Electromagnetc Theory: PHYS330

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Summary

Week 3 Summary

1. Laplace's Equation
   - One-dimension
   - Two-dimensions, three dimensions, uniqueness, boundaries
2. Separation of Variables: Boundary-value problems
   - Cartesian coordinates
   - Spherical coordinates
3. Multipole Expansions
   - Far-fields
   - Monopole and dipole terms
   - Electric Field of a Dipole

Laplace’s Equation: One Dimension

Laplace's Equation: One dimension

Laplace's Equation in one dimension:

$$\frac{d^2V}{dx^2} = 0 \tag{1}$$

What is the solution?

$$V(x) = mx + b \tag{2}$$

What is the magnitude of the E-field?

- A: $V(x)$
- B: x
- C: b
- D: m

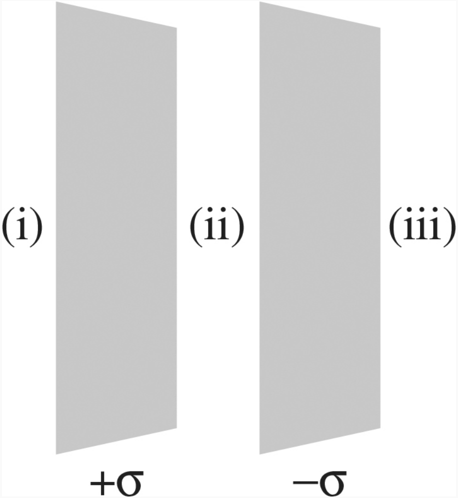

Figure 1: The setup of a parallel plate capacitor.

Laplace's Equation: One dimension

Suppose the negative side of the parallel plate capacitor is grounded, and the positive side is at a potential V_0. Let the separation between the plates be x_0. Further, let the positive plate occupy the yz plane, passing through the origin. Find the E-field magnitude and direction by solving Laplace's equation.

Laplace's Equation: One dimension

Show that the potential of a point charge at the origin satisfies Laplace's Equation for $r \neq 0$. *Use the form of the Laplacian in spherical coordinates.*

Boundary Conditions

Boundary Conditions

Let $V(x) = mx + b$. If $V(-a) = V_0$, and $V(a) = -V_0$, what are valid expressions for m and b?

- A: $b = 0$, and $m = -2V_0$
- B: $b = a$, and $m = V_0/a$
- C: $b = 0$, and $m = -V_0/a$
- D: $b = V_0$, and $m = -V_0/a$

Boundary Conditions

Let $V(x) = mx + b$. If $V(-a) = V_0$, and $V(a) = -V_0$, what is the electric field?

- A: $\frac{V_0}{a}\hat{x}$
- B: $-\frac{V_0}{a}\hat{x}$
- C: $V_0\hat{x}$
- D: $-V_0\hat{x}$

Boundary Conditions

Suppose a potential function $V(x, y) \propto (A \exp(-kx) + B \exp(kx))$. Which of the following is true, if $V \to 0$ as $x \to \infty$?

- A: A is 0
- B: B is 0
- C: A and B are 0
- D: Neither A nor B is 0

Boundary Conditions

Suppose a potential function $V(x, y) \propto (A \sin(kx) + B \cos(kx))$. Which of the following is true, if $V = 0$ as $x = 0$, and $V = 0$ as $x = a$?

- A: B is 0, and $k = n\pi$
- B: A is 0, and $k = n\pi/(2a)$
- C: A and B are 0
- D: B is 0, and $k = n\pi/a$

Boundary Conditions

Hyperbolic trigonometric functions:

- $\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$
- $\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$
- $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

Which of the following is zero?

- A: $\sinh(0)$
- B: $\cosh(0)$
- C: $\tanh(0)$
- D: None

Which of the following is one?

- A: $\sinh(0)$
- B: $\cosh(0)$
- C: $\tanh(0)$
- D: None

Hyperbolic trigonometric functions are solutions to which equation?

- A: $\frac{df}{dx} = k$
- B: $\frac{d^2f}{dx^2} = kx$
- C: $\frac{d^2f}{dx^2} = k^2 f$
- D: $\frac{d^2f}{dx^2} = 0$

Boundary Conditions

Fourier's Trick: Imagine a vector with n components:

$$\vec{v} = \sum_{i=1}^{n} c_i \hat{x}_i \tag{3}$$

In words, how do you solve for some c_m?

- A: Divide by $\hat{x}_i$
- B: Take the dot product of both sides with $\hat{x}_m$
- C: Take the dot product $\vec{v}$ and $\vec{u}$, and the sum the series
- D: Integrate both sides with respect to x

Fourier's Trick: Imagine a vector with n components:

$$\vec{v} = \sum_{i=1}^{n} c_i \hat{x}_i \tag{4}$$

In words, how do you solve for some c_m? Note that:

$$\vec{v} \cdot \hat{x}_m = \sum_{i=1}^{n} c_n \hat{x}_i \cdot \hat{x}_m = c_m \tag{5}$$

Why? Because

$$\hat{x}_i \cdot \hat{x}_j = 0 \tag{6}$$

$$\hat{x}_i \cdot \hat{x}_i = 1 \tag{7}$$

Boundary Conditions

Fourier's Trick: Imagine a known function that happens to be equal to a sum:

$$f(x) = \sum_{i=1}^{\infty} c_i g_i(x) \tag{8}$$

In words, how do you solve for some c_m?

- A: Multiply both sides by $g_m(x)$
- B: Multiply both sides by $g_m(x)$ and integrate both sides with respect to x
- C: Sum the infinite series and solve for c_m with algebra
- D: Integrate both sides with respect to x

Boundary Conditions

If it's true that a function can be written as an infinite series of functions with coefficients:

$$f(x) = \sum_{i=1}^{\infty} c_i g_i(x) \tag{9}$$

Then the functions $g_n(x)$ are said to be **complete**, or a complete basis (just like vectors are a sum of basis vectors. Examples of complete sets of functions:

- sines and cosines (Fourier series)
- exponentials with the right rates multiplying x
- Hyperbolic trigonometric functions (follows from exponentials)
- Taylor series (polynomials with derivatives as coefficients)

Boundary Conditions

The functions $f_n(x)$ are said to be **orthogonal** for $x \in [a, b]$ if

$$\int_a^b f_n(y) f_m(y) dy = \delta_{n,m} \tag{10}$$

One example:

$$I_{n,m} = \int_{-L}^{L} \frac{\sin(n\pi x/L)}{\sqrt{L}} \frac{\sin(m\pi x/L)}{\sqrt{L}} dx \tag{11}$$

What is the result of this integral? How would you approach solving this?

Boundary Conditions

The Fourier series representation of a function $f(x)$ is written:

$$S(x) = \frac{A_0}{2} + \sum_{i=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx)) \tag{12}$$

with

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx \tag{13}$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx \tag{14}$$

Boundary Conditions

Let's obtain the Fourier series coefficients A_n and B_n for a square-wave signal:

$$f(x) = 1, \quad 0 \le x \le \pi, \quad 0, \pi < x \le 2\pi \tag{15}$$

(Observe on board). The result: $A_0 = 1.0$, all other $A_n = 0$, odd B_n values follow $2/(n\pi)$, even $B_n = 0$ as well.

Separation of Variables

Separation of Variables

Laplaces' Equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{16}$$

Assume the solution follows

$$V(x, y, z) = X(x)Y(y)Z(z) \tag{17}$$

The Laplace equation then breaks into three separate ordinary differential equations. Application of boundary conditions to solve them (Asynchronous video content on Moodle).

Separation of Variables

Laplaces' Equation in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\left(\frac{\partial^2 V}{\partial\phi^2}\right)=0 \tag{18}$$

Assuming *azimuthal symmetry* means $V(r,\theta,\phi)=V(r,\theta)$ and $\partial V/\partial\phi=0$. Thus, Eq. 18 reduces and admits general solutions:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0 \tag{19}$$

Let the general solutions be separable:

$$V(r,\theta)=R(r)\Theta(\theta) \tag{20}$$

Separation of Variables

The radial equation is

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1) \tag{21}$$

Exercise: show that the solution is

$$R(r) = Ar^l + Br^{-(l+1)} \tag{22}$$

(The derivative operator distributes over addition, so the two solutions can be checked separately, or together).

What are the units of $R(r)$? What are the units of A and B?

Separation of Variables

The polar equation is

$$\frac{1}{\Theta(\theta)} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) = -l(l+1) \sin\theta \tag{23}$$

The solutions are complete, and orthogonal, and known as Legendre polynomials:

$$\Theta(\theta) = P_l(\cos\theta) \tag{24}$$

Defined by the *Rodrigues formula*:

$$P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2 - 1)^l \tag{25}$$

Exercise: show that

$$P_3(x) = (5x^3 - 3x)/2 \tag{26}$$

What is the result of the following integrals?

$$I_1 = \int_{-1}^{1} P_1(x)P_2(x)dx \tag{27}$$

$$I_2 = \int_{-1}^{1} P_2(x)P_2(x)dx \tag{28}$$

The general solution is a sum of individual solutions:

$$V(r,\theta) = \sum_{i=0}^{\infty} \left(Ar^l + B/r^{l+1} \right) P_l(\cos\theta) \tag{29}$$

The coefficients may be found via Fourier's Trick.

Examples 3.8 and 3.9: Professor on Board

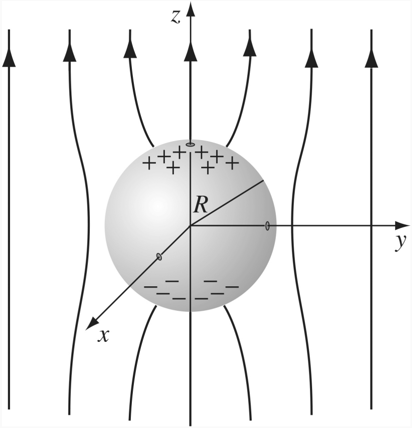

Figure 2: A metal spherical shell in an external field $\mathbf{E} = E_0\hat{z}$.

Examples 3.8 and 3.9: Professor on Board

A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere.

1. Inside the sphere, $B = 0$ to avoid a singularity at the origin (center of sphere).
2. Outside the sphere, $A = 0$ to ensure $V \to 0$ as $r \to \infty$.
3. General boundary conditions at $r = R$: potential is continuous ($-\int \vec{E} \cdot d\vec{l} = 0$).
4. Coefficients of same order l have a relationship.
5. E-field has a discontinuity at the boundary.
6. Fourier's trick to get the coefficients, after specifying $\sigma_0(\theta)$.

Multipole Expansion

Multipole Expansion

Imagine a *physical dipole* with q at $\hat{r}' = d/2\hat{z}$ and $-q$ at $\hat{r}' = -d/2\hat{z}$. Show that (professor example)

$$V(r,\theta) = \frac{kqd}{r^2} P_1(\cos\theta) \tag{30}$$

1. Far-field on script-r's
2. Subtract
3. Simplify
4. Note that $P_1(x) = x$.

Multipole Expansion

Can't you break *any* charge distribution into a collection of monopoles, dipoles, quadrupoles, … ? We can show in general that:

$$\boxed{\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n} P_n(\cos\theta)} \tag{31}$$

Can’t you break *any* charge distribution into a collection of monopoles, dipoles, quadrupoles, … ?

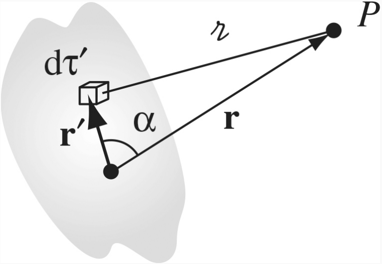

Figure 3: The general scheme for the multipole expansion.

Multipole Expansion

Find the Law of Cosines from the definition of the separation vector:

Then we let

$$\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) \tag{32}$$

Multipole Expansion

Find the Taylor series of $f(\epsilon) = (1+\epsilon)^{-1/2}$:

Remember that

$$\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) \tag{33}$$

Multipole Expansion

1. After computing the Taylor series, substitute $\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$
2. Collecting like powers of $\left(\frac{r'}{r}\right)$ together will lead to

$$\boxed{\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n} P_n(\cos\theta)} \tag{34}$$

Multipole Expansion

Recall that the potential for *any* charge distribution is

$$V(\vec{r}) = \int \frac{k\rho(\vec{r'})}{\mathscr{r}} d\tau' \tag{35}$$

Substitute the expansion for $1/\mathscr{r}$, and reverse the order of summation and integration:

$$V(\vec{r}) = k \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\alpha) \rho(\vec{r'}) d\tau' \tag{36}$$

The Monopole and Dipole Terms

The Monopole and Dipole Terms

Using the $\rho(\vec{r'})$ of a dipole oriented along the z-axis, reproduce Eq. 30 using the multipole expansion.

Hint: obtain the first few terms, but which ones vanish and why?

The Monopole and Dipole Terms

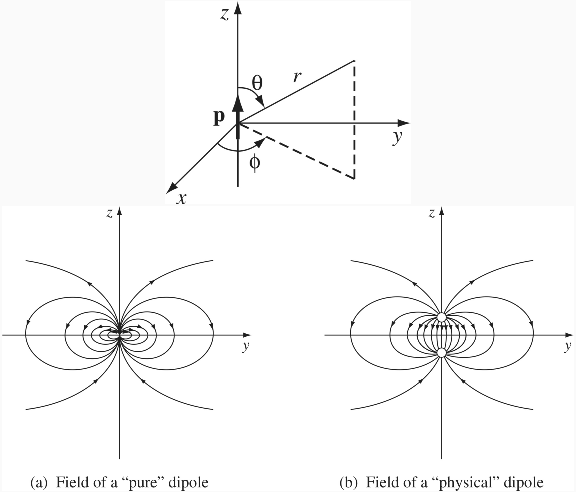

Figure 4: Fields of a $\hat{z}$-oriented (left) pure dipole and (right) physical dipole.

Conclusion

Week 3 Summary

1. Laplace's Equation
    - One-dimension
    - Two-dimensions, three dimensions, uniqueness, boundaries
2. Separation of Variables: Boundary-value problems
    - Cartesian coordinates
    - Spherical coordinates
3. Multipole Expansions
    - Far-fields
    - Monopole and dipole terms
    - Electric Field of a Dipole