Riley Sullivan

Homework 3

$\boxed{3.3}$ (I) general sol. to Laplace's Equation in Spher. Coordinates where V depends only on r,

Then... (II) for Cyl. Coords where V only has s dependence

(I) $\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$, $r^2 \frac{dV}{dr} = \underset{\text{constant}}{K} \rightarrow \frac{dV}{dr} = \frac{K}{r^2}$

$$\boxed{V = -\frac{K}{r} + C}$$

(II) $\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{\partial V}{\partial s}\right) = 0$, $s\frac{\partial V}{\partial s} = \underset{\text{another constant}}{C} = \frac{C}{s} \Rightarrow \boxed{V = C\ln s + K}$

3.5 | Prove that field is uniquely determined when ρ is given & either V or normal derivative $\frac{\partial V}{\partial n}$

- Very similar to 3.6, 2nd Uniqueness theorem proof, until integrating over 3.7,

(a) $\int_V \nabla \cdot (V_3 E_3)\, d\tau = \oint_S V_3 E_3 \cdot da = -\int_V (E_3^2)\, d\tau$

but on every surface in 3.6, $V_3 = 0$ when V had a set value on it's surface, or else $E_{3\,norm} = 0$, if $\frac{\partial V}{\partial n} = -E_{norm}$,

$\int_V E_3^2 = 0$, $E_2 = E$.

3.6 | Better Proof of 2nd Uniqueness Theorem Using Green's Identity

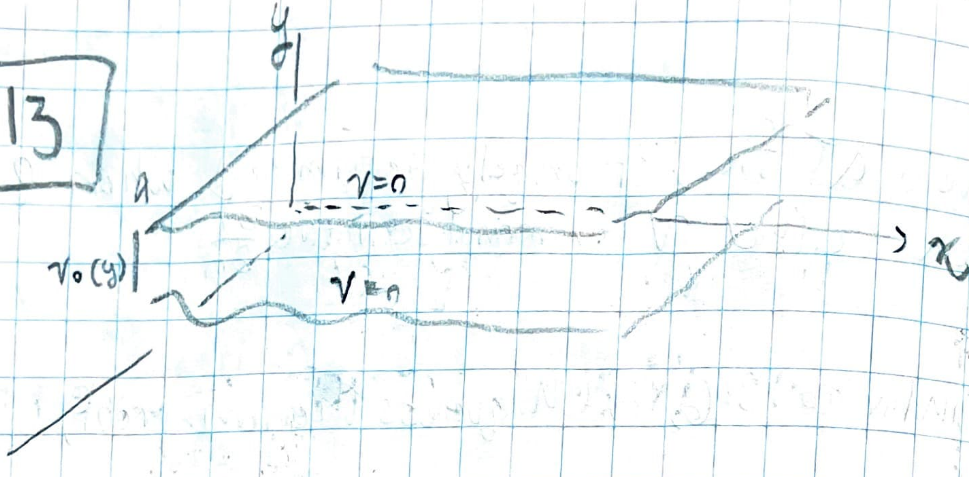
$$\int_V [V_3 \nabla^2 V_3 + \nabla V_3 \cdot \nabla V_3]\, d\tau = \oint_S V_3 \nabla V_3 \cdot da,$$

$$\nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\varepsilon_0} + \frac{\rho}{\varepsilon_0} = 0,$$

$$\nabla V_3 = -E_3$$

So, $\int_V E_3^2\, d\tau = -\oint_S V_3 E_3 \cdot da$

3.13



I

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi \frac{x}{a}} \sin\left(n\pi \frac{y}{a}\right)$$

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy$$

$$V_0(y) = \begin{cases} +V_0 & \text{for } 0<y<\frac{a}{2} \\ -V_0 & \text{for } \frac{a}{2}<y<a \end{cases},$$

$$c_n = \frac{2}{a}V_0\left\{\int_0^{a/2}\sin(n\pi y/a)\,dy - \int_{a/2}^{a}\sin(n\pi y/a)\,dy\right\}$$

$$= \frac{2V_0}{a}\left\{\frac{\cos(n\pi y/a)}{(n\pi/a)}\Bigg|_0^{\frac{a}{2}} + \frac{\cos(n\pi y/a)}{(n\frac{\pi}{a})}\Bigg|_{\frac{a}{2}}^{a}\right\}$$

$$= \frac{2V_0}{n\pi}\left\{-\cos\left(\frac{n\pi}{2}\right) + \cos(0) + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right\} =$$

$$= \frac{2V_0}{n\pi}\left\{1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right\}$$

$$\begin{cases} n=1: & 1-1-2\cos\left(\frac{\pi}{2}\right)=0 \\ n=2: & 1+1-2\cos(\pi)=4 \\ n=3: & 1-1-2\cos\left(\frac{3\pi}{2}\right)=0 \\ n=4: & 1+1-2\cos(2\pi)=0 \end{cases}$$

$$\therefore\ C_n = \begin{cases} \frac{8V_0}{n\pi} & n=2,6,10 = 4j\ldots \text{ for } j=0,\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore\ V(x,y) = \frac{8V_0}{\pi}\sum_{n=2,6,10\ldots\text{etc}} \frac{e^{-n\pi\frac{x}{a}}\sin[n\pi y/a]}{n}$$

$$= \boxed{\frac{8V_0}{\pi}\sum_{j=0}^{\infty}\frac{e^{-(4j+2)\pi x/a}\sin[(4j+2)\pi y/a]}{(4j+2)}}$$

$\boxed{3.14}$ for ex 3.3 again, determine $\sigma(y)$ on $x=0$, assuming it is a conductor of V_0 potential

Using

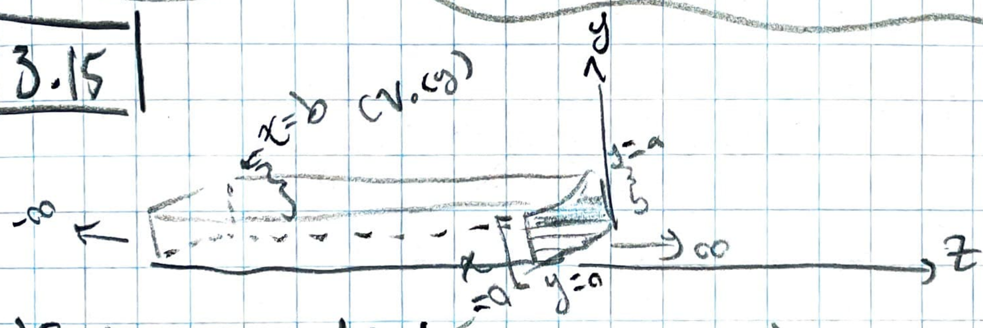
$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5\ldots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a)$$

$$\therefore \quad \sigma = \varepsilon_0 \frac{\partial V}{\partial n}$$

$$\sigma(y) = -\varepsilon_0 \frac{\partial}{\partial x}\left\{\frac{4V_0}{\pi}\sum \frac{1}{n} e^{-n\pi\frac{x}{a}} \sin(n\pi y/a)\right\}\Big|_{x=0}$$

$$= -\varepsilon_0 \frac{4V_0}{\pi}\sum \frac{1}{n}\left(\frac{-n\pi}{a}\right) e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)\Big|_{x=0} = \boxed{\frac{4\varepsilon_0 V_0}{a}\sum_{n=1,3,5\ldots} \sin(n\pi y/a)}$$

$\boxed{3.15}$



a) find general V formula for inside pipe

Boundary Condit's
$$\begin{cases} ① \; V(x,0)=0 \\ ② \; V(x,a)=0 \\ ③ \; V(0,y)=0 \\ ④ \; V(b,y)=V_0(y) \end{cases}$$

$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$

① $D=0$, ③ $B=-A$

ka is int. mult of π

$$V(x,y) = AC(e^{n\pi x/a} - e^{-n\pi x/a})\sin(n\pi y/a) = (2AC)\sinh(n\pi x/a)\sin(n\pi y/a)$$

$$V(x,y) \sum_{n=1}^{\infty} C_n \sinh(n\pi x/a)\sin(n\pi y/a)$$

$$\sum C_n \sinh(n\pi b/a)\sin(n\pi y/a) = V_0(y)$$ [By Fourier's trick]

$$C_n \sinh(n\pi b/a) = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy$$

Cont. $\Longrightarrow$

3.15 cont.

So again, by fourier's trick,

$$\boxed{C_n = \frac{2}{a\sinh(n\pi b/a)} \int_0^a V_0(y)\sin(n\pi y/a)\,dy}$$

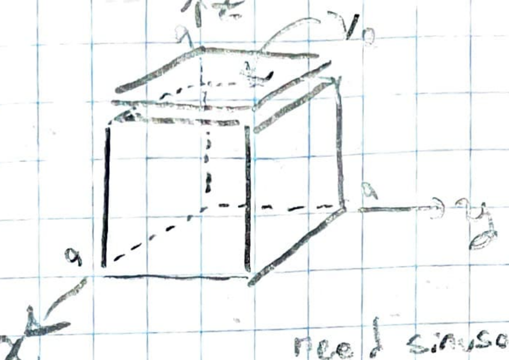
b) find the Potential Explicitly, for $V_0(y) = V_0$ ← constant

$$C_n = \frac{2}{a\sinh(n\pi b/a)} \int_0^a \sin(n\pi y/a)\,dy = \frac{2V_0}{n\sinh(n\pi b/a)} \times$$

- if n is even, 0
- if odd, $\frac{2a}{n\pi}$

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}}$$

$\boxed{3.16}$



Find V inside

; Whats V at center?

need sinusoid for x, y, exp. in z

① $V=0$ @ $x=0$
② $V=0$ @ $x=a$
③ $V=0$ @ $y=0$
④ $V=0$ @ $y=a$
⑤ $V=0$ @ $z=0$
⑥ $V=V_0$ @ $z=a$

$X(x) = A\sin(kx) + B\cos(kx)$

$y(y) = C\sin(ly) + D\cos(ly)$

$Z(z) = E e^{\sqrt{k^2+l^2}\,z} + F e^{-\sqrt{k^2+l^2}\,z}$

by ①, $B=0$, $k=\frac{n\pi}{a}$ by ③, $D=0$, $l=m\pi/a$

$E+F=0$

$\therefore$

$$Z(z) = 2E\sinh\left(\pi\sqrt{n^2+m^2}\,\frac{z}{a}\right)$$

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\pi\frac{\sqrt{n^2+m^2}}{a}z\right)$$

Eq 3.50 $C_{n,m} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)dy\,dz$

3.51 $C_{n,m} = \frac{4V_0}{ab}\int_0^a \sin(n\pi y/a)\int_0^b \sin\left(\frac{m\pi z}{b}\right)dz = \begin{cases} 0 & \text{if } n,m \text{ even} \\ \frac{16V_0}{\pi^2 nm} & \text{if } n,m \text{ odd}\end{cases}$

3.51 $C_{n,m}\sinh\left(\pi\sqrt{n^2+m^2}\right) = \left(\frac{2}{a}\right)^2 V_0\int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dx\,dy$

$$\begin{cases} 0, & \text{if } n \text{ or } m \text{ is even} \\ \frac{16V_0}{\pi^2 nm} & \text{if both } n,m \text{ odd}\end{cases}$$

$$\therefore V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n=1,3,5}\sum_{m=1,3,5}\frac{1}{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\cdot\frac{\sinh\left(\pi\sqrt{n^2+m^2}\,z/a\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}$$

$$V\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) = \frac{16V_0}{\pi^2}\sum_{n=1,3,5}\sum_{m=1,3,5\ldots}\frac{1}{nm}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right)\frac{\sinh\pi\sqrt{n^2+m^2}/2}{\sinh\pi\sqrt{n^2+m^2}}$$

$n=2i+1$, $m=2j+1$, $\sinh(2u) = 2\sinh(u)\cosh(u)$

$S = \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{(2i+1)(2j+1)}\operatorname{sech}\left[\pi\frac{\sqrt{(2i+1)^2+(2j+1)^2}}{2}\right]$, $V\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) = \frac{V_0}{6}$

3.19

Pot. at the surface of a sphere of rad. R

$V_0 = k\cos 3\theta$

find potential inside & out, and $\sigma(\theta)$

- any 3rd order - polynomial can be ex. as linear combo of Legendre Poly's, odd poly, so only 1 and 3 are needed,

$$V_0(\theta) = k[4\cos^3\theta - 3\cos\theta] = k[\alpha P_3(\cos\theta) + \beta P_1(\cos\theta)]$$

$$4\cos^3\theta - 3\cos\theta = \alpha[\tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)] + \beta\cos\theta = \frac{5\alpha}{2}\cos^3 \quad (\beta - \tfrac{3}{2}\alpha)\cos$$

$\downarrow$

$$4 = \frac{5\alpha}{2}, \quad \alpha = \frac{8}{5} \quad ; \quad -3 = \beta - \frac{3}{2}\alpha = \beta - \frac{12}{5} - 3 \text{ or } \beta = -\frac{3}{5}$$

$$\therefore \quad V_0(\theta) = \frac{k}{5}[8P_3(\cos\theta) - 3P_1(\cos\theta)]$$

$$V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & \text{for } r \le R \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) & \text{for } r \ge R \end{cases}$$

(from eq's 3.66, 3.72) and 3.69

$\downarrow$

$$A_l = \frac{2l+1}{2R^l}\int_0^\pi V_0(\theta) P_l(\cos\theta)\sin\theta \, d\theta$$

$$= \frac{k}{5}\frac{1}{R^l}[8\delta_{l3} - 3\delta_{l1}] \Rightarrow = \begin{cases} \frac{8k}{5R^3} & \text{when } l = 3 \\ \frac{-3k}{5R} & \text{when } l = 1 \\ \text{otherwise} & 0 \end{cases}$$

$$\therefore V(r,\theta) = -\frac{3k}{5R} r P_1(\cos\theta) + \frac{8k}{5R^3} r^3 P_3(\cos\theta),$$

$$V(r,\theta) = \frac{k}{5}\frac{r}{R}\cos\theta\left\{4\left(\frac{r}{R}\right)^2[5\cos^2\theta - 3] - 3\right\} \text{ when } r \le R$$

$$B_l = A_l R^{2l+1}$$

$$\therefore B_l = \begin{cases} \frac{8kR^4}{5} & \text{if } l=3 \\ \frac{-3kR^2}{5} & \text{if } l=1 \\ \text{or zero} & \text{otherwise} \end{cases}$$

$\downarrow$

$$V(r,\theta) = \frac{-3kR^2}{5} \frac{1}{r^2} P_1(\cos\theta) + \frac{8kr^4}{5} \frac{1}{r^4} P_3(\cos\theta) = \frac{k}{5}\left[8\left(\frac{R}{r}\right)^4 P_3(\cos\theta) - 3\left(\frac{R}{r}\right)^2 P_1(\cos\theta)\right]$$

$\downarrow$

$$\boxed{V(r,\theta) = \frac{k}{5}\left(\frac{R}{r}\right)^2 \cos\theta \left\{4\left(\frac{R}{r}\right)^2 \left[5\cos^2\theta - 3\right] - 3\right\}}$$

from 3.83

$$\sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\varepsilon_0} \sigma_0(\theta)$$

$$\sigma(\theta) = \varepsilon_0 \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta) = \varepsilon_0 \left[3A_1 P_1 + 7A_3 R^2 P_3\right]$$

$$\sigma(\theta) = \varepsilon_0 \left[-9\cos\theta + \frac{56}{2}(5\cos^3\theta - 3\cos\theta)\right]$$

$$\boxed{\sigma(\theta) = \frac{\varepsilon_0 k}{5R} \cos\theta \left[140\cos^2\theta - 93\right]}$$

$\boxed{3\text{-}22}$ $V(r,\theta) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2} - r\right)$

a) $\underline{3.72} \rightarrow V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

find 1st 3 terms in expansion

b) $\underline{3\text{-}66}$ $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

find pot. for $r<R$

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a) $V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l \cos\theta \; (r>R)$

$V(r,0) = \frac{B_l}{r^{l+1}} P_l(1) = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r^2+R^2} - r\right]$

since $r>R$ here,

$\sqrt{r^2+R^2} = r\sqrt{1+\left(\frac{R}{r}\right)^2} = r\left[1+\frac{1}{2}\left(\frac{R}{r}\right)^2 - \frac{1}{8}\left(\frac{R}{r}\right)^4 + \ldots\right]$,

$\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} = \frac{\sigma}{2\varepsilon_0} r\left[1 + \frac{1}{2}\frac{R^2}{r^2} - \frac{1}{8}\frac{R^4}{r^4} + \ldots - 1\right] =$

$\frac{\sigma}{2\varepsilon_0}\left(\frac{R^2}{2r} - \frac{R^4}{8r^3} + \ldots\right)$

$B_0 = \frac{\sigma R^2}{4\varepsilon_0}, B_1 = 0, B_2 = -\frac{\sigma R^4}{16\varepsilon_0}$

$\therefore V(r,\theta) = \frac{\sigma R^2}{4\varepsilon_0}\left[\frac{1}{r} - \frac{R^2}{4r^3} P_2(\cos\theta) + \ldots\right]$

$= \frac{\sigma R^2}{4\varepsilon_0 r}\left[1 - \frac{1}{8}\left(\frac{R}{r}\right)^2 (3\cos^2\theta - 1) + \ldots\right]$ again, for $r>R$

b)

now,

$\sum_{l=0}^{\infty} A$

$A_0 = \frac{\sigma}{2\varepsilon_0} R$

$= \frac{\sigma R}{2\varepsilon_0}$

for bot

$V(r,$

$\bar{A}_1 = ($

so,

$V($

b) $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r^2+R^2} - r\right]$

now, $r < R$, so $\sqrt{r^2+R^2} = R\left[1 + \frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{8}\left(\frac{r}{R}\right)^4 + \cdots\right]$ $\therefore$

$$\sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\varepsilon_0}\left[R + \frac{1}{2}\frac{r^2}{R} - \frac{1}{8}\frac{r^4}{R^3} + \cdots - r\right]$$

$A_0 = \frac{\sigma}{2\varepsilon_0}R$, $A_1 = -\frac{\sigma}{2\varepsilon_0}$, $A_2 = \frac{\sigma}{4\varepsilon_0 R}$ so, $V(r,\theta) = \frac{\sigma}{2\varepsilon_0}\left[R - rP_1(\cos\theta) + \frac{1}{2R}r^2 P_2(\cos\theta) \cdots\right]$

$$= \boxed{\frac{\sigma R}{2\varepsilon_0}\left[1 - \left(\frac{r}{R}\right)\cos\theta + \frac{1}{4}\left(\frac{r}{R}\right)^2(3\cos^2\theta - 1) + \cdots\right]}$$ (for $r<R$) top hemisph.

for bottom half of sphere... $\theta = \pi$, $P_l(-1) = (-1)^l$

$$V(r,\pi) = \sum_{l=0}^{\infty} (-1)^l \bar{A}_l r^l = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r^2+R^2} - r\right]$$

different $\bar{A}$ for bottom half

$\bar{A}_1 = (\sigma/2\varepsilon_0)$, $\bar{A}_0 = A_0$, $\bar{A}_2 = A_2$

so, $V(r,\theta) = \frac{\sigma}{2\varepsilon_0}\left[R + rP_1(\cos\theta) + \frac{1}{2R}r^2 P_2(\cos\theta) + \cdots\right]$

$$\boxed{V(r,\theta) = \frac{\sigma R}{2\varepsilon_0}\left[1 + \left(\frac{r}{R}\right)(\cos\theta) + \frac{1}{4}\left(\frac{r}{R}\right)^2(3\cos^2\theta - 1) + \cdots\right]}$$

3.24 Solve Laplace's by sep. vars in Cyl. coords no z dependence (cyl. symmetry)

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}=0$$

Sol will be $V(s,\phi)=S(s)\,\Phi(\phi)$

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=0$$

$$C_1=\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right),\quad C_2=\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2},\quad C_1+C_2=0$$

$$C_2=-k^2;\quad \frac{d^2\Phi}{d\phi^2}=-k^2\Phi\rightarrow\Phi=A\cos k\phi+B\sin k\phi$$

k must be an integer, $k=0,1,2,3$

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right)=k^2$$

$$s\frac{d}{ds}\left(s\,n\,s^{n-1}\right)=ns\frac{d}{ds}(s^n)=n^2 s s^{n-1}=n^2 s^n=k^2 S,\quad n=\pm k$$

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right)=0\Rightarrow s\frac{dS}{ds}=\text{const.}\,(C),\quad\frac{dS}{ds}=\frac{C}{s},\quad dS=C\frac{ds}{s}$$

$S=C\ln s+D$

$\Phi=A$ when $k=0$,

$$\frac{d^2\Phi}{d\phi^2}=0,\quad\frac{d\Phi}{d\phi}=\text{Const}(B),\quad\Phi=B\phi+A$$

not good, doesn't return to init when ϕ goes by 2π,

$$V(s,\phi)=a_0+b_0\ln s+\sum_{k=1}^{\infty}\Big[s^k(a_k\cos k\phi+b_k\sin k\phi)$$
$$+s^{-k}(c_k\cos k\phi+d_k\sin k\phi)\Big]$$

3-26 Chg. Density $\sigma(\phi) = a \sin 5\phi$

[a is constant] over surface of an inf. cylinder

Potential for inside & outside

from Inside,

$$V(s,\phi) = a_0 + \sum_{k=1}^{\infty} s^k (a_k \cos k\phi + b_k \sin k\phi)$$

In this region ln s and s^{-k} go to ∞ @ $s=0$

outside:

$$V(s,\phi) = \overset{\text{a outside}}{a^o} + \sum_{k=1}^{\infty} \frac{1}{s^k}(c_k \cos k\phi + d_k \sin k\phi)$$

ln s & s^{-k} are not valid $s \to \infty$

$$\sigma = -\varepsilon_0 \left(\frac{\partial V_o}{\partial s} - \frac{\partial V_i}{\partial s}\right)\Big|_{s=R} \quad \text{(from 2.36)}$$

$$\frac{\partial V_a}{\partial n} - \frac{\partial V_b}{\partial n} = -\frac{1}{\varepsilon_0}\sigma$$

above below

$\therefore$

$$a \sin 5\phi = -\varepsilon_0 \sum_{k=1}^{\infty} \left\{ \frac{k}{R^{k+1}}(c_k \cos k\phi + d_k \sin k\phi) - kR^{k-1}(a_k \cos k\phi + b_k \sin k\phi) \right\}$$

$a_k = c_k = 0$, $b_k = d_k = 0$, but $k=a$, $a = 5\varepsilon_0\left(\frac{1}{R^6}d_5 + R^4 b_5\right)$, V is cont @ $s=R$

$a_0 + R^5 b_5 \sin 5\phi = a^o_0 + \frac{1}{R^5} d_5 \sin 5\phi$, $a_0 = a^o_0$,

$R^5 b_5 = 5^{-5} d_5$ and $= R^{10} b_5$,

$a = 5\varepsilon_0 (R^4 b_5 + R^4 b_5) = 10\,\varepsilon_0 R^4 b_5$

$b_5 = \frac{a}{10\varepsilon_0 R^4}$, $d_5 = \frac{aR^6}{10\varepsilon_0}$

$\therefore$

$$\boxed{V(s,\phi) = \frac{a \sin 5\phi}{10\,\varepsilon_0} \begin{cases} s^5/R^4 & \text{for } s<R, \\ R^6/s^5 & \text{for } s>R \end{cases}}$$