Solutions for Homework 1

Dr. Jordan Hanson - Whittier College Dept. of Physics and Astronomy

February 21, 2022

1 Problem 1.54

Verify the divergence theorem for $\vec{v} = r^2 \cos\theta \hat{r} + r^2 \cos\phi \hat{\theta} - r^2 \cos\theta \sin\phi \hat{\phi}$ over the octant of the sphere of radius R with the center at the origin.

Break the problem into manageable pieces. (a) What is the divergence of the field? (b) What is the volume integral of it?

Divergence: $4r\cos\theta$.
Volume integral of the divergence:

$$\int_0^R \int_0^{\pi/2} \int_0^{\pi/2} 4r\cos\theta \; r^2 \sin\theta dr d\theta d\phi = \frac{\pi R^4}{4} \tag{1}$$

The surface integral has four parts. (a) Outer curved surface with $d\vec{a} = r^2 \sin\theta d\theta d\phi \hat{r}$, and the result is $\pi R^4/4$. (b) The lower side is defined by $\theta = \pi/2$, and $d\vec{a} = -r^2 dr d\phi \hat{z}$. What is $\hat{z}$ here ... $\hat{\theta}$. Consult back page of the book for conversions and set $\theta = \pi/2$. The result is $R^4/4$. (c) The left side is described by $\phi = 0$, and $d\vec{a} = rdrd\theta(-\hat{y}) = -rdrd\theta\hat{\phi}$. However, the $\hat{\phi}$-component is zero for $\phi = 0$, so the surface integral is zero. (d) The right side has $d\vec{a} = rdrd\theta\hat{\phi}$, and $\phi = \pi/2$. This time, the surface integral for $\phi = \pi/2$ is not zero, and the result is $-R^4/4$. Summing all the pieces, we find

$$\oint \vec{v} \cdot d\vec{a} = \frac{\pi R^4}{4} \tag{2}$$

2 Problem 1.55

Break the problem into the following pieces: (a) What is the curl of $\vec{v}$? (b) What is the surface integral of the curl? (c) How do we approach the line integral?

The curl may be evaluated in Cartesian coordinates: $\nabla \times \vec{v} = (b-a)\hat{k}$. Form the surface integral:

$$\int (\nabla \times \vec{v}) \cdot d\vec{a} = (b-a)\pi R^2 \tag{3}$$

The integrand is a constant, and parallel to the area vector. Thus, the constant moves outside the integral and we have just the area of the circle. What is $d\vec{l}$ on the circle of radius R? *Cylindrical coordinates* work best to describe the situation: $d\vec{l} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z}$. However, $dz = 0$ and $ds = 0$, so we are left with $d\vec{l} = sd\phi\hat{\phi}$. That makes the line integral ($s = R$):

$$\oint \vec{v} \cdot d\vec{l} = \int_0^{2\pi} ay\hat{x} \cdot Rd\phi\hat{\phi} + \int_0^{2\pi} bx\hat{y} \cdot Rd\phi\hat{\phi} \tag{4}$$

Here are some useful conversions:

- $x = R\cos\phi$
- $y = R\sin\phi$
- $\hat{x} = ... - \sin\phi\hat{\phi}$
- $\hat{y} = ... + \cos\phi\hat{\phi}$

The ... will be zero because those pieces are proportional to $\hat{s}$ and will be dotted with $\hat{\phi}$. Substituting all of that into Eq. 4 gives $(b-a)\pi R^2$.

3 Problem 1.56

Break the problem into pieces. First, address the closed-path line integral. For the first path,

$$d\vec{l} = dy\hat{y} \tag{5}$$
$$\vec{v} \cdot d\vec{l} = yz^2 dy \tag{6}$$
$$z = 0 \tag{7}$$
$$\int \vec{v} \cdot d\vec{l} = 0 \tag{8}$$

For the other straight piece,

$$d\vec{l} = dz\hat{z} \tag{9}$$
$$\vec{v} \cdot d\vec{l} = (3y + z)dz \tag{10}$$
$$x = y = 0 \tag{11}$$
$$\int \vec{v} \cdot d\vec{l} = -\int_0^2 (3y + z)dz = -2 \tag{12}$$

For the diagonal piece, the path has $x = 0$, and $z = 2 - 2y$, with $dz = -2dy$. We have

$$d\vec{l} = dy\hat{y} + dz\hat{z} \tag{13}$$
$$\int \vec{v} \cdot d\vec{l} = \int_1^0 dy \left(4y^3 - 8y^2 + 2y - 4\right) = \frac{14}{3} \tag{14}$$
$$\tag{15}$$

In total, the close-path line integral is $\boxed{-2 + 14/3 = 8/3}$. The curl of the field is

$$\nabla \times \vec{v} = (3 - 2yz)\hat{x} + ... \tag{16}$$

We don't need the other components of the curl because the area vector will just cancel them: $d\vec{a} = dydz\hat{x}$.

$$\int (\nabla \times \vec{v}) \cdot d\vec{a} = \int_0^1 \int_0^{2-2y} dzdy(3 - 2yz) \tag{17}$$
$$\int_0^1 dy \left(4y^3 - 8y^2 + 10y - 6\right) = \frac{8}{3} \tag{18}$$

Notice in Eq. 17 that we integrate z from 0 to z_{max}, where z_{max} is determined by the relationship between z and y. Thus, Stoke's theorem checks out.

4 Problem 1.57

This exercise helps us practice with coordinate systems besides Cartesian. The line integral involves four pieces. The first is in the x-direction. In spherical coordinates:

$$d\vec{l} = dr\hat{r} \tag{19}$$
$$\phi = 0, \ \theta = \pi/2 \tag{20}$$
$$\vec{v} \cdot d\vec{l} = r\cos^2\theta \ dr = 0 \tag{21}$$

The second piece is in the xy-plane, with $\theta = \pi/2$, and $r = 1$. In spherical coordinates:

$$d\vec{l} = rd\phi\hat{\phi} = d\phi\hat{\phi} \tag{22}$$
$$\theta = \pi/2 \tag{23}$$
$$\vec{v} \cdot d\vec{l} = 3rd\phi = 3d\phi \quad (r = 1) \tag{24}$$

The result is

$$\int \vec{v} \cdot d\vec{l} = \int_0^{\pi/2} 3rd\phi = \frac{3\pi}{2} \quad (r = 1) \tag{25}$$

The third piece is in the z-direction, with $y=1$ and $x=0$. We have

$$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} \tag{26}$$
$$\vec{v}\cdot d\vec{l} = r\cos^2\theta dr - r^2\cos\theta\sin\theta d\theta \tag{27}$$
$$y = r\sin\theta = 1 \quad (r = 1/\sin\theta) \tag{28}$$
$$dr = -\frac{\cos\theta}{\sin^2\theta}d\theta \tag{29}$$
$$\vec{v}\cdot d\vec{l} = (-\cot^3\theta - \cot\theta)d\theta \tag{30}$$

The line integral can therefore be cast in terms of θ only, and integrated from $\theta = \pi/2$ to $\tan^{-1}(1/2)$. The result is

$$\int \vec{v}\cdot d\vec{l} = -\frac{1}{2}\left.\frac{1}{\sin^2\theta}\right|_{\pi/2}^{\tan^{-1}(1/2)} = 2 \tag{31}$$

For the last piece, the path is along r, while $\phi = \pi/2$ *and* $\theta = \theta_0 = \tan^{-1}(1/2)$ remain fixed. We find

$$d\vec{l} = dr\hat{r} \tag{32}$$
$$\vec{v}\cdot d\vec{l} = \cos^2\theta_0 r dr \tag{33}$$
$$\int \vec{v}\cdot d\vec{l} = \cos^2\theta_0\int_{\sqrt{5}}^{0} rdr = -2 \tag{34}$$

Totaling the four contributions to the line integral: $\boxed{3\pi/2 + 2 - 2 = 3\pi/2}$. Checking Stoke's theorem requires the curl in spherical coordinates:

$$\nabla\times\vec{v} = 3\cot\theta\,\hat{r} - 6\hat{\theta} \tag{35}$$

The surface integral of the bottom face is ($d\vec{a} = -rdrd\phi\hat{\theta}$):

$$\int \nabla\times\vec{v}\cdot d\vec{a} = \int_0^{\pi/2}\int_0^1 6rdrd\phi = \frac{3\pi}{2} \tag{36}$$

For the back face, $d\vec{a} = da\hat{\phi}$. But the curl does not have a $\hat{\phi}$-component, so that surface integral is zero. Thus, Stoke's Theorem checks out.

5 Problem 1.59

First, find the divergence using spherical coordinates:

$$\nabla\cdot\vec{v} = 4r\cot\theta\cos\theta \tag{37}$$

Integrate over the slice of the sphere with radius R and opening angle $\theta = \pi/6$.

$$\int_0^R\int_0^{2\pi}\int_0^{\pi/6} 4r\cot\theta\cos\theta\, r^2\sin\theta drd\theta d\phi = 2\pi R^4\int_0^{\pi/6}\cos^2\theta d\theta = \boxed{\frac{\pi R^4}{12}(2\pi + 3\sqrt{3})} \tag{38}$$

The closed surface integral must be broken into the "cone" portion, and the "top" portion. For the top, we have

$$d\vec{a} = R^2\sin\theta d\theta d\phi\hat{r} \tag{39}$$
$$\vec{v}\cdot d\vec{a} = R^4\sin^2\theta d\theta d\phi \tag{40}$$
$$\int \vec{v}\cdot d\vec{a} = 2\pi R^4\int_0^{\pi/6}\sin^2\theta d\theta \tag{41}$$
$$\int \vec{v}\cdot d\vec{a} = \frac{\pi R^4}{12}(2\pi - 3\sqrt{3}) \tag{42}$$

For the cone portion:

$$d\vec{a} = \frac{1}{2}rdrd\theta d\phi\hat{\theta} \tag{43}$$
$$\int \vec{v}\cdot d\vec{a} = \int_0^1\int_0^{2\pi}\sqrt{3}r^3 drd\phi = \frac{\pi\sqrt{3}R^4}{2} \tag{44}$$

Summing the top and the cone, we find the surface integral total is $\boxed{\frac{\pi R^4}{12}(2\pi + 3\sqrt{3})}$.

6 Problem 1.62

- (a) First, note that $d\vec{a} = R^2 \sin\theta d\theta d\phi \hat{r}$. Integrating just $d\vec{a}$ should yield a vector, which can be broken into x, y, and z-components. By symmetry, there should be no x or y-components. Just the z-component of $\hat{r}$ is $\cos\theta \hat{z}$ (back cover of the textbook). Integrating:

$$\vec{a} = 2\pi R^2 \hat{z} \int_0^{\pi/2} \sin\theta \cos\theta \; d\theta = \pi R^2 \hat{z} \tag{45}$$

In other words, we find the projected cross-sectional area, that of a circle and not of a hemisphere.

- (b) Note that Problem 1.61 (a) says that

$$\int_{\mathcal{V}} (\nabla T) d\tau = \oint_{\mathcal{S}} T d\vec{a} \tag{46}$$

This is the type of formula that follows from the other fundamental theorems of calculus. It says that the volume integral over a vector field that is the gradient of a scalar is equal to the closed surface integral of the scalar. However, we can let $T(x, y, z) = 1$ so that the right hand side is

$$\oint_{\mathcal{S}} d\vec{a} = \int_{\mathcal{V}} (\nabla \; 1) d\tau = 0 \tag{47}$$

Thus, all closed surface integrals of constants are zero.

- (c) Suppose there are two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ that share the same boundary line. Adding the surface integrals:

$$\oint_{\mathcal{S}_1} d\vec{a} + \oint_{\mathcal{S}_2} d\vec{a} = \vec{a}_{\text{total}} \tag{48}$$

But the two surfaces now form a closed surface, so $\vec{a}_{\text{total}} = \vec{0}$ (part b). Further, the normal directions of $\mathcal{S}_1$ and $\mathcal{S}_2$ differ by a minus sign, so we find

$$\oint_{\mathcal{S}_1} d\vec{a} - \oint_{\mathcal{S}_2} d\vec{a} = 0 \tag{49}$$

$$\oint_{\mathcal{S}_1} d\vec{a} = \oint_{\mathcal{S}_2} d\vec{a} \tag{50}$$

- (d) For the kind of triangle described in the hint, $d\vec{a} = \frac{1}{2}\vec{r} \times d\vec{l}$, since the cross product can be interpreted as the area of a parallelogram and we need one half of that parallelogram. Totalling all of the triangles around the surface:

$$\vec{a} = \oint d\vec{a} = \oint \frac{1}{2}\vec{r} \times d\vec{l} \tag{51}$$

- (e) Letting $T = \vec{c} \cdot \vec{r}$ in 1.61 (e), we find

$$-\oint (\vec{c} \cdot \vec{r}) d\vec{l} = \int_{\mathcal{S}} \nabla(\vec{c} \cdot \vec{r}) \times d\vec{a} \tag{52}$$

From the reading, we need a product rule for the gradient on the left side:

$$\nabla(\vec{c} \cdot \vec{r}) = \vec{c} \times (\nabla \times \vec{r}) + (\vec{c} \cdot \nabla)\vec{r} \tag{53}$$

$$\nabla(\vec{c} \cdot \vec{r}) = (\vec{c} \cdot \nabla)\vec{r} = \vec{c} \tag{54}$$

$$(\nabla \times \vec{r} = 0) \tag{55}$$

Using that result gives

$$\oint (\vec{c} \cdot \vec{r}) d\vec{l} = -\int_{\mathcal{S}} \vec{c} \times d\vec{a} = -\vec{c} \times \vec{a} = \vec{a} \times \vec{c} \tag{56}$$

Reversing the order of the cross-product removes the minus sign in the final step.

$$\vec{a} \times \vec{c} = \oint (\vec{c} \cdot \vec{r}) d\vec{l} \tag{57}$$

7 Problem 1.63

- (a) If $\vec{v} = \hat{r}/r$, then

$$\nabla \cdot \vec{v} = \frac{1}{r^2} \tag{58}$$

If we perform the surface integral of $\vec{v}$ over the sphere of radius R, we get $4\pi R$, and 4π if $R = 1$. If we perform the volume integral over the sphere of radius with the divergence as the integrand, we also find $4\pi R$ (and 4π if $R = 1$). So this result appears free of the problems we find with the divergence of $\hat{r}/r^2$. The general form is

$$\nabla \cdot (\hat{r} r^n) = (n+2) r^{n-1} \tag{59}$$

However, if $n = -2$, then the divergence theorem won't work unless $\nabla \cdot \hat{r}/r^2 = 4\pi\delta^3(\vec{r})$.

- Plugging the field into the spherical version of the curl neatly gives 0. However, note that using 1.61 (b) invites a surface integral over $\vec{v} \times d\vec{a}$. However, $\vec{v}$ and $d\vec{a}$ are parallel, if we consider the surface to be a spherical surface. Since we get to choose the surface, the integrand is zero.

8 Problem 1.64

- (a) Using the definition of the Laplacian in spherical coordinates, we find

$$D(r, \epsilon) = \frac{3\epsilon^2}{4\pi}(r^2 + \epsilon^2)^{-5/2} \tag{60}$$

- (b) Setting $r = 0$, we have (as $\epsilon \to 0$)

$$D(0, \epsilon) = \frac{3\epsilon^2}{4\pi}(\epsilon^2)^{-5/2} \propto \epsilon^{-3} \to \infty \tag{61}$$

- (c) In the numerator there is one factor of ϵ^2, so as $\epsilon \to 0$, $D \to 0$ as long as the other term in the denominator (r^2) is not zero.

- (d) Using a trigonometric substitution $r = \epsilon \tan\theta$, one could show that

$$\int_{\text{space}} d\tau D(r, \epsilon) = 3 \int_0^{\pi/2} \tan^2\theta \cos^3\theta d\theta \tag{62}$$

Notice that when $r \to \infty$, $\theta \to \pi/2$. Notice that the integral is equivalent to

$$\int_{\text{space}} d\tau D(r, \epsilon) = 3 \int_0^{\pi/2} \sin^2\theta \cos\theta d\theta \tag{63}$$

Let $u = \sin\theta$, so that $du = \cos\theta d\theta$. If $\theta = 0$, then $\sin\theta = 0$, and if $\theta = \pi/2$, then $\sin\theta = 1$. The integral becomes

$$\int_{\text{space}} d\tau D(r, \epsilon) = 3 \int_0^1 u^2 du = 1 \tag{64}$$

Thus, the $D(r, \epsilon)$ function obeys all the properties of a 3D delta function when $\epsilon \to 0$.