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02/07/22

EMT HW # 1

1.54 Divergence theorem of

$$v = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$$

$$\nabla\cdot v = \frac{1}{r^2}\frac{d}{dr}\left(r^2 r^2\cos\theta\right) + \frac{1}{r\sin\theta}\frac{d}{d\theta}\left(\sin\theta\, r^2\cos\phi\right) + \frac{1}{r\sin\theta}\frac{d}{d\phi}\left(-r^2\cos\theta\sin\phi\right)$$

$$= \frac{1}{r^2}4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta\, r^2\phi + \frac{1}{r\sin\theta}\left(-r^2\cos\theta\cos\phi\right)$$

$$\nabla\cdot v = \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta + \cos\phi - \cos\phi\right] \Rightarrow \boxed{\nabla\cdot v = 4r\cos\theta}$$

$$\int(\nabla\cdot v)\,d\tau = \int(4r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 4\int_0^R r^3dr\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi$$

$$\Rightarrow (R^4)\left(\tfrac{1}{2}\right)\left(\tfrac{\pi}{2}\right) \Rightarrow \boxed{\pi R^4/4}$$

Check: 4 sides: Curved part, Left, Back, & Bottom

Curved $\Rightarrow da = r^2\sin\theta\,d\theta\,d\phi\,\hat{r} \Rightarrow r = R$

$$\Rightarrow v\cdot da = (R^2\cos\theta)(R^2\sin\theta\,d\theta\,d\phi)$$

$$\Rightarrow \int v\cdot da = R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi$$

$$= R^4\left(\tfrac{1}{2}\right)\left(\tfrac{\pi}{2}\right) = \boxed{\frac{\pi R^4}{4}}$$

1.54 cont'd

Left $\Rightarrow da = -r\,dr\,d\theta\,\hat{\phi}$

$\phi = 0$

$\Rightarrow V \cdot da = -(r^2\cos\theta\sin(0))(r\,dr\,d\theta) = \underline{0}$

$\int V \cdot da = 0$

Back side $\Rightarrow da = r\,dr\,d\theta\,\phi$

$\phi = \pi/2$

$\Rightarrow V \cdot da = (-r^2\cos\theta\sin(\frac{\pi}{2}))(r\,dr\,d\theta) = -r^3\cos\theta\,dr\,d\theta$

$\Rightarrow \int V \cdot da = \int_0^R r^3\,dr \int_0^{\pi/2}\cos\theta\,d\theta$

$\Rightarrow -(\frac{1}{4}R^4)(1) = \underline{-\frac{R^4}{4}}$

Bottom $\Rightarrow da = r\sin\theta\,dr\,d\phi\,\hat{\theta}$

$\theta = \pi/2 \Rightarrow V \cdot da = (r^2\cos\phi)(r\,dr\,d\phi)$

$\Rightarrow \int V \cdot da = \int_0^R r^3\,dr \int_0^{\pi/2}\cos\phi\,d\phi = \underline{\frac{R^4}{4}}$

Entire Sphere:

$\oint V \cdot da = \frac{\pi R^4}{4} + 0 \cancel{- \frac{R^4}{4}} \cancel{+ \frac{R^4}{4}}$

$\boxed{= \frac{\pi R^4}{4}}$

1.55| Check Stokes' theorem using the function $v = ay\hat{x} + bx\hat{y}$ (a & b are constants) and the circular path of radius R, centered at the origin in the xy plane.

$$\nabla \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ ay & bx & 0 \end{vmatrix} = \hat{k}(b-a)$$

$$\int (\nabla \times v)\, da = (b-a)\pi R^2$$

$$v \cdot dl = (ay\hat{i} + bx\hat{j}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$
$$= (ay\,dx + bx\,dy)$$

$$\Rightarrow x^2 + y^2 = R^2 \Rightarrow 2x\,dx + 2y\,dy = 0$$

$$\Rightarrow dy = \left(-\frac{x}{y}\right) dx.$$

$$v \cdot dl = ay\,dx + bx\left(-\frac{x}{y}\right) dx$$
$$= \frac{1}{y}(ay^2 - bx^2)\, dx.$$

Semicircle: $y = \sqrt{R^2 - x^2}$, & replace

$$\Rightarrow \frac{a(R^2 - x^2) - bx^2}{\sqrt{R^2 - x^2}}\, dx$$

1.55 (cont)

$$\Rightarrow \int v \cdot dl = \int_{R}^{-R} \frac{aR^2-(a+b)x^2}{\sqrt{R^2-x^2}}\,dx$$

$$= \left\{ aR^2 \sin^{-1}\left(\frac{x}{R}\right) - (a+b)\left[-\frac{x}{2}\sqrt{R^2-x^2} + \frac{R^2}{2}\sin^{-1}\left(\frac{x}{R}\right)\right]\right\}\Big|_{-R}^{+R}$$

$$\Rightarrow \frac{1}{2}R^2(a-b)\sin^{-1}\left(\frac{x}{R}\right)\Big|_{+R}^{-R}$$

$$\Rightarrow \frac{1}{2}R^2(a-b)\left(\sin^{-1}(-1)-\sin^{-1}(1)\right)$$

$$\Rightarrow \frac{1}{2}R^2(a-b)\left(-\frac{\pi}{2}-\frac{\pi}{2}\right) = \frac{1}{2}R^2(a-b)\left(-\frac{\pi}{2}-\frac{\pi}{2}\right)$$

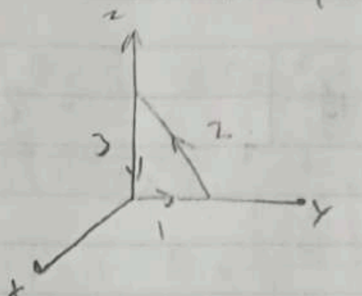
$$= \frac{1}{2}\pi R^2(b-a)$$

Lower semicircle $\Rightarrow$ reversed integral signs so

$$\boxed{\oint v \cdot dl = \pi R^2(b-a)}$$

1.56) Compute line integral of

$$V = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$$



Side 1): $x = 0,\ z = 0,\ dx = 0,\ dz = 0$

$y \to 1$

$V \cdot dl = (yz^2)dy = 0 \Rightarrow \underline{\int V \cdot dl = 0.}$

Side 2): $x = 0;\ z = 2-2y,\ dz = -2dy,\ y \to 0$

$V \cdot dl = (yz^2)dy + (3y+z)dz$

$\Rightarrow y(2-2y)^2 dy - (3y+2-2y)2dy;$

$$\Rightarrow \int V \cdot dl = 2\int_1^0 (2y^3 - 4y^2 + y - 2)\, dy$$

$$\Rightarrow 2\left(\frac{y^4}{2} - \frac{4y^3}{3} + \frac{y^2}{2} - 2y\right)\Big|_1^0$$

$$= \underline{\frac{14}{3}}$$

Side 3): $x=0,\ y=0,\ dx=0,\ dy=0,\ z \to 2 \to 0$

$$\Rightarrow v \cdot dl = (3y+z)\,dz = z\,dz$$

$$\Rightarrow \int v \cdot dl = \int_2^0 z\,dz = \frac{z^2}{2}\Big|_2^0 = \underline{-2}$$

Entire Triangle: $0 + \frac{14}{3} - 2 = \boxed{\frac{8}{3}}$

Stokes' Theorem: $\oint v \cdot dl = \int (\nabla \times v)\,da$

$da = dy\,dz\,\hat{x}$

$$(\nabla \times v) = \frac{d}{dy}(3y+z) - \frac{d}{dz}(yz^2) = 3 - 2yz$$

$$\Rightarrow \int (\nabla \times v)\cdot da = \iint (3-2yz)\,dy\,dz$$

$$\Rightarrow \int_0^1 \left[\int_0^{2-2y} (3-2yz)\,dz\right] dy \Rightarrow \int_0^1 \left[3(2-2y) - 2y\tfrac{1}{2}(2-2y)^2\right] dy$$

$$= \int_0^1 \left[\int_0^{2-2y} (3-2yz)\,dz\right] dy$$

$$= \int_0^1 \left[3(2-2y) - 2y\tfrac{1}{2}(2-2y)^2\right] dy$$

$$= \int_0^1 (-4y^3 + 8y^2 - 10y + 6)\,dy$$

$$= \left(-y^4 + \frac{8}{3}y^3 - 5y^2 + 6y\right)\Big|_0^1 \Rightarrow$$

$$-1 + \frac{8}{3} - 5 + 6 = \boxed{\frac{8}{3}}$$

1.57] Compute the line integral of

$$v = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}$$

Side 1): $\theta = \frac{\pi}{2}$, $\phi = 0$; $r = 0 \to 1$

$v \cdot dl = (r\cos^2\theta)(dr) = 0$, $dr = 0 \Rightarrow \underline{\int v \cdot dl = 0}$

Side 2): $r = 1$, $\theta = \frac{\pi}{2}$, $\phi = 0 \to \frac{\pi}{2}$

$v \cdot dl = (3r)(r\sin\theta\, d\phi) = 3\, d\phi$

$\int v \cdot dl = 3\int_0^{\pi/2} d\phi \Rightarrow \underline{\frac{3\pi}{2}}$

Side 3): $\phi = \frac{\pi}{2}$, $r\sin\theta = 1$, $y = 1$

$r = \frac{1}{\sin\theta}$, $dr = \frac{-1}{\sin^2\theta}\cos\theta\, d\theta$, $\theta = \frac{\pi}{2} \to \theta_0 \Rightarrow \tan^{-1}(\frac{1}{2})$

$v \cdot dl = (r\cos^2\theta)(dr) - (r\cos\theta\sin\theta)(r\, d\theta)$

$\Rightarrow \frac{\cos^2\theta}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\right)d\theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}\, d\theta$

$\Rightarrow -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right)d\theta = -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)d\theta$

$= \frac{-\cos\theta}{\sin^3\theta}\, d\theta$

$\Rightarrow \int v \cdot dl = -\int_{\pi/2}^{\theta_0} \frac{\cos\theta}{\sin^3\theta}\, d\theta = \frac{1}{2\sin^2\theta}\Big|_{\pi/2}^{\theta_0}$

$= \frac{1}{2(\frac{1}{5})} - \frac{1}{2(1)} = \frac{5}{2} - \frac{1}{2} = \underline{2}$

Side 4): $\theta = \theta_0,\ \phi = \frac{\pi}{2},\ r \to \sqrt{5} \to 0$

$\Rightarrow v \cdot dl = (r\cos^2\theta)(dr) = \frac{4}{5} r\,dr$

$\Rightarrow \int v \cdot dl = \frac{4}{5}\int_{\sqrt{5}}^{0} r\,dr = \frac{4}{5}\frac{r^2}{2}\Big|_{\sqrt{5}}^{0} = -\frac{4}{5}\cdot\frac{5}{2} = \underline{-2}$

All sides: $= 0 + \frac{3\pi}{2} + 2 - 2 = \boxed{\frac{3\pi}{2}}$

$\nabla \times v = \frac{1}{r\sin\theta}\left[\frac{d}{d\theta}(\sin\theta\, 3r) - \frac{d}{d\phi}(-r\sin\theta\cos\theta\right]\hat{r} +$

$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{d}{d\phi}(r\cos^2\theta) - \frac{d}{dr}(r\,3r)\right]\hat{\theta}$

$+ \frac{1}{r}\left[\frac{d}{dr}(-rr\cos\theta\sin\theta) - \frac{d}{d\theta}(r\cos^2\theta)\right]\hat{\phi}$

$= 3\cos\theta\,\hat{r} - 6\hat{\theta}$

Back): $da = -r\,dr\,d\theta\,\hat{\phi}$

$(\nabla \times v)\cdot da = 0 \qquad \int(\nabla \times v)\cdot da = 0$

Bottom): $da = -r\sin\theta\,dr\,d\phi\,\hat{\theta}$

$(\nabla \times v)\cdot da = 6r\sin\theta\,dr\,d\phi$

$\theta = \frac{\pi}{2}$

$,\ (\nabla \times v)\cdot da = 6r\,dr\,d\phi$

$\int(\nabla \times v)\cdot da = \int_0^1 6r\,dr \int_0^{\pi/2} d\phi =$

$= 6\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \boxed{\frac{3\pi}{2}}$

1.59) Check the divergence theorem for the function

$$v = r^2 \sin\theta \,\hat{r} + 4r^2 \cos\theta \,\hat{\theta} + r^2 \tan\theta \,\hat{\phi}$$

$$\nabla \cdot v = \frac{1}{r^2} \frac{d}{dr}\left(r^2 r^2 \sin\theta \right) + \frac{1}{r\sin\theta} \frac{d}{d\theta}\left(\sin\theta \, 4r^2 \cos\theta \right)$$

$$+ \frac{1}{r\sin\theta} \frac{d}{d\phi}\left(r^2 \tan\theta \right)$$

$$= \frac{1}{r^2} 4r^3 \sin\theta + \frac{1}{r\sin\theta} 4r^2 \left(\cos^2\theta - \sin^2\theta \right)$$

$$= \frac{4r}{\sin} \left(\sin^2\theta + \cos^2\theta - \sin^2\theta \right)$$

$$= 4r \frac{\cos^2\theta}{\sin\theta} \Rightarrow \int (\nabla \cdot v) \, d\tau$$

$$\Rightarrow \int \left(4r \frac{\cos^2\theta}{\sin\theta} \right) \left(r^2 \sin\theta \, dr \, d\theta \, d\phi \right)$$

$$\Rightarrow \int_0^R 4r^3 dr \int_0^{\pi/6} \cos^2\theta \, d\theta \int_0^{2\pi} d\phi = (R^4)(2\pi) \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right] \Bigg|_0^{\pi/6}$$

$$= 2\pi R^4 \left(\frac{\pi}{12} + \frac{\sin(60)}{4} \right)$$

$$\Rightarrow \frac{\pi R^4}{6} \left(\pi + 3\frac{\sqrt{3}}{2} \right) = \frac{\pi R^4}{12} \left(2\pi + 3\sqrt{3} \right)$$

1.59 cont

Surfaces: Ice cream & Cone

Ice-Cream): $r = R$, $\phi = 0 \to 2\pi$, $\theta = \frac{\pi}{6}$,

$da = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$,

$$v \cdot da = (R^2 \sin\theta)(R^2 \sin\theta \, d\theta \, d\phi) = R^4 \sin^2\theta \, d\theta \, d\phi$$

$$\int v \cdot da = R^4 \int_0^{\pi/6} \sin^2\theta \, d\theta \int_0^{2\pi} d\phi = (R^4)(2\pi)\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= 2\pi R^4 \left(\frac{\pi}{12} - \frac{1}{4}\sin 60\right) = \underline{\frac{\pi R^4}{6}\left(\pi - 3\frac{\sqrt{3}}{2}\right)}$$

Cone): $\theta = \frac{\pi}{6}$, $\phi = 0 \to 2\pi$, $r = 0 \to R$

$$da = r \sin\theta \, d\phi \, dr \, \hat{\theta} = \frac{\sqrt{3}}{2} r \, dr \, d\phi \, \hat{\theta}$$

$$v \cdot da = \sqrt{3} r^3 \, dr \, d\phi$$

$$\int v \cdot da = \sqrt{3} \int_0^R r^3 dr \int_0^{2\pi} d\phi = \sqrt{3} \cdot \frac{R^4}{4} \cdot 2\pi = \underline{\frac{\sqrt{3}}{2}\pi R^4}$$

$$\int v \cdot da = \frac{\pi R^4}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \sqrt{3}\right)$$

$$\boxed{= \frac{\pi R^4}{12}\left(2\pi + 3\sqrt{3}\right)}$$

1.62) The integral $a = \int_S da$

a) Vector area of a hemispherical bowl of radius R

Sphere: $d\vec{a} = R^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}$

$\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$

$\theta = 0 \to \frac{\pi}{2}$

$\vec{a} = \int R^2 \sin^2\theta \, d\theta \, d\varphi \cos\varphi \,\hat{x} + \int R^2 \sin^2\theta \sin\varphi \, d\theta \, d\varphi \,\hat{y}$

$+ \int R^2 \sin\theta\cos\theta \, d\theta \, d\varphi$

$\vec{a} = R^2 \cdot 2\pi \cdot \int_0^{\pi/2} \sin\theta\cos\theta \, d\theta \,\hat{z}$

$a = \pi R^2 \int_0^{\pi/2} \sin 2\theta \, d\theta \,\hat{z} \Rightarrow \boxed{\vec{a} = \pi R^2 \hat{z}}$

b) Show that $a = 0$ for any closed surface

$1.61\,c) = \int \nabla T = \oint T\, d\vec{a}$, $\boxed{\text{so } T = 1,\ \nabla T = 0}$

$\nabla T = 0 \Rightarrow \boxed{\oint d\vec{a} = 0}$

c)

d) $a = \frac{1}{2}\oint r \times dl$

were $\frac{1}{2}\pi r$?

?

e) $\oint (c \cdot r)\, dl = a \times c$

$T = \vec{c} \cdot \vec{r} \Rightarrow \vec{\nabla}T = \vec{\nabla}(\vec{c} \cdot \vec{r})$

$\Rightarrow \vec{c} \times (\vec{\nabla} \times \vec{r}) + (\vec{c} \cdot \vec{\nabla})\vec{r}$

$\Rightarrow \vec{\nabla} \times \vec{r} = 0$

$\Rightarrow (\vec{c} \cdot \vec{\nabla})\vec{r} = c_x\hat{x} + c_y\hat{y} + c_z\hat{z} = \vec{c}$

$\underline{\vec{\nabla}T = \vec{c}}$

$\oint T\, d\vec{l} = \oint (\vec{c} \cdot \vec{r})\, dl = -\int (\vec{\nabla}T) \times d\vec{a}$

$= -\int \vec{c} \times d\vec{a} \Rightarrow -\vec{c} \times \int d\vec{a} = -\vec{c} \times \vec{a}$

$\boxed{\Rightarrow \vec{a} \times \vec{c}}$

1.63) a) Find the divergence of the function

$$v = \frac{\hat{r}}{r}$$

$$\nabla \cdot v = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \cdot \frac{1}{r}\right) = \frac{1}{r^2}\frac{d}{dr}(r) = \frac{1}{r^2}$$

Sphere:

$$\int v \cdot da = \int \left(\frac{1}{R}\hat{r}\right) \cdot \left(R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right) = R\int \sin\theta \, d\theta \, d\phi = 4\pi R$$

$$\int (\nabla \cdot v) \, d\tau = \int \left(\frac{1}{r^2}\right)\left(r^2 \sin\theta \, dr \, d\theta \, d\phi\right) = \left(\int_0^R dr\right)\left(\int \sin\theta \, d\theta \, d\phi\right) = 4\pi R$$

$$\nabla \times (r^n \hat{r}) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 r^n\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^{n+2}\right) = \frac{1}{r^2}(n+2)r^{n+1}$$

$$= (n+2)r^{n-1}$$

Unless $n = -2$, then $= 4\pi\delta^3(r)$

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(r)$$

b) Curl of $r^n\hat{r}$

Using Prob 1.61(b)

If $(\nabla \times (r^n\hat{r})) = 0$, then $\int (\nabla \times v) \, d\tau = 0 = -\oint v \times da$

Since $v = r^n\hat{r}$ & $da = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$, are in the $\hat{r}$,

then $\boxed{v \times da = 0}$

02/04/22

Warm Up 1

Suppose we have an electron traveling at 1 percent of the speed of light, with a velocity vector

$$\vec{v} = \frac{|\vec{v}|}{\sqrt{2}}\hat{i} + \frac{|\vec{v}|}{\sqrt{2}}\hat{j}$$

The electron is traveling is traveling in a $\vec{B}$-field of 0.5 Gauss (0.5×10^{-4} Tesla) that is pointed in the z-direction. What is the force on the electron?
[Hint: $\vec{F} = q\vec{v} \times \vec{B}$]

$$\vec{v} = a\hat{i} + b\hat{j}$$
$$\Rightarrow \vec{v} = a(\hat{i} + \hat{j})$$
$$\vec{B} = B\hat{k}$$
$$\vec{v} \times B = aB(\hat{i} + \hat{j}) \times \hat{k} \Rightarrow aB(\hat{i} \times \hat{k} + \hat{j} \times \hat{k})$$
$$\Rightarrow aB(\hat{i} \times \hat{k} + \hat{i}) \Rightarrow aB(\hat{i} - \hat{j})$$
$$\Rightarrow \frac{|\vec{v}|}{\sqrt{2}} B(\hat{i} - \hat{j})$$

Warm-Up 2

Suppose we have a function that describes the PE of a system

$$U(x,y,z) = mgz$$

Take the gradient & multiply by minus one, what do you get?

[Hint: $\vec{F} = -\nabla U(x,y,z)$]

Answer

$$-\nabla U(x,y,z) = \left(\hat{i}\frac{d}{dx} + \hat{j} + \hat{k}\right)mgz$$

$$= \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)mgz$$

$$\nabla = \left(\frac{\hat{i}\,d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)$$

Break the problem into pieces

$$\frac{dr}{dx} = \qquad \frac{dr}{dx} = \frac{d}{dx}\left(x^2+\dots\right)^{\frac{1}{2}}$$

$$\Rightarrow \frac{1}{2}\left(x^2+\dots\right)^{-1/2}(2x)$$

$$\frac{dr}{dy} = \frac{d}{dy}\left(y^2+\dots\right)^{\frac{1}{2}} = \frac{1}{2}\left(x^2+\dots\right)^{-1/2}(2y)$$

$$\frac{dr}{dz} = \frac{d}{dz}\left(z^2+\dots\right)^{\frac{1}{2}} = \frac{1}{2}\left(z^2+\dots\right)^{-1/2}(2z)$$

$$\nabla r = \hat{r}$$