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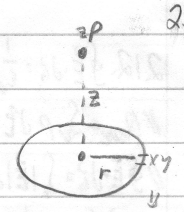
11/9/20 PHYS

HW2

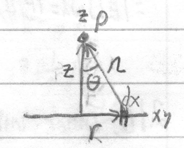
2.5, 6, 9, 12, 16, 18, 25, 29

(2.5) $\vec{E} = \oint d\vec{E}\cos\theta\,\hat{z}$ // $\cos\theta = \frac{z}{\pi} = \frac{z}{\sqrt{z^2+r^2}}$, $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{\pi^2}\frac{k\,dq}{z^2+r^2}$



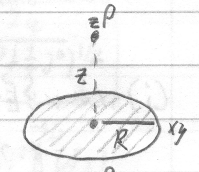
$\vec{E} = \oint \frac{kz\,dq}{(z^2+r^2)^{3/2}}\hat{z}$ // $dq = r\lambda\,d\phi$

$\vec{E} = \int_0^{2\pi}\frac{kzr\lambda}{(z^2+r^2)^{3/2}}d\phi\,\hat{z} = \frac{kzr\lambda\hat{z}}{(z^2+r^2)^{3/2}}\int_0^{2\pi}d\phi$



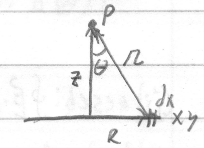
$\vec{E} = \frac{kzr\lambda\hat{z}}{(z^2+r^2)^{3/2}}(2\pi) \Rightarrow \vec{E} = \frac{1}{\cancel{2}4\pi\epsilon_0}\cdot\frac{zr\lambda\cancel{2\pi}\hat{z}}{(z^2+r^2)^{3/2}} \Rightarrow \boxed{\vec{E} = \frac{zr\lambda}{2\epsilon_0(z^2+r^2)^{3/2}}\hat{z}}$

(2.6) $\vec{E}_{loop} = \frac{1}{4\pi\epsilon_0}\cdot\frac{2\pi zr\lambda}{(z^2+r^2)^{3/2}}\hat{z}$ // $\vec{E} = d\vec{E}$, $\lambda \to \sigma\,dr$



$u = z^2 + r^2 \Rightarrow du = 2r\,dr$, $dr = \frac{du}{2r}$

$d\vec{E}_{disk} = \frac{1}{4\pi\epsilon_0}\cdot\frac{2\pi zr\sigma\,dr}{(z^2+r^2)^{3/2}}\hat{z} \Rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0}\int_0^R\frac{2\pi zr\sigma\,dr}{(z^2+r^2)^{3/2}}\hat{z}$



$\vec{E} = \frac{\cancel{2\pi}z\sigma}{2\cancel{4\pi}\epsilon_0}\hat{z}\int_0^R\frac{r}{(z^2+r^2)^{3/2}}dr \Rightarrow \vec{E} = \frac{z\sigma}{2\epsilon_0}\hat{z}\int_0^R\frac{r\,dr}{(z^2+r^2)^{3/2}}$ // $u = z^2+r^2$, $du = 2r\,dr$, $dr = \frac{du}{2r}$

$\vec{E} = \frac{z\sigma}{2\epsilon_0}\hat{z}\int_0^R\frac{du}{2(u)^{3/2}} \Rightarrow \vec{E} = \frac{z\sigma}{4\epsilon_0}\hat{z}\int_0^R u^{-3/2}du \Rightarrow \vec{E} = \frac{z\sigma}{4\epsilon_0}\hat{z}\left[-2u^{-1/2}\right]_0^R \Rightarrow \vec{E} = \frac{z\sigma}{4\epsilon_0}\hat{z}\left[\frac{-2}{\sqrt{z^2+r^2}}\right]_0^R$

$\vec{E} = \frac{z\sigma}{4\epsilon_0}\hat{z}\left[\frac{-2}{\sqrt{z^2+R^2}} + \frac{2}{z}\right] \Rightarrow \vec{E} = -\frac{z\sigma}{2\epsilon_0}\hat{z}\left[\frac{1}{\sqrt{z^2+R^2}} - \frac{1}{z}\right] \Rightarrow \boxed{\vec{E} = \frac{z\sigma}{2\epsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right)\hat{z}}$

When $R \to \infty$, $\vec{E} = \frac{\cancel{z}\sigma}{2\epsilon_0}\left(\frac{1}{\cancel{z}} - \cancel{\frac{1}{\infty}}\right)\hat{z} \Rightarrow \boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{z}}$ When $R \to \infty$

When $z \gg R$, $\vec{E} = \frac{z\sigma}{2\epsilon_0}\left(\frac{1}{z} - \frac{1}{z\sqrt{1+(R/z)^2}}\right)\hat{z} = \frac{\sigma\cancel{z}}{2\epsilon_0}\cdot\frac{1}{\cancel{z}}\left(1 - \frac{1}{\sqrt{1+(R/z)^2}}\right)\hat{z}$ // Approx $\Rightarrow \frac{\sigma}{2\epsilon_0}\left(\cancel{1} - \cancel{1} + \frac{1}{2}\frac{R^2}{z^2}\right)$

$\vec{E} = \frac{\sigma}{4\epsilon_0}\frac{R^2}{z^2} \Rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\pi R^2\sigma}{z^2}$ // $Q = \pi R^2\sigma \Rightarrow \boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}}$ ✓

(2.9) $\vec{E} = kr^3\hat{r}$ (in spherical coordinates) //

(a) $\rho = ?$ // $\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\rho$ //

$\rho = \epsilon_0(\nabla\cdot\vec{E}) \Rightarrow \epsilon_0\left[\frac{1}{r^2}\frac{d}{dr}(r^2\cdot kr^3)\right] = \epsilon_0\left[\frac{1}{r^2}\frac{d}{dr}(kr^5)\right] = \epsilon_0\left[\frac{1}{\cancel{r^2}}\cdot 5kr^{\cancel{4}2}\right] \Rightarrow$

$\boxed{\rho = 5k\epsilon_0 r^2}$

(b) $\oint\vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{enc} \Rightarrow$ // $d\vec{a} = 4\pi r^2\,\hat{r} \Rightarrow Q = \epsilon_0(kR^3)(4\pi R^2) \Rightarrow \boxed{Q = 4\pi k\epsilon_0 R^5}$ ✓ (// $r = R$ //)

$Q_{enc} = \int_V \rho\,d\tau \Rightarrow$ // $d\tau = 4\pi r^2 dr \Rightarrow Q = \int_0^R 5k\epsilon_0 r^2\cdot 4\pi r^2 dr \Rightarrow 20\pi k\epsilon_0\int_0^R r^4 dr \Rightarrow 20\pi k\epsilon_0\left[\frac{1}{5}r^5\right]_0^R$

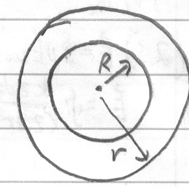
$\boxed{Q = 4\pi k\epsilon_0 R^5}$ ✓

2.12. $\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc} \Rightarrow E\cdot 4\pi r^2 = \frac{1}{\epsilon_0} Q_{enc} \Rightarrow$ // $Q_{enc} = \int_0^r \rho\, 4\pi r'^2\, dr'$

// $Q_{enc} = \int_V \rho\, d\tau$ $E\cdot \cancel{4\pi r^2} = \frac{1}{\epsilon_0}\cdot\frac{4}{3}\pi r^{\cancel{3}}\rho$ $= 4\pi\rho\left[\frac{1}{3}r'^3\right]_0^r \Rightarrow Q_{enc} = \frac{4}{3}\pi r^3 \rho$

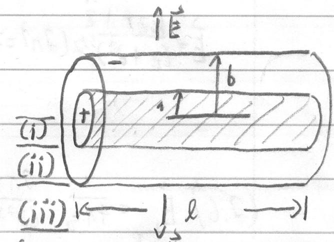
// $\oint \vec{E}\cdot d\vec{a} = \int_S |\vec{E}|\, da$

$= |\vec{E}|\int_S da = |E| 4\pi r^2$ $E = \frac{\rho r}{3\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\rho r}{3\epsilon_0}\hat{r}}$



2.16. Inner: uniform volume charge, ρ density, radius a.

Outer: uniform surface charge, radius b.



(i) $s<a$: $\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{enc}$ // $da = 2\pi s l$, $Q_{enc} = \int_0^s \rho 2\pi s' l\, ds' = 2\pi\rho l\left[\frac{1}{2}s'^2\right]_0^s \Rightarrow Q = \pi\rho s^2 l$

$E\cdot 2\cancel{\pi} s l = \frac{1}{\epsilon_0}\cdot\cancel{\pi}\rho s^2 l \Rightarrow E = \frac{\rho s}{2\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\rho s}{2\epsilon_0}\hat{s}}$

(ii) $a<s<b$: $\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{enc} \Rightarrow E\cdot 2\cancel{\pi} s l = \frac{1}{\epsilon_0}\cdot\cancel{\pi}\rho a^2 l \Rightarrow E = \frac{\rho a^2}{2s\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\rho a^2}{2s\epsilon_0}\hat{s}}$

// $da = 2\pi s l$

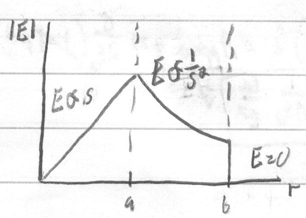
// $Q_{enc} = \int_V \rho\, d\tau = \int_0^a \rho 2\pi s' l\, ds'$

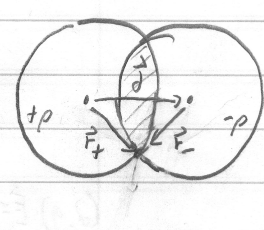
$= \cancel{2}\pi\rho l\left[\frac{1}{\cancel{2}}s'^2\right]_0^a = \pi\rho a^2 l$

(iii) $s>b$: $\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{enc}$ // $da = 2\pi s l$, $Q_{enc} = \int_V \rho\, d\tau$; no $\rho \to$ so, $Q_{enc} = \int_V (0)\, d\tau = 0$

$\vec{E}\cdot 2\pi s l = \frac{1}{\epsilon_0}(0) \Rightarrow \boxed{\vec{E} = 0}$

Plot:





2.18 $\vec{E} = \frac{\rho r}{3\epsilon_0}\hat{r}$ (from P2.12) $\Rightarrow \vec{E}_+ = \frac{\rho}{3\epsilon_0}\vec{r}_+\hat{r}$ $\Rightarrow \vec{E}_{tot} = \frac{\rho}{3\epsilon_0}\vec{r}_+ - \frac{\rho}{3\epsilon_0}\vec{r}_-$

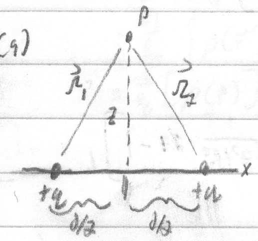
$\vec{E}_- = -\frac{\rho}{3\epsilon_0}\vec{r}_-\hat{r}$ $= \frac{\rho}{3\epsilon_0}(\vec{r}_+ - \vec{r}_-)$

// $\vec{d} = ? \Rightarrow \vec{d} = \vec{r}_+ + (-\vec{r}_-) \Rightarrow \vec{d} = \vec{r}_+ - \vec{r}_-$ // $\boxed{\vec{E}_{tot} = \frac{\rho}{3\epsilon_0}\vec{d}}$, constant

2.25 $V(\vec r)=\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$; $V=\frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\vec r')}{\mathscr{r}}dl'$; $V=\frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\vec r')}{\mathscr{r}}da'$

collection of charges ; line charge integral ; surface charge integral

(a)



q_1: $\vec r = z\hat z$, $\vec{\mathscr r}_1=\vec r-\vec r_1 = z\hat z+\frac{d}{2}\hat x$; q_2: $\vec r = z\hat z$, $\vec{\mathscr r}_2 = z\hat z-\frac{d}{2}\hat x$

$\vec r' = -\frac{d}{2}\hat x$, $\mathscr r_1^2 = z^2+\frac14 d^2$; $\vec r'=+\frac{d}{2}\hat x$, $\mathscr r_2^2=z^2+\frac14 d^2$

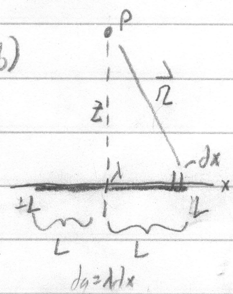
$\mathscr r_1=(z^2+\frac14 d^2)^{1/2}$, $\mathscr r_1=\mathscr r_2=\mathscr r$, $\mathscr r_2=(z^2+\frac14 d^2)^{1/2}$

$$V=k\sum_{i=1}^{2}\frac{q_i}{\mathscr r_i} = k\left[\frac{+q}{\mathscr r_1}+\frac{+q}{\mathscr r_2}\right] = k\left[\frac{2q}{\mathscr r}\right] \Rightarrow \boxed{V=\frac{1}{4\pi\epsilon_0}\frac{2q}{(z^2+\frac14 d^2)^{1/2}}}$$

$$\vec E=-\nabla V = \cancel{\frac{\partial}{\partial x}V_x}^{0}+\cancel{\frac{\partial}{\partial y}V_y}^{0}+\frac{\partial}{\partial z}V_z = \frac{-2q}{4\pi\epsilon_0}\cdot\frac{\partial}{\partial z}\left((z^2+\tfrac14 d^2)^{-1/2}\right) = \frac{+1}{4\pi\epsilon_0}2q\left(\frac{+1}{\cancel{2}}\right)(z^2+\tfrac14 d^2)^{-3/2}(\cancel{2}z)$$

$\boxed{\vec E=\frac{1}{4\pi\epsilon_0}\frac{2qz}{(z^2+1/4\,d^2)^{3/2}}}$ ✓ (same to Ex 2.1)

(b)



$\vec r = z\hat z$

dq: $\vec r'=x\hat x$, $dl'=dx$, $\vec{\mathscr r}=\vec r-\vec r' = z\hat z-x\hat x$, $\mathscr r=(z^2+x^2)^{1/2}$

$$V=k\int\frac{\lambda\, dl'}{\mathscr r} \Rightarrow k\int_{-L}^{L}\frac{\lambda\, dx}{(z^2+x^2)^{1/2}} \Rightarrow k\lambda\int_{-L}^{L}\frac{dx}{(z^2+x^2)^{1/2}}$$ // integral identity: $\int\frac{dx}{\sqrt{a^2+x^2}}=\ln|x+\sqrt{a^2+x^2}|$

$$V=k\lambda\left[\ln|x+\sqrt{z^2+x^2}|\right]_{x=-L}^{x=L} \Rightarrow k\lambda\left[\ln|L+\sqrt{z^2+L^2}|-\ln|-L+\sqrt{z^2+L^2}|\right]$$ ✓

$$\boxed{V=\frac{\lambda}{4\pi\epsilon_0}\ln\left|\frac{L+\sqrt{z^2+L^2}}{-L+\sqrt{z^2+L^2}}\right|}$$

$$\vec E=-\nabla V=\frac{-\partial}{\partial z}V_z=(-k\lambda)\frac{\partial}{\partial z}\left[\ln|L+\sqrt{z^2+L^2}|-\ln|-L+\sqrt{z^2+L^2}|\right]\hat z$$

check // $L\to\infty$, set Gauss's Law

$$=(\cdots)\left[\frac{1}{L+\sqrt{z^2+L^2}}\left(\frac{1}{\cancel{2}}\right)\frac{1}{\sqrt{z^2+L^2}}(\cancel{2}z)-\frac{1}{-L+\sqrt{z^2+L^2}}\left(\frac{1}{\cancel{2}}\right)\frac{1}{\sqrt{z^2+L^2}}(\cancel{2}z)\right]\hat z$$

$$=(\cdots)\cdot\frac{z}{(z^2+L^2)^{1/2}}\left[\frac{1}{L+(z^2+L^2)^{1/2}}-\frac{1}{-L+(z^2+L^2)^{1/2}}\right]\hat z \Rightarrow (\cdots)\frac{z}{(z^2+L^2)^{1/2}}\left[\frac{(z^2+L^2)^{1/2}-L}{((z^2+L^2)^{1/2}-L)\cdot((z^2+L^2)^{1/2}+L)}+\frac{-(z^2+L^2)^{1/2}-L}{((z^2+L^2)^{1/2}-L)((z^2+L^2)^{1/2}+L)}\right]\hat z$$

$$=(\cdots)\frac{z}{(z^2+L^2)^{1/2}}\left[\frac{-2L}{(z^2+L^2)-L^2}\right]\hat z \Rightarrow \frac{+\lambda}{4\pi\epsilon_0}\frac{\cancel{z}}{(z^2+L^2)^{1/2}}\left(\frac{+2L}{z^{\cancel{2}}}\right)\hat z \Rightarrow$$

$$\boxed{\vec E=\frac{2\lambda L}{4\pi\epsilon_0}\cdot\frac{1}{z\sqrt{z^2+L^2}}\hat z}$$ ✓ (same as Ex. 2.2)

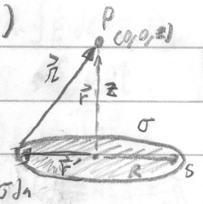
$\vec E=-\nabla V=\frac{\partial}{\partial z}V_z$

For 2.25(a), if $q_1=-q$: $V=k\left[\frac{-q}{\mathscr r_1}+\frac{q}{\mathscr r_2}\right]=k\left[\frac{-q}{\mathscr r}+\frac{q}{\mathscr r}\right]=0$. So, $\vec E=-\nabla V=0$. // Compare to 2.2, explain discrepancy.

This is contradictory to Prob. 2.2, where $\vec E\neq 0$, but $\vec E=(\cdots)\hat x$. The discrepancy lies in the assumption that $E_x=E_y=0$ due to the symmetry of 2.25(a). Now with a different charge, the symmetry no longer applies. So, to determine the true potential & field, we lack the new $\hat x$-component, which would solve the discrepancy between Prob. 2.25(a) and Prob. 2.2.

Cylindrical Coordinates
ϕ constant.

(c)



$dq = \sigma\, da$

$da = s\, ds\, d\phi = 2\pi s\, ds$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = z\hat{z} - s\hat{s}$

$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{z\hat{z} - s\hat{s}}{(z^2+s^2)^{1/2}}$

$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')}{\mathscr{r}}\, da' \Rightarrow k\int \frac{\sigma(2\pi)s\,ds}{(z^2+s^2)^{1/2}} \Rightarrow \frac{2\pi\sigma}{\cancel{2}4\pi\varepsilon_0}\int_0^R \frac{s}{(z^2+s^2)^{1/2}}\, ds$ // U-sub: $u = z^2 + s^2$,

$= \frac{\sigma}{2\varepsilon_0}\int \frac{1}{(u)^{1/2}} \frac{s}{2s}\, du = \frac{\sigma}{4\varepsilon_0}\int u^{-1/2}\, du$ // $du = 2s\,ds$, $ds = \frac{du}{2s}$

$= \frac{\sigma}{4\varepsilon_0}\left[2\sqrt{u}\right] \Rightarrow \frac{\sigma}{2\varepsilon_0}\left[\sqrt{z^2+s^2}\right]_0^R \Rightarrow \frac{\sigma}{2\varepsilon_0}\left[\sqrt{z^2+R^2} - z\right]$

$\boxed{V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2} - z\right)}$

$E = -\nabla V = \frac{-\partial}{\partial z}(V_z) = \frac{-\sigma}{2\varepsilon_0}\frac{\partial}{\partial z}\left(\sqrt{z^2+R^2} - z\right) \Rightarrow \frac{-\sigma}{2\varepsilon_0}\left[\frac{1}{\cancel{2}} \cdot \frac{1}{\sqrt{z^2+R^2}} \cdot \cancel{2}z - 1\right]$

$E = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \Rightarrow$

$\boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{1}{(z^2+R^2)^{1/2}}\right)\hat{z}}$ (same as Prob. 2.6) ✓

2.29 $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}} d\tau'$ satisfies $\nabla^2 V = \frac{-\rho}{\varepsilon_0}$. (Use $\nabla^2 V = 0$ & $\nabla^2 \frac{1}{\mathscr{r}} = -4\pi\,\delta^3(\vec{\mathscr{r}})$)

$\nabla^2 V = \nabla^2\left(\frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}} d\tau'\right)$

$= k\,\nabla^2 \int \rho(\vec{r}')\frac{1}{\mathscr{r}} d\tau' \Rightarrow k\int \rho(\vec{r}')\,\nabla^2\frac{1}{\mathscr{r}}\,d\tau' \Rightarrow \frac{1}{\cancel{4}\pi\varepsilon_0}\int \rho(\vec{r}')(-\cancel{4\pi}\,\delta^3(\vec{\mathscr{r}}))d\tau'$ // $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$= \frac{-1}{\varepsilon_0}\int \rho(\vec{r}')\,\delta^3(\vec{r}-\vec{r}')\,d\tau$ // $\int_{-\infty}^{\infty} f(x)\,\delta^3(x-a)\,d\tau = f(a)$, $f(x) = \rho(\vec{r}')$, $f(a) = \rho(\vec{r})$

$= \frac{-1}{\varepsilon_0}[\rho(\vec{r})] \Rightarrow \boxed{\nabla^2 V = \frac{-\rho(\vec{r})}{\varepsilon_0}}$