Electromagnetic Theory Mid-Term!

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1 (a) $(A\cdot\nabla)B \left[\left(\sum_{i=1}^{3} \delta_i A_i\right)\cdot\left(\sum_{j=1}^{3}\delta_j \frac{\partial}{\partial x_j}\right)\right]\left(\sum_{k=1}^{3}\delta_k B_k\right)$

$$= \left[\sum_{i=1}^{3}\sum_{j=1}^{3}(\delta_i\cdot\delta_j)A_i\frac{\partial}{\partial x_j}\right]\left(\sum_{k=1}^{3}\delta_k B_k\right)$$

$$= \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}A_i\frac{\partial}{\partial x_j}\right)\left(\sum_{k=1}^{3}\delta_k B_k\right)$$

$$= \sum_{i=1}^{3}\sum_{k=1}^{3}\delta_k A_i\frac{\partial B_k}{\partial x_i}$$

$$= \delta_1 A_1\frac{\partial B_1}{\partial x_1} + \delta_2 A_1\frac{\partial B_2}{\partial x_1} + \delta_3 A_1\frac{\partial B_3}{\partial x_1}$$

$$+ \delta_1 A_2\frac{\partial B_1}{\partial x_2} + \delta_2 A_2\frac{\partial B_2}{\partial x_2} + \delta_3 A_2\frac{\partial B_3}{\partial x_3}$$

$$+ \delta_1 A_3\frac{\partial B_1}{\partial x_3} + \delta_2 A_3\frac{\partial B_2}{\partial x_3} + \delta_3 A_3\frac{\partial B_3}{\partial x_3}$$

$$= \boxed{\hat{x}\left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right) + \hat{y}\left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right) + \hat{z}\left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right)}$$

1) b) $(\hat{r}\cdot\nabla)\hat{r} = \frac{r}{|r|}\cdot\left(\frac{\partial}{\partial r}\right)\hat{r} = \boxed{0}$! $\hat{r}$ is constant because it is a basis vector

so $\boxed{(\hat{r}\cdot\nabla)\hat{r} = 0}$

c) $F = (p\cdot\nabla)E$, $p = qd = qd\hat{x}$

& $V(r) = V_0 r^2 + V_1$

$\vec{p}\cdot\nabla = qd\frac{\partial}{\partial x}$ $\vec{F} = (qd\frac{\partial}{\partial x})\vec{E}$, $\vec{E} = -\nabla V = -\frac{\partial}{\partial r}(V_0 \vec{r}^2 + V_1)$

$\vec{F} = (qd\frac{\partial}{\partial x})(-2V_0\vec{r})$ $= -2qd\frac{\partial}{\partial x}V_0(r_x\hat{x} + r_y\hat{y} + r_z\hat{z})$

$r = r_x\hat{x} + r_y\hat{y} + r_z\hat{z}$

$= \boxed{-2qdV_0\frac{\partial r_x}{\partial x}\hat{x}}$

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$$J = \int_V e^{-r} \left(\nabla \cdot \frac{\hat{r}}{r^2} \right) d\tau \quad , \quad \nabla \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^3(r)$$

$$J = \int_V e^{-r} \left(4\pi \delta^3(r) \, d\tau \right)$$ over all space

from delta function

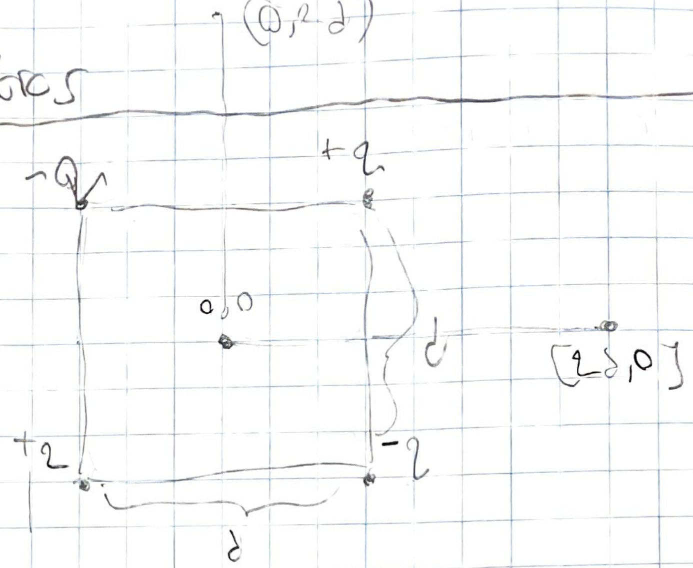
$$J = 4\pi \int_{-\infty}^{\infty} e^{-r} \delta^3(r) \, d\tau = 4\pi e^0$$

$$\boxed{= 4\pi}$$ super neat!

[if this is correct!]

Electrostatics

2.1



a) for (0,0), due to superpositron $E = 0$, as all charges "cancel", same dist, opposite charge (treating as 2 dipoles)

b) $V_{dip}(r,\theta) = \dfrac{\hat{r} \cdot p}{4\pi\varepsilon_0 r^2} = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$

taking negative gradient for field

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\theta = -\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} \ [\phi \text{ independent}] \ E\phi = 0$$

$$E_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$p_1 = \tfrac{1}{2} q d \hat{x} + \left(-\tfrac{1}{2} q \cdot d \cdot -\hat{z}\right) = q d \hat{z}$$

$p_2 = q d \hat{x}$ as well

$$E_{tot} = E_{dip\,1} + E_{dip\,2}$$

$$E_{dip\,1} = \frac{qd}{4\pi\varepsilon_0(3.375d)}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$E_{dip\,2} = \frac{qd}{4\pi\varepsilon_0(15.625d)}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$E_{tot} = \left[\frac{q}{4\pi\varepsilon_0(3.375)} + \frac{q}{4\pi\varepsilon_0(15.625)}\right]\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\boxed{E_{tot} \approx \frac{5.63q}{62.5\pi\varepsilon_0}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$

for 0, 2d, it will be the same, it will just be in $\hat{y}$ direction, but all else will be the same in spherical coords by symmetry

so again,

$$E_{tot} \cong \frac{5.63q}{62.5\pi\varepsilon_0}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Electrostatics

(2) $V(r) = A\dfrac{e^{-\lambda r}}{r}$ $\quad E = -\nabla\left(A\dfrac{e^{-\lambda r}}{r}\right)$

$$= -\frac{d}{dr}\left(A\frac{e^{-\lambda r}}{r}\right)$$

$$= -\left(\frac{-A\lambda r^{2} e^{-\lambda r} - Ae^{-\lambda r}}{r^{2}}\right)$$

$$\vec{E} = Ae^{-\lambda r}\left(\frac{\lambda r+1}{r^{2}}\right)\hat{r}$$

$$\rho = \varepsilon_0 A\, 4\pi \delta^3 \hat{r} - \frac{\lambda^2 e^{-\lambda r}}{r}$$

$\nabla \cdot E = \dfrac{\rho}{\varepsilon}$ ← Maxwell's Eqn

$\nabla \cdot E = \left(Ae^{-\lambda r}(\lambda r+1)\dfrac{\hat{r}}{r}\right)$

$+ A\dfrac{\hat{r}}{r^2} \cdot \nabla\left[e^{-\lambda r}(1+\lambda r)\right]$

1st: $Ae^{-\lambda r}(1+\lambda r)\, 4\pi\delta^3(r) = A(4\pi\,\delta^3(r))$

2nd: $A\dfrac{\hat{r}}{r^2} \cdot \dfrac{\partial}{\partial r}\left[e^{-\lambda r}(1+\lambda r)\right]\hat{r} = A\dfrac{\hat{r}}{r^2} \cdot \left[-\lambda e^{-\lambda r}(1+\lambda r) + e^{-\lambda r}\lambda\right]\hat{r}$

$= A\dfrac{\hat{r}}{r^2}\left[-\lambda^2 r e^{-\lambda r}\right]\hat{r} = \boxed{-A\dfrac{\lambda^2}{r}e^{-\lambda r}}$

$$\boxed{\rho = \varepsilon_0 A\left(4\pi\delta^3(r) - \frac{\lambda^2}{r}e^{-\lambda r}\right)}$$

$Q = \int \rho\, d\tau = 4A\varepsilon_0\pi \int \delta^3(\vec{r})\, d\tau - A\varepsilon_0\lambda^2 \int_0^\infty \dfrac{e^{-\lambda r}}{r}(4\pi r^2)$

$= 4A\pi\varepsilon_0 - 4\pi A\varepsilon_0\lambda^2\left(\dfrac{1}{\lambda^2}\right) = 0 \Rightarrow \boxed{Q = 0}$

Electrostatics

(3) a) $EA = \frac{q}{\varepsilon_0}$ Charge density λ $A = 2\pi s l$, $E(2\pi s l) = \frac{\lambda l}{\varepsilon_0}$

$$\vec{E} = \frac{\lambda}{2\pi s \varepsilon_0}\hat{s}$$

pt chg q, with mass m

b) $F = ma$, $ma = qE$

$$ma = \frac{q\lambda}{2\pi s \varepsilon_0}, \quad \underbrace{a}_{\frac{d^2 s}{dt^2}} = \frac{q\lambda}{2\pi m s \varepsilon_0}, \quad \underbrace{v}_{\frac{ds}{dt}} = \int \frac{q\lambda}{2\pi m s \varepsilon_0}\, dt$$

$$= \frac{q\lambda}{2\pi m \varepsilon_0}\int \frac{1}{s}\, dt = \frac{q\lambda}{2\pi m \varepsilon_0}\frac{t}{s} = v = \frac{ds}{dt}$$

$$s(t) = \frac{q\lambda}{2\pi m \varepsilon_0}\int \frac{t}{s}\, dt = \boxed{\frac{q\lambda}{4\pi m \varepsilon_0 s}t^2 = s(t)}$$

POTENTIALS

① (a) $\sigma(\theta) = \frac{\varepsilon_0}{2R} \sum_{l=0}^{\infty} (2l+1)^2 C_l P_l(\cos\theta)$

where $C_l = \int_0^{\pi} V_0(\theta) P_l(\cos(\theta)) \sin\theta \, d\theta$

$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$

when $r<R \Rightarrow V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l \cos\theta$

when $r>R$ $V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l \cos\theta$

$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

$B_l = A_l R^{2l+1}$

$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} \right)_{r=R} = \frac{\sigma_0(\theta)}{\varepsilon_0}$

$\sum_{l=0}^{\infty} (l+1) \frac{B_l}{R^{l+2}} P_l \cos\theta - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l \cos\theta = \frac{\sigma_0}{\varepsilon_0}$

$\therefore B_l = A_l R^{2l+1}$

$\frac{\sigma(\theta)}{\varepsilon_0} = \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l \cos\theta$

$\therefore A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$

$\frac{\sigma(\theta)}{\varepsilon_0} = \sum_{l=0}^{\infty} \frac{(2l+1)^2}{2R} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$

$\sigma\theta = \frac{\varepsilon_0}{2R} \sum_{l=0}^{\infty} (2l+1)^2 C_l P_l(\cos\theta)$

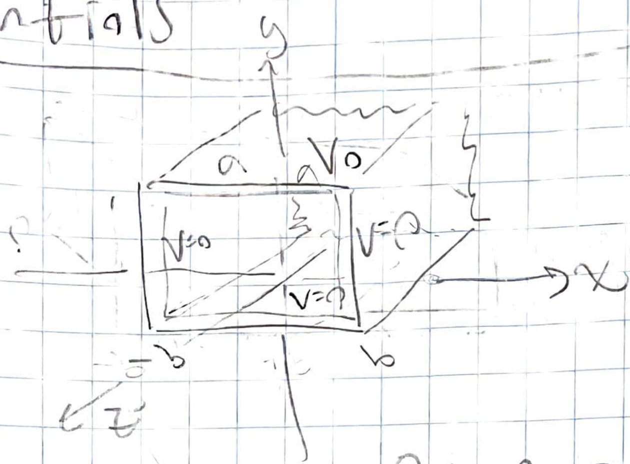
Potentials

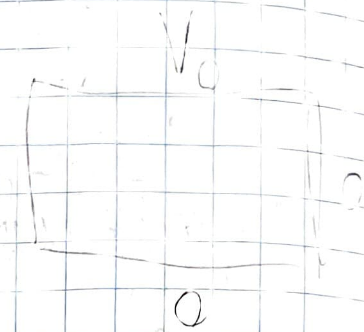
(b) produce results for $\sigma(\theta)$ w $V_0(\theta) = P_2(\cos\theta)$

$$\sigma(\theta) = \frac{\varepsilon_0}{2R} \sum_{l=0}^{\infty} \frac{(2l+1)^2}{2R} \int_0^{\pi} P_2(\cos\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

Potentials

2





① $V=0$ @ $y=0$
② $V=0$ @ $x=b$
③ $V=0$ @ $x=-b$
④ $V=V_0$ @ $y=a$

$$Y\frac{\partial^2 V}{\partial x^2} + X\frac{\partial^2 V}{\partial y^2} = 0$$

Need $X(x)$, $Y(y)$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

from ...

condit

$$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda, \quad \lambda + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \qquad \frac{d^2X}{dx^2} = \lambda X, \quad \frac{d^2Y}{dy^2} = -\lambda Y$$

letting $\lambda = -k^2$ $\quad \frac{d^2X}{dx^2} = -k^2 X$, $X(x) = A\cos(kx) + B\sin(kx)$

boundary condit 2 means $X(b) = 0 = A$,

$X(x) = B\sin(kx)$, $x=-b$, $X(-b) = 0 = B\sin(ka)$

$$\boxed{X(x) = B\sin\left(\frac{n\pi}{a}x\right)} \qquad \lambda_{now} = -\frac{n^2\pi^2}{a}$$

$$\frac{d^2Y}{dy^2} = -\lambda Y = k^2 Y, \quad Y(y) = Ce^{ky} + De^{-ky}$$

$$V(x,y) = B\sin\left(\frac{n\pi}{a}x\right)\left[C_1 e^{\frac{n\pi}{a}y} + De^{-\frac{n\pi}{a}y}\right.$$

$$V(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right)\left[C_n e^{\frac{n\pi}{a}y} + D_n e^{-\frac{n\pi}{a}y}\right]$$

$$V(x,0) = 0 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right)[C_n + D_n]$$

$$C_n = -D_n$$

$$V(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) C_n \overbrace{\left[e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right]}^{\approx \sinh}$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) \underbrace{C_n \widetilde{\sinh}\left(\frac{n\pi}{a}y\right)}$$

Redefining as K_0, all constants here

$$V_0 = \sum_{n=1}^{\infty} K_0 \sin\left(\frac{n\pi}{a}x\right)$$

$$b_n = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{n\pi}{a}x\right)dx \quad \begin{cases} \frac{4}{n\pi}V_0 & n \text{ odd} \\ 0 & \text{when } n \text{ even} \end{cases}$$

$$C_n = \frac{b_n}{\sinh\left(\frac{n\pi}{a}b\right)} = \begin{cases} \frac{4V_0}{n\pi \sinh\left(\frac{n\pi}{a}b\right)} & \text{odd} \\ 0 & \text{even} \end{cases}$$

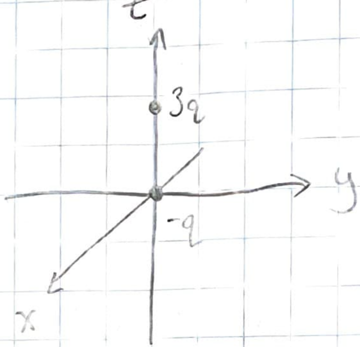
$$\boxed{V(x,y) = \sum_{n=1}^{\infty} \frac{4}{n\pi} V_0 \sin\left(\frac{n\pi}{a}x\right) \frac{\sinh\left(\frac{n\pi}{a}y\right)}{\sinh\left(\frac{n\pi}{a}b\right)}}$$

Yay!

POTENTIALS

3

pt. 1



Mono. Moment =

$Q = 2q$

Dipole Moment

$p = [3qa\hat{z} + -q(0)]$

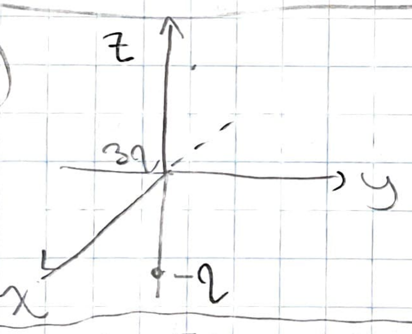
$p = 3qa\hat{z}$

$V(r) = V_{\text{Mono. moment}} + V_{\text{Dipole moment}}$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} + \frac{1}{4\pi\varepsilon_0}\frac{p\cdot\hat{r}}{r^2}$$

$$\boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right]}$$

b



Monopole moment

$Q = 2q$

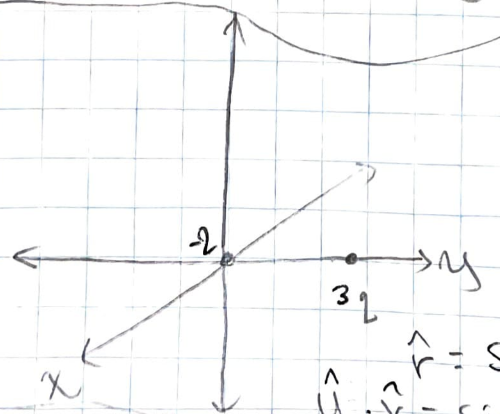
Dipole Moment

$p = [3q(0) + -qa\hat{z}]$

$p = -qa\hat{z}$, $p = qa\hat{z}$

$$\boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right]}$$

c



$Q = 2q$

$p = [3qa\hat{y} + -q(0)]$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{1}{4\pi\varepsilon_0}\frac{p\cdot\hat{r}}{r^2}\right)$$

$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$\hat{y}\cdot\hat{r} = \sin\theta\sin\phi$

$$\boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right]}$$