11/02/20

HW problems # 1.54, 1.55, 1.56, 1.57, 1.59, 1.62, 1.63, 1.64

1.54 $\vec{v} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

$\nabla\cdot\vec{v} = 2r\cos\theta + \cancel{0} - r^2\cos\theta\cos\phi$

$\int_V (2r\cos\theta - r^2\cos\theta\cos\phi)\,dr\,d\theta\,d\phi$

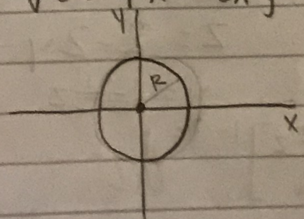
$= \int_0^{\frac{\pi}{4}}\left[\int_0^{\frac{\pi}{8}}\left(\int_0^R 2r\cos\theta - r^2\cos\theta\cos\phi\,dr\right)d\theta\right]d\phi = \int_0^{2\pi}\int_0^{\pi}\left[\frac{\cancel{2}r^2\cos\theta}{\cancel{2}} - \frac{r^3\cos\theta\cos\phi}{3}\right]_0^R d\theta\,d\phi$

$= \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{8}} R^2\cos\theta - \frac{R^3\cos\theta\cos\phi}{3}\,d\theta\,d\phi = \int_0^{\frac{\pi}{4}}\left[R^2\sin\theta - \frac{R^3\sin\theta\cos\phi}{3}\right]_0^{\frac{\pi}{8}}$

$= \int_0^{\frac{\pi}{4}}\left(R^2\sin\frac{\pi}{8} - \frac{R^3}{3}\sin\frac{\pi}{8}\cos\phi\right)d\phi \rightarrow \int_0^{\frac{\pi}{4}}\left(\frac{\pi R^2}{8} - \frac{\pi R^3}{24}\cos\phi\right)d\phi$

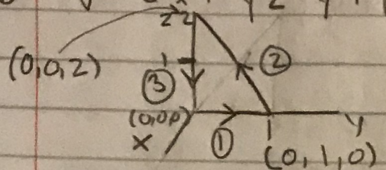
$= \frac{\pi R^2}{8}(\phi) - \frac{\pi R^3}{24}\sin\phi\Big|_0^{\frac{\pi}{4}} = \frac{\pi R^2}{8}\frac{\pi}{4} - \frac{\pi R^3}{24}\sin\frac{\pi}{4} = \frac{\pi^2 R^2}{32} - \frac{\pi^2 R^4}{96}$

1.55 $\vec{v} = ay\,\hat{x} + bx\,\hat{y}$ $\qquad \int_S (\nabla\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$



$$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ay & bx & 0 \end{vmatrix} = \cancel{\hat{x}(0-0)} - \cancel{\hat{y}(0-0)} + \hat{z}(a+b)$$

1.56 $\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$



path 1: $x, z = 0 \qquad d\vec{l} = +dy\,\hat{y}$

$\vec{v}\cdot d\vec{l} = yz^2\,dy \rightarrow \cancel{y(0)^2}\,dy$

$\int_0^1 0\,dy = 0$

path 2: $z = 2-2y \longrightarrow dz = -2dy$

$d\vec{l} = dy\,\hat{y} + dz\,\hat{z}$

$\vec{v}\cdot d\vec{l} = yz^2\,dy + (3y+z)\,dz \rightarrow y(2-2y)^2dy + (3y+(2-2y))(-2dy)$

$= y(4-4y-4y+4y^2)\,dy + (y+2)(-2dy) = (4y^3-8y^2+4y-2y-4)\,dy$

$= 4y^3-8y^2+2y-4\,dy$

$= \int_1^0 (4y^3-8y^2+2y-4)\,dy = \left[\frac{\cancel{4}y^3}{\cancel{4}} - \frac{8y^3}{3} + \frac{\cancel{2}y^2}{\cancel{2}} - 4y\right]_1^0$

$= \emptyset - (1^3 - \frac{8}{3}(1)^3 + (1)^2 - 4(1)) = -(-2-\frac{8}{3}) = \frac{8}{3} + \frac{6}{3} = \frac{14}{3}$

path 3: $x, y = 0 \qquad d\vec{l} = +dz\,\hat{z}$

$\vec{v}\cdot d\vec{l} = (3y+z)\,dz \rightarrow (3\cancel{(0)}+z)\,dz$

$= \int_2^0 z\,dz = [z^2/2]_2^0 = \emptyset - z^2/2 \qquad -2$

$\oint \vec{v}\cdot d\vec{l} = 14/3 - 6/3 = \boxed{8/3}$

$$|\nabla\times\vec{v}| = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 6 & yz^2 & (3y+z) \end{vmatrix} = \hat{x}(3-2yz) - \cancel{\hat{y}(0)} - \cancel{\hat{z}(0)} = (3-2yz)\hat{x}$$

$d\vec{a} = +dy\,dz\,\hat{x} \qquad z = 2-2y \rightarrow 2y = 2-z$

$y = 1-\frac{1}{2}z$

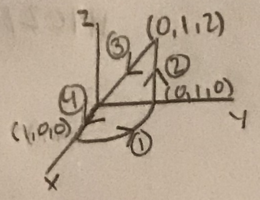
$\int|\nabla\times\vec{v}|\cdot d\vec{a} = \iint(3-2yz)\,dy\,dz$

$\int_0^2\int_0^{1-\frac{1}{2}z}(3-2yz)\,dy\,dz = \int_0^2\left[3y - \frac{\cancel{2}y^2z}{\cancel{2}}\right]_0^{1-z/2}dz$

$= \int_0^2 3(1-\frac{z}{2}) - (1-\frac{z}{2})^2 z\,dz = \int_0^2 3 - \frac{3z}{2} - (z - z^2 + \frac{z^3}{4})\,dz$

$= \int_0^2 -\frac{z^3}{4} + z^2 - \frac{5z}{2} + 3\,dz = \left[-\frac{z^4}{16} + \frac{z^3}{3} - \frac{5z^2}{4} + 3z\right]_0^2$

$\frac{-(2)^4}{16} + \frac{2^3}{3} - \frac{5(2^2)}{4} + 3(2) = \cancel{1} + \frac{8}{3} - \frac{\cancel{20}}{\cancel{4}} + \cancel{6} = \boxed{\frac{8}{3}}$



1.57 $\vec{v} = (r\cos^2\theta)\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}$

$(1,0,0) \longrightarrow (r\sin\theta\cos\phi, 0, 0)$

$(0,1,0) \longrightarrow (0, r\sin\theta\sin\phi, 0)$

$(0,1,2) \longrightarrow (0, r\sin\theta\sin\phi, 2r\cos\theta)$

1.59 $\vec{v} = r^2 \sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r^2\tan\theta\,\hat{\phi}$

$$\nabla \times \vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, r^2\tan\theta) - \cancel{\frac{\partial(4r^2\cos\theta)}{\partial\phi}}\right]\hat{r} + \frac{1}{r}\left[\cancel{\frac{1}{\sin\theta}\frac{\partial(r^2\sin\theta)}{\partial\phi}} - \frac{\partial}{\partial r}(r^3\tan\theta)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r^3\cdot 4\cos\theta) - \frac{\partial(r^2\sin\theta)}{\partial\theta}\right]\hat{\phi}$$

$$= \frac{1}{\cancel{r}\cancel{\sin\theta}}\left[r^{\cancel{2}}(\cos\theta\tan\theta + \cancel{\sin\theta}\sec^2 x)\right]\hat{r} + \frac{1}{\cancel{r}}\left[-3r^{\cancel{2}}\tan\theta\right]\hat{\theta}$$

$$+ \frac{1}{\cancel{r}}\left[\underbrace{3r^{\cancel{2}}\cdot 4\cos\theta - r^{\cancel{2}}\cos\theta}_{11r\cos\theta}\right]\hat{\phi} = r(\underbrace{1+\sec^2 x}_{1+1+\tan^2 x})\hat{r} - 3\tan\theta\,\hat{\theta} + 11\,r\cos\theta\,\hat{\phi}$$

$$\nabla \times \vec{v} = r(2+\tan^2 x)\hat{r} - 3\tan\theta\,\hat{\theta} + 11\,r\cos\theta\,\hat{\phi}$$

1.62 $a \equiv \int_S d\vec{a}$

a)

1.63 a) $\vec{v} = \hat{r}/r$

$\nabla \cdot \vec{v} = \frac{1}{r}\hat{r}(\partial r\,\hat{r}) = \boxed{1}$

- Yes there's a delta function @ the origin.
- $\nabla \cdot r^n \hat{r} = \partial/\partial r\; r^n\; \hat{r}\cdot\hat{r}$

$= \boxed{n\, r^{n-1}}$

b)

$$|\nabla \times r^n \hat{r}| = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ r^n & 0 & 0 \end{vmatrix} = \cancel{\hat{r}(0-0)} - \cancel{\hat{\theta}(0-0)} + \cancel{\hat{\phi}(0-0)} = \boxed{0}$$

1.64 $\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^3(\vec{r})$

$$D(r,\epsilon) \equiv \frac{-1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\epsilon^2}}$$

a)