$T = c \cdot r$

$\nabla T = \nabla(c \cdot r) = c \times (\nabla \times r) + (c \cdot \nabla) r$

$\nabla \times r = 0$

$c = (c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z})$

$(c \cdot \nabla) r = (c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z})(x\hat{x} + y\hat{y} + z\hat{z})$

$= c_x \hat{x} + c_y \hat{y} + c_z \hat{z} = c$

$\int T \, dl = \oint (c \cdot r) \, dl$

$\oint T \, dl = \int_S \nabla T \times da$

$\oint (c \cdot r) \, dl = -\int_S c \times da = -c \times \int_S da$

$\oint (c \cdot r) \, dl = -c \times a = \boxed{a \times c}$

1.63) a) $v = \frac{\hat{r}}{r}$ $\quad v = v_r \hat{r} + v_\theta \hat{\theta} + v_\phi \hat{\phi}$

$v_r = \frac{1}{r} \quad v_\theta = 0 \quad v_\phi = 0$

$\nabla \cdot v = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_r) = \frac{1}{r^2} \frac{\partial}{\partial r}(r) = \boxed{\frac{1}{r^2}}$

$\int \nabla \cdot v \, d\tau = \int (\frac{1}{r^2}) 4\pi r^2 dr = 4\pi \int_0^R dr = 4\pi R$

$\int v \cdot da = (\frac{1}{R}\hat{r}) \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$

$= R \int \sin\theta \, d\theta \, d\phi = R \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi$

$= R(-\cos\theta|_0^\pi)(\phi|_0^{2\pi}) = 4\pi R$

$\int (\nabla \cdot v) d\tau = \int v \cdot da$ no delta function at origin

1.63 cont) a) $v_r = r^n$ $v_\theta = 0$ $v_\phi = 0$

$$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)$$

$$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+2})$$

$$= \frac{1}{r^2}(n+2)r^{n+1} = (n+2)r^{n-1}$$

$\boxed{\text{divergence of } r^n \hat{r} = (n+2)r^{n-1} \text{ for } n \neq -2}$

b) $$\nabla \times v = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right)\hat{\theta}$$

$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right)\hat{\phi}$$

$v_r = r^n$

$v_\theta = 0$

$v_\phi = 0$

$= 0$ $\boxed{\text{curl of } v = r^n \hat{r} = 0}$

$$\int(\nabla \times v)\,d\tau = -\oint v \times da$$

$= 0$

$\boxed{v \times da = 0 \quad \text{result is verified}}$

4) $$D(r,\varepsilon) \equiv -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\varepsilon^2}}$$

a) $$\nabla r^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

$$\nabla^2 \frac{1}{\sqrt{r^2+\varepsilon^2}} = \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right](r^2+\varepsilon^2)^{-1/2}$$

$$\frac{\partial}{\partial r}\left((r^2+\varepsilon^2)^{-1/2}\right) = -\frac{1}{2}(r^2+\varepsilon^2)^{-3/2}\,2r = -r(r^2+\varepsilon^2)^{-3/2}$$

$$\frac{\partial^2}{\partial r^2}\left((r^2+\varepsilon^2)^{-1/2}\right) = -(r^2+\varepsilon^2)^{-3/2} + \frac{3r}{2}(r^2+\varepsilon^2)^{-5/2}\,2r$$

$$= -(r^2+\varepsilon^2)^{-3/2} + 3r^2(r^2+\varepsilon^2)^{-5/2}$$

Dane Goodman

Electromagnetic Theory HW #1

1.54) $\mathbf{v} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

$\int(\nabla\cdot v)\,d\tau = \oint v\cdot da$

$\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

$\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r^2\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$

$= 4r\cos\theta + r\cos\phi\cot\theta - r\cot\theta\cos\phi$

$= 4r\cos\theta$

$\int(\nabla\cdot v)\,d\tau = \int_0^R\int_0^{\pi/2}\int_0^{\pi/2}(4r\cos\theta)r^2\,dr\,\sin\theta\,d\theta\,d\phi$

$= 4\int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi$

$= 4\left(\frac{1}{4}R^4\right)\left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{\pi/2}\left(\frac{\pi}{2}\right)$

$= \frac{\pi R^4}{4}$

Curved: $da = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

$r = R$

$v\cdot da = (R^2\cos\theta)(R^2\sin\theta\,d\theta\,d\phi)$

$\int v\cdot da = R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi$

$= R^4\left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{\pi/2}\left(\frac{\pi}{2}\right)$

$= \frac{\pi R^4}{4}$

Left: $da = -r\,dr\,d\theta\,\hat{\phi}$ $\quad \phi = 0$

$v\cdot da = (r^2\cos\theta\sin\phi)(r\,dr\,d\theta)$

$v\cdot da = (r^2\cos\theta\sin(0))(r\,dr\,d\theta) = 0$

Back: $da = r\,dr\,d\theta\,\hat{\phi}$ $\quad \phi = \frac{\pi}{2}$

$v\cdot da = (-r^2\cos\theta\sin\phi)(r\,dr\,d\theta)$

$= (-r^2\cos\theta\sin(\frac{\pi}{2}))(r\,dr\,d\theta)$

$= -r^3\cos\theta\,dr\,d\theta$

$\int v\cdot da = -\int_0^R r^3\,dr\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta$

$= -\left(\frac{R^4}{4}\right)\sin\theta\Big|_0^{\pi/2} = -\frac{1}{4}R^4$

1.54 cont)

Bottom: $da = r\,dr\,d\phi\,\hat{\theta}$ $\qquad \theta = \frac{\pi}{2}$

$v \cdot da = (r^2\cos\phi)(r\,dr\,d\phi)$

$\int v \cdot da = \int_0^R r^3\,dr \int_0^{\frac{\pi}{2}} \cos\phi\,d\phi \quad = \frac{1}{4}R^4$

total $\int v \cdot da = \frac{\pi R^4}{4} - 0 - \frac{R^4}{4} + \frac{R^4}{4} = \frac{\pi R^4}{4}$

$\int (\nabla \cdot v)\,d\tau = \int v \cdot da$ ✓

1.55) $v = ay\hat{x} + bx\hat{y}$ $\qquad x = R\cos\theta \qquad y = R\sin\theta$

$$\nabla \times V = \begin{vmatrix} x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & bx & 0 \end{vmatrix} = (b-a)\hat{z}$$

$da = \pi R^2\hat{z} \Rightarrow \int(\nabla \times V)\cdot da = \int (b-a)\hat{z} \cdot \pi R^2\hat{z}$

$= \pi R^2(b-a) \longrightarrow$ Eq 1

$dl = dx\hat{x} + dy\hat{y}$ $\qquad dx = -R\sin\theta\,d\theta \qquad dy = R\cos\theta\,d\theta$

$dl = -R\sin\theta\,d\theta\,\hat{x} + R\cos\theta\,d\theta\,\hat{y}$

$\int v \cdot dl = \int (ay\hat{x} + bx\hat{y})\cdot(-R\sin\theta\,d\theta\,\hat{x} + R\cos\theta\,d\theta\,\hat{y})$

$= -\int_0^{2\pi} aR^2\sin^2\theta\,d\theta + b\int_0^{2\pi} R^2\cos^2\theta\,d\theta$

$= -\frac{aR^2}{2}\int_0^{2\pi}(1-\cos 2\theta)\,d\theta + \frac{bR^2}{2}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta$

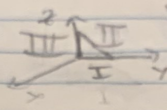
$= \frac{-aR^2}{2} \times (2\pi) + \frac{bR^2}{2} \times (2\pi)$

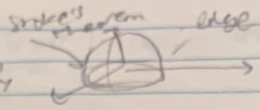
$= \pi R^2(b-a) \longrightarrow$ Eq 2

$\int(\nabla \times v)\cdot da = \oint v \cdot dl$

1.56) $V = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$

$\int \nabla \times \vec{v} \cdot d\vec{a} = \oint \vec{v} \cdot d\vec{l}$





Path I: $x=0$ $z=0$ $dx = dz = 0$ $d\vec{l} = +dy\,\hat{y}$

$\int_0^1 yz^2 dy = \int_I \vec{v} \cdot d\vec{l}$

$\int_I \vec{v} \cdot d\vec{l} = 0$

Path II: $z - z_1 = m(y - y_1)$

$m = \frac{z_2 - z_1}{y_2 - y_1} = \frac{2-0}{0-1} = -2$

$z - 0 = -2(y-1)$

$z = 2(1-y)$

$dz = -2dy$

$\int_{II} \vec{v} \cdot d\vec{l} = \int yz^2 dy + (3y+z)dz$

$= \int_0^1 [4y(1-y)^2 dy - [3y + 2(1-y)]2dy]$

$= y^2 + y^4 - \frac{8}{3}y^3 - 4y \Big|_1^0$

$= 2 + \frac{8}{3} = \frac{14}{3}$

Path III: $x=0$ $y=0$ $dx = dy = 0$ $dl = dz\,\hat{z}$

$\int_{III} \vec{v} \cdot d\vec{l} = \int_2^0 (3y+z)\,dz$

$= \int_2^0 3((0)+z)\,dz$

$= \int_2^0 z\,dz$

$= \frac{1}{2}z^2 \Big|_2^0 = -2$

total: Path I + Path II + Path III

$= 0 + \frac{14}{3} - 2 = \boxed{\frac{8}{3}}$

$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 6 & yz^2 & 3y+z \end{vmatrix}$

$= \left(\frac{\partial}{\partial y}(3y+z) - \frac{\partial}{\partial z}(yz^2)\right)\hat{x} - \left(\frac{\partial}{\partial x}(3y+z) - \frac{\partial}{\partial z}(6)\right)\hat{y} + \left(\frac{\partial}{\partial x}(yz^2) - \frac{\partial}{\partial y}(6)\right)\hat{z}$

$= (3 - 2yz)\hat{x}$

1.56 cont) $\int_S \nabla \times \vec{v} \cdot d\vec{a} = \iint (3-2yz)\,\hat{x} \cdot (dy\,dz)\,\hat{x}$

$= \int_0^{2(1-y)} \int_0^{y} (3-2yz)\,dy\,dz$

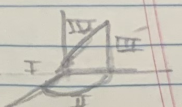
$= \int_0^1 \left[3z - yz^2\right]_0^{2(1-y)} dy$

$= \int_0^1 \left(3(2(1-y)) - y(2(1-y))^2\right) dy$

$= \int_0^1 (6 - 10y - 4y^3 + 8y^2)\,dy$

$= 6y - 5y^2 - y^4 + \frac{8}{3}y^3 \Big|_0^1 = \boxed{\frac{8}{3}}$

1.57) $v = (r\cos^2\theta)\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}$



I: $\theta = \frac{\pi}{2},\ \phi = 0$

$v \cdot dl = \left[(r\cos^2\theta)\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}\right] \cdot \left[dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}\right]$

$= (r\cos^2\theta)\,dr - (r^2\cos\theta\sin\theta)\,d\theta + (3r^2\sin\theta)\,d\phi$

$v \cdot dl\ (\frac{\pi}{2}) = 0$

$\int_0^1 v \cdot dl = 0$

II: $r = 1,\ \theta = \frac{\pi}{2},\ \phi : 0 \to \frac{\pi}{2}$

$v \cdot dl = ((1)\cos^2(\frac{\pi}{2}))\,dr - ((1)^2\cos(\frac{\pi}{2})\sin(\frac{\pi}{2}))\,d\theta + (3r^2\sin(\frac{\pi}{2}))\,d\phi$

$= 3\,d\phi$

$\int_0^{\frac{\pi}{2}} 3\,d\phi = \frac{3\pi}{2}$

III: $r = \frac{1}{\sin\theta} \longrightarrow dr = -\frac{1}{\sin^2\theta}\cos\theta\,d\theta,\ \phi = \frac{\pi}{2},\ \theta = \frac{\pi}{2} \to \tan^{-1}(\frac{1}{2})$

$v \cdot dl = (r\cos^2\theta)\,dr - (r^2\cos\theta\sin\theta)\,d\theta + 0 = \left(\frac{1}{\sin\theta}\cos^2\theta\right)\left(\frac{-\cos\theta}{\sin^2\theta}d\theta\right) - \left(\frac{1}{\sin^2\theta}\cos\theta\sin\theta\right)d\theta$

$= -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right)d\theta = -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)d\theta = -\frac{\cos\theta}{\sin^3\theta}\,d\theta$

$\int v \cdot dl = -\int \frac{1}{x^3}\,dx \qquad x = \sin\theta \to dx = \cos\theta\,d\theta$

$= \frac{1}{2x^2} = \frac{1}{2\sin^2\theta}$

1.57 cont) $\int v \cdot dl = \frac{1}{2\sin^2\theta}\Big|_{\frac{\pi}{2}}^{\tan^{-1}(\frac{1}{2})}$

$= \frac{1}{2(0.2)} - \frac{1}{2} = 2$

IV : $r = \sqrt{5} \to 0$, $\theta = \tan^{-1}(\frac{1}{2})$, $\phi = \frac{\pi}{2}$

$v \cdot dl = (r\cos^2(\tan^{-1}(\frac{1}{2})))dr = 0.8\,r\,dr$

$\int v \cdot dl = \int_{\sqrt{5}}^{0} 0.8\,r\,dr = 0.8\int_{\sqrt{5}}^{0} r\,dr = 0.8(\frac{1}{2}r^2)\Big|_{\sqrt{5}}^{0} = -2$

total $= 0 + \frac{3\pi}{2} + 2 - 2 = \boxed{\frac{3\pi}{2}}$

Stokes Theorem: $\nabla \times v = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right)\hat{r}$

$+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right)\hat{\theta}$

$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right)\hat{\phi}$

$= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta(3r)) - \frac{\partial(r\cos\theta\sin\theta)}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial(r\cos^2\theta)}{\partial\phi} - \frac{\partial}{\partial r}(r(3r))\right)\hat{\theta}$

$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r(r\cos\theta\sin\theta)) - \frac{\partial(r\cos^2\theta)}{\partial\theta}\right)\hat{\phi}$

$= \frac{1}{r\sin\theta}(3r)(\cos\theta)\hat{r} + \frac{1}{r}(-6r)\hat{\theta} + 0 = 3\cot\theta\,\hat{r} - 6\hat{\theta}$

$\int(\nabla\times v)\cdot da = 0 + \int_0^1 6r\,dr + \int_0^{\frac{\pi}{2}} d\phi = 6(\frac{1}{2}r^2)\Big|_0^1 (\phi)\Big|_0^{\frac{\pi}{2}} = \boxed{\frac{3\pi}{2}}$

$\oint v \cdot dl = \int(\nabla \cdot v)\cdot da$ ✓

1.59) $v = r^2\sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r^2\tan\theta\,\hat{\phi}$

$\int(\nabla\cdot v)d\tau = \oint v \cdot da$

$\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

$\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2(r^2\sin\theta)) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}((\sin\theta)(4r^2)(\cos\theta)) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\tan\theta)$

$= \frac{1}{r^2}(4r^3\sin\theta) + \frac{1}{r\sin\theta}4r^2(\cos 2\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\tan\theta)$

$= \frac{4r}{\sin\theta}(\sin^2\theta + \cos^2\theta - \sin^2\theta) = 4r\frac{\cos^2\theta}{\sin\theta}$

$\int(\nabla\cdot v)d\tau = \int(4r\frac{\cos 2\theta}{\sin\theta})r^2\sin\theta\,dr\,d\theta\,d\phi$

$= \int_0^R 4r^3dr\int_0^{\frac{\pi}{6}}\cos^2\theta\,d\theta\int_0^{2\pi}d\phi = 2\pi R^4[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta]\Big|_0^{\frac{\pi}{6}}$

$= 2\pi R^4(\frac{\pi}{12} + \frac{1}{4}\sin 60) = 2\pi R^4\left(\frac{\pi + 3\sqrt{3}}{12}\right)$

1.59 cont) $\frac{1}{6}\pi R^4\left(\pi + \frac{3\sqrt{3}}{2}\right)$ ①

Ice cream: $r = R$, $\phi: 0 \to 2\pi$, $\theta: 0 \to \frac{\pi}{6}$

$da = R^2 \sin\theta \, d\phi \, d\theta \, \hat{r}$

$v \cdot da = (R^2 \sin\theta) R^2 \sin\theta \, d\theta \, d\phi$

$\int v \cdot da = \int_0^{\frac{\pi}{6}} \int_0^{2\pi} R^4 \sin^2\theta \, d\theta \, d\phi = 2\pi R^4 \int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$

$= 2\pi R^4 \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right|_0^{\frac{\pi}{6}}$

$= 2\pi R^4 \left[\frac{\pi}{12} - \frac{1}{4}\sin 60\right] = \frac{1}{6}\pi R^4\left(\pi - \frac{3\sqrt{3}}{2}\right)$

cone: $r: 0 \to R$, $\phi: 0 \to 2\pi$, $\theta = \frac{\pi}{6}$

$da = r\sin\theta \, d\phi \, dr \, \hat{\theta}$

$v \cdot da = 4r^3 \cos\theta \sin\theta \, d\phi \, dr$

$\int v \cdot da = 4\int_0^R r^3 \, dr \int_0^{2\pi} \cos\theta \sin\theta \, d\phi \, dr$

$= \sqrt{3} \, 2\pi \int_0^R r^3 \, dr = \frac{\sqrt{3}}{2}\pi R^4$

$\int v \cdot da = \frac{1}{6}\pi R^4\left(\pi - \frac{3\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\pi R^4$

$= \frac{1}{6}\pi R^4\left(\pi - \frac{3\sqrt{3}}{2} + 3\sqrt{3}\right) = \frac{1}{6}\pi R^4\left(\pi + \frac{3\sqrt{3}}{2}\right)$ ②

$\boxed{① = ② \Rightarrow \int (\nabla \cdot v) \, d\tau = \int v \cdot da}$

1.62) a) $da = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$a = \int R^2 \sin\theta \, d\theta \, d\phi \, \hat{z} = 2\pi R^2 \hat{z} \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \, d\theta$

$\boxed{a = \pi R^2 \hat{z}}$

b) $\int_V (\nabla T) \, d\tau = \oint T \, da$

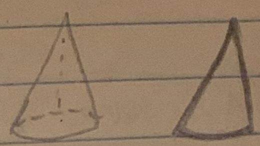
$\nabla T = 0$, $\oint da = 0$

$\boxed{a = 0}$

1.62 cont) c) $a_1 \neq a_2$

$$\boxed{\oint da = a_1 - a_2 \neq 0}$$

d) $da = \frac{1}{2}(r \times dl)$



$$\boxed{a = \frac{1}{2}\oint r \times dl}$$

e) $\int_S \nabla T \cdot da = -\oint T\,dl$

$T = c \cdot r$

$\nabla T = \nabla(c \cdot r) = c \times (\nabla \times r) + (c \cdot \nabla) r$

$\nabla \times r = 0$

$c = (c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z})$

$(c \cdot \nabla) r = (c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z})(x\hat{x} + y\hat{y} + z\hat{z})$

$= c_x \hat{x} + c_y \hat{y} + c_z \hat{z} = c$

$\int T\,dl = \oint (c \cdot r)\,dl$

$\oint T\,dl = \int_S \nabla T \times da$

$\oint (c \cdot r)\,dl = -\int_S c \times da = -c \times \int_S da$

$\oint (c \cdot r)\,dl = -c \times a = \boxed{a \times c}$

1.63) a) $v = \frac{\hat{r}}{r}$ $\quad v = v_r \hat{r} + v_\theta \hat{\theta} + v_\phi \hat{\phi}$

$v_r = \frac{1}{r} \quad v_\theta = 0 \quad v_\phi = 0$

$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r) = \boxed{\frac{1}{r^2}}$

$\int \nabla \cdot v\,d\tau = \int (\frac{1}{r^2}) 4\pi r^2\,dr = 4\pi \int_0^R dr = 4\pi R$

$\int v \cdot da = (\frac{1}{R}\hat{r}) \cdot (R^2 \sin\theta\,d\theta\,d\phi\,\hat{r})$

$= R \int \sin\theta\,d\theta\,d\phi = R \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi$

$= R(-\cos\theta)\Big|_0^\pi (\phi)\Big|_0^{2\pi} = 4\pi R$

$\int (\nabla \cdot v)\,d\tau = \int v \cdot da$ no delta function at origin

1.63 cont) a) $v_r = r^n \quad v_\theta = 0 \quad v_\phi = 0$

$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)$

$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+2})$

$= \frac{1}{r^2}(n+2)r^{n+1} = (n+2)r^{n-1}$

$\boxed{\text{divergence of } r^n \hat{r} = (n+2)r^{n-1} \text{ for } n \neq -2}$

b) $\nabla \times v = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right)\hat{\theta}$

$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right)\hat{\phi}$

$v_r = r^n$

$v_\theta = 0$

$v_\phi = 0$

$= 0 \quad \boxed{\text{curl of } v = r^n \hat{r} = 0}$

$\int(\nabla \times v)d\tau = -\int v \times da$

$= 0$

$\boxed{v \times da = 0 \qquad \text{result is verified}}$

1.64) $D(r,\varepsilon) \equiv -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\varepsilon^2}}$

a) $\nabla r^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$

$\nabla^2 \frac{1}{\sqrt{r^2+\varepsilon^2}} = \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right](r^2+\varepsilon^2)^{-1/2}$

$\frac{\partial}{\partial r}\left((r^2+\varepsilon^2)^{-1/2}\right) = -\frac{1}{2}(r^2+\varepsilon^2)^{-3/2}2r = -r(r^2+\varepsilon^2)^{-3/2}$

$\frac{\partial^2}{\partial r^2}\left((r^2+\varepsilon^2)^{-1/2}\right) = -(r^2+\varepsilon^2)^{-3/2} + \frac{3r}{2}(r^2+\varepsilon^2)^{-5/2}2r$

$= -(r^2+\varepsilon^2)^{-3/2} + 3r^2(r^2+\varepsilon^2)^{-5/2}$

$D(r,\varepsilon) = -\frac{1}{4\pi}\left[-(r^2+\varepsilon^2)^{-3/2} + 3r^2(r^2+\varepsilon^2)^{-5/2} - 2(r^2+\varepsilon^2)^{-3/2}\right]$

$= \boxed{\frac{3\varepsilon^2}{4\pi}(r^2+\varepsilon^2)^{-5/2}}$

1.64 (cont) b) $D(0,\varepsilon) = \frac{3\varepsilon^2}{4\pi}\varepsilon^{-5} = \boxed{\frac{3}{4\pi\varepsilon^3}}$

$\boxed{\text{as } \varepsilon \to 0,\ D(0,\varepsilon) \to \infty}$

c) $\boxed{\text{as } \varepsilon \to 0 : \frac{3\varepsilon^2}{4\pi} = 0 \therefore D(r,\varepsilon) \to 0}$

d) as $\varepsilon \to 0$, $D(r,\varepsilon) = \delta^3(r)$

$\int D(r,\varepsilon)\,dr = \int \delta^3(r)\,dr = \boxed{1}$