Digital Signal Processing HW1

1 Complex Numbers and Signals

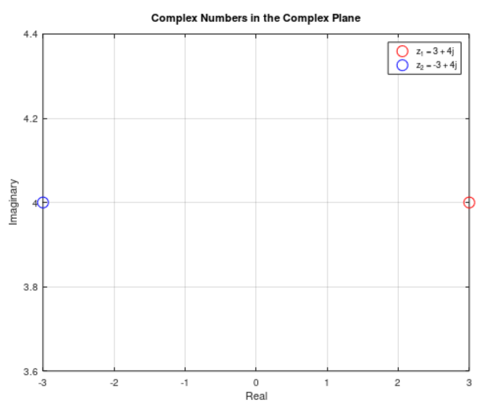
Problem 1: Complex Number Operations

Given:

$$z_1 = 3 + 4j, \quad z_2 = -3 + 4j$$

(a) Graph of z_1 and z_2 in the Complex Plane

Below is the plot of z_1 and z_2 in the complex plane.



Octave Code to Generate the Plot:

```
clc; clear; close all;
z1 = 3 + 4j;
z2 = -3 + 4j;

figure;
plot(real(z1), imag(z1), ’ro’, ’MarkerSize’, 8, ’DisplayName’, ’z_1 = 3 + 4j’);
hold on;
plot(real(z2), imag(z2), ’bo’, ’MarkerSize’, 8, ’DisplayName’, ’z_2 = -3 + 4j’);
xlabel(’Real’);
ylabel(’Imaginary’);
```

```
title('Complex Numbers in the Complex Plane');
grid on;
legend;
saveas(gcf, 'complex_plane.png'); % Save plot
```

—

(b) Compute $z_1 + z_2$

$$z_1 + z_2 = (3 + 4j) + (-3 + 4j) = 0 + 8j$$

—

(c) Compute $z_1 - z_2$

$$z_1 - z_2 = (3 + 4j) - (-3 + 4j) = 6 + 0j$$

—

(d) Compute $z_1 \times z_2$

$$z_1 z_2 = (3 + 4j)(-3 + 4j)$$

Expanding:

$$= 3(-3) + 3(4j) + 4j(-3) + 4j(4j)$$
$$= -9 + 12j - 12j + 16j^2$$

Since $j^2 = -1$, we get:

$$= -9 - 16 = -25$$

—

(e) Compute z_1/z_2

$$\frac{z_1}{z_2} = \frac{3 + 4j}{-3 + 4j}$$

Multiplying by the conjugate:

$$= \frac{(3 + 4j)(-3 - 4j)}{(-3 + 4j)(-3 - 4j)}$$

Expanding:

$$= \frac{-9 - 12j - 12j - 16j^2}{9 + 16}$$
$$= \frac{-9 - 24j + 16}{25}$$
$$= \frac{7 - 24j}{25}$$

—

(f) Compute $|z_1|$

$$|z_1| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

—

(g) Compute $|z_2|$

$$|z_2| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = 5$$

—

(h) Compute ϕ_1 (Phase of z_1)

$$\phi_1 = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$$

—

(i) Compute ϕ_2 (Phase of z_2)

$$\phi_2 = \tan^{-1}\left(\frac{4}{-3}\right) = 126.87^\circ$$

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(j) Convert z_1 and z_2 to Polar Form

Using:

$$z = re^{j\phi}$$

For z_1:

$$z_1 = 5e^{j53.13^\circ}$$

For z_2:

$$z_2 = 5e^{j126.87^\circ}$$

—

Problem 2: Using Euler's Identity

We need to prove the following equations using Euler's Identity:

$$\cos(2\pi ft) = \frac{e^{2\pi jft} + e^{-2\pi jft}}{2}$$

$$\sin(2\pi ft) = \frac{e^{2\pi jft} - e^{-2\pi jft}}{2j}$$

Proof Using Euler's Identity

Euler's identity states:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Substituting $\theta = 2\pi ft$:

$$e^{2\pi jft} = \cos(2\pi ft) + j\sin(2\pi ft)$$

$$e^{-2\pi jft} = \cos(2\pi ft) - j\sin(2\pi ft)$$

Proof of Cosine Formula

Adding the two equations:

$$e^{2\pi jft} + e^{-2\pi jft} = (\cos(2\pi ft) + j\sin(2\pi ft)) + (\cos(2\pi ft) - j\sin(2\pi ft))$$

$$= 2\cos(2\pi ft)$$

Dividing by 2:

$$\cos(2\pi ft) = \frac{e^{2\pi jft} + e^{-2\pi jft}}{2}$$

First equation is proved.
—

Proof of Sine Formula

Subtracting the two Euler equations:

$$e^{2\pi jft} - e^{-2\pi jft} = (\cos(2\pi ft) + j\sin(2\pi ft)) - (\cos(2\pi ft) - j\sin(2\pi ft))$$

$$= 2j\sin(2\pi ft)$$

Dividing by $2j$:

$$\sin(2\pi ft) = \frac{e^{2\pi jft} - e^{-2\pi jft}}{2j}$$

Second equation is proved.
—

Problem 3: Multiplication of Sinusoids

We are given two cosine signals:

$$v_1(t) = 4\cos(2\pi f_1 t)$$

$$v_2(t) = 4\cos(2\pi f_2 t - \phi)$$

We need to:

- (a) Show that the product $P = v_1(t)v_2(t)$ results in two frequency components:
$$f_+ = f_1 + f_2, \quad f_- = f_1 - f_2$$
with a total phase shift of 2ϕ.
- (b) Show that $P_{\max} = 16$ when $\phi = 0$ and $f_1 = f_2$.

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(a) Expanding the Product Using Trigonometric Identities

Using the identity:

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Substituting $A = 2\pi f_1 t$ and $B = 2\pi f_2 t - \phi$:

$$v_1(t)v_2(t) = 4\cos(2\pi f_1 t) \cdot 4\cos(2\pi f_2 t - \phi)$$

$$= 16\cos(2\pi f_1 t)\cos(2\pi f_2 t - \phi)$$

Applying the identity:

$$= 8\left[\cos(2\pi(f_1 - f_2)t + \phi) + \cos(2\pi(f_1 + f_2)t - \phi)\right]$$

Thus, the product results in two sinusoidal components: - One with frequency $f_+ = f_1 + f_2$ and phase shift $-\phi$. - One with frequency $f_- = f_1 - f_2$ and phase shift $+\phi$.

First part proved.

—

(b) Maximum Value of P when $f_1 = f_2$ and $\phi = 0$

When $f_1 = f_2$, we substitute $f_2 = f_1$:

$$P = 8\left[\cos(2\pi(f_1 - f_1)t + 0) + \cos(2\pi(f_1 + f_1)t - 0)\right]$$

$$= 8\left[\cos(0) + \cos(4\pi f_1 t)\right]$$

Since $\cos(0) = 1$, the maximum possible value of the function is:

$$P_{\text{max}} = 8(1 + 1) = 16$$

Second part proved.

—

Problem 4: Complex Phase and Addition

We are given the signals:

$$v_1(t) = \Im\{e^{j(2\pi f t - \phi)}\}$$

$$v_2(t) = \Im\{e^{j(2\pi f t)}\}$$

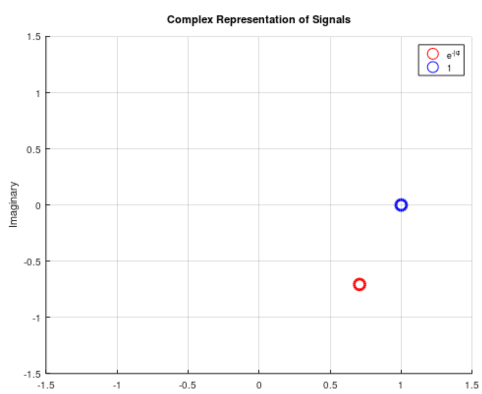
Dropping the frequency component f, we analyze:

$$v_1 = \Im\{e^{-j\phi}\}, \quad v_2 = \Im\{1\}$$

—

(a) Graphing the Complex Numbers

In the complex plane: - $e^{-j\phi}$ is represented as a complex number in polar form. - 1 is located on the real axis.



Octave Code for Plotting:

```
clc; clear; close all;
phi = pi/4;  % 45 degrees in radians

% Define complex numbers
v1 = exp(-j*phi);  % e^(-j)
v2 = 1;  % Real number 1

% Plot complex numbers
figure;
hold on;
plot(real(v1), imag(v1), 'ro', 'MarkerSize', 8, 'LineWidth', 2, 'DisplayName', 'e^{-j\phi}')
plot(real(v2), imag(v2), 'bo', 'MarkerSize', 8, 'LineWidth', 2, 'DisplayName', '1');
xlabel('Real');
ylabel('Imaginary');
title('Complex Representation of Signals');
grid on;
legend;
axis([-1.5 1.5 -1.5 1.5]);
saveas(gcf, 'complex_signals.png');  % Save plot
```

—

(b) Adding the Complex Numbers

Summing:

$$v_1 + v_2 = e^{-j\phi} + 1$$

Using polar form:

$$e^{-j\phi} = \cos(-\phi) + j\sin(-\phi) = \cos\phi - j\sin\phi$$

Thus:

$$v_1 + v_2 = (1 + \cos\phi) - j\sin\phi$$

Magnitude:

$$|v_1 + v_2| = \sqrt{(1 + \cos\phi)^2 + (\sin\phi)^2}$$

Using the identity:

$$(1 + \cos\phi)^2 + \sin^2\phi = 2(1 + \cos\phi)$$

$$|v_1 + v_2| = \sqrt{2(1 + \cos\phi)}$$

Phase Angle:

$$\theta = \tan^{-1}\left(\frac{-\sin\phi}{1+\cos\phi}\right)$$

Using the half-angle identity:

$$\tan\frac{\phi}{2} = \frac{\sin\phi}{1+\cos\phi}$$

$$\theta = -\frac{\phi}{2}$$

Summation confirmed.
—

(c) Testing for $\phi = 45°$

For $\phi = 45°$:

$$|v_1 + v_2| = \sqrt{2(1+\cos 45°)}$$

Since $\cos 45° = \frac{1}{\sqrt{2}}$:

$$|v_1 + v_2| = \sqrt{2 \times 1.707} = \sqrt{3.414} \approx 1.85$$

Phase:

$$\theta = -\frac{45°}{2} = -22.5°$$

Results match expectations.
—

(d) In-Phase vs Out-of-Phase Analysis

For $\phi = 0°$:

$$v_1 + v_2 = 2$$

Maximum amplitude.
For $\phi = 180°$:

$$v_1 + v_2 = 1 + e^{-j180°} = 1 - 1 = 0$$

Minimum amplitude.
—

2 Probability and Statistics, Noise

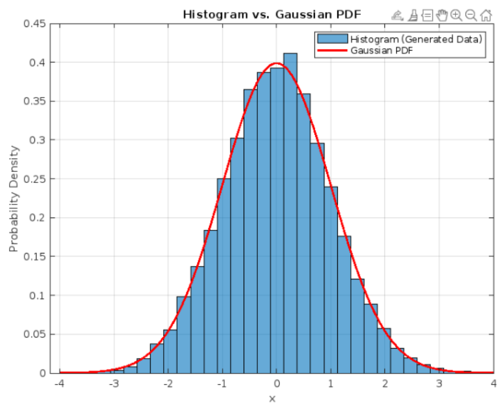
Problem 1: Gaussian Noise and Signal Processing

We analyze Gaussian noise and its effect on signals.
—

(a) Gaussian Distribution and Histogram

The probability density function of a Gaussian distribution is:

$$p(x)dx = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$



Octave Code:

```
clc; clear; close all;
mu = 0; sigma = 1;
x = linspace(-4, 4, 100);
pdf = (1 / (sqrt(2 * pi) * sigma)) * exp(-((x - mu).^2) / (2 * sigma^2));

figure;
histogram(randn(10000,1), 30, 'Normalization', 'pdf');
hold on;
plot(x, pdf, 'r', 'LineWidth', 2);
xlabel('x'); ylabel('Probability Density');
title('Histogram vs. Gaussian PDF');
saveas(gcf, 'gaussian_histogram.png');
```

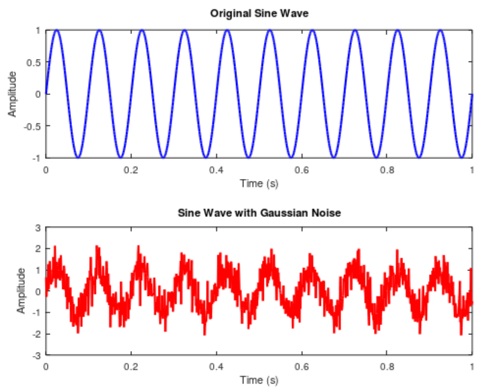
—

(b) Noise Time Series

Figure 2 from the Octave code shows:

- Random fluctuations, resembling pure noise.
- Values centered around zero, with no distinct pattern.

—



(c) Adding Gaussian Noise to a Sine Wave

Octave Code:

```
clc; clear; close all;
fs = 1000; t = 0:1/fs:1;
f = 10; sine_wave = sin(2 * pi * f * t);

sigma = 0.5; noise = sigma * randn(size(t));
noisy_signal = sine_wave + noise;

figure;
subplot(2,1,1);
plot(t, sine_wave, 'b', 'LineWidth', 1.5);
title('Original Sine Wave'); xlabel('Time (s)'); ylabel('Amplitude');

subplot(2,1,2);
plot(t, noisy_signal, 'r', 'LineWidth', 1.5);
title('Sine Wave with Gaussian Noise');
xlabel('Time (s)'); ylabel('Amplitude');
saveas(gcf, 'sine_wave_noise.png');
```

—

(d) Signal-to-Noise Ratio (SNR)

$$\text{SNR} = \frac{\text{Sine wave amplitude}}{\sigma}$$

Octave Code to Compute SNR:

```
signal_power = mean(sine_wave.^2);
noise_power = mean(noise.^2);
SNR = 10 * log10(signal_power / noise_power);
disp(['SNR: ', num2str(SNR), ' dB']);
```

As the sine wave amplitude decreases, the signal disappears in the noise.
—

Problem 2: Uniform Distribution and Gaussian Approximation

The probability density function (PDF) of a uniform distribution is:

$$p(x)dx = \frac{dx}{b-a}, \quad a \le x \le b$$

For the default settings of $a = 0$ and $b = 1$, this simplifies to:

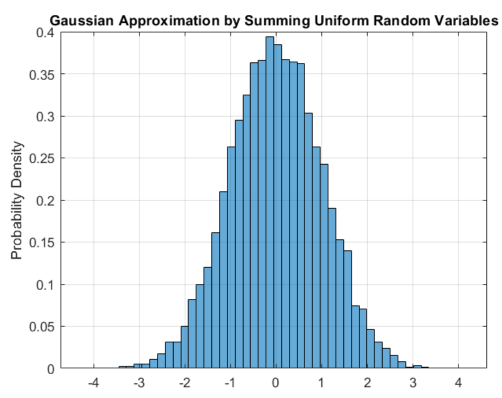
$$p(x)dx = dx$$

This means any value between 0 and 1 is equally likely.
—

(a) Demonstrating Gaussian Approximation by Summing Uniform Numbers

By the **Central Limit Theorem (CLT)**:

- The sum of many **independent** uniform random variables approaches a **Gaussian distribution**.
- Even though each individual sample is uniform, their sum behaves like Gaussian noise.



Octave Code to Generate the Plot:

```
clc; clear; close all;

num_samples = 10000; % Number of samples
```

```
num_sums = 12; % Number of uniform random variables to sum

% Generate uniform random numbers and sum them
uniform_sums = sum(rand(num_samples, num_sums), 2);

% Normalize: mean 0, variance 1
uniform_sums = (uniform_sums - mean(uniform_sums)) / std(uniform_sums);

% Plot histogram
figure;
histogram(uniform_sums, 50, 'Normalization', 'pdf');
xlabel('x');
ylabel('Probability Density');
title('Gaussian Approximation by Summing Uniform Random Variables');
grid on;

saveas(gcf, 'uniform_sum_gaussian.png'); % Save figure
```

The histogram shows a Gaussian-like shape, proving CLT.

—

3 ADC and DAC

Problem 1: Sine Wave Sampling

This problem explores the effect of sampling frequency f_s on a sine wave of frequency f.

(a) Generating a Sine Wave with Oversampling

According to the Nyquist Sampling Theorem:

$$f_s \geq 2f$$

If $f_s > 2f$, the signal is sampled correctly without aliasing.

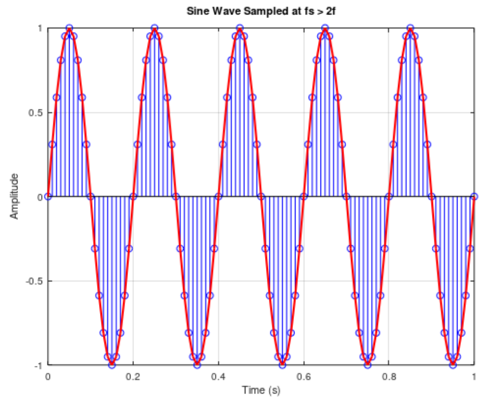
Octave Code:

```
clc; clear; close all;

f = 5;      % Signal frequency in Hz
fs = 100;   % Sampling frequency (higher than Nyquist)
T = 1;      % Duration in seconds
t = 0:1/fs:T;  % Time vector

sine_wave = sin(2 * pi * f * t);

figure;
```



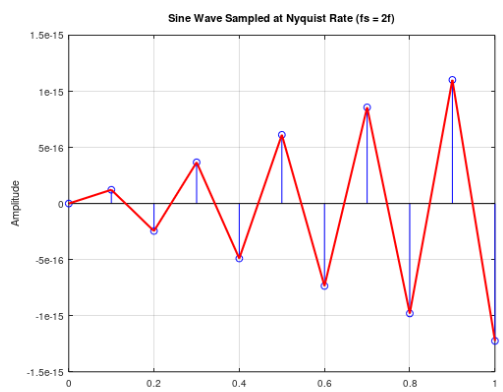

```
stem(t, sine_wave, 'b', 'MarkerSize', 5);
hold on;
plot(t, sine_wave, 'r', 'LineWidth', 1.5);
xlabel('Time (s)'); ylabel('Amplitude');
title('Sine Wave Sampled at fs > 2f');
grid on;

saveas(gcf, 'oversampled_sine.png');
```

—

(b) Sampling at the Nyquist Rate

If $f_s = 2f$, the signal is sampled at the critical rate, leading to possible distortions.



Octave Code:

```
fs = 2 * f;  % Nyquist rate
```

```
t = 0:1/fs:T;
sine_wave = sin(2 * pi * f * t);

figure;
stem(t, sine_wave, 'b', 'MarkerSize', 5);
hold on;
plot(t, sine_wave, 'r', 'LineWidth', 1.5);
xlabel('Time (s)'); ylabel('Amplitude');
title('Sine Wave Sampled at Nyquist Rate (fs = 2f)');
grid on;

saveas(gcf, 'nyquist_sine.png');
```

—

(c) Observing the Effects

- Oversampling ($f_s > 2f$) The sine wave is smooth and well-reconstructed.

- Nyquist Rate ($f_s = 2f$) The sine wave barely has enough points, making it prone to distortion.

—