HW 1

1.54

$$v = r^2 \cos\theta \,\hat{r} + r^2 \cos\phi \,\hat{\theta} - r^2 \cos\theta \sin\phi \,\hat{\phi}$$

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$$\int v = \int r^2 \cos\theta \,\hat{r} + \int r \cos\phi \,\hat{\theta} - \int r^2 \cos\theta \sin\phi \,\hat{\phi} \Rightarrow$$

$$\int_0^{\phi}\int_0^{\theta}\int_0^{\hat{r}} \left(r^2 \cos\theta \,\hat{r} + r^2 \cos\phi \,\hat{\theta} - r^2 \cos\theta \sin\phi \,\hat{\phi}\right) dr\, d\theta\, d\phi$$

$$\int_0^{\phi}\int_0^{\theta} r^2 \cos\theta \,\hat{r} + r^2 \cos\phi \,\hat{\theta} - r^2 \cos\theta \sin\phi \,\phi \Big|_0^r d\theta\, d\phi \Rightarrow$$

$$\int_0^{\phi}\int_0^{\theta} \frac{1}{3} r^3 \cos\theta \,\hat{r} + r^2 \cos\phi \,\hat{\theta} - \frac{1}{3} r^3 \cos\theta \sin\phi \,\hat{\phi} \Big|_0^r d\theta\, d\phi \Rightarrow$$

$\theta, \phi = \frac{\pi}{2}$

$$\int_0^{\phi}\int_0^{\theta} \frac{1}{3} r^3 \cos\left(\frac{\pi}{2}\right) \hat{r} + r^2 \cos\phi - \cos\frac{\pi}{2} \sin\phi \,\hat{\phi} \Big|_0^{\theta} \Rightarrow$$

$$\int_0^{\phi} \frac{1}{3} r^3 \cos\frac{\pi}{2} \,\hat{r} + r^2 \cos\left(\frac{\pi}{2}\right) - \cancel{\cos\frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) \hat{\phi}} \Big|_0^{\phi} \Rightarrow$$

$\rightarrow 0$

$$\frac{1}{3} r^3 \cos\left(\frac{\pi}{2}\right) + r^2 \cos\frac{\pi}{2} \Big|_0^R$$

$$\frac{1}{3} R^3 \left(\frac{-1}{2}\right) + R^2 \left(\frac{-1}{2}\right)$$

$$\frac{-R^3}{6} + \frac{-R^2}{2} = \frac{+2R^3}{12} + \frac{+6R^2}{12} \Rightarrow$$

$$\frac{8R^5}{12} = \frac{2R^5}{4}$$

1.55

$V = ay\hat{x} + bx\hat{y}$, $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

$\vec{r}(t) = a\sin(t)\hat{x} + b\cos(t)\hat{y} \Rightarrow$

$\int_0^R \vec{F}(\vec{r}(t))\cdot\vec{r}'dt \Rightarrow$

$\vec{r}' = \cos(t)\hat{x} + -\sin(t)\hat{y}$

$\vec{F}(\vec{r}(t)) = \cancel{ab\sin(t)\cos(t)\hat{y} + \cos(t)\sin(t)}$

$a^2\sin(t)\hat{x} + b^2\cos(t)\hat{y}$

$\vec{F}(\vec{r}(t))\cdot\vec{r}' = a^2\cos(t)\sin(t)\hat{x} + -b^2\sin(t)\cos(t)\hat{y}$

$\int_0^R a^2\cos(t)\sin(t)\hat{x} - b^2\sin(t)\cos(t)\hat{y}\,dt$

$a^2\frac{\sin^2(x)}{2} - b^2\frac{\sin^2(x)}{2}\Big|_0^R$

$\frac{\sin^2(R)}{2}(a^2 - b^2) \Rightarrow$

$\pi R^2(a^2 - b^2)$

1.56

$v = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$, $\int f(x,y,z)\,ds$

$l = \int_a^b ds$, $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \Rightarrow$

$\cancel{ds = \sqrt{(0)^2 + (z^2)^2 + (1)^2}}$

$\cancel{ds = (z^2+1)}$

$\cancel{\int 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}\,(z^2+1)}\,dt$

$ds = \sqrt{0^2 + (z^2)^2 + (3\sin(t))^2} \Rightarrow 3\sin t + z^2$

$\int_2^1 (6\hat{x} + \int_0^1 \sin(t)\,(\frac{\sin}{\cos} - \cos)^2\hat{y} + \int_1^2 3\sin + (\sin - \cos))\,3\sin t + z^2\,dt$

$\hat{x} = \int_2^0 18\sin^2 t + \sin - (3\sin\cos) = -\frac{\sin 2x - 2x}{4} \cdot 18$

$\hat{y} = \int_0^1 \sin = \int_0^1 \sin(t) \cdot 3\sin t \Rightarrow -3\frac{\sin 2x - 2x}{4}\Big|_0^1 \rightarrow 0$

$\hat{z} = \int_1^2 9\cancel{\sin^2} + (3\cos(t)\sin(t) - \cancel{3\sin^2}) \Rightarrow$

$\int_1^2 6 + 3\sin(t)\cos(t) = 3\frac{\sin^2 x}{2}\Big|_1^2$

$\cancel{108\sin^2 - 2x}$

$2\theta \sin^2 x - 2$

$\frac{\sin^2 x}{2}\Big|_1^2 = \frac{\sin^2(2)}{2} - \frac{\sin^2(1)}{2} \Rightarrow$

$\sin^2(2) - \sin^2(1)$

1.57

$v = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}$

$v = (r\cos^2\theta)(\cancel{\sin\theta\cos\phi\,\hat{x}} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}) -$

$(r\cos\theta\sin\theta)(\cancel{\cos\theta\cos\phi\,\hat{x}} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}) +$

$3r(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) \Rightarrow$

$v = -3r\sin\phi\,\hat{x} + 3r\cos\phi\,\hat{y} + [r(1-\sin^2\theta)\cos\theta + r\sin^2\theta\cos\theta]\hat{z} \Rightarrow$

$-3r\sin\phi\,\hat{x} + 3r\cos\phi\,\hat{y} + r\cos\theta\,\hat{z} \Rightarrow -3r\sin\phi\,\hat{x} + 3r\cos\phi\,\hat{y} + z\hat{z} \Rightarrow$

$\oint_S V\cdot dl = \int_{L_1} V\cdot dl + \int_{L_2} V\cdot dl + \int_{L_3} V\cdot dl + \int_{L_4} V\cdot dl + \int_{L_5} V\cdot dl +$
$\int_{L_6} V\cdot dl \Rightarrow$

$\int_0^2 t\,dt + \int_0^1 (-2(2-2t))\,dt + \int_0^1 0 + \int_0^1 0 + \int_0^{\pi/2} (3\sin^2 + 3\cos^2) \Rightarrow$
$+ \int 0$

$\int_0^2 t\,dt + 4\int_0^1 (t-1) + 3\int_0^{\pi/2} \Rightarrow 2 + 4(-.5) + (3\frac{\pi}{2}) =$
$\frac{3\pi}{2}$

1.59

$V = r^2 \sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r\tan\theta\,\hat{\phi}$

$\iiint_D \nabla\cdot v\,dV = \int_0^{\pi/6}\int_0^{2\pi}\int_0^R \nabla\cdot v\,dV$

$\nabla\cdot V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \Rightarrow$

$\frac{1}{r^2}\frac{\partial}{\partial r}[r^2(r^2\sin\theta)] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[4r^2\cos\theta(\sin\theta)] + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$
$[r^2\tan\theta] \Rightarrow$

$= \frac{1}{r^2}[4r^3\sin\theta] + \frac{1}{r\sin\theta}[-4r^2\sin\theta(\sin\theta) + 4r^2\cos\theta(\sin\theta)] + \cancel{\frac{1}{r\sin\theta}} \xrightarrow{} 0$

$\Rightarrow \cancel{4r\sin\theta} + -\cancel{4r\sin\theta} + 4r(\cos^2\theta/(\sin\theta)) \Rightarrow$

$\nabla\cdot V = 4r\frac{\cos^2\theta}{\sin\theta}$

$\int_0^{\pi/6}\int_0^{2\pi}\int_0^R \left(4r\frac{\cos^2\theta}{\cancel{\sin\theta}}\right) r^2\cancel{\sin\theta}\,dr\,d\phi\,d\theta \Rightarrow$

$\iiint_D 4r^3\cos^2\theta\,dr\,d\phi\,d\theta \Rightarrow$

$4\left(\int_0^{\pi/6}\cos^2\theta\,d\theta\right)\left(\int_0^{2\pi}d\phi\right)\left(\int_0^R r^3\,dr\right) \Rightarrow$

$4\left[\int_0^{\pi/6}\frac{1}{2}(1+\cos 2\theta)\,d\theta\right]\left(\int_0^{2\pi}d\phi\right)\left(\int_0^R r^3\,dr\right) \Rightarrow$

$4\left[\frac{1}{24}(2\pi + 3\sqrt{3})\right]2\pi\left(\frac{R^4}{4}\right) \Rightarrow$

$\frac{\pi R^4}{12}(2\pi + 3\sqrt{3})$

1.62

a) $a \equiv \int_S da$

$a = \int_0^{\pi/2}\int_0^{2\pi} \hat{r} R^2 \sin\theta \, d\phi \, d\theta \Rightarrow$

$\int_0^{\pi/2}\int_0^{2\pi} (\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}) R^2 \sin\theta \, d\phi \, d\theta \Rightarrow$

$= R^2 \left[\int_0^{\pi/2}\int_0^{2\pi} (\hat{x}\sin\cos\phi + \hat{y}\sin^2\theta\sin\phi + \hat{z}\sin\theta\cos\theta) \, d\phi \, d\theta \right]$

$= R^2 \left(\hat{x}\int_0^{\pi/2}\int_0^{2\pi} \sin^2\theta\cos\phi \, d\phi \, d\theta + \hat{y}\int_0^{\pi/2}\int_0^{2\pi} \sin^2 \sin\phi \, d\phi \, d\theta + \right.$

$\left. \hat{z}\int_0^{\pi/2}\int_0^{2\pi} \sin\theta\cos\theta \, d\phi \, d\theta \right) \Rightarrow$

$= R^2 \left[\cancel{\left(\int_0^{\pi/2} \sin^2\theta \, d\theta\right)\left(\int_0^{2\pi}\cos\phi \, d\phi\right)}^{\to 0} + \cancel{\left(\int_0^{\pi/2}\sin^2\theta \, d\theta\right)\left(\int_0^{\pi/2}\sin\phi \, d\phi\right)}^{\to 0} + \right.$

$\left. \hat{z}\underbrace{\left(\int_0^{\pi/2} \sin\theta\cos\theta \, d\theta\right)}_{1/2}\underbrace{\left(\int_0^{2\pi} d\phi\right)}_{2\pi} \right] \Rightarrow$

$\pi R^2 \hat{z}$

b) let $T = 1$, $\iiint_D dS = \iiint_D (\nabla 1) \, dV \Rightarrow$

$\iiint_D \underbrace{\left(\frac{\partial}{\partial x}(1), \frac{\partial}{\partial y}(1), \frac{\partial}{\partial z}(1)\right)}_{0} dV \Rightarrow$

$\iiint_D 0 \, dV = 0$

c) $a = \iint_S ds = \frac{1}{2}\oint_S r \times dl$

Regardless of the surface for S integration, a is dependent on boundary line for surface

d) $c \cdot \frac{1}{2}\oint r \times dl = \frac{1}{2}\oint (c \times r) \cdot dl \Rightarrow c \cdot \frac{1}{2}\oint r \times dl = \frac{1}{2}\iint_S \nabla \times [c \times r] \cdot ds \Rightarrow$

$= \frac{1}{2}\iint_S \sum_{i=1}^{3} \delta_i \frac{\partial}{\partial x_i} \times \left[\left(\sum_{j=1}^{3} \delta_j c_j\right) \times \left(\sum_{k=1}^{3} \delta_k x_k\right) \right] \cdot ds \Rightarrow$

Skipped over some simplification process to save space

$= \frac{1}{2}\iint \sum_{i=1}^{3} \delta_i \frac{\partial}{\partial x_i} \cdot \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_l \varepsilon_{jkl} c_j x_k \right) \cdot ds \Rightarrow$

$= \frac{1}{2}\iint \left[\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \delta_m \varepsilon_{ilm}\varepsilon_{jkl} \left(\frac{\partial c_j}{\partial x_i} x_k + c_j \frac{\partial x_k}{\partial x_i} \right) \right] \cdot ds$

$= \frac{1}{2}\iint \left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \delta_m \delta_{mj} \delta_{kk} c_j - \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \delta_m \delta_{mk} \delta_{kj} c_j \right) \cdot ds$

$= \frac{1}{2}\iint \left[\left(\sum_{k=1}^{3} \delta_{kk}\right) \sum_{j=1}^{3} \delta_j c_j - \sum_{j=1}^{3} \delta_j c_j \cdot ds \right] \Rightarrow$

$= \frac{1}{2}\iint_S \left[2 \sum_{j=1}^{3} \delta_j c_j \right) \cdot ds$

$= \iint_S \sum_{j=1}^{3} \delta_j c_j) \cdot ds$

$= \iint_S c \cdot ds \Rightarrow c \cdot \iint_S ds \Rightarrow \iint ds = \frac{1}{2}\oint r \times dl$

e) $\oint_S (c\cdot r)\, dl = -\iint_S \nabla(c\cdot r)\times dS =$

$-\iint_S \left(\sum_{i=1}^{3} \delta_i \frac{\partial}{\partial x_i}\right)\left[\left(\sum_{j=1}^{3}\delta_j c_j\right)\cdot\left(\sum_{k=1}^{3}\delta_k x_k\right)\right]\times dS$

$= -\iint_S \sum_{i=1}^{3}\delta_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\delta_{jk} c_j x_k\right)\times dS$

$= -\iint_S \sum_{i=1}^{3}\sum_{j=1}^{3}\delta_i\left(\frac{\partial}{\partial x_i} c_j x_j\right)\times dS$

$= -\iint_S \left[\sum_{i=1}^{3}\sum_{j=1}^{3}\delta_i\left[(0)x_j + c_j\delta_{ij}\right]\right]\times dS$

$= -\iint_S \sum_{i=1}^{3}\sum_{j=1}^{3}\delta_i c_j \delta_{ij}\cdot dS$

$= -\iint_S \sum_{i=1}^{3}\delta_i c_i \times dS$

$= -\iint_S c\times dS = -c\times\iint_S dS = -c\times a = a\times c$

1.63

a) $\iiint_D \nabla\cdot v\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{R}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r}\right)\right] r^2\sin\theta\, dr\, d\phi\, d\theta \Rightarrow$

$= \int_0^{\pi}\int_0^{2\pi}\int_0^{R}\left[\frac{1}{r^2}(1)\right] r^2 \sin\theta$

$= \int_0^{\pi}\sin\theta\, d\theta\left(\int_0^{2\pi} d\phi\right)\int_0^{R} dr$

$= 2(2\pi)R = 4\pi R$

$\nabla\cdot\frac{\hat{r}}{r} = \frac{1}{r^2} \Rightarrow \quad \nabla\cdot r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot r^n) + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial}{\partial\theta}(0\cdot\sin\theta)} +$

$\frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial}{\partial\phi}(0)} \Rightarrow$

$\frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+2}) \Rightarrow$

$= \frac{1}{r^2}(n+2)r^{n+1}$

$= (n+2)r^{n-1}$

b) $\nabla\times v = \cancel{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta}\ \frac{1}{r\sin\theta}\left[\cancelto{0}{\frac{\partial}{\partial\theta}\,0\cdot\sin\theta - \frac{\partial}{\partial\phi}(0)}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - 0\right]\hat{\theta} +$

$\frac{1}{r}\left[\cancelto{0}{\frac{\partial}{\partial r}(r\cdot 0)} - \frac{\partial}{\partial\theta}(r^n)\right]\hat{\phi} = 0$

$\iiint_D (\nabla\times v)\, dV = -\oiint_D v\times dS$

$\int_0^{\pi}\int_0^{2\pi}\int_0^{R}\left[\nabla\times r^n\hat{r}\right](r^2\sin\theta\, dr\, d\phi\, d\theta) = -\int_0^{\pi}\int_0^{2\pi}(r^n\hat{r})\Big|_{r=R}\times\hat{r}\, R^2\sin\theta\, d\phi\, d\theta \Rightarrow$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{R} \cancel{D(g\ldots)} = -\int_0^{\pi}\int_0^{2\pi} \cancel{R^n \hat{\theta} \times (\hat{r} R^2 \sin\theta \, d\phi \, d\theta)}$$

$\theta = 0$

1.64

a) $D(r,\varepsilon) \equiv -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\varepsilon^2}} \Rightarrow$

$$= -\frac{1}{4\pi}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{1}{\sqrt{r^2+\varepsilon^2}}\right)\right] + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial}{\partial\theta}\left(\frac{1}{\sqrt{r^2+\varepsilon^2}}\right)\right]} \to 0\right.$$

no θ so $\frac{\partial}{\partial\theta} = 0$

$$\left. + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left(\frac{1}{\sqrt{r^2+\varepsilon^2}}\right)} \to 0\right.$$

no θ ϕ so $\frac{\partial}{\partial\phi} \to 0$

$$= \frac{-1}{4\pi r^2}\frac{\partial}{\partial r}\left\{r^2\left[-\frac{r}{(r^2+\varepsilon^2)^{3/2}}\right.\right. \Rightarrow$$

$$= \frac{-1}{4\pi r^2}\frac{\partial}{\partial r}\left[-\frac{r^3}{(r^2+\varepsilon^2)^{3/2}}\right] = -\frac{1}{4\pi r^2}\left[\frac{3\varepsilon^2 r^2}{(r^2+\varepsilon^2)^{5/2}}\right]$$

$$= \frac{3\varepsilon^2}{4\pi(r^2+\varepsilon^2)^{5/2}}$$

b) $\lim_{\varepsilon\to 0} D(0,\varepsilon) = \lim_{\varepsilon\to 0}\frac{3\varepsilon^2}{4\pi(\varepsilon^2)^{5/2}} = \lim_{\varepsilon\to 0}\frac{3\varepsilon^2}{4\pi\varepsilon^5} = \frac{\varepsilon^2}{\varepsilon^5} = \infty$

Bottom function goes to 0 quicker

c) $$\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\frac{3\varepsilon^2}{4\pi(r^2+\varepsilon^2)^{5/2}}(r^2\sin\theta \, dr \, d\phi \, d\theta) \Rightarrow$$

$$= \frac{3\varepsilon^2}{4\pi}\left(\int_0^{\pi}\sin\theta \, d\theta\right)\left(\int_0^{2\pi}d\phi\right)\left(\int_0^{\infty}\frac{r^2 \, dr}{(r^2+\varepsilon^2)^{5/2}}\right)$$

d) $r = \varepsilon\tan\theta \Rightarrow r^2+\varepsilon^2 = \varepsilon^2(1+\tan^2\theta) = \varepsilon^2\sec^2\theta \, d\theta$

$dr = \varepsilon\sec^2\theta \, d\theta$

$$\iiint D(r,\varepsilon)\,dV = \frac{3\varepsilon^2}{4\pi}2(2\pi)\int_0^{\pi/2}\frac{(\varepsilon\tan\theta)^2(\varepsilon\sec^2\theta \, d\theta)}{(\varepsilon^2\sec^2\theta)^{5/2}} \Rightarrow$$

$$= 3\cancel{\varepsilon^2}\int_0^{\pi/2}\frac{(\cancel{\varepsilon^2}\tan^2\theta)\,\varepsilon\sec^2\theta \, d\theta}{\varepsilon^5\sec^5\theta}$$

$$= 3\int_0^{\pi}\tan^2\theta\cos^3\theta \, d\theta \Rightarrow 3\int_0^{\pi/2}\sin^2\theta\cos\theta \, d\theta \Rightarrow$$

$$3\int_0^{\pi/2}\sin^2\theta \, d(\sin\theta)$$

$$= 3\left.\frac{\sin^3\theta}{3}\right|_0^{\pi/2} =$$

$$3\frac{\sin^3(\pi/2)}{3} = 1$$