Midterm Solutions for Electromagnetic Theory (PHYS330)

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Abstract

This exam may be completed at home, and covers chapters 1-3 of the course text and in-class examples. Class notes and the course text may be used (open book), but no internet sources are allowed. The daily warm-up exercises are good study materials for this exam.

1 Math Bootcamp

1. (a) If $\mathbf{A}$ and $\mathbf{B}$ are two vector functions, what does the expression $(\mathbf{A}\cdot\nabla)\mathbf{B}$ mean? That is, what are its x, y, and z components, in terms of the Cartesian components of $\mathbf{A}$, ∇, and $\mathbf{B}$? (b) Compute $(\hat{r}\cdot\nabla)\hat{r}$, where $\hat{r}$ is $\mathbf{r}/r$. (c) One can show that the *force* on a dipole induced by a non-uniform field is

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \tag{1}$$

Compute the force on a physical dipole located at the origin with $\mathbf{p} = q\mathbf{d} = qd\ \hat{\mathbf{x}}$ in a field with associated potential $V(\mathbf{r}) = V_0 r^2 + V_1$.

- (a) $(\mathbf{A}\cdot\nabla)\mathbf{B} = A_x\frac{\partial\vec{B}}{\partial x} + A_y\frac{\partial\vec{B}}{\partial y} + A_z\frac{\partial\vec{B}}{\partial z}$
- (b) $(\hat{\mathbf{r}}\cdot\nabla)\hat{\mathbf{r}} = \mathbf{0}$
- (c) $\mathbf{F} = -2V_0\mathbf{p}$. The units are equivalent to qE, a force. The direction makes sense if you visualize the dipole in a bowl-shaped potential (negative charge rolls uphill). The potential is a bowl-shape to first-order.

2. Evaluate the following integral using (a) the three-dimensional Dirac delta function, or (b) integration by parts. Solving both earns a bonus point.

$$J = \int_{\mathcal{V}} e^{-r}\left(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}\right) \tag{2}$$

- (a) The integral is equivalent to evaluating the integrand at the origin, since it involves the Diract delta-function in three dimensions.

$$J = \int_{\mathcal{V}} e^{-r}4\pi\delta^3(\mathbf{r})d\tau' = 4\pi \tag{3}$$

- (b) Integration by parts, taking the surface and volume corresponding to a sphere of radius R:

$$e^{-r}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right) = \nabla\cdot\left(e^{-r}\frac{\hat{r}}{r^2}\right) + \frac{e^{-r}}{r^2} \tag{4}$$

$$J = \oint_{\mathcal{S}} e^{-r}r^{-2}\hat{r}\cdot d\mathbf{a} + \int_{\mathcal{V}}\frac{e^{-r}}{r^2}d\tau' \tag{5}$$

$$J = 4\pi\left(e^{-R}R^{-1} + e^{-r}\Big|_R^0\right),\quad R\to\infty \tag{6}$$

$$J = 4\pi \tag{7}$$

Thus the two methods give the same answer.

2 Electrostatics

1. Suppose two dipoles, each with dipole moment $\mathbf{p}$ pointed in opposite directions, form a square with alternating positive and negative charges and side length d. Calculate the field $\mathbf{E}_{\mathrm{tot}}$ at the following points P: (a) $P = (0,0)$, (b) $P = (2d, 0)$, and $P = (0, 2d)$. Check units and take limits[1].

- (a) $\mathbf{E} = \mathbf{0}$, by symmetry.
- (b) Break the problem into pieces, assuming $+q$'s are in first and third quadrants, and $-q$'s are in the second and fourth quadrants:

$$\begin{align} \mathbf{E} &= \sum_i \frac{kq_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i = kq\left(\frac{\hat{\mathscr{r}}_1}{\mathscr{r}_1^2} - \frac{\hat{\mathscr{r}}_2}{\mathscr{r}_2^2} + \frac{\hat{\mathscr{r}}_3}{\mathscr{r}_3^2} - \frac{\hat{\mathscr{r}}_4}{\mathscr{r}_4^2}\right) \tag{8}\\ \hat{\mathscr{r}}_{1,2} &= \left(3\sqrt{2}\hat{x} \mp \sqrt{2}\hat{y}\right)/\sqrt{20} \tag{9}\\ \hat{\mathscr{r}}_{3,4} &= \left(\sqrt{25}\hat{x} \pm \hat{y}\right)/\sqrt{26} \tag{10}\\ \mathscr{r}_{1,2}^2 &= \frac{5}{2}d^2 \tag{11}\\ \mathscr{r}_{3,4}^2 &= \frac{13}{2}d^2 \tag{12}\\ & \tag{13} \end{align}$$

- Note that summing the four contributions eliminates the $\hat{x}$ components. This is expected from symmetry considerations:

$$\mathbf{E} = \frac{4kq}{d^2}\hat{y}\left(\frac{1}{13\sqrt{26}} - \frac{\sqrt{10}}{25}\right) \approx -\frac{4kq}{9d^2}\hat{y} \tag{14}$$

- (c) By symmetry, **the field should be identical**, except in the $-\hat{x}$ direction.

2. The electric potential of some configuration is given by the expression

$$V(\mathbf{r}) = A\frac{e^{-\lambda r}}{r} \tag{15}$$

In Eq. 15, A and λ are constants. Find the field $\mathbf{E}(\mathbf{r})$, the charge density ρ and the total charge Q in terms of A and λ. *Hint:* $\rho = \epsilon_0 A(4\pi\delta^3(\mathbf{r}) - \lambda^2\exp(-\lambda r)/r)$. **Bonus:** compute the total energy stored in the field over all space.

- (a) The gradient leads to the $\mathbf{E}$-field, and just the $\hat{\mathbf{r}}$-component is necessary:

$$\mathbf{E} = A\hat{\mathbf{r}}\left(\frac{e^{-\lambda r}}{r^2} + \frac{\lambda e^{-\lambda r}}{r}\right) \tag{16}$$

- Note that, by Gauss' Law, $\rho = \epsilon_0\nabla\cdot\mathbf{E}$. Taking the appropriate derivatives leads to

$$\rho = -\epsilon_0 A\lambda^2\frac{e^{-\lambda r}}{r} \tag{17}$$

  The trouble is when $r = 0$, and Problem 1.2 should be a clue. The divergence *itself* blows up as $r \to 0$, so we should first take the limit of $\mathbf{E}$ as $r \to 0$. The result is (to lowest order):

$$\mathbf{E} = A\hat{\mathbf{r}}r^{-2} \tag{18}$$

  And when we take the divergence of Eq. 18, we encounter the three-dimensional Dirac δ-function. Collecting the results together gives

$$\rho = \epsilon_0 A\left(4\pi\delta^3(\mathbf{r}) - \frac{\lambda^2 e^{-\lambda r}}{r}\right) \tag{19}$$

  The units check out because $\epsilon_0 A$ has units of Coulombs, and A has units of Volt-meters.
- The total charge is the volume integral of ρ, and the results *turns out to be zero.* This is a highly intriguing result, because it must describe something about a spherically symmetry atomic structure with a positive center and negative outer shell, or a *shielded* positive charge with overall neutrality.
- **Bonus:** Given that there is a Dirac δ-function in the charge density, the energy density should diverge if we include the origin. This can be checked numerically, and represents a good final project idea.

[1] This object is an electrostatic quadrupole.

3. (a) Use Gauss' Law to compute the field $\mathbf{E}$ as a function of the distance s from a long straight wire with positive charge density λ. (b) Calculate the position versus time of a positive point charge q with mass m if it is released a distance s from the wire.

Using Gauss' Law, we can show that the field is

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s} \tag{20}$$

Newton's 2nd law tells us that

$$\frac{d^2 s}{dt^2} = \frac{C}{s} \tag{21}$$

$$C = \frac{q}{m}\frac{\lambda}{2\pi\epsilon_0} \tag{22}$$

Note that the constant C has units of acceleration times distance. This is a rather difficult differential equation. What if we assume that $s(t)$ is a power series, and take it to quadratic order in t? This would treat the acceleration as a constant, which is not inaccurate, due to the shape of the field. All the acceleration occurs *near* the charge distribution, and weakens as the particle moves away. Let

$$s(t) = \sum_{n=0}^{\infty} a_n t^n \approx a_0 + a_1 t + a_2 t^2 \tag{23}$$

Note that Eq. 21 simplifies to

$$s\frac{d^2 s}{dt^2} = 2a_2(a_0 + a_1 t + a_2 t^2) \tag{24}$$

Boundary conditions:

- At $t = 0$, $s = s_i$, the initial position $\to a_0 = s_i$
- At $t = 0$, $ds/dt = 0$, zero initial velocity, and $\to a_1 = 0$
- At $t = 0$, use Eq. 21 to find $a_2 = C/(2s_i)$

The approximate equation of motion becomes

$$s(t) = s_i + \frac{1}{2}\frac{C}{s_i}t^2 \tag{25}$$

The acceleration increases when either charge or charge distribution increases in strength. The units check out, because C has units of acceleration times distance. All boundary conditions are followed.

3 Potentials

1. Suppose the potential $V_0(\theta)$ at the surface of a sphere of radius R is specified, and there is no charge inside or outside the sphere. (a) Show that the charge density on the sphere is given by

$$\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta) \tag{26}$$

$$C_l = \int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta d\theta \tag{27}$$

(b) Produce the specific result for $\sigma(\theta)$ with $V_0(\theta) = P_2(\cos\theta)$.

Start with the basic solution to the Laplacian in spherical coordinates:

$$V_{in}(r,\theta) = \sum_l A_l r^l P_l(\cos\theta) \tag{28}$$

$$V_{out}(r,\theta) = \sum_l B_l r^{-(l+1)} P_l(\cos\theta) \tag{29}$$

There are two boundary conditions that must be satisfied: *continuity* in potential, and *discontinuity* in the $\mathbf{E}$-field. In this situation, the boundary condions lead to the following:

$$B_l = A_l R^{2l+1} \tag{30}$$

$$\frac{\sigma}{\epsilon_0} = -\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{out}}{\partial r}\right) \tag{31}$$

Combining these two facts, and letting the remaining coefficient be C_l, leads to

$$\sigma(\theta) = \frac{\epsilon_0}{R}\sum_l C_l(2l+1)R^l P_l(\cos\theta) \tag{32}$$

Using **Fourier's Trick** to isolate C_l in the V_{in} equation at $r = R$ gives

$$C_l = \frac{2l+1}{2}R^{-l}\int V_0(\theta)P_l(\cos\theta)\sin\theta d\theta \tag{33}$$

We can absorb the constants in front of the integral into our formula for $\sigma(\theta)$, and we find the following results:

$$\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta) \tag{34}$$

$$C_l = \int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta d\theta \tag{35}$$

If $V_0(\theta) = P_2(\theta)$, then $C_2 = 2/5$ and the other coefficients are zero. That makes the charge density

$$\sigma(\theta) = \frac{5\epsilon_0}{R}P_2(\cos\theta) \tag{36}$$

2. For the infinite rectangular pipe in Example 3.4 from the text, suppose the constant potential V_0 is now only on one side. That is, at $y = 0$ and $x = \pm b$, the potential is zero. At $y = a$, the potential is V_0. Find the potential $V(x, y)$ inside the pipe. *Square pipes are examples of electromagnetic waveguides often used in microwave electronics.*

   A general solution that matches boundary conditions, according to the procedures of Ch. 3, is

$$V(x,y) = \sum_{\text{n odd}} C_n \cos\left(\frac{n\pi x}{2b}\right)\sinh\left(\frac{n\pi y}{2b}\right) \tag{37}$$

   A version of **Fourier's Trick** for odd n, m is

$$\int_{-b}^{b}\cos(n\pi x/2b)\cos(m\pi x/2b)dx = b\delta_{n,m} \tag{38}$$

   The integral is zero if $n \neq m$. The result for C_n, with n odd is

$$C_n = \frac{V_0}{b\sinh(n\pi a/2b)}\int_{-b}^{b}\cos(n\pi x/2b)dx \tag{39}$$

   Change $n \to 2n+1$ to model the oddness, and performing the integral gives

$$C_n = \frac{4V_0\cos(n\pi)}{\pi(2n+1)\sinh((2n+1)\pi a/2b)} \tag{40}$$

   The final solution is then

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=0}\frac{\cos(n\pi)\sinh((2n+1)\pi y/2b)}{(2n+1)\sinh((2n+1)\pi a/2b)}\cos\left(\frac{(2n+1)\pi x}{2b}\right) \tag{41}$$

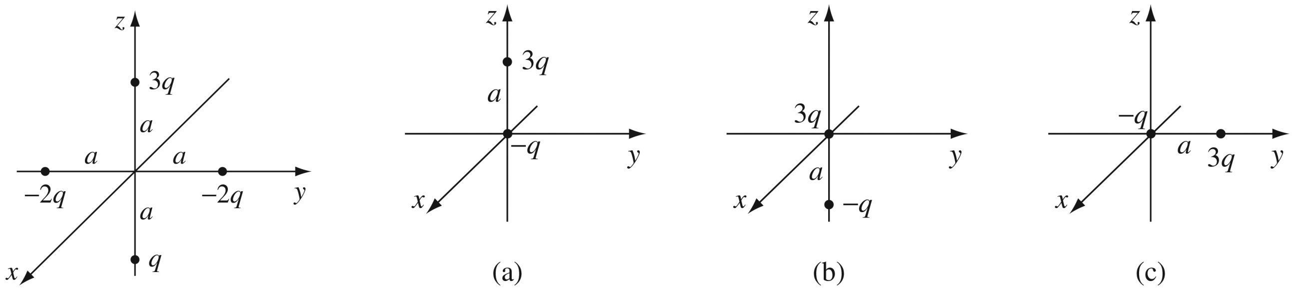

Figure 1: (Left) An arrangement of four charges near the origin. (Right, a-c) An arrangement of two charges near the origin, oriented three different ways.

3. Consider Fig. 1. Using the monopole and dipole potentials in the multipole expansion, find the approximate potential in spherical coordinates for each charge arrangement, far from the origin. *Note: these arrangements may or may not have a monopole moment in addition to the dipole moment.*

   - (a) The far-left configuration has a total charge (monopole moment) of zero. The monopole term in the multipole expansion vanishes and we're left with the dipole moment. The y-component of the dipole moment vanishes by symmetry and we have

     $$\mathbf{p} = (3qa - qa)\hat{\mathbf{z}} = 2qa\hat{\mathbf{z}} \tag{42}$$

     This makes the potential in the far-field

     $$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2} \tag{43}$$

   - For the rest, note that each configuration has a monopole moment of $2q$. The dipole moments of the three configurations, from left to right, are $3qa\hat{\mathbf{z}}$, $qa\hat{\mathbf{z}}$, and $3qa\hat{\mathbf{y}}$. The dipole moments are the sum of charge times distance vector (Eq. 3.100 in Ch. 3). The far-field potentials are then:
     - Configuration a ($\hat{z}\cdot\hat{r} = \cos\theta$):

       $$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right) \tag{44}$$

     - Configuration b ($\hat{z}\cdot\hat{r} = \cos\theta$):

       $$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right) \tag{45}$$

     - Configuration c ($\hat{y}\cdot\hat{r} = \sin\theta\sin\phi$):

       $$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right) \tag{46}$$