Homework

3.3 $\nabla^2 V = 0$

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \cancel{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right)} + \cancel{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}} = 0$$

~~By Separa~~

$$\cancel{\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right)}$$

$$\cancel{= \frac{1}{R}\left(r^2 \frac{\partial R^2}{\partial^2 r}\right)}$$

$$\cancel{\frac{2}{R}}$$

$$\cancel{\frac{1}{R^2}}$$

r^{-2} $\qquad \frac{r^{-1}}{-1}$

$$r^2 \frac{dV}{dr} = K$$

$$\int \frac{dV}{dr} = \int \frac{K}{r^2}$$

$$\boxed{V = -\frac{K}{r} + C}$$

Cylindrical

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{\partial V}{\partial s}\right) = 0$$

$$s \frac{\partial V}{\partial s} = K$$

$$\int \frac{\partial V}{\partial s} = \int \frac{K}{s}$$

$$\boxed{V = K \ln|s| + C}$$

3.5

Proof

$\nabla \cdot E_1 = -\frac{\rho}{\varepsilon_0}$

$\nabla \cdot E_2 = -\frac{\rho}{\varepsilon_0}$

$\bar{E}_3 = \bar{E}_2 - \bar{E}_1$

$\nabla \cdot E_3 = 0 \quad \rightarrow \quad \bar{E}_3 = -\nabla V_3$

$\nabla \cdot (V_3 \bar{E}_3) = V_3 (\nabla \cdot \bar{E}_3) + \bar{E}_3 (\nabla V_3) = -(E_3)^2$

$\int \nabla \cdot (V_3 E_3) d\tau = \oint_S V_3 E_3 da = -\int E_3^2 d\tau$

$\int E_3^2 d\tau = 0$

$\boxed{E_2 = E_1}$ $\quad E_3 = 0$

3.6

3.13.

Using Ex) 3.3 $[0, a/2]$ & $[a/2, a]$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy$$

$$C_n = \frac{2V_0}{a}\left(\int_0^{a/2} \sin(n\pi y/a)\,dy - \int_{a/2}^{a}\sin(n\pi y/a)\,dy\right)$$

$$= \frac{2V_0}{n\pi}\left(1 + (-1)^n - 2\cos(n\pi/2)\right)$$

$$C_n = \frac{8V_0}{n\pi} \quad \text{where } n = (4j+2),\ j = 0,1,2$$

$$V(x,y) = \frac{8V_0}{\pi}\sum_{j=0}^{\infty} \frac{e^{-(4j+2)\pi x/a}\sin((4j+2)\pi y/a)}{4j+2}$$

3.14

Conductor so,

$$\frac{\partial V}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=\text{odd}} \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a)$$

$$\sigma(y) = -\varepsilon_0 \frac{\partial V}{\partial x}\Big|_{x=0}$$

$$= \frac{4\varepsilon_0 V_0}{a}\sum_{n=0}^{\infty} \sin((2n+1)\pi y/a)$$

3.15) Sep of var.

- $V=0$ when $y=0=a$
- $V=0=x=$
- $V=V_0(y)$ when $x=b$

$$\frac{1}{X(x)}\frac{d^2X}{dx^2}=\pm k^2$$

$$\frac{1}{Y(y)}\frac{d^2Y}{dy^2}=\pm k^2$$

Ose

$$V(x,y)=(A\sin(ky)+B\cos(ky))(Ce^{kx}+De^{-kx})$$

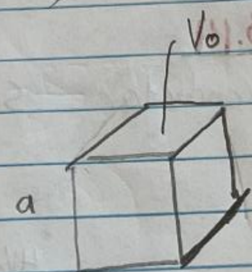
$$=\sum_{n=0}^{\infty} C_n \sin(n\pi y/a)\sinh(n\pi x/a)$$

Using Fourier's trick

$$C_n=\frac{2}{a\sinh(\pi n b/a)}\int_0^a V_0(y)\sin(n\pi y/a)\,dy$$

V_0

[redacted] $\boxed{C_n=\frac{4V_0}{n\pi\sinh(n\pi)}}$



3.16)

$$\frac{1}{X(x)}\frac{d^2X}{dx^2}=-k^2$$

$$\frac{1}{Y(y)}\frac{d^2Y}{dy^2}=-l^2$$

$$\frac{1}{Z(z)}\frac{d^2Z}{dz^2}-(k^2+l^2)$$

$X(x)=A\cos(kx)+B\sin(ky)$

$X(x)=C\cos(lx)+D\sin(ly)$

$Z(z)=E\exp(-z\sqrt{k^2+l^2}+F\exp(z\sqrt{k^2+l^2}$

boundary conditions

$A=C=0$ $\quad B\neq 0$ $\quad D\neq 0$ $\quad k=n\pi/a$ $\quad l=m\pi/a$

$E+F=0$

$$V(x,y,z)=\sum_n\sum_m C_{n,m}\sin(n\pi x/a)\sin(m\pi y/a)\,(\pi\sqrt{n^2+m^2}\,z/a)$$

Fourier's tricks

$$C_{n,m} \sinh\left(\pi\sqrt{n^2+m^2}\right) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a \int_0^a \sin(n\pi x/a)\, \sin(m\pi y/a)\, dx\, dy$$

$$= \frac{16V_0}{n^2 nm}, \quad \text{when } n, m \text{ are odd}$$

$$V(x,y,z) = \frac{16V_0}{\pi^2} = \frac{16V_0}{\pi^2} \sum_{n\,odd} \sum_{m\,odd} \frac{\sin(n\pi x/a)\sin(m\pi y/a)}{nm} \frac{\sinh\left(\pi\sqrt{n^2+m^2}\,(z/a)\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}$$

3.19

3.22

a) Legendre $V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{(l+1)}} P_l(\cos\theta)$

$$V(r,0) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2} - r\right) = \sum_{l=0}^{\infty} \frac{B_l}{r^{(l+1)}} P_l(1)$$

$$= \sum_{l=0}^{\infty} \frac{B_l}{r^{(l+1)}}$$

3.14

$$\sqrt{r^2+R^2} = r\left(1 + \frac{1}{2}\left(\frac{R}{r}\right)^2 - \frac{1}{8}\left(\frac{R}{r}\right)^4\right)$$

1) $B_0 = \frac{\sigma R^2}{4\varepsilon_0}$

2) $B_1 = 0$

$B_2 = \frac{\sigma R^4}{16\varepsilon_0}$

$$V(r,\theta) = \frac{\sigma R^2}{4\varepsilon_0}\left(\frac{1}{r} - \frac{R^2}{4r^3} P_2(\cos(\theta))\right)$$

b) $V_A = \frac{\sigma}{2\varepsilon_0}\left(R - rP_1(\cos\theta) + \frac{r^2}{2R} P_2(\cos(\theta))\right)$

$V_B = \frac{\sigma}{2\varepsilon_0}\left(R + rP_1(\cos\theta) + \frac{r^2}{2R} P_2\cos(\theta)\right)$

3.24

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right)=0$$

$$\frac{1}{\Phi}\frac{d^2\Phi(\phi)}{d\phi^2}=-K^2$$

$$\frac{s}{S(s)}\frac{d}{ds}\left(s\frac{dS(s)}{ds}\right)=K^2$$

$S(s)=s^n$

So

$$S(s)=Cs^K+Ds^{-K}$$
$$S(s)=C\ln(s)+D$$

General

$$V(s,\phi)=a_0+b_0\ln(s)+\sum_{K=1}^{\infty}\left[s^K(a_K\cos(K\phi)+b_K\sin(K\phi))+s^{-K}(c_K\cos(K\phi)+d_K\sin(K\phi))\right]$$

3.26) $\sigma(\phi) = a\sin(5\phi)$

$$V_{in}(s,\phi) = a_0 + \sum_{k=1}^{\infty} s^k(a_k\cos(k\phi) + b_k\sin(k\phi))$$

$$V_{out}(s,\phi) = a_0' + \sum_{k=1}^{\infty} \frac{1}{s^k}(c_k\cos(k\phi) + d_k\sin(k\phi))$$

$$\sigma = -\varepsilon_0\left(\frac{\partial V_{out}}{\partial n} - \frac{\partial V_{in}}{\partial n}\right)\Big|_{s=R}$$

$$V(s,\phi) = \frac{a\sin 5\phi}{10\varepsilon_0}\frac{s^5}{R^4}, \quad s \le R$$

$$V(s,\phi) = \frac{a\sin 5\phi}{10\varepsilon_0}\frac{R^6}{s^5}, \quad s \ge R$$