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11/17/20 PHYS

Ex. 4.2 (reproduce results, show all steps, using Ex. 3.9 worked in class)

Prob. 4.1, 7, 10, 15, 18

HW4

$\boxed{4.1}$ $\vec{p} = qd = \alpha\vec{E} \Rightarrow d = \frac{\alpha\vec{E}}{q}$

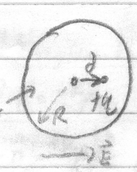
$// \alpha = 4\pi(0.667\times10^{-30}\ m^3)(8.85\times10^{-12}\ \frac{C^2}{N\cdot m^2})$

$\alpha = 7.418\times10^{-41}\ \frac{mC^2}{N}$

$d = \frac{\alpha\vec{E}}{q} = \frac{(7.418\times10^{-41})(5\times10^5)}{(1.6\times10^{-19})}$

$d = 2.318\times10^{-16}\ m.$

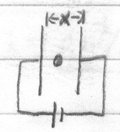
$\frac{d}{R} = \frac{2.318\times10^{-16}\ m}{5\times10^{-11}\ m} \Rightarrow \boxed{\frac{d}{R} = 4.64\times10^{-6}}$



$\alpha = (0.667\times10^{-30}\ m^3)(4\pi\epsilon_0)$

$\epsilon_0 = 8.85\times10^{-12}\ \frac{C^2}{N\cdot m^2}$, $q = 1.6\times10^{-19}\ C.$

$R = \frac{1}{2}Å = 5\times10^{-11}\ m.$



$x = 1\times10^{-3}\ m.$ $\quad V = Ex \Rightarrow \vec{E} = V/x$

$V = 500V.$ $\quad \vec{E} = \frac{500V.}{1\times10^{-3}m.} \Rightarrow \vec{E} = 5\times10^5\ V/m.$

For ionize, $d = R$. So, $R = \frac{\alpha\vec{E}}{q} \Rightarrow R = \frac{\alpha V}{qx} \Rightarrow V = \frac{Rqx}{\alpha} \Rightarrow V = \frac{(5\times10^{-11})(1.6\times10^{-19})(1\times10^{-3})}{(7.418\times10^{-41})} \Rightarrow \boxed{V = 1.08\times10^8\ V.}$

$\boxed{4.7}$ Show energy of ideal dipole $\vec{p}$ in E-field $\vec{E}$ is given by: $[U = -\vec{p}\cdot\vec{E}]$

$// V = \int_{\infty}^{\vec{r}} \vec{E}\cdot d\vec{l}$

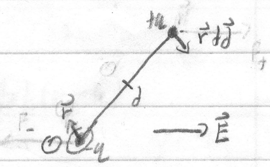
$= \int_{\infty}^{\vec{r}+\vec{d}} \vec{E}\cdot d\vec{l}$

$U = qV$

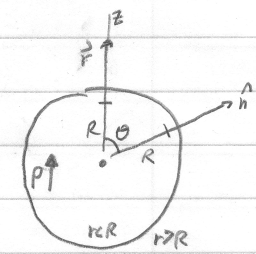
$U = qV$, $V = \int \vec{E}\cdot d\vec{l} \Rightarrow U = q(V(\vec{r}) - V(\vec{r}+\vec{d}))$

$U = q\left[\int_{\vec{r}+\vec{d}}^{\vec{r}} \vec{E}\cdot d\vec{l}\right] \Rightarrow -q\left(\int_{\vec{r}}^{\vec{r}+\vec{d}} \vec{E}\cdot d\vec{l}\right) \Rightarrow U = -q(\vec{E}\cdot\vec{d})$ $// \vec{p} = q\vec{d}$

$\boxed{U = -\vec{p}\cdot\vec{E}}$



$\boxed{4.10}$ $\vec{P} = k\vec{r} \Rightarrow \vec{P} = kR\hat{r}$



(a) $\sigma_b = \vec{P}\cdot\hat{n} = kR(\cancel{\hat{r}}\cdot\hat{r}) \Rightarrow \boxed{\sigma_b = kR}$

$\rho_b = -\nabla\cdot\vec{P} = \frac{-1}{r^2}\frac{d}{dr}(r^2\cdot kr) = \frac{-1}{r^2}(3r^2k) \Rightarrow \boxed{\rho_b = -3k}$

$// SA = 4\pi r^2$

(b) For $r<R$, $\oint\vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{enc}$ $//$ No σ_b, so $Q_{enc} = \int\rho_b\, d\tau'$; $\oint\vec{E}\cdot d\vec{a} = \int|\vec{E}|\,d\vec{a} = |\vec{E}|\int_s d\vec{a} = E4\pi r^2$ $// V = \frac{4}{3}\pi r^3$

$E4\pi r^2 = \frac{1}{\epsilon_0}\cdot\int\rho_b\, d\tau' \Rightarrow E = \frac{1}{4\pi\epsilon_0}\cdot\frac{1}{r^2}\int_0^r \rho_b 4\pi r'^2\, dr' = \frac{1}{4\pi\epsilon_0}\cdot\frac{1}{r^2}\cdot\rho_b 4\pi\left[\frac{1}{3}r'^3\right]_0^r$

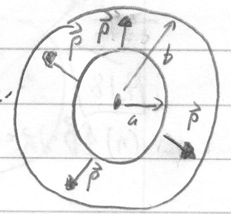
$E = \frac{1}{\epsilon_0 r^2}\cdot\rho_b\left(\frac{1}{3}r^3\right) \Rightarrow E = \frac{\rho_b r}{3\epsilon_0} \Rightarrow \vec{E} = \frac{\rho_b r}{3\epsilon_0}\hat{r}$ $// \rho_b = -3k \Rightarrow \boxed{\vec{E} = -\frac{kr}{\epsilon_0}\hat{r}}$

For $r>R$, $\oint\vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{enc}$ $// Q_{enc} = \int_V\rho_b\, d\tau' + \int_s\sigma_b\, da' = \rho_b(\frac{4}{3}\pi r^3) + \sigma_b(4\pi r^2)$

$E = \frac{1}{4\pi\epsilon_0 r^2}Q_{enc}$ $\quad = (-3k)(\frac{4}{3}\pi r^3) + (kr)(4\pi r^2)$

$\boxed{\vec{E} = 0}$ $\quad // Q_{enc} = -4\pi kr^3 + 4\pi kr^3 = 0 \Rightarrow \underline{Q_{enc} = 0}$

4.15 $\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$. Find $\vec{E}$-field in all 3 regions by 2 diff. methods. No free charge.



$// V = \frac{4}{3}\pi r^3$

$// SA = 4\pi r^2$

a) $\sigma_b = \vec{P}\cdot\hat{n} = (\frac{k}{r}\hat{r})\cdot(\hat{r}) = \frac{k}{r} \Rightarrow r=a: -\vec{P}\cdot\hat{n} = -k/a$ ($\vec{P}$ point inward).

$r=b: +\vec{P}\cdot\hat{n} = +k/b$ ($\vec{P}$ point outward)

$\sigma_b = -k/a,\ r=a\ ;\ = +k/b,\ r=b$

$\rho_b = -\nabla\cdot\vec{P} = \frac{-1}{r^2}\frac{\partial}{\partial r}(r^2\frac{k}{r}) = \frac{-1}{r^2}\frac{\partial}{\partial r}(rk) = \frac{-1}{r^2}k \Rightarrow \underline{\rho_b = -k/r^2}$

$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}Q_{enc}$

$E\cdot 4\pi r^2 = \frac{1}{\varepsilon_0}Q_{enc} \Rightarrow \underline{E = \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{r^2}}$

For $r<a$, $Q_{enc}=0 \Rightarrow \boxed{E=0}$

For $a<r<b$, $Q_{enc} = \int_S \sigma_b\, da' + \int_V \rho_b\, d\tau' = (\frac{-k}{a})(4\pi a^2) - \int_a^r \frac{k}{r^2}\cdot 4\pi r^2\, dr = -4\pi k a - 4\pi k(r-a)$

$Q_{enc} = \cancel{-4\pi ka} - 4\pi kr + \cancel{4\pi ka} \Rightarrow \underline{Q_{enc} = -4\pi kr}$

$E = \frac{1}{\cancel{4\pi}\varepsilon_0}\cdot\frac{1}{r^{\cancel{2}}}(\cancel{-4\pi}kr) \Rightarrow E = \frac{-k}{\varepsilon_0 r} \Rightarrow \boxed{\vec{E} = \frac{-k}{\varepsilon_0 r}\hat{r}}$

For $r>b$, $Q_{enc}=0 \Rightarrow \boxed{E=0}$

b) $\oint \vec{D}\cdot d\vec{a} = Q_{f_{enc}}$ // No free charge $\Rightarrow \oint \vec{D}\cdot d\vec{a} = 0 \Rightarrow \underline{\vec{D}=0}$

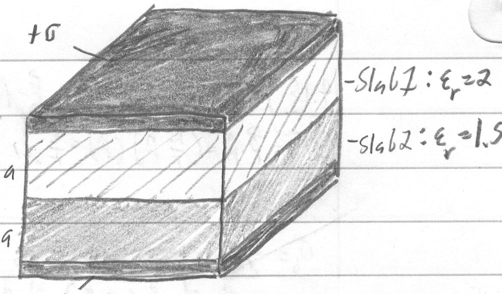
$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = 0 \Rightarrow \vec{E} = \frac{-\vec{P}}{\varepsilon_0}$ // $\vec{P} = \frac{k}{r}\hat{r} \Rightarrow \boxed{\vec{E} = \frac{-k}{\varepsilon_0 r}\hat{r}}$ $\boxed{\text{For } a<r<b}$

For $a<r<b$

For $r<a$, $\vec{P}=0 \Rightarrow \boxed{\vec{E}=0}$

For $r>b$, $\vec{P}=0 \Rightarrow \boxed{\vec{E}=0}$

$\frac{-1}{2}+\frac{2}{3}$ $\frac{-3}{6}+\frac{4}{6}$



4.18

(a) $\oint \vec{D}\cdot d\vec{a} = Q_{f_{enc}} \Rightarrow DA = +\sigma A \Rightarrow \boxed{\vec{D}_1 = +\sigma\hat{z},\ \vec{D}_2 = +\sigma\hat{z}}$

(b) $\vec{D} = \epsilon_0\vec{E} \Rightarrow \vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{1}{\epsilon_0\epsilon_r}\vec{D}$ // $\epsilon_{r,1} = 2$; $\epsilon_{r,2} = \frac{3}{2}$ //

$\vec{E}_1 = \frac{1}{2\epsilon_0}+\sigma\hat{z} \Rightarrow \boxed{\vec{E}_1 = \frac{+\sigma}{2\epsilon_0}\hat{z}}$; $\vec{E}_2 = \frac{2}{3\epsilon_0}\cdot +\sigma\hat{z} \Rightarrow \boxed{\vec{E}_2 = \frac{+2\sigma}{3\epsilon_0}\hat{z}}$

(c) $\vec{D}_1 = \epsilon_0\vec{E}_1 + \vec{P}_1 \Rightarrow +\sigma = \frac{\sigma}{2} + \vec{P}_1 \Rightarrow \boxed{\vec{P}_1 = \frac{\sigma}{2}\hat{z}}$

$\vec{D}_2 = \epsilon_0\vec{E}_2 + \vec{P}_2 \Rightarrow +\sigma = +\frac{2}{3}\sigma + \vec{P}_2 \Rightarrow \vec{P}_2 = \sigma\left(+1-\frac{2}{3}\right) \Rightarrow \boxed{\vec{P}_2 = +\frac{1}{3}\sigma\hat{z}}$

(d) $V = -\left(\int \vec{E}_1\cdot d\vec{l} + \int \vec{E}_2\cdot d\vec{l}\right) = -(E_1 a + E_2 a) = a\left(-\frac{\sigma}{2\epsilon_0} - \frac{2\sigma}{3\epsilon_0}\right) = \frac{\sigma a}{\epsilon_0}\left(\frac{-1}{2} - \frac{2}{3}\right) = \boxed{\frac{-7}{6}\frac{\sigma a}{\epsilon_0} = V}$

(e) $\nabla\cdot\vec{P} = -\rho_b$, but $\nabla\cdot\vec{P} = 0$, so $\rho_b = 0$. No bound charges, but $\sigma_b = \vec{P}\cdot\hat{n}$:

At top of Slab 1, $-P_1 \Rightarrow \boxed{\sigma_b = -\sigma/2}$ At top of slab 2, $-P_2 \Rightarrow \boxed{\sigma_b = -\sigma/3}$

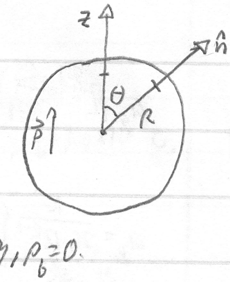
At bottom of Slab 1, $+P_1 \Rightarrow \boxed{\sigma_b = \sigma/2}$ At bottom of slab 2, $+P_2 \Rightarrow \boxed{\sigma_b = \sigma/3}$

(f) $E = \frac{1}{\epsilon_0}\sigma$

For slab 1: $\vec{E}_1 = \frac{1}{\epsilon_0}\left(\sigma_{top} - \sigma_{below}\right) = \frac{1}{\epsilon_0}\left(\sigma - \frac{1}{2}\sigma\right) \Rightarrow \boxed{\vec{E}_1 = \frac{1}{2\epsilon_0}\sigma\hat{z}}$

For slab 2: $\vec{E}_2 = \frac{1}{\epsilon_0}\left(\sigma_{top} - \sigma_{below}\right) = \frac{1}{\epsilon_0}\left(\sigma - \frac{1}{3}\sigma\right) \Rightarrow \boxed{\vec{E}_2 = \frac{2}{3\epsilon_0}\sigma\hat{z}}$

Ex. 4.2 (using Ex. 3.9)



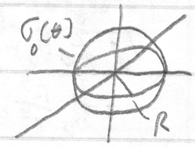
Find $\vec{E}$-field produced by uniformly distributed polarized sphere, radius R

Sol'n: Choose z-axis to be w/ polarization direction. The volume bound charge density, $\rho_b = 0$.

because $\vec{P}$ is uniform.

$\sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta$

Find potential inside & outside the sphere.



$\rightarrow$ sol'n: $\nabla^2 V = 0$ (ϕ dependent) $\Downarrow$

$//\sigma_0(\theta) = \sigma_b$

In sphere: $V_i(r,\theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$

Out sphere: $V_o(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$

Boundary conditions: 1) Voltage continuous, $r=R$
2) $\Delta V_{normal} \rightarrow \sigma_0(\theta)$ $\Downarrow$

1) $V_i(R,\theta) = V_o(R,\theta) \Rightarrow \sum_{l=0}^{\infty} a_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} B_l R^{-(l+1)} P_l(\cos\theta)$ $\Downarrow$

$a_l R^l \cancel{P_l(\cos\theta)} = B_l R^{-(l+1)} \cancel{P_l(\cos\theta)} \Rightarrow a_l R^l R^{+(l+1)} = B_l \Rightarrow \underline{B_l = a_l R^{2l+1}}$

2) $\vec{E}_o^{\perp} - \vec{E}_i^{\perp} = \frac{\sigma_0}{\varepsilon_0}$ $\Downarrow$ // Gauss's Law, at surface charge density, make Gaussian Pillbox // $\Downarrow$

$(\vec{E} = -\nabla V)$ $\left(\frac{\partial V_o}{\partial r} - \frac{\partial V_i}{\partial r}\right)\Big|_{r=R} = -\frac{\sigma_0}{\varepsilon_0}$ $\Downarrow$

$V_i = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta) \Rightarrow \frac{\partial V_i}{\partial r} = \sum_{l=0}^{\infty} a_l l r^{l-1} P_l(\cos\theta) \Rightarrow \frac{\partial V_i}{\partial r}\Big|_R = \sum_{l=0}^{\infty} a_l l R^{l-1} P_l(\cos\theta)$

$V_o = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) \Rightarrow \frac{\partial V_o}{\partial r} = \sum_{l=0}^{\infty} B_l (-(l+1)) r^{-l-2} P_l(\cos\theta) \Rightarrow \frac{\partial V_o}{\partial r}\Big|_R = -\sum_{l=0}^{\infty} B_l (l+1) R^{-(l+2)} P_l(\cos\theta)$

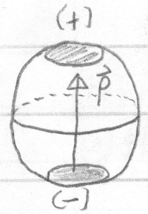
// apply boundary condition (2) $\Downarrow$

$V_o - V_i \Rightarrow +\sum_{l=0}^{\infty} B_l (l+1) R^{-(l+2)} P_l(\cos\theta) + \sum_{l=0}^{\infty} a_l l R^{l-1} P_l(\cos\theta) = +\frac{\sigma_0(\theta)}{\varepsilon_0}$ $\Downarrow$ // use $B_l = a_l R^{2l+1}$ from B.C. (1) $\Downarrow$

$\sum_{l=0}^{\infty} a_l R^{2l+1} (l+1) P_l(\cos\theta) + a_l l R^{l-1} P_l(\cos\theta) = \frac{\sigma_0(\theta)}{\varepsilon_0}$ $\Downarrow$

$\sum_{l=0}^{\infty} a_l R^{l-1} \{l+1+l\} P_l(\cos\theta) = \frac{\sigma_0(\theta)}{\varepsilon_0} \Rightarrow \sum_{l=0}^{\infty} a_l R^{l-1} (2l+1) P_l(\cos\theta) = \frac{\sigma_0(\theta)}{\varepsilon_0}$ // multiply by $\sin\theta$ & $P_{l'}$, then integrate $\int_0^{\pi} d\theta$ $\Downarrow$

$$\sum_{l=0}^{\infty} (2l+1) a_l R^{l-1} \underbrace{\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta}_{=\frac{2}{2l+1},\ \text{if } l=l'} = \int_0^{\pi} P_{l'}(\cos\theta)\sin\theta \frac{\sigma_0(\theta)}{\varepsilon_0}\, d\theta$$ $\Downarrow$



$$a_\ell R^{\ell-1}\frac{2}{2\ell+1}(2\ell+1)=\int_0^\pi P_{\ell'}(\cos\theta)\sin\theta\,\frac{\sigma_0(\theta)}{\varepsilon_0}\,d\theta \Rightarrow a_\ell=\frac{1}{2\varepsilon_0 R^{\ell-1}}\int_0^\pi P_\ell(\cos\theta)\,\sigma_0(\theta)\sin\theta\,d\theta$$

// Assume $\sigma_0(\theta)=k\,P_1(\cos\theta)$, & $k=P$ ✓

$$a_\ell=\frac{k}{2\varepsilon_0 R^{\ell-1}}\int_0^\pi P_\ell(\cos\theta)\,P_1(\cos\theta)\sin\theta\,d\theta \quad // \ \ell=1, \text{ if } \sigma_0(\theta) \text{ is true. } ✓$$

$$a_1=\frac{k}{\cancel{2}\varepsilon_0 R^{(1-1)}}\left(\frac{\cancel{2}}{2(1)+1}\right)\Rightarrow \underline{a_1=\frac{k}{3\varepsilon_0}} ✓$$

$$V_i=\sum_{\ell=0}^{\infty}a_\ell r^\ell P_\ell(\cos\theta)\Rightarrow V_i(r,\theta)=\frac{k}{3\varepsilon_0}r\cos\theta$$

$$V_o=\sum_{\ell=0}^{\infty}B_\ell r^{-(\ell+1)}P_\ell(\cos\theta)\ //\ B_1=a_1R^{2(1)+1}\ //\Rightarrow V_o(r,\theta)=B_1r^{-2}\cos\theta\ //\ B_1=\frac{k}{3\varepsilon_0}R^3\ // ✓$$

$$=\frac{k}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta ✓ \quad // \text{ w/ } P=k\ //$$

$$\underline{V(r,\theta)=\begin{cases}\dfrac{P}{3\varepsilon_0}r\cos\theta, & \text{for } r\le R\\[2mm] \dfrac{P}{3\varepsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for } r\ge R\end{cases}}$$

Since $r\cos\theta=z$, $\vec{E}$-field in sphere is uniform: $\vec{E}=-\nabla V=\frac{-P}{3\varepsilon_0}\hat{z}\Rightarrow\vec{E}=\frac{-1}{3\varepsilon_0}\vec{P}$, for $r<R$.

Outside sphere, potential identical to perfect dipole at the origin: $V=\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$, for $r\ge R$

So,

$$\boxed{\vec{E}=\frac{-1}{3\varepsilon_0}\vec{P},\ \text{for } r<R}$$