Digital Signal Processing HW3

1 Linear Systems and Commutativity

Problem 1: Identifying Linearity and Commutativity

(a): Identifying the Linear System

Given systems: - System A: $A\{x[n]\} = 2x[n] - 1$ - System B: $B\{x[n]\} = 0.5x[n]$

A system is linear if it satisfies:

1. Homogeneity: $S\{kx[n]\} = kS\{x[n]\}$ 2. Additivity: $S\{x_1[n] + x_2[n]\} = S\{x_1[n]\} + S\{x_2[n]\}$

Checking System B ($B\{x[n]\} = 0.5x[n]$):

$$B\{kx[n]\} = 0.5(kx[n]) = kB\{x[n]\} \quad \text{(Homogeneity)}$$

$$B\{x_1[n] + x_2[n]\} = 0.5(x_1[n] + x_2[n]) = B\{x_1[n]\} + B\{x_2[n]\} \quad \text{(Additivity)}$$

System B is **linear**.

Checking System A ($A\{x[n]\} = 2x[n] - 1$):

$$A\{kx[n]\} = 2(kx[n]) - 1 = k(2x[n]) - 1 \neq kA\{x[n]\} \quad \text{(Fails Homogeneity)}$$

$$A\{x_1[n]+x_2[n]\} = 2(x_1[n]+x_2[n])-1 = 2x_1[n]+2x_2[n]-1 \neq A\{x_1[n]\}+A\{x_2[n]\} \quad \text{(Fails Additivit}$$

System A is **nonlinear**.

(b): Modifying the System and Checking Commutativity

To make system A linear, we remove the constant term:

$$A_{\text{linear}}\{x[n]\} = 2x[n]$$

This satisfies both homogeneity and additivity.

Checking Commutativity: A and B commute if:

$$A\{B\{x[n]\}\} = B\{A\{x[n]\}\}$$

Applying B first:

$$B\{x[n]\} = 0.5x[n]$$

$$A\{B\{x[n]\}\} = A\{0.5x[n]\} = 2(0.5x[n]) = x[n]$$

Applying A first:

$$A\{x[n]\} = 2x[n]$$

$$B\{A\{x[n]\}\} = B\{2x[n]\} = 0.5(2x[n]) = x[n]$$

Since both paths yield the same result, A and B commute after modifying A.

Final Answer:

- System A is originally nonlinear (fails homogeneity).
- System B is linear.
- Modified system A ($A_{\text{linear}}\{x[n]\} = 2x[n]$) makes it linear.
- A and B commute after modification.

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Problem 2: Evaluating Integrals Involving the Dirac Delta Function

We are given the function:

$$f(t) = a_1 \cos(2\pi f_1 t) + a_2 \cos(2\pi f_2 t)$$

where:
- $T_1 = \frac{1}{f_1}$ - $T_2 = \frac{1}{f_2}$ - $f_2 = 2f_1$

We need to evaluate the integrals:

$$\int_{-\infty}^{\infty} f(t)\delta(t - T_1)dt$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - T_2)dt$$

Using the Sifting Property of the Dirac Delta Function:

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

Applying this:

$$\int_{-\infty}^{\infty} f(t)\delta(t - T_1)dt = f(T_1)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - T_2)dt = f(T_2)$$

Evaluating $f(T_1)$

$$f(T_1) = a_1 \cos(2\pi f_1 T_1) + a_2 \cos(2\pi f_2 T_1)$$

Since $T_1 = \frac{1}{f_1}$, we substitute:

$$f(T_1) = a_1 \cos(2\pi) + a_2 \cos(4\pi)$$

Since $\cos(2\pi) = 1$ and $\cos(4\pi) = 1$:

$$f(T_1) = a_1 + a_2$$

Evaluating $f(T_2)$

$$f(T_2) = a_1 \cos(2\pi f_1 T_2) + a_2 \cos(2\pi f_2 T_2)$$

Since $T_2 = \frac{1}{f_2} = \frac{1}{2f_1}$, we substitute:

$$f(T_2) = a_1 \cos(\pi) + a_2 \cos(2\pi)$$

Since $\cos(\pi) = -1$ and $\cos(2\pi) = 1$:

$$f(T_2) = -a_1 + a_2$$

Final Answers

$$\int_{-\infty}^{\infty} f(t)\delta(t - T_1)dt = a_1 + a_2$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - T_2)dt = -a_1 + a_2$$

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Problem 3: Fourier Transform of an Impulse Function

We are given:

$$f(t) = a\delta(t - t_0)$$

We need to show:
1. The **magnitude** of the Fourier Transform of this impulse is a.
2. The **phase angle** ϕ is $-2\pi f t_0$.
3. The **group delay** τ_g satisfies $\tau_g = -\frac{d\phi}{d\omega}$ and is equal to t_0.

(a): Compute the Fourier Transform

The Fourier Transform is given by:

$$F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt$$

Substituting $f(t) = a\delta(t - t_0)$:

$$F(f) = \int_{-\infty}^{\infty} a\delta(t - t_0)e^{-j2\pi ft}dt$$

Using the sifting property of the delta function:

$$F(f) = ae^{-j2\pi ft_0}$$

Taking the **magnitude**:

$$|F(f)| = |a| \cdot |e^{-j2\pi ft_0}|$$

Since $|e^{-j2\pi ft_0}| = 1$, we get:

$$|F(f)| = a$$

Thus, the magnitude of the Fourier transform is a.

(b): Compute the Phase Angle ϕ

From our Fourier transform:

$$F(f) = ae^{-j2\pi ft_0}$$

The phase angle $\phi(f)$ is:

$$\phi(f) = \arg(F(f)) = \arg(ae^{-j2\pi ft_0})$$

Since $e^{-j2\pi ft_0}$ has a phase of $-2\pi ft_0$, and assuming a is real and positive:

$$\phi(f) = -2\pi ft_0$$

Thus, the phase angle is $-2\pi ft_0$.

(c): Compute the Group Delay τ_g

The group delay is defined as:

$$\tau_g = -\frac{d\phi}{d\omega}$$

Since $\omega = 2\pi f$, rewriting the phase:

$$\phi(\omega) = -t_0\omega$$

Differentiating:

$$\frac{d\phi}{d\omega} = -t_0$$

Thus:

$$\tau_g = -(-t_0) = t_0$$

Final Answers:
1. Magnitude: a
2. Phase Angle: $-2\pi f t_0$
3. Group Delay: t_0
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Problem 4: Impulse and Step Response of a System

We are given the system:

$$y[n] = S[x[n]] = x[n] - 0.5x[n-2]$$

where $\delta[n]$ represents a discrete-time impulse:

$$\delta[n] = [1, 0, 0, 0, 0, 0, 0]$$

(a): Compute $S[\delta[n]]$

Applying the system equation:

$$h[n] = S[\delta[n]] = \delta[n] - 0.5\delta[n-2]$$

Expanding the impulse function:
- $\delta[n] = 1$ at $n = 0$, and 0 elsewhere.
- $\delta[n-2] = 1$ at $n = 2$, and 0 elsewhere.
Thus:

$$h[n] = [1, 0, -0.5, 0, 0, 0, 0]$$

Final Answer: Impulse Response: $h[n] = [1, 0, -0.5, 0, 0, 0, 0]$.

(b): Compute the Step Response $S[s[n]]$

The step function is:

$$s[n] = [0, 1, 1, 1, 1, 1, 1]$$

Applying the system equation:

$$y[n] = S[s[n]] = s[n] - 0.5s[n-2]$$

Expanding:

$$s[n] = [0, 1, 1, 1, 1, 1, 1]$$

Shifting by 2 steps:

$$s[n-2] = [0, 0, 0, 1, 1, 1, 1]$$

Now computing $y[n]$:

$$y[n] = [0, 1, 1, 1, 1, 1, 1] - 0.5 \times [0, 0, 0, 1, 1, 1, 1]$$

$$y[n] = [0, 1, 1, 1, 1, 1, 1] - [0, 0, 0, 0.5, 0.5, 0.5, 0.5]$$

$$y[n] = [0, 1, 1, 0.5, 0.5, 0.5, 0.5]$$

Final Answer: Step Response: $y[n] = [0, 1, 1, 0.5, 0.5, 0.5, 0.5]$.
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2 Fourier Transforms and Basic Filters

Problem 1: Analysis of Low-Pass and High-Pass Filter Responses to an Impulse Signal

We are given a low-pass filter with the response:

$$H(f) = \frac{1}{1 + j\omega\tau}$$

where $\omega = 2\pi f$ and $\tau = RC$.

(a): Compute Fourier Transform $S(f)$ for $s(t) = a\delta(t - t_0)$

The Fourier Transform of the Dirac delta function is:

$$\mathcal{F}\{\delta(t - t_0)\} = e^{-j2\pi f t_0}$$

Thus, if $s(t) = a\delta(t - t_0)$, its Fourier transform is:

$$S(f) = ae^{-j2\pi f t_0}$$

(b): Find the Output Magnitude for the Low-Pass Filter

The output of the low-pass filter is given by:

$$Y(f) = S(f)H(f) = \left(ae^{-j2\pi f t_0}\right)\left(\frac{1}{1 + j\omega\tau}\right)$$

$$Y(f) = \frac{ae^{-j2\pi f t_0}}{1 + j\omega\tau}$$

Now, computing the magnitude:

$$|Y(f)| = \left|\frac{ae^{-j2\pi f t_0}}{1 + j\omega\tau}\right|$$

Since $e^{-j2\pi f t_0}$ has magnitude 1:

$$|Y(f)| = \frac{|a|}{\sqrt{1 + (\omega\tau)^2}}$$

Final Answer:

$$|Y(f)| = \frac{|a|}{\sqrt{1 + (2\pi f\tau)^2}}$$

(c): Compute the Output for the High-Pass Filter

For a high-pass filter, the response is:

$$H_{\text{HP}}(f) = \frac{j\omega\tau}{1 + j\omega\tau}$$

The output is:

$$Y_{\text{HP}}(f) = S(f)H_{\text{HP}}(f) = \left(ae^{-j2\pi f t_0}\right)\left(\frac{j\omega\tau}{1 + j\omega\tau}\right)$$

Now, computing the magnitude:

$$|Y_{\text{HP}}(f)| = \left|\frac{j\omega\tau}{1 + j\omega\tau}\right| |a|$$

$$|Y_{\text{HP}}(f)| = \frac{|a||\omega\tau|}{\sqrt{1 + (\omega\tau)^2}}$$

Substituting $\omega = 2\pi f$:

$$|Y_{\text{HP}}(f)| = \frac{|a||2\pi f\tau|}{\sqrt{1 + (2\pi f\tau)^2}}$$

Final Answer:

$$|Y_{\text{HP}}(f)| = \frac{|a||2\pi f\tau|}{\sqrt{1 + (2\pi f\tau)^2}}$$

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Problem 2: Group Delay Calculation for Low-Pass and High-Pass Filters

Definition of Group Delay

The group delay τ_g is defined as:

$$\tau_g = -\frac{d\phi(f)}{d\omega}$$

where $\phi(f)$ is the phase response of the system.

Low-Pass Filter Phase Response

The transfer function of a low-pass RC filter is given by:

$$H_{\text{LP}}(f) = \frac{1}{1 + j\omega\tau}$$

where $\tau = RC$ and $\omega = 2\pi f$.

Finding the Phase Response

Rewriting the transfer function in polar form:

$$H_{\mathrm{LP}}(f) = \frac{1}{\sqrt{1+(\omega\tau)^2}} e^{-j\tan^{-1}(\omega\tau)}$$

From this, the phase response is:

$$\phi_{\mathrm{LP}}(f) = -\tan^{-1}(\omega\tau)$$

Computing the Group Delay

Using the definition:

$$\tau_g = -\frac{d}{d\omega}\left(-\tan^{-1}(\omega\tau)\right)$$

$$\tau_g = \frac{d}{d\omega}\tan^{-1}(\omega\tau)$$

Using the derivative identity:

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

we obtain:

$$\tau_g = \frac{\tau}{1+(\omega\tau)^2}$$

Thus, the group delay for a low-pass filter is:

$$\tau_g = \frac{RC}{1+(2\pi f RC)^2}$$

High-Pass Filter Phase Response

The transfer function of a high-pass RC filter is given by:

$$H_{\mathrm{HP}}(f) = \frac{j\omega\tau}{1+j\omega\tau}$$

Rewriting in polar form:

$$H_{\mathrm{HP}}(f) = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}} e^{j(\pi/2-\tan^{-1}(\omega\tau))}$$

From this, the phase response is:

$$\phi_{\text{HP}}(f) = \frac{\pi}{2} - \tan^{-1}(\omega\tau)$$

Computing the Group Delay

Using the definition:

$$\tau_g = -\frac{d}{d\omega}\left(\frac{\pi}{2} - \tan^{-1}(\omega\tau)\right)$$

$$\tau_g = -\left(-\frac{d}{d\omega}\tan^{-1}(\omega\tau)\right)$$

$$\tau_g = -\frac{\tau}{1+(\omega\tau)^2}$$

Thus, the group delay for a high-pass filter is:

$$\tau_g = -\frac{RC}{1+(2\pi fRC)^2}$$

Final Answer

- **Low-pass filter group delay:**

$$\tau_g = \frac{RC}{1+(2\pi fRC)^2}$$

- **High-pass filter group delay:**

$$\tau_g = -\frac{RC}{1+(2\pi fRC)^2}$$

The negative sign in the high-pass case indicates a **negative group delay**, meaning the phase shift behaves differently from the low-pass case.

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Problem 3: Inverse Fourier Transform of Given Spectra

(a): Cosine Function

The inverse Fourier transform is given by:

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df$$

Substituting $S(f) = \frac{a}{2}(\delta(f - f_0) + \delta(f + f_0))$:

$$s(t) = \int_{-\infty}^{\infty} \frac{a}{2}(\delta(f - f_0) + \delta(f + f_0)) e^{j2\pi ft} df$$

Using the sifting property:

$$\int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi ft} df = e^{j2\pi f_0 t}$$

$$\int_{-\infty}^{\infty} \delta(f + f_0) e^{j2\pi ft} df = e^{-j2\pi f_0 t}$$

Thus, we get:

$$s(t) = \frac{a}{2} \left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t} \right)$$

Using Euler's identity:

$$e^{jx} + e^{-jx} = 2 \cos x$$

we obtain:

$$s(t) = a \cos(2\pi f_0 t)$$

Thus, the inverse Fourier transform results in a cosine function.

(b): Sine Function

Applying the inverse Fourier transform:

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df$$

Substituting $S(f) = \frac{a}{2j}(\delta(f - f_0) - \delta(f + f_0))$:

$$s(t) = \int_{-\infty}^{\infty} \frac{a}{2j}(\delta(f - f_0) - \delta(f + f_0)) e^{j2\pi ft} df$$

Using the sifting property:

$$\int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi ft} df = e^{j2\pi f_0 t}$$

$$\int_{-\infty}^{\infty} \delta(f + f_0) e^{j2\pi f t} df = e^{-j2\pi f_0 t}$$

Thus:

$$s(t) = \frac{a}{2j} \left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t} \right)$$

Using Euler's identity:

$$e^{jx} - e^{-jx} = 2j \sin x$$

we obtain:

$$s(t) = a \sin(2\pi f_0 t)$$

Thus, the inverse Fourier transform results in a sine function.

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Convolution and Octave Code

Problem 1: Impulse Response and Shift Property

(a): Impulse Response

The discrete-time convolution sum is given by:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

If $x[n] = \delta[n]$, then using the delta function property:

$$\delta[n-k] = \begin{cases} 1, & \text{if } n = k \\ 0, & \text{otherwise} \end{cases}$$

Substituting this into the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\delta[n-k]$$

Applying the sifting property:

$$y[n] = h[n]$$

Thus, the system's response to an impulse input is the impulse response $h[n]$.

(b): Shift Property

If the input is shifted such that $x[n] = \delta[n - n_0]$, then:

$$y[n] = h[n] * \delta[n - n_0]$$

Using the shifting property of the delta function:

$$\sum_{k=-\infty}^{\infty} h[k]\delta[(n - n_0) - k] = h[n - n_0]$$

Thus, the output is:

$$y[n] = h[n - n_0]$$

which confirms that shifting the input shifts the output by the same amount.

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Problem 2: Convolution of a Sine Wave with an Impulse

Concept

The convolution of a sine wave $x[n]$ with an impulse function $\delta[n-n_0]$ results in a time shift:

$$x[n] * \delta[n - n_0] = x[n - n_0]$$

This means that convolving the sine wave with the impulse moves the sine wave forward or backward in time by n_0.

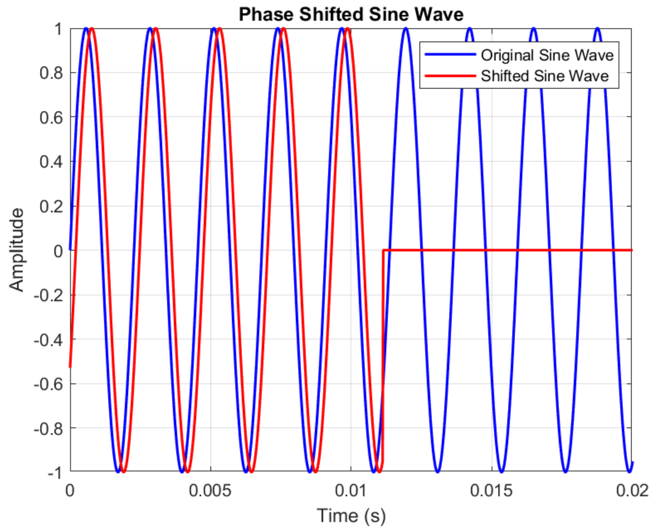
Octave Code

The following Octave script generates a 440 Hz sine wave, creates an impulse function at n_0, and convolves them to shift the sine wave phase:

```
% Sampling parameters
fs = 44100;  % Sampling frequency in Hz
t = 0:1/fs:0.02;  % Time vector (20 ms duration)

% Generate a 440 Hz sine wave
f_sine = 440;
```

```
sine_wave = sin(2 * pi * f_sine * t);

% Define impulse with shift n0
n0 = 50;  % Number of samples shift
impulse = zeros(1, length(t));
impulse(n0) = 1;  % Impulse at n0

% Convolve sine wave with shifted impulse
shifted_sine_wave = conv(sine_wave, impulse, 'same');

% Plot original and shifted sine waves
figure;
plot(t, sine_wave, 'b', 'LineWidth', 1.5); hold on;
plot(t, shifted_sine_wave, 'r', 'LineWidth', 1.5);
legend('Original Sine Wave', 'Shifted Sine Wave');
xlabel('Time (s)');
ylabel('Amplitude');
title('Phase Shifted Sine Wave');
grid on;

% Save the plot
saveas(gcf, 'shifted_sine_wave.png');

% Save the audio files
audiowrite('original_sine.wav', sine_wave, fs);
audiowrite('shifted_sine.wav', shifted_sine_wave, fs);
```

Plot of Shifted Sine Wave



Download Audio Files

The generated audio files can be downloaded from the following link:
Original Sine Wave: Download Original Sine Wave
Shifted Sine Wave: Download Shifted Sine Wave

Conclusion

This confirms that convolution with an impulse function effectively shifts the phase of the sine wave.

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