# Electromagnetc Theory: PHYS330

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# Summary

# Week 3 Summary

1. Laplace's Equation
    - One-dimension
    - Two-dimensions, three dimensions, uniqueness, boundaries
2. Separation of Variables: Boundary-value problems
    - Cartesian coordinates
    - Spherical coordinates
3. Multipole Expansions
    - Far-fields
    - Monopole and dipole terms
    - Electric Field of a Dipole

# Laplace’s Equation: One Dimension

# Laplace's Equation: One dimension

## Laplace's Equation in one dimension:

$$\frac{d^2V}{dx^2} = 0 \tag{1}$$

What is the solution?

$$V(x) = mx + b \tag{2}$$

What is the magnitude of the E-field?

- A: $V(x)$
- B: $x$
- C: $b$
- D: $m$

# Laplace's Equation: One dimension

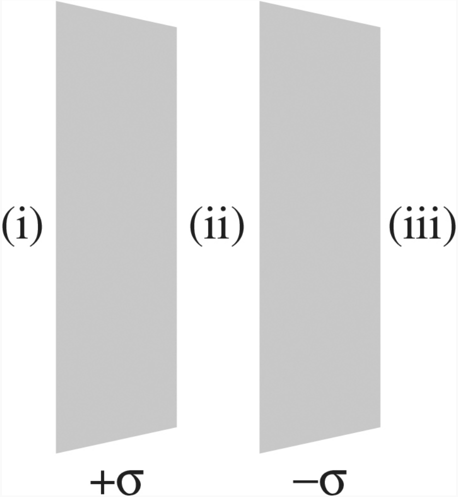


Figure 1: The setup of a parallel plate capacitor.

# Laplace's Equation: One dimension

Suppose the negative side of the parallel plate capacitor is grounded, and the positive side is at a potential $V_0$. Let the separation between the plates be $x_0$. Further, let the positive plate occupy the yz plane, passing through the origin. Find the E-field magnitude and direction by solving Laplace's equation.

# Laplace’s Equation: One dimension

Show that the potential of a point charge at the origin satisfies Laplace’s Equation for $r \neq 0$. *Use the form of the Laplacian in spherical coordinates.*

# Boundary Conditions

# Boundary Conditions

Let $V(x) = mx + b$. If $V(-a) = V_0$, and $V(a) = -V_0$, what are valid expressions for $m$ and $b$?

- A: $b = 0$, and $m = -2V_0$
- B: $b = a$, and $m = V_0/a$
- C: $b = 0$, and $m = -V_0/a$
- D: $b = V_0$, and $m = -V_0/a$

# Boundary Conditions

Let $V(x) = mx + b$. If $V(-a) = V_0$, and $V(a) = -V_0$, what is the electric field?

- A: $\frac{V_0}{a}\hat{x}$
- B: $-\frac{V_0}{a}\hat{x}$
- C: $V_0\hat{x}$
- D: $-V_0\hat{x}$

# Boundary Conditions

Suppose a potential function $V(x, y) \propto (A\,\mathrm{exp}(-kx) + B\,\mathrm{exp}(kx))$. Which of the following is true, if $V \to 0$ as $x \to \infty$?

- A: $A$ is 0
- B: $B$ is 0
- C: $A$ and $B$ are 0
- D: Neither $A$ nor $B$ is 0

## Boundary Conditions

Suppose a potential function $V(x, y) \propto (A\sin(kx) + B\cos(kx))$. Which of the following is true, if $V = 0$ as $x = 0$, and $V = 0$ as $x = a$?

- A: $B$ is 0, and $k = n\pi$
- B: $A$ is 0, and $k = n\pi/(2a)$
- C: $A$ and $B$ are 0
- D: $B$ is 0, and $k = n\pi/a$

# Boundary Conditions

Hyperbolic trigonometric functions:

- $\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$
- $\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$
- $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

Which of the following is zero?

- A: $\sinh(0)$
- B: $\cosh(0)$
- C: $\tanh(0)$
- D: None

Which of the following is one?

- A: $\sinh(0)$
- B: $\cosh(0)$
- C: $\tanh(0)$
- D: None

Hyperbolic trigonometric functions are solutions to which equation?

- A: $\frac{df}{dx} = k$
- B: $\frac{d^2f}{dx^2} = kx$
- C: $\frac{d^2f}{dx^2} = k^2f$
- D: $\frac{d^2f}{dx^2} = 0$

# Boundary Conditions

Fourier's Trick: Imagine a vector with $n$ components:

$$\vec{v} = \sum_{i=1}^{n} c_i \hat{x}_i \tag{3}$$

In words, how do you solve for some $c_m$?

- A: Divide by $\hat{x}_i$
- B: Take the dot product of both sides with $\hat{x}_m$
- C: Take the dot product $\vec{v}$ and $\vec{u}$, and the sum the series
- D: Integrate both sides with respect to $x$

# Boundary Conditions

**Fourier's Trick**: Imagine a vector with $n$ components:

$$\vec{v} = \sum_{i=1}^{n} c_i \hat{x}_i \tag{4}$$

In words, how do you solve for some $c_m$? Note that:

$$\vec{v} \cdot \hat{x}_m = \sum_{i=1}^{n} c_n \hat{x}_i \cdot \hat{x}_m = c_m \tag{5}$$

Why? Because

$$\hat{x}_i \cdot \hat{x}_j = 0 \tag{6}$$

$$\hat{x}_i \cdot \hat{x}_i = 1 \tag{7}$$

# Boundary Conditions

Fourier's Trick: Imagine a known function that happens to be equal to a sum:

$$f(x) = \sum_{i=1}^{\infty} c_i g_i(x) \tag{8}$$

In words, how do you solve for some $c_m$?

- A: Multiply both sides by $g_m(x)$
- B: Multiply both sides by $g_m(x)$ and integrate both sides with respect to $x$
- C: Sum the infinite series and solve for $c_m$ with algebra
- D: Integrate both sides with respect to $x$

# Boundary Conditions

If it's true that a function can be written as an infinite series of functions with coefficients:

$$f(x) = \sum_{i=1}^{\infty} c_i g_i(x) \tag{9}$$

Then the functions $g_n(x)$ are said to be **complete**, or a complete basis (just like vectors are a sum of basis vectors. Examples of complete sets of functions:

- sines and cosines (Fourier series)
- exponentials with the right rates multiplying $x$
- Hyperbolic trigonometric functions (follows from exponentials)
- Taylor series (polynomials with derivatives as coefficients)

# Boundary Conditions

The functions $f_n(x)$ are said to be orthogonal for $x \in [a, b]$ if

$$\int_a^b f_n(y) f_m(y) dy = \delta_{n,m} \tag{10}$$

One example:

$$I_{n,m} = \int_{-L}^{L} \frac{\sin(n\pi x/L)}{\sqrt{L}} \frac{\sin(m\pi x/L)}{\sqrt{L}} dx \tag{11}$$

What is the result of this integral? How would you approach solving this?

The Fourier series representation of a function $f(x)$ is written:

$$S(x) = \frac{A_0}{2} + \sum_{i=1}^{\infty} \left(A_n \cos(nx) + B_n \sin(nx)\right) \tag{12}$$

with

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx \tag{13}$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx \tag{14}$$

Let's obtain the Fourier series coefficients $A_n$ and $B_n$ for a square-wave signal:

$$f(x) = 1, \quad 0 \leq x \leq \pi, \quad 0, \pi < x \leq 2\pi \tag{15}$$

(Observe on board). The result: $A_0 = 1.0$, all other $A_n = 0$, odd $B_n$ values follow $2/(n\pi)$, even $B_n = 0$ as well.

# Separation of Variables

# Separation of Variables

Laplaces' Equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{16}$$

Assume the solution follows

$$V(x, y, z) = X(x)Y(y)Z(z) \tag{17}$$

The Laplace equation then breaks into three separate ordinary differential equations. Application of boundary conditions to solve them (Asynchronous video content on Moodle).

# Separation of Variables

Laplaces' Equation in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\left(\frac{\partial^2 V}{\partial\phi^2}\right)=0 \tag{18}$$

Assuming *azimuthal symmetry* means $V(r,\theta,\phi)=V(r,\theta)$ and $\partial V/\partial\phi=0$. Thus, Eq. 18 reduces and admits general solutions:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0 \tag{19}$$

Let the general solutions be separable:

$$V(r,\theta)=R(r)\Theta(\theta) \tag{20}$$

The radial equation is

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1) \tag{21}$$

Exercise: show that the solution is

$$R(r) = Ar^l + Br^{-(l+1)} \tag{22}$$

(The derivative operator distributes over addition, so the two solutions can be checked separately, or together).

**What are the units of $R(r)$? What are the units of $A$ and $B$?**

# Separation of Variables

The polar equation is

$$\frac{1}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta \tag{23}$$

The solutions are complete, and orthogonal, and known as Legendre polynomials:

$$\Theta(\theta) = P_l(\cos\theta) \tag{24}$$

Defined by the *Rodrigues formula*:

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \tag{25}$$

Exercise: show that

$$P_3(x) = (5x^3 - 3x)/2 \tag{26}$$

What is the result of the following integrals?

$$I_1 = \int_{-1}^{1} P_1(x)P_2(x)dx \tag{27}$$

$$I_2 = \int_{-1}^{1} P_2(x)P_2(x)dx \tag{28}$$

The general solution is a sum of individual solutions:

$$V(r,\theta) = \sum_{i=0}^{\infty} \left( Ar^{l} + B/r^{l+1} \right) P_l(\cos\theta) \tag{29}$$

The coefficients may be found via Fourier's Trick.

# Example 3.9: Professor on Board

A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius $R$. Find the resulting potential inside and outside the sphere.

1. Inside the sphere, $B = 0$ to avoid a singularity at the origin (center of sphere).
2. Outside the sphere, $A = 0$ to ensure $V \to 0$ as $r \to \infty$.
3. General boundary conditions at $r = R$: potential is continuous ($-\int \vec{E} \cdot d\vec{l} = 0$).
4. Coefficients of same order $l$ have a relationship.
5. E-field has a discontinuity at the boundary.
6. Fourier's trick to get the coefficients, after specifying $\sigma_0(\theta)$.

# Multipole Expansion

# Multipole Expansion

Imagine a *physical dipole* with $q$ at $\hat{r}' = d/2\hat{z}$ and $-q$ at $\hat{r}' = -d/2\hat{z}$. Show that (professor example)

$$V(r, \theta) = \frac{kqd}{r^2} P_1(\cos\theta) \tag{30}$$

1. Far-field on script-r's
2. Subtract
3. Simplify
4. Note that $P_1(x) = x$.

# Multipole Expansion

Can't you break *any* charge distribution into a collection of monopoles, dipoles, quadrupoles, ... ? We can show in general that:

$$\boxed{\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n} P_n(\cos\theta)} \tag{31}$$

# Multipole Expansion

Can't you break *any* charge distribution into a collection of monopoles, dipoles, quadrupoles, … ?

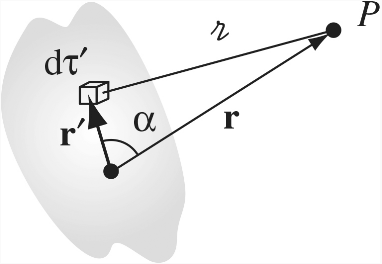


Figure 2: The general scheme for the multipole expansion.

Find the Law of Cosines from the definition of the separation vector:

Then we let

$$\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) \tag{32}$$

# Multipole Expansion

Find the Taylor series of $f(\epsilon) = (1+\epsilon)^{-1/2}$:

Remember that

$$\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) \tag{33}$$

# Multipole Expansion

1. After computing the Taylor series, substitute $\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$
2. Collecting like powers of $\left(\frac{r'}{r}\right)$ together will lead to

$$\boxed{\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n} P_n(\cos\theta)} \tag{34}$$

# Multipole Expansion

Recall that the potential for *any* charge distribution is

$$V(\vec{r}) = \int \frac{k\rho(\vec{r'})}{\mathscr{r}} d\tau' \tag{35}$$

Substitute the expansion for $1/\mathscr{r}$ , and reverse the order of summation and integration:

$$V(\vec{r}) = k \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\alpha)\rho(\vec{r'})d\tau' \tag{36}$$

# The Monopole and Dipole Terms

# The Monopole and Dipole Terms

Using the $\rho(\vec{r'})$ of a dipole oriented along the z-axis, reproduce Eq. 30 using the multipole expansion.

*Hint: obtain the first few terms, but which ones vanish and why?*

# Conclusion

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    - Cartesian coordinates
    - Spherical coordinates
3. Multipole Expansions
    - Far-fields
    - Monopole and dipole terms
    - Electric Field of a Dipole