Midterm   Jracksen Dnd

1.1) a) It's the divergence of A multiplied by B.

$$(\vec{A}\cdot\nabla)\vec{B} = (A_x\hat{x}+A_y\hat{y}+A_z\hat{z})\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)B$$

$$\Rightarrow \left(A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z}\right)\vec{B}, \quad \vec{B}=(B_x\hat{x}+B_y\hat{y}+B_z\hat{z})$$

$$\Rightarrow \left(A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}\right)\hat{x}+\left(A_x\frac{\partial B_y}{\partial x}+A_y\frac{\partial B_y}{\partial y}+A_z\frac{\partial B_y}{\partial z}\right)\hat{y}+\left(A_x\frac{\partial B_z}{\partial x}+A_y\frac{\partial B_z}{\partial y}+A_z\frac{\partial B_z}{\partial z}\right)\hat{z}$$

b) Since $\hat{r}=\frac{\vec{r}}{r}$ let $\hat{r}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, $(\hat{r}\cdot\nabla)\hat{r}$

$$(r_x\cdot\nabla)r_x = \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)+\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{d}{dy}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)+$$

$$\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{d}{dz}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\Rightarrow\frac{xy^2+xz^2-xy^2-xz^2}{(x^2+y^2+z^2)^{3/2}}\Rightarrow 0$$

$\therefore (r_y\cdot\nabla)r_y=0$ & $(r_z\cdot\nabla)r_z=0$ $\therefore$ $(\hat{r}\cdot\nabla)\hat{r}=0$

c) $F=(p\cdot\nabla)E$, $E=-\frac{dV}{dx}=-2V_0 r$, $V(r)=V_0r^2+V_1$, $p=qd\hat{x}$

$$\Rightarrow\left(qd\frac{d}{dx}\hat{x}+0\frac{d}{dy}+0\frac{d}{dz}\right)(-2V_0\vec{r})$$

$$\Rightarrow\left(-2V_0r_x\frac{d}{dx}\hat{x}+0\frac{d}{dy}(-2V_0r_y)+0\frac{d}{dz}(-2V_0r_z)\right)$$

$$\therefore F=-2V_0r_x\frac{d}{dx}\hat{x}$$

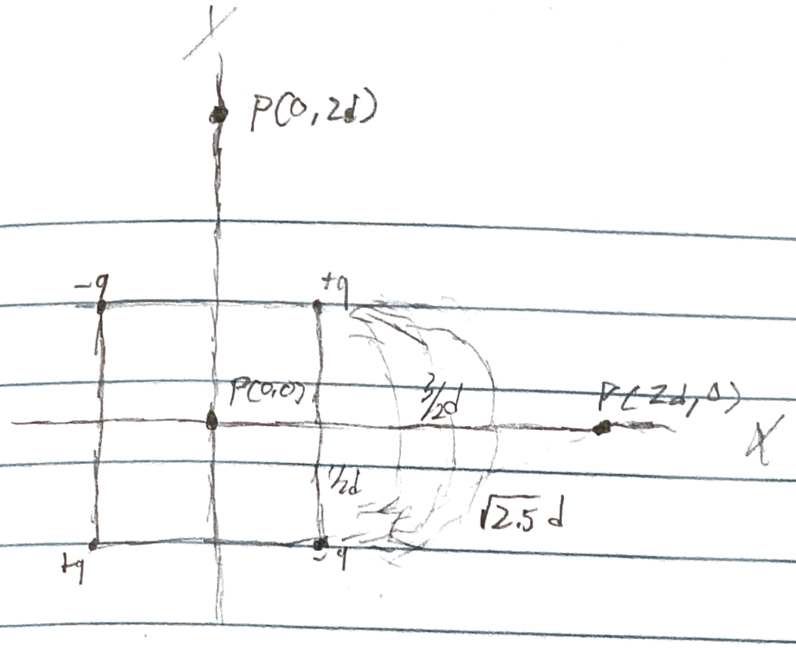
1.2) ~~three dimensional dirac delta function~~

$$J = \int_V e^{-r}\left(\nabla \cdot \frac{\hat{r}}{r^2}\right) \qquad \nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\,\delta^3(r)$$

$$J \Rightarrow \int_V e^{-r}\left(4\pi\,\delta^3(r)\right) \Rightarrow 4\pi \int_V e^{-r}\,\delta^3(r) \Rightarrow 4\pi \cdot 1 = \boxed{4\pi}$$

eh

2.1



$\sqrt{.5^2 + 1.5^2} = \sqrt{2.5}$

a) Since P(0,0) is located at the center of both of the dipoles, the charges will cancel such that $\vec{E} = \vec{0}$.

b) P(0,2d) & P(2d,0) are perpendicular so the answers are the same with respect to the perpendicular axis.

$E_{tot} = \sum E_i = \frac{KQ_i}{r_i^2}$ $\quad |r_i| = \sqrt{2.5}\,d$, each charge is the same distance away.

$Q_i = q_i\left(\frac{3d}{2}\hat{x} - \frac{d}{2}\hat{y}\right)$ the charge would be on this vector

P(0,2d): $E_{tot} = \sum E_i = \frac{K q_i \left(\frac{3d}{2}\hat{x} - \frac{d}{2}\hat{y}\right)}{2.5d}$

&

P(2d,0): $E_{tot} = \sum E_i = \frac{K q_i \left(\frac{3d}{2}\hat{y} - \frac{d}{2}\hat{x}\right)}{2.5d}$

2.7) $V(r) = A\dfrac{e^{-\lambda r}}{r}$, $E = -\nabla V$

$$E = -\frac{\partial}{\partial r} A\frac{e^{-\lambda r}}{r} \Rightarrow A\frac{\lambda r e^{-\lambda r} + e^{-\lambda r}}{r^2} \Rightarrow Ae^{-\lambda r}\left(\frac{\lambda r + 1}{r^2}\right)\hat{r}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \quad \rho = \varepsilon_0 A\left(4\pi\delta^3(r) - \lambda^2\frac{e^{-\lambda r}}{r}\right)$$

$$\frac{\rho}{\varepsilon_0} = \nabla \cdot Ae^{-\lambda r}\left(\frac{\lambda r + 1}{r^2}\right)\hat{r} = \underbrace{\nabla \cdot \frac{\hat{r}}{r^2}}_{\to 4\pi\delta^3 r}\left(Ae^{-\lambda r}(\lambda r + 1)\right)$$

$$\boxed{\rho = \varepsilon_0 A\left(4\pi\delta^3(r) - \lambda^2\frac{e^{-\lambda r}}{r}\right)}$$

$$Q = \int \rho \, d\tau$$

$$Q = \int \varepsilon_0 A\left(4\pi\delta^3(r) - \lambda^2\frac{e^{-\lambda r}}{r}\right)d\tau$$

$4\pi r^2$ for area of sphere

$$\Rightarrow \varepsilon_0 A\left(4\pi\left(\int\delta^3(r)d\tau - \int\lambda^2\frac{e^{-\lambda r}}{r}d\tau\right)\right)$$

$$\Rightarrow \varepsilon_0 A\left(4\pi(1) - 4\pi\lambda^2\int\frac{e^{-\lambda r}}{r}(r^2)\,dr\right)$$

$$\Rightarrow \varepsilon_0 A\left(4\pi - 4\pi\lambda^2\int r e^{-\lambda r}\,dr\right)$$

$$\Rightarrow \varepsilon_0 A 4\pi - \varepsilon_0 A 4\pi\lambda^2\left[\frac{-re^{-\lambda r}}{\lambda} - \frac{e^{-\lambda r}}{\lambda^2}\right]$$

$$\Rightarrow \varepsilon_0 A 4\pi - \varepsilon_0 A 4\pi \cancel{\lambda^2}\frac{1}{\cancel{\lambda^2}}$$

$$\boxed{Q = 0}$$

2.3

a) $\oint \vec{E} \cdot da = \frac{Q_{enc}}{\varepsilon_0}$, $Q_{enc} = L\lambda$ let L be length wire

$EA = \frac{L\lambda}{\varepsilon_0}$, $A = 2\pi s L$, $E(2\pi s L) = \frac{L\lambda}{\varepsilon_0}$

So $\hat{da}$ $\boxed{\vec{E} = \frac{\lambda}{\varepsilon_0 2\pi s}\hat{s}}$

b) $\vec{F} = Q\vec{E}$

$ma = \frac{q\lambda}{2\pi s \varepsilon_0}$, $a \to \frac{dv}{dt} = \frac{d^2 s}{dt^2}$ so $\frac{d^2 s}{dt^2} = \frac{q\lambda}{2\pi m s \varepsilon_0}$

$\frac{ds}{dt} = \int \frac{q\lambda}{2\pi m s \varepsilon_0} dt \Rightarrow \frac{q\lambda}{2\pi m \varepsilon_0} \int \frac{1}{s} dt \Rightarrow \frac{q\lambda}{2\pi m \varepsilon_0} \frac{t}{s}$

$\frac{ds}{dt} \Rightarrow s(t) = \frac{q\lambda}{2\pi m s \varepsilon_0} \int t\, dt = \frac{q\lambda}{2\pi m s \varepsilon_0} \cdot \frac{t^2}{2} + C$

$\boxed{s(t) = \frac{q\lambda t^2}{4\pi m s \varepsilon_0} + C}$

3.1

a) $\sigma(\theta) = \frac{\varepsilon_0}{2R}\sum_{L=0}^{\infty}(2L+1)^2 C_L P_L(\cos\theta)$

where $C_L = \int_0^{\pi} V_0(\theta) P_L(\cos\theta)\sin\theta \, d\theta$

let $V(r,\theta) = \sum_{L=0}^{\infty}\left(A_L r^L + \frac{B_L}{r^{(L+1)}}\right) P_l(\cos\theta)$

where for $r<R$ $V(r,\theta) = \sum_{L=0}^{\infty}(A_L r^L) P_l(\cos\theta)$

& $r>R$ $V(r,\theta) = \sum_{L=0}^{\infty}\left(\frac{B_L}{r^{(L+1)}}\right) P_l(\cos\theta)$

Because there is no charge inside or outside the sphere

$V(r,\theta)$ where $r<R$ must be $=$ $r>R$

$A_L r^L = \frac{B_L}{r^{(L+1)}} \rightarrow A_L r^L (r^{(L+1)}) = B_L$

$A_L r^{(2L+1)} = B_L$

So, $V(r,\theta) = \sum_{L=0}^{\infty}\left(A_L r^L + \frac{A_L r^{(2L+1)}}{r^{(L+1)}}\right) P_l(\cos\theta)$

$\Rightarrow \sum_{L=0}^{\infty}\left(A_L r^L \left(1 + \frac{r^{(L+1)}}{r^{(L+1)}}\right)\right) P_l(\cos\theta)$

$\Rightarrow \sum_{L=0}^{\infty}(2A_L r^L) P_l(\cos\theta)$

Since $\frac{\sigma(\theta)}{\varepsilon_0} = -\frac{dV}{dr}$, (note $r=R$), so $\frac{dV}{dr}\left(\sum_{L=0}^{\infty}(2A_L R^L) P_L(\cos\theta)\right)$

$-\frac{\sigma(\theta)}{\varepsilon_0} = \sum_{L=0}^{\infty}(-2LA_L R^{L-1}) P_l(\cos\theta)$

let $A_L = \frac{2L+1}{4R}\int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta \, d\theta$ such that

$\sigma(\theta) = \varepsilon_0 \sum_{L=0}^{\infty} 2L\left(\frac{2L+1}{4R}\right)\int_0^{\pi} V_0(\theta) P_L(\cos\theta)\sin\theta \, d\theta$

$\Rightarrow \frac{\varepsilon_0}{2R}\sum_{L=0}^{\infty}(2L+1)^2 C_L P_L(\cos\theta)$, with C_L from before

∴ charge density formula holds

$C_L \downarrow$

3.1 b) Since $\sigma(\theta) = \frac{\epsilon_0}{2R} \sum_{L=0}^{\infty} (2L+1)^2 P_L \int_0^{\pi} V_0(\theta) P_L(\cos\theta)\sin\theta \, d\theta$

$\cancel{V_0(\theta) = P_2(\cos\theta)}$

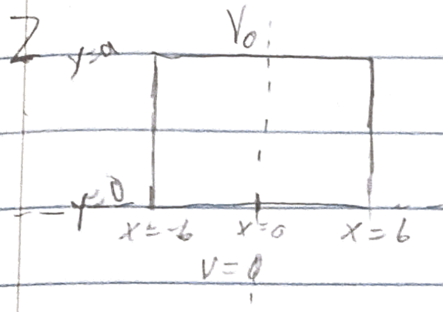
so $\sigma(\theta) = \frac{\epsilon_0}{2R} \sum_{L=0}^{\infty} (2L+1)^2 P_L(\cos\theta) \int_0^{\pi} P_2(\cos\theta) P_L(\cos\theta) \sin\theta \, d\theta$

~~→ note, the cos will be orthogonal~~

~~so $C_L \to 0$ & $\sigma(\theta) = 0$~~

~~when $V_0(\theta) = P_2(\cos\theta)$~~

3.2



$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$, $V(x,y) = X(x)Y(y)$

so $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$

let $\frac{\partial^2 X}{\partial x^2} = -k^2 X$ & $\frac{\partial^2 Y}{\partial y^2} = k^2 Y$

$\cancel{\text{So } X(x) = A\cos(kx) + B\sin(kx)}$, $k = \frac{n\pi}{b}$

if $x = \pm b$ then $0 = A\cos(kx) + B\sin(kx)$

$A\cos(kx) = -B\sin(kx)$

$A = -B\left(\frac{\sin(kx)}{\cos(kx)}\right) \Rightarrow -B(0)$, $A \Rightarrow 0$

So, $X(x) = 0 + B\sin(kx)$, then $Y(y) = Ce^{ky} + De^{-ky}$

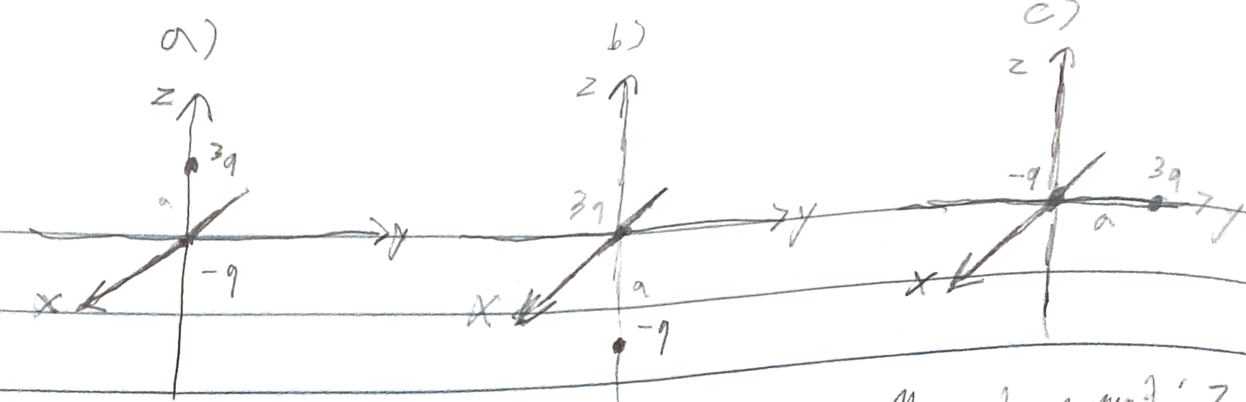
$Y(y) = Ce^{\frac{n\pi y}{b}} + De^{\frac{-n\pi y}{b}}$ $C = -D$

so $V(x,y) = X(x)Y(y) = B\sin\left(\frac{n\pi x}{b}\right) C\left[e^{\frac{n\pi x}{b}} - e^{\frac{-n\pi y}{b}}\right]$ B & C both const. BC → C

$V(x,y) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi x}{b}\right) C_n \sinh\left(\frac{n\pi y}{b}\right)$

to add n_s

3.3



Monopole moment: 2q   Monopole moment: 2q   Monopole moment: 2q

a) Dipole moment; $\bar{p} = \sum_{i=1}^{2} q_i \bar{r}_i = -q(0) + 3qa\hat{z} \Rightarrow 3qa\hat{z}$

$$V = V_{mon} + V_{dip} = \frac{Q_{tot}}{4\pi\varepsilon_0 r} + \frac{\bar{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}, \quad \hat{r} = \cos\theta$$

$$\therefore V = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right]$$

b) $V_{mon} = 2q$, $V_{Dip} = p = \sum_{i=1}^{2} q\bar{r}_i = 3q(0) + -q(a)(-\hat{z}) = qa\hat{z}$

$$V = \frac{2q}{4\pi\varepsilon_0 r} + \frac{qa\hat{z}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right]$$

c) $V_{mon} = 2q$, $p = \sum_{i=1}^{2} q_i\bar{r}_i = -q(0) + 3qa\hat{y} = 3qa\hat{y}$

$$V = \frac{2q}{4\pi\varepsilon_0 r} + \frac{3qa\hat{y}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right]$$

$\hat{r}\cdot\hat{y} = \sin\theta\sin\phi\,\hat{y}$