Homework 1, Unit 0: Foundations and Fundamentals

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January 23, 2025

1 Memory Bank

- $\sqrt{-1} = j$ ... The fundamental imaginary unit.
- $z = x + jy$ ... A complex number.
- $\Re\{z\} = x$, $\Im\{z\} = y$ ... Real and imaginary parts.
- $z^* = x - jy$ ... The complex conjugate of z.
- $|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$ ... The magnitude of z.
- $\tan\phi = y/x$ ... The phase angle of z.
- $|z| = r$, so $x = r\cos\phi$, and $y = r\sin\phi$.
- **Taylor Series:** Let $f(t)$ be a continuous, differentiable function. Let $f^n(t)$ be the n-th derivative of $f(t)$, with $f^0(t) = f(t)$. The Taylor series is an infinite series, equivalent to $f(t)$, given by

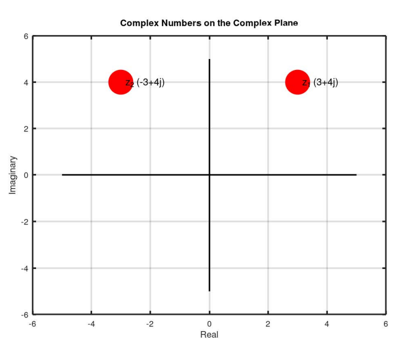
$$f(t) = \sum_{n=0}^{\infty} \frac{f^n(t_0)}{n!}(t - t_0)^n \tag{1}$$

- **Euler's Identity:** $e^{j\phi} = \cos\phi + j\sin\phi$

2 Complex Numbers and Signals

1. Let $z_1 = 3 + 4j$, and $z_2 = -3 + 4j$. Evaluate:
   (a) Graph z_1 and z_2 in the complex plane.
   (b) $z_1 + z_2 = 0 + 8j = \boxed{8j}$
   (c) $z_1 - z_2 = 6 + 0j = \boxed{6}$
   (d) $z_1 * z_2 = (3+4j)(-3+4j) = -9 + \cancel{12j} - \cancel{12j} + 16j^2 = -9 + 16(-1) = \boxed{-25}$
   (e) $z_1/z_2 = \frac{(3+4j)(-3-4j)}{(-3+4j)(-3-4j)} = \frac{-9-24j+16}{9+16} = \frac{7-24j}{25} = \boxed{\frac{7}{25} - \frac{24j}{25}}$
   (f) $|z_1| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = \boxed{5}$
   (g) $|z_2| = \sqrt{(-3)^2+4^2} = \sqrt{9+16} = \sqrt{25} = \boxed{5}$
   (h) $\phi_1 = \arctan\left(\frac{4}{3}\right) \approx \boxed{.927}$
   (i) $\phi_2 = \arctan\left(-\frac{4}{3}\right) = \pi - \arctan\left(\frac{4}{3}\right) \approx \boxed{2.214}$
   (j) Write z_1 and z_2 in polar form.
   $z_1 = 5e^{j.927}$ $\quad z_2 = 5e^{j2.214}$

a)



2. Use Euler's Identity to show that

$$\cos(2\pi ft) = \frac{e^{2\pi jft} + e^{-2\pi jft}}{2} \tag{2}$$

$$\sin(2\pi ft) = \frac{e^{2\pi jft} - e^{-2\pi jft}}{2j} \tag{3}$$

$e^{j\theta} = \cos\theta + j\sin\theta$
$e^{-j\theta} = \cos\theta - j\sin\theta$

$e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft)$
$e^{-j2\pi ft} = \cos(2\pi ft) - j\sin(2\pi ft)$
$e^{j2\pi ft} + e^{-j2\pi ft} = \cos(2\pi ft) + \cancel{j\sin(2\pi ft)} + \cos(2\pi ft) - \cancel{j\sin(2\pi ft)} = 2\cos(2\pi ft)$
$\boxed{\cos(2\pi ft) = \frac{e^{j2\pi ft} + e^{-j2\pi ft}}{2}}$
$e^{j2\pi ft} - e^{-j2\pi ft} = \cancel{\cos(2\pi ft)} + j\sin(2\pi ft) - (\cancel{\cos(2\pi ft)} - j\sin(2\pi ft)) = 2j\sin(2\pi ft)$
$\sin(2\pi ft) = \boxed{\frac{e^{j2\pi ft} - e^{-j2\pi ft}}{2j}}$

3. Let $v_1(t) = 4\cos(2\pi f_1 t)$, $v_2(t) = 4\cos(2\pi f_2 t - \phi)$. Use the results of the previous exercise in the following questions. (a) Show that $P = v_1(t)v_2(t)$ is a pair of sinusoids with frequencies $f_+ = f_1 + f_2$ and $f_- = f_1 - f_2$, offset by a total phase shift of 2ϕ. (b) Show that $P_{\text{max}} = 16$, if $\phi = 0$ and $f_1 = f_2$. Why is 16 the correct number?[1]

a) $v_1(t) = 4\left(\frac{e^{j2\pi f_1 t} + e^{-j2\pi f_1 t}}{2}\right)$, $v_2(t) = \left(\frac{e^{j2\pi f_2 t - \phi} + e^{-j2\pi f_2 t - \phi}}{2}\right)$
$P = v_1(t)v_2(t) = 16 \cdot \left(\frac{e^{j2\pi f_1 t} + e^{-j2\pi f_1 t}}{}\right)\left(\frac{e^{j2\pi f_2 t - \phi} - e^{-j2\pi f_2 t - \phi}}{}\right)$
$= 4\left(e^{j(2\pi f_+ t - \phi)} + e^{j(2\pi f_- t + \phi)} + e^{-j(2\pi f_+ t - \phi)} + e^{-j(2\pi f_- t + \phi)}\right)$
$= 8\left(\cos(2\pi f_+ t - \phi) + \cos(2\pi f_- t + \phi)\right)$

b) $f_- = 0$, $f_+ = 2f_1$
$\cos(0) = 1$, $P = 8(1 + \cos(4\pi f_1 t))$
max of $\cos(4\pi f_1 t) = 1$ so,

$P_{max} = 8(1+1) = 16$
$\boxed{P_{max} = 16}$

4. Suppose that

$$v_1(t) = \Im\{\exp(j(2\pi ft - \phi))\} \tag{4}$$

$$v_2(t) = \Im\{\exp(2\pi jft)\} \tag{5}$$

Drop the portion of the complex phase containing the frequency f, and represent the signals with just $\exp(-j\phi)$ and 1. (a) Graph these signals by treating the 1 and $\exp(-j\phi)$ as complex numbers in polar form. (b) Add the complex numbers, and obtain formulas for the new magnitude and phase angle. (c) Test your formulas for $\phi = 45$ degrees. When you add two signals of the same frequency offset by a phase, you should obtain a new

[1] The product of two mixed signal voltages, divided by the resistance, is the power (in Watts). The formula is $P = v^2/R$.

signal at the same frequency with a new phase and amplitude. What happens when the signals are in phase ($\phi = 0$ degrees) and out of phase ($\phi = 180$ degrees)?

b) $V_1 = e^{-j\phi}$, $V_2 = 1$

$V = e^{-j\phi} + 1$

$V = 1 + \cos\phi - j\sin\phi$

$RE(v) = 1 + \cos\phi$, $Im(v) = -\sin\phi$

$|V| = \sqrt{(1+\cos\phi)^2 + (-\sin\phi)^2}$

$= \sqrt{1 + 2\cos\phi + \cos^2\phi + \sin^2\phi}$

$= \sqrt{2(1+\cos\phi)}$

$\theta = \arctan\left(\frac{-\sin\theta}{1+\cos\theta}\right)$

$\theta = -\frac{\phi}{2}$

$|V| = 2\cos\left(\frac{\phi}{2}\right)$, $\theta = -\frac{\phi}{2}$

c) $\phi = 45°$

$|V| = 2\cos\left(\frac{45}{2}\right)$

$= 2\cos\left(\frac{22.5}{2}\right)$

$= 2 * .924 = 1.848$

magnitud ≈ 1.848

Phaseshift $= -22.5°$

$\phi = 0°$

$|V| = 2\cos(0°)$

$= 2(1) = 2$

$\theta = -\frac{0°}{2} = 0°$

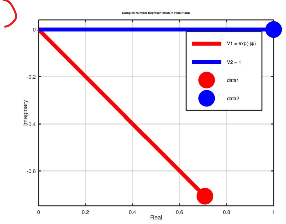
$\phi = 180°$

$|V| = 2\cos\left(\frac{180}{2}\right)$

$= 2\cos(90) = 2(0) = 0$

$\theta = -\frac{180}{2} = -90°$

a)



3 Probability and Statistics, Noise

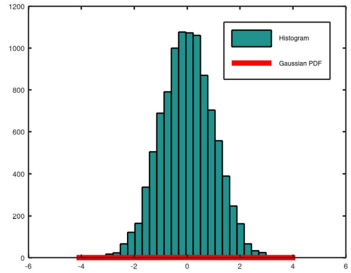
1. Consider the following octave code:

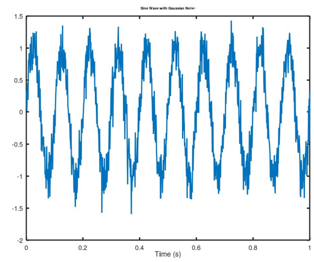
```
clear;
close;
home;

x = randn(10000,1);
figure(1)
hist(x,30);
figure(2);
plot(x)
axis([-1 10001 -10 10]);
```

The octave workspace is cleared, and a vector of data `x` is created. This vector contains pseudo-random numbers drawn from *the Gaussian distribution*, with mean μ and standard deviation σ:

$$p(x)dx = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx \tag{6}$$

(a) Graph Eq. 6, and compare to Figure 1 created by the code. This figure contains a *histogram*, that counts how often the pseudo-random numbers fall into each of 30 bins. Does the histogram resemble Eq. 6? (b) Examine Figure 2 created by the code. If the numbers represent digitized, sampled noise voltages, they appear to be pure noise. (c) Write code that adds gaussian noise to a sine wave. (d) Notice that, as the amplitude of the sine wave decreases, the signal appears to be lost in the noise. The ratio of sine wave amplitude divided by σ in Eq. 6 is called the signal-to-noise ratio (SNR).





2. The octave function `rand` gives pseudo-random numbers drawn from a *uniform distribution*:

$$p(x)dx = \frac{dx}{b-a}, \quad a \le x \le b \tag{7}$$

This PDF is flat between a and b, where any number between these is equally likely to occur. The `rand` function has default settings of $b = 1$ and $a = 0$. Write an octave code that demonstrates that the sum of a large set of numbers drawn from `rand` is distributed according to Eq. 6. That is, we get gaussian noise from the repeated addition of many uniform random numbers.

4 ADC and DAC

1. Create an octave code that graphs a sine wave of frequency `f` and sampling frequency `fs` (see Code Lab 1 on Moodle for examples). Now tune the sampling frequency to with a factor of 2 of the signal frequency. Qualitatively, what happens to the signal graph?

As fs increases the sine wave will get more accurate and smoother and if it is below 2*f it will be less accurate and not look like the original sine wave.