Homework 2, Unit 0: Foundations and Fundamentals

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February 6, 2025

1 Memory Bank

- $\bar{x} = \frac{1}{N}\sum_{i=0}^{N-1} x_i$ ... Sample mean.
- $\overline{x^2} = \frac{1}{N}\sum_{i=0}^{N-1} x_i^2$ ... Sample mean of the square.
- $s = \frac{1}{N-1}\sum_{i=0}^{N-1}(x_i - \bar{x})^2$ ... Sample std. deviation.
- $s^2 = \overline{x^2} - \bar{x}^2$ ... Formula for the variance.
- Let a **histogram** be defined by M bins i, with the data organized into M *frequencies* H_i.
- Total number of data points in a histogram: $N = \sum_{i=0}^{M-1} H_i$
- (1) Sample mean and (2) variance from histograms:
  1. $\bar{x} = \frac{1}{N}\sum_{i=0}^{M-1} iH_i$
  2. $s = \frac{1}{N-1}\sum_{i=0}^{M-1}(i - \bar{x})^2 H_i$
- For the following two formulas: $\omega = 2\pi f$, $\tau = RC$.
- **Low-pass filter response**, as a function of frequency:

$$R(f) = \frac{1}{1 + j\omega\tau} \tag{1}$$

- **High-pass filter response**, as a function of frequency:

$$R(f) = \frac{j\omega\tau}{1 + j\omega\tau} \tag{2}$$

2 Probability and Statistics, Noise

1. Consult Fig. 2-6 in Ch. 2 of the course text. (a) Write three functions in `octave` that produce the following: a square wave, a triangle wave, and gaussian noise. (b) Write code that creates histograms of the outputs of the three functions. (c) **Normalize** the histograms by dividing the frequencies by the total number of data samples, N. (d) Graph the histograms to demonstrate that each process matches the PDFs in Fig. 2-6. (e) Compute the mean and standard deviation of each PDF.[1]

3 ADC and DAC

1. Complete the following exercises about the precision of ADC and DAC components:
   - ADC:
     (a) What is the ΔV (voltage per level) of an ADC with signals in the [0,2.55] V range with 255 levels, plus zero (8-bit precision)?

     (b) What is the ΔV (voltage per level) of an ADC with signals in the [0,4.095] V range with 4095 levels, plus zero (12-bit precision)?

     (c) How many bits of precision, or how many voltage levels, are required for $\Delta V < 1$ mV, if signals are in the [0,12] V range?

     (d) What is the digital amplitude (in counts) of a 2.52 V signal, if signals are in the [0,5] V range, and there are 2048 levels?
   - DAC:
     (a) If the digital amplitude of a signal is 256 counts, and signals are in the [0,5] V range with 9.8 mV per level, what is the signal amplitude in volts?

     (b) If the digital amplitude of a signal is 2048 counts, and signals are in the [0,5] V range with max counts 4095, what is the signal amplitude in volts?

     (c) If the digital amplitude of a signal is 128 counts, the max counts is 511, and the analog output is 0.25 V, what is the maximum voltage?

[1] *Hint: (1) square waves with amplitudes of 0 and 1 should have a mean of 0.5, (2) this is also true of flat PDFs, which also have a standard deviation of $1/\sqrt{12}$, and (3) Eq. 2-6 in the course text gives the Gaussian PDF, which has a std. dev. of σ.*

2. For the following exercises, refer to Fig. 3-4 in Ch. 3 of the course text.

   (a) If the sampling rate is 500 kHz, and the analog signal frequency is 50 kHz, what is the digital signal frequency?

   (b) If the sampling rate is 500 kHz, and the analog signal frequency is 250 kHz, what is the digital signal frequency?

   (c) If the sampling rate is 500 kHz, and the analog signal frequency is 750 kHz, what is the digital signal frequency?

   (d) If the sampling rate is 500 kHz, and the analog signal frequency is 1000 kHz, what is the digital signal frequency?

3. Consider Fig. 3-10 in the course text. The single-pole, low-pass RC filter is depicted in the top middle section of Fig. 3-10. (a) Suppose a signal has an amplitude of 3.3 V and a frequency of 25 MHz, while $R = 10$ kΩ. What value of C is necessary to filter the signal to 0.33 V?

4. Consider again Fig. 3-10. The single-pole, high-pass RC filter is similar to the depiction in the top middle section of Fig. 3-10, but with the C and R switched. (a) Suppose a signal has an amplitude of 3.3 V and a frequency of 10 MHz, while $R = 10$ kΩ. What value of C is necessary to filter the signal to 0.33 V?

5. **Bonus Point:** What is the phase shift introduced by the filters in the previous two exercises?

1. Consult Fig. 2-6 in Ch. 2 of the course text. (a) Write three functions in `octave` that produce the following: a square wave, a triangle wave, and gaussian noise. (b) Write code that creates histograms of the outputs of the three functions. (c) **Normalize** the histograms by dividing the frequencies by the total number of data samples, N. (d) Graph the histograms to demonstrate that each process matches the PDFs in Fig. 2-6. (e) Compute the mean and standard deviation of each PDF.[1]

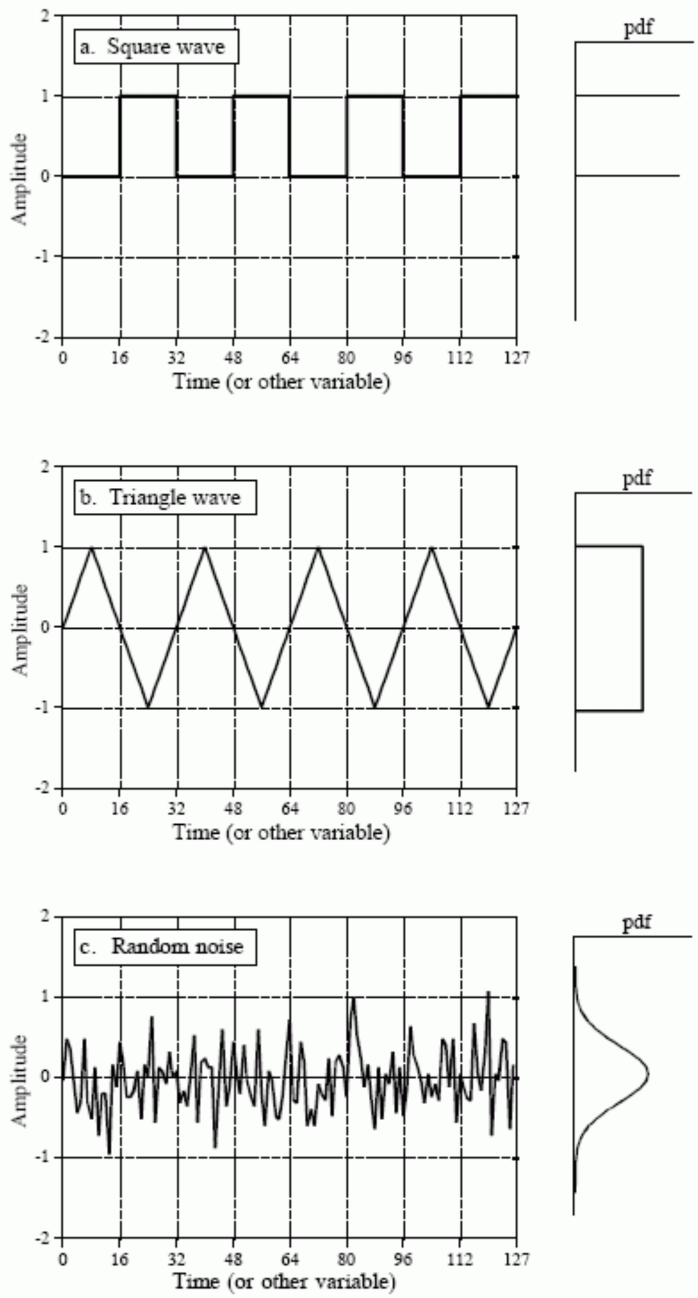
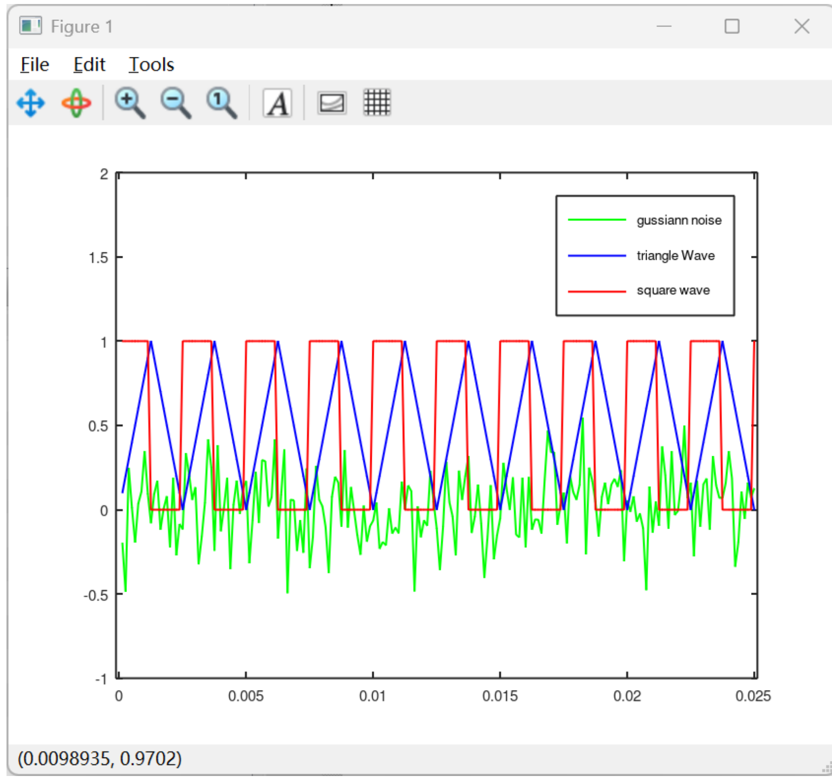


FIGURE 2-6
Three common waveforms and their probability density functions. As in these examples, the pdf graph is often rotated one-quarter turn and placed at the side of the signal it describes. The pdf of a square wave, shown in (a), consists of two infinitesimally narrow spikes, corresponding to the signal only having two possible values. The pdf of the triangle wave, (b), has a constant value over a range, and is often called a *uniform* distribution. The pdf of random noise, as in (c), is the most interesting of all, a bell shaped curve known as a *Gaussian*.

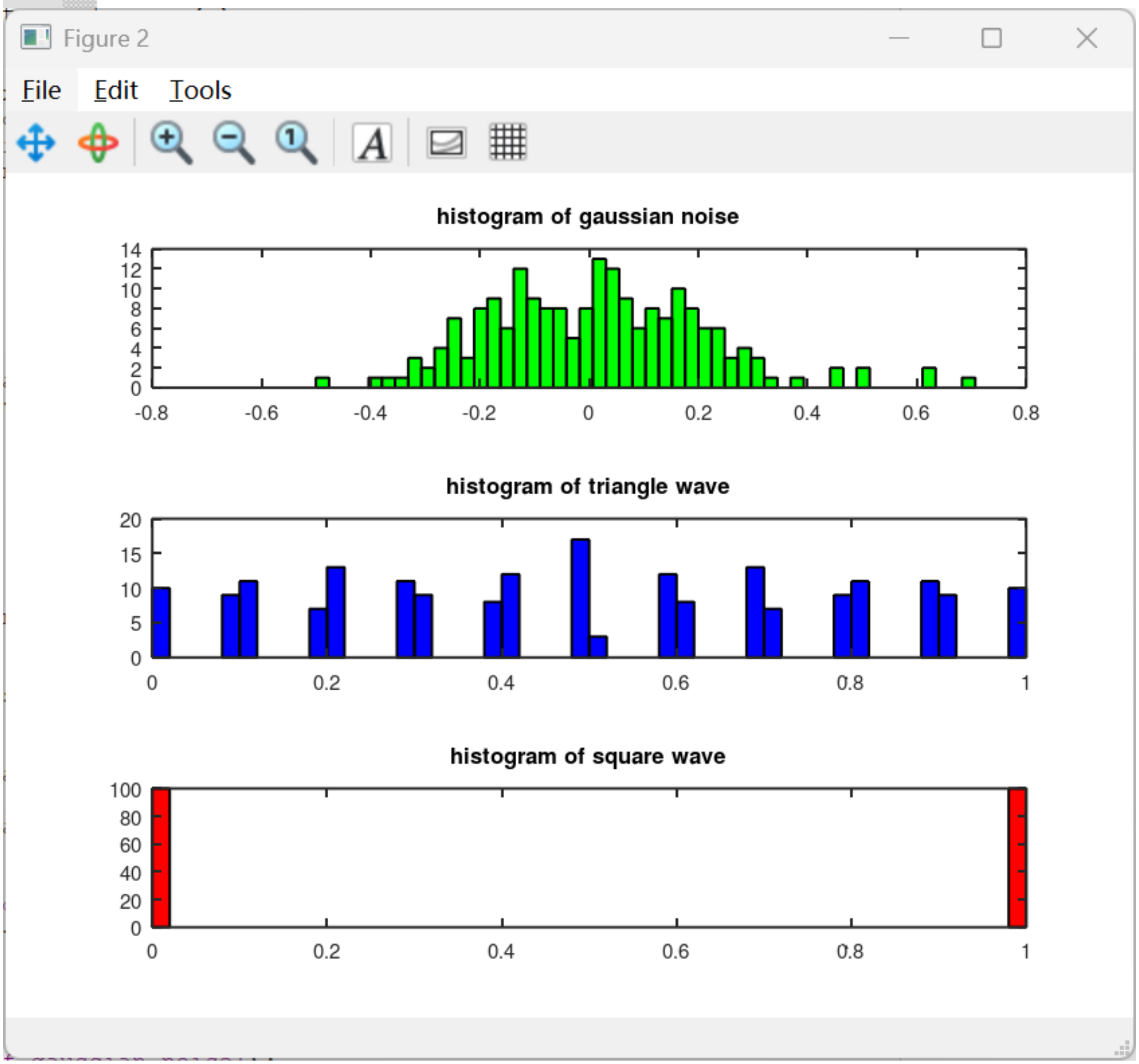
a). implement based on 'Square Wave Example Code' on Moodle

```
clear;
close;
home;

function retval = square_wave(x)
  n_max = length(x);
  retval = zeros(size(x));
  for i=[1:n_max]
    n = floor(x(i)/pi);
    if(mod(n,2))
      retval(i)=0;
    else
      retval(i)=1;
    endif
  endfor
endfunction

function retval = triangle_wave(x)
    n_max = length(x);
    retval = zeros(size(x));
    for i = 1:n_max
        pi_n = floor(x(i)/pi);
        frac = (x(i)/pi - pi_n);
        if mod(pi_n, 2) == 0
            retval(i) = frac;
        else
            retval(i) = 1 - frac;
        endif
    endfor
endfunction

function noise = gaussian_noise(N, sigma)
    noise = sigma * randn(1, N);
endfunction
f = 400.0;
T = 1/f;
fs = 8000.0;
dt = 1/fs;
t_start = dt;
t_end = 10*T;
t = t_start:dt:t_end;
x = 2*pi*f*t;
sigma = 0.2;
noise_signal = gaussian_noise(length(t), sigma);

plot(t, noise_signal, 'g');
hold on;
plot(t, triangle_wave(x), 'b')
hold on;
plot(t,square_wave(x),'r')
legend('gussiann noise', 'triangle Wave', 'square wave');
axis([-dt 10*T+dt -1 2])
```



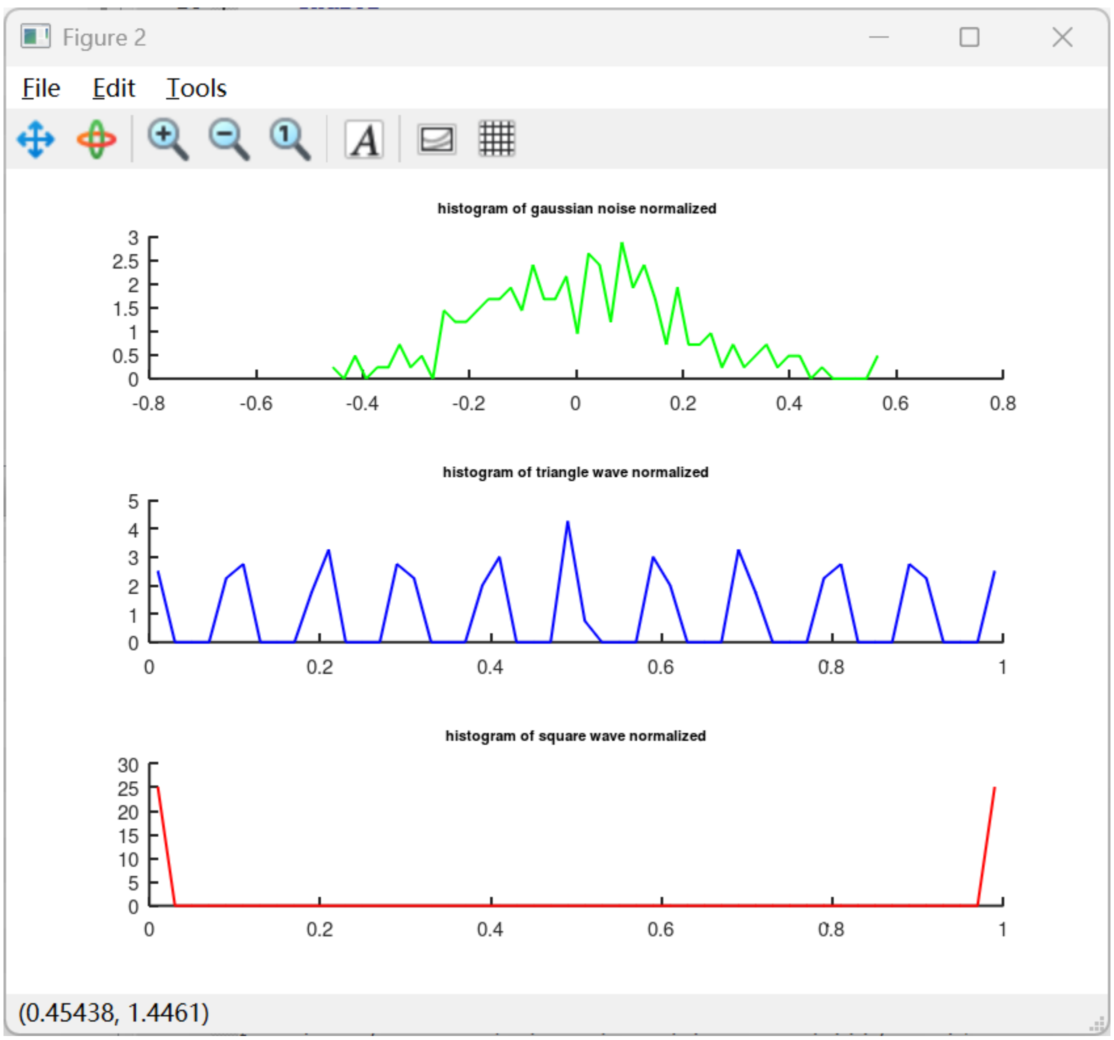
b).

```
figure(2);
subplot(3, 1, 1);
hist(noise_signal, 50, 'g');
title('histogram of gaussian noise');


subplot(3, 1, 2);
hist(triangle_wave(x), 50, 'b');
title('histogram of triangle wave');

subplot(3, 1, 3);
hist(square_wave(x), 50, 'r');
title('histogram of square wave');
drawnow;
```



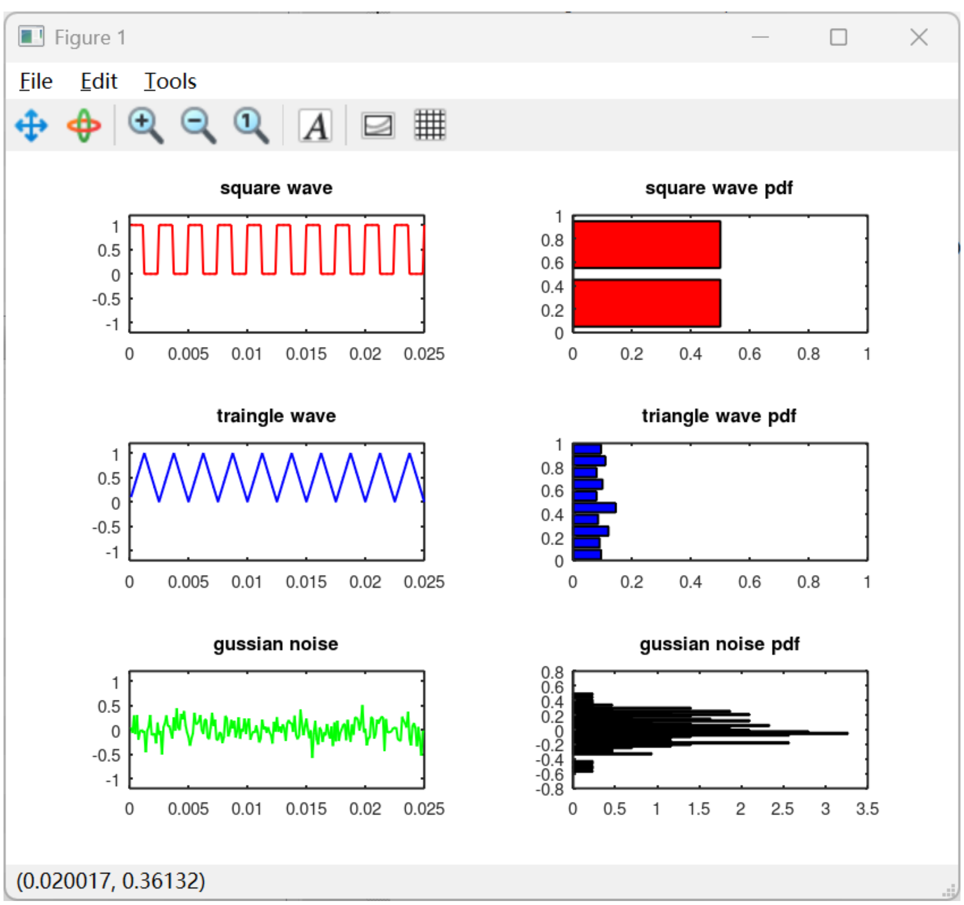
c).

```
figure(2);
N = length(t);

subplot(3, 1, 1);
[counts, bins] = hist(noise_signal, 50);
#hist(noise_signal, 50);
hold on;
plot(bins, counts / (N * (bins(2) - bins(1))), 'g');
title(' histogram of gaussian noise normalized');

subplot(3, 1, 2);
[counts, bins] = hist(triangle_wave(x), 50);
#hist(triangle_wave(x), 50);
hold on;
plot(bins, counts / (N * (bins(2) - bins(1))), 'b');
title('histogram of triangle wave normalized');

subplot(3, 1, 3);
[counts, bins] = hist(square_wave(x), 50);
#hist(square_wave(x), 50);
hold on;
plot(bins, counts / (N * (bins(2) - bins(1))), 'r');
title('histogram of square wave normalized');

drawnow;
```



d).

```
figure(1);
subplot(3, 2, 1);
plot(t, square_wave(x), 'r');
title('square wave');
axis([0 t_end -1.2 1.2]);

subplot(3, 2, 3);
plot(t, triangle_wave(x), 'b');
title('traingle wave');
axis([0 t_end -1.2 1.2]);

subplot(3, 2, 5);
plot(t, noise_signal, 'g');
title('gussian noise');
axis([0 t_end -1.2 1.2]);

subplot(3, 2, 2);
[counts, bins] = hist(square_wave(x), 2);
barh(bins, counts / N, 'r');
title('square wave pdf');
xlim([0, 1]);

subplot(3, 2, 4);
[counts, bins] = hist(triangle_wave(x), 10);
barh(bins, counts / N, 'b');
title('triangle wave pdf');
xlim([0, 1]);

subplot(3, 2, 6);
[counts, bins] = hist(noise_signal, 50);
barh(bins, counts / (N * (bins(2) - bins(1))), 'g');
title('gussian noise pdf');
drawnow;
```



e)

```
%(e)std of each pdf
mu_square = mean(square_wave(x));
sigma_square = std(square_wave(x));
disp(['square wave mean: ', num2str(mu_square), ' Std: ', num2str(sigma_square)]);

mu_triangle = mean(triangle_wave(x));
sigma_triangle = std(triangle_wave(x));
disp(['triangle wave mean: ', num2str(mu_triangle), ' Std: ', num2str(sigma_triangle)]);

mu_gaussian = mean(noise_signal);
sigma_gaussian = std(noise_signal);
disp(['gaussian noise mean: ', num2str(mu_gaussian), ' Std: ', num2str(sigma_gaussian)]);
```

gnu.octave.9.3.0

```
square wave mean: 0.5 Std: 0.50125
triangle wave mean: 0.5 Std: 0.29228
gaussian noise mean: 0.010042 Std: 0.21385
>>
```

3 ADC and DAC

1. Complete the following exercises about the precision of ADC and DAC components:

   - ADC:

     (a) What is the ΔV (voltage per level) of an ADC with signals in the [0,2.55] V range with 255 levels, plus zero (8-bit precision)?

     $$\Delta V = \frac{2.55V}{2^8} = \frac{2.55V}{256} = 9.96 \times 10^{-3} V \approx 10 mV$$

     (b) What is the ΔV (voltage per level) of an ADC with signals in the [0,4.095] V range with 4095 levels, plus zero (12-bit precision)?

     $$\Delta V = \frac{4.095V}{2^{12}} = \frac{4.095V}{4096} = 0.9998 \times 10^{-3} V \approx 1 mV$$

     (c) How many bits of precision, or how many voltage levels, are required for $\Delta V < 1$ mV, if signals are in the [0,12] V range?

     $$\Delta V \leq 1mV \Rightarrow 2^n > \frac{12V}{1 \times 10^{-3} V}$$

     $$n = \log_2\left(\frac{12}{0.001}\right) \simeq 14$$

     14 bits needed

     (d) What is the digital amplitude (in counts) of a 2.52 V signal, if signals are in the [0,5] V range, and there are 2048 levels?

     $$\text{Digital Value} = \frac{V_{signal}}{\Delta V} = \frac{2.52V}{\Delta V} = \frac{2.52V}{\frac{5V}{2048}} = 1032$$

- DAC:
  (a) If the digital amplitude of a signal is 256 counts, and signals are in the [0,5] V range with 9.8 mV per level, what is the signal amplitude in volts?

  $$V_{signal} = 256 \times 9.8 \times 10^{-3} V = \underline{2.5088 V}$$

  (b) If the digital amplitude of a signal is 2048 counts, and signals are in the [0,5] V range with max counts 4095, what is the signal amplitude in volts?

  $$\Delta V = \frac{5V}{4095} = 1.221 \times 10^{-3} V$$

  $$V_{signal} = 2048 \cdot 1.221 \times 10^{-3} V = \underline{2.5006 V}$$

  (c) If the digital amplitude of a signal is 128 counts, the max counts is 511, and the analog output is 0.25 V, what is the maximum voltage?

  $$V_{max} = \frac{0.25V \cdot 511}{128}$$

  $$= 0.998 V$$

2. For the following exercises, refer to Fig. 3-4 in Ch. 3 of the course text.

   (a) If the sampling rate is 500 kHz, and the analog signal frequency is 50 kHz, what is the digital signal frequency?

   $$f_{ans} = 50kHz \leqslant \frac{500kHz}{2}$$

   $$f_{digital} = f_{ans} = \underline{50kHz}$$

   (b) If the sampling rate is 500 kHz, and the analog signal frequency is 250 kHz, what is the digital signal frequency?

   $$f_{ans} = 250kHz = \frac{500kHz}{2}$$

   $$f_{digital} = f_{ans} = \underline{250kHz}$$

   (c) If the sampling rate is 500 kHz, and the analog signal frequency is 750 kHz, what is the digital signal frequency?

   $$f_{ans} = 750kHz > \frac{500kHz}{2}$$

   $$f_{digital} = |f_{ans} - 1 \cdot 500kHz| = \underline{250kHz}$$

   (d) If the sampling rate is 500 kHz, and the analog signal frequency is 1000 kHz, what is the digital signal frequency?

   $$f_{digital} = |1000kHz - 2 \cdot 500kHz| =$$

   $$= \underline{\underline{0 Hz}}$$

   Always Aliasing

3. Consider Fig. 3-10 in the course text. The single-pole, low-pass RC filter is depicted in the top middle section of Fig. 3-10. (a) Suppose a signal has an amplitude of 3.3 V and a frequency of 25 MHz, while $R = 10$ kΩ. What value of C is necessary to filter the signal to 0.33 V?

Low pass: $\omega = 2\pi f, \ \tau = RC$

$$R(f) = \frac{1}{1 + j\omega\tau} = \frac{1}{1 + j2\pi f RC}$$

$$|R(f)| = \frac{V_{out}}{V_{in}} = \frac{0.33\,V}{3.3\,V} = 0.1$$

$$= \sqrt{\left(\frac{1}{1+j2\pi fRC}\right)\left(\frac{1}{1-j2\pi fRC}\right)}$$

$$0.1 = \frac{1}{\sqrt{1+(2\pi fRC)^2}}$$

$$\frac{1}{0.1^2} = 1 + 2\pi fRC$$

$$100 - 1 = 4\pi^2 f^2 R^2 C^2$$

$$C = \sqrt{\frac{99}{(2\pi \cdot 25\times10^6\,Hz \times 10\times10^3\,\Omega)^2}}$$

$$C = 6.33\times10^{-12}\,F$$

4. Consider again Fig. 3-10. The single-pole, high-pass RC filter is similar to the depiction in the top middle section of Fig. 3-10, but with the C and R switched. (a) Suppose a signal has an amplitude of 3.3 V and a frequency of 10 MHz, while $R = 10\ \mathrm{k\Omega}$. What value of C is necessary to filter the signal to 0.33 V?

high pass: $\omega = 2\pi f, \ \tau = RC$

$$R(f) = \frac{j\omega\tau}{1 + j\omega\tau} = \frac{j2\pi fRC}{1 + j2\pi fRC}$$

$$|R(f)| = \frac{V_{out}}{V_{in}} = \frac{0.33\text{V}}{3.3\text{V}} = 0.1$$

$$0.1 = \frac{2\pi fRC}{\sqrt{1+(2\pi fRC)^2}}$$

$$0.1^2 = \frac{(2\pi fRC)^2}{1+(2\pi fRC)^2}$$

```
var('C')
f = 10e6
R = 10e3
Rf = 0.1
eq = ((2 * pi * f * R * C)^2) / (1 + (2 * pi * f * R * C)^2) == Rf^2
solution = solve(eq, C)
Cv = solution[1].rhs().n()
print(Cv)
```

Evaluate

```
1.59956736292783e-13
```

solve C using sage math

$$C = 1.6 \times 10^{-13}\ \text{F}$$

5.

phase shift of 3 (low pass filter)

$$\theta(f) = \tan^{-1} \frac{\mathrm{Im}\ R(f)}{\mathrm{Re}\ R(f)} = -\tan^{-1}\left(\frac{2\pi f RC}{1}\right)$$

$f = 25\times10^{6}\ Hz$

$R = 10\times10^{3}\ \Omega$

$C = 6.33\ pF$

$$= -\tan^{-1}\left(2\pi \cdot 25\times10^{6} \times 10\times10^{3} \times 6.33\times10^{-12}\right)$$

$$= -84.3^{\circ}$$

phase shift of 4 (low pass filter)

$f = 10\times10^{6}\ Hz$

$R = 10\times10^{3}\ \Omega$

$C = 0.16\ pF$

$$|R(f)| = \frac{(j2\pi f RC)(1-j2\pi f RC)}{(1+j2\pi f RC)(1-j2\pi f RC)} = \tan^{-1}\left(\frac{\cancel{\frac{2\pi f RC}{1+(2\pi f RC)^2}}}{\frac{(2\pi f RC)^{\cancel{2}}}{\cancel{1+(2\pi f RC)^2}}}\right)$$

$$= \frac{j2\pi f RC - j^2(2\pi f RC)^2}{1-j2\pi f RC + j2\pi f RC - (j^2(2\pi f RC)^2)} = \tan^{-1}\left(\frac{1}{2\pi f RC}\right)$$

$$\approx 84.3^{\circ}$$

$$= \frac{(2\pi f RC)^2 + j2\pi f RC}{1+(2\pi f RC)^2}$$