Electromagnetc Theory: PHYS330

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Summary

Week 3 Summary

1. Laplace's Equation
   - One-dimension
   - Two-dimensions, three dimensions, uniqueness, boundaries
2. Separation of Variables: Boundary-value problems
   - Cartesian coordinates
   - Spherical coordinates
3. Multipole Expansions
   - Far-fields
   - Monopole and dipole terms
   - Electric Field of a Dipole

Laplace's Equation: One Dimension

Laplace's Equation: One dimension

Laplace's Equation in one dimension:

$$\frac{d^2V}{dx^2} = 0 \tag{1}$$

What is the solution?

$$V(x) = mx + b \tag{2}$$

What is the magnitude of the E-field?

- A: $V(x)$
- B: x
- C: b
- D: m

Laplace's Equation: One dimension

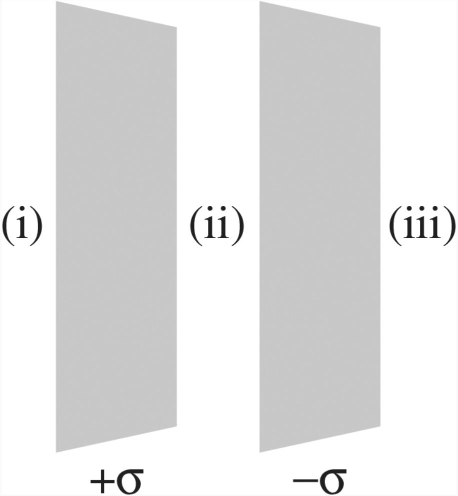

Figure 1: The setup of a parallel plate capacitor.

Laplace's Equation: One dimension

Suppose the negative side of the parallel plate capacitor is grounded, and the positive side is at a potential V_0. Let the separation between the plates be x_0. Further, let the positive plate occupy the yz plane, passing through the origin. Find the E-field magnitude and direction by solving Laplace's equation.

Laplace's Equation: One dimension

Show that the potential of a point charge at the origin satisfies Laplace's Equation for $r \neq 0$. *Use the form of the Laplacian in spherical coordinates.*

Boundary Conditions

Boundary Conditions

Let $V(x) = mx + b$. If $V(-a) = V_0$, and $V(a) = -V_0$, what are valid expressions for m and b?

- A: $b = 0$, and $m = -2V_0$
- B: $b = a$, and $m = V_0/a$
- C: $b = 0$, and $m = -V_0/a$
- D: $b = V_0$, and $m = -V_0/a$

Boundary Conditions

Let $V(x) = mx + b$. If $V(-a) = V_0$, and $V(a) = -V_0$, what is the electric field?

- A: $\frac{V_0}{a}\hat{x}$
- B: $-\frac{V_0}{a}\hat{x}$
- C: $V_0\hat{x}$
- D: $-V_0\hat{x}$

Boundary Conditions

Suppose a potential function $V(x, y) \propto (A \exp(-kx) + B \exp(kx))$. Which of the following is true, if $V \to 0$ as $x \to \infty$?

- A: A is 0
- B: A is 0
- C: A and B are 0
- D: Neither A nor B is 0

Boundary Conditions

Suppose a potential function $V(x,y) \propto (A\sin(kx) + B\cos(kx))$. Which of the following is true, if $V = 0$ as $x = 0$, and $V = 0$ as $x = a$?

- A: B is 0, and $k = n\pi$
- B: A is 0, and $k = n\pi/(2a)$
- C: A and B are 0
- D: B is 0, and $k = n\pi/a$

Boundary Conditions

Hyperbolic trigonometric functions:

- $\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$
- $\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$
- $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

Which of the following is zero?

- A: sinh(0)
- B: cosh(0)
- C: tanh(0)
- D: None

Which of the following is one?

- A: sinh(0)
- B: cosh(0)
- C: tanh(0)
- D: None

Hyperbolic trigonometric functions are solutions to which equation?

- A: $\frac{df}{dx} = k$
- B: $\frac{d^2 f}{dx^2} = kx$
- C: $\frac{d^2 f}{dx^2} = k^2 f$
- D: $\frac{d^2 f}{dx^2} = 0$

Boundary Conditions

Fourier's Trick: Imagine a vector with n components:

$$\vec{v} = \sum_{i=1}^{n} c_n \hat{x}_i \tag{3}$$

In words, how do you solve for some c_m?

- A: Divide by $\hat{x}_i$
- B: Take the dot product of both sides with $\hat{x}_m$
- C: Take the dot product $\vec{v}$ and $\vec{u}$, and the sum the series
- D: Integrate both sides with respect to x

Boundary Conditions

Fourier's Trick: Imagine a vector with n components:

$$\vec{v} = \sum_{i=1}^{n} c_n \hat{x}_i \tag{4}$$

In words, how do you solve for some c_m? Note that:

$$\vec{v} \cdot \hat{x}_m = \sum_{i=1}^{n} c_n \hat{x}_i \cdot \hat{x}_m = c_m \tag{5}$$

Why? Because

$$\hat{x}_i \cdot \hat{x}_j = 0 \tag{6}$$

$$\hat{x}_i \cdot \hat{x}_i = 1 \tag{7}$$

Boundary Conditions

Fourier's Trick: Imagine a known function that happens to be equal to a sum:

$$f(x) = \sum_{i=1}^{\infty} c_n g_n(x) \tag{8}$$

In words, how do you solve for some c_m?

- A: Multiply both sides by $g_m(x)$
- B: Multiply both sides by $g_m(x)$ and integrate both sides with respect to x
- C: Sum the infinite series and solve for c_m with algebra
- D: Integrate both sides with respect to x

Boundary Conditions

If it's true that a function can be written as an infinite series of functions with coefficients:

$$f(x) = \sum_{i=1}^{\infty} c_n g_n(x) \tag{9}$$

Then the functions $g_n(x)$ are said to be **complete**, or a complete basis (just like vectors are a sum of basis vectors. Examples of complete sets of functions:

- sines and cosines (Fourier series) with the right frequencies
- exponentials with the right rates multiplying x
- Hyperbolic trigonometric functions (follows from exponentials)
- Taylor series (polynomials with special coefficients: derivatives).

Boundary Conditions

The functions $g_n(x)$ are said to be **orthogonal** if

$$\int_0^a f_n(y) f_m(y) dy = \delta_{n,m} 0 \tag{10}$$

One example:

$$I_{n,m} = \int_L^{-L} \frac{\sin(n\pi x/L)}{\sqrt{L}} \frac{\sin(m\pi x/L)}{\sqrt{L}} dx \tag{11}$$

What is the result of this integral? How would you approach solving this?

Boundary Conditions

The **Fourier series** representation of a function $f(x)$ is written:

$$S(x) = \frac{A_0}{2} + \sum_{i=1}^{\infty} \left(A_n \cos(nx) + B_n \sin(nx) \right) \tag{12}$$

with

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx \tag{13}$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx \tag{14}$$

Boundary Conditions

Let's obtain the **Fourier series** coefficients A_n and B_n for a square-wave signal:

$$f(x) = 1, \quad 0 \leq x \leq \pi, \quad 0, \pi < x \leq 2\pi \tag{15}$$

(Observe on board). The result: $A_0 = 1.0$, all other $A_n = 0$, odd B_n values follow $2/(n\pi)$, even $B_n = 0$ as well.

Create octave code that plots this (see Moodle for example). Initially, plot a solution that has the first 5 coefficients.

Plot a solution with the first 20 coefficients. *Hint: can you find how to use the for-loop in octave?*

Separation of Variables

Separation of Variables

Laplaces' Equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{16}$$

Assume the solution follows

$$V(x, y, z) = X(x)Y(y)Z(z) \tag{17}$$

Laplaces' equation then breaks into three separate ordinary differential equations. Application of boundary conditions to solve them (Asynchronous video content on Moodle).

Conclusion

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