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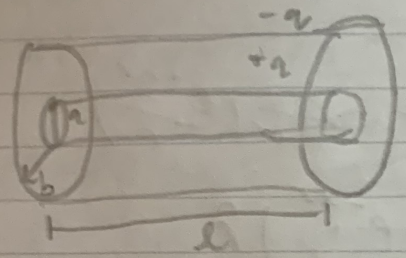
ET Homework #3

2.43, 2.50, 3.1, 3.3, 3.13, 3.14, 3.15

2.43)



$E = \frac{q}{2\pi s l \epsilon_0}$ when $a < s < b$

$$V(a) - V(b) = -\int_b^a \left(\frac{q}{2\pi s l \epsilon_0}\right) ds$$

$$= -\frac{q}{2\pi l \epsilon_0} \left[\log s\right]_b^a = -\frac{q}{2\pi l \epsilon_0}\left(\log(a) - \log(b)\right)$$

$$= \frac{q}{2\pi l \epsilon_0} \ln\left(\frac{b}{a}\right)$$

$V = \frac{q}{C}$

$C = \frac{q}{V}$ $\qquad C = \frac{\cancel{q}}{\left(\frac{\cancel{q}}{2\pi l \epsilon_0} \ln\left(\frac{b}{a}\right)\right)} = \frac{2\pi l \epsilon_0}{\ln\left(\frac{b}{a}\right)}$

$$\boxed{\frac{C}{l} = \frac{2\pi \epsilon_0}{\ln\left(\frac{b}{a}\right)}}$$

2.50) a) $V(r) = A\frac{e^{-\lambda r}}{r}$

$$E = -\nabla V$$
$$= -\frac{\partial}{\partial r}\left(A\frac{e^{-\lambda r}}{r}\right) = -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)$$
$$= -A\left(\frac{-r\lambda e^{-\lambda r} - e^{-\lambda r}}{r}\right)\hat{r}$$
$$= A\left(r\lambda e^{-\lambda r} + e^{-\lambda r}\right)\frac{\hat{r}}{r^2}$$
$$E = Ae^{-\lambda r}(r\lambda + 1)\frac{\hat{r}}{r^2}$$

$\nabla \cdot E = \frac{1}{\varepsilon_0}\rho$ $\qquad \rho = \varepsilon_0 \nabla \cdot E$

$$\rho = \varepsilon_0 \nabla \cdot \left(Ae^{-\lambda r}(r\lambda+1)\frac{\hat{r}}{r^2}\right)$$
$$= \varepsilon_0\left(\left(Ae^{-\lambda r}(r\lambda+1)\right)\left(\nabla \cdot \frac{\hat{r}}{r^2}\right) + \left(\frac{\hat{r}}{r^2}\right)\nabla\left(Ae^{-\lambda r}(r\lambda+1)\right)\right)$$

$$\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(r)$$

$$\left(e^{-\lambda r}(r\lambda+1)\right)\left(4\pi\delta^3(r)\right) = 4\pi\delta^3(r)$$
$$= \varepsilon_0\left(A4\pi\delta^3(r) + \left(\frac{\hat{r}}{r^2}\right)\nabla\left(Ae^{-\lambda r}(r\lambda+1)\right)\right)$$

$$\nabla\left(Ae^{-\lambda r}(r\lambda+1)\right) = \hat{r}\frac{\partial}{\partial r}\left(Ae^{-\lambda r}(r\lambda+1)\right)$$
$$= \hat{r}A\frac{\partial}{\partial r}\left(e^{-\lambda r}(r\lambda+1)\right) = \hat{r}A\left((r\lambda+1)\frac{\partial}{\partial r}(e^{-\lambda r}) + e^{-\lambda r}\frac{\partial}{\partial r}(r\lambda+1)\right)$$
$$= \hat{r}A\left((r\lambda+1)(-\lambda)e^{-\lambda r} + e^{-\lambda r}(\lambda)\right)$$
$$= \hat{r}A\left(-\lambda(r\lambda+1)e^{-\lambda r} + \lambda e^{-\lambda r}\right)$$
$$= \hat{r}A\left(\lambda e^{-\lambda r} - \lambda e^{-\lambda r}(r\lambda+1)\right) = \hat{r}A\left(-\lambda e^{-\lambda r}(r\lambda+1) + \lambda e^{-\lambda r}\right)$$
$$= \hat{r}A\left(\lambda e^{-\lambda r}(1 - r\lambda - 1)\right) = \hat{r}A\left(\lambda e^{-\lambda r}(1-(r\lambda+1))\right)$$
$$= \hat{r}A\left(\lambda e^{-\lambda r}(-r\lambda)\right) = \hat{r}A\left(-\lambda^2 r e^{-\lambda r}\right)$$

3.1) $V_{ave} = V_{center} + \frac{Q}{4\pi\varepsilon_0 R}$

$V = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}$

$r^2 = z^2 + R^2 - 2Rz\cos\theta$

$\sqrt{r^2} = \sqrt{z^2 + R^2 - 2Rz\cos\theta}$

$V = \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{z^2+R^2-2Rz\cos\theta}}$

$V_{ave} = \frac{\int V \cdot da}{4\pi R^2}$ $\quad da = R^2 \sin\theta \, d\theta \, d\phi$

$V_{ave} = \frac{1}{4\pi R^2} \frac{q}{4\pi\varepsilon_0} \int (z^2+R^2-2Rz\cos\theta)^{-1/2} R^2 \sin\theta \, d\theta \, d\phi$

$= \frac{q}{4\pi\varepsilon_0} \frac{1}{2zR} \sqrt{z^2+R^2-2zR\cos\theta}\Big|_0^{\pi}$

$= \frac{1}{4\pi\varepsilon_0} \frac{q}{R}$

$V_{center} = V_{ave}$

$$\boxed{V_{ave} = V_{center} + \frac{1}{4\pi\varepsilon_0} \frac{Q_{enc}}{R}}$$

3.13 $c_n(\text{cont}) = \frac{2V_0}{n\pi}\left\{1+(-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right\}$

if $n = \text{odd}$, $c_n = 0$

if $n = 4, 8, 12, \ldots$, $c_n = 1+1-2(1) = 0$

if $n = 2, 6, 10, \ldots$, $c_n = 1+1-2(-1) = 4$

$c_n = \frac{8V_0}{n\pi}$, $n = 2, 6, 10, 14, \ldots$

$V(x,y) = \sum_{n=2,6,10\ldots} c_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$

$$\boxed{V(x,y) = \frac{8V_0}{\pi} \sum_{n=2,6,10\ldots}^{\infty} \frac{1}{n} \cdot e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)}$$

3.14) $V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$

$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$

$\sigma(y) = -\varepsilon_0 \left(\frac{\partial V}{\partial x}\right)\Big|_{x=0}$

$\sigma(y) = -\varepsilon_0 \frac{\partial}{\partial x}\left\{\frac{4V_0}{\pi} \sum \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)\right\}\Big|_{x=0}$

$= -\varepsilon_0 \frac{4V_0}{\pi} \sum \frac{1}{n}\left(-\frac{n\pi}{a}\right) e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)\Big|_{x=0}$

$$\boxed{= \frac{4\varepsilon_0 V_0}{a} \sum_{n=1,3,5\ldots}^{\infty} \sin\left(\frac{n\pi y}{a}\right)}$$

3.3) a) $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$

$V = V(r) \Rightarrow \frac{\partial V}{\partial\theta} = \frac{\partial V}{\partial\phi} = 0$

$r^2 \frac{dV}{dr} = C_1 \Rightarrow V = \int dV = \int \frac{C_1}{r^2} dr$

$\boxed{= -\frac{C_1}{r} + C_2}$

b) $\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial \phi^2} = 0$

$V = V(s) \Rightarrow \frac{\partial V}{\partial\phi} = \frac{\partial V}{\partial z} = 0$

$s \frac{dV}{ds} = C_3 \Rightarrow V = \int dV = \int \frac{C_3}{s} ds$

$\boxed{V = C_3 \ln(s) + C_4}$

3.13) $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad V(x,y) = X(x)Y(y)$

$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$

$C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$

$V(0,y) = \begin{cases} +V_0 & 0 < y < a/2 \\ -V_0 & a/2 < y < a \end{cases}$

$C_n = \frac{2}{a}\left[\int_0^{a/2} V_0 \sin\left(\frac{n\pi y}{a}\right)dy - \int_{a/2}^{a} V_0 \sin\left(\frac{n\pi y}{a}\right)dy\right]$

$C_n = \frac{2V_0}{\cancel{a}} \cdot \frac{\cancel{a}}{n\pi}\left[-\cos\left(\frac{n\pi y}{a}\right)\right]\Big|_0^{a/2} + \left[\cos\left(\frac{n\pi y}{a}\right)\right]\Big|_{a/2}^{a}$

$\int \sin\left(\frac{n\pi y}{a}\right)dy = \frac{a}{n\pi} \cdot \cos\left(\frac{n\pi y}{a}\right)$

$= \frac{2V_0}{n\pi}\left\{-\cos\left(\frac{n\pi}{2}\right) + \cos(0) + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right\}$

$= \frac{2V_0}{n\pi}\left\{1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right\}$

2.50 cont) $\left(\frac{\hat{r}}{r^2}\right)\cdot\nabla\left(Ae^{-\lambda r}(r\lambda+1)\right)$

$= \frac{1}{r^2}\left(A(-\lambda^2 r e^{-\lambda r}\right) = -\frac{A\lambda^2 e^{-\lambda r}}{r}$

$\rho = \varepsilon_0\left(A4\pi\delta^3(r) - \frac{A\lambda^2 e^{-\lambda r}}{r}\right)$

$\boxed{= A\varepsilon_0\left(4\pi\delta^3(r) - \frac{\lambda^2 e^{-\lambda r}}{r}\right)}$

b) $Q = \int \rho\, d\tau$

$= \int A\varepsilon_0\left(4\pi\delta^3(r) - \frac{\lambda^2 e^{-\lambda r}}{r}\right) d\tau$

$= \int A\varepsilon_0 4\pi\delta^3(r)\, d\tau - \int A\varepsilon_0 \frac{\lambda^2 e^{-\lambda r}}{r}\, d\tau$

$= A\varepsilon_0 4\pi \int \delta^3(r)\, d\tau - A\varepsilon_0\lambda^2 \int \frac{e^{-\lambda r}}{r}\, d\tau$

$= A\varepsilon_0 4\pi - A\varepsilon_0\lambda^2 \int \frac{e^{-\lambda r}}{r}(4\pi r^2\, dr)$

$= A\varepsilon_0 4\pi - 4\pi A\varepsilon_0\lambda^2 \int_0^\infty r e^{-\lambda r}\, dr$

$= A\varepsilon_0 4\pi - 4\pi A\varepsilon_0\lambda^2 \left(-\frac{re^{-\lambda r}}{\lambda} - \frac{e^{-\lambda r}}{\lambda^2}\right|_0^\infty$

$= A\varepsilon_0 4\pi - 4\pi A\varepsilon_0 \cancel{\lambda^2}\left(\frac{1}{\cancel{\lambda^2}}\right)$

$= \boxed{0}$

3.15) a) $\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0$

$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$

$0 = (Ae^{kx} + Be^{-kx})D$

$D = 0$

$0 = (A+B)C\sin(ky)$

$A = -B$

$V(x,y) = AC(e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}})\sin(\frac{n\pi y}{a})$

$= 2AC\sinh(\frac{n\pi x}{a})\sin(\frac{n\pi y}{a})$

$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh(\frac{n\pi x}{a})\sin(\frac{n\pi y}{a})$

$V_0(y) = \sum_{n=1}^{\infty} C_n \sinh(\frac{n\pi b}{a})\sin(\frac{n\pi y}{a})$

$C_n\sinh(\frac{n\pi b}{a}) = \frac{2}{a}\int_0^a V_0(y)\sin(\frac{n\pi y}{a})dy$

$$\boxed{C_n = \frac{2}{a\sinh(\frac{n\pi b}{a})}\int_0^a V_0(y)\sin(\frac{n\pi y}{a})dy}$$

b) $C_n = \frac{2}{a\sinh(\frac{n\pi b}{a})} V_0 \int_0^a \sin(\frac{n\pi y}{a})dy$

$V_0(y) = V_0$

$V_0(y) = \frac{2V_0}{a\sinh(\frac{n\pi b}{a})}\frac{a}{n\pi}[-\cos(\frac{n\pi y}{a})]\Big|_0^a$

$= \frac{2V_0}{a\sinh(\frac{n\pi b}{a})}\frac{a}{n\pi}[-\cos(n\pi)+1]$

n is even: 0

n is odd: 2

$C_n = \frac{4V_0}{n\pi\sinh(\frac{n\pi b}{a})}$

$$\boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5\ldots}\frac{\sinh(\frac{n\pi x}{a})\sin(\frac{n\pi y}{a})}{n\sinh(\frac{n\pi b}{a})}}$$