HW #1

1.54 $\mathbf{v} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$ ans: $\frac{\pi R^4}{4}$

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$r = R \qquad 0 \le \theta \le \frac{\pi}{2} \qquad 0 < \phi \le \frac{\pi}{2}$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r^2\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$$

$$= 4r\cos\theta + \frac{\cos\theta}{r\sin\theta}r^2\cos\phi + \left(\frac{-r}{\sin\theta}\cos\phi\cos\theta\right)$$

$$r\cos\theta\left(4 + \cancel{\frac{\cos\phi}{\sin\theta}} - \cancel{\frac{\cos\phi}{\sin\theta}}\right)$$

$$= 4r\cos\theta = (\nabla\cdot\mathbf{v})$$

$d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\int(\nabla\cdot\mathbf{v})\,d\tau = \int 4r\cos\theta\,d\tau$$

$$= 4\int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi$$

[illegible]

$$= 4\left(\frac{R^4}{4}\right)(0.5)\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4} \checkmark$$

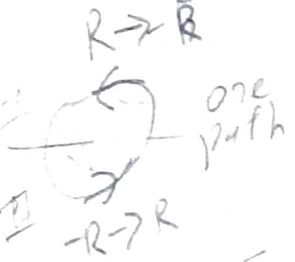
1.55 $v = ay\hat{x} + bx\hat{y}$ $\qquad r = R$ $\qquad$ ans $= \pi R^2(b-a)$

$$\nabla \cdot v \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ ay & bx & 0 \end{vmatrix} = (0-0)\hat{x} - (0-0)\hat{y} + (b-a)\hat{z}$$

circle

$$\therefore \int \nabla \times v \cdot da = \boxed{(b-a)\pi R^2} \qquad x^2+y^2=R^2$$

$\hat{z} = 0$



$$v \cdot dl = ay\,dx + bx\,dy$$

$$da = 2x\,dx + 2y\,dy = 0$$

$$dy = \frac{-x}{y}\,dx$$

$y = +\sqrt{R^2 - x^2}$

$$\frac{y}{y}\left(ay\,dx + bx\left(-\frac{x}{y}\right)dx\right) = v \cdot dl$$

$$\int_{R}^{-R} v \cdot dl = \int (ay^2 - bx^2)\frac{1}{y}\,dx$$

$$\int_{+R}^{-R} \frac{a(R^2-x^2) - bx^2}{\sqrt{R^2-x^2}}\,dx \qquad \text{wolfram pro}$$

$$= 0.5(\pi R^2(b-a))$$

II $\quad y = -\sqrt{R^2-x^2}$

$R \to -R$ $\qquad$ I = II

$$\int_{-R}^{+R} \nabla v \cdot da = \frac{1}{2}\pi R^2(b-a) \qquad = \boxed{\pi R^2(b-a)}$$

1.56 $v = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$ $= \frac{8}{3}$

$dx = 0 \quad dz = 0$

1. $dy = z^2 \qquad \int_0^1 z^2\,dy = 0$

2. $dx = 0$

$z = 2-2y$

$dz = -2dy$

$\int y(2-2y)^2 + (3y + 2 - 2y)\,dz$

$y(4y^2-8y+4)\,dy - 2(y+2)\,dy$

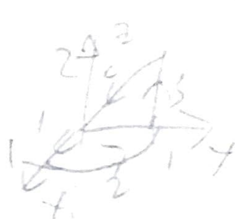
$\int_1^0 4y^3 - 8y^2 + 4y - 2y - 2\,dy$

$= y^4 - \frac{8}{3}y^3 + y^2 - 4y \Big]_1^0 = \frac{14}{3}$

3. $v\,dl = (3y+z)\,dz = z\,dz$

$\int_2^0 z\,dz = -(2)$ $= \frac{8}{3}$

1.57 $v = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}$



$x: 0 \to 1 \qquad r: 0 \to 1$

$\int_0^1 \cos^2\theta\,dr = 0$

$y = r\sin\theta$

$r=1 : \theta = \frac{\pi}{2} \qquad \phi : 0 \to \frac{\pi}{2} \qquad \int_0^{\frac{\pi}{2}}$

$\phi : (3r)(r\sin\theta\, d\phi) = 3\, d\phi$

$d\theta$

$\int_0^{\pi/2} 3\, d\phi = 3\frac{\pi}{2}$

$= \approx 0.14$

3) $\phi = \frac{\pi}{2} \quad y = r\sin\theta = 1 \qquad \theta \quad \frac{\pi}{2} \to$

$r = \frac{y}{\sin\theta} \qquad -\frac{1}{x^2}\, dx$

$dr = -\frac{1}{\sin^2\theta}\cos\theta\, d\theta$

$\int_{\frac{\pi}{2}}^{0.1411} (r\cos^2(\theta))\, dr - (r\cos\theta\sin\theta)\, d\theta \quad$ wolfram

$\to -\frac{\cos\theta}{\sin^3\theta}$

$\int -\frac{\cos\theta}{\sin^3\theta}\, d\theta$ wolfram

$\frac{1}{2}\left(\frac{1}{\sin^2\theta}\Big|_{\pi/2}^{0.1411}\right) = 2$

4) $r = \sqrt{5} \to 0 \quad , \phi = \frac{\pi}{2} \quad \theta \approx 0.1411$

$v \cdot dl = (r\cos^2\theta)(dr) - \quad r\cos^2(0.1411) \approx \frac{9}{5} r\, dr$

$\int v \cdot dl = \frac{9}{5}\frac{r^2}{2}\Big]_{\sqrt{5}}^{0} = -2$

total) $\boxed{\frac{3\pi}{2}}$ ~~$+2-2$~~

1.59 $v = r^2\sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r^2\tan\theta\,\hat{\phi}$ θ 30° $= \frac{\pi}{6}$

$r: 0\to R$ $\phi: 0\to 2\pi$ $\theta: 0\to\frac{\pi}{6}$

$\int (\nabla\cdot v)\,d\tau$

$\nabla\cdot v = \frac{1}{r^2}\partial_r(r^4\sin\theta) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\, 4r^2\cos\theta)$

$+ \frac{1}{r\sin\theta}\partial_\phi(r^2\tan\theta)$ $\sec = \frac{1}{\cos}$

$= 4r\sin\theta + \frac{4r}{\sin\theta}\cos^2\theta - \frac{4r}{\sin\theta}\sin^2\theta$

$= 4r\frac{\cos^2\theta}{\sin\theta}$

$4\int_V \frac{r\cos^2\theta}{\sin\theta}\,dV$

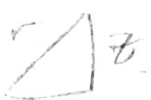
$= 4\int_0^R r^3\,dr \int_0^{2\pi} d\phi \int_0^{\pi/6}\cos^2\theta\,d\theta$ wolfram

R^4

$= R^4\, 2\pi\left(\frac{\pi/6}{2} + \frac{\sin(\frac{\pi}{3})}{4}\right)$

1.62 $a \equiv \int_S da$ 

a) $r=R$ $\int x = 0 = \int y$ $da = R^2\sin\theta\, d\theta\, d\phi$ all cancel out to 0

 $\hat{z} = \cos\theta\,\hat{r}$

a) $\int \cos\theta\, R^2 \sin\theta \, d\theta\, d\phi\, \hat{z}$

$= R^2 \hat{z} \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$

└ wolfram

$= 2\pi R^2 \left(\frac{1}{2}\right) \hat{z} \quad = \pi R^2 \hat{z}$

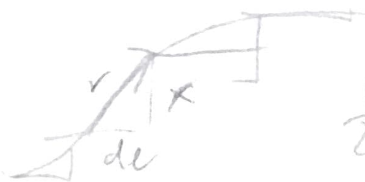
b) $a = 0$ closed surfaces don't have a border so stokes says it must be 0

$\int \nabla \cdot v \, da = 0$

c) $\oint da$ $\quad a_1 = a_2$

$da_1 = da_2$ $\quad \oint_S da_1 = \oint_S da_2$

d) $a = \frac{1}{2} \oint r \times dl$



$\frac{1}{2} dl \times r$

= triangle area

$\int \frac{1}{2} dl \times r = a$

e) $\oint (c \cdot r) dl = a \times c$

$\oint T dl = a \times c$

$\nabla (c \cdot r) = \nabla c \cdot r + \nabla r \cdot c$

$\left(\frac{\partial}{\partial x} c_x + \frac{\partial}{\partial y} c_y + \frac{\partial}{\partial z} c_z\right) \cdot r$

└ $x\hat{x} + y\hat{y} + z\hat{z}$

$= c_x \hat{x} + c_y \hat{y} + c_z \hat{z} \quad = \hat{c}$

1.63 $v = \frac{\hat{r}}{r}$ $\phi = 0, \theta = 0$

$\nabla \cdot V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{1}{r} \right) = \frac{1}{r^2} (1) = \frac{1}{r^2}$

$\int V \cdot da = \int \frac{1}{r} \hat{r} \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$

$\int \frac{1}{r^n} \hat{r} \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$ no charge at origin like with dirac delta function

b) $r^n \hat{r}$ $\nabla \times r^n \hat{r} = 0$

curl is 0 as all radii point out from the same direction

1.64 $D(r, \epsilon) \equiv -\frac{1}{4\pi} \nabla^2 \frac{1}{\sqrt{r^2 + \epsilon^2}}$ $\delta^3(r)$ as $\epsilon \to 0$

a) $D(r, \epsilon) = (3\epsilon^2 / 4\pi)(r^2 + \epsilon^2)^{-3/2}$ $\nabla = ?$

b) $D(0, \epsilon) \to \infty$, as $\epsilon \to 0$ $r \to 0$

$D(0, \epsilon) = \frac{3\epsilon^2}{4}$