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HW 5

4) a) $\rho \approx 0$, near axis so...

$B_z(\rho, z) = b_0(z) + \rho b_1(z) + \frac{1}{2}\rho^2 b_2(z) \ldots \Rightarrow \sum_{n=0}^{\infty} \frac{\rho^n}{n!} b_n(z)$

$B_\rho(\rho, z) = c_0(z) + \rho c_1(z) + \frac{1}{2}\rho^2 c_2(z) \ldots \Rightarrow \sum_{n=0}^{\infty} \frac{\rho^n}{n!} c_n(z)$

$B_\phi(\rho, z) = 0$

if there's no current then $\vec{\nabla} \cdot \vec{B} = 0$ & $\vec{\nabla} \times \vec{B} = 0$

$0 = \vec{\nabla} \cdot \vec{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho} B_\rho \rho + \frac{\partial}{\partial z} B_z \Rightarrow$

$\sum_{n=0}^{\infty} \left(\frac{(n+1)\rho^{n-1}}{n!} c_n(z) + \frac{\rho^n}{n!} b_n'(z) \right)$

let $n = n+1$ to get...

$\sum_{n=0}^{\infty} \left(\frac{(n+1)+1}{(n+1)!} \rho^{(n+1)-1} \quad c_{n+1}(z) + \frac{\rho^{n+1}}{(n+1)!} b_n'(z) \right) \Rightarrow$

$0 = \frac{1}{\rho} c_0(z) + \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \left(\frac{n+2}{(n+1)} c_{n+1}(z) + \frac{\rho^{n+1}}{(n+1)} b_n'(z) \right)$

$c_0(z) = 0$, $c_{n+1} = -\frac{n+1}{n+2} b_n'(z)$

Now for $\vec{\nabla} \times \vec{B}$

$0 = |\vec{\nabla} \times \vec{B}|_\phi = \frac{\partial}{\partial z} B_\rho - \frac{\partial}{\partial \rho} B_z = \sum_{n=0}^{\infty} \left(\frac{\rho^n}{n!} c_n'(z) - \frac{\rho^{n-1}}{(n-1)!} b_n(z) \right)$

again let $n = n+1$ so that

$\sum_{n=0}^{\infty} \left(\frac{\rho^{\cancel{n+1}}}{(\cancel{n+1})!} c_n'(z) - \frac{\rho^{(\cancel{n+1-1})}}{((\cancel{n+1})-1)!} b_n(z) \right)$

$\Rightarrow$

$\left(\frac{\rho^n}{n!} c_n'(z) \frac{\rho^n}{n!} b_n(z) \right) \Rightarrow$

$0 = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \left(c_n'(z) - b_{n+1}(z) \right) \Rightarrow$

$c_n'(z) = (b_{n+1}(z)) \Rightarrow$

$$b_{n+1}(z) = c_n'(z) = -\frac{n}{n+1} b_{n-1}''(z)$$

if $b_1(z) = 0$ then $c_0(z) = 0$ but $b_0(z)$ is ~~[illegible]~~ unknown

$\Rightarrow$

$$b_n(z) = (-1)^{n/2} \frac{(n-1)(n-3)\cdots 3\cdot 1}{n(n-2)\cdots 4\cdot 2} b_0^{(n)}(z) \quad (\text{even})$$

&

$$c_{n+1}(z) = \frac{-1^{n/2+1}}{2^n} \frac{(n+1)!}{(n+2)((n/2)!)^2} b_0^{n+1}(z) \quad (\text{odd})$$

These 2 equation in the taylor expansion result in ...

$$B_z(\rho, z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \frac{\rho^{2k}}{(k!)^2} \left[\frac{\partial^{2k} B_z(0,z)}{\partial z^{2k}} \right] \quad \leftarrow \text{even}$$

$$B_\rho(\rho, z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4^k} \frac{\rho^{2k+1}}{(2k+2)(k!)^2} \left[\frac{\partial^{2k+1} B_z(0,z)}{\partial z^{2k+1}} \right] \quad \leftarrow \text{odd}$$

b)

if $\vec{B}(\rho, z)$ ~~an~~ $\sim \frac{\rho^n}{2^n[(n/2)!]^2} \left[\frac{\partial^n B_z(0,z)}{\partial z^n} \right]$

and ~~without~~ we know the series is

$$\frac{a_{n+2}}{a_n} \sim \rho^2 \left[\frac{\partial^{n+2} B_z(0,z)/\partial z^{n+2}}{\partial^n B_z(0,z)/\partial z^n} \right] \quad \text{so}$$

for convergence then

$$\rho^2 \left[\frac{\partial^{n+2} \cdots}{\partial^n \cdots} \right] \Rightarrow$$

$$\sqrt{\left[\frac{\partial^n \cdots}{\partial^{n+2} \cdots} \right]} = \rho \quad \& \quad \sqrt{\left[\frac{\partial^n \cdots}{\partial^{n+2} \cdots} \right]} \gg \rho$$

7) $\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}\int_V \rho r d\tau \Rightarrow \underbrace{\int\left(\frac{\partial \rho}{\partial t}\right) r d\tau = -\int(\nabla\cdot J) r d\tau}_{\text{continuity}}$

$\nabla\cdot(xJ) = x(\nabla\cdot J) + J\cdot(\nabla x) \Rightarrow$

$\nabla x = \hat{x} \Rightarrow$

$\nabla\cdot(xJ) = x(\nabla\cdot J) + J_x \Rightarrow$

$\underbrace{\int_V(\nabla\cdot J)\, x d\tau}_{} = \int_V \nabla\cdot(xJ) d\tau - \int_V J_x d\tau$

$\int_S xJ\cdot da$ & J is in V & is 0 on surface $\Rightarrow$

$\int_V(\nabla\cdot J) x d\tau = -\int_V J_x d\tau$

11) Using eq. 5.41 :

$B = \frac{\mu_0 n I}{2}\int \frac{a^2}{(a^2+z^2)^{3/2}} dz$, $z = a\cot\theta$

so $dz = -\frac{a}{\sin^2\theta} d\theta$, & $\frac{1}{(a^2+z^2)^{3/2}} = \frac{\sin^3\theta}{a^3} \Rightarrow$

$B = \frac{\mu_0 n I}{2}\int \frac{a^2 \sin^3\theta}{a^3 \sin^2\theta}(-a d\theta) = \frac{-\mu_0 n I}{2}\int \sin\theta d\theta \Rightarrow$

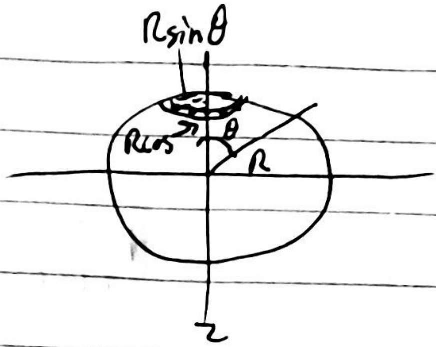
$\frac{\mu_0 n I}{2}\cos\theta\Big|_{\theta_1}^{\theta_2} = \frac{\mu_0 n I}{2}(\cos\theta_2 - \cos\theta_1) \Rightarrow$

$\theta_2 = 0$ & $\theta_1 = \pi \Rightarrow$

$(\cos 0 - \cos\pi) \Rightarrow 1 - (-1) = 2 \Rightarrow$

$B = \frac{\mu_0 n I}{\cancel{2}}\cancel{2} = \mu_0 n I$

12) if



By eq. 5.41

$$dB = \frac{\mu_0 dI}{2} \frac{(R\sin\theta)^2}{((R\sin\theta)^2 + (R\cos\theta)^2)^{3/2}} = \frac{\mu_0}{2R}\sin^2\theta \, dI$$

$$\Rightarrow$$

$$dI = KR\,d\theta, \quad K = \sigma v, \quad \sigma = \frac{Q}{4\pi R^2}, \quad v = \omega R\sin\theta \Rightarrow$$

$$dI = \frac{Q}{4\pi R^2}\omega R\sin\theta R\,d\theta = \frac{Q\omega}{4\pi}\sin\theta\, d\theta \Rightarrow$$

$$B = \frac{\mu_0}{2R}\frac{Q\omega}{4\pi}\int_0^\pi \sin^3\theta\, d\theta = \frac{\mu_0 Q\omega}{8\pi R}\left(\frac{4}{3}\right)$$

$$B = \frac{\mu_0 Q\omega}{6\pi R}\hat{z}$$

16)

a) $A_\phi^{loop} = \frac{\mu_0 I a}{2}\sum_{l\,odd}\frac{1}{l(l+1)}\frac{r_<^l}{r_>^{l+1}}P_l^1(\cos\theta)$, loop = cylinder

if $\nabla^2 A = 0$, then by symmetry $\Rightarrow$

$$A_\phi = \sum\frac{1}{(l+1)l}\left(\frac{r^l}{r^{l+1}} + \alpha_l r^l\right)P_l^1(0)P_l^1(\cos\theta)$$

if $r = b$ for $\vec{B} \to 0$ then

$$B_\theta(r=b) = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)$$

$$= \frac{\mu_0 I a}{2}\sum\frac{1}{l(l+1)}\left(l\frac{a^l}{r^{l+2}} - (l+1)\alpha_l r^{l-1}\right)P_l^1 P_l^1\cos\theta$$

$$\Rightarrow$$

$$\frac{\mu_0 I a}{2}\sum_{l\,odd}\frac{b^{l-1}}{l(l+1)}\left(l\frac{a^l}{b^{2l+1}} - (l+1)\alpha_l\right)P_l^1(0)P_l^1\cos\theta$$

for $r^l = a$ & $r^{l+1} > r^l$

$$\alpha_l = \frac{l}{l+1}\frac{a^l}{b^{2l+1}} \Rightarrow$$

$$A_\phi = \frac{\mu_0 I a}{2}\sum_{l\,odd}\frac{1}{l(l+1)}\left(\frac{r^l}{r^{l+1}} + \frac{l}{l+1}\frac{(ar)^l}{b^{2l+1}}\right)P_l^1(0)P_l^1\cos\theta$$

b) $B_z = \frac{\mu_0 I}{2R} \Rightarrow$

* needed help with this part

$$\frac{\mu_0 I}{2a}\left(1+\frac{1}{2}\left(\frac{a}{b}\right)^3\right) = \frac{\mu_0 I}{2a} + \frac{\mu_0 I}{2(2b^3/a^2)}$$

$\Rightarrow$

$$R = \frac{2b^3}{a^2}$$

4) a) $\phi = -\frac{1}{4\pi}\int_V \frac{\vec{\nabla}\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' + \frac{1}{4\pi}\oint_S \frac{\hat{n}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}da'$

$$\vec{M} = M_0\hat{z}$$

$$\phi = \frac{M_0}{4\pi}\left[\int_{L/2}\frac{1}{|\vec{x}-\vec{x}'|}da' - \int_{-L/2}\frac{1}{|\vec{x}-\vec{x}'|}da'\right] \Rightarrow$$

$$\frac{M_0}{4\pi}\int\left(\frac{1}{\sqrt{\rho^2+(z+L/2)^2}} - \frac{1}{\sqrt{\rho^2+(z-L/2)^2}}\right)\rho\, d\rho\, d\phi$$

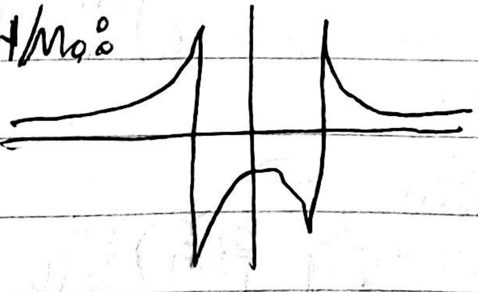
$\Rightarrow$

$$\frac{M_0}{2}\left[\sqrt{a^2+(z+L/2)^2} - \sqrt{a^2+(z-L/2)^2} - |z-L/2| + |z+L/2|\right]$$

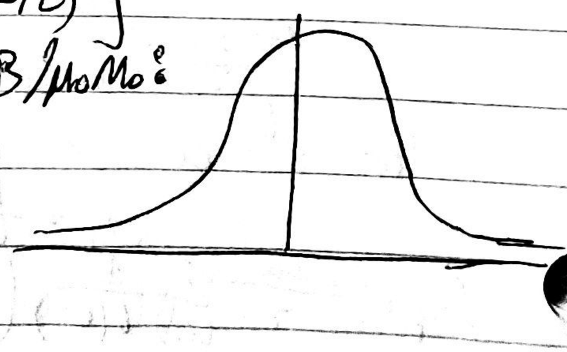
Given $H_z = -\frac{\partial}{\partial z}\phi \Rightarrow$

$$-\frac{M_0}{2}\left[\frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} - \frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - (z-L/2) + z+L/2\right]$$

b) H/M_0:



$B/\mu_0 M_0$:



21) Show $\int \vec{B} \cdot \vec{H} \, d^3x = 0$

a) $\int \vec{H} \cdot (\vec{\nabla} \times \vec{A}) d^3x \Rightarrow$

$$\int \vec{A} \cdot (\vec{\nabla} \times \vec{H}) d^3x + \int \vec{\nabla} \cdot (\vec{A} \times \vec{H}) d^3x$$
$$= \underbrace{\int \vec{J} \cdot \vec{A} \, d^3x}_{J=0} + \int_\infty \vec{A} \times \vec{H} \cdot d\vec{a} = 0$$

b) $W = \frac{\mu_0}{2} \int \vec{H} \cdot \vec{H} d^3x = -\frac{\mu_0}{2} \int \vec{M} \cdot \vec{H} d^3x$

$U = -\vec{m} \cdot \vec{B} \quad \Rightarrow$

$$W = -\sum \vec{m}_j \cdot B_i$$

$B_i = \frac{\mu_0}{4\pi} \frac{3(\vec{m}_i \cdot \hat{x})\hat{x} - \vec{m}_i}{|\vec{x}|^3} \quad \Rightarrow$

$$\vec{m}_j \cdot \vec{B}_i = \vec{m}_i \cdot \vec{B}_j \quad \Rightarrow$$

$W = -\frac{1}{2} \sum \vec{m}_j \cdot \vec{B}_i$

$W = -\frac{1}{2} \int \vec{M} \cdot \vec{B} d^3x$

if $\vec{B} = \mu_0 (\vec{H} + \vec{M}) \Rightarrow$

$W = -\frac{\mu_0}{2} \int \vec{M} \cdot (\vec{H} + \vec{M}) d^3x = W_0 = \frac{\mu_0}{2} \int \vec{M} \cdot \vec{H} d^3x$

if $\vec{M} = \frac{1}{\mu_0} \vec{B} - \vec{H}$, then

$W = W_0 - \frac{\mu_0}{2} \int \left(\frac{1}{\mu_0} \vec{B} - \vec{H}\right) \cdot \vec{H} d^3x$

23) $A = \frac{\mu_0}{4\pi}\int \frac{I\hat{z}}{r}dz = \frac{\mu_0 I}{4\pi}\hat{z}\int_{z_1}^{z_2}\frac{dz}{\sqrt{z^2+s^2}}$

$= \frac{\mu_0 I}{4\pi}\hat{z}\left[\ln\left(z+\sqrt{z^2+s^2}\right)\right]\Big|_{z_1}^{z_2} = \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{(z_2)^2+s^2}}{(z_1)+\sqrt{(z_1)^2+s^2}}\right]\hat{z}$

$B = \nabla\times A = -\frac{\partial A}{\partial s}\hat{\phi} = \frac{-\mu_0 I}{4\pi}\left[\frac{1}{z_2+\sqrt{z_2^2+s^2}}\frac{s}{\sqrt{z_2^2+s^2}} - \frac{1}{\sqrt{z_1^2+s^2}+z_1}\frac{s}{\sqrt{z_1^2+s^2}}\right]$

$\Rightarrow \frac{\mu_0 I s}{4\pi}\left[\frac{z_2-\sqrt{z_2^2+s^2}}{z_2^2-z_2^2+s^2}\frac{1}{\sqrt{z_2^2+s^2}} - \frac{z_1-\sqrt{z_1^2+s^2}}{z_1^2-[z_1^2+s^2]}\frac{1}{\sqrt{z_1^2+s^2}}\right]\hat{\phi}$

$\Rightarrow -\frac{\mu_0 I s}{4\pi}\left(\frac{-1}{s^2}\right)\left[\frac{z_2}{\sqrt{z_2^2+s^2}} - 1 - \frac{z_1}{z_1^2+s^2}+1\right]\hat{\phi} \Rightarrow$

$\frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{z_2^2+s^2}} - \frac{z_1}{\sqrt{z_1^2+s^2}}\right]\hat{\phi}$

27) $\vec{J} = \begin{cases} (I/\pi b^2)\hat{z} & \rho < b \\ 0 & \text{else} \end{cases}$

$\vec{B} = \begin{cases} \frac{\mu I \rho}{2\pi b^2}\hat{\phi} & \rho < b \\ \frac{\mu_0 I}{2\pi\rho} & b < \rho < a \\ 0 & \rho > a \end{cases}$

$W/l = \frac{1}{2}\int_0^a \vec{B}\cdot\vec{H}\,2\pi\rho\,d\rho = \frac{I^2}{4\pi}\left[\mu\int_0^b \frac{\rho^3}{b^4}d\rho + \mu_0\int_b^a \frac{1}{\rho}d\rho\right]$

$= \frac{I^2}{4\pi}\left[\frac{\mu}{4} + \mu_0 \log\frac{a}{b}\right]$

if $LI^2\frac{1}{2}$ is set to energy then

$L/l = \frac{\mu_0}{4\pi}\left[\frac{\mu_r}{2} + \log\frac{a^2}{b^2}\right]$ so since

$\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$ fo $b < \rho < a$ then $L/l = \frac{\mu_0}{4\pi}\log\frac{a^2}{b^2}$