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EM Midterm.

1.1 1) a) Support A and B are 2 vector functions, what is $(A\cdot\nabla)B$?

$$(A\cdot(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}))B = (A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z})(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})$$

$$\boxed{=\hat{x}\left(A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_y}{\partial y}+A_z\frac{\partial B_z}{\partial z}\right)+\hat{y}\left(A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_y}{\partial y}+A_z\frac{\partial B_z}{\partial z}\right)+\hat{z}\left(A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_y}{\partial y}+A_z\frac{\partial B_z}{\partial z}\right)}$$

Note that A is not derivatived because operator order.

b) $(\hat{r}\cdot\nabla)\hat{r}$ $\quad \hat{r}=\frac{\vec{r}}{||r||}$. $\left(\frac{\partial}{\partial r}\right)\hat{r}=0$ Basis vector is a constant.

$$\boxed{(\hat{r}\cdot\nabla)\hat{r}=0}$$

c) $\vec{F}=(\vec{p}\cdot\nabla)\vec{E}$ with $\vec{p}=q\vec{d}=qd\hat{x}$ with $V(\vec{r})=V_0r^2+V_1$

$\vec{p}\cdot\nabla=qd\frac{\partial}{\partial x}$. $\quad \vec{F}=(qd\frac{\partial}{\partial x})\vec{E}$. $\quad \vec{E}=-\nabla V.=-\frac{\partial}{\partial r}(V_0r^2+V_1)$

$\vec{F}=(qd\frac{\partial}{\partial x})(-2V_0\vec{r})$ $\quad \vec{E}=-2V_0\vec{r}$

$=-2qdV_0\frac{\partial}{\partial x}\vec{r}=-2qdV_0\frac{\partial}{\partial x}(r_x\hat{x}+r_y\hat{y}+r_z\hat{z})=-2qdV_0\frac{\partial r_x}{\partial x}\hat{x}$

$$\boxed{\vec{F}=-2qdV_0\frac{\partial r_x}{\partial x}\hat{x}}$$

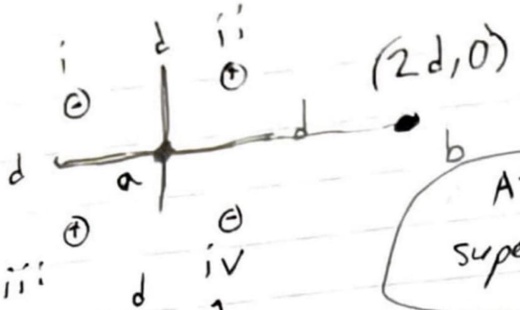
If $r=||r||$ then $\vec{F}=0$ because $\frac{\partial}{\partial x}$ constant $=0$.

1.2 2) $J=\int_V e^{-r}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)=\int_V e^{-r}(4\pi\delta^3(\vec{r}))d\vec{r}=4\pi\int_V e^{-r}\delta^3(r)dr$

Integrate over all space. ← Spike at $r=0$.

$J=4\pi\int_{-\infty}^{\infty}e^{-r}\delta^3(r)dr=4\pi(e^0)=4\pi \Rightarrow \boxed{J=4\pi}$

2.1 3]



At $P=(0,0)$ $\vec{E}=0$ by superposition and destructive interference.

Now $P=(2d,0)$

i $|\vec{\mathscr{R}}| = \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{5}{2}d\right)^2} = \sqrt{\frac{d^2}{4} + \frac{25d^2}{4}}$ $\quad \vec{E}_i = k\frac{-q}{\frac{d^2}{4}+\frac{25d^2}{4}}\hat{\mathscr{R}}$

So $\vec{E}_i = -\frac{kq\,(3d\hat{x} - \frac{d}{2}\hat{y})}{\left(\frac{d^2}{4}+\frac{25}{4}d^2\right)^{3/2}} = \boxed{\vec{E}_i = -\frac{kq\left(\frac{5}{2}d\hat{x} - \frac{d}{2}\hat{y}\right)}{\left(\frac{26d^2}{4}\right)^{3/2}}}$ $\quad \vec{\mathscr{R}} = 3d\hat{x} - \frac{d}{2}\hat{y}$

$\hat{\mathscr{R}} = \frac{3d\hat{x} - \frac{d}{2}\hat{y}}{\sqrt{\frac{d^2}{4}+\frac{25}{4}d^2}}$

E_{ii} time. $|\mathscr{R}| = \sqrt{\left(\frac{d}{2}\right)^2+\left(\frac{3d}{2}\right)^2} = \sqrt{\frac{d^2}{4}+\frac{9d^2}{4}}$

$|\mathscr{R}| = \sqrt{\frac{5d^2}{2}}$ $\quad = \sqrt{\frac{10d^2}{4}} = \sqrt{\frac{5d^2}{2}}$

$\vec{E}_{ii} = \frac{kq}{\frac{5d^2}{2}}\hat{\mathscr{R}}$ $\quad \hat{\mathscr{R}} = \frac{\vec{R}}{\|R\|} = \frac{\frac{3}{2}d\hat{x} - \frac{d}{2}\hat{y}}{\sqrt{\frac{5d^2}{2}}}$

$\boxed{\vec{E}_{ii} = \frac{kq\left(\frac{3}{2}d\hat{x} - \frac{d}{2}\hat{y}\right)}{\left(\frac{5d^2}{2}\right)^{3/2}}}$

E_{iii}: $|\mathscr{R}| = \sqrt{\left(\frac{d}{2}\right)^2+\left(\frac{5}{2}d\right)^2} = \sqrt{\frac{26d^2}{4}}$

$\vec{E}_{iii} = \frac{kq}{\frac{26d^2}{4}}\hat{\mathscr{R}}$ $\quad \hat{\mathscr{R}} = \frac{\frac{5}{2}d\hat{x} + \frac{d}{2}\hat{y}}{|\mathscr{R}|}$ $\quad \boxed{\vec{E}_{iii} = \frac{kq\left(\frac{5}{2}d\hat{x} + \frac{d}{2}\hat{y}\right)}{\left(\frac{26d^2}{4}\right)^{3/2}}}$

$\vec{E}_{iv}$. $|\mathscr{R}| = \sqrt{\left(\frac{d}{2}\right)^2+\left(\frac{3d}{2}\right)^2} = \sqrt{\frac{d^2}{4}+\frac{9d^2}{4}} = \sqrt{\frac{10d^2}{4}} = \sqrt{\frac{5d^2}{2}}$ Same as ii.

$\vec{E}_{iv} = -\frac{kq}{\frac{5d^2}{2}}\hat{\mathscr{R}} = -\frac{kq\left(\frac{3}{2}d\hat{x}+\frac{d}{2}\hat{y}\right)}{|\mathscr{R}|}$ $\quad \boxed{\vec{E}_{iv} = -\frac{kq\left(\frac{3}{2}d\hat{x}+\frac{d}{2}\hat{y}\right)}{\left(\frac{5d^2}{2}\right)^{3/2}}}$

$\boxed{\vec{E}_T = E_i + E_{ii} + E_{iii} + E_{iv}}$

$$\vec{E}_T = -\frac{kq\left(\frac{5}{2}d\hat{x}-\frac{d}{2}\hat{y}\right)}{\left(\frac{13}{2}d^2\right)^{3/2}} + \frac{kq\left(\frac{3}{2}d\hat{x}-\frac{d}{2}\hat{y}\right)}{\left(\frac{5d^2}{2}\right)^{3/2}} + \frac{kq\left(\frac{5}{2}d\hat{x}+\frac{d}{2}\hat{y}\right)}{\left(\frac{13}{2}d^2\right)^{3/2}} - \frac{kq\left(\frac{3}{2}d\hat{x}+\frac{d}{2}\hat{y}\right)}{\left(\frac{5d^2}{2}\right)^{3/2}}$$

$\boxed{\vec{E}_T = \frac{kqd}{\left(\frac{13}{2}d^2\right)^{3/2}}\hat{y} - \frac{kqd}{\left(\frac{5d^2}{2}\right)^{3/2}}\hat{y}}$ At $(0,2d)$ we have (rotate your paper 90°) what was − is now +

so $\boxed{\vec{E}_T = -\frac{kqd}{\left(\frac{13}{2}d^2\right)^{3/2}}\hat{x} + \frac{kqd}{\left(\frac{5d^2}{2}\right)^{3/2}}\hat{x}}$

2.2 4] $V(\vec{r}) = A\frac{e^{-\lambda r}}{r}$. $E = -\nabla V$. $E = A\left(\frac{+\lambda e^{-\lambda r} r + e^{-\lambda r}}{r^2}\right)\hat{r}$

$\vec{E} = Ae^{-\lambda r}\left(\frac{+\lambda r + 1}{r^2}\right)$ $\boxed{\vec{E} = +Ae^{-\lambda r}\left(\frac{\lambda r + 1}{r^2}\right)\hat{r}}$

$\rho = \epsilon_0 A\left(4\pi\delta^3(\vec{r}) - \frac{\lambda^2 e^{-\lambda r}}{r}\right)$ $\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$.

$\nabla\cdot\vec{E} = \left(Ae^{-\lambda r}(\lambda r+1)\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)\right) + A\frac{\hat{r}}{r^2}\cdot\nabla[e^{-\lambda r}(1+\lambda r)]$

1st term $= Ae^{-\lambda r}(1+\lambda r)4\pi\delta^3(r) = A(4\pi\delta^3(r))$

2nd term $= A\frac{\hat{r}}{r^2}\cdot\frac{\partial}{\partial r}[e^{-\lambda r}(1+\lambda r)]\hat{r} = A\frac{\hat{r}}{r^2}\cdot[-\lambda e^{-\lambda r}(1+\lambda r) + e^{-\lambda r}\lambda]\hat{r}$

$= A\frac{\hat{r}}{r^2}\cdot[-\lambda^2 r e^{-\lambda r}]\hat{r} = -A\frac{\lambda^2}{r}e^{-\lambda r}$

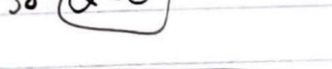
$$\boxed{\rho = \epsilon_0 A\left(4\pi\delta^3(r) - \frac{\lambda^2 e^{-\lambda r}}{r}\right)}$$

Now $Q = \int\rho\, d\tau = 4A\epsilon_0\pi\overset{1}{\cancel{\int\delta^3(\vec{r})\,d\tau}} - A\epsilon_0\lambda^2\int_0^\infty \frac{e^{-\lambda r}}{r}(4\pi r^2)\,dr$

$= 4A\pi\epsilon_0 - 4\pi A\epsilon_0\lambda^2\int_0^\infty re^{-\lambda r}\,dr = 4A\pi\epsilon_0 - 4\pi A\epsilon_0\lambda^2\left(\frac{1}{\lambda^2}\right) = 0$

So $\boxed{Q = 0}$

2.3 5]



$\oint E\cdot dA = \frac{Q_{enc}}{\epsilon_0} \Rightarrow EA = \frac{Q_{enc}}{\epsilon_0}$ $Q_{enc} = \lambda l$

$E(2\pi s\cancel{l}) = \frac{\lambda\cancel{l}}{\epsilon_0} \Rightarrow 2\pi s E = \frac{\lambda}{\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\lambda}{2\pi s\epsilon_0}\hat{s}}$

$\vec{F} = Q\vec{E}$

$m\dot{v} = QE$

$m\dot{v} = \frac{q\lambda}{2\pi s\epsilon_0}$

$\dot{v} = \frac{q\lambda}{2\pi s m\epsilon_0}$

So $\frac{d^2s}{dt^2} = \frac{q\lambda}{2\pi s m\epsilon_0}$

$\int\frac{d^2s}{dt^2} = \frac{q\lambda}{2\pi m\epsilon_0}\int\frac{1}{s}\,dt$

$\frac{ds}{dt} = \frac{q\lambda}{2\pi m\epsilon_0}\frac{t}{s}$

$\boxed{s(t) = \frac{q\lambda}{4\pi m\epsilon_0 s}t^2 + s}$

Makes sense because charge should fly away due to + and +.

3.1 6] On the surface of the sphere we have from 3.83 in the book,

$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta)=\frac{1}{\epsilon_0}\sigma_0(\theta) \text{ so}$$

$$\sigma(\theta)=\epsilon_0\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta)$$

But consider 3.69

$$A_l=\frac{2l+1}{2R^l}\int_0^{\pi}V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta$$

$$\text{So } A_l=\frac{2l+1}{2R^l}C_l$$

$$\sigma(\theta)=\frac{\epsilon_0}{2}\sum_{l=0}^{\infty}(2l+1)^2\frac{1}{R}C_l P_l(\cos\theta)$$

$$\boxed{\sigma(\theta)=\frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta)}$$

Now $V_0(\theta)=P_2(\cos\theta)$ $\quad C_l=\int_0^{\pi}P_2(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta$

All cancel except $l=2$ because orthogonal, orthogonal.

$$\text{So } C_2=\int_0^{\pi}(P_2(\cos\theta))^2\sin\theta\,d\theta=\frac{2}{5}$$

SAGEMATH (Nice)

$$\sigma(\theta)=\frac{\epsilon_0}{\cancel{2}R}(\cancel{5})^2\left(\frac{\cancel{2}}{\cancel{5}}\right)P_2(\cos\theta)=\frac{\epsilon_0}{R}(5)P_2(\cos\theta)=\frac{5\epsilon_0}{R}P_2(\cos\theta)$$

$$\boxed{\sigma(\theta)=\frac{5\epsilon_0}{R}\left[\frac{3}{2}\cos^2 x-\frac{1}{2}\right]}$$

3.2 7] Laplace's Equation $\nabla^2 V = 0$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

$$\begin{cases} i & V=0 \quad y=0 \\ ii & V=0 \quad x=-b \\ iii & V=0 \quad x=b \\ iv & V=V_0 \quad y=a \end{cases}$$

"The Physicist's First Line of Defense is a Seperable Solution" - Griffiths.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \Rightarrow \frac{d^2X}{dx^2} = -k^2X \quad \frac{d^2Y}{dy^2} = k^2Y$$

$\Rightarrow X = C\sin kx + D\cos kx \quad Y = Ae^{ky} + Be^{-ky}$

$i \Rightarrow A = -B.$ $\quad ii \Rightarrow -C\sin kb + D\cos kb = 0$ $\quad iii \Rightarrow C\sin kb + D\cos kb = 0$

$ii = iii \Rightarrow 2D\cos kx = 0 \Rightarrow D = 0.$ Now $C\sin kb = 0$

$kb = n\pi, n \in \mathbb{Z}.$

$k = \frac{n\pi}{b}$ Discrete k values.

With this we have

$$V(x,y) = A(e^{ky} - e^{-ky})(C\sin(\tfrac{n\pi}{b}x))$$

$$V(x,y) = A\sinh(\tfrac{n\pi}{b}y)C\sin(\tfrac{n\pi}{b}x)$$

$$V(x,y) = C_n\sinh(\tfrac{n\pi}{b}y)\sin(\tfrac{n\pi}{b}x).$$

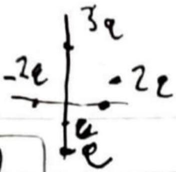
$$C_n\sinh(\tfrac{n\pi}{b}a) = \frac{2}{b}\int_0^b V_0\sin(\tfrac{n\pi}{b}x)\,dx = \frac{-2V_0}{n\pi}\left(\cos(\tfrac{n\pi}{b}x)\right)\Big|_0^b$$

$\cos(n\pi) = (-1)^n.$

$$\boxed{C_n = \begin{cases} \dfrac{4V_0}{n\pi\sinh(\frac{n\pi}{b}a)} & n = 2k+1 \quad k \in \mathbb{Z}. \\ 0 & \text{else} \end{cases}}$$

$$\boxed{\text{So } V(x,y) = \frac{4V_0}{\pi}\sum_{n=odd}\frac{\sinh(\frac{n\pi}{b}y)\sin(\frac{n\pi}{b}x)}{n\sinh(\frac{n\pi}{b}a)}}$$

3.3 8] Arrangement 1.



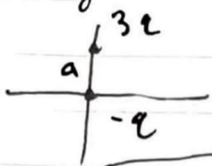
$$\vec{p} = (3qa - qa)\hat{z} + (-2qa + 2qa)\hat{y}$$

$$\vec{p} = 2qa\hat{z}$$

$$\boxed{V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}}$$

$$\boxed{V_{mon} = 0}$$

Arrangement 2.



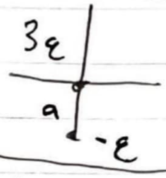
$$\boxed{V_{mon} = \frac{1}{4\pi\epsilon_0}\frac{2q}{r}}$$

$$\vec{p} = 3qa\hat{z}$$

$$\boxed{V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{3qa\cos\theta}{r^2}}$$

$$\boxed{V_{mon} + V_{dip} = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right]}$$

Arrangement 3.



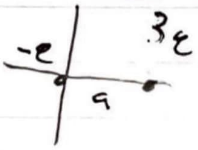
$$\boxed{V_{mon} = \frac{1}{4\pi\epsilon_0}\frac{2q}{r}}$$

$$\vec{p} = qa\hat{z}$$

$$\boxed{V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{qa\cos\theta}{r^2}}$$

$$\boxed{V_{mon} + V_{dip} = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right]}$$

Arrangement 4.



$$\boxed{V_{mon} = \frac{1}{4\pi\epsilon_0}\frac{2q}{r}}$$

$$\vec{p} = 3qa\cancel{\hat{z}}\hat{y}$$

$$\boxed{V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{3qa\sin\theta\sin\phi}{r^2}}$$

$$\boxed{V_{mon} + V_{dip} = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right]}$$