t).$$

4. Convolution and Octave Code

Problem 1.

$$y[i] = h[n] * x[n] = \sum_{j=0}^{M-1} h[j]\,x[i-j].$$

(a) If $x[n] = \delta[n]$, then

$$y[i] = \sum_{j=0}^{M-1} h[j]\, \delta[i-j].$$

This is nonzero only when $i - j = 0$, so

$$y[i] = h[i].$$

(b) If $x[n] = \delta[n - n_0]$, then

$$y[i] = \sum_{j=0}^{M-1} h[j]\, \delta[i-j-n_0].$$

This is nonzero only when $i - j - n_0 = 0$, so

$$y[i] = h[i - n_0].$$

Problem 2. My Octave code to convolve a 440 Hz sine wave with a delayed impulse is:

```
Fs = 8000;
t = 0:1/Fs:0.005;
x = sin(2*pi*440 * t);
n0 = 40;
imp = zeros(1, length(x));
imp(n0+1) = 1;
y = conv(x, imp);
```

This shifts the sine wave by n0 samples.