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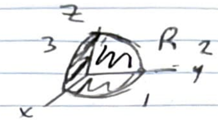
Electromagnetic Theory HW 1.

1.54]

$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$ $\quad \int_V (\nabla\cdot v)\,d\tau = \oint_S v\cdot da$



4 is sphere surface part.

$$\vec{v} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$$

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r^2\cos\phi) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\cos\theta\sin\phi)$$

$$= 4r\cos\theta + r\cos\phi\cot\theta - r\cot\theta\cos\phi = 4r\cos\theta$$

So $$\int_0^R\int_0^{\pi/2}\int_0^{\pi/2}(4r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi = \int_0^R\int_0^{\pi/2}\int_0^{\pi/2} 4r^3\cos\theta\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{\pi}{2}\int_0^R\int_0^{\pi/2} 4r^3\cos\theta\sin\theta\,dr\,d\theta = \frac{\pi}{2}\left[\int_0^R 4r^3\,dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\right] = \frac{\pi}{2}\left[R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\right]$$

$$= \frac{\pi R^4}{2}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta = \frac{\pi R^4}{2}\left[\frac{1}{2}\cos^2\theta\right]\Big|_{\pi/2} = \frac{\pi R^4}{2}\left[\frac{1}{2} - 0\right] \quad \boxed{= \frac{\pi R^4}{4}}$$

Now surface integral.

1. We have ~~area~~ sketched by R and ϕ, so $da = dr\,d\phi \rightarrow da = r\sin\theta\,dr\,d\phi$ at $\theta = \frac{\pi}{2}$.
$da = r\,dr\,d\phi$. $\quad \vec{v}\cdot d\vec{a} = (r^2\cos\phi)(r\,dr\,d\phi)$ Only θ parts. $\vec{v}\cdot d\vec{a} = r^3\cos\phi\,dr\,d\phi$.

$$\int_1 r^3\cos\phi\,dr\,d\phi = \int_0^R r^3\,dr\int_0^{\pi/2}\cos\phi\,d\phi = \frac{r^4}{4}\Big|_0^R \sin\phi\Big|_0^{\pi/2} = \frac{R^4}{4}$$

$$I1 = \frac{R^4}{4}$$

2. Sketched by R and θ, so $da = r\,dr\,d\theta\,\hat{\phi}$ $\quad \vec{v}\cdot d\vec{a} = (-r^2\cos\theta\sin\phi)(r\,dr\,d\theta)$
$= -r^3\cos\theta\sin\phi\,dr\,d\theta \quad \phi = \frac{\pi}{2}$
$= -r^3\cos\theta$

$$\int_2 = -\int_0^R\int_0^{\pi/2} r^3\cos\theta\,dr\,d\theta = -\int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\,d\theta$$

$$= -\frac{R^4}{4}\sin\theta\Big|_0^{\pi/2} \qquad I2 = -\frac{R^4}{4}$$

I 3. In R and θ direction w/ $\phi = 0$. $-\theta$ direction so

$d\vec{a} = -r\,dr\,d\theta\,\hat{\phi}$ $\quad \vec{v}\cdot d\vec{a} = (-r^2\cos\theta\sin\phi)(-r\,dr\,d\theta)$

$\vec{v}\cdot d\vec{a} = r^3\cos\theta\sin\phi\,dr\,d\theta. = 0.$ Nice!

$I3 = 0$

I4. The orthogonal vector is in $\hat{r}$, so $d\vec{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

$$\vec{v}\cdot d\vec{a} = (R^2\cos\theta)(R^2\sin\theta\,d\theta\,d\phi) = R^4\cos\theta\sin\theta\,d\theta\,d\phi$$

$$I4 = R^4\int_0^{\pi/2}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\,d\phi = R^4\frac{\pi}{2}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta = \frac{\pi R^4}{2}\left[\frac{1}{2}\cos\theta\right]_{\pi/2}^{0}$$

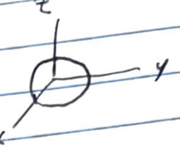
$$I4 = \frac{\pi R^4}{4}$$

$$I_{Total} = I1 + \dots + I4 = \frac{R^4}{4} - \frac{R^4}{4} + 0 + \frac{\pi R^4}{4} = \frac{\pi R^4}{4}$$

$$\boxed{\frac{\pi R^4}{4} = \frac{\pi R^4}{4}}$$

Divergence Theorem works.

1.55] Stoke's Theorem w/ $\vec{v} = ay\hat{x} + bx\hat{y}$ w/ circular path R centered at origin in xy plane.



$$\int_S (\nabla\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$$

$\int_S (\nabla\times\vec{v})\cdot d\vec{a}$. $\quad d\vec{a} = dx\,dy\,\hat{z}$

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ V_x & V_y & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial}{\partial x}bx - \frac{\partial}{\partial y}ay\right)$$

$$= (b-a)\hat{k}$$

$\int_S (b-a)\hat{k}\cdot dx\,dy\,\hat{k} = \int_S (b-a)\,dx\,dy = (b-a)\int_S dx\,dy.$ The area enclosed is a straight up circle so $\int da = A = \pi R^2$

$$\int_S (\nabla\times\vec{v})\cdot d\vec{a} = \pi R^2(b-a)$$

$\oint_P \vec{v}\cdot d\vec{l}$.

$\vec{v}\cdot d\vec{l} = (ay\hat{x} + bx\hat{y})(dx\hat{x} + dy\hat{y}) = ay\,dx + bx\,dy$

~~Do for a quarter circle.~~ $x = R\cos\theta$ and $y = R\sin\theta$ $\quad \theta: 0 \to \frac{\pi}{2}$

$dx = -R\sin\theta\,d\theta \qquad dy = R\cos\theta\,d\theta$

$\vec{v}\cdot d\vec{l} = -aR^2\sin^2\theta\,d\theta + bR^2\cos^2\theta\,d\theta = R^2(b\cos^2\theta - a\sin^2\theta)\,d\theta$

So now... $R^2\int_0^{2\pi}(b\cos^2\theta - a\sin^2\theta)\,d\theta = R^2\left[b\int_0^{2\pi}\cos^2\theta\,d\theta - a\int_0^{2\pi}\sin^2\theta\,d\theta\right]$

$= R^2[b\pi - a\pi] \qquad = \pi R^2(b-a)$

So $\pi R^2(b-a) = \pi R^2(b-a)$.

Stoke's Theorem is verified.

1.56 $\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$ w/ a line integral and triangular path in 1.49.

$\oint \vec{v}\cdot d\vec{l}$. Path 1 is from $(0,0,0) \to (0,1,0)$.

So $\vec{v}\cdot d\vec{l} = yz^2 dy$ But $z=0$, so $\vec{v}\cdot d\vec{l} = 0$.

$I1 = 0$

Path 2 is $(y = -2x+2)$ so $(0,1,0) \to (0,0,-2y+2)$.

$z = -2y+2$

$$\int \vec{v}\cdot d\vec{l} = \int_l yz^2 dy + (3y+z)dz$$

$z = (-2y+2)$ and $dz = -2dy$

$$= \int_l \left[y(-2y+2)^2 dy + (y+2)(-2dy)\right] \quad (2-2y)(2-2y) = 4-4y-4y+4y^2 = 4y^2-8y+4$$

$$= \int_l [4y^3 - 8y^2 + 4y - 2y - 4]dy = \int_1^0 (4y^3 - 8y^2 + 2y - 4)dy = \left[y^4 - \frac{8}{3}y^3 + y^2 - 4y\right]\Big|_1^0$$

$$= \left[\frac{8}{3}y^3 + 4y - y^4 - y^2\right]\Big|_0^1 = \frac{8}{3} + 4 - 1 - 1 = \frac{8}{3} + 2$$

$I2 = \frac{8}{3} + 2$

Path 3. $(0,0,2) \to (0,0,0)$. $\vec{v}\cdot d\vec{l} = \int_l (3y+z)dz = \int_2^0 z\,dz = \frac{z^2}{2}\Big|_2^0 = -\frac{z^2}{2}\Big|_0^2$

$I3 = -2$

$\sum I_j = 0 + \frac{8}{3} + 2 - 2 = \frac{8}{3}$. $\boxed{\oint_l \vec{v}\cdot d\vec{l} = \frac{8}{3}}$

Stoke's Theorem.

$$\int_S (\nabla \times \vec{v})\cdot d\vec{a}. \quad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6 & yz^2 & 3y+z \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}(3y+z) - \frac{\partial}{\partial z}(yz^2)\right)$$

$$-\hat{y}\left(\frac{\partial}{\partial x}(3y+z) - \frac{\partial}{\partial z}(6)\right)$$

$$+\hat{z}\left(\frac{\partial}{\partial x}(yz^2) - \frac{\partial}{\partial y}(6)\right)$$

$$= \hat{x}(3 - 2yz)$$

$d\vec{a} = -dydz\,\hat{x}$

$(\nabla \times \vec{v})\cdot d\vec{a} = (3-2yz)(dydz)$

$= \int_S (3-2yz)dydz$ Can't use area trick since stuff is there.

$\boxed{= \frac{8}{3}.}$

$y: 0 \to 1$ $z: 0 \to -2y+2$

$$= \int_0^1 \int_0^{-2y+2} (3-2yz)dydz = \int^1 (3z - yz^2)\Big|_0^{-2y+2} dy = \int^1 [3(-2y+2) - y(-2y+2)^2]dy$$

$$= \int_0^1 [-6y + 6 - 4y^3 + 8y^2 - 4y]dy = \int_0^1 (-4y^3 + 8y^2 - 10y + 6)dy = \left[-y^4 + \frac{8}{3}y^3 - 5y^2 + 6y\right]\Big|_0^1$$

$= 8/3.$

1.57 Line integral $\vec{v} = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}$

w/ path in 1.50. $\int \vec{v}\cdot d\vec{l}$

I1. $\theta = \frac{\pi}{2}$. $\phi = 0$. $r: 0 \to 1$.

So $\vec{v}\cdot d\vec{l} = r\cos^2\theta dr = 0$. $I1 = 0$.

I2. $r = 1$. $\phi: 0 \to \frac{\pi}{2}$. $\theta = \frac{\pi}{2}$. $\vec{v}\cdot d\vec{l} = (3r)(r\sin\theta)d\phi = 3r^2\sin\theta d\phi$

$= 3r^2 d\phi$

$= 3\int_0^{\pi/2} d\phi = \frac{3\pi}{2}$ $I2 = \frac{3\pi}{2}$ $= 3d\phi$

I3. $\phi = \frac{\pi}{2}$. $r: 1 \to \sqrt{5}$. $\theta = \frac{\pi}{2} \to \arctan(\frac{1}{2}) =$

$y = r\sin\theta$ and $y = 1$, so $r = \frac{1}{\sin\theta}$. $dr = -\frac{\cos\theta}{\sin^2\theta} d\theta$

$\vec{v}\cdot d\vec{l} = r\cos^2\theta\, dr - r\cos\theta\sin\theta\, r d\theta = r\cos^2\theta dr - r^2\cos\theta\sin\theta d\theta$

$= -\frac{\cos^2\theta}{\sin\theta}\left(\frac{\cos\theta}{\sin^2\theta}\right)d\theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}d\theta = -\frac{\cos^3\theta}{\sin^3\theta}d\theta - \frac{\cos\theta}{\sin\theta}d\theta$

$-\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right)d\theta = -\left(\frac{\cos^3\theta + \cos\theta\sin^2\theta}{\sin^3\theta}\right)d\theta$

$= -\frac{\cos\theta}{\sin^3\theta}(\cos^2\theta + \sin^2\theta)d\theta = -\frac{\cos\theta}{\sin^3\theta}d\theta$

$-\int_{\pi/2}^{\arctan(\frac{1}{2})} \frac{\cos\theta}{\sin^3\theta}d\theta$. Proof by Wolfram. $-(-2) = 2$. $I3 = 2$.

I4. $\phi = \frac{\pi}{2}$. $r: \sqrt{5} \to 0$. $\theta = \arctan(\frac{1}{2})$

$\vec{v}\cdot d\vec{l} = r\cos^2\theta dr = \frac{4}{5} r dr$

$= \frac{4}{5}\int_{\sqrt{5}}^{0} r dr = \frac{-2}{5} r^2 \Big|_0^{\sqrt{5}} = -2$

So integral is $0 + \frac{3\pi}{2} + 2 - 2 = \boxed{\frac{3\pi}{2}}$

Now Stoke's Theorem.

$$\int (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint \vec{v}\cdot d\vec{l}$$

$$\vec{v} = r\cos^2\theta\,\hat{r} - r\cos\theta\sin\theta\,\hat{\theta} + 3r\,\hat{\phi}$$

$$\nabla\times\vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta}$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(3r\sin\theta) + \frac{\partial}{\partial\phi}(r\cos\theta\sin\theta)\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\cos^2\theta) - \frac{\partial}{\partial r}(3r^2)\right]\hat{\theta}$$

$$+\frac{1}{r}\left[-\frac{\partial}{\partial r}(r^2\cos\theta\sin\theta) - \frac{\partial}{\partial\theta}(r\cos^2\theta)\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}\left[3r\cos\theta\right]\hat{r} + \frac{1}{r}\left[-6r\right]\hat{\theta}$$

$$+\frac{1}{r}\left[-2r\cos\theta\sin\theta + 2r\cos\theta\sin\theta\right]\hat{\phi}$$

$$\vec{\nabla}\times\vec{v} = 3\cot\theta\,\hat{r} - 6\hat{\theta}$$

$da_1 = -r\,dr\,d\theta\,\hat{\phi}$ $\quad$ $da_2 = -r\sin\theta\,dr\,d\phi\,\hat{\theta}$

$(\nabla\times\vec{v})\cdot da_1 = 0$ Nice. $\quad$ $(\nabla\times\vec{v})\cdot da_2 = 6r\sin\theta\,dr\,d\phi$ w/ $\theta = \frac{\pi}{2}$.

$$= 6r\,dr\,d\phi$$

$$\int_S (\nabla\times\vec{v})\cdot da_2 = \int_0^1\int_0^{\pi/2} 6r\,dr\,d\phi = 3\pi\int_0^1 r\,dr = \frac{3\pi}{2}r^2\Big|_0^1 = \frac{3\pi}{2}.$$

$$\boxed{\int_S (\nabla\times\vec{v})\cdot d\vec{a} = \frac{3\pi}{2}}$$

1.59 $\quad \vec{v} = r^2\sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r^2\tan\theta\,\hat{\phi}$

Volume integral $\qquad$ Figure 1.52

$$\int_V (\nabla\cdot\vec{v})\,d\tau = \oint_S \vec{v}\cdot d\vec{a} \qquad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v_\phi$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(4r^2\cos\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\tan\theta)$$

$$= 4r\sin\theta + \frac{4r}{\sin\theta}(-\sin^2\theta+\cos^2\theta) = 4r\sin\theta - 4r\sin\theta + \frac{4r\cos^2\theta}{\sin\theta} = \frac{4r\cos^2\theta}{\sin\theta}$$

$$= 4r\frac{\cos^2\theta}{\sin\theta}$$

$$\int_V \left(4r\frac{\cos^2\theta}{\sin\theta}\right) r^2\sin\theta\,dr\,d\theta\,d\phi = 4\int_V r^3\cos^2\theta\,dr\,d\theta\,d\phi$$

$$= 4\int_0^R\int_0^{\pi/6}\int_0^{2\pi} r^3\cos^2\theta\,dr\,d\theta\,d\phi = R^4\int_0^{\pi/6}\int_0^{2\pi}\cos^2\theta\,d\theta\,d\phi = 2\pi R^4\int_0^{\pi/6}\cos^2\theta\,d\theta$$

$$= 2\pi R^4\left(\frac{1}{24}\,3\sqrt{3}+2\pi\right) \qquad \boxed{= \frac{\pi R^4}{12}(2\pi+3\sqrt{3})}$$

Surface Integral $\quad \oint_S \vec{v}\cdot d\vec{a} \qquad \theta = \frac{\pi}{6} \quad \phi: 0\to 2\pi \quad r: 0\to R \quad d\vec{a} = r\sin\theta\,d\phi\,dr\,\hat{\theta}$

$$d\vec{a} = \frac{\sqrt{3}}{2}r\,d\phi\,dr\,\hat{\theta}$$

$$\vec{v}\cdot d\vec{a} = (4r^2\cos\theta)\left(\frac{\sqrt{3}}{2}r\,d\phi\,dr\right) = 2\sqrt{3}r^3\cos\theta\,d\phi\,dr = \sqrt{3}r^3\,d\phi\,dr$$

$$\sqrt{3}\int_S r^3\,d\phi\,dr = \sqrt{3}\int_0^R\int_0^{2\pi} r^3\,d\phi\,dr = \sqrt{3}\,2\pi\frac{R^4}{4} = \frac{\sqrt{3}}{2}\pi R^4$$

$$r=R \quad \phi: 0\to 2\pi \quad \theta: 0\to\frac{\pi}{6} \qquad d\vec{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$$

$$\vec{v}\cdot d\vec{a} = (r^2\sin\theta)(r^2\sin\theta\,d\theta\,d\phi) = r^4\sin^2\theta\,d\theta\,d\phi = R^4\sin^2\theta\,d\theta\,d\phi$$

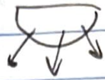
$$= \int_S R^4\sin^2\theta\,d\theta\,d\phi = R^4\int_0^{\pi/6}\int_0^{2\pi}\sin^2\theta\,d\theta\,d\phi = 2\pi R^4\int_0^{\pi/6}\sin^2\theta\,d\theta$$

$$= 2\pi R^4\left(\frac{1}{24}(2\pi-3\sqrt{3})\right) = \pi R^4\left(\frac{1}{12}(2\pi-3\sqrt{3})\right)$$

So $\pi R^4\left(\frac{\sqrt{3}}{2}+\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right) = \frac{\pi R^4}{12}(6\sqrt{3}+2\pi-3\sqrt{3}) \quad \boxed{= \frac{\pi R^4}{12}(2\pi+3\sqrt{3})}$

1.62) $a = \int_S d\vec{a}$ a) Get vector area of bowl radius R.

We have $d\vec{a} = R^2 \sin\theta \, d\theta d\phi \hat{r}$ In $\hat{r}$.



$$a = \int R^2 \sin\theta\cos\theta d\theta d\phi$$

$$a = R^2 \int_0^{\pi/2}\int_0^{2\pi} \sin\theta\cos\theta d\theta d\phi = 2\pi R^2 \int_0^{\pi/2} \sin\theta\cos\theta d\theta \overset{S}{=} 2\pi R^2 \left(\frac{\sin^2\theta}{2}\right)\Big|_0^{\pi/2} \boxed{= \pi R^2}$$

b) $a = 0$ for any closed surface! Using 1.61a corollary where $\int_V (\nabla T) d\tau = \oint_S T d\mathbf{a}$.

This is case where $T = 1$. $\nabla T = 0$ (derivative of constant)!

$\Rightarrow a = 0$.

c) Show that a is the same for all surfaces with the same boundary.

Consider any 2 surfaces with vector areas a_1 and a_2. The vector area of the combined surface is not 0.

d) $a = \frac{1}{2}\oint \vec{r} \times d\vec{l}$ For one of the triangles described, the area would be $\frac{1}{2}(\vec{r} \times d\vec{l})$

So we have $d\vec{a} = \frac{1}{2}(\vec{r} \times d\vec{l})$. The integral serves to add all of that up to a total vector area $\vec{a}$.

e) $\oint (\vec{c}\cdot\vec{r}) d\vec{l} = \vec{a} \times \vec{c}$ Let $T = \vec{c}\cdot\vec{r}$ $\nabla T = \nabla(c\cdot r)$

So we have $\nabla(c\cdot r) = c \times (\nabla \times r) + (c\cdot\nabla)\cdot r$

$\nabla \times r = 0$ because $\vec{r}$ causes no curl. $\nabla T = \nabla(c\cdot r) = (c\cdot\nabla)\cdot r$

$$= \left(c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z}\right)\cdot(x\hat{x} + y\hat{y} + z\hat{z}) = c_x\hat{x} + c_y\hat{y} + c_z\hat{z}$$

However, this is $\vec{c}$ because it is a constant vector. So $\nabla T = \vec{C}$

Then $-\int \nabla T \times d\vec{a} = -\int \vec{c} \times d\vec{a} = -\vec{c} \times \vec{a} \boxed{= \vec{a} \times \vec{c}}$

1.63 a) $\vec{v} = \frac{\hat{r}}{r}$ What is divergence.

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r) = \frac{1}{r^2}.$$

Divergence Theorem.

$$\int_V (\nabla\cdot\vec{v})\,d\tau = \int_S \vec{v}\cdot d\vec{a}$$

The measure we will use is a sphere w/ radius R. (Normal for spherical coordinates).

$$\int_V (\nabla\cdot\vec{v})\,d\tau = \int_V \left(\frac{1}{r^2}\right)(r^2\sin\theta\,dr\,d\theta\,d\phi) = \int_V \sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^R\int_0^\pi\int_0^{2\pi} \sin\theta\,dr\,d\theta\,d\phi = 2\pi R\int_0^\pi \sin\theta\,d\theta = 2\pi R\Big[-\cos\theta\Big]_0^\pi = 2\pi R[1+1]$$

$$\boxed{= 4\pi R}$$

$$\int_S \vec{v}\cdot d\vec{a} = \int_S \left(\frac{1}{r}\right)\cdot(r^2\sin\theta\,d\theta\,d\phi) = R\int_S \sin\theta\,d\theta\,d\phi = R\int_0^\pi\int_0^{2\pi} \sin\theta\,d\theta\,d\phi$$

Same integral as before. $\boxed{= 4\pi R}$

Seems like there is not a delta function because everything worked as it should!

General form of $r^n\hat{r}$ divergence.

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+2}) = \frac{1}{r^2}(n+2)r^{n+1} = \frac{r^{n+1}}{r^2}(n+2)$$

$$= \frac{r^n}{r}(n+2) = r^{n-1}(n+2) = (n+2)r^{n-1}$$

So $\boxed{\nabla\cdot r^n\hat{r} = (n+2)r^{n-1}}$

$$\nabla\times r^n\hat{r} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} \cdot \frac{1}{r\sin\theta}[0] = 0.$$

$$\boxed{\nabla\times r^n\hat{r} = 0}$$

$\boxed{1.64}$ $\quad V \quad \nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})$

$r \to \sqrt{r^2+\epsilon^2}$ and let $\epsilon \to 0$.

$D(r,\epsilon) = -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\epsilon^2}}$ To show as $\epsilon \to 0$, this $\to \delta^3(\vec{r})$.

a) Show that $D(r,\epsilon) = \frac{3\epsilon^2}{4\pi}(r^2+\epsilon^2)^{-5/2}$

$$= -\frac{1}{4\pi}\nabla^2\left((r^2+\epsilon^2)^{-1/2}\right) = -\frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left[-r^3(r^2+\epsilon^2)^{-3/2}\right]$$

$$= \frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3(r^2+\epsilon^2)^{-3/2}\right) = \frac{1}{4\pi r^2}\left[3r^2(r^2+\epsilon^2)^{-3/2} - 3(r^2+\epsilon^2)^{-5/2}r^4\right]$$

$$= \frac{1}{4\pi r^2}\left[\frac{3r^2}{(r^2+\epsilon^2)^{3/2}} - \frac{3r^4}{(r^2+\epsilon^2)^{5/2}}\right] = \frac{3}{4\pi}\left[\frac{1}{(r^2+\epsilon^2)^{3/2}} - \frac{r^2}{(r^2+\epsilon^2)^{5/2}}\right]$$

$$= \frac{3}{4\pi(r^2+\epsilon^2)^{5/2}}(r^2+\epsilon^2-r^2) = \boxed{\frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}}}$$

$$r \to 0 \quad = \frac{3\epsilon^2}{4\pi\epsilon^5} = \boxed{\frac{3}{4\pi\epsilon^3}.\ \text{As } \epsilon \to 0, \text{ this blows up}}$$

$\boxed{\text{As } \epsilon \to 0 \text{ w/ } r \to 0 \text{ b/c } D \propto \epsilon^2 \text{ in numerator.}}$

$$= \int_0^\infty \frac{3\epsilon^2 4\pi r^2}{4\pi(r^2+\epsilon^2)^{5/2}}dr = \frac{3\epsilon^2}{4\pi}\int_0^\infty \frac{4\pi r^2}{(r^2+\epsilon^2)^{5/2}}dr = 3\epsilon^2\int_0^\infty \frac{r^2}{(r^2+\epsilon^2)^{5/2}}dr$$

Multiply by $4\pi R^2$ because surface area

$3\epsilon^2\left(\frac{1}{3\epsilon^2}\right) \boxed{=1}$ Integral done in Sagemath.

These all are properties that $\delta^3(\vec{r})$ has, so it is safe to say that I hath been convinced.