Electromagnetc Theory: PHYS330

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Summary

Week 2 Summary

1. Homework discussions
    - Proofs! Glorious proofs.
    - Exercises with *checking* fundamental theorems
2. Electrostatics and Coulomb forces
    - Charge distributions, superposition, and the Coulomb force
    - A note about the *far-field*
    - Setting up integrals, taking limits, checking units
    - The divergence of electric fields
    - The curl of electric fields
3. Electric Potential
    - Definitions, fundamental theorem for gradients
    - Reference points
    - Laplace equation ...
4. Work, energy, and conductors

Homework

Homework, Week 2

Unlike last week, these exercises come from *within* the chapter. Ideally, you should look at all of the problems within the chapter as you study.

- Exercise 2.5
- Exercise 2.6
- Exercise 2.9
- Exercise 2.12
- Exercsie 2.16
- Exercise 2.18
- Exercise 2.25
- Exercise 2.29

Charge distributions, Superposition, and the Coulomb Force

Charge distributions, Superposition, and the Coulomb Force

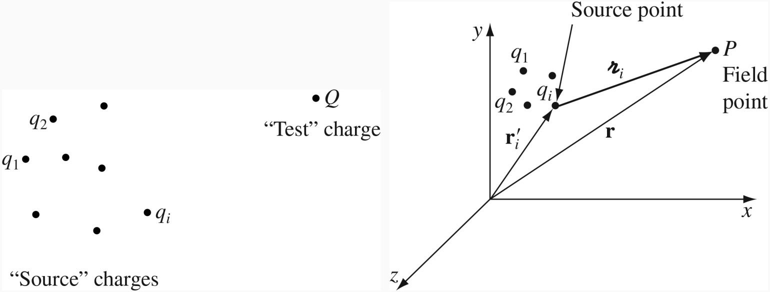

Figure 1: The basic problem of electrostatics. Note the definition of the separation vector, and the vectors to the field point and to all the source charges.

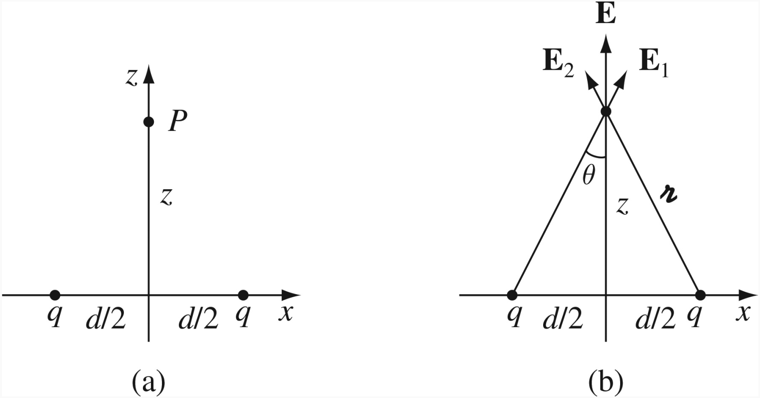

Figure 2: Begin with a dipole, and then a *physical* dipole.

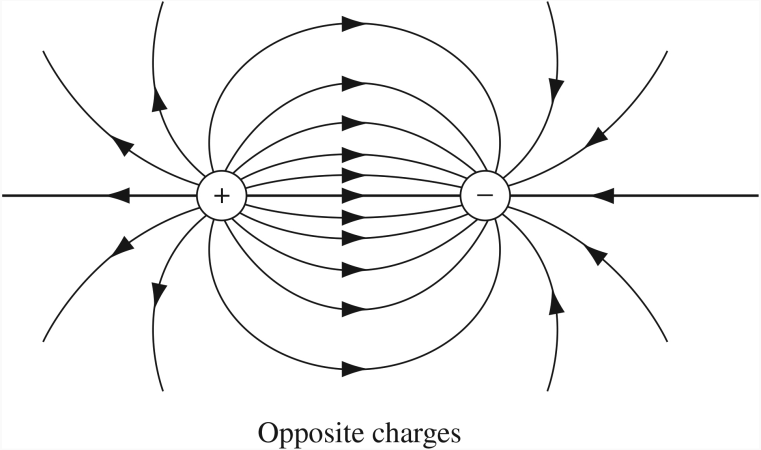

Figure 3: Field of a *physical* dipole.

Charge distributions, Superposition, and the Coulomb Force

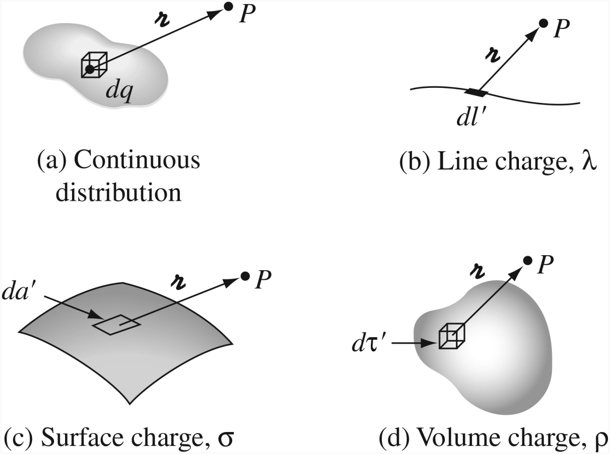

Figure 4: The continuous limit implies a variety of symmetries and geometries over which we integrate, rather than sum.

Charge distributions, Superposition, and the Coulomb Force

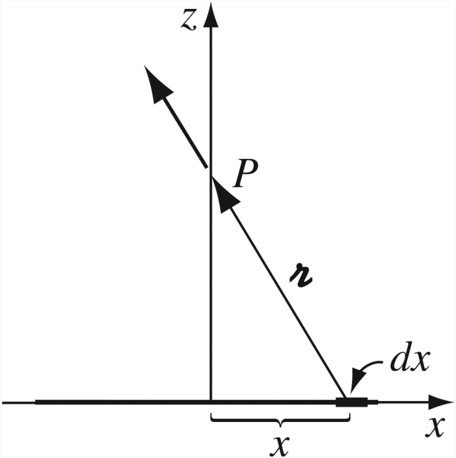

Figure 5: A coninuous line density of charge. Integration yields the electric field.

Charge distributions, Superposition, and the Coulomb Force

Useful calculations:

1. Continuous line charge, length L.
2. Continuous plane of charge, radius R.
3. Loop of charge, radius R, a distance z above the center.

Why are these interesting? One example is that these shapes are used as *antennas*. Give some alternating current at the right voltage and impedance to a shape of metal, then you've got your antenna that radiates a certain way.

Professor do these examples.[1]

[1] Remember from PHYS180? Remember? Yeah...good times.

A Note about the Far-Field

The Far-Field

One way to express the **far-field** approximation (compare to Fraunhofer and Fresnel limits in diffraction):

$$\begin{aligned}
\vec{r} &= \vec{r'} + \vec{\mathscr{r}} && (1)\\
\vec{\mathscr{r}} &= \vec{r} - \vec{r'} && (2)\\
\mathscr{r} &= \sqrt{r^2 - 2\vec{r}\cdot\vec{r'} + r'^2} && (3)\\
\mathscr{r} &= r\sqrt{1 - 2\vec{r}\cdot\vec{r'}r^{-2} + r'^2 r^{-2}} && (4)\\
\mathscr{r} &\approx r\sqrt{1 - 2\vec{r}\cdot\vec{r'}r^{-2}} && (5)\\
\mathscr{r} &\approx r\left(1 - \vec{r}\cdot\vec{r'}r^{-2}\right) && (6)\\
\mathscr{r} &\approx r\left(1 - \hat{r}\cdot\vec{r'}r^{-1}\right) && (7)\\
\mathscr{r} &\approx r - \hat{r}\cdot\vec{r'} && (8)
\end{aligned}$$

The Far-Field

Repeat the charged loop calculation, but replace $\mathscr{r} \approx r - \hat{r} \cdot \vec{r'}$ at the outset. What happens?

The Divergence of $\vec{E}$-fields

The Divergence of $\vec{E}$-fields

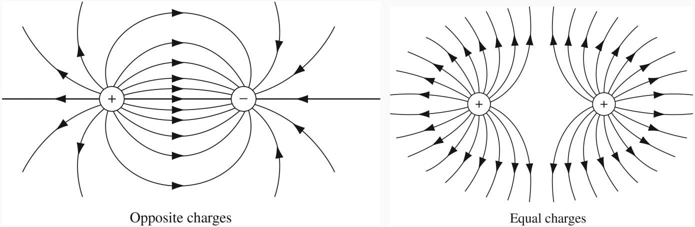

Figure 6: Field lines are an important theoretical concept.

The Divergence of $\vec{E}$-fields

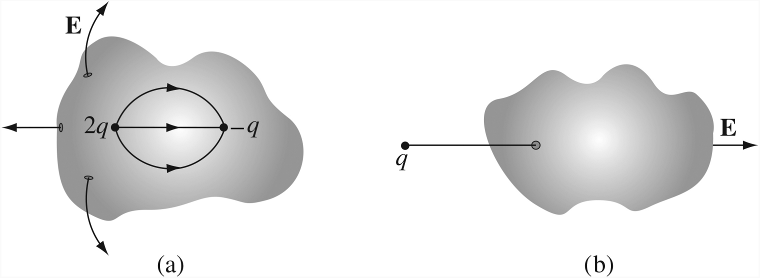

Figure 7: The concept of a closed Gaussian surface.

The Divergence of $\vec{E}$-fields

$$\boxed{\oint \vec{E}_i \cdot d\vec{a} = \frac{1}{4\pi\epsilon_0} \int_0^{\pi} \int_0^{2\pi} \frac{q_i \hat{r}}{r^2} \cdot r^2 \sin\theta d\theta d\phi \hat{r} = \frac{q_i}{\epsilon_0}} \tag{9}$$

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i \tag{10}$$

$$\oint \vec{E} \cdot d\vec{a} = \sum_{i=1}^{n} \left(\oint \vec{E}_i \cdot d\vec{a} \right) \tag{11}$$

$$\oint \vec{E} \cdot d\vec{a} = \sum_{i=1}^{n} \left(\frac{q_i}{\epsilon_0} \right) \tag{12}$$

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{tot}}{\epsilon_0} \tag{13}$$

Gauss' Law: the total flux is proportional to the contained charge.

The Divergence of $\vec{E}$-fields

The divergence theorem:

$$\oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \int_{\mathcal{V}} (\nabla \cdot \vec{E}) d\tau \tag{14}$$

Remark that the total charge is the integral over the 3D charge density:

$$\frac{Q_{tot}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho d\tau \tag{15}$$

This implies

$$\oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \int_{\mathcal{V}} (\nabla \cdot \vec{E}) d\tau = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho d\tau \tag{16}$$

Looking at the last two expressions:

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}} \tag{17}$$

The Divergence of $\vec{E}$-fields

Differential form of Gauss' Law:

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}} \tag{18}$$

Consider a different argument:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho(\vec{r}') d\tau' \tag{19}$$

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left(\frac{\hat{\hat{\boldsymbol{\mathscr{r}}}}}{\mathscr{r}^2} \right) \rho(\vec{r}') d\tau' \tag{20}$$

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi\delta^3(\boldsymbol{\mathscr{r}}) \rho(\vec{r}') d\tau' \tag{21}$$

$$\nabla \cdot \vec{E} = \frac{4\pi}{4\pi\epsilon_0} \int \delta^3(\vec{r} - \vec{r}') \rho(\vec{r}') d\tau' \tag{22}$$

$$\nabla \cdot \vec{E} = \rho(\vec{r})/\epsilon_0 \tag{23}$$

The Divergence of $\vec{E}$-fields

Symmetry in the Application of Gauss' Law:
If the E-field and the area element are *always* orthogonal,

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{tot}}{\epsilon_0} \tag{24}$$

$$|\vec{E}|A = \frac{Q_{tot}}{\epsilon_0} \tag{25}$$

$$\vec{E} = \frac{Q_{tot}}{\epsilon_0 A}\hat{n} \tag{26}$$

This trick can be used even when the charge distribution is not uniform, but *does exhibit* symmetry.

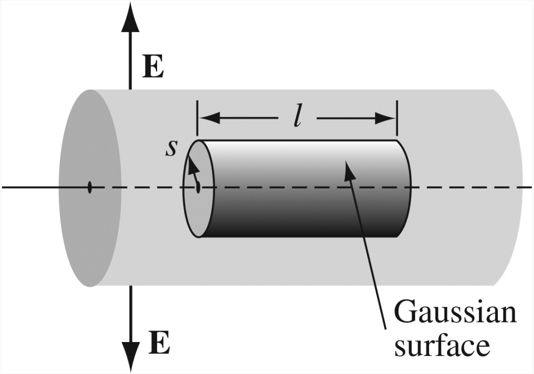

Figure 8: Use cylindrical symmetry to apply Gauss' Law. The charge distribution function is $\rho(s) = ks$. Obtain the field (a) *inside* the object, then (b) outside the object.

The Divergence of $\vec{E}$-fields

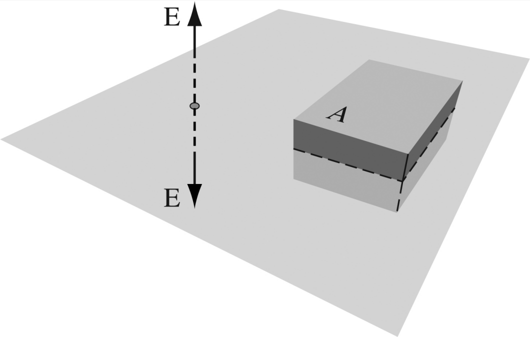

Figure 9: Use Cartesian symmetry to apply Gauss' Law. The charge distribution function is $\rho(x, y) = +\sigma$. Obtain the field above (or below) the charged plane.

The Divergence of $\vec{E}$-fields

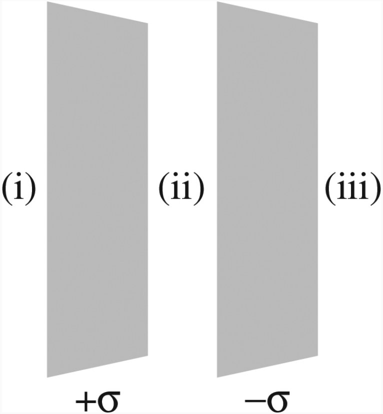

Figure 10: Combinations of "Gaussian charged objects." What about the field between two line charges?

The Curl of $\vec{E}$-fields

The Curl of $\vec{E}$-fields

The $\vec{E}$-field of a point charge at the origin ($\vec{r}' = 0$), and the line element in spherical coordinates:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{27}$$

$$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi} \tag{28}$$

$$\vec{E}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr \tag{29}$$

$$\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \tag{30}$$

$$\oint \vec{E}\cdot d\vec{l} = 0 \quad (\vec{a} = \vec{b}) \tag{31}$$

$$\nabla \times \vec{E} = 0 \tag{32}$$

Any combination of point charges will also lead to zero curl. Why? Superposition.

The Curl of $\vec{E}$-fields

Define a function, then, that encapsulates *path-independence*:

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E} \cdot d\vec{l} \tag{33}$$

- $\mathcal{O}$ is a reference point, naturally taken to be ∞ (far from origin)
- $V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$
- $0 = V(\vec{a}) - V(\vec{a}) = -\oint \vec{E} \cdot d\vec{l} = 0$

Fundamental theorem for gradients:

$$V(\vec{b}) - V(\vec{a}) = \int_{\vec{a}}^{\vec{b}} \nabla V \cdot d\vec{l} = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l} \tag{34}$$

$$\vec{E} = -\nabla V \tag{35}$$

The Curl of $\vec{E}$-fields

Path independence:

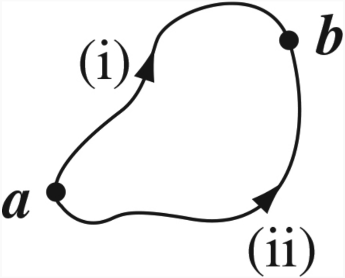

Figure 11: If the line integral was not path-independent, then path i minus path ii would not be zero. Path i and ii form a closed line integral.

The Curl of $\vec{E}$-fields

Two more ideas:

$$\vec{E} = -\nabla V \tag{36}$$

$$\nabla \cdot \vec{E} = -\nabla^2 V \tag{37}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{38}$$

Equation 38 is known as the *Poisson Equation*. If $\rho = 0$ in some region:

$$\boxed{\nabla^2 V = 0} \tag{39}$$

Equation 39 is known as the Laplacian of the potential or Laplace's Equation. Solving it is the subject of Ch. 3.

The Curl of $\vec{E}$-fields

Show that the potential of a point charge q at the origin is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{40}$$

Indeed, collections of point charges lead to

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i} \tag{41}$$

A collection of *many* point charges smoothed into a continuous distribution leads to

$$V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho' d\tau'}{\mathscr{r}} \tag{42}$$

The Curl of $\vec{E}$-fields

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho' d\tau'}{\mathcal{r}} \tag{43}$$

(a) Derive the potential due to a line charge of length L, directly above the center of the line. (b) Take the gradient in cylindrical coordinates to obtain $\vec{E}$.

Brief Remark about Boundary Conditions

Boundary Conditions

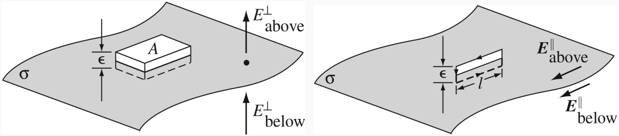

Figure 12: (Left) By Gauss' Law, we may find the discontinuity in the orthogonal E-field component. (Right) Via a closed-loop line integral, we may find that the parallel component of the E-field is continuous.

$$(\nabla V_{top} - \nabla V_{bottom}) \cdot \hat{n} = -\frac{\sigma}{\epsilon_0} \tag{44}$$

$$(\nabla V_{top} - \nabla V_{bottom}) \cdot \hat{s} = 0 \tag{45}$$

Boundary Conditions

Which of the following functions has a discontinuity *in the derivative* but not the function itself, at $x = 0$?

- A: $\tan(x)$
- B: $|x| = abs(x)$
- C: $\sqrt{x}$
- D: $\theta(x)$ (Heaviside step-function)

Boundary Conditions

Consult your notes regarding the charge distribution with spherical symmetry.

- What was the formula for the electric field as a function of r, when $r < R$?
- What is the correct formula for the electric field as a function of r, when $r \geq R$? (*Think of Gauss' Law*).
- What is the voltage just outside the radius R?
- What is the voltage just inside the radius R?

Energy and Work

Energy and Work

The work required to assemble point charges by bring each in from infinity is

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j\neq i}^{n} \frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}_{ij}} \right) \tag{46}$$

The parentheses contains the potential due to all charges except q_i:

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{\mathscr{r}}_i) \tag{47}$$

Note that it doesn't matter when charge q_i arrived; we have to pay for the energy used to keep q_i there as everyone else arrives as well.

Energy and Work

Continuous limit:

$$\boxed{W = \frac{1}{2}\int \rho(r')V(r')d\tau'} \tag{48}$$

This result can be re-written:

$$W = \frac{\epsilon_0}{2}\left\{-\int \vec{E}\cdot(\nabla V)d\tau + \oint V\vec{E}\cdot d\vec{a}\right\} \tag{49}$$

Like any integration by parts, there is an integral term and a surface term (the second term). Use $-\vec{E} = \nabla V$, and expand the Gaussian surface so far that all contributions from the second term vanish:

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau \tag{50}$$

Energy and Work

Since we do not specify the shape of the volume, we can speak about the *energy per unit volume*:

$$\boxed{u_E = \frac{\epsilon_0}{2} E^2} \tag{51}$$

- E-fields contain energy
- This equation is valid even if E-field values vary in space (of course)
- Consider the simple example of a capacitor ...

Energy and Work

Start with the E-field inside a parallel-plate capacitor:

$$E = \frac{\sigma}{\epsilon_0} \tag{52}$$

$$u_E = \frac{1}{2\epsilon_0}\sigma^2 \tag{53}$$

$$\mathcal{E} = \frac{Ad}{2\epsilon_0}\sigma^2 \tag{54}$$

$$\sigma = \frac{Q}{A} \tag{55}$$

$$C = \frac{\epsilon_0 A}{d} \tag{56}$$

$$\mathcal{E} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = U \tag{57}$$

Energy conservation.

Conclusion

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