Solutions for Homework 5

Dr. Jordan Hanson - Whittier College Dept. of Physics and Astronomy

April 28, 2022

1 Problem 5.4

Suppose that the magnetic field in some region has the form

$$\mathbf{B} = kz\hat{\mathbf{x}} \tag{1}$$

(where k is a constant). Find the force on a square loop (side a), lying in the yz-plane and centered at the origin, if it arries a current I, flowing counterclockwise, when you look down the x axis.

Using $\mathbf{F} = I\mathbf{L} \times \mathbf{B}$, the Lorentz force for current in a magnetic field, we find

$$\mathbf{F}_{\text{net}} = Ika^2\hat{\mathbf{z}} \tag{2}$$

2 Problem 5.7

For a configuration of charges and currents confined within a volume $\mathcal{V}$, show that

$$\int \mathbf{J} d\tau = \frac{d\mathbf{p}}{dt} \tag{3}$$

where $\mathbf{p}$ is the total dipole moment. [Hint: evaluate $\int_{\mathcal{V}} \nabla \cdot (x\mathbf{J}) d\tau$].

Folliwing the hint, keeping in mind that $\mathcal{V}$ contains all currents and charges (so none penetrate the surface $\mathcal{S}$ enclosing $\mathcal{V}$):

$$\nabla \cdot (x\mathbf{J}) = x(\nabla \cdot \mathbf{J}) + \mathbf{J} \cdot (\nabla x) = -x\frac{\partial \rho}{\partial t} + J_x \tag{4}$$

$$\int_{\mathcal{V}} \nabla \cdot (x\mathbf{J}) d\tau = \int_{\mathcal{V}} \left(-x\frac{\partial \rho}{\partial t} + J_x\right) d\tau \tag{5}$$

$$\int_{\mathcal{V}} \nabla \cdot (x\mathbf{J}) d\tau = \oint_{\mathcal{S}} (x\mathbf{J}) \cdot d\mathbf{a} = 0 \tag{6}$$

$$\int_{\mathcal{V}} \left(-x\frac{\partial \rho}{\partial t} + J_x\right) d\tau = 0 \tag{7}$$

$$\int_{\mathcal{V}} x\frac{\partial \rho}{\partial t} d\tau = \int_{\mathcal{V}} J_x d\tau \tag{8}$$

$$\int_{\mathcal{V}} y\frac{\partial \rho}{\partial t} d\tau = \int_{\mathcal{V}} J_y d\tau \tag{9}$$

$$\int_{\mathcal{V}} z\frac{\partial \rho}{\partial t} d\tau = \int_{\mathcal{V}} J_z d\tau \tag{10}$$

Combine the same arguement used for x with the copies of it for y and z. Multiply by the corresponding unit vector on both sides of each each equation, and sum. Then, switch the order of the time-derivative with the volume integration:

$$\int_{\mathcal{V}} \mathbf{J} d\tau = \frac{d}{dt}\int \mathbf{r}\rho d\tau = \frac{d\mathbf{p}}{dt} \tag{11}$$

In words, the volume integration of all current densities in a closed space is the time-derivative of the dipole moment of all charge.