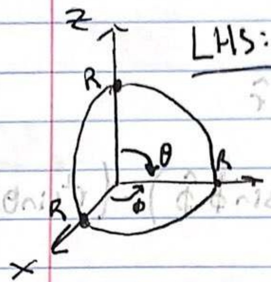
HW 1: 1.54, 1.55, 1.56, 1.57, 1.59, 1.62, 1.63, 1.64

1.54.) $V = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$



LHS: $\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

$$\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$$

$$= \frac{4r^{\cancel{3}}\cos\theta}{\cancel{r^2}} + \frac{r^{\cancel{2}}\cos\theta\cos\phi}{\cancel{r}\sin\theta} + \left(-\frac{r^{\cancel{2}}\cos\theta\cos\phi}{\cancel{r}\sin\theta}\right)$$

$$\nabla\cdot v = 4r\cos\theta + \overset{\emptyset}{\cancel{\frac{2r\cos\theta\cos\phi}{\sin\theta}}}$$

whoops forgot negative 'till later!

$d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\int_V (\nabla\cdot v)\,d\tau = \int_V \left(4r\cos\theta + \overset{\emptyset}{\cancel{\frac{2r\cos\theta\cos\phi}{\sin\theta}}}\right) r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_V \left(4r^3\sin\theta\cos\theta + \overset{\emptyset}{\cancel{2r^3\cos\theta\cos\phi}}\right) dr\,d\theta\,d\phi$$

$\downarrow$

$$\int_0^R\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} 4r^3\sin\theta\cos\theta\,dr\,d\theta\,d\phi = \frac{4\pi}{2}\int_0^R\int_0^{\frac{\pi}{2}} r^3\sin\theta\cos\theta\,dr\,d\theta$$

let $u = \sin\theta \Rightarrow du = \cos\theta\,d\theta$

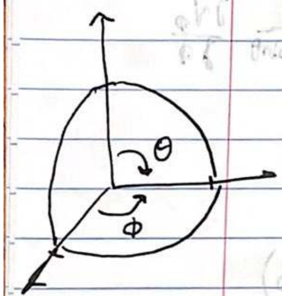
$$\frac{4\pi}{2}\int_0^R\int_0^{\frac{\pi}{2}} r^3 u\,dr\,du = \frac{4\pi}{2}\int_0^R r^3\left[\frac{u^2}{2}\right]_0^{\frac{\pi}{2}} dr$$

$$= \frac{4\pi}{2}\int_0^R r^3\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} dr = \frac{4\pi}{2}\int_0^R \frac{1}{2}r^3\,dr$$

$$= \pi\int_0^R r^3\,dr = \boxed{\frac{\pi R^4}{4}}$$

~~1.55.) $\nabla \times \vec{v} = \ldots$~~

1.54.) RHS: $r = \text{constant}$, so,



$$d\vec{a} = dl_\theta dl_\phi \hat{r} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$\vec{v} \cdot d\vec{a} = \left(r^2 \cos\theta \hat{r} + r^2 \cos\phi \, \hat{\theta} - r^2 \cos\theta \sin\phi \, \hat{\phi}\right) \cdot \left(r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right)$$

$$= r^4 \sin\theta \cos\theta \, d\theta \, d\phi$$

$$\oint_S \vec{v} \cdot d\vec{a} = \int_0^{\pi/2}\int_0^{\pi/2} r^4 \sin\theta \cos\theta \, d\theta \, d\phi$$

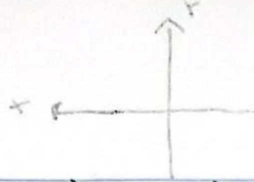
$* \; r = R$

$$= \frac{\pi R^4}{2} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

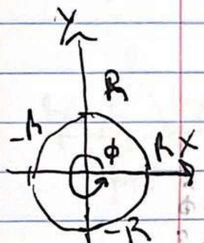
$$= \frac{\pi R^4}{2} \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{\pi R^4}{2} \left(\frac{1}{2} - 0\right)$$

$$\boxed{= \frac{\pi R^4}{4}}$$

LHS = RHS so Gauss' thm is verified. ✓



1.55.) $\vec{v} = ay\hat{x} + bx\hat{y}$



LHS: $(\nabla\times\vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (ay) & (bx) & 0 \end{vmatrix} = \hat{i}\left(0-\cancel{\frac{\partial}{\partial z}bx}\right) - \hat{j}\left(0-\cancel{\frac{\partial}{\partial z}ay}\right) + \hat{k}\left(\frac{\partial}{\partial x}bx - \frac{\partial}{\partial y}ay\right)$

$\vec{v} = (b-a)\hat{k} = (b-a)\hat{z}$

using cylindrical coordinates -- since $z = \text{const.} = 0$.

$$d\vec{a} = dl_s\, dl_\phi = s\,ds\,d\phi\,\hat{z}$$

now $(\nabla\times\vec{v})\cdot d\vec{a} = (b-a)\hat{z} \cdot s\,ds\,d\phi\,\hat{z}$

$$= s(b-a)\,ds\,d\phi$$

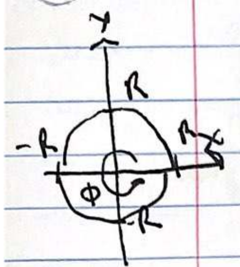
$$\int_S (\nabla\times\vec{v})\cdot d\vec{a} = (b-a)\int_S s\,ds\,d\phi$$

$$= (b-a)\int_0^R\int_0^{2\pi} s\,ds\,d\phi$$

$$= 2\pi(b-a)\int_0^R s\,ds$$

$$= 2\pi(b-a)\left[\frac{s^2}{2}\right]_0^R = \boxed{\pi R^2(b-a)}$$

1.55.) RHS: $\vec{v} = ay\hat{x} + bx\hat{y}$



$s = R,\ z = \phi \Rightarrow d\vec{l} = d\vec{l}_\phi = s\,d\phi\,\hat{\phi}$

$\hat{x} = \cos\phi\,\hat{s} - \sin\phi\,\hat{\phi}$ $\qquad y = s\sin\phi$

$\hat{y} = \sin\phi\,\hat{s} + \cos\phi\,\hat{\phi}$ $\qquad x = s\cos\phi$

$\Rightarrow \vec{v} = a(s\sin\phi)(\cos\phi\,\hat{s} - \sin\phi\,\hat{\phi}) + b(s\cos\phi)(\sin\phi\,\hat{s} + \cos\phi\,\hat{\phi})$

$= as\sin\phi\cos\phi\,\hat{s} - as\sin^2\phi\,\hat{\phi} + bs\cos\phi\sin\phi\,\hat{s} + bs\cos^2\phi\,\hat{\phi}$

$= (a+b)s\sin\phi\cos\phi\,\hat{s} + (bs\cos^2\phi - as\sin^2\phi)\,\hat{\phi}$

$\vec{v}\cdot d\vec{l} = (bs^2\cos^2\phi - as^2\sin^2\phi)\,d\phi$

$* \ \cos^2\phi = 1 - \sin^2\phi$

$= (bs^2(1-\sin^2\phi) - as^2\sin^2\phi)\,d\phi$

$= (bs^2 - bs^2\sin^2\phi - as^2\sin^2\phi)\,d\phi$

$= (bs^2 - (b+a)s^2\sin^2\phi)\,d\phi$

$\oint_P \vec{v}\cdot d\vec{l} = \oint_P (bs^2 - (b+a)s^2\sin^2\phi)\,d\phi$

$= \oint_P bs^2\,d\phi - \oint_P (b+a)s^2\sin^2\phi\,d\phi$ $\qquad * \ s = R$

$= \int_0^{2\pi} bs^2\,d\phi - \int_0^{2\pi} (b+a)s^2\sin^2\phi\,d\phi$

$= (2\pi bR^2) - (b+a)R^2 \int_0^{2\pi} \sin^2\phi\,d\phi$ $\qquad$ solved via integral table

$= 2\pi bR^2 - (b+a)R^2\left[\frac{\phi}{2} - \frac{1}{4}\sin(2\phi)\right]_0^{2\pi}$

1.55.) cont.

$$= 2\pi b R^2 - (b+a) R^2 \left[\frac{2\pi}{2}\right]$$

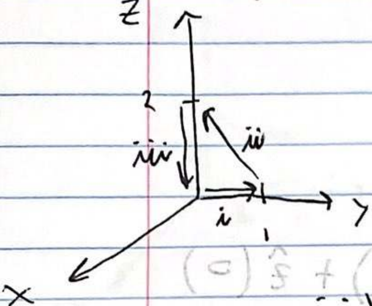
$$= 2\pi b R^2 - \pi b R^2 - \pi a R^2$$

$$= \pi b R^2 - \pi a R^2$$

$$= (b-a)\pi R^2$$

LHS = RHS thus Stokes' thm is verified.

1.56.) $\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$



i.) $x=0,\ z=0 \quad d\vec{l} = dy\,\hat{y}$

$$\vec{v}\cdot d\vec{l} = yz^2\,dy = 0\,dy$$

$$\int \vec{v}\cdot d\vec{l} = \emptyset.$$

ii.) $x=0,\ z=-2y+2$

$dx=0 \quad dz=-2dy \Rightarrow d\vec{l} = dy\,\hat{y} + dz\,\hat{z} = d\hat{y} + (-2dy)\hat{z}$

$$\vec{v}\cdot d\vec{l} = yz^2dy + (3y+z)(-2dy)$$

$$= y(-2y+2)^2\,dy + (-6y+4y-4)\,dy$$

$$= (y(4y^2+4-8y) - 2y - 4)\,dy$$

$$= (4y^3 - 8y^2 + 2y - 4)\,dy$$

$$\int \vec{v}\cdot d\vec{l} = \int_1^0 (4y^3 - 8y^2 + 2y - 4)\,dy = \left[y^4 - \frac{8}{3}y^3 + y^2 - 4y\right]_1^0$$

$$= -1 + \frac{8}{3} - 1 + 4 = \frac{8}{3} + 2$$

iii.) $x=0,\ y=0,\ dx=0,\ dy=0 \Rightarrow d\vec{l} = dz\,\hat{z}$

$$\vec{v}\cdot d\vec{l} = (3y+z)\,dz = z\,dz$$

$$\int_2^0 z\,dz = -\int_0^2 z\,dz = -\left[\frac{z^2}{2}\right]_0^2 = -2$$

So, $0 + \frac{8}{3} + 2 + (-2) = \boxed{\frac{8}{3}}$

1.56.) cont. $\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$

$$\int_S (\nabla \times \vec{v}) \cdot d\vec{a}$$

$x=0 \Rightarrow dx = 0 \Rightarrow d\vec{a} = dz\,dy\,\hat{x}$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 6 & yz^2 & 3y+z \end{vmatrix} = \hat{x}(3 - 2yz) - \hat{y}(0) + \hat{z}(0) = (3-2yz)\hat{x}$$

$$(\nabla \times \vec{v}) \cdot d\vec{a} = (3-2yz)\,dy\,dz$$

$y: 0 \to 1 \quad z: 0 \to -2y+2$

$$\int_S (\nabla \times \vec{v} \cdot d\vec{a}) = \int_0^1 \int_0^{-2y+2} (3-2yz)\,dy\,dz$$

$$= \int_0^1 \left[3z - yz^2\right]_0^{-2y+2} dy$$

$$= \int_0^1 \left((-6y+6) - y(4y^2+4-8y)\right) dy$$

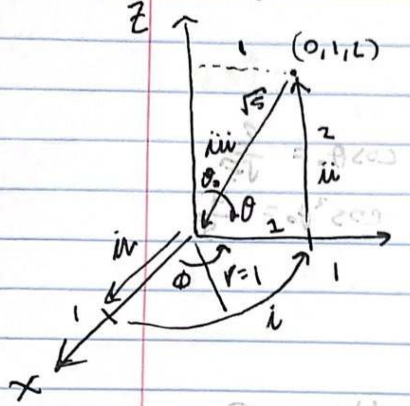
$$= \int_0^1 (-4y^3 + 8y^2 - 10y + 6)\,dy$$

$$= \left[-y^4 + \frac{8}{3}y^3 - 5y^2 + 6y\right]_0^1 = -1 + \frac{8}{3} - 5 + 6$$

$$= \boxed{\frac{8}{3}}$$

∴ Stoke's thm verified

1.57.) $\vec{v} = (r\cos^2\theta)\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\hat{\phi}$.



i.) $r = 1\,,\ \theta = \frac{\pi}{2} \Rightarrow dr = 0\,,\ d\theta = 0$

$\Rightarrow d\vec{l} = dl_\phi\,\hat{\phi} = r\sin\theta\, d\phi\,\hat{\phi}$

$\vec{v}\cdot d\vec{l} = 3r^2\sin\theta\, d\phi = 3\,d\phi$

$$\int_0^{\frac{\pi}{2}} 3\,d\phi = \frac{3\pi}{2}$$

ii.) $\phi = \frac{\pi}{2}\,,\ \cancel{\cos\theta = \frac{2}{r} \Rightarrow r = \frac{2}{\cos\theta}}\,,\quad \sin\theta = \frac{1}{r} \Rightarrow r = \frac{1}{\sin\theta}$

$d\phi = 0\,,\quad dr = -\frac{1}{\sin^2\theta}\cos\theta\, d\theta$

$d\vec{l} = dl_r\,\hat{r} + dl_\theta\,\hat{\theta} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$

$= -\frac{1}{\sin^2\theta}\cos\theta\, d\theta\,\hat{r} + \frac{1}{\sin\theta}\,d\theta\,\hat{\theta}$

$\vec{v}\cdot d\vec{l} = -\frac{r\cos^3\theta}{\sin^2\theta}\,d\theta + (-r\cos\theta)\,d\theta = -\frac{\cos^3\theta}{\sin^3\theta}\,d\theta + \left(-\frac{\cos\theta}{\sin\theta}\right)d\theta$

$= -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta}{\sin^2\theta} + 1\right)d\theta$

$= -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)d\theta = -\frac{\cos\theta}{\sin\theta}\left(\frac{1}{\sin^2\theta}\right)d\theta$

$= \frac{-\cos\theta}{\sin^3\theta}\,d\theta$

$$\int \vec{v}\cdot d\vec{l} = \int_{\frac{\pi}{2}}^{\theta_0} -\frac{\cos\theta}{\sin^3\theta}\,d\theta = -\int_{\frac{\pi}{2}}^{\theta_0} -\frac{1}{u^3}\,du = -\left[-\frac{1}{2}\frac{1}{u^2}\right]_{\frac{\pi}{2}}^{\theta_0}$$

$u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$= -\frac{1}{2}\left[-\frac{1}{\sin^2\theta}\right]_{\frac{\pi}{2}}^{\theta_0}$ $*\ \sin\theta_0 = \frac{1}{\sqrt{5}}$

$= -\frac{1}{2}\left[-\frac{1}{\sin^2\theta_0} + \frac{1}{\sin^2(\frac{\pi}{2})}\right]$

$= -\frac{1}{2}\left[-5 + 1\right] = +\frac{4}{2} = 2$

1.57.) cont. iii) $\phi = \frac{\pi}{2}$, $\theta = \theta_0$ $\Rightarrow d\phi = 0$, $d\theta = \theta_0$

$$\Rightarrow d\vec{l} = dl_r \hat{r} = dr\,\hat{r}$$

$$\vec{v} \cdot d\vec{l} = r\cos^2\theta\, dr$$
$$= \frac{4}{5} r\,dr$$

* $\cos\theta_0 = \frac{2}{\sqrt{5}}$

$\cos^2\theta_0 = \frac{4}{5}$

$$\int_{\sqrt{5}}^{0} \frac{4}{5} r\,dr = \frac{4}{5}\left[\frac{r^2}{2}\right]_{\sqrt{5}}^{0}$$
$$= \frac{4}{5}\left[-\frac{5}{2}\right] = -\frac{4}{2} = -2$$

iv.) ~~[illegible]~~ $\phi = 0$, $\theta = \frac{\pi}{2} \Rightarrow d\phi = d\theta = 0$

$$\Rightarrow d\vec{l} = dl_r \hat{r} = dr\,\hat{r}$$

$$\vec{v} \cdot d\vec{l} = r\cos^2\theta\, dr = 0$$

* $\cos\theta = \cos\frac{\pi}{2} = 0$

$$\int \vec{v} \cdot d\vec{l} = \phi.$$

So, $\frac{3\pi}{2} + 2 - 2 + \phi = \boxed{\frac{3\pi}{2}}$

1.57.) cont. $\vec{v} = r\cos^2\theta\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}$

$$\nabla \times \vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, 3r) - \frac{\partial}{\partial\phi}(r\cos\theta\sin\theta)\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\cos^2\theta) - \frac{\partial}{\partial r}(r\cdot 3r)\right]\hat{\theta}$$

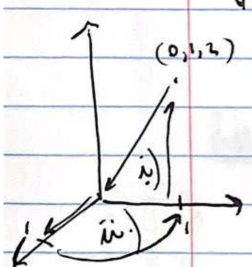
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r(-r)\cos\theta\sin\theta) - \frac{\partial}{\partial\theta}(r\cos^2\theta)\right]\hat{\phi}$$

$$= \frac{1}{\cancel{r}\sin\theta}\left[3\cancel{r}\cos\theta - 0\right]\hat{r} + \frac{1}{\cancel{r}}\left[0 - 6\cancel{r}\right]\hat{\theta}$$

$$+ \frac{1}{\cancel{r}}\left[\cancel{-2r\cos\theta\sin\theta + 2r\cos\theta\sin\theta}\right]\hat{\phi}$$

$$= 3\frac{\cos\theta}{\sin\theta}\,\hat{r} - 6\,\hat{\theta} + \cancel{\ldots\,\hat{\phi}}$$

$$\nabla \times \vec{v} = 3\frac{\cos\theta}{\sin\theta}\,\hat{r} - 6\,\hat{\theta}$$



i.) $d\vec{a} = -dl_r\, dl_\theta\, \hat{\phi} = -r\,dr\,d\theta\,\hat{\phi}$

$(\nabla \times \vec{v} \cdot d\vec{a}) = 0 \Rightarrow \int \nabla \times \vec{v} \cdot d\vec{a} = 0$

ii.) $d\vec{a} = -dl_r\, dl_\phi\, \hat{\theta} = -r\sin\theta\, dr\, d\phi\, \hat{\theta}$

$\theta = \frac{\pi}{2} \rightarrow \quad = -r\,dr\,d\phi\,\hat{\theta}$

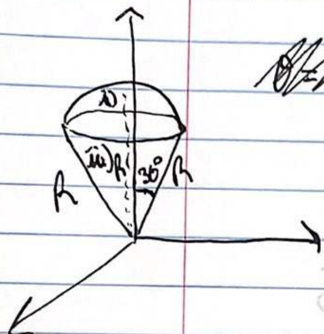
$(\nabla \times \vec{v} \cdot d\vec{a}) = 6r\,dr\,d\phi$

$$\int_S (\nabla \times \vec{v} \cdot d\vec{a}) = \int_0^1\int_0^{\frac{\pi}{2}} 6r\,dr\,d\phi = \frac{6\pi}{2}\int_0^1 r\,dr$$

$$= 3\pi\left[\frac{r^2}{2}\right]_0^1 = \frac{3\pi}{2}$$

So, $0 + \frac{3\pi}{2} = \boxed{\frac{3\pi}{2}}$ and Stoke's thm is verified.

1.59.) $\vec{v} = r^2\sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r^2\tan\theta\,\hat{\phi}$



$d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$\cancel{\nabla\cdot\vec{v} = 2r\sin\theta + (-4r^2\sin\theta) + 0 = 2r\sin\theta - 4r^2\sin\theta}$ $\cancel{\theta = 30°}$

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,4r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\tan\theta)$$

$$= \frac{4r^3}{r^2}\sin\theta + \frac{4r}{\sin\theta}(\cos^2\theta - \sin^2\theta) + 0$$

$$= 4r\sin\theta + 4r\frac{\cos^2\theta}{\sin\theta} - 4r\sin\theta$$

$$\nabla\cdot\vec{v} = 4r\frac{\cos^2\theta}{\sin\theta}$$

$$\int_V(\nabla\cdot\vec{v})\,d\tau = \int_V 4r\frac{\cos^2\theta}{\sin\theta}\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int 4r^3\cos^2\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^R\int_0^{\pi/6}\int_0^{2\pi} 4r^3\cos^2\theta\,dr\,d\theta\,d\phi = 2\pi\int_0^R\int_0^{\pi/6} 4r^3\cos^2\theta\,dr\,d\theta$$

1.59.) cont.

$$= 2\pi \left[r^4 \right]_0^R \int_0^{\pi/6} \cos^2\theta \, d\theta$$

$* \cos^2\theta = \frac{1}{2}\left(1+\cos(2\theta)\right)$

$$= \frac{2\pi R^4}{2} \int_0^{\pi/6} \left(1+\cos 2\theta\right) d\theta$$

$$= \frac{2\pi R^4}{2} \left[\frac{\pi}{6} + \int_0^{\pi/6} \cos(2\theta)\, d\theta \right]$$

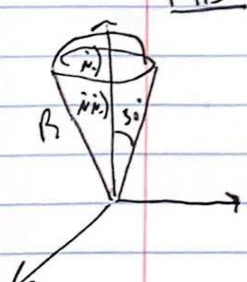
$u = 2\theta \Rightarrow du = 2d\theta$
$d\theta = \frac{1}{2} du$

$$= \frac{2\pi R^4}{2} \left[\frac{\pi}{6} + \frac{1}{2} \left[\sin(2\theta) \right]_0^{\pi/6} \right]$$

$$= \pi R^4 \left(\frac{\pi}{6} + \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 0 \right) \right)$$

$$= \boxed{\pi R^4 \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right)}$$

RHS: $\vec{V} = r^2 \sin\theta\, \hat{r} + 4r^2 \cos\theta\, \hat{\theta} + r^2 \tan\theta\, \hat{\phi}$



i) $r = R \Rightarrow dr = 0$

$d\vec{a} = dl_\theta dl_\phi\, \hat{r} = r^2 \sin\theta\, d\theta\, d\phi$

$\vec{V} \cdot d\vec{a} = \left(r^2 \sin\theta\right)^2 d\theta\, d\phi = r^4 \sin^2\theta\, d\theta\, d\phi = R^4 \sin^2\theta\, d\theta\, d\phi$

$$\int_S \vec{v} \cdot d\vec{a} = R^4 \int_0^{\pi/6}\int_0^{2\pi} \sin^2\theta\, d\theta\, d\phi = 2\pi R^4 \int_0^{\pi/6} \sin^2\theta\, d\theta$$

$$= 2\pi R^4 \int_0^{\pi/6} \frac{1}{2}\left(1-\cos(2\theta)\right) d\theta$$

$* \sin^2\theta = \frac{1}{2}\left(1-\cos(2\theta)\right)$

$$= \pi R^4 \left[\frac{\pi}{6} - \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/6} \right] = \pi R^4 \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right)$$

1.59.) cont. iii.) $\theta = 30° \Rightarrow d\theta = 0$

$$\Rightarrow d\vec{a} = dl_r dl_\phi \, \hat{\theta} = r\sin\theta \, dr \, d\phi \, \hat{\theta}$$

$$= r/2 \, dr \, d\phi \, \hat{\theta}$$

$$\vec{v} \cdot d\vec{a} = 24r^2 \cos\theta \left(\tfrac{r}{2} \, dr \, d\phi\right) = 2r^3 \left(\tfrac{\sqrt{3}}{2}\right) dr \, d\phi$$

$$= \sqrt{3} \, r^3 \, dr \, d\phi$$

$$\int_S \vec{v} \cdot d\vec{a} = \sqrt{3} \int_0^R \int_0^{2\pi} r^3 \, dr \, d\phi$$

$$= 2\sqrt{3}\pi \left[\frac{r^4}{4}\right]_0^R$$

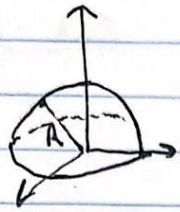
$$= \frac{\sqrt{3}\pi R^4}{2}$$

So, $\pi R^4 \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \frac{\sqrt{3}\pi R^4}{2} = \pi R^4 \left(\frac{\pi}{6} + \frac{2\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right)$

$$= \pi R^4 \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)$$

So, LHS = RHS and Green's Thm. is verified.

1.62.) a.)



$r = R \Rightarrow dr = 0$

$\Rightarrow d\vec{a} = dl_\phi \, dl_\theta = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$$a = \int_S d\vec{a} = \int_0^{\pi/2}\int_0^{2\pi} R^2 \sin\theta \, d\theta \, d\phi$$

$$= 2\pi R^2 \int_0^{\pi/2} \sin\theta \, d\theta$$

$$= 2\pi R^2 \left[-\cos\theta\right]_0^{\pi/2}$$

$$= 2\pi R^2 \left[0 - (-1)\right] = \boxed{2\pi R^2 \, \hat{r}}$$

b.) $\oint_S T d\vec{a} = \int_V (\nabla T) d\tau$ according to the hint.

If $T = 1 \Rightarrow \oint_S d\vec{a}$ is $\vec{a}$ of a closed surface

$$\oint_S d\vec{a} = \int_V \nabla(1) \, d\tau = 0.$$

c.) Suppose $a_1 \neq a_2$ for two surfaces sharing the same boundary. Now imagine adding those surfaces to create a closed surface, then

$$\oint_S d\vec{a} = a_1 - a_2 = 0, \text{ but}$$

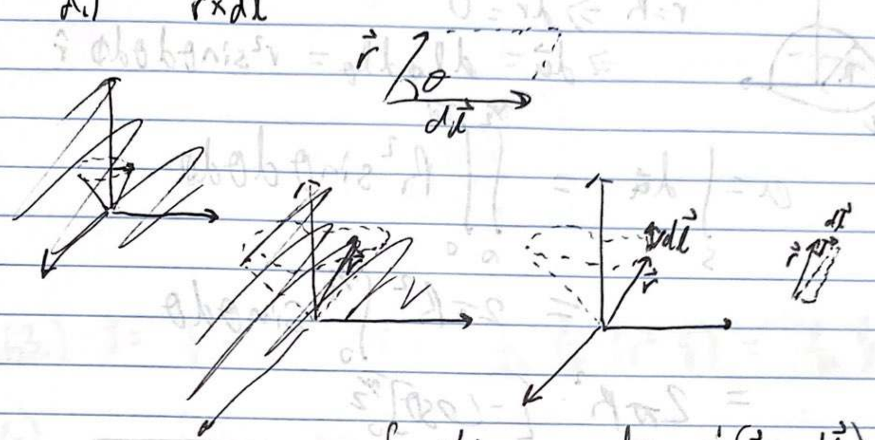
$$a_1 \neq a_2.$$

Contradiction!

Thus $a_1 = a_2$.

1.62) cont.

d.) $\vec{r} \times d\vec{l}$



for this case $d\vec{a} = \frac{1}{2}(\vec{r} \times d\vec{l})$

since $\vec{r} \times d\vec{l}$ is the angle of the rectangle formed (since there is a $90°$ angle btw $\vec{r}, d\vec{l}$)

If we ~~integrate~~ sum all these...

$$\oint d\vec{a} = \oint \frac{1}{2}(\vec{r} \times d\vec{l}).$$

e) $\oint (\vec{c} \cdot \vec{r}) d\vec{l}$ Following the hint: $\int_S \nabla T \times d\vec{a} = -\oint_P T \, d\vec{l}$

Let $T = \vec{c} \cdot \vec{r}$

$$\nabla T = \nabla(\vec{c} \cdot \vec{r}) = \vec{c} \times (\nabla \times \vec{r}) + \vec{r} \times (\nabla \times \vec{c}) + (\vec{c} \cdot \nabla)\vec{r} + (\vec{r} \cdot \nabla)\vec{c}$$

$* \ \nabla \times \vec{r} = \phi$

$$= 0 + \vec{r} \times \cancel{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ c_x & c_y & c_z \end{vmatrix}}^{\phi} + \left(c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z}\right)(\vec{r}) + \cancel{\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)(\vec{c})}^{\phi}$$

$\leftarrow$ zero since $\vec{c}$ is const.

$$= \left(c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z})$$

$$\nabla T = c_x \hat{x} + c_y \hat{y} + c_z \hat{z} = \vec{c}$$

1.62.) cont. e.) So, $\oint (\vec{c}\cdot\vec{r})\,dl = -\int \vec{c}\times d\vec{a}$

$$= -\vec{c}\times\int d\vec{a} \quad \text{since } \vec{c} \text{ const. vec.}$$

$$= -\vec{c}\times\vec{a}$$

$$= \vec{a}\times\vec{c}.$$

1.63.) $\vec{v} = \frac{1}{r}\hat{r}$ $\quad \boxed{\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r) = \boxed{\frac{1}{r^2}}}$

e.) $$\int_V (\nabla\cdot v)\,d\tau = \int_V \frac{1}{r^2}(r^2\sin\theta)\,dr\,d\theta\,d\phi$$

Let V be a sphere of radius R:

$$\int_0^R\int_0^\pi\int_0^{2\pi} \cancel{\frac{1}{r^2}}\cancel{r^2}\sin\theta\,dr\,d\theta\,d\phi = 2\pi R\int_0^\pi \sin\theta\,d\theta$$

$$= 2\pi R\left[-\cos\theta\right]_0^\pi = 2\pi R[1+1]$$

$$= 4\pi R.$$

$\oint_S \vec{v}\cdot d\vec{a}$ $\quad r=R \Rightarrow dr=0 \Rightarrow d\vec{a} = dl_\theta\,dl_\phi\,\hat{r}$

$$= r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$$

$$\vec{v}\cdot d\vec{a} = \frac{1}{r}r^2\sin\theta\,d\theta\,d\phi$$

$$\oint_S \vec{v}\cdot d\vec{a} = R\int_0^\pi\int_0^{2\pi}\sin\theta\,d\theta\,d\phi = 2\pi R\int_0^{\pi}\sin\theta\,d\theta$$

$$= 4\pi R.$$

So thm. checks out.

$$\boxed{\nabla\cdot(r^n\hat{r}) = \frac{\partial}{\partial r}(r^n) = \underline{\underline{n r^{n-1}}}}$$

1.63.) b.) $\vec{v} = r^n \hat{r}$

$$\nabla \times \vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, \cancel{v_\phi}) - \cancel{\frac{\partial v_\theta}{\partial\phi}}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial v_r}{\partial\phi}} - \frac{\partial}{\partial r} r(\cancel{v_\phi})\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\cancel{v_\theta}) - \cancel{\frac{\partial v_r}{\partial\theta}}\right]\hat{\phi}$$

$$\nabla \times \vec{v} = \boxed{0 = \nabla \times r^n \hat{r}}$$

1.64.) $D(r,\epsilon) \equiv -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\epsilon^2}}$

a.) $$\nabla^2 \frac{1}{\sqrt{r^2+\epsilon^2}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right)\right)$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(r^2\left(-\frac{1}{\cancel{2}}\frac{1}{(r^2+\epsilon^2)^{3/2}}\cancel{2}r\right)\right)$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(-\frac{r^3}{(r^2+\epsilon^2)^{3/2}}\right)$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(-r^3(r^2+\epsilon^2)^{-3/2}\right)$$

$$= \frac{1}{r^2}\left(-3r^2(r^2+\epsilon^2)^{-3/2} + (\cancel{-}r^3)(\cancel{-}\frac{3}{\cancel{2}})(r^2+\epsilon^2)^{-5/2}\cancel{2}r\right)$$

$$= \frac{1}{\cancel{r^2}}\left(\frac{-3\cancel{r^2}(r^2+\epsilon^2)}{(r^2+\epsilon^2)^{5/2}} + \frac{3r^{\cancel{4}\,2}}{(r^2+\epsilon^2)^{5/2}}\right)$$

$$= \frac{3r^2 - 3r^2 - 3\epsilon^2}{(r^2+\epsilon^2)^{5/2}} = \frac{-3\epsilon^2}{(r^2+\epsilon^2)^{5/2}}$$

$$\Rightarrow D(r,\epsilon) = \frac{-1}{4\pi}\left(\frac{-3\epsilon^2}{(r^2+\epsilon^2)^{5/2}}\right) = \boxed{\frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}}}$$

1.64.) cont.

b.) $D(0,\epsilon) = \frac{3\epsilon^2}{4\pi\,\epsilon^5} = \frac{3}{4\pi\epsilon^3}$

$$\lim_{\epsilon\to 0} \frac{3}{4\pi\epsilon^3} = \infty \checkmark$$

c.) $D(r,\epsilon) = \frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}}$

$$\lim_{\epsilon\to 0} D(r,\epsilon) = \frac{0}{4\pi\, r^5} = 0. \checkmark$$

d.) $\int_0^\infty D(r,\epsilon)\, 4\pi r^2\, dr = \int_0^\infty \frac{3\epsilon^2\, \cancel{4\pi} r^2}{\cancel{4\pi}(r^2+\epsilon^2)^{5/2}}\, dr$

$$= 3\epsilon^2 \int_0^\infty \frac{r^2}{(r^2+\epsilon^2)^{5/2}}\, dr$$

done using sage $\rightarrow$ $= 3\epsilon^2\left(\frac{1}{3\epsilon^2}\right) = 1$