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Midterm

1) a) $A \cdot \nabla = \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz} \Rightarrow$

$$(A \cdot \nabla) B = \frac{dA_x}{dx}\vec{B} + \frac{dA_y}{dy}\vec{B} + \frac{dA_z}{dz}\vec{B}$$

b) $\hat{r} = \frac{r}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{(x^2+y^2+z^2)}} \Rightarrow$

$(\hat{r} \cdot \nabla)\hat{r} \Rightarrow$

$$\frac{dx\hat{x}}{d\sqrt{x^2+y^2+z^2}}(\hat{r}) + \frac{dy\hat{y}}{d\sqrt{x^2+y^2+z^2}}(\hat{r}) + \frac{dz\hat{z}}{d\sqrt{x^2+y^2+z^2}}(\hat{r})$$

c)

2) $J = \int_V e^{-r} \left(\nabla \cdot \frac{\hat{r}}{r^2}\right)$

$\nabla \cdot \frac{\hat{r}}{r^2} = \frac{1}{r^2}\frac{d}{dr}\left[\cancel{r^2}\left(\frac{\hat{r}}{\cancel{r^2}}\right)\right] \Rightarrow \frac{1}{r^2}\cancel{\frac{d}{dr}(1)} = 0 \Rightarrow$

$\int \frac{1}{R^2}\hat{r} \cdot d\bar{a} = \int_0^{\pi}\int_0^{2\pi} \cancel{\frac{1}{R^2}}\cancel{\hat{r}} \cdot \cancel{R^2} \sin\theta \, d\theta \, d\phi \, \cancel{\hat{r}} \Rightarrow$

$\int_0^{\pi} \sin\theta \, d\theta \cdot \int_0^{2\pi} d\phi \Rightarrow$

$[\underbrace{\sin(\pi)}_{2} - \underbrace{\sin 0}_{0}] \cdot [2\pi - 0] \Rightarrow 2 \cdot 2\pi = 4\pi \Rightarrow$

$4\pi \delta^3(r) \Rightarrow$

$\int_V e^{-r} 4\pi \delta^3(r) \Rightarrow 4\pi \int_V e^{-r} \delta^3(r) \Rightarrow$

$4\pi [\underbrace{e^{-0}}_{1}]|_V \Rightarrow$

$4\pi [1]_V \Rightarrow$

4π

Electrostatics

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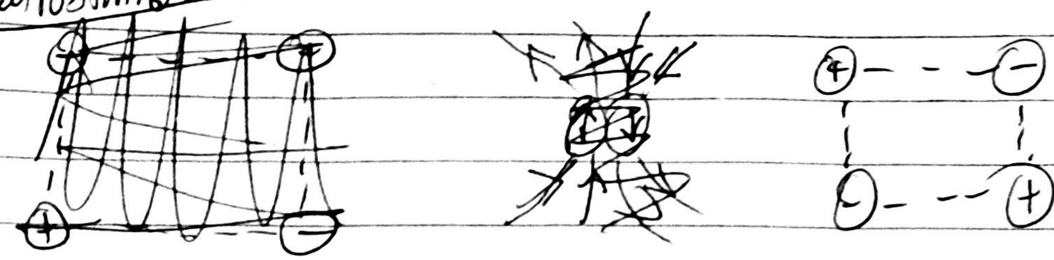

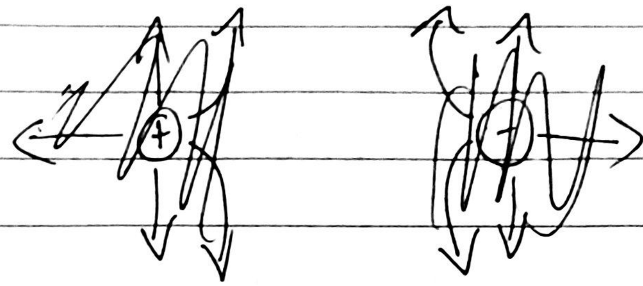

$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r+(\frac{r}{2})^2}$

$E_{tot} = E_1 + E_2 + \dots \Rightarrow$

a) $2(E_1+E_2) \Rightarrow$

$$\begin{cases} E_x = E_1\cos(0) + \cancel{E_2\cos(0)} = 1 \\ E_y = E_1\sin(0) + \cancel{E_2\cos(0)} = 0 \end{cases}$$

$E_{tot} = \cancel{2(2,0) = (4,0)}\ 4\left[\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{2+0}\right)\right] \Rightarrow \frac{4q}{8\pi\varepsilon_0} = \frac{q}{2\pi\varepsilon_0}$

~~2 pairs of dipoles~~

b) $P(2d)$ $$\begin{cases} E_x = E_1\cos(2d) + \cancel{E_2\cos(2d)} = 2d \\ E_y = E_1\sin(0) = 0 \end{cases} \Rightarrow$$

$4\left[\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{2d+0}\right)\right] \Rightarrow \frac{q}{2d\pi\varepsilon_0}$

$P(0,2d) \Rightarrow \frac{4q}{4d^2\pi\varepsilon_0}$

2) $$\vec{E} = -\nabla V = -\hat{r}\,\partial_r A\frac{e^{-\lambda r}}{r} = -\hat{r}A\left(-\lambda\frac{e^{-\lambda r}}{r} - \frac{e^{-\lambda r}}{r^2}\right) \Rightarrow$$

$$-\hat{r}A\,\frac{e^{-\lambda r}}{r}\left(-\lambda - \frac{1}{r}\right)$$

$$\frac{\rho}{\varepsilon_0} = \nabla\cdot\vec{E} = -Ae^{-\lambda r}\left(-\lambda-\frac{1}{r}\right)\left(\nabla\cdot\frac{\hat{r}}{r}\right) \Rightarrow$$

$\left(\nabla\cdot\frac{\hat{r}}{r}\right) \Rightarrow$ prev. problem $= 4\pi\delta^3(r) \Rightarrow$

$$-Ae^{-\lambda r}\left(-\lambda-\frac{1}{r}\right)4\pi\delta^3(r) +$$

$$\underbrace{-A\,\nabla e^{-\lambda r}\left(-\lambda-\frac{1}{r}\right)\left(\frac{\hat{r}}{r}\right)} \Rightarrow$$

$$-A\left(\frac{\hat{r}}{r}\right)\left[\lambda e^{-\lambda r}\left(-\lambda-\frac{1}{r}\right)+e^{-\lambda r}\lambda\right]\hat{r} \Rightarrow$$

$$-A\left(\frac{\hat{r}}{r}\right)-\lambda e^{-\lambda r}\left(-\lambda-\frac{1}{r}\right)+e^{-\lambda r}\lambda+\left(-Ae^{-\lambda r}\right)\left(-\lambda-\frac{1}{r}\right)4\pi\delta^3(r)$$

3 $\oint E \cdot ds \cos\theta$, $\phi = q/\varepsilon_0$

$\cancel{\Delta A\, E\cos\theta}$

a) $\phi = \oint E\, ds\, \underbrace{\cancel{\cos\theta}}_{1} \Rightarrow$

$\phi = \oint E \cdot ds \Rightarrow$

ds = surface area of wire part that curves = $2\pi r l$

For Gaussian surface = $q = \lambda l \Rightarrow$

$$E(2\pi r l) = \frac{\lambda l}{\varepsilon_0} \Rightarrow E = \frac{\lambda \cancel{l}}{2\pi r \cancel{l} \varepsilon_0} \Rightarrow$$

$$E = \frac{\lambda}{2\pi r \varepsilon_0}$$

b) $\frac{dx}{dt} = \frac{qE}{m}$ $\cancel{[\ldots]}$ $\Rightarrow \frac{q\lambda}{2\pi r \varepsilon_0 m}$

I can't from here, sorry

Potentials

1)

a) $\sigma(\theta) = \varepsilon_0 \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta)$

&

$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta \Rightarrow$

$\sigma(\theta) = \varepsilon_0 \sum_{l=0}^{\infty} (2l+1) \cancel{R^{l-1}} \frac{2l+1}{\cancel{2R^l}} \left(\int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta \right) P_l \cos\theta$

$\Rightarrow$

$\sigma_\theta = \frac{\varepsilon_0}{2R} \sum_{l=0}^{\infty} \frac{(2l+1)^2}{1} \underbrace{\int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta}_{C_l} P_l \cos\theta \Rightarrow$

$\sigma_\theta = \frac{\varepsilon_0}{2R} \sum_{l=0}^{\infty} (2l+1)^2 C_l P_l \cos\theta$

b) $C_l = \int_0^{\pi} P_2(\cos\theta) P_l(\cos\theta) \sin\theta \, d\theta$

$\sigma_\theta = \frac{\varepsilon_0}{2R} \sum_{l=0}^{\infty} (2l+1)^2 \int_0^{\pi} P_2(\cos\theta) P_l(\cos\theta) \sin\theta \, d\theta$

$\int_0^{\pi} \underbrace{P_2(\cos\theta) P_l(\cos\theta)}_{\text{if this is a legendre polynomial}} d\theta \sin\theta \Rightarrow$ then $\begin{cases} 0 & \text{if } l'=0 \\ \frac{2}{2l+1} & \text{if } l=l \end{cases}$

~~$P_2(\cos\theta) = \frac{1}{2}(3(\cos\theta)^2 - 1)$~~

so $\Rightarrow$

$\int_0^{\pi} P_2 \cos\theta \, P_1 \cos\theta = 0 \Rightarrow$

$\sigma_\theta = \frac{\varepsilon_0}{2R} \sum_{l=0}^{\infty} (2l+1)^2 \cdot 0 \Rightarrow \sigma_\theta = 0$

2) Boundary $\begin{cases} V=0 \text{ @ } y=0 \\ V=V_0 \text{ @ } y=a \\ V=0 \text{ @ } x=b \\ V=0 \text{ @ } x=-b \end{cases}$

$$\frac{d^2X}{dx^2} = k^2 X, \quad \frac{d^2Y}{dy^2} = -k^2 Y$$

$$X(x) = Ax + B, \quad Y(y) = Cy + D$$

$$V(x,y) = \sum_{1}^{\infty} C_n \cosh(n\pi x/a) \sin(n\pi y/a)$$

$V=0$ @ $x = \pm b \Rightarrow V(\pm b, y) = 0$

$V=0$ @ $y=0 \Rightarrow V(x,0) = 0$

$V = V_0$ @ $y=a \Rightarrow V(x,a) \Rightarrow$

$\sum_1^{\infty}$ ~~$C_n \cosh n$~~

$V(x,y) = C(e^{-kx} + e^{kx}) \sin(ky)$ where $y = a \Rightarrow$

$$V(x,a) = C(e^{-kx} + e^{kx}) \underbrace{\sin(ka)}_{\sin(ka)}$$

$\sin(ka) \Rightarrow k = \frac{n\pi}{a} \Rightarrow$

~~$\sin\left(\left(\frac{n\pi}{a}\right)a\right) \Rightarrow \sin(n\pi)$~~

$\int V(x,a) = \int C(\dots) \Rightarrow$

$= \int y \sin(ka)\,dy \Rightarrow \int_0^a y \sin\left(\frac{n\pi y}{a}\right) dy \Rightarrow$

$\left[\left(\frac{a}{n\pi}\right)^2 \sin(n\pi y/a) - \left(\frac{ay}{n\pi}\right) \cos(n\pi y/a)\right]\Big|_0^a \Rightarrow$

$\left[\left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi a}{a}\right) - \left(\frac{aa}{n\pi}\right) \cos(n\pi a/a)\right] \Rightarrow$

$\sin(\pi) = 0$

$$\left[\left(\frac{a}{n\pi}\right)^2 \cancel{\sin(n\pi)} - \left(\frac{a^2}{n\pi}\right)\cos(n\pi)\right]$$

$$V(x,y) = C(e^{-kx} + e^{kx})\left[-\left(\frac{a^2}{n\pi}\right)\cos(n\pi)\right]$$

3) $V_{mon} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$, $Q = \int \rho \, d\tau$

$$V_{dip} = \frac{1}{4\pi\varepsilon_0}\frac{p \cdot \hat{r}}{r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

a)

$Q = 3q - q = 2q \Rightarrow$

$$V_{mon} = \frac{1}{\cancel{4}\pi\varepsilon_0}\frac{\cancel{2}q}{r} = \frac{q}{2\pi\varepsilon_0 r}$$

$$V_{dip} = \frac{\vec{p}\cos\theta}{4\pi\varepsilon_0 r^2}, \quad \vec{p} = (a\hat{z})3q + (0\hat{z})(-q) \Rightarrow$$

$$\vec{p} = 3qa\hat{z}, \quad \hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta} \Rightarrow$$

$$V_{dip} = \frac{3qa(\cos\theta\hat{r} - \sin\theta\hat{\theta})\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$V_{approx.} = V_{dip} + V_{mon} = \frac{q}{2\pi\varepsilon_0 r} + \frac{3qa}{4\pi\varepsilon_0 r^2}(\cos\theta\hat{r} - \sin\theta\hat{\theta})\cos\theta \Rightarrow$$

$$V_{approx} = \frac{q}{2\pi\varepsilon_0 r}\left[1 + \frac{3a}{2}(\cos\theta\hat{r} - \underbrace{\sin\theta\hat{\theta}}_{0})\cos\theta\right]$$

no y-term $\Rightarrow$

$$V_{approx} = \frac{q}{2\pi\varepsilon_0 r}\left[1 + \frac{3a}{2}(\cos\theta\hat{r})\cos\theta\right]$$

b) $Q = 3q + (-q) = 2q$

$V_{mon} = \frac{2q}{4\pi\varepsilon_0 r} = \frac{q}{2\pi\varepsilon_0 r}$

$\vec{p} = a(-q)(-\hat{z}) + \cancel{3q(a=0)} \to 0$

$\vec{p} = qa\hat{z}$

$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \Rightarrow$

$V_{dip} = \frac{qa(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\cos\theta}{4\pi\varepsilon_0 r^2}$

$V_{approx} = V_{mon} + V_{dip} =$

$\frac{q}{2\pi\varepsilon_0 r}\left[1 + \frac{a(\cos\theta\,\hat{r})\cos\theta}{2r}\right]$

c) $Q = 2q$

$V_{mon} = \frac{q}{2\pi\varepsilon_0 r}$

V_{dip} : $3qa\hat{y}$

$\hat{y} = \underbrace{\sin\theta\cos\theta\,\hat{r}}_{\text{sperical coords}} - \cancel{\sin\varphi\cos\theta\,\hat{\theta} + \cos\varphi\,\hat{\varphi}} \Rightarrow$

$3qa\sin\theta\cos\theta\,\hat{r} \Rightarrow$

$V_{dip} = \frac{(3qa\sin\theta\cos\theta\,\hat{r})\cos\theta}{4\pi\varepsilon_0 r^2}$

$V_{approx} = \frac{q}{2\pi\varepsilon_0 r} + \frac{(3qa\sin\theta\cos\theta\,\hat{r})\cos\theta}{4\pi\varepsilon_0 r^2} \Rightarrow$

$V_{approx} = \frac{q}{2\pi\varepsilon_0 r}\left[1 + \frac{(3a\sin\theta\cos\theta\,\hat{r})\cos\theta}{2r}\right]$