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3.3 3.5 3.6 3.13 3.14 3.15 3.16 3.19 3.22 3.24 3.26

HW 3.

3.3) General solution in spherical coordinates for $V \equiv V(r)$. Then cylindrical $V \equiv V(s)$.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{dV}{dr}\right) = 0 \text{ so } r^2\frac{dV}{dr} = a \quad \text{(a is a constant).}$$

$$\frac{dV}{dr} = \frac{a}{r^2} \Rightarrow \int dV = \int \frac{a}{r^2} dr \quad \boxed{V = -\frac{a}{r} + C}$$

Now cylindrical.

$$\nabla^2 V = \frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0. \text{ Once again, } s\frac{dV}{ds} = a \text{ so } \frac{dV}{ds} = \frac{a}{s}$$

$$\int dV = \int \frac{a}{s} ds \Rightarrow \boxed{V = a\ln s + C}$$

3.5) Show field is unique when ρ is given and V or $\frac{\partial V}{\partial n}$ is specified on each boundary surface.
Suppose there are 2 field with the conditions in the problem.

$$\nabla\cdot E_1 = \frac{1}{\epsilon_0}\rho \text{ and } \nabla\cdot E_2 = \frac{1}{\epsilon_0}\rho \text{ and } \oint E_1\cdot da = \frac{1}{\epsilon_0}Q_i \text{ and } \oint E_2\cdot da = \frac{1}{\epsilon_0}Q_i$$

for the enclosed conductors and the outer boundaries. $E_3 = E_1 - E_2$. with $\nabla\cdot E_3 = 0$, and with $\oint E_3\cdot da = 0$. Then with the product rule $\nabla\cdot(V_3 E_3) = V_3(\nabla\cdot E_3) + E_3\cdot(\nabla V_3) = -(E_3)^2$.
However, integrating over V we have $\int_V \nabla\cdot(V_3E_3)d\tau = \oint_S V_3 E_3\cdot da = -\int_V (E_3)^2 d\tau$.

This means $-\int_V (E_3)^2 d\tau = \oint_S V_3 E_3\cdot da$. We now have that $\int_V (E_3)^2 d\tau = 0$ because

$V_3 = 0$ or $E_{3_\perp} = 0$. This follows the initial conditions. Because $\int_V (E_3)^2 d\tau = 0$, we know that $E_1 = E_2$ because $E_3 = 0$.

QED

3.6) $T = U = V_3$. Green's identity yields $\int_V (V_3\nabla^2 V_3 + \nabla V_3\cdot\nabla V_3)d\tau = \oint_S V_3 \nabla V_3\cdot da$

Recall that by definition we have $\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0$, and $\nabla V_3 = -E_3$.
This means $\int_V (E_3)^2 d\tau = -\oint_S V_3 E_3\cdot da$, and for the same reason as before $\int_V (E_3)^2 d\tau = 0$

which implies $E_1 = E_2$.

QED.

3.13] Potential of infinite slot of ex 3.3. Boundary at $x=0$ is 2 metal strips. One from $y=0$ to $y=\frac{a}{2}$ at potential V_0. Other is $y=\frac{a}{2}$ to $y=a$ at $-V_0$.

$$V(x,y)=\sum_{n=1}^{\infty} C_n \exp\left(-\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \text{ with } C_n=\frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

Our potential function here is $V_0(y)=\begin{cases} V_0 & 0<y<\frac{a}{2} \\ -V_0 & \frac{a}{2}<y<a\end{cases}$ Quantum flashbacks.

So $C_n=\frac{2V_0}{a}\left(\int_0^{a/2}\sin\left(\frac{n\pi y}{a}\right)dy-\int_{a/2}^{a}\sin\left(\frac{n\pi y}{a}\right)dy\right)=\frac{2V_0}{a}\left(\cancel{\frac{a}{n\pi}\cos(}\right.$

$$=\frac{2V_0}{a}\left[\frac{a}{n\pi}\cos\left(\frac{n\pi y}{a}\right)\Big|_{a/2}^{0}+\frac{a}{n\pi}\cos\left(\frac{n\pi y}{a}\right)\Big|_{a/2}^{a}\right]=\frac{2V_0}{a}\left[\frac{a}{n\pi}-\frac{a}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{a}{n\pi}\cos(n\pi)-\frac{a}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right]$$

$$=\frac{2V_0}{n\pi}\left[1-2\cos\left(\frac{n\pi}{2}\right)+\cos(n\pi)\right]=\frac{2V_0}{n\pi}\left[1+(-1)^n-2\cos\left(\frac{n\pi}{2}\right)\right]$$

So what is THIS? If n is odd, this is 0. Also, if cosine is max, this is 0. So our C_n's are every other even integer for nonzero values.

This means that

$$C_n=\begin{cases}\frac{8V_0}{n\pi} & n=4k+2 \text{ with } k\in\mathbb{Z}. \\ 0 & \text{else}\end{cases}$$

So our general solution is

$$\boxed{V(x,y)=\frac{8V_0}{\pi}\sum_{i=\text{index}}^{\infty}\frac{\exp\left(-\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}{n} \text{ with index generated by index}=4k+2.}$$

3.14] In infinite slot, get charge density $\sigma(y)$ on strip at $x=0$ at potential V_0 and conductor.

$$V(x,y)=\frac{4V_0}{\pi}\sum_{n=1,3,5,\dots}^{\infty}\frac{\exp\left(-\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}{n} \quad \text{and } \sigma=-\epsilon_0\frac{\partial V}{\partial n}$$

Soo $\sigma(y)=-\frac{4\epsilon_0 V_0}{\pi}\frac{\partial}{\partial x}\sum\frac{\exp\left(-\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}{n}=-\frac{4\epsilon_0 V_0}{\pi}\sum\frac{\partial}{\partial x}\sim$

$$=\left(-\frac{4\epsilon_0 V_0}{\pi}\left(-\frac{n\pi}{a}\right)\sum\frac{\exp\left(-\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}{n}\right)_{x=0}$$

so $\boxed{\sigma(y)=\frac{4\epsilon_0 n}{a}\sum_{i=1,3,5,\dots}^{\infty}\frac{\sin\left(\frac{n\pi y}{a}\right)}{n}}$

3.15 A rectangular pipe parallel to z from $-\infty$ to ∞. 3 grounded metal sides at $y=0$, $y=a$, and $x=0$. Last side at $x=b$ has potential $V_0(y)$.

This means

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0.$$ Boundary conditions of $V(x,0)=0$. $V(x,a)=0$. $V(0,y)=0$. $V(b,y)=V_0(y)$.

Like the warm ups, we have $V(x,y)=(Ae^{kx}+Be^{-kx})(C\sin(ky)+D\cos(ky))$.

$V(x,0)=0=(Ae^{kx}+Be^{-kx})D$ so $D=0$.

$V(0,y)=0=(A+B)(C\sin(ky))$. $A+B$ must equal 0 because C cannot.

So $A=-B$.

Our potential reduces to $V(x,y)=AC(e^{kx}-e^{-kx})\sin(ky)$ so

$V(x,y)=2AC\sinh(kx)\sin(ky)$ with $k=\frac{n\pi}{a}$. $V(x,y)=C_n\sinh(\frac{n\pi x}{a})\sin(\frac{n\pi y}{a})$

$$\boxed{V(x,y)=\sum_{n=1}^{\infty}C_n\sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}$$ Now potential at $V_0(y)=V_0$.

$C_n\sinh(\frac{n\pi b}{a})=\frac{2}{a}\int_0^a V_0(y)\sin(\frac{n\pi y}{a})dy$ from Fourier's Trick.

So $C_n=\frac{2}{a\sinh(\frac{n\pi b}{a})}\int_0^a V_0(y)\sin(\frac{n\pi y}{a})dy$. At V_0.

$$C_n=\frac{2V_0}{a\sinh(\frac{n\pi b}{a})}\int_0^a\sin\left(\frac{n\pi y}{a}\right)dy=\sim\int_0^a\sin\left(\frac{n\pi y}{a}\right)dy=\frac{2V_0}{a\sinh(\sim)}\left(\frac{a}{n\pi}\right)\left(\cos\left(\frac{n\pi y}{a}\right)\right)\Big|_a^0$$

$$=\frac{2V_0}{n\pi\sinh(\sim)}\left[1-(-1)^n\right]$$ 0 if n is even.

$C_n=\frac{4V_0}{n\pi\sinh(\frac{n\pi b}{a})}$ with $n=2k+1$ with $k\in\mathbb{Z}$.

So $$\boxed{V(x,y)=\frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\sinh(\frac{n\pi x}{a})\sin(\frac{n\pi y}{a})}{n\sinh(\frac{n\pi b}{a})}}$$

3.16) All sheets are grounded except top at V_0. What is the potential in the box, and at the center.

Because boundary conditions.

$x=0 \Rightarrow V=0.$

$x=a \Rightarrow V=0.$

$y=0 \Rightarrow V=0$

$y=a \Rightarrow V=0$

$z=0 \Rightarrow V=0$

$z=a \Rightarrow V=V_0$

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

leads to the solution

$X(x) = A\sin(kx)+B\cos(kx)$

$Y(y) = C\sin(ly)+D\cos(ly)$

$Z(z) = E\exp(\sqrt{k^2+l^2}\,z)+\cancel{A}F\exp(-\sqrt{k^2+l^2}\,z)$

$B=0$ for same reason as previous.

$D=0$ for same reason as previous.

$k=\frac{n\pi}{a}$ $\quad l=\frac{m\pi}{a}$ $\quad m\neq n$ from orthonormal.

From $z=0$, $E=G$. $Z(z) = 2E\sinh\left(\frac{\pi\sqrt{n^2+m^2}\,z}{a}\right)$

For general solution then, we have

$$V(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_nC_m\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\pi\sqrt{n^2+m^2}\,z}{a}\right)$$

We need to get C_n and C_m. Use $z=a \Rightarrow V=V_0$.

$$V_0=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[C_nC_m\sinh(\pi\sqrt{n^2+m^2})\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\right].$$

$$C_nC_m\sinh(\pi\sqrt{n^2+m^2})=\left(\frac{2}{a}\right)^2V_0\int_0^a\int_0^a\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dx\,dy.$$ Fourier's trick again.

~~To get~~ m and n must be odd to get something nonzero.

Integral done on wolfram to reduce algebra.

$$\boxed{V(x,y,z)=\frac{16V_0}{\pi^2}\sum_{n\text{ odd}}\sum_{m\text{ odd}}\frac{1}{nm}\frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\pi\sqrt{n^2+m^2}\,z}{a}\right)}{\sinh(\pi\sqrt{n^2+m^2})}}$$

let $x,y,z \to \frac{a}{2}$ to get $$\frac{16V_0}{\pi^2}\sum_{n\text{ odd}}\sum_{m\text{ odd}}\frac{\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right)\sinh\left(\frac{\pi\sqrt{n^2+m^2}}{2}\right)}{\sinh(\pi\sqrt{n^2+m^2})}$$

Used sagemath.

This is [redacted] $\frac{V_0}{6}$

3.19 $\qquad V_0 = k\cos(3\theta)$

Get charge in, surface, and out.

Inside we want some good old polynomials.

$$V_0 = k\cos(3\theta) = k[4\cos^3\theta - 3\cos\theta] = k[AP_3(\cos\theta) + BP_1(\cos\theta)]$$

↑ Identity online.

So what are A and B.

$$4\cos^3\theta - 3\cos\theta = A\left(\frac{1}{2}(5\cos^3\theta - 3\cos\theta)\right) + B\cos\theta = \frac{5A\cos^3\theta}{2} - \frac{3A\cos\theta}{2} + B\cos\theta$$

$$= \left(\frac{5A}{2}\right)\cos^3\theta + \cos\theta\left(B - \frac{3A}{2}\right) \quad \text{so} \quad \frac{5A}{2} = 4 \quad \text{and} \quad B - \frac{3A}{2} = -3$$

$$5A = 8 \qquad B - \frac{3\left(\frac{8}{5}\right)}{2} = -3$$

$$A = \frac{8}{5} \qquad B = -\frac{3}{5}$$

$$V_0(\theta) = \frac{k}{5}[8P_3(\cos\theta) - 3P_1(\cos\theta)]$$

$$\text{So } V(r,\theta) = \begin{cases} \sum_{i=0}^{\infty} A_i r^i P_i(\cos\theta) & \text{with } r \le R \\ \sum_{i=0}^{\infty} \frac{B_i}{r^{i+1}} P_i(\cos\theta) & \text{with } r \ge R. \end{cases}$$

$$A_i = \frac{2i+1}{2R^i}\int_0^{\pi} V_0(\theta)P_i(\cos\theta)\sin\theta\,d\theta = \frac{2i+1}{2R^i}\frac{k}{5}\left[8\int_0^{\pi} P_3(\cos\theta)P_i(\cos\theta)\sin\theta\,d\theta - 3\int_0^{\pi} P_1(\cos\theta)P_i(\cos\theta)\sin\theta\,d\theta\right]$$

$$= \frac{2i+1}{2R^i}\frac{k}{5}\left(\frac{16}{2i+1} - \frac{6}{2i+1}\right) = \frac{k}{5}\frac{1}{R^i}[8\delta_{i3} - 3\delta_{i1}] = \begin{cases} -\frac{3k}{5R} & \text{when } i=1 \\ \frac{8k}{5R^3} & i=3 \end{cases} \quad \text{All else is 0.}$$

$$\text{So } V(r,\theta) = -\frac{3k}{5R} r P_1(\cos\theta) + \frac{8k}{5R^3} r^3 P_3(\cos\theta) = \boxed{\frac{k}{5}\left[-\frac{3r}{R}P_1(\cos\theta) + \frac{8r^3}{R^3}P_3(\cos\theta)\right]}$$

3.22 $\quad V(r,0) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2} - r\right)$

$$V(r,\theta) = \sum_{i=0}^{\infty}\frac{B_i}{r^{i+1}}P_i(\cos\theta) \text{ with } \theta = 0 \text{ we have } V(r,0) = \sum_{i=0}^{\infty}\frac{B_i}{r^{i+1}}P_i(1) = \sum_{i=0}^{\infty}\frac{B_i}{r^{i+1}}$$

$$\sum_{i=0}^{\infty}\frac{B_i}{r^{i+1}} = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2} - r\right) = \frac{\sigma}{2\epsilon_0}\left(\frac{R^2}{2r} - \frac{R^4}{8r^3} + \dots\right)$$

$$\boxed{V(r,\theta) = \frac{\sigma R^2}{4\epsilon_0 r}\left[1 - \frac{R^2}{8r^2}(3\cos^2\theta - 1) + \dots\right]}$$

$$V(r,\theta) = \sum_{i=0}^{\infty} A_i r^i P_i(\cos\theta) \text{ in } 0 \le \theta \le \tfrac{\pi}{2} \text{ region.}$$

$$V(r,0) = \sum_{i=0}^{\infty} A_i r^i = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right]$$ Same expansion as before.

$$\sum_{i=0}^{\infty} A_i r^i = \frac{\sigma}{2\epsilon_0}\left[R + \frac{r^2}{2R} - \frac{r^4}{8R^3} + \dots - r\right]$$ So we have

$$\boxed{V(r,\theta) = \frac{\sigma R}{2\epsilon_0}\left[1 - \frac{r\cos\theta}{R} + \frac{r^2}{4R^2}(3\cos^2\theta - 1) + \dots\right] \text{ in } 0 \le \theta \le \tfrac{\pi}{2}.}$$

Now in $\frac{\pi}{2} \le \theta \le \pi$ region: $V(r,\pi) = \sum_{i=0}^{\infty} A_i r^i P(-1) = \sum_{i=0}^{\infty} A_i r^i (-1)^i = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right]$

Once again, use the same expansion to get

$$\boxed{V(r,\theta) = \frac{\sigma R}{2\epsilon_0}\left[1 + \frac{r\cos\theta}{R} + \frac{r^2}{4R^2}(3\cos^2\theta - 1) + \dots\right] \text{ for } \tfrac{\pi}{2} \le \theta \le \pi}$$

3.24

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$ No spherical harmonics.

we want a seperable solution like $V(s,\phi) = S(s)\Phi(\phi)$.

$$\frac{1}{s}\Phi\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{s^2}S\frac{d^2\Phi}{d\phi^2} = 0 \implies \frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = C_1 \text{ and } \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = C_2 \qquad \frac{d^2\Phi}{d\phi^2} = -k^2\Phi \text{ so } \Phi = A\cos(k\phi) + B\sin(k\phi)$$

Now we have $s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = SC_1$ let $C_1 = k^2 \implies s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = k^2 S$. Looked up solution to this equation.

$S = C\ln(s) + D$

$$\boxed{\text{Solution} = (C\ln(s) + D)(A\cos(k\phi) + B\sin(k\phi))}$$

But now general is

$$\boxed{V(s,\phi) = A_0 + B_0\ln(s) + \sum_{i=1}^{\infty}\left[s^i(a_i\cos(k\phi) + b_i\sin(k\phi)) + s^{-i}(c_i\cos(k\phi) + d_i\sin(k\phi))\right]}$$

3.26] $\sigma(\phi) = a\sin(5\phi)$

Inside is $V(s,\phi) = a_0 + \sum_{i=1}^{\infty} s^i (a_i \cos(k\phi) + b_i \sin(k\phi))$ outside is

$$V(s,\phi) = a_0 + \sum_{i=1}^{\infty} s^{-i} (c_i \cos(k\phi) + d_i \sin(k\phi)) \quad \text{with } \sigma = -\epsilon_0 \left(\frac{\partial V_{outside}}{\partial s} - \frac{\partial V_{inside}}{\partial s} \right) \Bigg|_{s=R}$$

$$a\sin(5\phi) = -\epsilon_0 \sum_{i=1}^{\infty} \left[-\frac{i}{R^{i+1}} (c_i \cos(5\phi) + d_i \sin(5\phi)) - i R^{i-1} (a_i \cos(k\phi) + b_i \sin(k\phi)) \right]$$

From boundaries, $a = 5\epsilon_0 (R^4 b_5 + R^4 b_5) = 10\epsilon_0 R^4 b_5$ with $b_5 = \frac{a}{10\epsilon_0 R^4} = \frac{aR^6}{10\epsilon_0}$

$$\boxed{\begin{aligned} V(s,\phi) &= \frac{a\sin(5\phi)}{10\epsilon_0} \frac{s^5}{R^4} & s<R \\ &= \frac{a\sin(5\phi)}{10\epsilon_0} \frac{R^6}{s^5} & s>R. \end{aligned}}$$