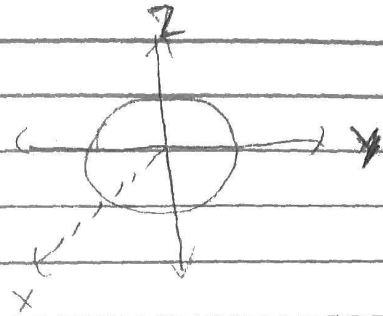
5.4, 5.5, 5.6, ~~5.7, 5.11, 6.1~~, 6.3, ~~6.4~~

Manuel Alvarez

11-2-20

HW #1

5.37 $\vec{v} = ay\hat{x} + bx\hat{y}$



Stokes Thm: $\int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$

surface integral

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ay & bx & 0 \end{vmatrix} = \hat{x}\left(\cancel{\partial/\partial y(0)}^{0} - \cancel{bx(\partial/\partial z)}^{0}\right) - \hat{y}\left(\cancel{\partial/\partial x(0)}^{0} - \cancel{ay(\partial/\partial z)}^{0}\right) + \hat{z}\left(\partial/\partial x(bx) - \partial/\partial y(ay)\right)$$

$$= \hat{z}(b-a)$$

$$\int_S (b-a)\hat{z} \cdot d\vec{a}\,\hat{z} = (b-a) \cdot \underbrace{(\text{area})}_{A = \pi R^2} = \boxed{\pi R^2 (b-a)}$$

Line Integral

$$\oint_P \vec{v} \cdot d\vec{l} = \oint_P (ay)\,dx + \oint_P (bx)\,dy$$

using polar coor.

$$= \int_0^{2\pi} (ay)\,dx + \int_0^{2\pi} bx\,dy$$

$y = r\sin\theta$

$x = r\cos\theta$

$$\int_0^{2\pi} a(\underbrace{r\sin\theta \cdot (-r\sin\theta\,d\theta)}_{-r^2\sin^2\theta}) + b(\underbrace{r\cos\theta\,(r\cos\theta\,d\theta)}_{r^2\cos^2\theta})$$

$$= \int_0^{2\pi} r^2(-a\sin^2\theta + b\cos^2\theta)\,d\theta = r^2\int_0^{2\pi}(-a+b)\,d\theta$$

$$= r^2(-a+b)[2\pi - 0]$$

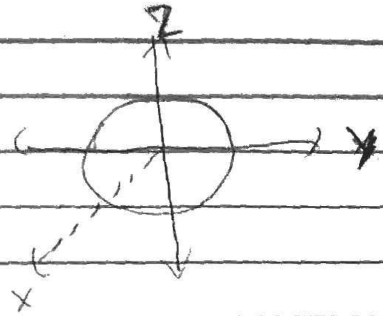
$$\boxed{= \pi r^2 (b-a)}$$

5/1, 5/5, ~~5/6~~, ~~5/7~~, ~~5/1~~, ~~X~~, ~~03~~, ~~X~~

Manuel Alvarez
11-2-10

HW #1

55) $\vec{v} = ay\hat{x} + bx\hat{y}$



Stokes Thm: $\int_S (\nabla \times v) \cdot da = \oint_P v \cdot dl$

surface integral

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ay & bx & 0 \end{vmatrix} = \hat{x}\left(\partial/\partial y(0) - bx(\partial/\partial z)\right) - \hat{y}\left(\partial/\partial x(0) - ay(\partial/\partial z)\right) + \hat{z}\left(\partial/\partial x(bx) - \partial/\partial y(ay)\right)$$

$$= \hat{z}(b-a)$$

$$\int_S (b-a)\hat{z} \cdot d\vec{a} \cdot \hat{z} = (b-a) \cdot (\text{area}) = \boxed{\pi R^2 (b-a)}$$

$A = \pi r^2$

Line Integral: $\oint_P V \cdot dl = \oint_P (ay)dx + \oint_P (bx)dy$

$$= \int_0^{2\pi} (ay)dx + \int_0^{2\pi} bx\,dy$$

using polar coord.

$y = r\sin\theta$
$x = r\cos\theta$

$$\int_0^{2\pi} a(r\sin\theta \cdot \underbrace{(-r\sin\theta\, d\theta)}_{-r^2\sin^2\theta}) + b(r\cos\theta(\underbrace{r\cos\theta}_{r^2\cos^2\theta}\, d\theta))$$

$$= \int_0^{2\pi} r^2(-a\sin^2\theta + b\cos^2\theta)\,d\theta = r^2\int_0^{2\pi}(-a+b)\,d\theta$$

$$= r^2(-a+b)[2\pi - 0]$$

$$\boxed{= \pi r^2(b-a)}$$

(3) a) Area of hemispherical bowl of R:

$x = r\sin\theta\cos\phi$ $\qquad$ $0 - \pi/2$

$y = r\sin\theta\cos\phi$ $\qquad$ $0 - 2\pi$

$z = r\cos\theta$

$\vec{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

A

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix} = \begin{matrix} \hat{x}\left(\partial/\partial\theta\,(\cos\theta) - (\partial/\partial\phi)\sin\theta\cos\theta\right) \\ \hat{y}\left(\partial/\partial r\,(\cos\theta) - \partial/\partial\phi\,(\cos\theta\sin\theta)\right) \\ \hat{z}\left(\partial/\partial r\,(\sin\theta\sin\theta) - \sin\theta\cos\theta(\partial/\partial\theta)\right) \end{matrix}$$

$\phi\,(-\cos\theta\sin\theta)\,z$

$$\int_0^{\pi/2}\int_0^{2\pi} r^2\sin\theta\cos\theta\,dr\,d\theta\,d\phi\,\hat{z} = r^2 2\pi\int_0^{\pi/2}\sin\theta\cos\theta\,dr\,d\theta\,\hat{z}$$

$=$

b) $\boxed{\oint_S d\vec{a} = 0}$, closed surface

c) $\oint_S d\vec{a} = 0$

its a closed surface

11-2-20

d) $a = \frac{1}{2} \oint_P r \times dl$

$da = \frac{1}{2}(r \times dl)$, thus $\int_S da = \boxed{a = \frac{1}{2} \int r \times dl}$

a

e) $\oint (c \cdot r) dl = a \times c$, $\oint \nabla T \times da = -\int_P T \cdot dl$

Hint: $T = c \cdot r$

$\nabla T = \nabla(c \cdot r) = c \times (\nabla \times r) \quad (c \cdot \nabla) r$

(3) $v = \vec{r}/r$, $r^n \vec{r}$

a)

(1.84) $\nabla \cdot \frac{\vec{v}}{r} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(\frac{r^2}{r}\right) = 1/r^2$

$R \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi = \int_0^{2\pi} d\phi \cdot \int_0^{\pi} \sin\theta$

$\boxed{= 4\pi R}$, no

$\nabla \cdot (r^n \vec{r}) = \frac{1}{r^2} \frac{\partial}{\partial r}(r^{n+2}) = (n+2) r^{n-1}$

$\boxed{\text{if and only if } n=-2, \text{ then it is a ~~delta~~ function}}$

b) $\boxed{\nabla \times r^n \vec{r} = 0}$

$\hat{\phi}/r \left[\frac{1}{r\sin\theta} \cancel{\frac{\partial}{\partial r}}^{0} (r \cdot 0) + \frac{1}{r\sin\theta} \cancel{\partial (r^n)}^{0} \right] \theta =$

0, thus

5(4) Divergence Thm.

$$v = r^2\cos\theta\,\vec{r} + r^2\cos\vec{\theta} + r^2\cos\theta\sin\phi\,\vec{\phi}$$

[Divergence] $\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\,\partial/_{\partial\theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(r^2\cos\theta\right)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(r^2\cos\phi\right)\right) +$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\theta\right)$$

$\boxed{\text{sphere } \mathcal{I} \text{ octant}}$

$$= 4r\cos\theta + \overline{r\cos\phi\cot\theta - r\cot\cos\phi}$$

$$= 4r\cos\theta$$

$\int(\nabla\cdot v)\,d = \int_0^R\int_0^{\pi/2}\int_0^{\pi/2}\left(4r\cos\theta\right) r^2\sin\theta\, d\theta\, d\phi$

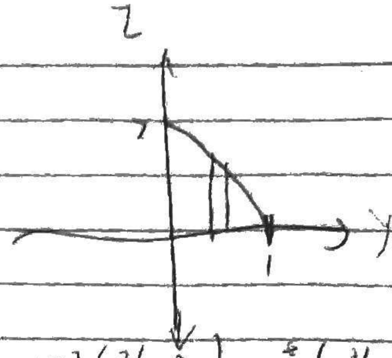
$$= 4\left[\frac{r^4}{4}\right]\Bigg|_0^R \int_0^{\pi/2}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\, d\phi$$

$$= R^4 \cancel{\int_0^{\pi/2}}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\,\cancel{d\phi} \rightarrow \pi/2$$

$$= R^4\left[\sin^2\theta/2\right]\Big|_0^{\pi/2}$$

$$\boxed{= \frac{\pi R^4}{4}}$$

5c) $\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$



L.I.: $\oint \vec{v}\cdot dl = \int \nabla\times\vec{v}\cdot d\vec{a}$

(surface integral)

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6 & yz^2 & 3y+z \end{vmatrix} = \hat{x}\left(\partial/\partial y(3y+z) - yz^2(\partial/\partial z)\right) - \hat{y}\left(\cancel{\partial/\partial x(3y+z)}^{0} - \cancel{(\partial/\partial z)}^{0}\right)$$
$$+ \left(\cancel{\partial/\partial x(yz^2)}^{0} - 6\cancel{(\partial/\partial z)}^{0}\right)$$

(curl) $\underline{\nabla\times\vec{v}} = \hat{x}(3-2yz^2)$

$\boxed{d\vec{a} = dy\,dz\,\hat{x}}$

$$\oint \nabla\times\vec{v}\cdot d\vec{a} = \int_0^1\int_0^{2-2y}(3-2yz^2)\hat{x}\,dy\,dx\,\hat{x}$$

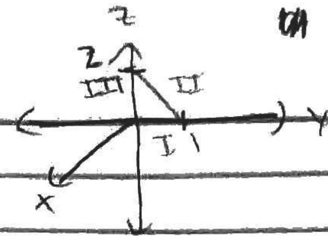
$$= 3\int_0^1\int_0^{2-2y} - 2\int_0^1\int_0^{2-2y} yz\,dy\,dz$$

$$= 3\int_0^1 (2-2y)\,dy = 3(2-1) = 3$$

$$= -2\int_0^1\int_0^{2-2y} z\,dz\,y\,dy = -2\int_0^1 \tfrac{1}{2}(2-2y)^2\,dy$$

$$\boxed{= 4/3}$$

5(c) Line Integral



$\int \vec{v} \cdot d\vec{l} =$

$\boxed{\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}}$

$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

Path(I): $x=0$ & $z=0$

Path(I): $\vec{v} \cdot d\vec{l} = \int_0^1 yz\,dy = \int_0^1 y(0)\,dy = \boxed{0}$ = Path(I)

Path(II): $\vec{v} \cdot d\vec{l} = \int_1^0 (yz\,dy + (3y+z)\,dz) = \int_1^0 yz\,dy + (3y+z)(-2dy)$

$z = (2-2y)$

$dz = -2dy$

$= \int_1^0 y(2-2y)^2\,dy + (-6y-4)\,dy$

$= \int_1^0 4y^3 - 8y^2 + 2y - 4)\,dy$

$= \left[y^4 - 8/3\,y^3 - \cancel{y} - 4y\right]\Big|_1^0$

$\boxed{= 14/3}$

Path(III): $x=0$, $y=0$

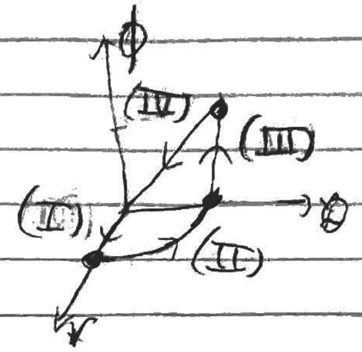
$\int_2^0 (3y+z)\,dz \Rightarrow \int_2^0 z\,dz = \left[\frac{z^2}{2}\right]\Big|_2^0 = \boxed{-2}$

$\frac{14}{3} - 2 = 14/3 - 6/3 = \boxed{8/3}$

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$dl =$

57) Line Integral: $\vec{v} = (r\cos^2\theta)\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}$



$v \cdot d\vec{l}$

Path(I):

$\phi =$ ~~const.~~

$\theta = \pi/2$

~~$r =$~~

$$= \int_0^1 (r\cos^2\theta\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}) \cdot dl$$

$$= \int_0^1 r\cancel{\cos^2\theta}^{\,\cos(\pi/2)=0}\,dr - r^2\cancel{\cos\theta}^{\,0}\sin\theta\,d\theta + 3r^2\cancel{\sin\theta}\,d\phi$$

$$= \int_0^1 0 = \boxed{0}$$

Path (II):

$r =$ ~~1~~

$\phi = \pi/2$

$\theta = \pi/2$

$$\int v\cdot dl = \int_0^{\pi/2} 3r^2\sin(\pi/2)\,d\phi = \int_0^{\pi/2} 3\,d\phi$$

$$= 3\phi = \boxed{3\pi/2}$$

Path(III):

$r = 1$

$\theta = \pi/2$

$\phi = 2$

$$\int v\cdot dl = \int_0^2 3$$

$\int (\nabla \times \vec{v}) \cdot dA$

57) Stoke's Thm $\vec{v} = (r\cos^2\theta)\vec{r} - (r\cos\theta\sin\theta)\vec{\theta} + 3r\vec{\phi}$

curl

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{r} & \vec{\theta} & \vec{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ r\cos^2\theta & r\cos\theta\sin\theta & 3r \end{vmatrix}$$

$\cdot \sin\theta$

$$= \frac{1}{r\sin\theta}\vec{r}\left(\partial/\partial\theta\,(3r) - \cancel{\partial/\partial\phi(r\cos\theta\sin\theta)}\right) - \vec{\theta}\left(\partial/\partial r(r3r) - \cancel{\partial/\partial\phi(r\cos^2\theta)}\right)$$

$$+ \vec{\phi}\left(\partial/\partial r\,(\cancel{r\cos\theta\sin\theta}) - \partial/\partial\theta\,(r\cos^2\theta)\right)$$

$$= (-6r)/r = -6\vec{\Theta}$$

$$= \frac{3r\cos\theta}{r\sin\theta}\vec{r} = 3\cot\theta\,\vec{r}$$

$$\nabla \cdot \vec{v} = \boxed{3\cot\theta\,\vec{r} - 6\vec{\theta}}$$

no ϕ = $da_1 = 0$ $\quad da_1$! $\quad da_2 = -r\,dr\,d\phi\,\vec{\theta}$

$$\int(\nabla \cdot v)\cdot dA = \cancel{\int_1 (\nabla \times V) \cdot da_1} + \int_2 (\nabla \times V)\cdot d_2$$

$$= \int_2 (\nabla \times V)\cdot da_2$$

$$= \int_0^{3} \int_0^{\pi/2} (-6)\,r\,dr\,d\phi\,\vec{\theta} = 6\left[\frac{r}{2}\right]\Bigg|_0^{1} + [r]\Bigg|_0^{\pi/2}$$

$$\frac{6}{2}\cdot\frac{\pi}{2} = \boxed{3\pi/2}$$

=

$$\boxed{\int_V (\nabla \cdot v)\, d\tau = \oint_S v \cdot da}$$

5a) Div. Thm: $\vec{v} = r^2 \sin\theta\, \hat{r} + 4r^2 \cos\theta\, \hat{\theta} + r^2 \tan\theta\, \hat{\phi}$

$$\nabla \cdot v = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 (r^2 \sin\theta \hat{r}) \right) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \cdot (4r^2 \cos\theta \hat{\theta}) \right) +$$

$$\cancel{\frac{1}{r\sin\theta} \frac{\partial (r^2 \tan\theta)}{\partial \phi}} \to 0$$

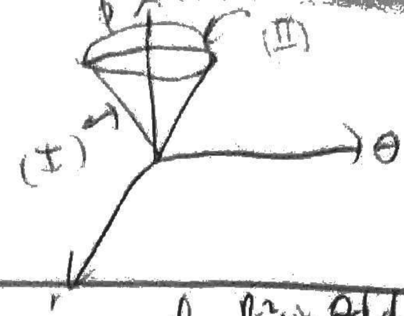
$$(2\cos^2\theta - 1) = (\cos^2\theta$$

$$= \frac{4r^3 \sin\theta}{r^2} + \frac{4r\,(\cos(2\theta))}{\sin\theta}$$

$$= 4r\sin\theta + \frac{4r(2\cos^2\theta - 1)}{\sin\theta}$$

$$= \int_0^R \int_0^{\pi/2} \int_0^{2\pi}$$

da



5a) $\int v \cdot da = \int R^2 \sin\theta \,(da)$ $\quad da = R^2 \sin\theta \, d\phi \, d\theta \, \hat{r}$

$r = R$
$\phi = 0 \to 2\pi$
$\theta = 0 \to \pi/6$

$$= \int_0^{2\pi} \int_0^{\pi/6} R^4 \sin^2\theta \, d\theta \, d\phi$$

PATH (I):

$$= 2\pi R^4 \int_0^{\pi/6} \sin^2\theta \, d\theta$$

$$= 2\pi R^4 \left(\frac{\cancel{\sin(\pi/3)} - \pi/3}{4} \right) \quad \pi - 3\sqrt{3}/2$$

$da = r \sin\theta \, d\phi \, dr \, \hat{\theta}$

$v \cdot da = 4r^3 \cos\theta \sin\theta \, d\phi \, dr$

Path (II) $= \int_0^R \int_0^{2\pi}$

$$= \int_0^R \int_0^{2\pi} 4r^3 \cos\theta \sin\theta \, d\phi \, dr = 2\pi \int_0^R 4r^3 \, dr$$

$$= 2\pi \left[\frac{4r^4}{4} \right]_0^R$$

$$= 2\pi R^4$$

(a) $D(r,\epsilon) = -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\epsilon^2}} = \left(\frac{3\epsilon^2}{4\pi}\right)\frac{1}{(r^2+\epsilon^2)^{5/2}}$

$\epsilon \to 0$, "

$\delta^3(r), \epsilon \to 0$

(a) $D(r,\epsilon) = \left(\frac{3\epsilon^2}{4\pi}\right)\left(r^2+\epsilon^2\right)^{-5/2}$

$\partial/\partial r$

$-\frac{1}{4\pi}\left(\nabla^2 \frac{1}{\sqrt{r^2+\epsilon^2}}\right) = \left[\frac{1}{r^2}\,\partial/\partial r\left(r^2 \cdot \frac{r}{\sqrt{r^2+\epsilon^2}}\right)\right] \cdot \frac{1}{4\pi}$

$= \frac{1}{4\pi r^2}\,\partial/\partial r\left[\frac{r^2}{(r^2+\epsilon^2)^{1/2}}\right] = \frac{1}{4\pi r^2}\,\partial/\partial r\left[\frac{2r}{(r^2+\epsilon)^{3/2}}\right]$

$=$

d) $D(0,\epsilon) \Rightarrow \infty$, @ $\epsilon \to 0$

$\int_0^\infty D(r,\epsilon)\,4\pi r^2\,dr = 3\epsilon^2\int_0^\infty \frac{1}{(r^2+\epsilon^2)^{5/2}}\,dr = 3\epsilon^2\left(\frac{1}{3\epsilon^2}\right)$

$\boxed{=1}$

b) $D(r,\epsilon) = D(0,\epsilon)$

$= \frac{1}{\sqrt{\epsilon^2}} \cdot \frac{-3\epsilon^2}{4\pi} \Rightarrow \boxed{\frac{-3\epsilon^2}{4\pi}}$

c) $r \neq 0$, $\epsilon \to 0$; $\boxed{\frac{3(0)^2}{4\pi}\,\frac{1}{(0)^2+(0)^2} = 0}$