# DIGITAL SIGNAL PROCESSING: COSC390

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1. *What is digital signal processing?*
2. *COSC330: Computer Logic and Digital Circuit Design*
3. Read the syllabus for a roadmap
4. *This course can be fast.*
5. **Data science project and presentation**
6. Textbook: `http://dspguide.com`
7. Download and install octave: `https://www.gnu.org/software/octave`

# LECTURE FORMAT, WITH MODIFICATIONS

- Theory and examples
- Programming with Octave
- Application
- Study hall
    1. Homework help
    2. Project and presentation development
    3. Special topics lectures

This lecture will cover:

1. Complex numbers 1: Arithmetic and some calculus (continuous and discete) … see Chapter 30 of text

Next lectures will cover:

1. Complex numbers 2: The Fourier series and Fourier transform (continuous and discrete)
2. *Time-permitting*: The Laplace transform (continuous and discrete)

# COMPLEX NUMBERS 1: THEORY AND EXAMPLES

A complex number is an expression for which one term is proportional to $j = \sqrt{-1}$:

$$z = x + jy \tag{1}$$

To call the *complex unit* $j$ is the convention in electrical engineering, and in physics it is often called $i$.

Example of complex numbers: $(3 + 4j)$, $(x_1 + x_2 j)$. Each number has a *real* part and an *imaginary* part.

Operations to learn:

1. Addition
2. Subtraction
3. Real part Re and Im
4. Multiplication
5. Conjugation
6. Magnitude/Norm
7. Division

Notations to learn:

1. Cartesian
2. Polar
3. Graphical

Addition follows the pattern of two-dimensional vectors:

$$z_1 = 3 + 4j \tag{2}$$

$$z_2 = -2 + 5j \tag{3}$$

$$z_1 + z_2 = 1 + 9j \tag{4}$$

Subtraction follows the pattern of two-dimensional vectors:

$$z_1 = 3 + 4j \tag{5}$$

$$z_2 = -2 + 5j \tag{6}$$

$$z_1 - z_2 = 5 - 1j \tag{7}$$

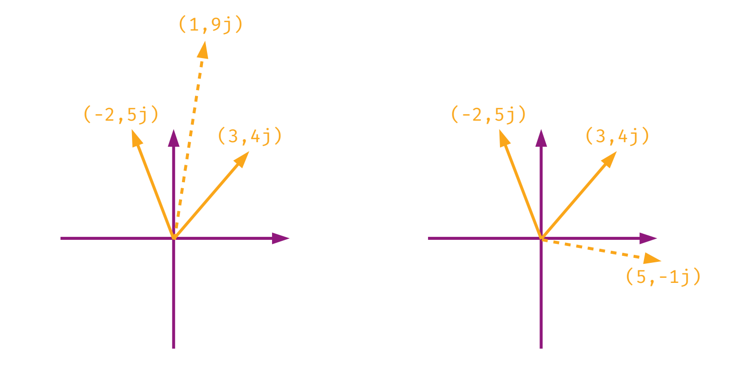


Figure 1: Complex addition and subtraction follows the pattern of two-dimensional vectors. (Left): Addition of $z_1$ and $z_2$. (Right): Subtraction of $z_1$ and $z_2$.

We also have the Re and Im operations:

$$z_1 = 3 + 4j \tag{8}$$

$$\mathrm{Re}\{z_1\} = 3 \tag{9}$$

$$\mathrm{Im}\{z_2\} = 4 \tag{10}$$

These are known as taking the *real*-part and the *imaginary*-part. The original complex number can be recovered by adding real and imaginary parts together:

$$z_1 = \mathrm{Re}\{z_1\} + j\,\mathrm{Im}\{z_1\} \tag{11}$$

When we add/subtract complex numbers, we combine the real parts and imaginary parts separately.

*Add or subtract, then simplify:*

1. $z_1 = 7 + 7j$, $z_2 = -6 + 3j$. $z_1 + z_2 =$
2. $z_1 = 2 + 2j$, $z_2 = 3 - 3j$. $z_1 - z_2 =$
3. $z_1 = 2x + 7j$, $z_2 = 2 + 4xj$. $z_1 + z_2 =$

Let $x = -1$ and $y = 1$. *Evaluate the following expressions:*

1. $z_1 = x + yj$, $z_2 = y + xj$. $z_1 + z_2 =$
2. $z_1 = x^2 + y^2j$, $z_2 = 2y^2 + x^2j$. $z_1 - z_2 =$

*Multiplication: Recall that* $j^2 = -1$.

$$z_1 = x_1 + jy_1 \tag{12}$$

$$z_2 = x_2 + jy_2 \tag{13}$$

$$z_1 \times z_2 = x_1 x_2 - y_1 y_2 + j(x_1 y_2 + x_2 y_1) \tag{14}$$

The cross-terms are straightforward, but remember the minus sign when multiplying the imaginary parts.

Another similarity with two-dimensional vectors?

$$z_1 = 4 - 1j \tag{15}$$

$$z_2 = 1 + 4j \tag{16}$$

$$z_1 \times z_2 = 8 + 15j \neq 0 \tag{17}$$

What would be the result if we were dealing with regular two-dimensional vectors?

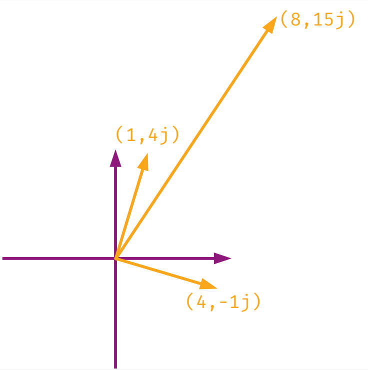


Figure 2: Complex multiplication resembles the *dot-product* for two-dimensional vectors, with key differences.

Complex conjugation: change the sign of the imaginary part.

$$z_1 = 4 - 1j \tag{18}$$

$$z_1^* = 4 + 1j \tag{19}$$

$$z_2 = = 2x + 1j \tag{20}$$

$$z_2^* = 2x - 1j \tag{21}$$

Is there a significance to $z_1 z_2^*$? What about $z_1 z_1^*$? What about $\sqrt{z_1 z_1^*}$?

Let $z = x + jy$. Compute the following:

1. $zz^* =$
2. $\sqrt{zz^*} =$

The second item on this list has a special name: the *magnitude* or *norm* of the complex number, $|z|$.

Compute the norm of the following complex numbers:

1. $2 + 2j$
2. $3 + 4j$

Division of complex numbers: remember that there are multiple steps.

$$z_1 = x_1 + jy_1 \tag{22}$$

$$z_2 = x_2 + jy_2 \tag{23}$$

$$\frac{z_2}{z_1} = \frac{x_2 + jy_2}{x_1 + jy_1} \tag{24}$$

$$\frac{z_2}{z_1} = \frac{z_2 z_1^*}{z_1 z_1^*} = \frac{z_2 z_1^*}{|z_1|^2} \tag{25}$$

$$\frac{z_2}{z_1} = \frac{\mathrm{Re}\{z_2 z_1^*\}}{|z_1|^2} + j\frac{\mathrm{Im}\{z_2 z_1^*\}}{|z_1|^2} \tag{26}$$

$$\frac{z_2}{z_1} = \frac{x_1 x_2 + y_1 y_2}{x_1^2 + y_1^2} + j\frac{x_1 y_2 - x_2 y_1}{x_1^2 + y_1^2} \tag{27}$$

Using Eq. 27, show that if $z_1 = z_2$, that $z_2/z_1 = 1$.

Evaluate the divisions below:

1. $z_1 = 1 + 4j$, $z_2 = 2 - 2j$. $z_2/z_1 =$
2. $z_1 = 1 + 1j$, $z_2 = -3 - 3j$. $z_2/z_1 =$

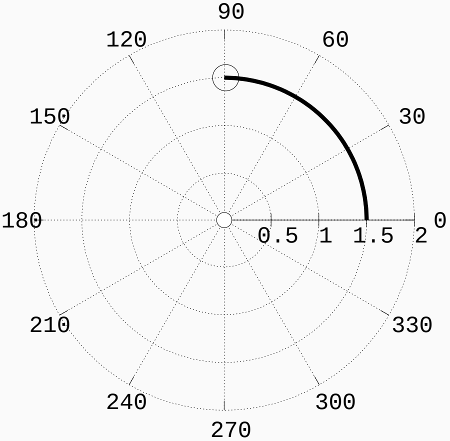


Figure 3: Polar coordinate systems rely on $(\rho, \phi)$ notation, rather than $(x, y)$ notation.

Polar notation for complex numbers: let $z = x + jy$.

$$z = r\exp(j\phi) \tag{28}$$

$$r = |z| = \sqrt{x^2 + y^2} \tag{29}$$

$$\phi = \tan^{-1}(y/x) \tag{30}$$

Useful for multiplication and division:

$$z_1 = r_1 \exp(j\phi_1) \tag{31}$$

$$z_2 = r_2 \exp(j\phi_2) \tag{32}$$

$$z_1 \times z_2 = r_1 r_2 \exp(j(\phi_1 + \phi_2)) \tag{33}$$

$$\frac{r_2}{r_1} = \frac{r_2}{r_1} \exp(j(\phi_2 - \phi_1)) \tag{34}$$

Convert these complex numbers from Cartesian to polar form:

1. $z_1 = 5 + 13j$
2. $z_2 = 7 - 24j$
3. $z_3 = 20 - 21j$

Divide:

1. $z_3/z_2$
2. $z_3/z_1$

Multiply:

1. $z_1 \times z_2$
2. $z_1 \times z_1^*$

Notice that the procedure for finding the modulus is evident in polar notation:

$$\sqrt{z_1 z_1^*} = \sqrt{r_1 r_1} \exp(j(\phi_1 - \phi_1)/2) = r_1 \tag{35}$$

Shouldn't we be saying $\pm r_1$? How does the square root function work in the complex plane?[1]

[1] Complex fractional-power functions are outside the scope of the course, as it turns out.

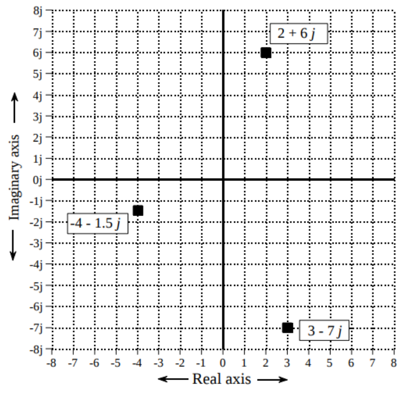


**Figure 4:** The real and imaginary axes are an *extension* of the real number line, allowing a broader representation of physical systems than just real numbers. A prime example is AC circuits.

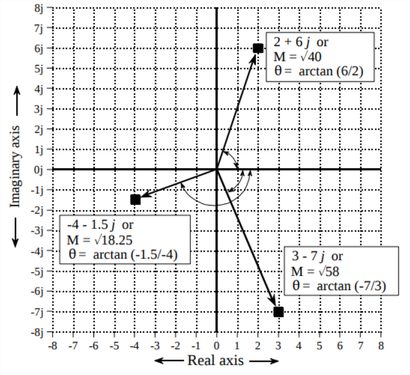


**Figure 5:** Consider the complex plane in light of trigonometry. Think of the magnitude of a complex number as the hypoteneuse of a right triangle. Then, $\mathrm{Re}\{z\} = |z| \cos\phi$, and $\mathrm{Im}\{z\} = |z| \sin\phi$, and $r = |z| = \sqrt{x^2 + y^2}$.

In the trigonometric picture:

$$z = |z|\cos\phi + j|z|\sin\phi \tag{36}$$

Proof of polar-notation relationship:

$$\exp(j\phi) = \sum_{i=0}^{\infty} \frac{(j\phi)^n}{n!} = \sum_{i=0}^{\infty} \frac{j^n\phi^n}{n!} \tag{37}$$

$$\exp(j\phi) = \sum_{even}^{\infty} \frac{j^n\phi^n}{n!} + \sum_{odd}^{\infty} \frac{j^n\phi^n}{n!} \tag{38}$$

$$\exp(j\phi) = \sum_{i=0}^{\infty}(-1)^n \frac{\phi^{2n}}{n!} + j\sum_{i=0}^{\infty} \frac{\phi^{2n+1}}{n!} \tag{39}$$

$$\exp(j\phi) = \cos\phi + j\sin\phi \tag{40}$$

$$|z|\exp(j\phi) = |z|\cos\phi + j|z|\sin\phi = z \tag{41}$$

Thus we have proven:

$$\boxed{z = x + jy = r\exp(i\phi)} \tag{42}$$

Corollary (prove these):

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \tag{43}$$

$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \tag{44}$$

Suppose we have now have a voltage signal

$$v_i(t) = A_i \cos(2\pi f_i t + \phi_i) \tag{45}$$

We may write

$$v_i(t) = A_i \operatorname{Re}\{\exp(j(2\pi f_i t + \phi_i))\} \tag{46}$$

What if we treat the signal as complex, but agree to take the real part at the end of our calculations?

$$v_i(t) \rightarrow A_i \exp(j(2\pi f_i t + \phi_i)) \tag{47}$$

As long as we take the real part of the right hand side, we'll have the original signal.

Now contemplate the addition of signals of the same frequency, but different amplitudes and phases. Let $x_i = 2\pi f t + \phi_i$. A signal comprised of $N$ sinusoids can be written

$$V_i(t) \rightarrow \sum_i^N a_i \exp(jx_i) \tag{48}$$

Remember that $x_i = 2\pi ft + \phi_i$. The sum of two sinusoids in the complex plane can then be written[2]

$$V(t) = a_1 \exp(x_1) + a_2 \exp(x_2) \tag{49}$$

1. Compute $|V|^2 = V^*V$, and $\phi_2 - \phi_1 = \pi$, $\phi_2 - \phi_1 = 0$.
2. What is $\phi_V = \tan^{-1}(\mathrm{Im}\{V\}/\,\mathrm{Re}\{V\})$ in each case?

Why do these results make sense? Thus, the complex numbers encapsulate the concepts of *constructive* and *destructive* interference.

[2] Notice that taking the real part distributes if the original signal is real.

What if we continue to add terms, putting in carefully chosen phases and amplitudes? We can represent *any* periodic signal[3].

This is known as the **Fourier series**:

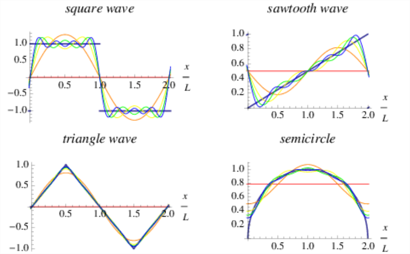


**Figure 6:** The Fourier series representing four different periodic functions.

[3] We will return to this in Unit 1.2.

# COMPLEX NUMBERS 1: PROGRAMMING WITH OCTAVE

Let's take the time to get octave installed on your systems:
`https://www.gnu.org/software/octave`.

If we cannot get it installed on your systems, we can always run it on the local desktops.

A good tutorial can be found here:
`https://en.wikibooks.org/wiki/Octave_Programming_Tutorial`

Octave is a high-level *scripting* programming language. Although it is possible to write packages and compile code in octave, the most common application is executing a script that performs some analysis on digital data.

```
a = 1+1i;
b = conj(a);
a * b
```

The result of this code should be 2.0. Why? We are defining a complex number in the first line, computing the complex conjugate, and multiplying them.

Octave naturally handles vectors of numbers and matrices. Let's define a vector of complex numbers.

```
a = [1 2 3 5 7 11];
size(a)
ans = 1   6
a = a';
size(a)
```

The code in the fourth line *transposes* the vector. This means trading the rows for the columns of the vector. What begins as a $1 \times 6$ vector (one row by six columns) ends as a $6 \times 1$ vector (six rows by one column).

Operations are as expected, but we need special notation for vectoral calculations:

```
a = 2.0;
b = 4.0;
b/a
ans = 2
b = [4.0 4.0 4.0]
b/a
ans =
2   2   2
a./b
ans =
0.5   0.5   0.5
```

Placing a dot (.) before a standard operation indicates that the operation is to be carried out in a vectoral-sense.

```
t1 = [1 2 3 4 5 6 7 8 9 10];
t2 = t1+1;
t1.*t2
ans =
2   6   12   20   30   42   56   72   90   110
```

The colon operator (:) represents iteration in octave. Consider three cases:

```
fs = 1000.0;
t = [0.0:1/fs:10.0];
plot(t,sin(2.0 * pi * 3.0 .* t));
```

Octave should produce a plot of a sine wave with a frequency of 3.0 Hz (if the time is in seconds). How can you tell?

Count the number of complete oscillations in 2.0 seconds. Do you see 6.0 oscillations? What is the significance of $f_s$ in the code?

```
fs = 1000.0;
t = [0.0:1/fs:10.0];
plot(t,sin(2.0 * pi * 3.0 .* t));
```

The axis command is useful for controlling the plotted region:

```
axis([0 2 -2 2]);
```

The colon operator (:) also refer to elements in a vector.

```
t(1)
ans =
0
t(2:5)
ans =
0.001   0.002   0.003   0.004
t(:)
ans =
(should be the whole vector)
t(1:end)
ans =
(should be the whole vector)
```

# CONCLUSION

Text