NAILYN
LOPEZ
JAN 31

Homework 1, Unit 0: Foundations and Fundamentals

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1 Memory Bank

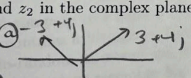
- $\sqrt{-1} = j$ ... The fundamental imaginary unit.
- $z = x + jy$ ... A complex number.
- $\Re\{z\} = x$, $\Im\{z\} = y$ ... Real and imaginary parts.
- $z^* = x - jy$ ... The complex conjugate of z.
- $|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$ ... The magnitude of z.
- $\tan\phi = y/x$ ... The phase angle of z.
- $|z| = r$, so $x = r\cos\phi$, and $y = r\sin\phi$.
- **Taylor Series:** Let $f(t)$ be a continuous, differentiable function. Let $f^n(t)$ be the n-th derivative of $f(t)$, with $f^0(t) = f(t)$. The Taylor series is an infinite series, equivalent to $f(t)$, given by

$$f(t) = \sum_{n=0}^{\infty} \frac{f^n(t_0)}{n!}(t - t_0)^n \tag{1}$$

- **Euler's Identity:** $e^{j\phi} = \cos\phi + j\sin\phi$

2 Complex Numbers and Signals

1. Let $z_1 = 3 + 4j$, and $z_2 = -3 + 4j$. Evaluate:
   (a) Graph z_1 and z_2 in the complex plane.
   (b) $z_1 + z_2$
   (c) $z_1 - z_2$
   (d) $z_1 * z_2$
   (e) z_1/z_2
   (f) $|z_1|$
   (g) $|z_2|$
   (h) ϕ_1
   (i) ϕ_2
   (j) Write z_1 and z_2 in polar form.

a 

b $(3+4j)+(-3+4j) = 0+8j = 8j$

c $(3+4j)-(-3+4j) = 3+4j+3-4j$
$= 6$

d $(3+4j)(-3+4)$

$3(-3)+3(4j)+4j(-3)+4j(4j)$
$-9+12j-12j+16j^2$
$j^2 = -1$
$-9+16(-1)$
$= -9-16 = -25$

e $\dfrac{3-4j}{-3+4j}$

f
$= \sqrt{(3)^2+(4)^2}$
$= 5$

g $\sqrt{(-3)^2+(4)^2}$
$\sqrt{25} = 5$

h $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
$\theta = \tan^{-1}\left(\frac{4}{3}\right)$
$= 53.13°$

i $\tan^{-1}\left(\frac{4}{-3}\right) = 180° - 53.13°$
$= 126.87$

2. Use Euler's Identity to show that

$$\cos(2\pi ft) = \frac{e^{2\pi jft} + e^{-2\pi jft}}{2} \tag{2}$$

$$\sin(2\pi ft) = \frac{e^{2\pi jft} - e^{-2\pi jft}}{2j} \tag{3}$$

$e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft)$
$e^{-j2\pi ft} = \cos(2\pi ft) - j\sin(2\pi ft)$
$e^{j2\pi ft} + e^{-j2\pi ft} = (\cos(2\pi ft) + j\sin(2\pi ft) + \cos(2\pi ft) - j\sin(2\pi ft))$
$= 2\cos(2\pi ft)$
$\cos(2\pi ft) = \dfrac{e^{j2\pi ft} + e^{-j2\pi ft}}{2}$

3. Let $v_1(t) = 4\cos(2\pi f_1 t)$, $v_2(t) = 4\cos(2\pi f_2 t - \phi)$. Use the results of the previous exercise in the following questions. (a) Show that $P = v_1(t)v_2(t)$ is a pair of sinusoids with frequencies $f_+ = f_1 + f_2$ and $f_- = f_1 - f_2$, offset by a total phase shift of 2ϕ. (b) Show that $P_{\max} = 16$, if $\phi = 0$ and $f_1 = f_2$. Why is 16 the correct number?[1].

a $P = v_1(t)v_2(t)$
$= 4\cos(2\pi f_1 t) \cdot 4\cos(2\pi f_2 t - \theta)$
$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$
$P = 16 \cdot \left(\frac{1}{2}\right)(\cos(2\pi f_1 t - (2\pi f_2 t - \theta)) + \cos(2\pi f_1 t + (2\pi f_2 t - \theta)))$
$= 8(\cos(2\pi(f_1 - f_2)t + \theta) + \cos(2\pi(f_1 + f_2)t - \theta))$
$f_+ = f_1 + f_2 \quad f_- = f_1 - f_2$

4. Suppose that

$$v_1(t) = \Im\{\exp(j(2\pi ft - \phi))\} \tag{4}$$

$$v_2(t) = \Im\{\exp(2\pi jft)\} \tag{5}$$

Drop the portion of the complex phase containing the frequency f, and represent the signals with just $\exp(-j\phi)$ and 1. (a) Graph these signals by treating the 1 and $\exp(-j\phi)$ as complex numbers in polar form. (b) Add the complex numbers, and obtain formulas for the new magnitude and phase angle. (c) Test your formulas for $\phi = 45$ degrees. When you add two signals of the same frequency offset by a phase, you should obtain a new

[1]The product of two mixed signal voltages, divided by the resistance, is the power (in Watts). The formula is $P = v^2/R$.

j $z = r(\cos\theta + j\sin\theta)$
$z_1 = 3+4j \quad z_2 = -3+4j$
$z = re^{j\theta}$
$z_2 = 5e^{j126.87°}$

② $\sin(2\pi f t) = \dfrac{e^{j2\pi f t} - e^{-j2\pi f t}}{2j}$

$\cos(2\pi f t) = \dfrac{e^{j2\pi f t} + e^{-j2\pi f t}}{2}$ / & $\sin(2\pi f t) = \dfrac{e^{j2\pi f t} - e^{-j2\pi f t}}{2j}$

③

ⓑ

$P = 8[\cos(2\pi(f_1 - f_1)t) + \cos(2\pi(f_1 + f_1)t)]$

$= 8(\cos(0) + \cos(4\pi f_1 t))$

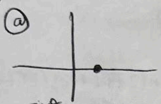
$P = 8(1 + \cos(4\pi f_1 t))$

$P = 8(1 + \cos(4\pi f_1 t)$

$P = 8(1+1) = 16$

④

ⓐ



$e^{-j\theta} = (\cos(-\theta), \sin(-\theta)) = (\cos\theta, -\sin\theta)$

ⓑ $e^{j0} + e^{-j\theta} = 1 + e^{-j\theta}$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$1 + e^{-j\theta} = 1 + \cos\theta - j\sin\theta$

$|1 + e^{-j\theta}| = \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}$

$= \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}$

$= \sqrt{2(1+\cos\theta}$

$1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

$|1 + e^{-j\theta}| = \sqrt{2(2\cos^2\frac{\theta}{2}} = 2\cos\frac{\theta}{2}$

$\tan\theta = \dfrac{(1+e^{j\theta})}{1+e^{-j\theta}} = \dfrac{-\sin\theta}{1+\cos\theta}$

$\tan\theta = \dfrac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$

$= -\tan\frac{\theta}{2}$

ⓒ $2\cos\frac{45^\circ}{2}$

$2\cos(22.5^\circ)$

$-\frac{45^\circ}{2} = -22.5^\circ$

1.

```
clear;
close all;

% (a) Generate and plot Gaussian distribution
mu = 0;       % Mean
sigma = 1;    % Standard deviation
x = randn(10000,1); % Generate Gaussian-distributed random numbers

figure(1);
hist(x,30); % Histogram with 30 bins
title('Histogram of Gaussian Random Numbers');
xlabel('x');
ylabel('Frequency');

% Compare with theoretical Gaussian PDF
hold on;
xx = linspace(min(x), max(x), 100);
pdf_gauss = (1 / (sqrt(2*pi) * sigma)) * exp(-((xx - mu).^2) / (2*sigma^2));
plot(xx, pdf_gauss * length(x) * (max(x) - min(x)) / 30, 'r', 'LineWidth', 2);
legend('Histogram', 'Theoretical Gaussian PDF');
hold off;

% (b) Plot the random numbers
figure(2);
plot(x);
axis([-1 10001 -10 10]);
title('Gaussian Noise Signal');
xlabel('Sample Index');
ylabel('Noise Amplitude');

% (c) Adding Gaussian noise to a sine wave
fs = 1000; % Sampling frequency
t = 0:1/fs:1; % Time vector
f = 5; % Frequency of sine wave
A = 1; % Amplitude of sine wave
sine_wave = A * sin(2 * pi * f * t);
noise = sigma * randn(size(t)); % Gaussian noise

noisy_signal = sine_wave + noise;

figure(3);
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plot(t, sine_wave, 'b', 'LineWidth', 2);
hold on;
plot(t, noisy_signal, 'r');
title('Sine Wave with Gaussian Noise');
xlabel('Time (s)');
ylabel('Amplitude');
legend('Clean Sine Wave', 'Noisy Sine Wave');
hold off;

% (d) Effect of decreasing amplitude on SNR
A_values = [1, 0.5, 0.2, 0.1]; % Different amplitudes
figure(4);
for i = 1:length(A_values)
   A = A_values(i);
   sine_wave = A * sin(2 * pi * f * t);
   noisy_signal = sine_wave + noise;

   subplot(2,2,i);
   plot(t, noisy_signal);
   title(['SNR: ', num2str(A/sigma)]);
   xlabel('Time (s)');
   ylabel('Amplitude');
end

2.clear;
close all;

% Parameters
N = 10000; % Number of samples
M = 12; % Number of uniform random variables to sum (higher M -> better Gaussian
approximation)

% Generate uniform random numbers and sum them
uniform_numbers = rand(N, M); % Each row has M uniform numbers
summed_values = sum(uniform_numbers, 2); % Sum along rows

% Normalize to zero mean and unit variance
mu = M * 0.5; % Mean of summed uniform distribution
sigma = sqrt(M / 12); % Standard deviation of summed uniform distribution
gaussian_approx = (summed_values - mu) / sigma; % Normalized to standard normal

% Plot histogram of summed values
figure;
hist(gaussian_approx, 50);
```

```
title(['Sum of ', num2str(M), ' Uniform Random Numbers']);
xlabel('Summed Value (Normalized)');
ylabel('Frequency');
hold on;

% Overlay theoretical Gaussian PDF
x = linspace(min(gaussian_approx), max(gaussian_approx), 100);
pdf_gauss = (1 / sqrt(2 * pi)) * exp(-x.^2 / 2);
plot(x, pdf_gauss * N * (max(gaussian_approx) - min(gaussian_approx)) / 50, 'r', 'LineWidth', 2);
legend('Histogram', 'Theoretical Gaussian PDF');
hold off;
```

3. clear;

```
close all;

% Parameters
f = 5;     % Frequency of sine wave (in Hz)
fs1 = 20;  % First sampling frequency (less than 2 * f)
fs2 = 2 * f; % Second sampling frequency (exactly 2 * f)

% Time vectors for the sine wave
t1 = 0:1/fs1:1; % Sampling period for fs1
t2 = 0:1/fs2:1; % Sampling period for fs2

% Generate sine waves
sine_wave1 = sin(2 * pi * f * t1); % Sine wave for fs1
sine_wave2 = sin(2 * pi * f * t2); % Sine wave for fs2

% Plot sine wave with sampling frequency fs1
figure(1);
subplot(2,1,1);
plot(t1, sine_wave1, 'b');
title(['Sine Wave with Sampling Frequency fs = ', num2str(fs1)]);
xlabel('Time (s)');
ylabel('Amplitude');
axis([0 1 -1.5 1.5]);

% Plot sine wave with sampling frequency fs2
subplot(2,1,2);
plot(t2, sine_wave2, 'r');
title(['Sine Wave with Sampling Frequency fs = ', num2str(fs2)]);
xlabel('Time (s)');
ylabel('Amplitude');
axis([0 1 -1.5 1.5]);
```