Solutions for Homework 3

Dr. Jordan Hanson - Whittier College Dept. of Physics and Astronomy

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1 Problem 3.3

The Laplacian in spherical coordinates, assuming $V(\mathbf{r})$ depends only on r, is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \tag{1}$$

- Multiply both sides by r, assuming that $r \neq 0$ (for the Laplacian has a potential singularity there).
- The quantity $(r^2 dV/dr)$ must be a constant because its derivative is zero.
- We have

$$r^2\frac{dV}{dr} = k \tag{2}$$
$$\frac{dV}{dr} = \frac{k}{r^2} \tag{3}$$
$$V(r) = -\frac{k}{r} + C \tag{4}$$

Thus the $1/r$ dependence of potential with spherical symmetry is a consequence of Laplace's Equation.

The Laplacian in cylindrical coordinates, assuming $V(\mathbf{r})$ depends only on s, is

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0 \tag{5}$$

Using the same approach as the steps above, we find that s times dV/ds is a constant and that

$$V(s) = k\ln(s) + C \tag{6}$$

Imagine the solution for the potential surrounding a line charge, and notice that it follows this pattern.

2 Problem 3.5

Prove that the field is uniquely determined when the charge density ρ is given and either V or the normal derivative $\partial V/\partial n$ is specified on each boundary surface. Do not assume the boundaries are conductors, or that V is constant over any given surface.

This proof follows closely the one on pp. 121-123, with one key difference. Start by assuming there are two solutions for the field within the volume $\mathcal{V}$ where we find ρ. There is only one ρ, but there could be many objects (conductors or otherwise) within the volume, and one global surface containing them all. The two solutions follow the original reasoning:

$$\nabla \cdot \mathbf{E}_1 = -\frac{\rho}{\epsilon_0} \tag{7}$$
$$\nabla \cdot \mathbf{E}_2 = -\frac{\rho}{\epsilon_0} \tag{8}$$
$$\mathbf{E}_3 = \mathbf{E}_2 - \mathbf{E}_1 \tag{9}$$
$$\nabla \cdot \mathbf{E}_3 = 0 \tag{10}$$
$$\oint \mathbf{E}_3 \cdot d\mathbf{a} = 0 \quad (all \quad surfaces) \tag{11}$$
$$\mathbf{E}_3 = -\nabla V_3 \tag{12}$$
$$\nabla \cdot (V_3\mathbf{E}_3) = V_3(\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3(\nabla V_3) = 0 - \mathbf{E}_3 \cdot \mathbf{E}_3 = -E_3^2 \tag{13}$$

Integrate the final line with respect to volume, over $\mathcal{V}$:

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{E}_3) d\tau = -\int E_3^2 d\tau \tag{14}$$

Use the divergence theorem on the left-hand side:

$$\oint V_3 \mathbf{E}_3 \cdot d\mathbf{a} = -\int E_3^2 d\tau \tag{15}$$

The integrand on the left hand is zero if:

- The potentials V_1 and V_2 are specified on each surface. Then, V_3 over each and every surface, and the integrand is zero.
- The normal derivatives $\partial V_1/\partial n$ and $\partial V_2/\partial n$ are specified. Then, $\partial V_3/\partial n = -E_{3,\perp} = 0$, and the integrand is zero.

Thus, either way, the integrand is zero and

$$\int E_3^2 d\tau = 0 \tag{16}$$

That means that $E_3 = 0$, and $\mathbf{E}_2 = \mathbf{E}_1$. The field is unique.

3 Problem 3.6

A more elegant proof of the second uniqueness theorem uses Green's identity (Problem 1.16c), with $T = U = V$. Supply the details.

And ... go!

$$\int_{\mathcal{V}} (V_3 \nabla^2 V_3 + \nabla V_3 \cdot \nabla V_3) d\tau = \oint (V_3 \nabla V_3) \cdot d\mathbf{a} \tag{17}$$

$$\int_{\mathcal{V}} (0 + \nabla V_3 \cdot \nabla V_3) d\tau = -\oint (V_3 \mathbf{E}_3) \cdot d\mathbf{a} \tag{18}$$

$$-\int_{\mathcal{V}} E_3^2 d\tau = -\oint (V_3 \mathbf{E}_3) \cdot d\mathbf{a} \tag{19}$$

$$\int_{\mathcal{V}} E_3^2 d\tau = \oint (V_3 \mathbf{E}_3) \cdot d\mathbf{a} \tag{20}$$

Using the same logic as the prior exercises completes the proof.

4 Problem 3.13

Find the potential in the infinite slot of Ex. 3.3 if the boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = a/2$, is held at a positive constant potential V_0, and the other, from $y = a/2$ to $y = a$, is held at a negative constant potential $-V_0$.

Using Ex. 3.3, one can arrive at

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a) \tag{21}$$

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a) dy \tag{22}$$

The Fourier coefficient is found by applying the boundary condition:

$$C_n = \frac{2V_0}{a} \left(\int_0^{a/2} \sin(n\pi y/a) dy - \int_{a/2}^{a} \sin(n\pi y/a) dy \right) = \frac{2V_0}{n\pi} (1 + (-1)^n - 2\cos(n\pi/2)) \tag{23}$$

How do we generalize the result into a sum of solutions? Notice that the coefficient is zero if (a) n is odd, or (b) n is a multiple of 4. Otherwise, it turns into 4. Thus:

$$C_n = \frac{8V_0}{n\pi}, \quad n = (4j+2), \; j = 0, 1, 2, ... \tag{24}$$

The solutions can be gathered into the series above like so:

$$V(x, y) = \frac{8V_0}{\pi} \sum_{j=0}^{\infty} \frac{e^{-(4j+2)\pi x/a} \sin((4j+2)\pi y/a)}{4j+2} \tag{25}$$