1.59

Check divergence theorem

$$\vec{v} = r^2 \sin\theta \,\hat{r} + 4r^2 \cos\theta \,\hat{\theta} + r^2 \tan\theta \,\hat{\phi}$$

$$\int_V (\nabla \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^4 \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(4r^2 \sin\theta\cos\theta\right)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r^2 \tan\theta\right)$$

$$= \frac{1}{r^2} \cdot 4r^3 \sin\theta + \frac{4r^2(2\cos^2(\theta) - 1)}{r\sin\theta}$$

$$= \frac{4r\sin\theta}{} + \frac{4r(2\cos^2(\theta) - 1)}{\sin\theta} = \frac{4r(2\cos^2\theta - 1 + \sin^2\theta)}{\sin\theta}$$

$$= \frac{4r(\cos^2(\theta))}{\sin\theta}$$

$$\int_0^R \int_0^{\pi/6} \int_0^{2\pi} \frac{4r^3 \cos^2(\theta)\sin\theta}{\sin\theta}\, dr\, d\theta\, d\phi$$

$$= R^4 \cdot 2\pi \int_0^{\pi/6} \cos^2\theta\, d\theta = R^4 2\pi \left[\frac{\sin\theta\cos\theta}{2} + \frac{\theta}{2}\right]_0^{\pi/6}$$

$$= R^4 2\pi \left[\frac{\sqrt{3}}{8} + \frac{\pi}{12}\right] = \boxed{\frac{\pi R^4}{12}\left(3\sqrt{3} + 2\pi\right)}$$

$$\oint \vec{V}\cdot d\vec{a} = \oint_{top} \vec{V}\cdot d\vec{a} + \oint_{cone} \vec{V}\cdot d\vec{a}$$

$r = R$

$$top = \oint_{top} \vec{V}\cdot d\vec{a} ; \quad \vec{V}\cdot d\vec{a} = \hat{r}\cdot r^2 \sin\theta \, d\theta \, d\phi$$

$$\int_0^{R}\int_0^{2\pi} r^4 \sin^2\theta \, d\theta \, d\phi = 2\pi R^4 \int_0^{\pi/6} \sin^2\theta \, d\theta = 2\pi R^4 \left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right]$$

$$= \frac{\pi R^4}{12}\left[2\pi - 3\sqrt{3}\right]$$

$$Cone = \oint_{cone} \vec{V}\cdot d\vec{a} ; \quad \vec{V}\cdot d\vec{a} = \hat{\theta}\cdot r\sin\theta \, dr \, d\phi =$$

$$\frac{1}{2} 4r^3 \cos\theta \, dr \, d\phi = \frac{\sqrt{3}}{4} 4r^3 \, dr \, d\phi = \sqrt{3}\, r^3 \, dr \, d\phi$$

$$\frac{4}{\sqrt{3}} \int_0^{R}\int_0^{2\pi} r^3 \, dr \, d\phi = \frac{\sqrt{3}R^4}{4} 2\pi = \frac{\pi R^4}{12}\left(6\sqrt{3}\right)$$

$$Cone + top = \boxed{\frac{\pi R^4}{12}\left[2\pi + 3\sqrt{3}\right]}$$

1.62 $\vec{a} \equiv \int_S d\vec{a}$

(a) $\vec{a}_{bot} \equiv \iint d\vec{a} = \int_0^{\pi/2}\int_0^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$

$= 2\pi \int_0^{\pi/2} r^2 \sin\theta \, d\theta = 2\pi R^2 [-(0-1)] = \boxed{2\pi R^2 \hat{a}}$

(b) $\vec{a} = \vec{0} = \vec{a}_{bot} + \vec{a}_{top} \Rightarrow \vec{0} = \pi R^2 \hat{z}(1-1)$

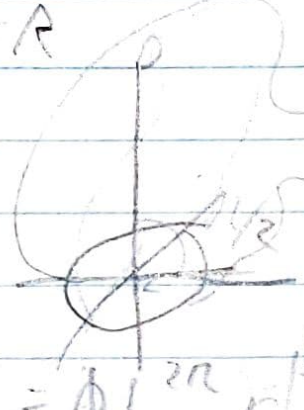
$\vec{a}_{top} = \int_{\pi/2}^{\pi}\int_0^{2\pi} R^2 \sin\theta \, d\theta \, d\phi = R^2 2\pi \int_{\pi/2}^{\pi} \sin\theta \, d\theta = -2\pi R^2 \hat{a}$

$= \vec{a}_{top} = -\vec{a}_{bot} \therefore \vec{a}_{bot} + \vec{a}_{top} = \boxed{\vec{0}}$ □

(c) Show $\vec{A}$ of any ~~boundary~~ surface with same boundary is same

boundary is circle with $r = R$

$\int_S f(r) \, d\vec{a} = \int_0^{\pi/2}\int_0^{2\pi} R^2 \sin\theta \, d\theta \, d\phi$
$= 2\pi R^2$



$f(r)$ is a $\hat{r}$ varying function, but $d\hat{a}$ is $\hat{\theta}$ and $\hat{\phi}$ □

$\theta\Big|_0^{\pi/2} = \phi\Big|_0^{2\pi} \; r\Big|_R^{?} \; F(r)$

1.63 Find divergence

$$\vec{v} = \frac{\hat{r}}{r}$$

(a) use Eq 1.84

$$\nabla \cdot \vec{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}\right) = \frac{1}{r^2} \cdot 1 = \boxed{\frac{1}{r^2}}$$

(b) Test using 1.85

$$\oint \vec{v} \cdot d\vec{a} = \int \frac{1}{R} \hat{r} R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$= 2\pi R \int_0^{\pi} \sin\theta \, d\theta = \boxed{4\pi R}$$

(c) Is there a delta function at the origin?

No. $1/r$ is one of those fancy functions that [illegible] behaves with limits so there is no need for $\delta(r)$ because $\lim_{r \to \infty} =$ DNE

(d) What is formula for divergence of $\vec{v} = r^n \hat{r}$

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r} r^{n+2} = (n+2)\, r^{n-1}$$

But if $n=-2$, then dirac $\delta^3(\vec{r}) \cdot 4\pi$

Negative div is weird: so do something else for $n<-2$

(1.64) (a) $D(r,\epsilon) \equiv -\frac{1}{4\pi}\nabla^2\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right)$

$-\nabla^2\frac{1}{\sqrt{r^2+\epsilon^2}} = -4\pi\,\delta^3\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right) = -4\pi\,\delta^3\left(\frac{1}{r\sqrt{1+\left(\frac{\epsilon}{r}\right)^2}}\right)$

as $\epsilon \to 0 \Rightarrow -4\pi\,\delta^3\left(\frac{1}{r}\right)$

$D(r,\epsilon) = \nabla^2\left(\frac{1}{r}\right) = \frac{-1}{4\pi}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{\sqrt{r^2+\epsilon^2}}\right)$

$= \frac{-1}{4\pi r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{-r}{(r^2+\epsilon^2)^{3/2}}\right) = \frac{-1}{4\pi r^2}\cdot\frac{-3r^2\epsilon^2}{(r^2+\epsilon^2)^{5/2}}$

$= \boxed{\frac{3\epsilon^2}{4\pi}\frac{1}{(r^2+\epsilon^2)^{5/2}}}$

(b) Check $D(0,\epsilon) \to \infty$ as $\epsilon \to 0$

$\lim_{\epsilon\to 0}\frac{3\epsilon^2}{4\pi}\frac{1}{(\epsilon^5)}$ ∞ ϵ^5 grows faster than ϵ^2 shrinks

$\square$

(c) Check that $D(r,\epsilon) \to 0$ as $\epsilon \to 0$ for $r \neq 0$

$\lim_{\epsilon\to 0}\frac{3\epsilon^2}{4\pi}\frac{1}{(r^2+\epsilon^2)^{5/2}} = \frac{3\cdot 0^2}{4\pi}\cdot\frac{1}{r^5} = 0$ if $r \neq 0$

(d) Check $\int D(r,\epsilon)$ everywhere is 1

$$\frac{3\epsilon^2}{4\pi}\int_{-\infty}^{\infty}\int_0^{\pi}\int_0^{2\pi}\frac{r^2\sin\theta}{(r^2+\epsilon^2)^{5/2}}\,dr\,d\theta\,d\phi$$

$$=\frac{4\pi\cdot 3\epsilon^2}{4\pi}\int_{-\infty}^{\infty}\frac{r^2}{(r^2+\epsilon^2)^{5/2}}\,dr$$

$$=3\epsilon^2\left(\frac{r(2r^2+3\epsilon^2)}{6(r^2+\epsilon^2)^{3/2}\epsilon^2}-\frac{r}{2\cdot(r^2+\epsilon^2)^{3/2}}\right)\Bigg|_{-\infty}^{\infty}$$

$$=\frac{r(2r^2+3\epsilon^2)}{2(r^2+\epsilon^2)^{3/2}}-\frac{3\epsilon^2}{2(r^2+\epsilon^2)^{3/2}}\Bigg|_{-\infty}^{\infty}$$

$$=\frac{2r^3}{2(r^2+\epsilon^2)^{3/2}}\Bigg|_{-\infty}^{\infty}\quad\frac{2r^3}{2r^3}\Bigg|_{-\infty}^{\infty}=\boxed{1}\ \square$$