

Electromagnetic Theory: PHYS330

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Summary

Week 2 Summary

1. Homework discussions
 - Proofs! Glorious proofs.
 - Exercises with *checking* fundamental theorems
2. Electrostatics and Coulomb forces
 - Charge distributions, superposition, and the Coulomb force
 - A note about the *far-field*
 - Setting up integrals, taking limits, checking units
 - The divergence of electric fields
 - The curl of electric fields
3. Electric Potential
 - Definitions, fundamental theorem for gradients
 - Reference points
 - Laplace equation ...
4. Work, energy, and conductors

Homework

Homework, Week 2

Unlike last week, these exercises come from *within* the chapter. Ideally, you should look at all of the problems within the chapter as you study.

- Exercise 2.5
- Exercise 2.6
- Exercise 2.9
- Exercise 2.12
- Exercise 2.16
- Exercise 2.18
- Exercise 2.25
- Exercise 2.29

Charge distributions, Superposition, and the Coulomb Force

Charge distributions, Superposition, and the Coulomb Force

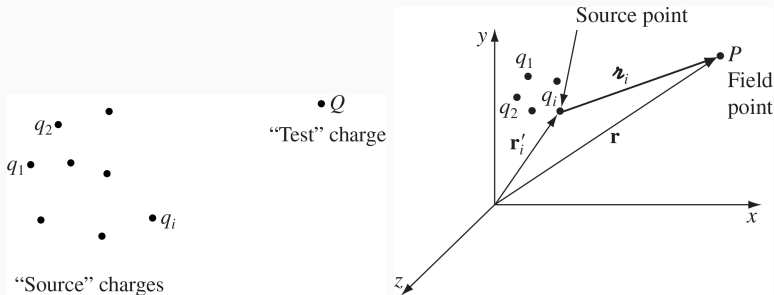


Figure 1: The basic problem of electrostatics. Note the definition of the separation vector, and the vectors to the field point and to all the source charges.

Charge distributions, Superposition, and the Coulomb Force

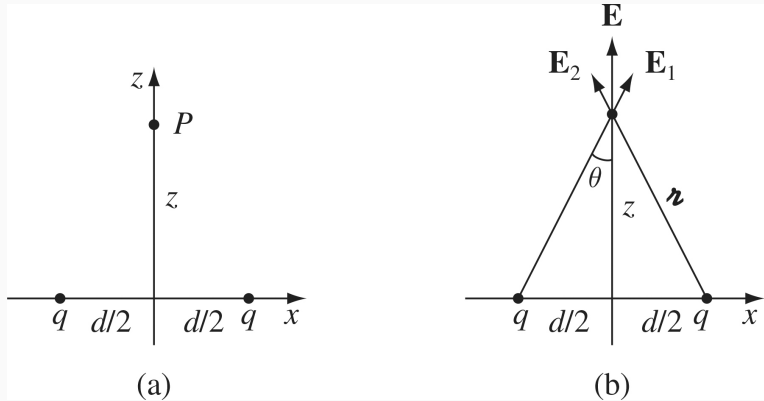


Figure 2: Begin with a dipole, and then a *physical* dipole.

Charge distributions, Superposition, and the Coulomb Force

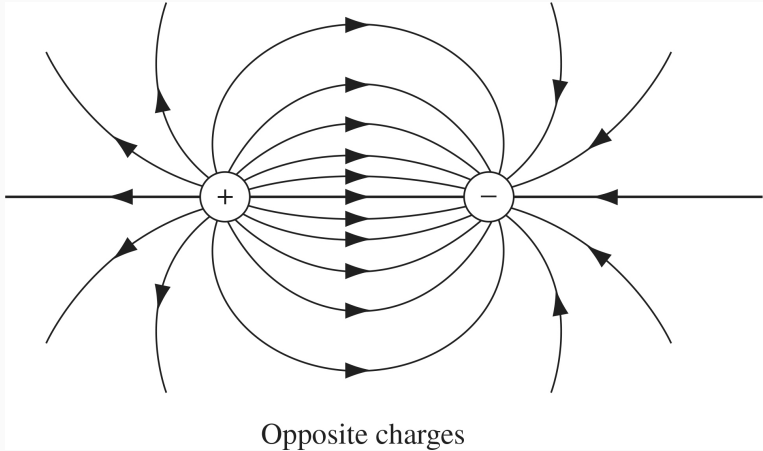


Figure 3: Field of a *physical* dipole.

Charge distributions, Superposition, and the Coulomb Force

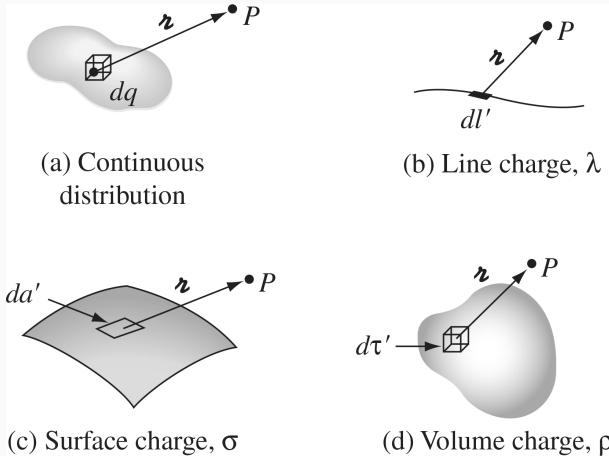


Figure 4: The continuous limit implies a variety of symmetries and geometries over which we integrate, rather than sum.

Charge distributions, Superposition, and the Coulomb Force

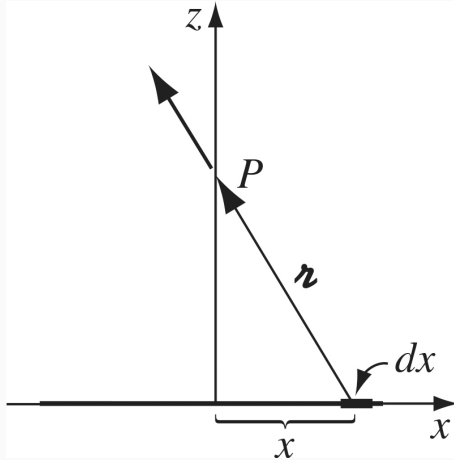


Figure 5: A continuous line density of charge. Integration yields the electric field.

Charge distributions, Superposition, and the Coulomb Force

Useful calculations:

1. Continuous line charge, length L .
2. Continuous plane of charge, radius R .
3. Loop of charge, radius R , a distance z above the center.

Why are these interesting? One example is that these shapes are used as *antennas*. Give some alternating current at the right voltage and impedance to a shape of metal, then you've got your antenna that radiates a certain way.

Professor do these examples.¹

¹Remember from PHYS180? Remember? Yeah...good times.

A Note about the Far-Field

The Far-Field

One way to express the **far-field** approximation (compare to Fraunhofer and Fresnel limits in diffraction):

$$\vec{r} = \vec{r'} + \vec{z} \quad (1)$$

$$\vec{z} = \vec{r} - \vec{r'} \quad (2)$$

$$z = \sqrt{r^2 - 2\vec{r} \cdot \vec{r'} + r'^2} \quad (3)$$

$$z = r\sqrt{1 - 2\vec{r} \cdot \vec{r'} r^{-2} + r'^2 r^{-2}} \quad (4)$$

$$z \approx r\sqrt{1 - 2\vec{r} \cdot \vec{r'} r^{-2}} \quad (5)$$

$$z \approx r\left(1 - \vec{r} \cdot \vec{r'} r^{-2}\right) \quad (6)$$

$$z \approx r\left(1 - \hat{r} \cdot \vec{r'} r^{-1}\right) \quad (7)$$

$$z \approx r - \hat{r} \cdot \vec{r'} \quad (8)$$

The Far-Field

Repeat the charged loop calculation, but replace $z \approx r - \hat{r} \cdot \vec{r}$ at the outset. What happens?

The Divergence of \vec{E} -fields

The Divergence of \vec{E} -fields

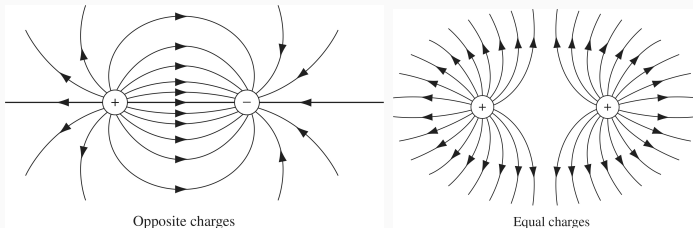


Figure 6: Field lines are an important theoretical concept.

The Divergence of \vec{E} -fields

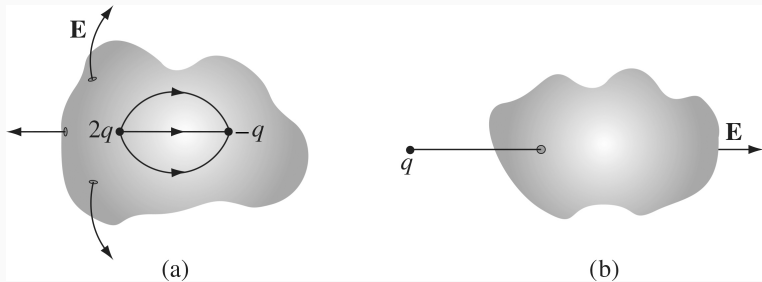


Figure 7: The concept of a closed Gaussian surface.

The Divergence of \vec{E} -fields

$$\oint \vec{E}_i \cdot d\vec{a} = \frac{1}{4\pi\epsilon_0} \int_0^\pi \int_0^{2\pi} \frac{q_i \hat{r}}{r^2} \cdot r^2 \sin\theta d\theta d\phi \hat{r} = \frac{q_i}{\epsilon_0} \quad (9)$$

$$\vec{E} = \sum_{i=1}^n \vec{E}_i \quad (10)$$

$$\oint \vec{E} \cdot d\vec{a} = \sum_{i=1}^n \left(\oint \vec{E}_i \cdot d\vec{a} \right) \quad (11)$$

$$\oint \vec{E} \cdot d\vec{a} = \sum_{i=1}^n \left(\frac{q_i}{\epsilon_0} \right) \quad (12)$$

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{tot}}{\epsilon_0} \quad (13)$$

Gauss' Law: the total flux is proportional to the contained charge.

The Divergence of \vec{E} -fields

The divergence theorem:

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{E}) d\tau \quad (14)$$

Remark that the total charge is the integral over the 3D charge density:

$$\frac{Q_{tot}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho d\tau \quad (15)$$

This implies

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{E}) d\tau = \frac{1}{\epsilon_0} \int_V \rho d\tau \quad (16)$$

Looking at the last two expressions:

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}} \quad (17)$$

The Divergence of \vec{E} -fields

Differential form of Gauss' Law:

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}} \quad (18)$$

Consider a different argument:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathbf{r}}}{r^2} \rho(\vec{r}') d\tau' \quad (19)$$

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) \rho(\vec{r}') d\tau' \quad (20)$$

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi\delta^3(\mathbf{r} - \mathbf{r}') \rho(\vec{r}') d\tau' \quad (21)$$

$$\nabla \cdot \vec{E} = \frac{4\pi}{4\pi\epsilon_0} \int \delta^3(\vec{r} - \vec{r}') \rho(\vec{r}') d\tau' \quad (22)$$

$$\nabla \cdot \vec{E} = \rho(\vec{r})/\epsilon_0 \quad (23)$$

The Divergence of \vec{E} -fields

(Refresh with delta-functions): Use this charge distribution and Eq. 19 to find the \vec{E} -field.

$$\rho(\vec{r}') = q\delta^3(\vec{r}' - x\hat{x}) - q\delta^3(\vec{r}' + x\hat{x}) \quad (24)$$

The Divergence of \vec{E} -fields

Symmetry in the Application of Gauss' Law:

If the E-field and the area element are *always* orthogonal,

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{tot}}{\epsilon_0} \quad (25)$$

$$|\vec{E}|A = \frac{Q_{tot}}{\epsilon_0} \quad (26)$$

$$\vec{E} = \frac{Q_{tot}}{\epsilon_0 A} \hat{n} \quad (27)$$

This trick can be used even when the charge distribution is not uniform, but *does exhibit* symmetry.

The Divergence of \vec{E} -fields

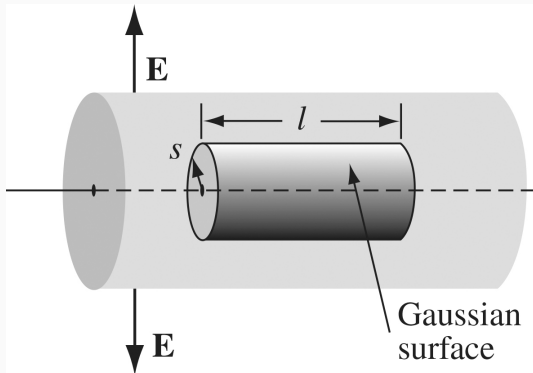


Figure 8: Use cylindrical symmetry to apply Gauss' Law. The charge distribution function is $\rho(s) = ks$. Obtain the field (a) *inside* the object, then (b) *outside* the object.

The Divergence of \vec{E} -fields

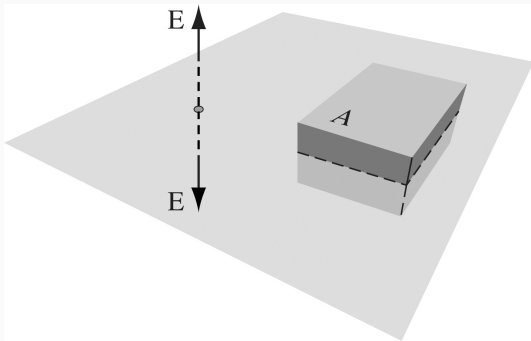


Figure 9: Use Cartesian symmetry to apply Gauss' Law. The charge distribution function is $\rho(x, y) = +\sigma$. Obtain the field above (or below) the charged plane.

The Divergence of \vec{E} -fields

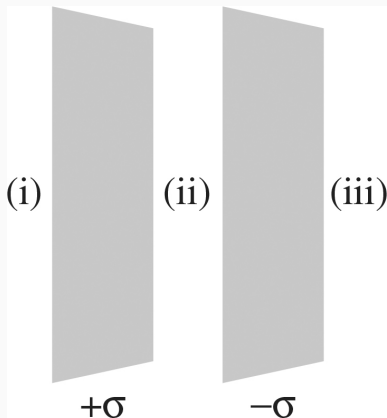


Figure 10: Combinations of “Gaussian charged objects.” What about the field between two line charges?

The Curl of \vec{E} -fields

The Curl of \vec{E} -fields

The \vec{E} -field of a point charge at the origin ($\vec{r} = 0$), and the line element in spherical coordinates:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \quad (28)$$

$$d\vec{l} = dr\hat{r} + r d\theta\hat{\theta} + r \sin\theta d\phi\hat{\phi} \quad (29)$$

$$\vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr \quad (30)$$

$$\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_a} - \frac{1}{r_b} \right) \quad (31)$$

$$\oint \vec{E} \cdot d\vec{l} = 0 \quad (\vec{a} = \vec{b}) \quad (32)$$

$$\nabla \times \vec{E} = 0 \quad (33)$$

Any combination of point charges will also lead to zero curl. Why? Superposition.

The Curl of \vec{E} -fields

Define a function, then, that encapsulates *path-independence*:

$$V(\vec{r}) = - \int_{\mathcal{O}}^{\vec{r}} \vec{E} \cdot d\vec{l} \quad (34)$$

- \mathcal{O} is a reference point, naturally taken to be ∞ (far from origin)
- $V(\vec{b}) - V(\vec{a}) = - \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$
- $0 = V(\vec{a}) - V(\vec{a}) = - \oint \vec{E} \cdot d\vec{l} = 0$

Fundamental theorem for gradients:

$$V(\vec{b}) - V(\vec{a}) = \int_{\vec{a}}^{\vec{b}} \nabla V \cdot d\vec{l} = - \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l} \quad (35)$$

$$\vec{E} = -\nabla V \quad (36)$$

The Curl of \vec{E} -fields

Path independence:

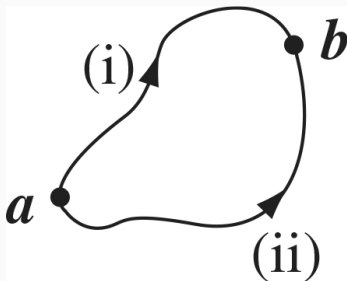


Figure 11: If the line integral was not path-independent, then path i minus path ii would not be zero. Path i and ii form a closed line integral.

The Curl of \vec{E} -fields

Two more ideas:

$$\vec{E} = -\nabla V \quad (37)$$

$$\nabla \cdot \vec{E} = -\nabla^2 V \quad (38)$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad (39)$$

Equation 39 is known as the *Poisson Equation*. If $\rho = 0$ in some region:

$$\boxed{\nabla^2 V = 0} \quad (40)$$

Equation 40 is known as the Laplacian of the potential or Laplace's Equation. Solving it is the subject of Ch. 3.

The Curl of \vec{E} -fields

Show that the potential of a point charge q at the origin is

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad (41)$$

Indeed, collections of point charges lead to

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_i} \quad (42)$$

A collection of *many* point charges smoothed into a continuous distribution leads to

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho' d\tau'}{r} \quad (43)$$

The Curl of \vec{E} -fields

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho' d\tau'}{r} \quad (44)$$

(a) Derive the potential due to a line charge of length L , directly above the center of the line. (b) Take the gradient in cylindrical coordinates to obtain \vec{E} .

Brief Remark about Boundary Conditions

Boundary Conditions

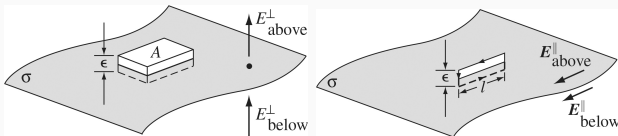


Figure 12: (Left) By Gauss' Law, we may find the discontinuity in the orthogonal E-field component. (Right) Via a closed-loop line integral, we may find that the parallel component of the E-field is continuous.

$$(\nabla V_{\text{top}} - \nabla V_{\text{bottom}}) \cdot \hat{n} = -\frac{\sigma}{\epsilon_0} \quad (45)$$

$$(\nabla V_{\text{top}} - \nabla V_{\text{bottom}}) \cdot \hat{s} = 0 \quad (46)$$

Boundary Conditions

Which of the following functions has a discontinuity *in the derivative* but not the function itself, at $x = 0$?

- A: $\tan(x)$
- B: $|x| = \text{abs}(x)$
- C: \sqrt{x}
- D: $\theta(x)$ (Heaviside step-function)

Boundary Conditions

Consult your notes regarding the charge distribution with spherical symmetry.

- What was the formula for the electric field as a function of r , when $r < R$?
- What is the correct formula for the electric field as a function of r , when $r \geq R$? (*Think of Gauss' Law*).
- What is the voltage just outside the radius R ?
- What is the voltage just inside the radius R ?

Energy and Work

Energy and Work

The work required to assemble point charges by bring each in from infinity is

$$W = \frac{1}{2} \sum_{i=1}^n q_i \left(\sum_{j \neq i}^n \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} \right) \quad (47)$$

The parentheses contains the potential due to all charges except q_i :

$$W = \frac{1}{2} \sum_{i=1}^n q_i V(\vec{r}_i) \quad (48)$$

Note that it doesn't matter when charge q_i arrived; we have to pay for the energy used to keep q_i there as everyone else arrives as well.

Energy and Work

Continuous limit:

$$W = \frac{1}{2} \int \rho(r') V(r') d\tau' \quad (49)$$

This result can be re-written:

$$W = \frac{\epsilon_0}{2} \left\{ - \int \vec{E} \cdot (\nabla V) d\tau + \oint V \vec{E} \cdot d\vec{a} \right\} \quad (50)$$

Like any integration by parts, there is an integral term and a surface term (the second term). Use $-\vec{E} = \nabla V$, and expand the Gaussian surface so far that all contributions from the second term vanish:

$$W = \frac{\epsilon_0}{2} \int E^2 d\tau \quad (51)$$

Energy and Work

Since we do not specify the shape of the volume, we can speak about the *energy per unit volume*:

$$u_E = \frac{\epsilon_0}{2} E^2 \quad (52)$$

- E-fields contain energy
- This equation is valid even if E-field values vary in space (of course)
- Consider the simple example of a capacitor ...

Energy and Work

Start with the E-field inside a parallel-plate capacitor:

$$E = \frac{\sigma}{\epsilon_0} \quad (53)$$

$$u_E = \frac{1}{2\epsilon_0} \sigma^2 \quad (54)$$

$$\mathcal{E} = \frac{Ad}{2\epsilon_0} \sigma^2 \quad (55)$$

$$\sigma = \frac{Q}{A} \quad (56)$$

$$C = \frac{\epsilon_0 A}{d} \quad (57)$$

$$\mathcal{E} = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2 = U \quad (58)$$

Energy conservation.

Conclusion

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