Homework 3, Unit 0: Foundations and Fundamentals

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1 Memory Bank

- **Homogeneous system:** Let k be a constant, and let $s_{\text{in}}(t)$ and $s_{\text{out}}(t)$ be the input and output signals to a system S, respectively. S is *homogeneous* if:

$$s_{\text{out}}(t) = S[s_{\text{in}}(t)] \tag{1}$$
$$ks_{\text{out}}(t) = S[ks_{\text{in}}(t)] \tag{2}$$

- **Additive system:** Let $s_1(t)$ and $s_2(t)$ be two input signals to a system S, with outputs $s_1'(t)$ and $s_2'(t)$. S is *additive* if:

$$s_1'(t) = S[s_1(t)] \tag{3}$$
$$s_2'(t) = S[s_2(t)] \tag{4}$$
$$s_1'(t) + s_2'(t) = S[s_1(t) + s_2(t)] \tag{5}$$

- **Shift-invariant system:** Let $s_{\text{in}}(t)$ and $s_{\text{out}}(t)$ be input and output signals to a system S, and let t_0 be a constant. S is *shift invariant* if:

$$s_{\text{out}}(t) = S[s_{\text{in}}(t)] \tag{6}$$
$$s_{\text{out}}(t - t_0) = S[s_{\text{in}}(t - t_0)] \tag{7}$$

- $F(f) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-2\pi jft}dt$ ... The Fourier Transform.
- $\mathcal{F}^{-1}\{F(f)\} = \int_{-\infty}^{\infty} F(f)e^{2\pi jft}df$ ... The Inverse Fourier Transform.
- The **Dirac δ-function** is a distribution defined by the following property:

$$f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt \tag{8}$$

  In words, the integral of a δ-function times a function f is the value of the function at t_0.

- **Convolution**: this is an operation that characterizes the response $h[n]$ of a linear system.

$$y[i] = h[n] * x[n] = \sum_{j=0}^{M-1} h[j]x[i-j] \tag{9}$$

  In words, the output at sample i is equal to the produce of the system response h and the input signal x, summed over the proceeding M samples (from $j = 0$ to $j = M - 1$).

2 Linear Systems

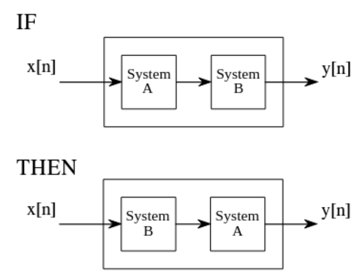

Figure 1: Linear systems **commute**.

1. Consider Fig. 1, which depicts two linear systems A and B. Symbolically, systems A and B **commute** if $A\{B\{x[n]\}\} = B\{A\{x[n]\}\}$. (a) Let $A\{x[n]\} = 2x[n] - 1$, and $B\{x[n]\} = 0.5x[n]$. Which system, A or B, is a linear system? For the system that is not linear, which linear property does it break? (b) Modify the non-linear system to make it linear, and show that A and B commute.

2. Consider Eq. 8 in the Memory Bank. Let $f(t) = a_1\cos(2\pi f_1 t) + a_2\cos(2\pi f_2 t)$, with $T_1 = 1/f_1$, $T_2 = 1/f_2$, and $f_2 = 2f_1$. Evaluate the following:
   - $\int_{-\infty}^{\infty} f(t)\delta(t - T_1)dt$
   - $\int_{-\infty}^{\infty} f(t)\delta(t - T_2)dt$

3. Let $f(t) = a\delta(t - t_0)$. (a) Show that the magnitude of the **Fourier transform** of this impulse is a. (b) Show that the phase angle, ϕ, is $-2\pi f t_0$. (c) Show that the group delay, $\tau_g = -d\phi/d\omega$ is t_0.

4. Let $\delta[n]$ represent a digital impulse: [1000 0000][1]. (a) If $y[n] = S[x[n]] = 0.5x[n-2]$, what is $S[\delta[n]]$? (b) $y[n]$ is the *impulse response* of S. What is the *step response*, if the step input is $s[n] = [01111111]$?

3 Fourier Transforms and Basic Filters

1. Suppose we pass a signal $s(t)$ into a low-pass filter. The signal as a function of frequency is $S(f)$, the Fourier transform of $s(t)$. The output of the low-pass filter will be $S(f)$ times $1/(1 + j\omega\tau)$, where $\omega = 2\pi f$, and $\tau = RC$. That is, the output will be $S(f)/(1 + j\omega\tau)$. (a) Calculate the Fourier transform $S(f)$, if $s(t) = a\delta(t - t_0)$ (as we did in class). (b) Suppose we pass our impulse $s(t)$ into a low-pass filter. What is the magnitude of the output, as a function of frequency? (c) Repeat this exercise, but with a high-pass filter response: $j\omega\tau/(1 + j\omega\tau)$.

[1] Let the index for data in this list of numbers start with $n = 0$.

2. For the output spectra of the previous exercise, low-pass and high-pass, calculate the group delays.[2]

3. (a) Show that the inverse Fourier transform of $S(f) = (a/2)(\delta(f - f_0) + \delta(f + f_0))$ is a cosine function. (b) Show that the inverse Fourier transform of $S(f) = (a/2j)(\delta(f - f_0) - \delta(f + f_0))$ is a sine function.

4 Convolution and Octave Code

1. For the following exercises, use Eq. 9. Let the digital impulse be $\delta[n]$ which is 1 for $n = 0$, and 0 if $n \neq 0$. For example, $\delta[n-5]$ is 1 when $n = 5$. (a) Show that if $x[n] = \delta[n]$, $y[n] = h[n] * x[n] = h[n]$. That is, if the input data is an impulse, the output is the system response. (b) Show that if the input impulse is shifted ($x[n] = \delta[n - n_0]$), the output is $h[n]$, shifted by the same amount.

2. In `octave`, use the `conv` function to convolve a 440 Hz sine wave with a $\delta[n - n_0]$ impulse. Shift the phase of the sine output by varying n_0.

[2] Hint: multiply the numerator and denominator of ratios by the complex conjugate of the denominator, to aid in splitting the complex expression into real and imaginary parts.

2.1

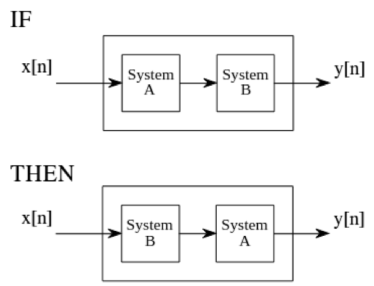

Figure 1: Linear systems **commute**.

1. Consider Fig. 1, which depicts two linear systems A and B. Symbolically, systems A and B **commute** if $A\{B\{x[n]\}\} = B\{A\{x[n]\}\}$. (a) Let $A\{x[n]\} = 2x[n] - 1$, and $B\{x[n]\} = 0.5x[n]$. Which system, A or B, is a linear system? For the system that is not linear, which linear property does it break? (b) Modify the non-linear system to make it linear, and show that A and B commute.

a). To be linear system, k is constant

① $S\{kx[n]\} = kS\{x[n]\}$

② $S\{x_1[n]+x_2[n]\} = S\{x_1[n]\} + S\{x_2[n]\}$

consider $A\{x[n]\} = 2x[n]-1$

① $A\{kx[n]\} = 2(kx[n])-1$

$= 2kx[n]-1$

$\neq$

$kA\{x[n]\} = k(2x[n]-1)$

$= 2kx[n]-k$

Thus $A\{x[n]\}$ not linear.

Fails Homogeneity property

consider $B\{x[n]\} = 0.5x[n]$

① $B\{kx[n]\} = 0.5kx[n]$

$=$

$kB\{x[n]\} = 0.5kx[n]$

② $B\{x_1[n]+x_2[n]\} = 0.5(x_1[n]+x_2[n])$

$= 0.5x_1[n]+0.5x_2[n]$

$=$

$B\{x_1[n]\}+B\{x_2[n]\} = 0.5x_1[n]+0.5x_2[n]$

Thus. System B is linear

b). Change $A\{x[n]\} = 2x[n]$

①. $A\{kx[n]\} = 2kx[n]$

$= kA\{x[n]\} = 2kx[n]$

②. $A\{x_1[n]+x_2[n]\} = 2x_1[n]+2x_2[n]$

$A\{x_1[n]\}+A\{x_2[n]\} = 2x_1[n]+2x_2[n]$

Now A is linear system.

Check A . B commutativity.

$A\{B\{x[n]\}\} = B\{A\{x[n]\}\}$

$= A\{0.5x[n]\}$	$= B\{2x[n]\}$
$= 2\cdot(0.5x[n])$	$= 0.5(2x[n])$
$= \underline{\underline{x[n]}}$	$= \underline{\underline{x[n]}}$

LHS=RHS. Thus Commutative

2.2.

2. Consider Eq. 8 in the Memory Bank. Let $f(t) = a_1 \cos(2\pi f_1 t) + a_2 \cos(2\pi f_2 t)$, with $T_1 = 1/f_1$, $T_2 = 1/f_2$, and $f_2 = 2f_1$. Evaluate the following:

- $\int_{-\infty}^{\infty} f(t)\delta(t - T_1)dt$
- $\int_{-\infty}^{\infty} f(t)\delta(t - T_2)dt$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0) = f(t_0)$$

$$T_1 = \frac{1}{f_1}$$

$$T_2 = \frac{1}{f_2} \Rightarrow T_2 = \frac{1}{2f_1} = \frac{T_1}{2}$$

①. $$\int_{-\infty}^{\infty} f(t)\,\delta(t-T_1)\,dt = f(T_1)$$

$$= a_1 \cos(2\pi f_1 T_1) + a_2 \cos(2\pi f_2 T_1)$$

$$= a_1 \cos(2\pi f_1 \cdot \tfrac{1}{f_1}) + a_2 \cos(2\pi\, 2f_1 \tfrac{1}{f_1})$$

$$= a_1 \underbrace{\cos(2\pi)}_{=1} + a_2 \underbrace{\cos(4\pi)}_{=1}$$

$$= a_1 + a_2$$

② $$\int_{-\infty}^{\infty} f(t)\,\delta(t-T_2)\,dt = f(T_2)$$

$$= a_1 \cos(2\pi f_1 T_2) + a_2 \cos(2\pi f_2 T_2)$$

$$= a_1 \cos(2\pi f_1 \cdot \tfrac{1}{2f_1}) + a_2 \cos(2\pi\, 2f_1 \cdot \tfrac{1}{2f_1})$$

$$= \underbrace{a_1 \cos(\pi)}_{=-1} + \underbrace{a_2 \cos(2\pi)}_{=1}$$

$$= a_2 - a_1$$

3. Let $f(t) = a\delta(t - t_0)$. (a) Show that the magnitude of the **Fourier transform** of this impulse is a. (b) Show that the phase angle, ϕ, is $-2\pi f t_0$. (c) Show that the group delay, $\tau_g = -d\phi/d\omega$ is t_0.

a). $F(f) = \int_{-\infty}^{\infty} f(t) e^{-2\pi j f t} dt = \int_{-\infty}^{\infty} a\delta(t-t_0)\, e^{-2\pi j f t}\, dt$

$= a \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-2\pi j f t}\, dt$

$= a\, e^{-2\pi j f t_0}$

$\left| a e^{-2\pi j f t} \right| = \sqrt{a\, \cancel{e^{-2\pi j f t_0}}\, a\, \cancel{e^{2\pi j f t_0}}}$

$= \sqrt{a^2 e^0}$

$= a$

b). $\phi = \mathrm{Im}\,(a\, e^{-2\pi j f t_0})$

$= -2\pi f t_0$

c). $T_g = -\dfrac{d\phi}{d\omega}$ $\qquad \omega = 2\pi f$

$\phi = -\omega t_0$

$-\left(\dfrac{d}{d\omega}(-\omega t_0)\right) = t_0$

2.4

4. Let $\delta[n]$ represent a digital impulse: $[1000\ 0000]^1$. (a) If $y[n] = S[x[n]] = 0.5x[n-2]$, what is $S[\delta[n]]$? (b) $y[n]$ is the *impulse response* of S. What is the *step response*, if the step input is $s[n] = [01111111]$?

a).
$$\delta[n-2] = \begin{cases} 1 & n=2 \\ 0 & n \neq 2 \end{cases}$$

$$S[\delta[n]] = 0.5\,\delta[n-2]$$

$$= [0,0,0.5,0,0,0,0,0]$$

b).
$$h[n-2] = \begin{cases} 1 & n \geq 2 \\ 0 & n < 2 \end{cases}$$

$$S[h[n]] = [0,0,0.5,0.5,0.5,0.5,0.5,0.5]$$

3.1

1. Suppose we pass a signal $s(t)$ into a low-pass filter. The signal as a function of frequency is $S(f)$, the Fourier transform of $s(t)$. The output of the low-pass filter will be $S(f)$ times $1/(1 + j\omega\tau)$, where $\omega = 2\pi f$, and $\tau = RC$. That is, the output will be $S(f)/(1 + j\omega\tau)$. (a) Calculate the Fourier transform $S(f)$, if $s(t) = a\delta(t - t_0)$ (as we did in class). (b) Suppose we pass our impulse $s(t)$ into a low-pass filter. What is the magnitude of the output, as a function of frequency? (c) Repeat this exercise, but with a high-pass filter response: $j\omega\tau/(1 + j\omega\tau)$.

a).

$$S(f) = F\{s(t)\} = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t}\, dt$$

$$s(t) = a\delta(t - t_0)$$

$$F\{s(t)\} = a\int_{-\infty}^{\infty} \delta(t - t_0) e^{-2\pi j f t}\, dt$$

$$= a e^{-2\pi j f t}$$

b) $H_{lp} = \dfrac{1}{1 + j\omega\tau}$ $\qquad \omega = 2\pi f$

Output: $S(f) \cdot H_{LP}$

$$= a e^{-2\pi j f t} \cdot \frac{1}{1 + j2\pi f\tau}$$

$$|\text{output}| = \left|a e^{-2\pi j f t}\right| \left|\frac{1}{1 + j2\pi f\tau}\right|$$

$$= |a| \cdot \frac{1}{\sqrt{1 + (2\pi f\tau)^2}}$$

$$= \frac{a}{\sqrt{1 + (2\pi f\tau)^2}}$$

c). $H_{hp} = \dfrac{j\omega\tau}{1 + j\omega\tau} = \dfrac{j2\pi f\tau}{1 + j2\pi f\tau}$

$$\text{output} = a e^{-2\pi j f t} \cdot \frac{j2\pi f\tau}{1 + j2\pi f\tau}$$

$$|\text{output}| = |a| \cdot \left|\frac{j2\pi f\tau}{1 + j2\pi f\tau}\right|$$

$$= a \cdot \frac{2\pi f\tau}{\sqrt{1 + (2\pi f\tau)^2}}$$

$$= \frac{2a\pi f\tau}{\sqrt{1 + (2\pi f\tau)^2}}$$

3.2

2. For the output spectra of the previous exercise, low-pass and high-pass, calculate the group delays.[2]

① LPF:

$$H_{LP} = \frac{1}{1+j\omega\tau}$$

$$= \frac{1}{1+j\omega\tau} \times \frac{1-j\omega\tau}{1-j\omega\tau}$$

$$= \frac{1-j\omega\tau}{1+(\omega\tau)^2}$$

$$\phi_{Lp}(\omega) = Im(H_{LP}) = \tan^{-1}(-\omega\tau)$$

$$\tau_g = -\frac{d}{d\omega}\tan^{-1}(-\omega\tau)$$

$$= -\frac{-\tau}{1+(\omega\tau)^2}$$

$$= \frac{\tau}{1+(\omega\tau)^2} = \frac{\tau}{1+(2\pi f\tau)^2}$$

② HPF:

$$H_{HP} = \frac{j\omega\tau}{1+j\omega\tau}$$

$$= \frac{j\omega\tau}{1+j\omega\tau} \cdot \frac{1-j\omega\tau}{1-j\omega\tau}$$

$$= \frac{j\omega\tau(1-j\omega\tau)}{1+(\omega\tau)^2}$$

$$\phi_{Hp}(\omega) = Im(H_{HP}) = \tan^{-1}\left(\frac{\omega\tau}{1+(\omega\tau)^2}\right)$$

$$T_g = -\frac{d}{d\omega}\tan^{-1}\left(\frac{\omega\tau}{1+(\omega\tau)^2}\right)$$

$$= \frac{\tau}{1+(\omega\tau)^2} = \frac{\tau}{1+(2\pi f\tau)^2}$$

3. (a) Show that the inverse Fourier transform of $S(f) = (a/2)(\delta(f - f_0) + \delta(f + f_0))$ is a cosine function. (b) Show that the inverse Fourier transform of $S(f) = (a/2j)(\delta(f - f_0) - \delta(f + f_0))$ is a sine function.

$$s(t) = F^{-1}\{S(f)\} = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df$$

a) $S(f) = \frac{a}{2}(\delta(f-f_0) + \delta(f+f_0))$

$$s(t) = \int_{-\infty}^{\infty} \frac{a}{2}(\delta(f-f_0) + \delta(f+f_0)) e^{j2\pi ft} df$$

$$= \frac{a}{2}\left(e^{2\pi j f_0 t} + e^{-2\pi j f_0 t}\right)$$

$$= \frac{a}{2}\left(e^{j\theta} + e^{-j\theta}\right) \Rightarrow \theta = 2\pi f_0 t$$

$$= \frac{a}{2} \cdot 2\cos\theta$$

$$= a\cos(2\pi f_0 t)$$

b) $S(f) = \frac{a}{2j}(\delta(f-f_0) - \delta(f+f_0))$

$$s(t) = \int_{-\infty}^{\infty} \frac{a}{2j}(\delta(f-f_0) - \delta(f+f_0)) e^{2\pi jft} df$$

$$= \frac{a}{2j}\left(e^{2\pi j f_0 t} - e^{-2\pi j f_0 t}\right)$$

$$= \frac{a}{2j} \cdot (2j\sin\theta) \Rightarrow \theta = 2\pi f_0 t$$

$$s(t) = a\sin(2\pi f_0 t)$$

4 Convolution and Octave Code

1. For the following exercises, use Eq. 9. Let the digital impulse be $\delta[n]$ which is 1 for $n = 0$, and 0 if $n \neq 0$. For example, $\delta[n-5]$ is 1 when $n = 5$. (a) Show that if $x[n] = \delta[n]$, $y[n] = h[n] * x[n] = h[n]$. That is, if the input data is an impulse, the output is the system response. (b) Show that if the input impulse is shifted ($x[n] = \delta[n - n_0]$), the output is $h[n]$, shifted by the same amount.

$$y[n] = h[n] \cdot x[n] = \sum_{j=0}^{M-1} h[j]\, x[n-j]$$

① $x[n] = \delta[n]$

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$

$$y[n] = \sum_{j=0}^{M-1} h[j]\,\delta[n-j]$$

$$\sum_j g(j)\,\delta(n-j) = g(n)$$

Thus $y[n] = h[n]$ when $j=n$, else $=0$

② $x[n] = \delta[n-n_0]$

$$y[n] = \sum_{j=0}^{M-1} h[j]\,\delta[n-n_0-j] \Rightarrow \sum_{j=0}^{M-1} h[j]\,\delta[(n-n_0)-j]$$

$y[n] = h[n-n_0]$ when $j = n-n_0$, else $= 0$

2. In `octave`, use the `conv` function to convolve a 440 Hz sine wave with a $\delta[n - n_0]$ impulse. Shift the phase of the sine output by varying n_0.