Warm-up for Electromagnetic Theory (PHYS330)

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1 Problem 1.54

Verify the divergence theorem for $\vec{v} = r^2 \cos\theta \hat{r} + r^2 \cos\phi \hat{theta} - r^2 \cos\theta \sin\phi \hat{\phi}$ over the octant of the sphere of radius R with the center at the origin.

Break the problem into manageable pieces. (a) What is the divergence of the field? (b) What is the volume integral of it?

Divergence: $4r\cos\theta$.
Volume integral of the divergence:

$$\int_0^R \int_0^{\pi/2} \int_0^{\pi/2} 4r\cos\theta \; r^2 \sin\theta dr d\theta d\phi = \frac{\pi R^4}{4} \tag{1}$$

The surface integral has four parts. (a) Outer curved surface with $d\vec{a} = r^2 \sin\theta d\theta d\phi \hat{r}$, and the result is $\pi R^4/4$. (b) The lower side is defined by $\theta = \pi/2$, and $d\vec{a} = -r^2 dr d\phi \hat{z}$. What is $\hat{z}$ here ... $\hat{\theta}$. Consult back page of the book for conversions and set $\theta = \pi/2$. The result is $R^4/4$. (c) The left side is described by $\phi = 0$, and $d\vec{a} = r dr d\theta(-\hat{y}) = -r dr d\theta \hat{\phi}$. However, the $\hat{\phi}$-component is zero for $\phi = 0$, so the surface integral is zero. (d) The right side has $d\vec{a} = r dr d\theta \hat{\phi}$, and $\phi = \pi/2$. This time, the surface integral for $\phi = \pi/2$ is not zero, and the result is $-R^4/4$. Summing all the pieces, we find

$$\oint \vec{v} \cdot d\vec{a} = \frac{\pi R^4}{4} \tag{2}$$

2 Problem 1.55

Break the problem into the following pieces: (a) What is the curl of $\vec{v}$? (b) What is the surface integral of the curl? (c) How do we approach the line integral?

The curl may be evaluated in Cartesian coordinates: $\nabla \times \vec{v} = (b-a)\hat{k}$. Form the surface integral:

$$\int (\nabla \times \vec{v}) \cdot d\vec{a} = (b-a)\pi R^2 \tag{3}$$

The integrand is a constant, and parallel to the area vector. Thus, the constant moves outside the integral and we have just the area of the circle. What is $d\vec{l}$ on the circle of radius R? *Cylindrical coordinates* work best to describe the situation: $d\vec{l} = ds\hat{s} + s d\phi \hat{\phi} + dz\hat{z}$. However, $dz = 0$ and $ds = 0$, so we are left with $d\vec{l} = s d\phi \hat{\phi}$. That makes the line integral ($s = R$):

$$\oint \vec{v} \cdot d\vec{l} = \int_0^{2\pi} ay\hat{x} \cdot R d\phi \hat{\phi} + \int_0^{2\pi} bx\hat{y} \cdot R d\phi \hat{\phi} \tag{4}$$

Here are some useful conversions:

- $x = R\cos\phi$
- $y = R\sin\phi$
- $\hat{x} = 0 - \sin\phi \hat{\phi}$
- $\hat{y} = 0 + \cos\phi \hat{\phi}$

Substituting all of that into Eq. 4 gives $(b-a)\pi R^2$.