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PHYS 330

HW #1 1.54, 1.55, 1.56, 1.57, 1.59, 1.62, 1.63, 1.64

1.54) Check the divergence theorm for the function

$$v = r^2\cos\theta\hat{r} + \hat{r}\cos\phi\hat{\theta} - r^2\cos\theta\sin\phi\hat{\phi}$$

using as your volume one octant of the sphere of radius R.

$$\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta r^2\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$$

$$= \frac{1}{r^2}4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta r^2\cos\phi + \frac{1}{r\sin\theta}(-r^2\cos\theta\cos\phi)$$

$$= \frac{r\cos\theta}{\sin\theta}[4\sin\theta + \cos\phi - \cos\phi]$$

$$= 4r\cos\theta$$

$$\int(\nabla\cdot v)d\tau \Rightarrow \int(4r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 4\int_0^R r^3 dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi$$

$$= R^4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

$$\boxed{= \frac{\pi R^4}{4}}$$

1.55) Check Stokes' Theorem using the function $v = ay\hat{x} + bx\hat{y}$ (a & b are constants) and the circular path of radius R, centered at the origin in the xy plane.

$$\nabla\times V = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ ay & bx & 0 \end{vmatrix} = \cancel{b\hat{x}(0) - \hat{y}(0)} + \hat{z}(b-a)$$

$$= \hat{z}(b-a)$$

so $\int(\nabla\times v)\cdot da = (b-a)\pi R^2$

$$v\cdot dl = (ay\hat{x} + bx\hat{y})\cdot(dx\hat{x} + dy\hat{y} + dz\hat{z})$$
$$= ay\,dx + bx\,dy$$

$$\Rightarrow v\cdot dl = ay\,dx + bx\left(-\frac{x}{y}\right)dx$$
$$= \frac{1}{y}(ay^2 - bx^2)dx$$

$$x^2 + y^2 = R^2 \Rightarrow 2x\,dx + 2y\,dy = 0$$
$$dy = -\left(\frac{x}{y}\right)dx$$

<u>upper semi-circle</u>

$$y = \sqrt{R^2 - x^2}$$

$$\Rightarrow v\cdot dl = \frac{a(R^2 - x^2) - bx^2}{\sqrt{R^2 - x^2}}dx$$

$$\int v\cdot dl = \int_R^{-R}\frac{aR - (a+b)x^2}{(R^2 - x^2)^{1/2}}dx = \left\{aR^2\sin^{-1}\left(\frac{x}{R}\right) - (a+b)\left[\frac{-x}{2}(R^2-x^2)^{1/2} + \frac{R^2}{2}\sin^{-1}\left(\frac{x}{R}\right)\right]\right\}\Big|_{-R}^{+R}$$

$$= \frac{1}{2}R^2(a-b)\sin^{-1}\left(\frac{x}{R}\right)\Big|_{+R}^{-R} = \frac{1}{2}R^2(a-b)(\sin^{-1}(-1) - \sin^{-1}(+1))$$

$$= \frac{1}{2}R^2(a-b)\left(-\frac{\pi}{2} - \frac{\pi}{2}\right)$$

$$\boxed{= \frac{1}{2}\pi R^2(b-a)}$$

→

1.56) Compute the line integral of $v = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$ along the triangular path in Fig. 1.49

① $x = z = 0$; $dx = dz = 0$; $y: 0 \to 1$

$v \cdot dl = (yz^2)dy = 0$; $\int v \cdot dl = 0$

② $x = 0$; $z = 2-2y$; $dz = -2dy$; $y: 1 \to 0$

$v \cdot dl = (yz^2)dy + 3(y+z)dz = y(2-2y)^2 dy - (3y + 2 - 2y)\,2dy$

$$\int v \cdot dl = 2\int_1^0 (2y^3 - 4y^2 + y - 2)\,dy = 2\left(\frac{y^4}{2} - \frac{4y^3}{3} + \frac{y^2}{2} - 2y\right)\Big|_1^0$$

$$= 14/3$$

③ $x = y = 0$; $dx = dy = 0$; $z: 2 \to 0$

$v \cdot dl = \int_2^0 z\,dz = \frac{z^2}{2}\Big|_2^0 = 2$

Total: $\oint v \cdot dl = 0 + \frac{14}{3} - 2 = \frac{8}{3}$

Check:

$(\nabla \times v)_x = \frac{\partial}{\partial y}(3y+z) - \frac{\partial}{\partial z}(yz^2) = 3 - 2yz$

$$\int (\nabla \times v) \cdot da = \iint (3 - 2yz)\,dy\,dz = \int_0^1 \left[\int_0^{2-2y} (3-2yz)\,dz\right] dy$$

$$= \int_0^1 \left[3(2-2y) - 2y\tfrac{1}{2}(2-2y)^2\right] dy = \int_0^1 (-4y^3 + 8y^2 - 10y + 6)\,dy$$

$$= \left(-y^4 + \frac{8}{3}y^3 - 5y^2 + 6y\right)\Big|_0^1 = -1 + 8/3 + 5 + 6$$

$$= 8/3 \checkmark$$

1.57) Compute the line integral of $v = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}$ around path in Fig. 1.50.

① $\theta = \frac{\pi}{2}$; $\phi = 0$; $r: 0 \to 1$; $v \cdot dl = (r\cos^2\theta)(dr) = 0$; $(dr) = 0$; $\int v \cdot dl = 0$

② $r = 1$; $\theta = \frac{\pi}{2}$; $\phi: 0 \to \frac{\pi}{2}$; $v \cdot dl = (3r)(r\sin\theta\, d\phi) = 3d\phi$; $\int v \cdot dl = 3\int_0^{\pi/2} d\phi = \frac{3\pi}{2}$

③ $\phi = \frac{\pi}{2}$; $r\sin\theta = y = 1$; $r = \frac{1}{\sin\theta}$; $dr = \frac{-1}{\sin^2\theta}\cos\theta\, d\theta$; $\theta: \frac{\pi}{2} \to \theta_0 \equiv \tan^{-1}(\frac{1}{2})$

$$v \cdot dl = (r\cos^2\theta)(dr) - (r\cos\theta\sin\theta)(r\,d\theta) = \frac{\cos^2\theta}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\right)d\theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}\,d\theta$$

$$\int v \cdot dl = -\int_{\frac{\pi}{2}}^{\theta_0} \frac{\cos\theta}{\sin^3\theta}\,d\theta = \frac{1}{2\sin^2\theta}\Big|_{\pi/2}^{\theta_0} = \frac{1}{2(1/5)} - \frac{1}{2(1)} = \frac{5}{2} - \frac{1}{2} = 2$$

④ $\theta = \theta_0$; $\phi = \frac{\pi}{2}$; $r: \sqrt{5} \to 0$; $v \cdot dl = (r\cos^2\theta)(dr) = \frac{4}{5}r\,dr$

$$\int v \cdot dl = \frac{4}{5}\int_{\sqrt{5}}^0 r\,dr = \frac{4}{5}\frac{r^2}{2}\Big|_{\sqrt{5}}^0 = -\frac{\cancel{4}^{2}}{\cancel{5}} \cdot \frac{\cancel{5}}{\cancel{2}} = -2$$

Total: $\oint v \cdot dl = \cancel{0} + \frac{3\pi}{2} + \cancel{2} - \cancel{2} = 3\pi/2$

1.59) Check divergence theorem for $\mathbf{v} = r^2 \sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r^2\tan\theta\,\hat{\phi}$ using the volume of the ice cream cone in Fig. 1.52.

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2 \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, 4r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\tan\theta)$$

$$= \frac{1}{r^2}4r^3\sin\theta + \frac{1}{r\sin\theta}4r^2(\cos^2\theta - \sin^2\theta) = \frac{4r}{\sin\theta}(\sin^2\theta + \cos^2\theta - \sin^2\theta)$$

$$= 4r\,\frac{\cos^2\theta}{\sin\theta}$$

$$\int(\nabla\cdot\mathbf{v})\,d\tau = \int\left(4r\frac{\cos^2\theta}{\sin\theta}\right)(r^2\sin\theta\,dr\,d\theta\,d\phi) = \int_0^R 4r^3dr\int_0^{\pi/6}\cos^2\theta\,d\theta\int_0^{2\pi}d\phi$$

$$= (R^4)(2\pi)\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/6}$$

$$= 2\pi R^4\left(\frac{\pi}{12} + \frac{\sin 60^\circ}{4}\right) = \frac{\pi R^4}{6}\left(\pi + 3\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi R^4}{12}(2\pi + 3\sqrt{3})}$$

1.62) (a) $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

$$\mathbf{a} = \int R^2\sin\theta\cos\theta\,d\theta\,d\phi\,\hat{z} = 2\pi R^2\hat{z}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = 2\pi R^2\hat{z}\,\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}$$

$$= \pi R^2\hat{z}$$

(b) $T = 1$ in (a)

$\nabla T = 0$, $\oint d\mathbf{a} = 0$

(c) suppose $\mathbf{a}_1 \neq \mathbf{a}_2$ put together to make closed surface

$\oint d\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2 \neq 0$

(d) one triangle, $d\mathbf{a} = \frac{1}{2}(\mathbf{r}\times d\mathbf{l})$

$\mathbf{a} = \frac{1}{2}\oint \mathbf{r}\times d\mathbf{l}$

(e) $T = \mathbf{c}\cdot\mathbf{r}$ $\quad \nabla T = \nabla(\mathbf{c}\cdot\mathbf{r}) = \mathbf{c}\times\underbrace{(\nabla\times\mathbf{r})}_{=0} + \underbrace{(\mathbf{c}\cdot\nabla)\mathbf{r}}$

$$= \left(c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z})$$

$$= c_x\hat{x} + c_y\hat{y} + c_z\hat{z}$$

$$= \mathbf{c}$$

$$\oint T\cdot d\mathbf{l} = \oint(\mathbf{c}\cdot\mathbf{r})\,d\mathbf{l} = -\int(\nabla T)\times d\mathbf{a} = -\int\mathbf{c}\times d\mathbf{a} = -\mathbf{c}\times\int d\mathbf{a} = -\mathbf{c}\times\mathbf{a} = \mathbf{a}\times\mathbf{c}$$

1.63) ① $\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r) = \frac{1}{r^2}$

$\int v \cdot da = \int\left(\frac{1}{R}\hat{r}\right)\cdot\left(R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}\right) = R\int \sin\theta\, d\theta\, d\phi = 4\pi R$

$\int(\nabla \cdot v)\, d\tau = \int\left(\frac{1}{r^2}\right)\left(r^2 \sin\theta\, d\theta\, d\phi\right) = \left(\int_0^R dr\right)\left(\int \sin\theta\, d\theta\, d\phi\right) = 4\pi R$

$\nabla \times (r^n \hat{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+2}) = \frac{1}{r^2}(n+2)r^{n-1} = (n+2)r^{n-1}$

$\nabla \cdot (\hat{r}/r^2) = 4\pi\delta^3(r)$

$\nabla \times (r^n \hat{r}) = 0 \rightarrow \int(\nabla \times v)\, d\tau = 0$

$\boxed{v \times da = 0}$

1.64) (a) $\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$

$D = \frac{-1}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(-\frac{1}{2}\right)\frac{2r}{(r^2+\varepsilon^2)^{5/3}}\right] = \frac{1}{4\pi r^2}\frac{3r^2}{(r^2+\varepsilon^2)^{5/2}}\left(\cancel{r^2}+\varepsilon^2-\cancel{r^2}\right)$

$= \frac{3\varepsilon^2}{4\pi(r^2+\varepsilon^2)^{5/2}}$

(b) $D(0,\varepsilon) = \frac{3\varepsilon^2}{4\pi\varepsilon^5} = \frac{3}{4\pi\varepsilon^3}$

(c) from (a) $D(r,0) = 0$ for $r \neq 0$

(d) $\int D(r,\varepsilon)4\pi r^2 dr = 3\varepsilon^2\int_0^\infty \frac{r^2}{(r^2+\varepsilon^2)^{5/2}}dr = \cancel{3\varepsilon^2}\left(\frac{1}{\cancel{3\varepsilon^2}}\right) = 1$ ✓