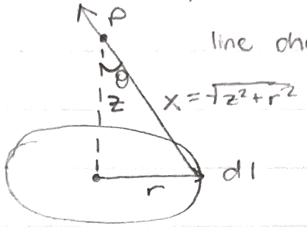
Kevin Alvarez

Homework 2.5, 2.6, 2.9, 2.12, 2.16, 2.18, 2.25, 2.29

2.5)



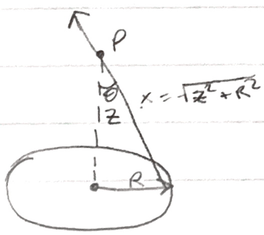
line charge λ, $E(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(r')}{\eta^2} \hat{\eta}\, dl'$

$x^2 = z^2 + r^2$ $\qquad dE = \frac{1}{4\pi\varepsilon_0} \frac{dq}{x^2} \cos\theta$

$\cos\theta = \frac{z}{\sqrt{r^2+z^2}}$ $\qquad dE = \frac{1}{4\pi\varepsilon_0} \frac{dq}{(z^2+r^2)} \left(\frac{z}{\sqrt{r^2+z^2}}\right) = \frac{1}{4\pi\varepsilon_0} \frac{z\, dq}{(z^2+r^2)^{3/2}}$

$$E = \int dE$$
$$= \int_0^{2\pi r} \frac{1}{4\pi\varepsilon_0} \frac{\lambda z}{(z^2+r^2)^{3/2}} dl' = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{\lambda z}{(z^2+r^2)^{3/2}}\right)(l)\Big|_0^{2\pi r}$$
$$= \left(\frac{1}{\cancel{4\pi}_{2}\varepsilon_0}\right)(\cancel{2\pi} r)\left(\frac{\lambda z}{(z^2+r^2)^{3/2}}\right) = \boxed{\left(\frac{1}{2\varepsilon_0}\right)\left(\frac{\lambda z}{(z^2+r^2)^{3/2}}\right)}$$

2.6)



σ $\qquad E(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(r')}{\eta^2} \hat{\eta}\, da'$ $\qquad dq = \sigma da'$

$x^2 = z^2 + R^2$

$dE = \frac{1}{4\pi\varepsilon_0} \frac{dq}{x^2} \cos\theta$ $\qquad$ Columb 7, 8, 9 videos

$\cos\theta = \frac{z}{\sqrt{z^2+R^2}}$ $\qquad dE = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{dq}{z^2+R^2}\right)\left(\frac{z}{\sqrt{z^2+R^2}}\right)$

$dE = \frac{1}{4\pi\varepsilon_0} \frac{z\, dq}{(z^2+R^2)^{3/2}}$

$E = \int dE$ $\qquad \sigma = \frac{dq}{(2\pi R)dR}$ $\qquad dq = \sigma(2\pi R)\, dR$

$$= \int_0^r \frac{1}{4\pi\varepsilon_0} \frac{z\, dq}{(z^2+R^2)^{3/2}}$$
$$= \int_0^r \frac{1}{\cancel{4\pi}\varepsilon_0} \frac{z\sigma(\cancel{2\pi} R)\, dR}{(z^2+R^2)^{3/2}}$$

$$= \frac{\sigma z}{2\varepsilon_0} \int_0^r \frac{R\, dR}{(z^2+R^2)^{3/2}}$$ $\qquad u = z^2+R^2$, $du = 2R\, dR$, $\frac{1}{2} du = R\, dR$ $\qquad = \frac{\sigma z}{4\varepsilon_0} \int_0^r u^{3/2} du$

$$= \frac{\sigma z}{2\varepsilon_0} \left(\frac{-1}{\sqrt{z^2+R^2}} \right)\Big|_0^r = \boxed{\frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+r^2}}\right)}$$

if $r \to \infty$

$$E = \frac{\sigma}{2\varepsilon_0}\left(1 - \cancel{\frac{z}{\sqrt{\infty}}}^{\,0}\right) = \boxed{\frac{\sigma}{2\varepsilon_0}}$$

$z \gg R$

2.9) $E = kr^3 \hat{r}$ spherical coordinates

a) charge density ρ

$$\nabla \cdot E = \frac{1}{\varepsilon_0}\rho \rightarrow \rho = (\nabla \cdot E)\,\varepsilon_0$$

$$\nabla \cdot E = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(E)\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(kr^3)\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^5\right) = \frac{1}{\cancel{r^2}}\left(5kr^{\cancel{4}\,2}\right) = 5kr^2$$

$$\boxed{\rho = (\nabla \cdot E)\,\varepsilon_0 = 5kr^2\varepsilon_0}$$

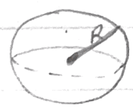
B) total charge contained in a sphere of radius R, centered at the origin

for an enclosed surface $\oint E \cdot da = \frac{1}{\varepsilon_0} Q_{enc}$.

$$Q_{enc} = \int_V \rho \, d\tau \qquad \rho = (\nabla \cdot E)\,\varepsilon_0$$

$$= \varepsilon_0 \int (kR^3) \cdot (4\pi R^2)$$

$$\boxed{= \varepsilon_0 (4\pi k R^5)}$$



$$d q = \rho \, d\tau \qquad \rho = 5kr^2\varepsilon_0$$

$$dq = (5kr^2\varepsilon_0)(4\pi r^2 dr)$$

$$Q_{enc} = \int \rho \, d\tau = \int_0^R (5kr^2\varepsilon_0)(4\pi r^2)\,dr$$

$$= 4\pi\varepsilon_0 \int_0^R 5kr^4 \, dr$$

$$= 4\pi\varepsilon_0 \left(kr^5 \Big|_0^R \right.$$

$$\boxed{= 4\pi\varepsilon_0 k R^5}$$

2.12) Gauss's law electric field inside uniformly charge solid sphere (charge density ρ)

solid sphere so it is an enclosed surface

surface area : $4\pi r^2$

$$\oint E \cdot da = \frac{1}{\varepsilon_0} Q_{enc}.$$

How? ⟶ $Q_{enc} = \int \rho \, d\tau$

$$= \rho \times \frac{4}{3}\pi r^3$$

$$\oint E \cdot da = \frac{1}{\varepsilon_0} Q_{enc}.$$

$$\cancel{\oint_S} |E| \cancel{d}a = \frac{1}{\varepsilon_0} Q_{enc}$$

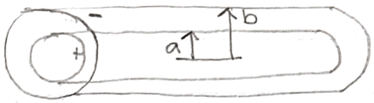
$$E(a) = \frac{1}{\varepsilon_0} Q_{enc}$$

$$E(\cancel{4}\cancel{\pi} r^{\cancel{2}}) = \frac{1}{\varepsilon_0}(\rho \times \cancel{\tfrac{4}{3}}\cancel{\pi} r^{\cancel{3}})$$

$$E = \frac{1}{\varepsilon_0} \frac{(\rho r)}{3}$$

$$\boxed{\vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r}}$$

2.16)



i) inside the inner cylinder ($s<a$)

enclosed surface ⟶ $\oint E \cdot da = \frac{1}{\varepsilon_0} Q_{enc}$

$$Q_{enc} = \int \rho \, d\tau$$

$$= \int_0^1 \int_0^{2\pi} \int_0^s \rho (s \, ds) \, d\phi \, dz$$

$$= \left(\int_0^s s \, ds\right)\left(\int_0^{2\pi} \rho \, d\phi\right)\left(\int_0^1 dz\right)$$

$$Q_{enc} = \left(\frac{s^2}{\cancel{2}}\right)(\cancel{2}\pi\rho)(1) = s^2\pi\rho l$$

$$\cancel{\oint} E \cdot \cancel{d}a = \frac{1}{\varepsilon_0} Q_{enc}$$

$$E(a) = \frac{1}{\varepsilon_0} Q_{enc}$$

$$E(2\cancel{\pi}\cancel{s}\cancel{l}) = \frac{s^{\cancel{2}}\cancel{\pi}\rho\cancel{l}}{\varepsilon_0}$$

$$\boxed{E = \frac{\rho s}{2\varepsilon_0}\hat{s}}$$

ii) between the cylinders $(a<s<b)$

$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$

$E(a) = \frac{1}{\varepsilon_0} Q_{enc}$

$E(2\cancel{\pi} s \cancel{l}) = \frac{1}{\varepsilon_0}(\rho a^2 \cancel{\pi} \cancel{l})$

$\boxed{E = \dfrac{\rho a^2}{2\varepsilon_0 s}\hat{s}}$

$Q_{enc} = \int \rho \, d\tau$

$= \int_0^1 \int_0^{2\pi} \int_0^a \rho s \, ds \, d\phi \, dz$

$= \left(\int_0^a \rho s \, ds\right)\left(\int_0^{2\pi} d\phi\right)\left(\int_0^1 dz\right)$

$= \left(\tfrac{1}{2}\rho a^2\right)(2\pi)(l) = \rho a^2 \pi l$

iii) outside the cable $(s>b)$

$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$

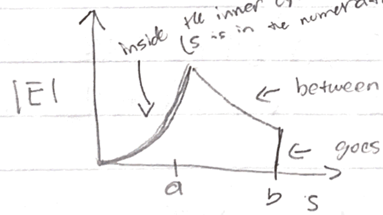
$E(a) = \frac{1}{\varepsilon_0} Q_{enc}$

$E(a) = 0$ $\quad \boxed{\vec{E} = 0}$

$Q_{enc} = 0$ ← no charge outside the cable

Plot $|E|$ as a function of s.

$E = \dfrac{\rho s}{2\varepsilon_0}\hat{s}$ $\qquad E = \dfrac{\rho a^2}{2\varepsilon_0 s}$

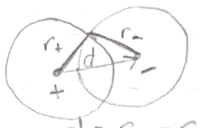


inside the inner cylinder (s is in the numerator)

← between the cylinders (s is in the denominator)

← goes to 0 for outside the cable

2.18)



Show that the overlapped field is constant

Answer from 2.12 $\vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r}$

$d = r_+ - r_-$

positive sphere : $\vec{E}_+ = \frac{+\rho}{3\varepsilon_0} r_+$

negative sphere $\vec{E}_- = \frac{-\rho}{3\varepsilon_0} r_-$

$E_{total} = E_+ + E_-$

$= \frac{+\rho}{3\varepsilon_0} r_+ + \left(\frac{-\rho}{3\varepsilon_0} r_-\right)$

$E_{total} = \frac{\rho}{3\varepsilon_0}(r_+ - r_-)$

$$\boxed{E_{total} = \frac{\rho}{3\varepsilon_0}(d)}$$

2.29) Eqn 2.29 $V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{𝓇}\,d\tau'$

$\nabla^2 \frac{1}{𝓇} = -4\pi\delta^3(𝓇)$ $\qquad 𝓇 = r - r'$

$\nabla^2 \frac{1}{𝓇} = -4\pi\delta^3(r-r')$

$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{𝓇}\,d\tau'$

$\nabla^2 V(r) = \frac{1}{4\pi\varepsilon_0}\int \nabla^2 \frac{\rho(r')}{𝓇}\,d\tau' = \frac{1}{4\pi\varepsilon_0}\int\left(\nabla^2\frac{1}{𝓇}\right)\rho(r')\,d\tau'$

$= \frac{1}{\cancel{4\pi}\varepsilon_0}\int\left(-\cancel{4\pi}\delta^3(r-r')\rho(r')\,d\tau'\right.$

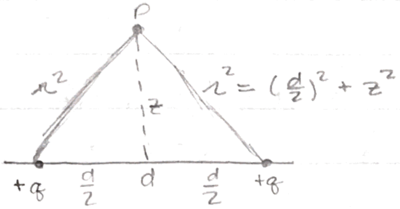
$= -\frac{1}{\varepsilon_0}\int \delta^3(r-r')\rho(r')\,d\tau'$

$$\boxed{\nabla^2 V(r) = -\frac{1}{\varepsilon_0}(\rho(r))}$$

same as Poisson's equation

2.25 $V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$ $\quad V = \frac{1}{4\pi\varepsilon_0}\int\frac{\lambda(r')}{r}\,dl'$ $\quad V = \frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(r')}{r}\,da'$

Figure A)



$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i} = \frac{1}{4\pi\varepsilon_0}\left(\frac{+q}{\sqrt{(\frac{d}{2})^2+z^2}} + \frac{+q}{\sqrt{(\frac{d}{2})^2+z^2}}\right)$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{2q}{\sqrt{(\frac{d}{2})^2+z^2}}\right)$$

$$E = -\nabla V$$

$$= -\left(\frac{\partial}{\partial x}V_x\hat{x} + \frac{\partial}{\partial y}V_y\hat{y} + \frac{\partial}{\partial z}V_z\hat{z}\right)$$

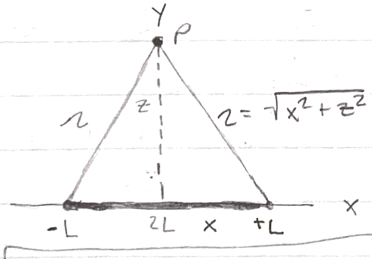
$$= -\left(0\hat{x} + 0\hat{y} + \frac{\partial}{\partial z}\left(\frac{1}{4\pi\varepsilon_0}\frac{2q}{\sqrt{(\frac{d}{2})^2+z^2}}\hat{z}\right)\right)$$

$$= -\frac{q}{2\pi\varepsilon_0}\left(\frac{\partial}{\partial z}\left((\tfrac{d}{2})^2+z^2\right)^{-1/2}\right)$$

$$= \cancel{-}\frac{q}{2\pi\varepsilon_0}\left(\left(\cancel{-}\frac{1}{\cancel{2}}\right)\left((\tfrac{d}{2})^2+z^2\right)^{-3/2}(\cancel{2}z)\right)$$

$$\boxed{E = \frac{qz}{2\pi\varepsilon_0}\left(\frac{1}{\left((\frac{d}{2})^2+z^2\right)^{3/2}}\right)\hat{z}}$$

Figure B)



$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{\lambda(r')}{r}\,dl'$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_{-L}^{+L}\frac{\lambda}{\sqrt{x^2+z^2}}\,dx$$

$$V = \frac{\lambda}{4\pi\varepsilon_0}\left(\ln\left(\sqrt{x^2+z^2}+x\right)\Big|_{-L}^{L}\right)$$

$$V = \frac{\lambda}{4\pi\varepsilon_0}\left(\ln\left(\sqrt{L^2+z^2}+L\right) - \ln\left(\sqrt{L^2+x^2}-L\right)\right)$$

$$V = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{\sqrt{L^2+z^2}+L}{\sqrt{L^2+z^2}-L}\right)$$

$$E = -\nabla V = \frac{\partial}{\partial x} V_x \hat{x} + \frac{\partial}{\partial y} V_y \hat{y} + \frac{\partial}{\partial z} V_z \hat{z} \qquad V = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{\sqrt{L^2+z^2}+L}{\sqrt{L^2+z^2}-L}\right)$$

$$= \left(0\hat{x} + 0\hat{y} + \frac{\partial}{\partial z}\left(\frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{\sqrt{L^2+z^2}+L}{\sqrt{L^2+z^2}-L}\right)\right)\hat{z}\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{\sqrt{L^2+z^2}+L}{\sqrt{L^2+z^2}-L}\right)\left(\frac{\left(\cancel{\sqrt{L^2+z^2}}-L\right)\left(\frac{\cancel{2}z}{\cancel{2}\sqrt{L^2+z^2}}\right)-\left(\cancel{\sqrt{L^2+z^2}}\overset{-L}{+L}\right)\left(\frac{\cancel{2}z}{\cancel{2}\sqrt{L^2+z^2}}\right)}{\left(\sqrt{L^2+z^2}-L\right)^2}\right)\hat{z}$$

$$= \frac{\lambda}{\cancel{4}_2\pi\varepsilon_0}\left(\frac{\cancel{\sqrt{L^2+z^2}-L}}{\sqrt{L^2+z^2}+L}\right)\left(\frac{-\cancel{2}Lz}{\sqrt{L^2+z^2}}\right)\frac{1}{\left(\sqrt{L^2+z^2}-L\right)^{\cancel{2}}}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{Lz}{\sqrt{L^2+z^2}}\right)\frac{1}{\left(\sqrt{L^2+z^2}+L\right)\left(\sqrt{L^2+z^2}-L\right)}$$

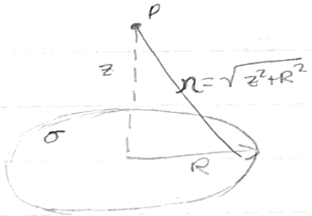
$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{Lz}{\sqrt{L^2+z^2}}\right)\frac{1}{\left(\cancel{\sqrt{L^2+z^2}}\right)^{\cancel{2}}-L^2}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{Lz}{\sqrt{L^2+z^2}}\right)\frac{1}{\cancel{L^2}+z^2\cancel{-L^2}}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{L\cancel{z}}{\sqrt{L^2+z^2}}\right)\frac{1}{z^{\cancel{2}}}$$

$$\boxed{= \left(\frac{\lambda}{2\pi\varepsilon_0}\right)\left(\frac{L}{z\sqrt{L^2+z^2}}\right)\hat{z}}$$

Figure Q



$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(r')}{\mathscr{r}}\, da' \qquad da' = 2\pi R\, dR$$

$$= \frac{1}{\cancel{4\pi\varepsilon_0}_{2\pi}}\int_0^R \frac{\sigma\, \cancel{2}\pi R}{\sqrt{z^2+R^2}}\, dR$$

$$u = z^2+R^2$$
$$du = 2R\, dr$$

$$= \frac{\sigma}{2\pi\varepsilon_0}\int_0^R \frac{1}{2}u^{-1/2}\, du$$

$$= \frac{\sigma}{2\pi\varepsilon_0}\left(\sqrt{z^2+R^2}\,\Big|_0^R\right.$$

$$= \frac{\sigma}{2\pi\varepsilon_0}\left(\sqrt{z^2+R^2} - \sqrt{z^2+0}\right)$$

$$= \frac{\sigma}{2\pi\varepsilon_0}\left(\sqrt{z^2+R^2} - z\right)$$

$$E = -\nabla V = \left(0 + 0 - \frac{\partial}{\partial z}\left(\frac{\sigma}{2\pi\varepsilon_0}\left(\sqrt{z^2+R^2} - z\right)\right)\right)\hat{z}$$

$$= -\frac{\sigma}{2\pi\varepsilon_0}\left(\frac{\partial}{\partial z}\left((z^2+R^2)^{1/2} - z\right)\right)\hat{z}$$

$$= -\frac{\sigma}{2\pi\varepsilon_0}\left(\cancel{\tfrac{1}{2}}(z^2+R^2)^{-1/2}(\cancel{2}z) - 1\right) = \boxed{\frac{\sigma}{2\pi\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)\hat{z}}$$