Electromagnetc Theory: PHYS330

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Summary

Week 5 Summary

1. Current density and continuity equation
2. The divergence and curl of $\vec{B}$-fields
3. The magnetic vector potential, $\vec{B} = \nabla \times \vec{A}$
   - Vector calculus theorems
   - Boundary conditions
   - Multipole expansion
4. Magnetic fields in matter
   - Magnetization
   - Field of a magnetized object
   - The auxiliary field, $\vec{H}$
   - Linear magnetic media

Current density and continuity equation

Current density and continuity equation

Let the *current density* $\vec{J}$ be defined by

$$\vec{J} = \rho \vec{v} \tag{1}$$

Units: current per unit area (other definitions available for different geometries). So it's reasonable to obtain the whole scalar current by integrating:

$$I = \int_{\mathcal{S}} \vec{J} \cdot d\vec{a} \tag{2}$$

If we want to account for the charge leaving a volume $\mathcal{V}$ through a closed surface $\mathcal{S}$ is

$$\oint_{\mathcal{S}} \vec{J} \cdot d\vec{a} = \int_{\mathcal{V}} (\nabla \cdot \vec{J}) d\tau \tag{3}$$

$$\int_{\mathcal{V}} (\nabla \cdot \vec{J}) d\tau = -\frac{d}{dt} \int_{\mathcal{V}} \rho d\tau = -\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau \tag{4}$$

Current density and continuity equation

This is true for *any* volume, so the integrands must be equal:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \tag{5}$$

This is called the continuity equation, and it also arises in quantum mechanics. If $\partial\rho/\partial t = 0$, then we have a **steady current.**

Suppose we have a current density $\vec{J}(\vec{r}) = I_0(t)\hat{r}/r^2$, with $I_0(t) = \delta(t - t_0)$. Find $\rho(t)$, the charge density as a function of time in the region containing $\vec{J}$. (Breakout rooms).

The Divergence of B-fields

The Divergence of B-fields

The Biot-Savart law states that

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau' \tag{6}$$

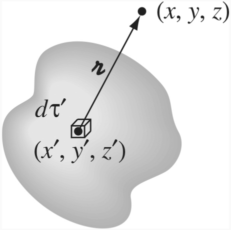

Figure 1: Definitions of coordinates in variables for derivation of divergence of B-fields. The gray region represents charges and current densities.

The Divergence of B-fields

Take the divergence of the Biot-Savart law, but then use a product rule for the integrand.

$$\nabla \cdot \vec{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\vec{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau' \tag{7}$$

$$\nabla \cdot \left(\vec{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\nabla \times \vec{J}) - \vec{J} \cdot \left(\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \tag{8}$$

- $\nabla \times \vec{J} = 0$, because this is like taking $df(x)/dx'$.
- We showed in Chapter 1 that $\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 0$. Is this visually obvious?

Thus,

$$\boxed{\nabla \cdot \vec{B} = 0} \tag{9}$$

The Divergence of B-fields

From warmup exercises, we know that we can therefore write

$$\vec{B} = \nabla \times \vec{A} \tag{10}$$

(Breakout rooms): create three divergence-less vector fields. One in Cartesian coordinates, one in cylindrical coodinates, and one in spherical. Exclude trivial cases like $\vec{B} = 0$.

Conclusion

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