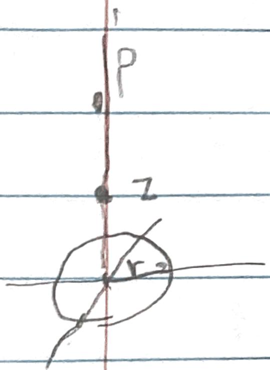
Homework 2   2.5, 2.6, 2.9, 2.12, 2.16, 2.18, 2.25, 2.29

2.5) Find the electric field distance z above the center loop carrying uniform line charge λ



Coulomb effect: $\frac{1}{4\pi\epsilon_0}\frac{dq'}{\mathscr{r}^2}\hat{\mathscr{r}} = dE$

$\mathscr{r} = r - r'$

$Q = 2\pi R\lambda$

$\Rightarrow z\hat{z} - R\hat{s}$

$\mathscr{r} = \sqrt{z^2+R^2} \Rightarrow z\sqrt{1+(R/z)^2} \Rightarrow z\sqrt{1+\epsilon^2}$, $\epsilon = \frac{R}{z}$

$\mathscr{r}^2 = z^2+R^2$, $dq = R\lambda\, d\phi'$

$\hat{\mathscr{r}} = \frac{\hat{z}-\epsilon\hat{s}}{\sqrt{1+\epsilon^2}}$, where $z = (\hat{z}-\epsilon\hat{s})^{-1}$

$$\int_0^{2\pi} dE = \int_0^{2\pi}\left(\frac{1}{4\pi\epsilon_0}\cdot\frac{dq'}{\mathscr{r}^2}\hat{\mathscr{r}}\right)$$

$$\Rightarrow \int \frac{1}{4\pi\epsilon_0}\cdot\frac{R\lambda\, d\phi}{z^2+R^2}\cdot\frac{\hat{z}-\epsilon\hat{s}}{\sqrt{1+\epsilon^2}}$$

& re-arrange

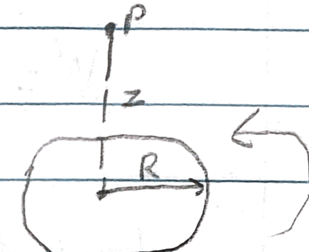
$$E \Rightarrow \frac{R\lambda}{4\pi\epsilon_0 z^2(1+\epsilon^2)^{3/2}}\int_0^{2\pi} d\phi\,(\hat{z}-\epsilon\hat{s})$$

$$E \Rightarrow \frac{2\pi R\lambda\hat{z}}{4\pi\epsilon_0 z^2(1+\epsilon^2)^{3/2}} = \frac{Q\,\hat{z}}{4\pi\epsilon_0 z^2(1+\epsilon^2)^{3/2}}$$

If $z=0$ the $\epsilon \to \infty$ making $E=0$. Work for far field too!

effect if $z \gg R$.

2.6) Find the electric field a distance z from the center of a circular loop of radius R that carries uniform surface charge σ. What happens when the limit $R \to \infty$



$$dE = k \frac{dq'}{r^2}\hat{r}, \quad k = \frac{1}{4\pi\epsilon_0}$$

$$\hat{r} = \frac{z\hat{z} - s\hat{s}}{\sqrt{z^2+s^2}} \qquad r = z\hat{z} - s\hat{s}$$

$$r^2 = z^2 + s^2 \qquad Q = \pi R^2 \sigma$$

total charge

$dq' = \sigma\, da' = s\,ds\,d\phi$ - cylindrical coordinates

$$\theta_0 = \tan^{-1}\left(\frac{R}{z}\right) \qquad s = z\tan\theta \qquad \cancel{ds = z\sec^2\theta\, d\theta}$$

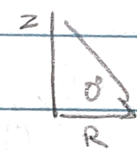
$$\int dE = E = \int_0^R \left(\frac{1}{4\pi\epsilon_0}\, \frac{s\,ds\,d\phi}{z^2+s^2}\, \frac{z\hat{z} - s\hat{s}}{\sqrt{z^2+s^2}}\right)$$

σ & $\hat{s}$

Such that

$$E = \frac{\sigma}{2\epsilon_0} z\hat{z} \int_0^R \frac{s\,ds}{(s^2+z^2)^{3/2}} \overset{\text{sub } z}{\Longrightarrow} \frac{\sigma}{2\epsilon_0} \cos\theta \Big|_0^{\theta_0} \hat{z}$$

$$E = \frac{\sigma}{2\epsilon_0}(1 - \cos\theta_0)\hat{z}$$



$$\rightarrow \hat{r} \Rightarrow \frac{z}{\sqrt{z^2+R^2}}$$

for z

$$E = \frac{\sigma\hat{z}}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$

if $R \to \infty$ in this form it will cancel out the second term

$$E = \frac{\sigma\hat{z}}{2\epsilon_0}, \quad R \to \infty$$

Continued

2.8 continued) To check the limit that $z >> R$

by factoring in a z such:

$$E = \frac{\sigma \hat{z}}{2\epsilon_0} z\left(z^{-1} - (z^2 + R^2)^{-1/2}\right) \Rightarrow \frac{\sigma \hat{z}}{2\epsilon_0} z\left(z^{-1} - z^{-1}(1 + (\tfrac{R}{z})^2)^{-1/2}\right)$$

Let $(1 + (\frac{R}{z})^2)^{-1/2} \approx (1 - (\frac{1}{2})(\frac{R}{z})^2)$

$$E = \frac{\sigma \hat{z}}{2\epsilon_0} z\left(\frac{1}{2z}\left(\frac{R}{z}\right)^2\right)$$ multiplied by $\frac{\pi}{\pi}$ w/ $Q = \pi R^2$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q\hat{z}}{z^2}$$ point charge field.

2.9) Suppose the electric field in some region is found to be $E = kr^3\hat{r}$, find charge density ρ, find the total charge

$$\nabla E = \nabla(kr^3\hat{r}) = \frac{\partial}{\partial r}\frac{1}{r^2}(r^2 E \cdot \hat{r}) + \ldots$$ no others in sum

$\nabla \cdot E = \frac{1}{\epsilon_0}\rho$ $\quad \nabla E = 5kr^2$ making the charge distribution

$$\rho(r) = 5k\epsilon_0 r^2$$

$$Q = \int \rho(r) d\tau$$

$\sin\theta \Rightarrow 0$

$$\Rightarrow \int_0^R \int_0^\pi \int_0^{2\pi} 5k\epsilon_0 r^4 \sin\theta \, dr\, d\theta\, d\phi \Rightarrow (4\pi)(5k\epsilon_0)\int_0^R r^4 dr$$

$5 \cdot \frac{1}{5} = 0$ cancels

$Q = 4\pi k\epsilon_0 R^5$ is the total charge

a second way using gaussian first law

$$Q = \epsilon_0 \oint E\, da \Rightarrow \epsilon_0 \int_0^{2\pi}\int_0^{\pi} kR^3\hat{r}\cdot\hat{r}R^2 d\theta d\phi$$

$\hat{r}$ goes away & R combines

simplifies to $Q = 4\pi k\epsilon_0 R^5$

2.12) Use Gauss's Law to find the electric field inside a uniformly charged solid sphere

$\nabla \cdot E = \frac{1}{\epsilon_0} \rho(r)$ - at radius r, E would be constant

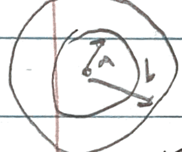
$\oint E \cdot da = E \cdot A$ - surface area oriented in $\hat{r}$ direction

$$E \cdot A = \frac{1}{\epsilon_0} \rho \int dr' \Rightarrow \frac{\rho}{\epsilon_0} (4\pi) \int_0^r r'^2 dr'$$

$$E \cdot A \Rightarrow \frac{\rho}{\epsilon_0} (4\pi) \frac{1}{3} r^3 \Big|_0^r \Rightarrow \frac{\rho}{\epsilon_0} \frac{4}{3} \pi r^3$$ divide by A

$$E(r) = \frac{\rho r \hat{r}}{3\epsilon_0}$$ e-field inside uniform sphere

2.16) Inner cylinder with radius A has a uniform charge density ρ, uniform surface charge density on the outer shell of radius b. As a whole it is electrically neutral.



Find the electric field in the 3 regions:

Inside the first cylinder ($s < a$)

Gauss's law can be simplified due to symmetry such that

Let $A = (2\pi s) z$

$$E \cdot A = \frac{Q}{\epsilon_0}, \quad E(2\pi s) z = \frac{1}{\epsilon_0} \int \rho' d\tau'$$

$$\Rightarrow \frac{1}{\epsilon_0} \int_0^s \rho s' ds' (2\pi) z \quad \therefore E = \rho \frac{s \hat{s}}{2\epsilon_0}$$ inner radius

Outside of the cylinder $E = 0$ since the cable is electrically neutral

Contin

2.16 continued) Between $(a < s < b)$

Gauss's law by symmetry is the same $E \cdot A = \frac{Q}{\epsilon_0}$

$Q = \rho(\pi a^2 z)$ volume of cylinder

$$E(2\pi s)z = \frac{\rho(\pi a^2 z)}{\epsilon_0} \quad \therefore \; E = \frac{\rho a^2}{2 s \epsilon_0}\hat{s}$$ Intermediate radius

2.18) Two spheres with radius R & carrying uniform volume charge densities $+\rho$ & $-\rho$. They partially overlap creating vector d from $+\rho$ to $-\rho$. Show that the overlapped region is constant by finding it's value.

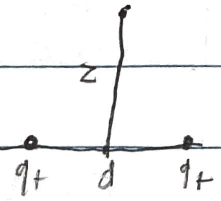
Let point P be in the center of vector d & r_+ & r_- be the displacements of the end points to P.

Since we know $E(r) = \frac{\rho \hat{r}}{3\epsilon_0}$ we can replace for r_+ & r_-

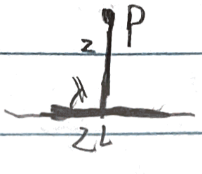
$$E(r) = \frac{\rho r_+}{3\epsilon_0} - \frac{\rho r_-}{3\epsilon_0} \quad \text{which is} \Rightarrow \frac{\rho d}{3\epsilon_0}$$

2.25) Using the equations describing the potential of point, line, & surface charges. Find the potential at a distance z above the center of:

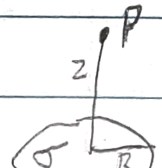
1- a pair of positive charges separated by distance d



2- a line charge of length 2L



3- a uniform surface charge spread across disc of radius R.



Compute $E = -\nabla V$ for each case. What happens if you make one of the point charges negative. What does it suggest.

Let $K = \frac{1}{4\pi\epsilon_0}$ for each distribution

1- The displacement between each charge and $P = z$ is the same as $r = \sqrt{z^2 + \frac{1}{4}d^2}$

$$V(z) = \frac{2Kq}{r} \Rightarrow \frac{2Kq}{(z^2 + \frac{1}{2}d^2)^{3/2}}$$

$$-\nabla \cdot V(z) \Rightarrow -\frac{dV}{dz}\hat{z} = \frac{2Kqz\hat{z}}{(z^2 + \frac{1}{2}d^2)^{3/2}}$$ which matches

2- $E = -\nabla V$, let the charge density be λ. $dq = \lambda dx'$

$$dV = \frac{K\lambda dx'}{\sqrt{x'^2 + z^2}}$$ using trig sub $\frac{dx'}{\sqrt{x^2 + z^2}}$

$$V(z) = K\lambda \ln\left(\frac{\sqrt{z^2 + L^2} + L}{\sqrt{z^2 + L^2} - L}\right) \Rightarrow K\lambda\left(\int \ln(\sqrt{z^2 + L^2} + L) - \int(\ln\sqrt{z^2+L^2} - L)\right)$$

$$E = -\frac{dV}{dz} = 2LK\lambda \frac{1}{z\sqrt{z^2 + L^2}}\hat{z}$$ this match up for the far field

2.25 cont'd

3- $E = -\nabla V$, radius R , density σ

$$dV = \frac{K\,dq'}{r} = \frac{K\sigma\,da'}{\sqrt{s'^2 + z^2}} \quad , \quad da' = s'\,ds\,d\phi$$

$$V(z) = \int_0^R \int_0^{\pi} \frac{K\sigma\, s'\,ds\,d\phi}{\sqrt{s'^2 + z^2}} \Rightarrow \frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2+z^2} - z\right)$$

$$-\nabla V = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)\hat{z}$$

If one was to swap the right charge to negative such that  then $V(z) = 0$ on the whole axis. $dV/dz = 0$ but if the field was taken in a different direction then it could be nonzero.

2.29) Check the laplacian of the volume integral satisfies Poisson's equation by showing it gives the charge density.

electrostatic potential

$$V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{r}\,d\tau' \quad , \quad \nabla^2 V(r) = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 V(r) = \frac{1}{4\pi\epsilon_0}\int \rho(r')\,\delta^3(r-r')\,d\tau' \quad , \quad \nabla^2 \frac{1}{r} = -4\pi\delta^3(r-r')$$

$$\Rightarrow -\frac{4\pi}{4\pi\epsilon_0}\rho(r) = \boxed{-\frac{\rho(r)}{\epsilon_0}}$$

$$\therefore \nabla^2 V(r) = -\rho/\epsilon_0$$