EMT Homework 4

Matthew Buchanan Garza

04/15/22

1 Problem 4.10

A sphere of radius R carries a polarization

$$\mathbf{P(r)} = \boldsymbol{k}\mathbf{r} \tag{1}$$

where k is a constant and $\mathbf{r}$ is the vector from the center

a) Calculate the bound charges σ_b and ρ_b.

$$\hat{n} = \hat{r} \tag{2}$$

$$\sigma_b = P \cdot \hat{n} \tag{3}$$

$$\sigma_b = kR\hat{r} \cdot \hat{r} == kR \tag{4}$$

Surface bound charge = kR;

$$P(r) = r^2 kr \tag{5}$$

We can then use the equation 4.12 to define the volume bound charge as:

$$\rho_b = -\nabla \cdot P \tag{6}$$

$$\rho_b = -\{\frac{1}{r^2}\frac{d}{dr}P\} \tag{7}$$

Sub r^2kr into P;

$$\rho_b = -\{\frac{1}{r^2}\frac{d}{dr}(r^2kr)\} \tag{8}$$

Then simplify;

$$\rho_b = -\{\frac{k}{r}(3r^2)\} \tag{9}$$

Volume bound charge = -3k;

b) Find the field inside and outside the plane.

For r < R;

$$E = \frac{1}{3\epsilon_0}\rho r\hat{r} \tag{10}$$

Using Gauss's Law and having $4\pi r^2$ as the surface area of the sphere of radius r

$$E(4\pi r^2) = \frac{q_{enclosed}}{\epsilon_0} \tag{11}$$

Sub in (3k)($\frac{4}{3}\pi r^3$) for $q_{enclosed}$ and Solve for E

So the E-field in the sphere is:

$$E = \frac{-kr}{\epsilon_0}\hat{r} \tag{12}$$

To figure out the total, we add the volume and the surface;

$$q_{total} = q_{volume} + q_{surface} \tag{13}$$

Using the answers we got before will result in:

$$q_{total} = 0 \tag{14}$$

Since $q_{total} = 0$, then;

$$E = 0 \tag{15}$$

2 Problem 4.14

When you polarize a neutral dielectric, the charge moves a bit, but the total remains zero. This fact should be reflected in the bound charges σ_b and ρ_b. Prove from Eqs. 4.11 and 4.12 that the total bound charge vanishes.

Stated in Prob. 4.10:

$\sigma_b = P \cdot \hat{n}$

$\rho_b = -\nabla \cdot P$

$$Q_b = \oint_{surface} \sigma_b da + \oint_{volume} \rho_b d\tau \tag{16}$$

Sub in $\sigma_b = P \cdot \hat{n}$ and $\rho_b = -\nabla \cdot P$;

$$\oint_V \rho_b d\tau = -\oint_V (-\nabla \cdot P) da = -\oint_S P \cdot da \tag{17}$$

Eventually will result in;

$$Q_b = \oint_S P \cdot da - \oint_S P \cdot da = 0 \tag{18}$$

3 Problem 4.15

A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization

$$P(r) = \frac{k}{r}\hat{r} \tag{19}$$

where k is a constant and r is the distance from the center (Fig. 4.18). (There is no free charge in the problem.) Find the electric field in all three regions by two different methods.

Method 1:)

$$\rho_b = -\nabla \cdot P = -\{\frac{1}{r^2}\frac{d}{dr}(r^2 kr)\} \tag{20}$$

$$\sigma_b = P \cdot \hat{n} = \{+P \cdot \hat{r} = k/b(at\, r = b), -P \cdot \hat{r} = -k/a(at\, r = a)\} \tag{21}$$

Using Gauss's Law: $E = \frac{1}{4\pi\epsilon_0}\ \frac{Q_{enc}}{r^2}$

From Problem 4.14:

For r < a,

$$Q_{enc} = 0,\ so\, E = 0 \tag{22}$$

For r > b,

$$Q_{enc} = 0,\ so\, E = 0 \tag{23}$$

By this then, for a < r < b then the result would be,

$$Q_{enc} = (\frac{k}{a})(4\pi a^2) + \int_a^r (\frac{-k}{\bar{r}^2}) 4\pi \bar{r}^2 d\bar{r},\ Simplified\ = -4\pi kr \tag{24}$$

Due to this then the E-field:

$$E = -\frac{k}{\epsilon_0 r}\hat{r} \tag{25}$$

Method 2:)

$$\oint D \cdot da = Q_{f_{enc}} = 0 \tag{26}$$

By this then:

$$D = 0\, everywhere \tag{27}$$

For r < a and r > b:

$$D = \epsilon_0 E + P = 0\ So, E = \frac{-1}{\epsilon_0}P \tag{28}$$

Therefore: E = 0

For a < r < b:

$$E = -\frac{k}{\epsilon_0 r}\hat{r} \tag{29}$$

4 Problem 4.18

The space between the plates of parallel-plate capacitor (Fig. 4.24) is filled with two slabs of linear dielectric material. Each slab has thickness a, so the total distance between the plates is 2a. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free density on the top plate is σ and on the bottom plate $-\sigma$

a) Find the electric displacement **D** in each slab.

$$Apply \int D \cdot da = Q_{f_{enc}}\, to\, the\, Gaussian\, Surface \tag{30}$$

$$DA = \sigma A => D = \sigma \tag{31}$$

D = 0 in both metal places, therefore;

D points down

b) Find the electric field **E** in each slab.

Slab 1)

$$D = \epsilon E => E = \frac{\sigma}{\epsilon_1} \tag{32}$$

Slab 2)

$$E = \frac{\sigma}{\epsilon_2} \tag{33}$$

Since $\epsilon = \epsilon_0 \epsilon_r$, so therefore:

$$E_1 = \frac{\sigma}{2\epsilon_0} \tag{34}$$

and

$$E_2 = \frac{2\sigma}{2\epsilon_0} \tag{35}$$

c) Find the polarization **P** in each slab.

Since we got E from b)

$$P = \frac{\epsilon_0 \chi_e d}{\epsilon_0 \epsilon_r} = (\frac{\chi_e}{\epsilon_r})\sigma, \tag{36}$$

Then,

$\chi_e = \epsilon_r - 1$, therefore;

$$P = (1 - \epsilon_r^{-1})\sigma \tag{37}$$

In result,

$$P_1 = \frac{\sigma}{2} \tag{38}$$

and

$$P_2 = \frac{\sigma}{3} \tag{39}$$

d) Find the potential difference between the plates.

Knowing $E_1\ and\ E_2$, then we know;

$$V = E_1 a + E_2 a = (\frac{\sigma a}{6\epsilon_0})(3 + 4) \tag{40}$$

Solving this, we know that V is equal to;

$$V = (\frac{7\sigma a}{6\epsilon_0}) \tag{41}$$

e) Find the location and the amount of all bound charges

Summarized: $\rho_b = 0$, therefore:

$\sigma_b = +P_1$ at the bottom slab of (1) $= \frac{\sigma}{2}$

$\sigma_b = -P_1$ at the top slab of (1) $= \frac{-\sigma}{2}$

$\sigma_b = +P_2$ at the bottom slab of (2) $= \frac{\sigma}{3}$

$\sigma_b = -P_2$ at the top slab of (2) $= \frac{-\sigma}{3}$

f) Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (b)

In Slab 1)

$$\{Total\ Surface\ Charge\ Above : \sigma - \frac{\sigma}{2} = \frac{\sigma}{2}, Total\ Surface\ Charge\ Below : \frac{\sigma}{2} - \frac{\sigma}{3} + \frac{\sigma}{3} - \sigma = -\frac{\sigma}{2}\} \tag{42}$$

Therefore:

$$E_1 = \frac{\sigma}{2\epsilon_0} \tag{43}$$

In Slab 2)

$$\{Total\ Surface\ Charge\ Above : \sigma - \frac{\sigma}{2} + \frac{\sigma}{2} - \frac{\sigma}{3} = \frac{2\sigma}{3}, Total\ Surface\ Charge\ Below : \frac{\sigma}{3} - \sigma = -\frac{2\sigma}{3}\} \tag{44}$$

Therefore:

$$E_2 = \frac{2\sigma}{3\epsilon_0} \tag{45}$$

5 Problem 4.26

A spherical conductor, of radius a, carries charge Q (Fig. 4.29). It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration (Eq. 4.58).

We are given the following variables:

a = Radius of the Spherical Conductor

Q = Charge of the Spherical Conductor

χ_e = Susceptibility of the dielectric material

b = Radius of the surrounding dielectric material

To find Energy, we must find W (from 4.58):

$$W = \frac{1}{2}\int_{enc} D \cdot E \tag{46}$$

$$D = \{0 \; at \; (r < a), \frac{Q}{4\pi r^2}\hat{r} \; at \; (r > a)\} \tag{47}$$

$$E = \{0 \; at \; (r < a), \frac{Q}{4\pi\epsilon r^2}\hat{r} \; at \; (a < r < b), \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \; at \; (r > b)\} \tag{48}$$

Plugging into the Work equation:

$$\frac{1}{2}\int (\frac{Q}{4\pi r^2}\hat{r})(0 + \frac{Q}{4\pi\epsilon r^2}\hat{r} + \frac{Q}{4\pi\epsilon_0 r^2}\hat{r})4\pi r^2 dr \tag{49}$$

$$\frac{1}{2}(\frac{Q}{(4\pi)})^2(4\pi)\left[\int_a^b \frac{1}{r^2}\frac{1}{\epsilon r^2} r^2 dr + \int_b^\infty \frac{1}{\epsilon_0 r^2} r^2 dr\right] \tag{50}$$

Reduces to:

$$W = \frac{Q^2}{8\pi}\left[\frac{1}{\epsilon}(\frac{1}{a} - \frac{1}{b}) + \frac{1}{\epsilon_0 b}\right] \tag{51}$$

From earlier, we sub $\epsilon = \epsilon_0(1 + \chi_e)$ in:

$$W = \frac{Q^2}{8\pi}\left[\frac{1}{\epsilon_0(1+\chi_e)}(\frac{1}{a} - \frac{1}{b}) + \frac{1}{\epsilon_0 b}\right] \tag{52}$$

Which reduces the energy to:

$$W = \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left[\frac{1}{a} + \frac{\chi_e}{b}\right] \tag{53}$$

6 Problem 4.35

A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?

Using Gauss's Law in Dielectrics;

$$\oint D \cdot da = Q_{f_{enc}} => D = \frac{q}{4\pi r^2}\hat{r} \tag{54}$$

Then we have the E-field:

$$(E) = \frac{1}{\epsilon}D = \frac{q}{4\pi\epsilon r^2}\hat{r} \tag{55}$$

Therefore the E-field is:

$$(E) = \frac{q}{4\pi\epsilon_0(1+\chi_e)r^2}\hat{r}, \epsilon = \epsilon_0(1+\chi_e) \tag{56}$$

Polarization:

$$(P) = \epsilon_0\chi_e E = \frac{q\chi_e}{4\pi(1+\chi_e)r^2}\hat{r} \tag{57}$$

Using $\rho_b = -\nabla \cdot P$ and $\sigma_b = P \cdot \hat{r}$:

$$\rho_b = -\frac{q\chi_e}{4\pi(1+\chi_e)}(\nabla \cdot \frac{\hat{r}}{r^2}) = -q\frac{\chi_e}{1+\chi_e}\delta^3(r) \tag{58}$$

$$\sigma_b = \frac{q\chi_e}{4\pi(1+\chi_e)r^2} \tag{59}$$

$Q_{surface}$ = Charge of the Spherical Conductor, is solved with σ_b

$$Q_{surface} = \sigma_b(4\pi r^2) = \frac{q\chi_e}{1+\chi_e} \tag{60}$$

Then the Compensatory negative charge towards the center:

$$\int \rho_b d\tau = \frac{-q\chi_e}{1+\chi_e} \int \delta^3(r) d\tau \tag{61}$$

Which in the end simplifies Negative Bound Charge to:

$$\int \rho_b d\tau = \frac{-q\chi_e}{1+\chi_e} \tag{62}$$

located in all space