DIGITAL SIGNAL PROCESSING: COSC390

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FOURIER SERIES OF A SAWTOOTH SIGNAL

Let the repeating sawtooth signal be defined like

$$f(x) = x \quad -\pi \leq x \leq \pi \tag{1}$$

We observe three things:

- This is a strictly-odd function $\rightarrow$ Half of the terms in Fourier series should vanish
- The amplitude is π $\rightarrow$ Amplitudes of sinusoids should reflect this
- It is centered on $y = 0$ (it has no constant offset) A_0 should be 0.0

We see that the even terms and the constant term should vanish in the Fourier series. The other terms are like:

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx) dx \tag{2}$$

We may do this integral a number of ways. The complex-exponential method turned out to be complicated, so let's try integration by-parts[1]:

$$\int_a^b u dv = uv|_a^b - \int_a^b v du \tag{3}$$

[1] Or just go to WolframAlpha and ask the oracle to tell you the answer :-)

Let $u = x$ (and $du = dx$), and $dv = \sin(nx)dx$. Now we can solve for v:

$$\frac{dv}{dx} = \sin(nx) \tag{4}$$

$$v = -\frac{1}{n}\cos(nx) \tag{5}$$

Now we can plug into Eq. 3:

$$b_n = -\frac{x\cos(nx)}{n\pi}\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos(nx)dx \tag{6}$$

The second term on the right side is zero, because it represents integrating a periodic function over one period[2].

[2] Think of the area under the curve: one period contains as much negative area as it does positive area.

Finally, we have:

$$b_n = -\frac{1}{n\pi}\left(\pi\cos(n\pi) + \pi\cos(n\pi)\right) = -\frac{2}{n\pi}\left(\pi\cos(n\pi)\right) \tag{7}$$

Cosine alternates between -1 and 1 for integer values of π. The even ones are 1.0, and the odd ones are -1.0. Thus,

$$b_n = -\frac{2}{n}(-1)^n \tag{8}$$

Finally, the Fourier series for the sawtooth in Eq. 1 is

$$s(x) = b_1\sin(x) + b_2\sin(2x) + ... = -2\sum_{i=1}^{\infty}\frac{(-1)^n}{n}\sin(nx) \tag{9}$$