Homework 1, Unit 0: Foundations and Fundamentals

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1 Memory Bank

- $\sqrt{-1} = j$ ... The fundamental imaginary unit.
- $z = x + jy$ ... A complex number.
- $\Re\{z\} = x$, $\Im\{z\} = y$ ... Real and imaginary parts.
- $z^* = x - jy$ ... The complex conjugate of z.
- $|z| = \sqrt{zz^*} = \sqrt{x^2+y^2}$ ... The magnitude of z.
- $\tan\phi = y/x$ ... The phase angle of z.
- $|z| = r$, so $x = r\cos\phi$, and $y = r\sin\phi$.
- **Taylor Series:** Let $f(t)$ be a continuous, differentiable function. Let $f^n(t)$ be the n-th derivative of $f(t)$, with $f^0(t) = f(t)$. The Taylor series is an infinite series, equivalent to $f(t)$, given by

$$f(t) = \sum_{n=0}^{\infty} \frac{f^n(t_0)}{n!}(t-t_0)^n \tag{1}$$

- **Euler's Identity:** $e^{j\phi} = \cos\phi + j\sin\phi$

2 Complex Numbers and Signals

1. Let $z_1 = 3+4j$, and $z_2 = -3+4j$. Evaluate:
   (a) Graph z_1 and z_2 in the complex plane.
   (b) $z_1 + z_2$
   (c) $z_1 - z_2$
   (d) $z_1 * z_2$
   (e) z_1/z_2
   (f) $|z_1|$
   (g) $|z_2|$
   (h) ϕ_1
   (i) ϕ_2
   (j) Write z_1 and z_2 in polar form.

2. Use Euler's Identity to show that

$$\cos(2\pi f t) = \frac{e^{2\pi j f t} + e^{-2\pi j f t}}{2} \tag{2}$$
$$\sin(2\pi f t) = \frac{e^{2\pi j f t} - e^{-2\pi j f t}}{2j} \tag{3}$$

3. Let $v_1(t) = 4\cos(2\pi f_1 t)$, $v_2(t) = 4\cos(2\pi f_2 t - \phi)$. Use the results of the previous exercise in the following questions. (a) Show that $P = v_1(t)v_2(t)$ is a pair of sinusoids with frequencies $f_+ = f_1 + f_2$ and $f_- = f_1 - f_2$, offset by a total phase shift of 2ϕ. (b) Show that $P_{\max} = 16$, if $\phi = 0$ and $f_1 = f_2$. Why is 16 the correct number?[1].

4. Suppose that

$$v_1(t) = \Im\{\exp(j(2\pi f t - \phi))\} \tag{4}$$
$$v_2(t) = \Im\{\exp(2\pi j f t)\} \tag{5}$$

Drop the portion of the complex phase containing the frequency f, and represent the signals with just $\exp(-j\phi)$ and 1. (a) Graph these signals by treating the 1 and $\exp(-j\phi)$ as complex numbers in polar form. (b) Add the complex numbers, and obtain formulas for the new magnitude and phase angle. (c) Test your formulas for $\phi = 45$ degrees. When you add two signals of the same frequency offset by a phase, you should obtain a new

[1] The product of two mixed signal voltages, divided by the resistance, is the power (in Watts). The formula is $P = v^2/R$.

signal at the same frequency with a new phase and amplitude. What happens when the signals are in phase ($\phi = 0$ degrees) and out of phase ($\phi = 180$ degrees)?

3 Probability and Statistics, Noise

1. Consider the following octave code:

```
clear;
close;
home;

x = randn(10000,1);
figure(1)
hist(x,30);
figure(2);
plot(x)
axis([-1 10001 -10 10]);
```

The octave workspace is cleared, and a vector of data `x` is created. This vector contains pseudo-random numbers drawn from *the Gaussian distribution*, with mean μ and standard deviation σ:

$$p(x)dx = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad (6)$$

(a) Graph Eq. 6, and compare to Figure 1 created by the code. This figure contains a *histogram*, that counts how often the pseudo-random numbers fall into each of 30 bins. Does the histogram resemble Eq. 6? (b) Examine Figure 2 created by the code. If the numbers represent digitized, sampled noise voltages, they appear to be pure noise. (c) Write code that adds gaussian noise to a sine wave. (d) Notice that, as the amplitude of the sine wave decreases, the signal appears to be lost in the noise. The ratio of sine wave amplitude divided by σ in Eq. 6 is called the signal-to-noise ratio (SNR).

2. The octave function `rand` gives pseudo-random numbers drawn from a *uniform distribution*:

$$p(x)dx = \frac{dx}{b-a}, \quad a \leq x \leq b \quad (7)$$

This PDF is flat between a and b, where any number between these is equally likely to occur. The `rand` function has default settings of $b = 1$ and $a = 0$. Write an octave code that demonstrates that the sum of a large set of numbers drawn from `rand` is distributed according to Eq. 6. That is, we get gaussian noise from the repeated addition of many uniform random numbers.

4 ADC and DAC

1. Create an octave code that graphs a sine wave of frequency `f` and sampling frequency `fs` (see Code Lab 1 on Moodle for examples). Now tune the sampling frequency to with a factor of 2 of the signal frequency. Qualitatively, what happens to the signal graph?

2 Complex Numbers and Signals

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   (a) Graph z_1 and z_2 in the complex plane.
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   (f) $|z_1|$
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   (h) ϕ_1
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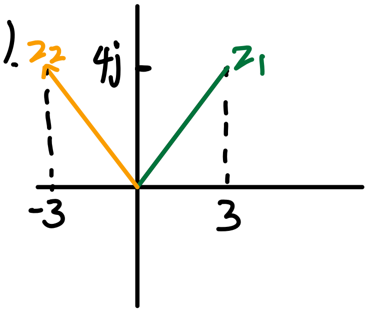
   (j) Write z_1 and z_2 in polar form.

a).



b). $z_1 + z_2 = 3+4j + (-3+4j)$
$= 8j$

c). $z_1 - z_2 = 3+4j - (-3+4j)$
$= 6$

d). $z_1 \times z_2 = (3+4j)(-3+4j)$
$= -9 + 12j - 12j + 16j^2$
$= -9 - 15$
$= -25$

e). $\frac{z_1}{z_2} = \frac{3+4j}{-3+4j}$
$= \frac{(3+4j)(3-4j)}{(-3+4j)(3-4j)}$
$= \frac{9+16}{-9+12j+12j-16j^2}$
$= \frac{25}{7+24j}$

f). $|z_1| = \sqrt{3^2+4^2} = \sqrt{25}$
$= 5$

g). $|z_2| = \sqrt{(-3)^2+4^2} = \sqrt{25}$
$= 5$

h). $\phi_1 = \tan^{-1}\left(\frac{4}{3}\right)$
$= 53.13°$

i). $\phi_2 = \tan^{-1}\left(\frac{4}{-3}\right)$
$= 126.87°$
* since -3 in second quordrant.

j). $|z_1| = 5$, $\phi_1 = 53.13°$
$z_1 = 5(\cos 53.13° + j\sin 53.13°)$

$|z_2| = 5$, $\phi_2 = 126.87°$
$z_2 = 5(\cos 126.87° + j\sin 126.87°)$

2. Use Euler's Identity to show that

$$\cos(2\pi ft) = \frac{e^{2\pi jft} + e^{-2\pi jft}}{2} \qquad (2)$$

$$\sin(2\pi ft) = \frac{e^{2\pi jft} - e^{-2\pi jft}}{2j} \qquad (3)$$

① $e^{j2\pi ft} = \cos(2\pi ft) + \cancel{j\sin(2\pi ft)}$

② $e^{-j2\pi ft} = \cos(2\pi ft) - \cancel{j\sin(2\pi ft)}$

①+②: $2\cos(2\pi ft)$

Thus. $\cos(2\pi ft) = \frac{①+②}{2} = \frac{e^{2\pi jft} + e^{-2\pi jft}}{2}$

③ $e^{j2\pi ft} = \cancel{\cos(2\pi ft)} + j\sin(2\pi ft)$

④ $e^{-j2\pi ft} = \cancel{\cos(2\pi ft)} - j\sin(2\pi ft)$

③-④ $= 2j\sin(2\pi ft) \Rightarrow \frac{③-④}{2j} = \sin(2\pi ft)$

Thus $\sin(2\pi ft) = \frac{e^{2\pi jft} - e^{-2\pi jft}}{2j}$

3. Let $v_1(t) = 4\cos(2\pi f_1 t)$, $v_2(t) = 4\cos(2\pi f_2 t - \phi)$. Use the results of the previous exercise in the following questions. (a) Show that $P = v_1(t)v_2(t)$ is a pair of sinusoids with frequencies $f_+ = f_1 + f_2$ and $f_- = f_1 - f_2$, offset by a total phase shift of 2ϕ. (b) Show that $P_{\max} = 16$, if $\phi = 0$ and $f_1 = f_2$. Why is 16 the correct number?[1].

a). $\cos(2\pi f_1 t) = \dfrac{e^{j2\pi f_1 t} + e^{-j2\pi f_1 t}}{2}$

$$\cos(2\pi f_2 t - \phi) = \frac{e^{j(2\pi f_2 t - \phi)} + e^{-j(2\pi f_2 t - \phi)}}{2}$$

$$P = \frac{e^{j2\pi f_1 t} + e^{-j2\pi f_1 t}}{2} \times \frac{e^{j(2\pi f_2 t - \phi)} + e^{-j(2\pi f_2 t - \phi)}}{2} \cdot 16$$

$$= 4\left(e^{j2\pi f_1 t} + e^{-j2\pi f_1 t}\right)\left(e^{j(2\pi f_2 t - \phi)} + e^{-j(2\pi f_2 t - \phi)}\right)$$

$e^{j[2\pi(\underline{f_1+f_2})t]-\phi}$ $\qquad$ $e^{j[2\pi(\underline{f_1-f_2})t]+\phi}$

$$= 4\left[e^{j(2\pi f_1 t + 2\pi f_2 t - \phi)} + e^{j(2\pi f_1 t - 2\pi f_2 t + \phi)}\right.$$

$$\left. + e^{-j(2\pi f_1 t + 2\pi f_2 t + \phi)} + e^{j(2\pi f_1 t - 2\pi f_2 t - \phi)}\right]$$

$e^{j[-2\pi(\underline{f_1+f_2})t]+\phi}$ $\qquad$ $e^{j[-2\pi(\underline{f_1-f_2})t]-\phi}$

$$\text{offset} = \phi - (-\phi) = 2\phi$$

b). $\phi=0$, $f_1=f_2=f$

$$P=4\cos(2\pi ft)\,4\cos(2\pi ft-\phi)$$

$$P=16\underbrace{\cos(2\pi ft)(\cos 2\pi ft-\phi)}_{\leq 1}$$

Thus, $P_{max}=16$

4. Suppose that

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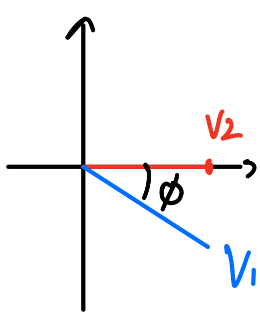
[1]The product of two mixed signal voltages, divided by the resistance, is the power (in Watts). The formula is $P = v^2/R$.

a). $V_1(t) = \Im\left(e^{j\cancel{(2\pi f t - \phi)}}\right) \quad = \quad e^{-j\phi}$

$V_2(t) = \Im\left(\cancel{e^{2\pi j f t}}\right) \quad = 1$



b) $V_1(t) = \mathbb{I}\left[\cos 2\pi ft - \phi + j\sin(2\pi ft - \phi)\right]$

$= \sin(2\pi ft - \phi)$

$V_2(t) = \mathbb{I}\left[\cos(2\pi ft) + j\sin(2\pi ft)\right]$

$= \sin(2\pi ft)$

$\sin(2\pi ft - \phi) + \sin(2\pi ft)$

$= 2\cos\left(\frac{\phi}{2}\right)\sin\left(2\pi ft - \frac{\phi}{2}\right)$

c) $\phi = 45°$

$\cos\frac{\phi}{2} = \cos(22.5°) = 0.924$

magnitude: $2\cos\frac{\phi}{2} = 1.848$

phase: $2\pi ft - 22.5° =$

$\phi = 0° \rightarrow V_1 + V_2 = \sin(2\pi ft) \cdot 2 = 2\sin(2\pi ft)$

magnitude = 2 · magnitude.

ϕ stay same.

$\phi = 180° \rightarrow V_1 + V_2 = 0$

magnitude = 0

$\phi = 0$.

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a).

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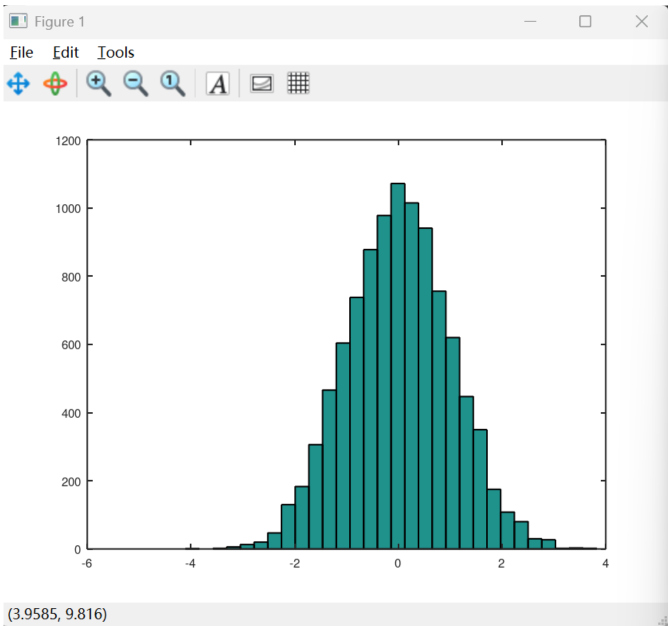
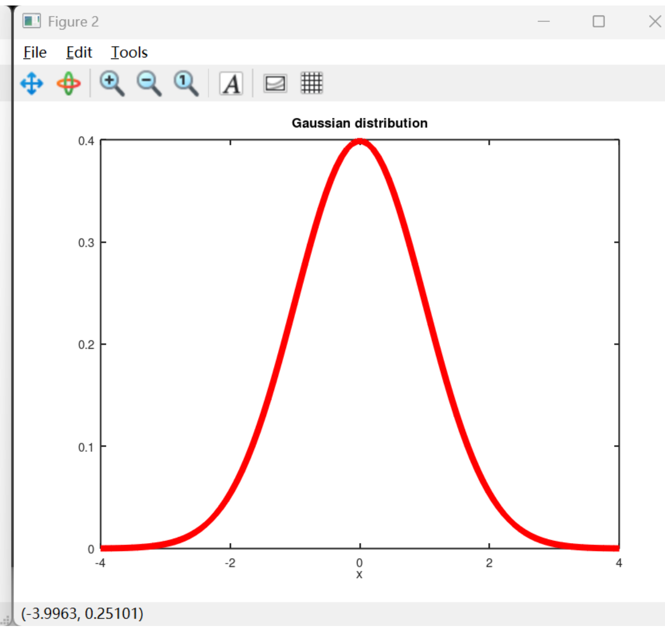
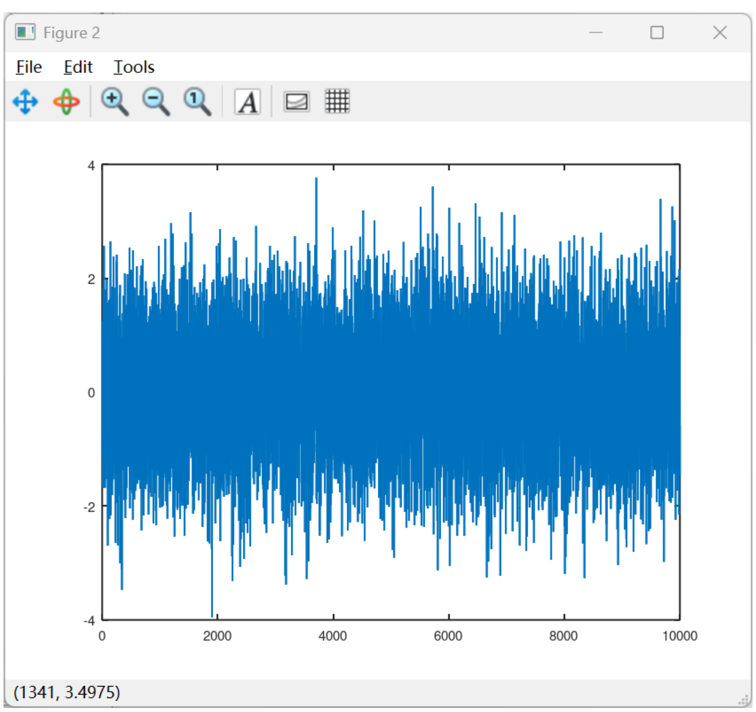
>>  x = randn(10000,1);
>>  figure(1)
>>  hist(x,30);
>>  figure(2);
>>  plot(x)
>>  axis([-1 10001 -10 10]);
>> xv = linspace(-4, 4, 100);
>> mu = 0;
>> sigma = 1;
>> eqn = (1 / (sqrt(2 * pi * sigma^2))) * exp(- (xv - mu).^2 / (2 * sigma^2));
>> plot(xv, eqn, 'r', 'LineWidth', 2);
>> title('Gaussian distribution');
>> xlabel('x');
>> ylabel('P Density');
```



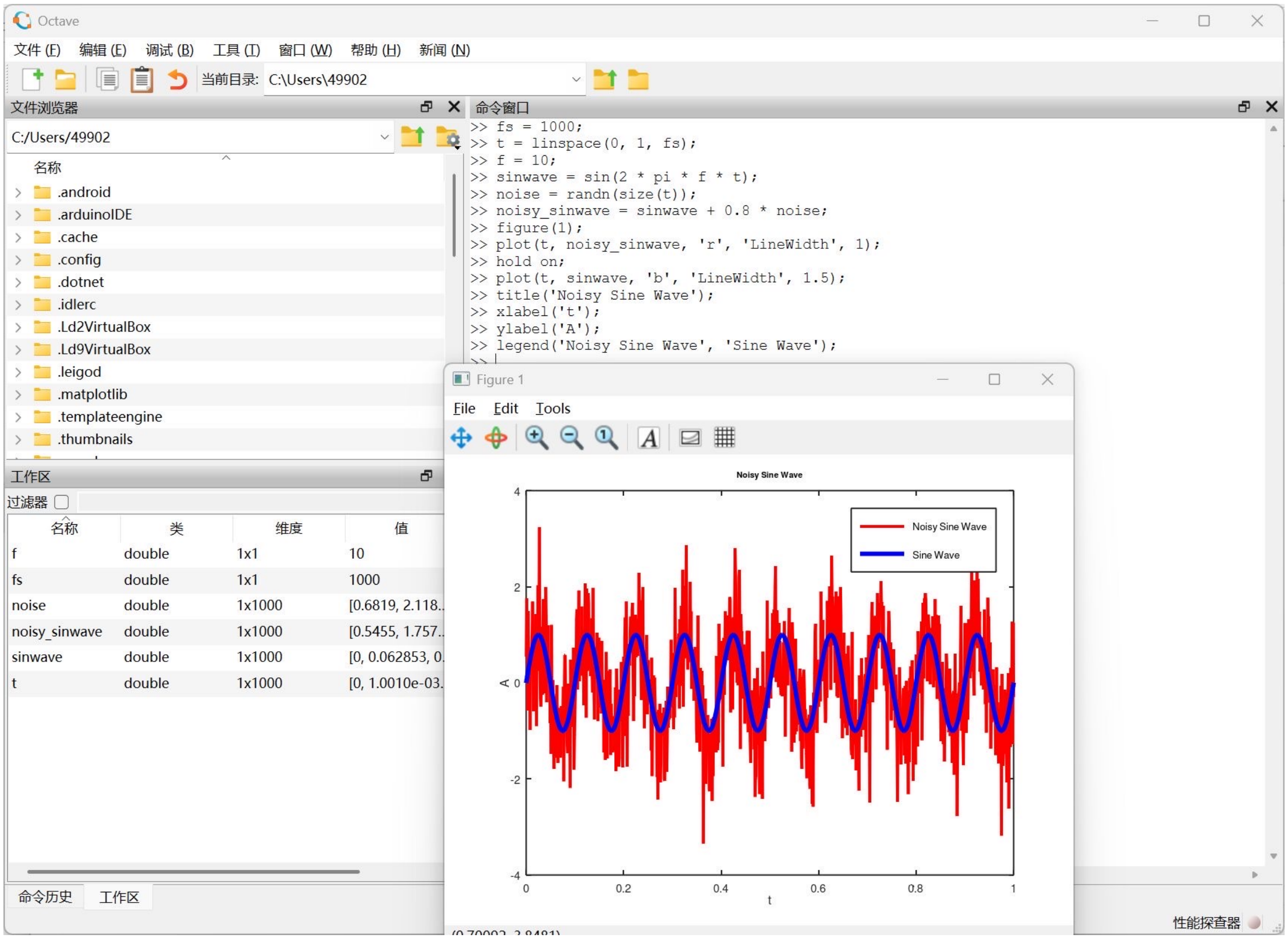


b.



looks noisy.

9).


Octave
文件 (F) 编辑 (E) 调试 (B) 工具 (T) 窗口 (W) 帮助 (H) 新闻 (N)
当前目录: C:\Users\49902
文件浏览器
C:/Users/49902
名称
.android
.arduinoIDE
.cache
.config
.dotnet
.idlerc
.Ld2VirtualBox
.Ld9VirtualBox
.leigod
.matplotlib
.templateengine
.thumbnails
工作区
过滤器
名称 类 维度 值
f double 1x1 10
fs double 1x1 1000
noise double 1x1000 [0.6819, 2.118..
noisy_sinwave double 1x1000 [0.5455, 1.757..
sinwave double 1x1000 [0, 0.062853, 0.
t double 1x1000 [0, 1.0010e-03.
命令历史 工作区
命令窗口
>> fs = 1000;
>> t = linspace(0, 1, fs);
>> f = 10;
>> sinwave = sin(2 * pi * f * t);
>> noise = randn(size(t));
>> noisy_sinwave = sinwave + 0.8 * noise;
>> figure(1);
>> plot(t, noisy_sinwave, 'r', 'LineWidth', 1);
>> hold on;
>> plot(t, sinwave, 'b', 'LineWidth', 1.5);
>> title('Noisy Sine Wave');
>> xlabel('t');
>> ylabel('A');
>> legend('Noisy Sine Wave', 'Sine Wave');
Figure 1
File Edit Tools
Noisy Sine Wave
Noisy Sine Wave
Sine Wave
A
t
性能探查器

2. The octave function **rand** gives pseudo-random numbers drawn from a *uniform distribution*:

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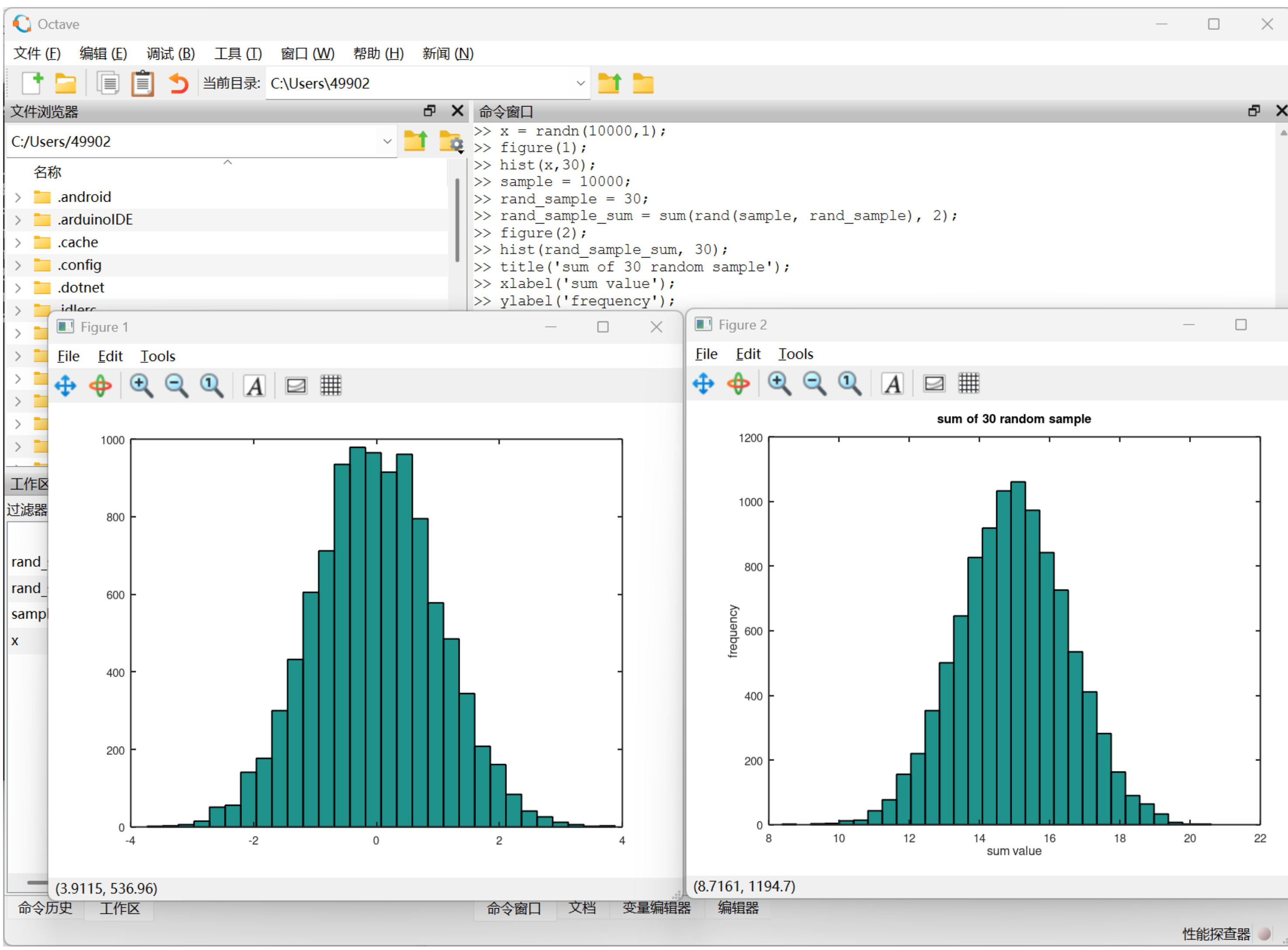
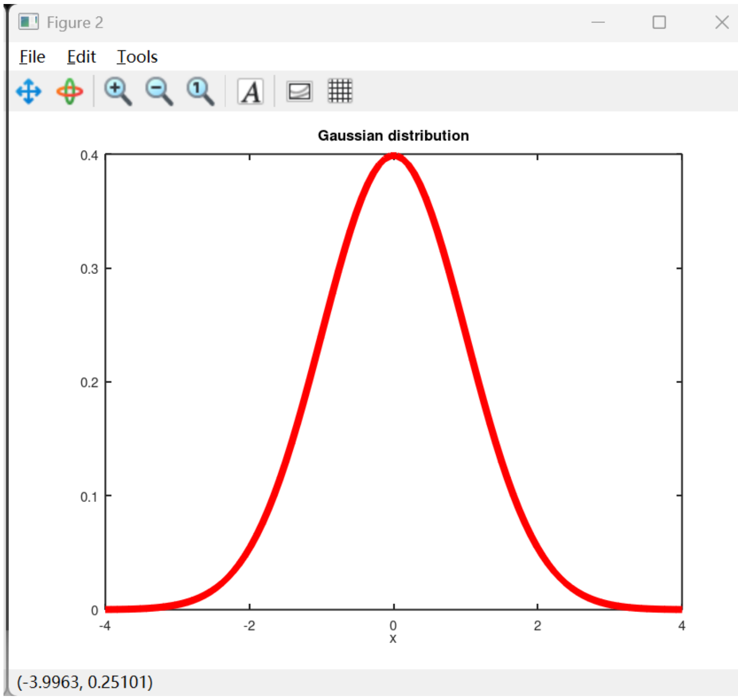
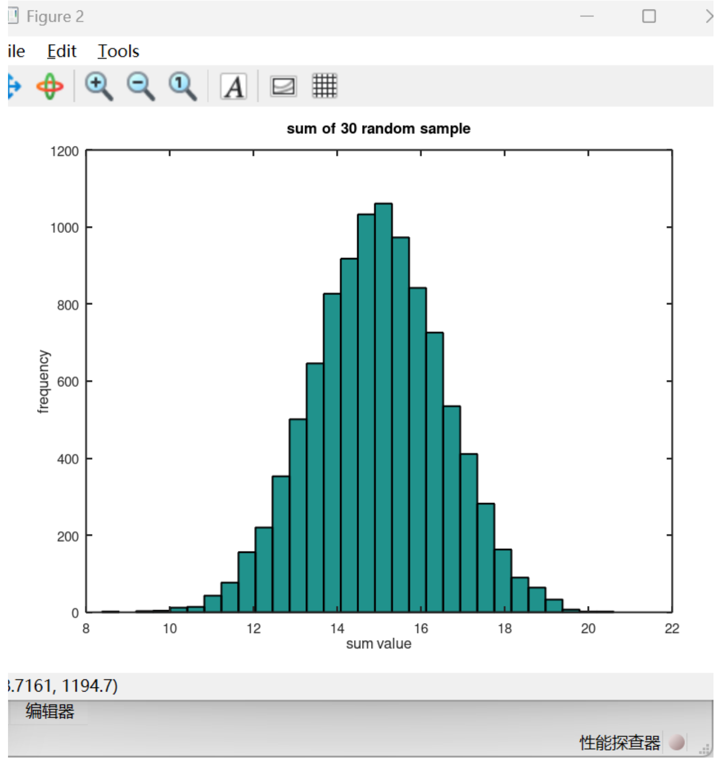
This PDF is flat between a and b, where any number between these is equally likely to occur. The **rand** function has default settings of $b = 1$ and $a = 0$. Write an octave code that demonstrates that the sum of a large set of numbers drawn from **rand** is distributed according to Eq. 6. That is, we get gaussian noise from the repeated addition of many uniform random numbers.



exp

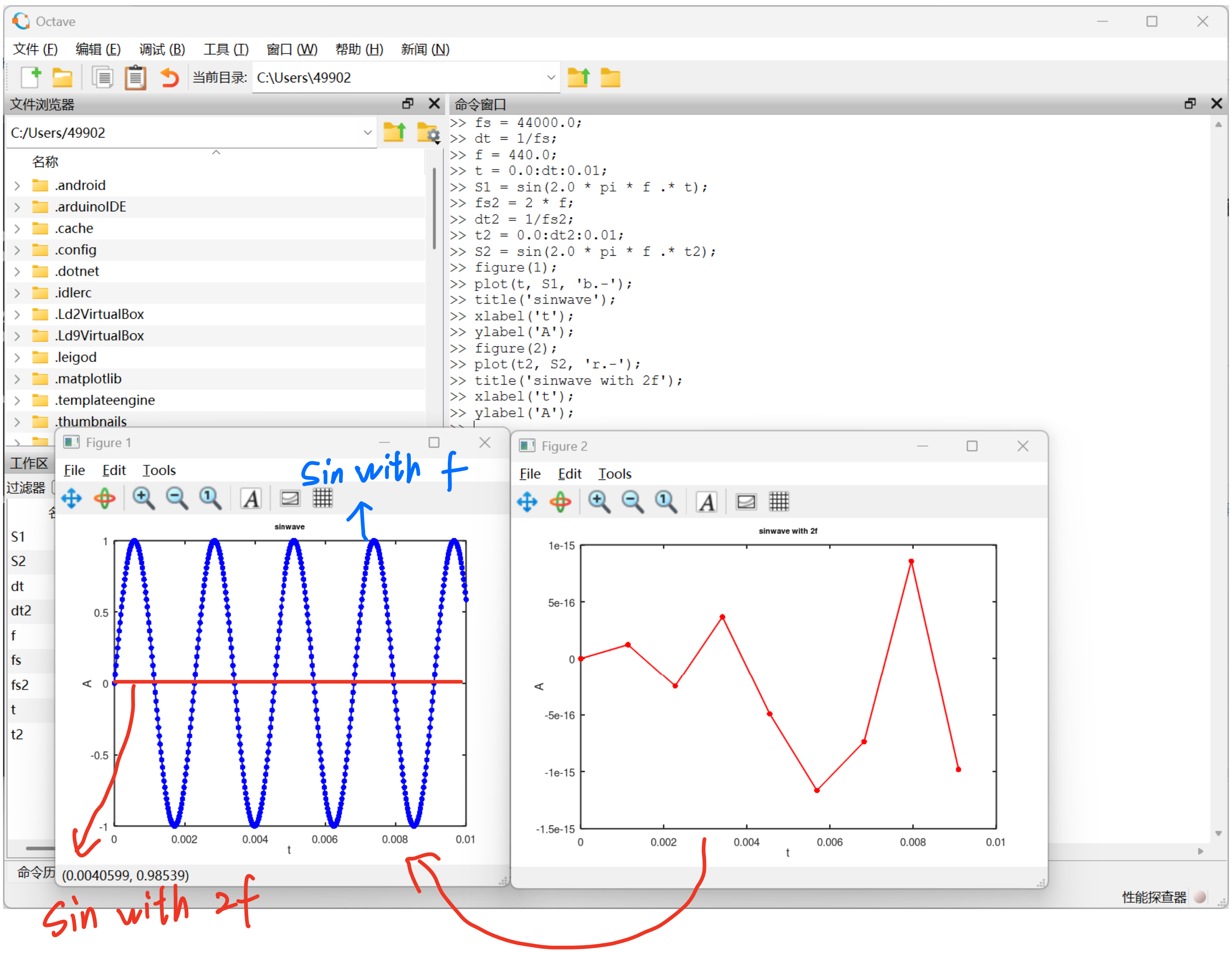




we can see the distribution of sum of 30 random sample is very close to gaussian distribution now.

4 ADC and DAC

1. Create an octave code that graphs a sine wave of frequency `f` and sampling frequency `fs` (see Code Lab 1 on Moodle for examples). Now tune the sampling frequency to with a factor of 2 of the signal frequency. Qualitatively, what happens to the signal graph?



sample point at maximum and minimum value of amplitude. (0 and 1)