# Homework 3, Unit 0: Foundations and Fundamentals

Prof. Jordan C. Hanson

February 19, 2025

## 1 Memory Bank

- **Homogeneous system:** Let $k$ be a constant, and let $s_{\text{in}}(t)$ and $s_{\text{out}}(t)$ be the input and output signals to a system $S$, respectively. $S$ is *homogeneous* if:

$$s_{\text{out}}(t) = S[s_{\text{in}}(t)] \tag{1}$$
$$ks_{\text{out}}(t) = S[ks_{\text{in}}(t)] \tag{2}$$

- **Additive system:** Let $s_1(t)$ and $s_2(t)$ be two input signals to a system $S$, with outputs $s_1'(t)$ and $s_2'(t)$. $S$ is *additive* if:

$$s_1'(t) = S[s_1(t)] \tag{3}$$
$$s_2'(t) = S[s_2(t)] \tag{4}$$
$$s_1'(t) + s_2'(t) = S[s_1(t) + s_2(t)] \tag{5}$$

- **Shift-invariant system:** Let $s_{\text{in}}(t)$ and $s_{\text{out}}(t)$ be input and output signals to a system $S$, and let $t_0$ be a constant. $S$ is *shift invariant* if:

$$s_{\text{out}}(t) = S[s_{\text{in}}(t)] \tag{6}$$
$$s_{\text{out}}(t - t_0) = S[s_{\text{in}}(t - t_0)] \tag{7}$$

- $F(f) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-2\pi jft}dt$ ... The Fourier Transform.
- $\mathcal{F}^{-1}\{F(f)\} = \int_{-\infty}^{\infty} F(f)e^{2\pi jft}df$ ... The Inverse Fourier Transform.
- The **Dirac $\delta$-function** is a distribution defined by the following property:

$$f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt \tag{8}$$

  In words, the integral of a $\delta$-function times a function $f$ is the value of the function at $t_0$.

- **Convolution**: this is an operation that characterizes the response $h[n]$ of a linear system.

$$y[i] = h[n] * x[n] = \sum_{j=0}^{M-1} h[j]x[i-j] \tag{9}$$

  In words, the output at sample $i$ is equal to the produce of the system response $h$ and the input signal $x$, summed over the proceeding $M$ samples (from $j = 0$ to $j = M - 1$).

## 2 Linear Systems

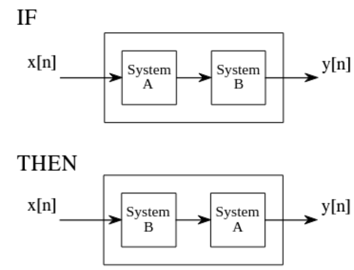


Figure 1: Linear systems **commute**.

1. Consider Fig. 1, which depicts two linear systems A and B. Symbolically, systems A and B **commute** if $A\{B\{x[n]\}\} = B\{A\{x[n]\}\}$. (a) Let $A\{x[n]\} = 2x[n] - 1$, and $B\{x[n]\} = 0.5x[n]$. Which system, A or B, is a linear system? For the system that is not linear, which linear property does it break? (b) Modify the non-linear system to make it linear, and show that A and B commute.

2. Consider Eq. 8 in the Memory Bank. Let $f(t) = a_1\cos(2\pi f_1 t) + a_2\cos(2\pi f_2 t)$, with $T_1 = 1/f_1$, $T_2 = 1/f_2$, and $f_2 = 2f_1$. Evaluate the following:
   - $\int_{-\infty}^{\infty} f(t)\delta(t - T_1)dt$
   - $\int_{-\infty}^{\infty} f(t)\delta(t - T_2)dt$

3. Let $f(t) = a\delta(t - t_0)$. (a) Show that the magnitude of the **Fourier transform** of this impulse is $a$. (b) Show that the phase angle, $\phi$, is $-2\pi f t_0$. (c) Show that the group delay, $\tau_g = -d\phi/d\omega$ is $t_0$.

4. Let $\delta[n]$ represent a digital impulse: [1000 0000][1]. (a) If $y[n] = S[x[n]] = 0.5x[n-2]$, what is $S[\delta[n]]$? (b) $y[n]$ is the *impulse response* of $S$. What is the *step response*, if the step input is $s[n] = [01111111]$?

# 3 Fourier Transforms and Basic Filters

1. Suppose we pass a signal $s(t)$ into a low-pass filter. The signal as a function of frequency is $S(f)$, the Fourier transform of $s(t)$. The output of the low-pass filter will be $S(f)$ times $1/(1 + j\omega\tau)$, where $\omega = 2\pi f$, and $\tau = RC$. That is, the output will be $S(f)/(1 + j\omega\tau)$. (a) Calculate the Fourier transform $S(f)$, if $s(t) = a\delta(t - t_0)$ (as we did in class). (b) Suppose we pass our impulse $s(t)$ into a low-pass filter. What is the magnitude of the output, as a function of frequency? (c) Repeat this exercise, but with a high-pass filter response: $j\omega\tau/(1 + j\omega\tau)$.

[1] Let the index for data in this list of numbers start with $n = 0$.

2. For the output spectra of the previous exercise, low-pass and high-pass, calculate the group delays.[2]

3. (a) Show that the inverse Fourier transform of $S(f) = (a/2)(\delta(f - f_0) + \delta(f + f_0))$ is a cosine function. (b) Show that the inverse Fourier transform of $S(f) = (a/2j)(\delta(f - f_0) - \delta(f + f_0))$ is a sine function.

# 4 Convolution and Octave Code

1. For the following exercises, use Eq. 9. Let the digital impulse be $\delta[n]$ which is 1 for $n = 0$, and 0 if $n \neq 0$. For example, $\delta[n-5]$ is 1 when $n = 5$. (a) Show that if $x[n] = \delta[n]$, $y[n] = h[n] * x[n] = h[n]$. That is, if the input data is an impulse, the output is the system response. (b) Show that if the input impulse is shifted ($x[n] = \delta[n - n_0]$), the output is $h[n]$, shifted by the same amount.

2. In `octave`, use the `conv` function to convolve a 440 Hz sine wave with a $\delta[n - n_0]$ impulse. Shift the phase of the sine output by varying $n_0$.

[2] Hint: multiply the numerator and denominator of ratios by the complex conjugate of the denominator, to aid in splitting the complex expression into real and imaginary parts.