Physics Homework chapter 1

1.54, 1.55, 1.56, 1.57, 1.59, 1.62, 1.63, 1.64

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10/28/20
Phys 330

1.54. Divergence theorem of

$v = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

Divergence $= \nabla\cdot\vec{v} = \frac{1}{r^2}\frac{d}{dr}(r^4\cos\theta) + \frac{1}{r\sin\theta}\frac{d}{d\theta}(\sin\theta\, r^2\cos\phi) - \frac{1}{r\sin\theta}\frac{d}{d\phi}(r^2\cos\theta\sin\phi)$

$\frac{1}{\cancel{r^2}}4r^{\cancel{3}}\cos\theta + \frac{1}{r\sin\theta}(\cos\theta\, r^2\cos\phi) - \frac{1}{r\sin\theta}(r^2\cos\theta\cos\phi)$

$4r\cos\theta +$ $\frac{1}{r\sin\theta}r^2\cos\phi\cos\theta$ $\quad \frac{r^2}{r\sin\theta}\cos\theta\cos\phi$

$\frac{r}{\sin\theta}\cos\theta\cos\phi$ $\quad \frac{r}{\sin\theta}\cos\theta\cos\phi$

$r\cdot\frac{\cos\theta}{\sin\theta}\cos\phi$ $\quad r\frac{\cos\theta}{\sin\theta}\cos\phi$

$r\cos\cot\phi$ $\quad r\cot\theta\cos\phi$

$4r\cos\theta + \cancel{r}\cos\phi\cot\theta - \cancel{r\cot\theta}\cos\phi = 4r\cos\theta$

$\int_0^R\int_0^{\pi/2}\int_0^{\pi/2} 4r\cos\theta\, r^2 dr \sin\theta\, d\theta\, d\phi = \frac{\pi}{2}R^4\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta$

cos
-sin
-cos
sin

$\int_0^R 4r\cdot r^2\,dr$

$\int_0^R 4r^3\,dr$

$\frac{4r^4}{4}\quad R^4 - 0$

$\int_{a=0}^{b=\pi/2}\cos\theta\sin\theta\,d\theta$ $\quad u = \sin(\theta)\ du = \cos$

$\sin\theta(\cos\theta)\Big|_0^{\pi/2}$ $\quad \int u\,du \quad \frac{u^2}{2} = \frac{1}{2}$

$\int_0^{\pi/2}\frac{1}{2}\sin^2\theta\Big|_0^{\pi/2}$ $\quad \frac{1}{2}\sin^2\frac{\pi}{2} - \frac{1}{2}\sin^2 0$

$= \frac{1}{2}$

$\int_0^{\pi/2} d\phi = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ $\quad R^4(\frac{1}{2})\cdot\frac{\pi}{2}$

$\boxed{\frac{R^4\pi}{4}}$

1.55). Check Stokes' theorem using $v = ay\hat{x} + bx\hat{y}$

Surface integral

$$\nabla \times \bar{v} = \frac{\partial}{\partial x} = ay \quad \frac{\partial}{\partial y} = bx \quad \frac{\partial}{\partial z} = 0 \quad = \hat{z}(b-a)$$

$$\int_S (b-a)\hat{z} \cdot da\hat{z} = (b-a)R^2\pi$$

$$\int_C v \cdot dl = \oint (ay\,dx + bx\,dy) \qquad x = R\cos\theta \qquad dx = -R\sin\theta\, d\theta$$

$$y = R\sin\theta \qquad dy = R\cos\theta\, d\theta$$

$$\int_0^{2\pi} R^2(-a\sin^2\theta + b\cos^2\theta)\,d\theta = R^2\int_0^{2\pi}(-a\pi + b\pi) = \boxed{R^2\pi(b-a)}$$

1.56). line integral along triangular path $v = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$

wrt $\hat{x}, \hat{y}, \hat{z}$

$$\nabla \times v = \frac{\partial}{\partial x} = 6$$

$$= \hat{x}(3-2yz)$$

$$\int_0^1\int_0^{2-2y}(3-2yz)\hat{x} \cdot dy\,dz\,\hat{x} = \qquad \frac{\partial}{\partial y} = yz^2$$

$$3\int_0^1\int_0^{2-2y} dz\,dy - 2\int_0^1\int_0^{2-2y} z\,dz\,dy \qquad \frac{\partial}{\partial z} = 3y+z$$

$$3\int_0^1\int_0^{2-2y} dz\,dy = 3\int_0^1 (2-2y)\,dy = 3(2-1) = 3$$

$$2y - \frac{2y^2}{2}$$

$$2(1) - \frac{2(1)^2}{2} - 0$$

$$2 \cdot \frac{2}{2} \qquad 2-1 \qquad 3(1)$$

$$-2\int_0^1\int_0^{2-2y} z\,dz\,dy = -2\int_0^1 \tfrac{1}{2}(2-2y)^2 y\,dy$$

$$= -\int_0^1 (4y - 4y^2 - 4y^3)\,dy = -\left(2 - \frac{4}{3} - \frac{4}{4}\right) = -\tfrac{1}{3}$$

$$\int_0^1 yz^2\,dy = 0 \quad z = 0 \quad 2-2y = z \quad dz = -2dy$$

$$\int_1^0 yz^2\,dy + (3y+z)\,dz = \int_0^1 [4y(1-y)^2\,dy - 2(2+y)\,dy$$

$$\int_1^0 (4y - 8y^2 + 4y^3 - 4 - 2y)\,dy = -\int_0^1 4y^3 - 8y^2 + 2y - 4\,dy$$

$$= -\left(1 - \frac{8}{3} + 1 - 4\right) = \frac{14}{3}$$

$$= \left(\frac{3}{3} - \frac{8}{3} + \frac{3}{3} - \frac{12}{3}\right.$$

1.57). line integral

$$v=(r\cos^2\theta)\hat{r}-(r\cos\theta\sin\theta)\hat{\theta}+3r\hat{\phi} \quad d\bar{l}=dr\hat{r} \quad \theta=\pi/2$$
$$\phi=0$$

$$\int_0^1 r\cos^2\theta\, dr=0$$

$$d\bar{l}=r\,d\phi\hat{\phi} \quad r=1 \quad \theta=\pi/2$$
$$\int_0^{\pi/2} 3r\,d\phi=3\cdot\pi/2=3\pi/2$$

$$\sin\theta=\frac{1}{r} \quad dr=-\frac{\cos\theta}{\sin^2\theta}d\theta \quad dl=dr\hat{r}+r\,d\theta\hat{\theta} \quad \phi=\pi/2$$
$$r=\frac{1}{\sin\theta}$$

$$\int_{\pi/2}^{\arcsin(1/\sqrt{5})} r\cos^2\theta\, dr - r\cos\theta\sin\theta\, r\, d\theta$$
$$=\int_{\pi/2}^{\arcsin(1/\sqrt{5})} = \frac{-\cos^3}{\sin^3}$$

wolfram calculation

$$-\int_{\pi/2}^{\arsin(1/\sqrt{5})} \frac{\cos\theta}{\sin^3\theta}d\theta=\frac{1}{2\sin^2\theta}\Big|_{\pi/2}^{\arcsin 1/\sqrt{5}} \quad 1/2(5-1)=2$$

$$\frac{4}{2}-\frac{4}{2}+\frac{\pi}{2}$$
$$\frac{3\pi}{2}+\frac{4}{2} \quad \frac{3\pi}{2}-\frac{4}{2}=\frac{3\pi}{2}$$

$$\theta=\arcsin 1/\sqrt{5} \quad \phi=\pi/2 \quad dl=dr\hat{r}$$
$$\int_{\sqrt{5}}^0 r\cos^2\theta\, dr=\int_{\sqrt{5}}^0 (1-\sin^2\theta)dr$$
$$1/2\,r^2\Big|_{\sqrt{5}}^0\left(1-\frac{1}{5}\right)=-2$$
$$\boxed{=\frac{3\pi}{2}}$$

1.59). divergence theorem $v=r^2\sin\theta\hat{r}+4r^2\cos\theta\hat{\theta}+r^2\tan\theta\hat{\phi}$

Ice cream cone: $r\sin\theta\cos\phi$, $r\sin\theta\sin\phi$, $r\cos\theta$ $(0,\pi/6)$ $(0,R)$

$$\int_V \nabla\cdot v\, d\tau=\oint_{\partial V} v\cdot\hat{n}\, dA$$
$$\nabla\cdot v=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta)+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$
$$=\frac{1}{r^2}\frac{\partial}{\partial r}(r^4\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(4r^2\sin\theta\cos\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\tan\theta)$$

$$=4r\sin\theta+\frac{4r}{\sin\theta}(\cos^2\theta-\sin^2\theta) \quad \frac{4r\cos^2\theta}{\sin\theta}$$

$$\int_V \nabla\cdot v\, d\tau=\int_0^{2\pi}\int_0^{\pi/6}\int_0^R \frac{4r\cos^2\theta}{\sin\theta}r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$=4\int_0^{2\pi}d\phi\int_0^{\pi/6}\cos^2\theta\, d\theta\int_0^R r^3 dr$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$$\int_V \nabla \cdot v \, d\tau = 4 \cdot 2\pi \cdot \frac{R^4}{4} \int_0^{\pi/6} \tfrac{1}{2}(\cos 2\theta + 1)\, d\theta$$

$$\pi R^4 \left(\tfrac{1}{2} \sin 2\theta \Big|_0^{\pi/6} + \frac{\pi}{6} \right)$$

$$\pi R^4 \left(\tfrac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\pi}{6} \right)$$

$$\boxed{\frac{\pi R^4}{12} (3\sqrt{3} + 2\pi)}$$

1.62). $a \equiv \int_S da$

a). Bowl of Radius R

$\hat{x} \, \& \, \hat{y} = \emptyset$ $\qquad \int_S dS = \int_0^{\pi/2} \int_0^{2\pi} R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$$R^2 \int_0^{\pi/2} \int_0^{2\pi} \sin\theta \cos\theta \, d\theta \, d\phi \, \hat{z} = 2\pi R^2 \int_0^{\pi/2} \tfrac{1}{2} \sin(2\theta) \, d\theta$$

$$= \pi/2 \, R^2 \left[-\cos(2\theta) \right]_0^{\pi/2} \hat{z} = \pi R^2 \hat{z}$$

b). From 1.61, $T = 1$ $\quad \nabla T = 0$, $\boxed{\oint da = 0}$

c). $\bar{S} = 0 \quad S_1 - S_2 = 0 \quad S_1 = S_2$

for closed surface $\quad a_1 = a_2$

D). $a = \tfrac{1}{2} \oint r \times dl \qquad da = \tfrac{1}{2}(r \times dl)$

entire surface $= a = \tfrac{1}{2} \oint r \times dl$

E). $T = \bar{c} \cdot \bar{r}$

$$\nabla T = c \times (\nabla \times r) + (c \cdot \nabla) r = (c \cdot \nabla) \bar{r}$$

$$(c_x \partial_x + c_y \partial_y + c_z \partial_z) = c_x x + c_y y + c_z z$$

$= c$

1.63). a). $v = \frac{\hat{r}}{r}$

$v = v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}$

$v_r = \frac{1}{r}$ $v_\theta = 0$ $v\phi = 0$

$\nabla \cdot v = \frac{1}{r^2}\frac{d}{dr}(r^2 v_r) = \frac{1}{r^2}\frac{d}{dr} = \frac{1}{r^2}$

$\int \nabla \cdot v \, d\tau = \int \frac{1}{r^2} 4\pi r^2 dr = 4\pi \int_0^R dr = 4\pi R$

$\int v \cdot da = \left(\frac{1}{R}\hat{r}\right) \cdot R^2 \sin\theta d\theta d\phi \hat{r})$

$= R\int \sin\theta d\theta d\phi = R\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$

$= R(-\cos\theta|_0^\pi (\phi))|_0^{2\pi} = 4\pi R$

$\int(\nabla \cdot v)d\tau = \int v \cdot da$

* Rest of 1.63 on last page

$v_r = r^n$ $v_\theta = 0$ $v_\phi = 0$

$\nabla \cdot v = \frac{1}{r^2}\frac{d}{dr}(r^2 v_r)$

$\nabla \cdot v = \frac{1}{r^2}\frac{d}{dr}(r^2 r^n) = \frac{1}{r^2}\frac{d}{dr}(r^{n+2})$

$= 1/r^2 (n+2) r^{n+1} = (n+2) r^{n-1}$

divergence of $r^n\hat{r} = (n+2)r^{n-1}$ when $n \neq 2$

1.64). $\nabla^2(1/r) = -4\pi\delta^3(r)$ $r' = 0$ $r = \sqrt{r^2+\varepsilon^2}$ $\varepsilon \to 0$

a).

$D(r,\varepsilon) = -\frac{1}{4\pi}\nabla^2\frac{1}{\sqrt{r^2+\varepsilon^2}}$

$D(r,\varepsilon) = -\frac{1}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left(r^2(-1/2)\frac{2r}{(r^2+\varepsilon^2)^{3/2}}\right) =$

$= 1/4\pi r^2 \frac{d}{dr}\left(\frac{r^3}{(r^2+\varepsilon^2)^{3/2}}\right)$

$= \frac{1}{4\pi r^2}\left(\frac{3r^2}{(r^2+\varepsilon^2)^{3/2}} - 3/2\frac{r^3 2r}{(r^2+\varepsilon^2)^{5/3}}\right) = \frac{1}{4\pi r^2}\frac{3r^2}{(r^2+\varepsilon^2)}$

$(r^2+\varepsilon^2-r^2) = \frac{3\varepsilon^2}{4\pi(r^2+\varepsilon^2)^{5/2}}$

b). D

$r=0 = \frac{3\varepsilon^2}{4\pi(r^2+\varepsilon^2)^{5/2}} = \frac{3\varepsilon^2}{4\pi(0+\varepsilon^2)^{5/2}}$ $\frac{3\varepsilon^2}{4\pi(\varepsilon^2)^{5/2}} = \frac{3\varepsilon^2}{4\pi\varepsilon^3} = \frac{3}{4\pi\varepsilon}$ wh

goes to infinity since denominator grows faster than numerator. $\varepsilon \to 0$ r for $D(0,\varepsilon)$

c). $D(r,0) = 0$ when $r \neq 0$

$\varepsilon \to 0$

approaches 0 for same reason as part B.

[redacted]

D). $D(r, \varepsilon)$ all space $= 1$

$$\int D(r,\varepsilon) 4\pi r^2 dr$$

by wolfram

$$= 3\varepsilon^2 \int_0^\infty \frac{r^2}{(r^2+\varepsilon^2)^{5/2}} dr = 3\varepsilon^2 \left(\frac{1}{3\varepsilon^2}\right) = \boxed{1}$$

1.63). continued

b). $$\nabla \times v = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right)\hat{\phi}$$

$v_r = r^n$
$v_\theta = 0$
$v_\phi = 0$

therefore curl $= 0$

$\boxed{r \times da = 0}$