Matthew Buchanan
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1.1

Midterm EMT

1) a) $(A\cdot\nabla)B$?

$A = A\hat{x} + A\hat{y} + A\hat{z}$ $\quad B = B\hat{x} + B\hat{y} + B\hat{z}$

$\nabla = \frac{d}{dx}\hat{x} + \frac{d}{dy}\hat{y} + \frac{d}{dz}\hat{z}$

$\Rightarrow (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\left(\frac{d}{dx}\hat{x} + \frac{d}{dy}\hat{y} + \frac{d}{dz}\hat{z}\right)(B\hat{x} + B\hat{y} + B\hat{z})$

$\Rightarrow A_x\frac{d}{dx}(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) + A_y\frac{d}{dy}(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$

$+ A_z\frac{d}{dz}(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$

$$\boxed{\Rightarrow \left(A_x\frac{dB_x}{dx} + A_y\frac{dB_x}{dy} + A_z\frac{dB_x}{dz}\right)\hat{x} + \left(A_x\frac{dB_y}{dx} + A_y\frac{dB_y}{dy} + A_z\frac{dB_y}{dz}\right)\hat{y} + \left(A_x\frac{dB_z}{dx} + A_y\frac{dB_z}{dy} + A_z\frac{dB_z}{dz}\right)\hat{z}}$$

b) Compute $(\hat{r}\cdot\nabla)\hat{r}$, where $\hat{r} = \bar{r}/r$

$\hat{r} = \frac{\bar{r}}{|r|} \Rightarrow \bar{r} = \langle x+y+z\rangle$ $\quad |r| = (x^2+y^2+z^2)^{1/2}$

$|r|^2 = x^2+y^2+z^2$

$\hat{r}^2 = \frac{\bar{r}^2}{|r^2|}$

$(\hat{r}\cdot\nabla) = \left(\hat{r}_x\frac{d}{dx} + \hat{r}_y\frac{d}{dy} + \hat{r}_z\frac{d}{dz}\right)$

$\Rightarrow \left(\hat{r}_x\frac{d}{dx} + \hat{r}_y\frac{d}{dy} + \hat{r}_z\frac{d}{dz}\right)\hat{r} \Rightarrow \left(\hat{r}_x^2\frac{d}{dx} + \hat{r}_y^2\frac{d}{dy} + \hat{r}_z^2\frac{d}{dz}\right)$

Sub $\hat{r}^2$

$\Rightarrow \left(\frac{\bar{r}^2}{|r^2|}\frac{d}{dx} + \frac{\bar{r}^2}{|r^2|}\frac{d}{dy} + \frac{\bar{r}^2}{|r^2|}\frac{d}{dz}\right)$

1.1 cont)

$$\Rightarrow = \bar{r}^{2}\left(\frac{1}{r^2}\frac{d}{dx} + \frac{1}{r^2}\frac{d}{dy} + \frac{1}{r^2}\frac{d}{dz}\right)$$

Goes to zero due to the Dirac delta

$\boxed{= 0}$

d) Non-uniform field $\Rightarrow F = (p \cdot \nabla) E$

$p = qd \Rightarrow qd\hat{x}$ with associated potential
$V(r) = V_0 r^2 + v_1$

$\bar{p} = \langle qd\hat{x} \rangle$

Using $V(r) = V_0 r^2 \times V_1)$, then $\nabla V = -E$.

$\nabla(V_0 r^2 + V_1) = -E \Rightarrow \boxed{\bar{E} = -2V_0 r}$

$F = (p \cdot \nabla) E \Rightarrow$ sub

$$\Rightarrow \left(qd\hat{x}\frac{d}{dr}\right)\left(-2V_0 r\right)$$

$\Rightarrow F = qd\hat{x} - 2V_0 \Rightarrow \boxed{F = -2V_0 qd\hat{x}}$

1.2)

1.2) Evaluate the following integrals

$$J = \int_V e^{-r} \left(\nabla \cdot \frac{\hat{r}}{r^2} \right)$$

Integration by parts

$\vec{V} = \frac{\hat{r}}{r^2} \Rightarrow (\nabla \cdot \vec{V}) = 4\pi \delta^3(r)$

Sub

$$\Rightarrow J = \int_V e^{-r} \left(4\pi \delta^3(r) \right) d\tau$$

$$\Rightarrow J = \int_V e^0 \Rightarrow \boxed{J = 0}$$

2.1)

2.1)

?

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2.2)

2.2) Electric Potential

$$V(r) = A\frac{e^{-\lambda r}}{r}$$

A & λ are constant

Find $E(r)$, charge density ρ & total charge Q in terms of A & λ.

$$\rho = \varepsilon_0 A\left(4\pi\delta^3(r) - \lambda^2\frac{e^{-\lambda r}}{r}\right)$$

Shown in 1c) $\underline{\nabla V = -E}$

$$\nabla = \left(\hat{r}\cdot\frac{d}{dr}\right) \Rightarrow \text{so in } \nabla V = -E$$

$$\Rightarrow \left(\hat{r}\cdot\frac{d}{dr}\right)\left(\frac{Ae^{-\lambda r}}{r}\right) \Rightarrow A\hat{r}\frac{d}{dr}\left(\frac{e^{-\lambda r}}{r}\right) = -E$$

Int by parts $\Rightarrow$ $\underline{u = -\lambda r dr \quad du = -\lambda dr}$

$$\Rightarrow A\hat{r}\left(\frac{-\lambda e^{-\lambda r}}{r} - \frac{e^{-\lambda r}}{r^2}\right) \Rightarrow \left(-\frac{A(\lambda r+1)e^{-\lambda r}}{r^2}\hat{r}\right) = -E$$

$$\Rightarrow \boxed{E = \frac{A(\lambda r+1)e^{-\lambda r}}{r^2}\hat{r}}$$

2.2 (cont)

Having E, we know $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$

$$E = \frac{A(\lambda r+1)e^{-\lambda r}}{r^2}\hat{r}$$

$$\nabla \cdot E = \rho/\varepsilon_0 \Rightarrow \rho = \varepsilon_0(\nabla \cdot E)$$

$$\Rightarrow \text{Sub } \rho = \varepsilon_0\left(\nabla \cdot \frac{A(\lambda r+1)e^{-\lambda r}}{r^2}\hat{r}\right)$$

$$\Rightarrow \varepsilon_0 A\left(\nabla \cdot \left(\frac{\hat{r}}{r^2} \cdot \frac{(\lambda r+1)e^{-\lambda r}}{r}\right)\right)$$

$\Rightarrow$ Given to us, we solve ρ to be:

$$\boxed{\rho = \varepsilon_0 A\left(4\pi\delta^3(r) - \frac{\lambda^2 e^{-\lambda r}}{r}\right)}$$

2.2 (cont)

Q_{total}? Int of ρ

$$Q_{tot} = \int_V \rho \, d\tau$$

$\int \delta^3(r) dr = 1$ by Eq 1.97

$d\tau = dr$

$$\Rightarrow \int \left(\varepsilon_0 A \left(4\pi \delta^3(r) - \frac{\lambda^2 e^{-\lambda r}}{r} \right) \right) dr$$

Split to two integrals

$$\Rightarrow \varepsilon_0 A \left(\left(4\pi \int \delta^3(r) \, dr \right) - \int \left(\frac{\lambda^2 e^{-\lambda r}}{r} dr \right) \right)$$

$$\Rightarrow \varepsilon_0 A \left(\left(4\pi \cdot (1) \right) - \lambda^2 \int \frac{e^{-\lambda r}}{r} dr \right)$$

2.3)

2.3) Gauss's Law $\oint \vec{E} \cdot da = Q/\varepsilon_0$

a)

$\lambda = \rho$

$A = da = 2\pi s h + 2\pi s^2$ ← surface area of a cylinder

Since the field is constant, we don't integrate, we get E alone.

$$E \cdot A = Q/\varepsilon_0 \Rightarrow E = Q/\varepsilon_0 A$$

From 2.2 $Q_{enc} = \int v \lambda \, d\tau \Rightarrow$ volume integral of cylinder

so

$Q_{enc} = \pi s^2 h \Rightarrow$ volume of cylinder

$$E = \frac{Q}{\varepsilon_0 A} \Rightarrow \frac{\pi s^2 h}{\varepsilon_0(2\pi s h + 2\pi s^2)}$$

$$\boxed{E = \frac{s h}{\varepsilon_0(2h + 2s)} \hat{s}}$$

2.3 cont

b) $F = ma$

$\Rightarrow a = F/m$

$F = QE$

In our case q with mass m so

$F = qE$

$\Rightarrow a = qE/m$

Using $\vec{a} \Rightarrow \vec{a} = \dfrac{d^2(\vec{r})}{dt^2}$

$\hat{r} = \hat{s}$

$$\Rightarrow \frac{d(s)^2}{dt^2} = \frac{qE}{m}$$

3.1)

3.1) ?

3.2)

3.2) Laplace's eq

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0$$ depend on z

Boundary cond are
$V=0$ at $x=b$
$V=0$ at $x=-b$
$V=0$ at $y=0$
$V=V_0$ at $y=a$

$V(x,y) = X(x)\,Y(y)$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$\Rightarrow \lambda = \frac{1}{X}\frac{d^2X}{dx^2}$$ and

$$\Rightarrow \lambda + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

so

$$\Rightarrow \frac{d^2X}{dx^2} = \lambda X \qquad \frac{d^2Y}{dy^2} = -\lambda Y$$

If $\lambda < 0$ in the boundary then $\lambda = -k^2$

$$-k^2 Y = \frac{d^2Y}{dy^2}$$

=>

3.2) cont.

solution to a diff eq

$$\Rightarrow Y(y) = A\cos(ky) + B\sin(ky)$$

$y=0$, $V(x,0)=0$ so
$X(x)Y(0) = Y(0) = \underline{0}$

$\Rightarrow$ $Y(0) = 0 = A$
$Y(y) = B\sin(ky)$

$y=a$, $V(x,a) = V_0$ so
$X(x)Y(a) = Y(a) = \underline{V_0}$

for $X(a)=0$ then $\Rightarrow ka = n\pi$

So
$$\boxed{\begin{aligned} Y(y) &= B\sin(ky), \text{ where } k = n\pi/a \\ X(x) &= Ce^{kx} + De^{-kx} \end{aligned}}$$

with $X(x)$ & $Y(y)$ we get,
$V(x,y) = X(x)Y(y)$

$$\Rightarrow B\sin(ky)\left[Ce^{kx} + De^{-kx}\right]$$

where $k = n\pi/a$

So

$$\boxed{\Rightarrow V(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{a}\right)\left[C_n \cosh\left(\frac{n\pi x}{a}\right)\right]}$$

3.3)

3.3) Using monopole ($Q = 3q - q = 2q$) & dipole ($p = [3qa\hat{z} + (-q)0] \Rightarrow p = 3qa\hat{z}$) find the approx potential in spherical coord.

a) $V(r) = V_{mono} + V_{dipo}$

$$\Rightarrow V_{mono} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

$$V_{dipo} = \frac{1}{4\pi\epsilon_0}\frac{p\cdot\hat{r}}{r^2}$$

$k = 4\pi\epsilon_0$

$$\text{so} \Rightarrow \frac{1}{k}\frac{Q}{r} + \frac{1}{k}\frac{p\cdot\hat{r}}{r^2}$$

$$\boxed{\Rightarrow V(r) = \frac{1}{k}\left(\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right)}$$

b) For the second charge arrangement $Q = 2q$ & $p = qa\hat{z}$

$V(r) = V_{mono} + V_{dipo}$

$k = 4\pi\epsilon_0$

so $\Rightarrow$

$$\boxed{V(r) = \frac{1}{k}\left(\frac{2q}{r} + q\frac{a\cos\theta}{r^2}\right)}$$

c) Third charge arrangement. & $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$

$Q = 2q$ & $p = 3qa\hat{y}$ then $\hat{y}\cdot\hat{r} = \sin\theta\sin\phi$

using above eq.

$$V(r) = \frac{1}{k}\frac{Q}{r} + \frac{1}{k}\frac{p\cdot\hat{r}}{r^2}$$

So: $\boxed{V(r) = \frac{1}{k}\left(\frac{2q}{r} + \frac{3qa\,\hat{y}\cdot\hat{r}}{r^2}\right)}$