Homework 3 2/28/2025 Dylan.Z

Linear systems

1.

$$A\{x[n]\} = 2x[n] - 1$$
$$B\{x[n]\} = 0.5[n]$$

Check linearity of system A

Let $x'[n] = ax[n]$

$$A\{x'[n]\} = 2(ax[n]) - 1 = 2ax[n] - 1$$

For A to be linear, this must equal $aA\{x[n]\}$

$$aA\{x[n]\} = a(2x[n] - 1) = 2ax[n] - a$$

Since $2ax[n] - 1 \neq 2ax[n] - a$ (except when $a = 1$) the system fails homogenity.

Additivity test

let $y[n] = x_1[n] + x_2[n]$ then,

$$A\{x_1[n] + x_2[n]\} = 2(x_1[n] + x_2[n]) - 1 = 2x_1[n] + 2x_2[n] - 1$$

$$A\{x_1[n]\} + A\{x_2[n]\} = (2x_1[n] - 1) + (2x_2[n] - 1) = 2x_1[n] + 2x_2[n] - 2$$

Since they are not equal, A fails additivity as well

A is non-linear and breaks both homogeneity and additivity.

Check linearity of system B

Homogeneity Test

For $x'[n] = ax[n]$:

$$B\{x'[n]\} = 0.5(ax[n]) = a(0.5x[n]) = aB\{x[n]\}$$

Additivity Test

For $y[n] = x_1[n] + x_2[n]$:

$$B\{x_1[n] + x_2[n]\} = 0.5(x_1[n] + x_2[n]) = 0.5x_1[n] + 0.5x_2[n]$$

$$B\{x_1[n]\} + B\{x_2[n]\} = 0.5x_1[n] + 0.5x_2[n]$$

(b) $A_{mod}\{x[n]\} = 2x[n]$

Check if A_{mod} and B commute

$$A_{mod}\{B\{x[n]\}\} = A_{mod}\{0.5x[n]\} = 2(0.5x[n]) = x[n]$$

$$B\{A_{mod}\{x[n]\}\} = B\{2x[n]\} = 0.5(2x[n]) = x[n]$$

$$A_{mod}\{B\{x[n]\}\} = B\{A_{mod}\{x[n]\}\}$$

2.

$f(t) = a_1 \cos(2\pi f_1 t) + a_2 \cos(2\pi f_2 t)$ $T_1 = 1/f_1$,

$T_2 = 1/f_2$ $f_2 = 2f_1$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t - T_1)\,dt$$

The Dirac Delta function $\delta(t - t_0)$ has the sifting property

$$\int_{-\infty}^{\infty} g(t)\,\delta(t - t_0)\,dt = g(t_0)$$

for any function $g(t)$

Subsititue $f(t)$

$$\int_{-\infty}^{\infty} \left(a_1 \cos(2\pi f_1 t) + a_2 \cos(2\pi f_2 t)\right)\delta(t - T_1)\,dt$$

Apply sifting property

$$a_1 \cos(2\pi f_1 T_1) + a_2 \cos(2\pi f_2 T_1)$$

Since $T_1 = 1/f_1$

$$\cos(2\pi f_1 T_1) = \cos(2\pi \cdot f_1 \cdot 1/f_1) = \cos(2\pi) = 1$$

$$\cos(2\pi f_2 T_1) = \cos(2\pi \cdot 2f_2 \cdot 1/f_2) = \cos(4\pi) = 1$$

Thus

$$a_1(1) + a_2(1) = a_1 + a_2$$

- $$\int_{-\infty}^{\infty} f(t)\,\delta(t - T_2)\,dt$$

Similar for second integral

$$\int_{-\infty}^{\infty} (a_1\cos(2\pi f_1 t) + a_2\cos(2\pi f_2 t))\,\delta(t - T_2)\,dt$$

$$T_2 = 1/f_2$$

$$a_1\cos(2\pi f_1 T_2) + a_2\cos(2\pi f_2 T_2)$$

$$T_2 = 1/f_2 = 1/(2f_1)$$

$$\cos(2\pi f_1 T_2) = \cos(2\pi f_1 \cdot 1/(2f_1)) = \cos(\pi) = -1$$

$$\cos(2\pi f_2 T_2) = \cos(2\pi f_2 \cdot 1/(2f_1)) = \cos(2\pi) = 1$$

$$a_1(-1) + a_2(1) = -a_1 + a_2$$

3.

a)

$$F(f) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ft} dt$$

$$f(t) = a\delta(t - t_0)$$

$$F(f) = \int_{-\infty}^{\infty} a\delta(t - t_0) e^{-j2\pi ft} dt$$

Sifting Property

$$\int_{-\infty}^{\infty} g(t)\delta(t - t_0) dt = g(t_0)$$

$$F(f) = ae^{-j2\pi ft_0}$$

$$|F(f)| = |ae^{-j2\pi ft_0}|$$

$$e^{-j\theta} = 1$$

$$F(f) = |a| \cdot 1 = |a|$$

b)

$$F(f) = a e^{-j2\pi f t_0}$$

$$\phi(f) = \arg(F(f))$$

$$\phi(f) = -2\pi f t_0 + \arg(a)$$

If a is real and positive then $\arg(a) = 0$

$$\phi(f) = -2\pi f t_0$$

c)

$$T_g = \frac{d\phi}{d\omega} \qquad \omega = 2\pi f$$

$$\phi(\omega) = -\omega t_0$$

$$\frac{d\phi}{d\omega} = -t_0$$

$$T_g = -(-t_0) = t_0$$

$$T_g = t_0$$

4. $y[n] = S[x[n]] = 0.5x[n-2]$ What is $S[\delta[n]]$?

$\delta[n]$ is 1 at $n=0$

Then it shifts by 2 to the right

Then scales the impulse by 0.5

So

$$\delta[n] = [1, 0, 0, 0, 0, 0, 0, 0]$$

Then

$$S[\delta[n]] = [0, 0, 0.5, 0, 0, 0, 0, 0]$$

5) If the step input is $s[n] = [0, 1, 1, 1, 1, 1, 1, 1]$

Then $S[s[n]] = [0, 0, 0, 0.5, 0.5, 0.5, 0.5, 0.5]$

Fourier Transforms and Basic Filters

$$F(f) = \mathcal{F}\{f(t)\} = \int^{\infty} f(t)\, e^{-2\pi j f t}\, dt$$

$-\infty$

$$s(t) = a\delta(t-t_0)$$

$$S(f) = \int_{-\infty}^{\infty} a\delta(t-t_0)e^{-2\pi jft}\,dt$$

$$= \int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$$

$$f(t) = e^{-2\pi jft}$$

$$S(f) = ae^{-2\pi jft_0}$$

b) $Y(f) = \dfrac{S(f)}{1+j\omega T}$

$\omega = 2\pi f$ $\quad T = RC$ $\qquad S(f) = ae^{-2\pi jft_0}$

$$Y(f) = \frac{ae^{-2\pi jft_0}}{1+j2\pi fT}$$

$$|Y(f)| = \left|\frac{ae^{-2\pi jft_0}}{1+j2\pi fT}\right| = \frac{|a|}{|1+j2\pi fT|}$$

$e^{-2\pi j f t_0} = 1$

$$|Y(f)| = \frac{a}{\sqrt{1+(2\pi f \tau)^2}}$$

b)

$$Y(f) = \frac{j\omega\tau}{1+j\omega\tau} S(f)$$

$S(f) = a e^{-2\pi j f t_0}$ $\qquad \omega = 2\pi f$

$$Y(f) = \frac{j 2\pi f \tau}{1 + j 2\pi f \tau} a e^{-2\pi j f t_0}$$

$$|Y(f)| = \left|\frac{j 2\pi f \tau}{1 + j 2\pi f \tau}\right| \cdot \left|a e^{-2\pi j f t_0}\right|$$

$$|Y(f)| = \frac{2\pi f \tau}{\sqrt{1+(2\pi f \tau)^2}} \cdot a$$

$$|Y(f)| = \frac{a \cdot 2\pi f \tau}{\sqrt{1+(2\pi f \tau)^2}}$$

2. Group delay is defined

$$T_g = -\frac{d\phi}{d\omega}$$

Frequency response for low-pass filter is

a) $$H_{LP}(f) = \frac{1}{1+j\omega T} \qquad \omega = 2\pi f \qquad T = RC$$

$$|H_{LP}| = \frac{1}{\sqrt{1+(\omega T)^2}}$$

$$\phi_{LP} = -\tan^{-1}(\omega T)$$

group delay is the negative derivative of the phase with respect to ω:

$$T_g = -\frac{d\phi_{LP}}{d\omega}$$

$$\frac{d\phi_{LP}}{d\omega} = -\frac{d}{d\omega}\left(\tan^{-1}(\omega T)\right) = -\frac{T}{1+(\omega T)^2}$$

$$T_g = -\left(-\frac{T}{1+(\omega T)^2}\right) = \frac{T}{1+(\omega T)^2}$$

Group delay for high-pass filter

$$H_{HP}(f) = \frac{j\omega T}{1 + j\omega T} \qquad \omega = 2\pi f \quad T = RL$$

$$|H_{HP}(f)| = \frac{\omega T}{\sqrt{1 + (\omega T)^2}}$$

$$\phi_{HP} = \frac{\pi}{2} - \tan^{-1}(\omega T)$$

group delay is the negative derivative

$$T_g = -\frac{d\phi_{HP}}{d\omega}$$

Taking the derivative of ϕ_{HP}:

$$\frac{d\phi_{HP}}{d\omega} = \frac{d}{d\omega}\left(\frac{\pi}{2} - \tan^{-1}(\omega T)\right) = -\frac{T}{1 + (\omega T)^2}$$

High-pass filter is.

$$T_g = -\left(-\frac{T}{1 + (\omega T)^2}\right) = \frac{T}{1 + (\omega T)^2}$$

3.

a) need to show the inverse Fourier transform of

$S(f) = \frac{a}{2}(\delta(f-f_0) + \delta(f+f_0))$ is a cosine function

Inverse Fourier

$$f(t) = F^{-1}\{F(f)\} = \int_{-\infty}^{\infty} F(f) e^{2\pi j f t} \, df$$

$$S(f) = \frac{a}{2}(\delta(f-f_0) + \delta(f+f_0))$$

Subsitute $S(f)$ into the inverse Fourier transform:

$$f(t) = \int_{-\infty}^{\infty} \frac{a}{2}(\delta(f-f_0) + \delta(f+f_0)) e^{2\pi j f t} \, df$$

Using sifting property of the Dirac delta funtion

$$\int_{-\infty}^{\infty} \delta(f-f_0) g(f) \, df = g(f_0)$$

$$f(t) = \frac{a}{2}\left(\int_{-\infty}^{\infty} \delta(f-f_0) e^{2\pi j f t} \, df + \int_{-\infty}^{\infty} \delta(f+f_0) e^{2\pi j f t} \, df \right)$$

Use sifting property

$$f(t) = \frac{a}{2}\left(e^{2\pi j f_0 t} + e^{-2\pi j f_0 t}\right)$$

$$e^{j\theta} + e^{-j\theta} = 2\cos(\theta)$$

$$f(t) = \frac{a}{2} \cdot 2\cos(2\pi f_0 t) = a\cos(2\pi f_0 t)$$

Thus Fourier transform of

$S(f) = \frac{a}{2}(\delta(f-f_0) + \delta(f+f_0))$ is

$$f(t) = a\cos(2\pi f_0 t)$$

b) Show that the inverse Fourier transform of

$S(f) = \frac{a}{2j}(\delta(f-f_0) - \delta(f+f_0))$ is a sine function

given

$$S(f) = \frac{a}{2j}(\delta(f-f_0) - \delta(f+f_0))$$

Sub $S(f)$ into the inverse fourier transform:

$$f(t) = \int_{-\infty}^{\infty} \frac{a}{2j}(\delta(f-f_0) - \delta(f+f_0))$$

$S(f)$ into Inverse fourier transform

$$f(t) = \int_{-\infty}^{\infty} \frac{a}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right) e^{2\pi j f t}\, df$$

Use sifting property

$$f(t) = \frac{a}{2j}\left(\int_{-\infty}^{\infty} \delta(f-f_0) e^{2\pi j f t}\, df - \int_{-\infty}^{\infty} \delta(f+f_0) e^{2\pi j f t}\, df\right)$$

$$f(t) = \frac{a}{2j}\left(e^{2\pi j f_0 t} - e^{-2\pi j f_0 t}\right)$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin(\theta)$$

$$f(t) = \frac{a}{2j} \cdot 2j\sin(2\pi f_0 t) = a\sin(2\pi f_0 t)$$

So the inverse Fourier transform of

$S(f) = \frac{a}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right)$ is

$$f(t) = a\sin(2\pi f_0 t)$$

3. Fourier Transforms and

Basic filters

a)

$$S(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi f t}\, dt$$

$$S(f) = \int_{-\infty}^{\infty} a\,\delta(t - t_0)\, e^{-j2\pi f t}\, dt$$

$$S(f) = a e^{-j2\pi f t_0}$$

b)

$$S_{out}(f) = \frac{S(f)}{1 + j\omega T}$$

$$S_{out}(f) = \frac{a e^{-j2\pi f t_0}}{1 + j2\pi f T}$$

$$|S_{out}(f)| = \frac{|a|}{\sqrt{1 + (2\pi f T)^2}}$$

c)

$$S_{out}(f) = \frac{j\omega T}{1 + j\omega T} S(f)$$

$$S_{out}(f) = \frac{j2\pi f\tau}{1+j2\pi f\tau} a e^{-j2\pi f t_0}$$

$$|S_{out}(f)| = \frac{|a|(2\pi f\tau)}{\sqrt{1+(2\pi f\tau)^2}}$$

2.

a) $$S_{out}(f) = \frac{a e^{-j2\pi f t_0}}{1+j2\pi f\tau}$$

$$\theta_{LP}(f) = -2\pi f t_0 + \tan^{-1}(-2\pi f\tau)$$

$$T_g^{LP}(f) = -\frac{d}{df}\left[-2\pi f_0 + \tan^{-1}(-2\pi f_\tau)\right]$$

$$T_g^{LP}(f) = 2\pi\tau \frac{1}{1+(2\pi f\tau)^2} + f_0$$

b)

$$S_{out}(f) = \frac{j2\pi f\tau}{1+j2\pi f\tau} a e^{-j2\pi f t_0}$$

$$\theta_{HP}(f) = -2\pi f t_0 + \tan^{-1}(2\pi f T)$$

$$T_g^{HP}(f) = \frac{-d}{df}\left[-2\pi f t_0 + \tan^{-1}(2\pi f T)\right]$$

$$T_g^{HP}(f) = -2\pi T \frac{1}{1+(2\pi f T)^2} + t_0$$

3.

a)

$$S(f) = \frac{a}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$$

$$s(t) = \int_{-\infty}^{\infty} \frac{a}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right] e^{j2\pi f t}\, df$$

$$s(t) = \frac{a}{2} e^{j2\pi f_0 t} + \frac{a}{2} e^{-j2\pi f_0 t}$$

$$e^{j\theta} + e^{-j\theta} = 2\cos\theta$$

$$s(t) = a\cos(2\pi f_0 t)$$

b) $$S(f) = \frac{a}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right]$$

$$S(t) = \int_{-\infty}^{\infty} \frac{a}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right] e^{j2\pi f t}\, df$$

$$S(t) = \frac{a}{2j} e^{j2\pi f_0 t} - \frac{a}{2j} e^{-j2\pi f_0 t}$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$

$$S(t) = a\sin(2\pi f_0 t)$$

4 Convolution and octave code

a)

$$y[n] = h[n] * x[n] = \sum_{j=-\infty}^{\infty} h[j]\, x[n-j]$$

$$y[n] = \sum_{j=-\infty}^{\infty} h[j]\, \delta[n-j]$$

$$y[n] = h[n]$$

b)

$$y[n] = \sum_{j=-\infty}^{\infty} h[j]\, \delta[n-j-n_0]$$

$$y[n] = h[n-n_0]$$