HW # 4 # 4.10, 4.14, 4.15, 4.18, 4.26, 4.35

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4.10) A sphere of radius R carries a polarization $\vec{P}(\vec{r}) = k\vec{r}$ where k is a constant and $\vec{r}$ is a vector from the center.

(a) Calculate the bound charges σ_b and ρ_b

$P(\vec{r}) = k\vec{r} = kr\hat{r}$ $\qquad \hat{r} = \frac{\vec{r}}{r} \qquad \vec{r} = r\hat{r}$

$\sigma_b = P \cdot \hat{n}$

$= kR\hat{r} \cdot \hat{n}$

$\boxed{\sigma_b = kR}$

$\rho_b = -\nabla \cdot P$

$= \frac{-1}{r^2}\frac{\partial}{\partial r}(r^2 kr)$

$= \frac{-1}{r^2} 3r^2 k$ $\quad \boxed{\rho_b = -3k}$

(b) Find the field inside and outside the sphere.

$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$ $\qquad Q_{enc} = \rho(\text{volume of sphere})$

$E(4\pi r^2) = \frac{1}{\varepsilon_0}(-3k \frac{4}{3}\pi r^3)$ $\qquad = -3k \frac{4}{3}\pi r^3$

$E(4\pi r^2) = \frac{1}{\varepsilon_0}(-4k\pi r^3)$

$E = \frac{1}{\varepsilon_0} kr$

$\boxed{\vec{E}_{in} = \frac{kr}{\varepsilon_0}\hat{r}}$

Outside: $Q_{total} = \sigma(\text{surface area of sphere}) + \rho(\text{volume of sphere})$

$= kR(4\pi R^2) + (-3k)\frac{4}{3}\pi R^3$

$\oint \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0}$ $\qquad = 4\pi R^3 k - 4\pi R^3 k$

$EA = \frac{0}{\varepsilon_0}$ $\qquad = 0$

$\boxed{E_{out} = 0}$

4.14) When you polarize a neutral dielectric, the charge moves a bit, but the total remains the same. Prove from eqs. 4.11 & 4.12 that the total bound charge vanishes.

$\sigma_b = \vec{P} \cdot \hat{n}$ $\quad \rho_b = -\nabla \cdot \vec{P}$

$Q_{total} = \oint_S \sigma_b \, da + \int_V \rho_b \, d\tau$

$= \oint_S (\vec{P} \cdot \hat{n}) \cdot da + \int_V (-\nabla \cdot \vec{P}) \cdot d\tau$

$= \oint_S \vec{P} \cdot da - \int_V (\nabla \cdot \vec{P}) \, d\tau$

Divergence theorem: $\int(\nabla \cdot A)\, d\tau = \oint A \cdot da$

$= \oint_S \vec{P} \cdot da - \oint_S \vec{P} \cdot da$

$= 0$

4.15) Thick spherical (inner radius a, outer radius b) is made of dielectric material with polarization $\vec{P}(r) = \frac{k}{r}\hat{r}$ where k is constant, r is dist. from center

(a) Locate all bound charge + use Gauss's law to calculate its field

$$\sigma_b = \vec{P}\cdot\hat{n} \qquad \hat{n} = \hat{r}$$
$$= \frac{k}{r}\hat{r}\cdot\hat{n}$$
$$\sigma_b = \frac{k}{r}$$

$$\rho_b = -\nabla\cdot P = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right)$$
$$\rho_b = -\frac{k}{r^2}$$

$E = \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{r^2}\hat{r}$ $\quad Q_{enc} = 0$ when $r > b$ or $r < a$ $\rightarrow E = 0$ when $r > b$ or $r < a$

No free charges here so $\rho = \rho_b$

$$Q_{enc} = \frac{k}{a}(4\pi a^2) + \int_a^r \left(\frac{-k}{r^2}\right) 4\pi r^2\, dr \quad \text{when } a < r < b$$
$$= 4\pi k a - \int_a^r 4\pi k\, dr = 4\pi k a - 4\pi k(r-a) = 4\pi k a - 4\pi k r - 4\pi k a$$

$Q_{enc} = -4\pi k r$ when $a < r < b$

$E = \frac{1}{4\pi\varepsilon_0}\frac{-4\pi k r}{r^2}\hat{r}$ when $a < r < b$

$E = \frac{-k}{\varepsilon_0 r}\hat{r}$

$$\boxed{E \begin{cases} = 0 & \text{when } r > b \\ = 0 & \text{when } r < a \\ = \frac{-k}{\varepsilon_0 r}\hat{r} & \text{when } a < r < b \end{cases}}$$

(b) Find $\vec{D}$ then get $\vec{E}$

$$\oint \vec{D}\cdot d\vec{a} = Q_{enc} = 0 \rightarrow \vec{D} = 0$$
$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$
$$0 = \varepsilon_0\vec{E} + \vec{P}$$
$$\vec{E} = -\frac{\vec{P}}{\varepsilon_0} = \frac{-k}{r\varepsilon_0}\hat{r}$$

$$\boxed{E \begin{cases} = 0 & \text{when } r < a \\ = 0 & \text{when } r > b \\ = \frac{-k}{r\varepsilon_0}\hat{r} & \text{when } a < r < b \end{cases}}$$

4.18) Space between plates of parallel-plate capacitor filled w/ two slabs of linear dielectric material. Each slab has thickness a so dist. between plates is 2a. Slab 1 has dielectric constant of 2, slab 2 has const. of 1.5. Free charge density of top plate is σ, bottom is $-\sigma$

(a) Find electric displacement $\vec{D}$ in each slab

$\oint \vec{D} \cdot d\vec{a} = Q_{enc}$ $\qquad$ $\oint \vec{D}_2 \cdot d\vec{a} = Q_{enc}$

$DA = \sigma A$ $\qquad$ $D_2 A = -\sigma A$

$\boxed{D_1 = \sigma}$ $\qquad$ $\boxed{D_2 = -\sigma}$ $\qquad$ $\vec{D} = \sigma\hat{z}$ for both

(b) Find the electric field $\vec{E}$ in each slab

$\vec{D} = \epsilon\vec{E} \rightarrow \vec{E} = \frac{1}{\epsilon}\vec{D}$ $\qquad$ $\epsilon = \epsilon_0\epsilon_r$ $\quad$ $\epsilon_r = \frac{\epsilon}{\epsilon_0}$ $\quad$ $\epsilon_{r_1} = 2$ $\quad$ $\epsilon_{r_2} = 1.5$

$\epsilon_1 = \epsilon_0 2$ $\quad$ $\epsilon_2 = \epsilon_0 1.5$

$$\boxed{\begin{aligned}\vec{E}_1 &= \frac{1}{2\epsilon_0}\vec{D} \\ \vec{E}_2 &= \frac{2}{3\epsilon_0}\vec{D}\end{aligned}}$$

(c) Find the polarization $\vec{P}$ in each slab

$\vec{D} = \epsilon_0\vec{E} + \vec{P} \rightarrow \vec{P} = \vec{D} - \epsilon_0\vec{E}$

$\vec{P}_1 = \sigma - \epsilon_0\left(\frac{1}{2\epsilon_0}\vec{D}_1\right)$ $\qquad$ $\vec{P}_2 = -\sigma - \epsilon_0\left(\frac{2}{3\epsilon_0}\vec{D}_2\right)$

$\vec{P}_1 = \sigma - \frac{1}{2}(\sigma)$ $\qquad$ $\vec{P}_2 = -\sigma - \frac{2}{3}(-\sigma)$

$\vec{P}_1 = \frac{\sigma}{2}$ $\qquad$ $\vec{P}_2 = \frac{-5}{3}\sigma$

or if using $\vec{D} = \sigma\hat{z}$: $\vec{P}_1 = \frac{\sigma}{2}\hat{z}$ & $\vec{P}_2 = \frac{\sigma}{3}\hat{z}$

(d) Find the potential difference between the two plates

$E = \frac{\Delta V}{d}$ $\qquad$ $\Delta V = Ed$

$$\begin{aligned}\Delta V &= E_1 a + E_2 a \\ &= \frac{1}{2\epsilon_0}\sigma a + \frac{2}{3\epsilon_0}\sigma a \\ &= \frac{3}{6\epsilon_0}\sigma a + \frac{4}{6\epsilon_0}\sigma a \\ &= \boxed{\frac{7\sigma a}{6\epsilon_0}}\end{aligned}$$

(e) Find the location and amount of all bound charge

$\sigma_b = \vec{P}\cdot\hat{n}$

$$\sigma_b \begin{cases} = \frac{\sigma}{2}\hat{z}\cdot\hat{n} = -\frac{\sigma}{2} & \text{@ top of slab 1} \\ = \frac{\sigma}{2}\hat{z}\cdot\hat{n} = \frac{\sigma}{2} & \text{@ bottom of slab 2} \\ = \frac{\sigma}{3}\hat{z}\cdot\hat{n} = -\frac{\sigma}{3} & \text{@ top of slab 2} \\ = \frac{\sigma}{3}\hat{z}\cdot\hat{n} = \frac{\sigma}{3} & \text{@ bottom of slab 2} \end{cases}$$

no ρ_b because $\vec{P}$ is constant $(-\nabla\cdot\vec{P} = 0)$

4.26) Spherical conductor of radius a carries a charge Q, surrounded by linear dielectric material of susceptibility χ_e out to radius b. Find energy

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\,d\tau$$

$$\vec{D} = 0 \quad \text{when } 0 \le r < a$$

$$= \frac{Q}{4\pi r^2} \quad \text{when } r > a$$

$$\vec{D} = \epsilon\vec{E} \qquad \epsilon = \varepsilon_0(1+\chi_e)$$

$$\vec{E} = \frac{\vec{D}}{\epsilon}$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0(1+\chi_e)}$$

when $0 \le r < a$, $\vec{D} = 0 \rightarrow \vec{E} = 0$

when $r > b$, no χ_e

$$\boxed{\vec{E} = \begin{cases} 0 & \text{when } 0 \le r < a \\ \dfrac{Q}{4\pi r^2\varepsilon_0(1+\chi_e)}\hat{r} & \text{when } a < r < b \\ \dfrac{Q}{4\pi r^2\varepsilon_0}\hat{r} & \text{when } r > b \end{cases}}$$

4.35) Point charge q is imbedded at center of sphere of linear dielectric material w/ susceptibility χ_e & radius R. Find electric field, polarization, bound charge densities, ρ_b & σ_b. Total bound charge on surface? Where is negative bound charge?

$$\oint \vec{D}\cdot d\vec{a} = Q \qquad \vec{D} = \epsilon\vec{E}$$

$$DA = Q \qquad \vec{E} = \frac{Q}{4\pi r^2\epsilon}\hat{r}$$

$$D = \frac{Q}{4\pi r^2}\hat{r} \qquad \boxed{\vec{E} = \frac{Q}{4\pi r^2\varepsilon_0(1+\chi_e)}\hat{r}}$$

$$\vec{P} = \epsilon_0\chi_e\vec{E}$$

$$= \epsilon_0\chi_e\left(\frac{Q}{4\pi r^2\varepsilon_0(1+\chi_e)}\hat{r}\right)$$

$$\boxed{\vec{P} = \frac{Q\chi_e}{4\pi r^2(1+\chi_e)}\hat{r}}$$

$$\sigma_b = \vec{P}\cdot\hat{n} \qquad \hat{n} = \hat{r}$$

$$= \frac{Q\chi_e}{4\pi r^2(1+\chi_e)}\hat{r}\cdot\hat{r}$$

$$\boxed{\sigma_b = \frac{Q\chi_e}{4\pi r^2(1+\chi_e)}}$$

$$\rho_b = -\nabla\cdot P$$

$$= -\nabla\cdot\frac{Q\chi_e}{4\pi r^2(1+\chi_e)}\hat{r}$$

$$= \frac{-Q\chi_e}{4\pi(1+\chi_e)}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right) \qquad = 4\pi\delta^3(r)$$

$$= \frac{-Q\chi_e(4\pi\delta^3(r))}{4\pi(1+\chi_e)}$$

$$\boxed{\rho_b = \frac{-Q\chi_e\delta^3(r)}{1+\chi_e}}$$

$$Q_{surface} = \sigma_b A$$

$$= \frac{Q\chi_e}{4\pi r^2(1+\chi_e)}(4\pi r^2)$$

$$\boxed{Q_{surface} = \frac{Q\chi_e}{1+\chi_e}}$$

The compensating charge would be at the center, and is $\frac{-Q\chi_e}{1+\chi_e}$