2.43, 2.5, 3.1, 3.3, 3.13, 3.14, 3.15

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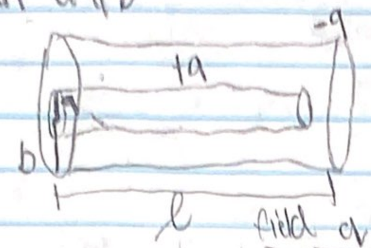
Phys 330

Electromagnetic Theory Homework 3

2.43). Capacitance per unit length

radii a & b

symmetry
Gauss
$\int E \cdot da$



field $E = \frac{q}{2\pi s l \varepsilon_0}\hat{s}$ for Efield

$$V(a)-V(b) = -\int_b^a \left(\frac{q}{2\pi s l \varepsilon_0}\right) ds$$

$V(a)-V(b)=V$

wrt s

$$V = -\int_b^a \left(\frac{q}{2\pi s l \varepsilon_0}\right) ds$$

$$= \frac{q}{2\pi\varepsilon_0 l}\log s \Big|_b^a$$

$\log a - \log b$

$$= -\frac{q}{2\pi\varepsilon_0 l}\log s$$

$$\frac{q}{2\pi\varepsilon_0 l}(\log a - \log b)$$

$$= \frac{+q}{2\pi\varepsilon_0 l}(\log a + \log b)$$

$$= \frac{q}{2\pi\varepsilon_0 l}\ln\left(\frac{b}{a}\right) \qquad V = \frac{q}{C}$$

compare

$$\boxed{C = \frac{2\pi\varepsilon_0 l}{\ln\left(\frac{b}{a}\right)}}$$

2.5). $V(r) = A\frac{e^{-\lambda r}}{r}$ A & λ are constants E field, charge ρ, total Q

Efield $= E = -\nabla V$

$$-\frac{\partial}{\partial r}\frac{Ae^{-\lambda r}}{r} = -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)$$

$$-A\left[\frac{re^{-\lambda r}(-\lambda) - e^{-\lambda r}(1)}{r^2}\right]\hat{r}$$

$$= -A\left[\frac{-r\lambda e^{-\lambda r} - e^{-\lambda r}}{r^2}\right]\hat{r} = A\left(r\lambda e^{-\lambda r} + e^{-\lambda r}\right)\frac{\hat{r}}{r^2}$$

$$\boxed{= Ae^{-\lambda r}(1+r\lambda)\frac{\hat{r}}{r^2}}$$

charge density
$\rho(r)$

$$\nabla E = \frac{1}{\varepsilon_0}\rho$$

$$\rho = \varepsilon_0 \nabla . E$$

$$= \varepsilon_0 \nabla \left(Ae^{-\lambda r}(1+r\lambda)\frac{\hat{r}}{r^2}\right)$$

$$\varepsilon_0\left(Ae^{-\lambda r}(1+r\lambda)\right)\left(\nabla\frac{\hat{r}}{r^2} + \frac{\hat{r}}{r^2}\nabla\left(Ae^{-\lambda r}(1+r\lambda)\right)\right)$$

$$\nabla\frac{\hat{r}}{r^2} = 4\pi\delta^3(r)$$

$$\rho(r) = \varepsilon_0\left(Ae^{-\lambda r}(1+r\lambda)\right)\left(4\pi\delta^3(r) + \frac{\hat{r}}{r^2}\nabla\left(Ae^{-\lambda r}(1+r\lambda)\right)\right)$$

$$= \varepsilon_0 A\, 4\pi\delta^3(r) + \left(\frac{\hat{r}}{r^2}\right)\nabla\left(Ae^{-\lambda r}(1+r\lambda)\right)$$

$$e^{-\lambda r}(1+r\lambda))(4\pi\delta^3(r)) = 4\pi\delta^3(r)$$

$$\nabla\left(Ae^{-\lambda r}(1+r\lambda)\right) = \hat{r}\frac{d}{dr}\left(Ae^{-\lambda r}(1+r\lambda)\right)$$

$$= \hat{r}A\left(\lambda e^{-\lambda r} - \lambda(1+\lambda r)e^{-\lambda r}\right)$$

$$= \hat{r}A\left(\lambda e^{-\lambda r} - \lambda e^{-\lambda r}(1+\lambda r)\right)$$

$$= \hat{r}A\left(\lambda e^{-\lambda r}(1-(1+\lambda r)\right)$$

$$= \hat{r}A\left(\lambda e^{-\lambda r}(1-1-\lambda r)\right)$$

$$= \hat{r}A\left(\lambda e^{-\lambda r}(-\lambda r)\right)$$

$$= \hat{r}A\left(-\lambda^2 r e^{-\lambda r}\right)$$

$$\cdot\left(\frac{\hat{r}}{r^2}\right)$$

$$\frac{\hat{r}}{r^2}\nabla\left(Ae^{-\lambda r}(1+r\lambda)\right) = \frac{\hat{r}}{r^2}\nabla\left(Ae^{-\lambda r}(1+r\lambda)\right)$$

$$= \frac{\hat{r}}{r^2}A\left(\lambda e^{-\lambda r} + (1+r\lambda)e^{-\lambda r}(-\lambda)\right)$$

$$= \frac{\hat{r}}{r^2}\left(A\left(\lambda e^{-\lambda r} - \lambda e^{-\lambda r} - r\lambda^2 e^{-\lambda r}\right)\right)\hat{r}$$

$$\frac{1}{r^2}A\left(-\lambda^2 r e^{-\lambda r}\right)$$

$$= -\frac{A\lambda^2}{r}e^{-\lambda r}$$

$$\rho(r) = \varepsilon_0\left(A4\pi\delta^3(r) + \frac{\hat{r}}{r^2}\left(Ae^{-\lambda r}(1+r\lambda)\right)\right)$$

$$\varepsilon_0\left(A\,4\pi\delta^3(r) - \frac{A\lambda^2}{r}e^{-\lambda r}\right)$$

$$\boxed{A\varepsilon_0\left(4\pi\delta^3(r) - \frac{\lambda^2}{r}e^{-\lambda r}\right)}$$

Total charge; Q $\qquad Q = \int \rho\, d\tau$

$$= \int A\varepsilon_0 \left(4\pi\delta^3(r) - \frac{\lambda^2}{r}e^{-\lambda r}\right)d\tau$$

$$= \int A\varepsilon_0 4\pi\delta^3(r)d\tau - \int A\varepsilon_0 \frac{\lambda^2}{r}e^{-\lambda r}d\tau$$

$$= 4\pi\varepsilon_0 A\int\delta^3(r)d\tau - A\varepsilon_0\lambda^2\int\frac{e^{-\lambda r}}{r}d\tau$$

$$= 4\pi\varepsilon_0 A(1) - A\varepsilon_0\lambda^2\int\frac{e^{-\lambda r}}{r}(4\pi r^2 dr)$$

$$= 4\pi\varepsilon_0 A - 4\pi A\varepsilon_0\lambda^2\int re^{-\lambda r}dr$$

$$= 4\pi\varepsilon_0 A - 4\pi A\varepsilon_0\lambda^2\int_0^\infty re^{-\lambda r}dr$$

$$= 4\pi\varepsilon_0 A - 4\pi A\varepsilon_0\lambda^2\left(\frac{1}{\lambda^2}\right)$$

$$= 4\pi\varepsilon_0 A - 4\pi A\varepsilon_0$$

$$= 0 \quad \boxed{\text{Total charge } Q=0}$$

3.1). Radius due to point charge q

$$V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\varepsilon_0 R} \quad Q = \text{total charge enclosed}$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad \begin{matrix} q\text{-charge} \\ r = \text{distance} \end{matrix}$$

$4\pi\varepsilon_0$ ← free space

$$z<R \quad r^2 = z^2+R^2-2Rz\cos\theta$$

$$r = \sqrt{z^2+R^2-2zR\cos\theta}$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{z^2+R^2-2z\cos\theta}} = \frac{q}{4\pi\varepsilon_0}\left(z^2+R^2-2z\cos\theta\right)^{-1/2}$$

$$V_{av} = \frac{\int V\,dA}{4\pi R^2} \quad \text{where } dA = R^2\sin\theta d\theta d\phi$$

$$= V_{av} = \frac{1}{4\pi R^2}\int\frac{q}{4\pi\varepsilon_0}\left(z^2+R^2-2zR\cos\theta\right)^{-1/2}R^2\sin\theta d\theta d\phi$$

$$z<R = R^{-1}$$

$$V_{av} = \frac{1}{4\pi R^2}\frac{qR^2}{4\pi\varepsilon_0}\int\frac{1}{R}\sin\theta d\theta d\phi$$

$$= \frac{q}{4\pi\varepsilon_0 R}$$

$$\underline{\underline{V_{cen} = V_{avg}}}$$

$\frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{R}$ $\boxed{V_{avg} = V_{cent} + \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{R}}$

3.3). Laplaces equation where V depends only on r

v depends only on s

in spherical coordinates, $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}}$

because v depends on r

$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$

$\underbrace{\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)}_{\text{constant}}$

$dV = \frac{c}{r^2}dr$

$\boxed{V = \frac{-c}{r} + \text{constant}}$

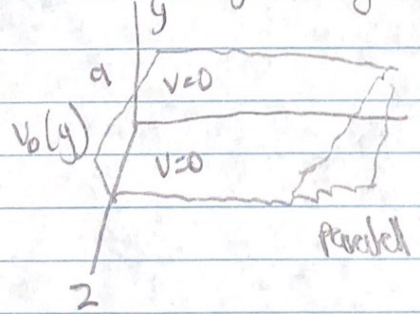
$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$

~~$\nabla^2 V$ depends on s~~, so $\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(\underbrace{s\frac{\partial V}{\partial s}}_{\text{constant}}\right) = 0$

$\frac{\partial V}{\partial s}$ ~~to~~ $= \frac{c}{s}$

$\boxed{dV = \frac{c}{s}ds}$

3.15). potential $x=0$ $y=0$ + $y=a/2$ V_0 $y=a/2$ $y=a$ $-V_0$



Laplace equations

$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

parallel to x,y plane

$V(x,y) = X(x)Y(y)$

boundary condition

$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin\frac{n\pi y}{a}$

$C_n = \frac{2}{a}\int_0^a V_0(y)\sin\frac{n\pi y}{a}\,dy$

3.13 continued   bounded by $0<y<a$ $0<y<a/2$ $a/2<y<a$

$$C_n = \frac{2}{a}\int_0^{a/2} V_0 \sin\frac{n\pi y}{a}\,dy - \int_{a/2}^{a} V_0 \sin\frac{n\pi y}{a}\,dy$$

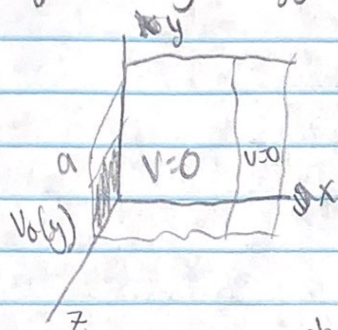
$$= \frac{2V_0}{a} \times \frac{a}{n\pi}\left(-\cos\frac{n\pi y}{a}\right)\Big|_0^{a/2} + \cos\frac{n\pi y}{a}\Big|_{a/2}^{a}$$

$$\frac{2V_0}{n\pi}\; -\cos\frac{n\pi}{2} + \cos 0 + \cos n\pi - \cos\frac{n\pi}{2}$$

$$C_n = \frac{2V_0}{n\pi}\left(1+(-1)^n - 2\cos\frac{n\pi}{2}\right)$$

when n is odd, $C_n = 0$

3.14). charge density $\sigma(y)$ $x=0$ constant potential V_0



$V(x,y)$

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5} \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\frac{n\pi y}{a}$$

charge density σ wrt electric potential

$$\sigma = -\varepsilon_0 \frac{dV}{dx} \qquad \sigma(y) = -\varepsilon_0\left(\frac{\partial V}{\partial x}\right)_{x=0}$$

$$\sigma(y) = -\varepsilon_0 \frac{d}{dx}\left(\frac{4V_0}{\pi}\sum \frac{1}{n} e^{\frac{n\pi x}{a}} \sin\frac{n\pi y}{a}\right)$$

$$= -\varepsilon_0 \frac{4V_0}{\pi}\frac{d}{dx}\sum \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\frac{n\pi y}{a}$$

$$= -\varepsilon_0 \frac{4V_0}{\pi}\sum \frac{1}{n}\,\frac{-n\pi}{a} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$$\boxed{\sigma(y) = \frac{4\varepsilon_0 V_0}{a}\sum_{odd} \sin\frac{n\pi y}{a}}$$

3.15). Rectangular pipe $x=0$ $y=0$ $y=a$ & $x=0$ $x=b$ $V_0(y)$

a). general potential formula

Laplacian wrt x & y = $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

Boundary condition
$V(x,0)=0$
$V=(\cancel{x},a)=0$
$V=(0,y)=0$
$V(b,y) = (V)y$

$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$

$V(x,0)=0 \rightarrow (Ae^{kx} + Be^{-kx})D=0$
$=D=0$

$0=(A+B)C\sin ky = A=-B$

$V(x,y) = AC(e^{\frac{n\pi x}{a}} - e^{\frac{-n\pi x}{a}})\sin\frac{n\pi y}{a}$
$= 2AC\sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}$

$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$

$V_0(y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$

*Fourier's theorem

$C_n\sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$

$C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$

$C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)} V_0\int_0^a \sin\left(\frac{n\pi y}{a}\right)dy$

$C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)} V_0\int_0^a \sin\left(\frac{n\pi y}{a}\right)dy$

$V_0(y) = V_0$

$V_0(y) = \frac{2V_0}{a\sinh\left(\frac{n\pi b}{a}\right)}\frac{a}{n\pi}\left(-\cos\frac{n\pi y}{a}\right)$

$\boxed{= \frac{2V_0}{a\sinh\left(\frac{n\pi b}{a}\right)}\frac{a}{n\pi}\left(-\cos n\pi + 1\right)}$