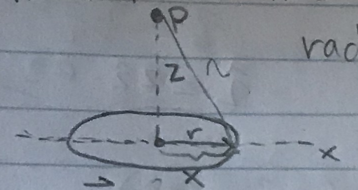
HW Set 2 #2.5, 2.6, 2.9, 2.12, 2.16, 2.18, 2.25, 2.29

2.5



radius: r, line charge: λ

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\, dl'$$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$, $\vec{r} = z\hat{z}$, $\vec{r}\,' = x\hat{x}$, $dl' = dx$

$$\hat{\mathscr{r}} = \frac{z\hat{z} - x\hat{x}}{\sqrt{z^2+x^2}} \qquad \mathscr{r} = \sqrt{z^2+x^2} \qquad \mathscr{r} = z^2+x^2$$

$$E = \frac{1}{4\pi\varepsilon_0}\int_0^r \frac{\lambda}{(z^2+x^2)}\left(\frac{z\hat{z} - x\hat{x}}{\sqrt{z^2+x^2}}\right)dx = \frac{\lambda}{4\pi\varepsilon_0}\int_0^r \frac{z\hat{z} - x\hat{x}}{(z^2+x^2)^{3/2}}\,dx$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\int_0^r \frac{z\hat{z}\,dx}{(z^2+x^2)^{3/2}} - \int_0^r \frac{x\hat{x}\,dx}{(z^2+x^2)^{3/2}}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{z\hat{z}}{z^2}\int_0^r \frac{z^2\,dx}{(z^2+x^2)^{3/2}} - \hat{x}\int_0^r \frac{x\,dx}{(z^2+x^2)^{3/2}}\right]$$

$u = z^2 + x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\frac{u^{-3/2+\frac{2}{2}}}{-\frac{3}{2}+\frac{2}{2}} = \frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}$$

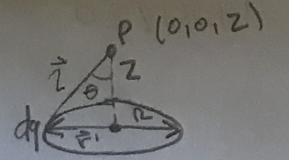
$-2u^{-1/2}$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{\hat{z}}{z}\left[\frac{x}{(z^2+x^2)^{1/2}}\right]_0^r - \hat{x}\int_{z^2}^{z^2+r^2}\frac{\frac{1}{2}dx}{u^{3/2}}\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{\hat{z}}{z}\left[\frac{r}{(z^2+x^2)^{1/2}} - \cancel{0}\right] - \frac{\hat{x}}{\cancel{2}}\left[-\cancel{2}u^{-1/2}\right]_{z^2}^{z^2+r^2}\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{r\hat{z}}{z(z^2+x^2)^{1/2}} + \hat{x}\left[\frac{1}{(z^2+r^2)^{1/2}} - \frac{1}{(z^{\cancel{2}})^{1/\cancel{2}}}\right]\right)$$

$$\boxed{\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{r}{z(z^2+x^2)^{1/2}}\hat{z} + \left(\frac{1}{(z^2+r^2)^{1/2}} - \frac{1}{z}\right)\hat{x}\right]}$$



2.6 $Q = \lambda(\overbrace{2\pi R}^{\text{Circumference}})$ $\qquad \vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ $\qquad \hat{\mathscr{r}} = \vec{\mathscr{r}}/\mathscr{r}$

$d\vec{E} = \frac{k\,dq\,\hat{\mathscr{r}}}{\mathscr{r}^2}$ $\qquad \vec{\mathscr{r}} = z\hat{z} - R\hat{s}$ $\qquad \hat{\mathscr{r}} = z\hat{z} - R\hat{s} \,/\, (z^2+R^2)^{1/2}$

$\mathscr{r}^2 = z^2 + R^2$

$$d\vec{E} = \frac{k\,dq}{(z^2+R^2)}\left(\frac{z\hat{z} - R\hat{s}}{(z^2+R^2)^{1/2}}\right) = \frac{k\,dq\,(z\hat{z} - R\hat{s})}{(z^2+R^2)^{3/2}}$$

arc length = $r\phi = R\phi$ $\qquad d\vec{E} = \frac{k(\lambda R\,d\phi)(z\hat{z} - \cancelto{0}{R\hat{s}})}{(z^2+R^2)^{3/2}}$

d(arc) = $R\,d\phi$

λ d(arc) = $\lambda R\,d\phi$ $\qquad \int_0^{2\pi} d\phi\,\hat{s} = \int_0^{2\pi} \hat{x}\cos\phi + \hat{y}\sin\phi = 0$

$dq = \lambda R\,d\phi$

$$\int d\vec{E} = \int_0^{2\pi} \frac{k\lambda R\, z\hat{z}\, d\phi}{(z^2+R^2)^{3/2}} \rightarrow \vec{E} = \left[\frac{k\lambda R z\hat{z}\,(\phi)}{(z^2+R^2)^{3/2}}\right]_0^{2\pi}$$

$$\vec{E} = \frac{(2\pi R)\,kz\hat{z}}{(z^2+R^2)^{3/2}} = \boxed{\frac{Qkz\hat{z}}{(z^2+R^2)^{3/2}}}$$

as $R \rightarrow \infty$: $\vec{E} \approx 0$

when $z \gg R$: $\vec{E} \sim Qkz\hat{z} \,/\, z^3$

2.9 $\vec{E} = kr^3(\hat{r})$ (in spherical coordinates)

a) $\nabla \cdot E = (1/\varepsilon_0)\rho \rightarrow \rho = (\nabla \cdot E)\,\varepsilon_0$

$(\nabla \cdot E) = 3kr^2 \rightarrow \boxed{\rho = 3k\varepsilon_0 r^2}$

b) $Q_{enc} = \int_V \rho\, d\tau$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} 3k\varepsilon_0 r^2\, dr\, d\theta\, d\phi = \cancel{3}k\varepsilon_0 \int_0^{2\pi}\int_0^{\pi}\left[\frac{r^3}{\cancel{3}}\right]_0^R d\theta\, d\phi$$

$$= k\varepsilon_0 r^3(2\pi \cdot \pi) = 2\pi^2 k\varepsilon_0 R^3$$

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

$$Q_{enc} = \left(\oint \vec{E} \cdot d\vec{a}\right)\varepsilon_0 = \int kr^3\,\cancel{\hat{r}} \cdot (r^2 \sin\theta\, d\theta\, d\phi\, \cancel{\hat{r}}) = \int kr^5 \sin\theta\, d\theta\, d\phi$$

$$= kr^5 \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = kr^5 \int_0^{2\pi} \left[-\cos\theta\right]_0^{\pi} d\phi = kr^5 \int_0^{2\pi} (\overbrace{-\cos\pi + \cos 0}^{1+1=2})\, d\phi$$

$$= kr^5 \int_0^{2\pi} 2\, d\phi = 2kr^5(2\pi) = 4\pi k\varepsilon_0 r^5 = 4\pi k\varepsilon_0 R^5$$

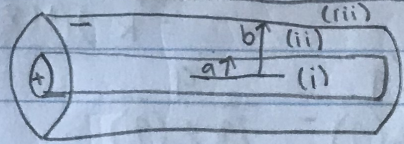
2.12 Gauss' Law: $\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$, $Q_{enc} = q$

$$\int |\vec{E}| da = |\vec{E}| \int da = |\vec{E}| (4\pi R^2)$$

$$|\vec{E}|(4\pi R^2) = \frac{Q_{enc}}{\varepsilon_0} = \frac{q}{\varepsilon_0}$$

$$\boxed{\vec{E} = \frac{q}{(4\pi R^2)\varepsilon_0}\hat{R}}$$

2.16



$$dE = \frac{k\,dq\,\hat{\imath}}{\imath^2} \qquad \hat{\imath} = \frac{\vec{\imath}}{\imath}$$

(i) inside the inner cylinder $(s < a)$

volume charge density: ρ

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$
$$\hat{z} = \hat{z}$$

$$\vec{E} = k\int \frac{\rho}{\imath^2}\hat{\imath}\,d\tau'$$

$$\vec{\imath} = \vec{r} - \vec{r}\,' = z\hat{z} - s\hat{s} \rightarrow \imath = (z^2+s^2)^{1/2}$$

$$\hat{\imath} = \frac{\vec{\imath}}{\imath} = \frac{z\hat{z} - s\hat{s}}{\sqrt{z^2+s^2}} \qquad Q = 2\pi R$$

$$\rho\,dQ = \rho s\,d\phi \rightarrow dq = \rho s\,d\phi$$

$$d\vec{E} = \frac{k(\rho s\,d\phi)\,(z\hat{z} - s\hat{s})}{(z^2+s^2)^1\,(z^2+s^2)^{1/2}} = \frac{(k\rho s\,d\phi)(\cancel{z\hat{z}} - s\hat{s})}{(z^2+s^2)^{3/2}}$$

$$\int d\vec{E} = -k\rho\int_0^{2\pi}\left(\frac{s^2\hat{s}\,d\phi}{(z^2+s^2)^{3/2}}\right) \rightarrow \boxed{\vec{E} = \frac{-2\pi k\rho\,s^2\hat{s}}{(z^2+s^2)^{3/2}}}$$

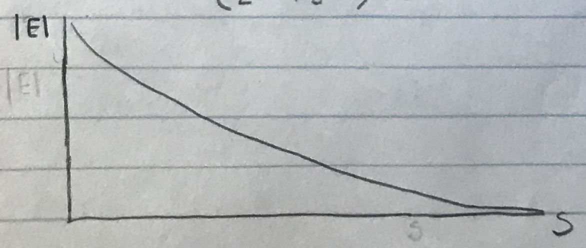
(ii) between the cylinders $(a < s < b)$

uniform surface charge: σ

$$\vec{E} = \frac{-2\pi k\sigma(b-a)^2\hat{s}}{(z^2+s^2)^{3/2}}$$

(iii) outside the cable $(s > b)$

$$\vec{E} = \frac{-2\pi k\sigma(b)^2\hat{s}}{(z^2+s^2)^{3/2}}$$



2.18 $\vec{E} = \frac{kq}{r^2}\hat{r} \rightarrow \frac{kq}{d^2}\hat{d}$ $\qquad q = \rho - (-\rho)$

$q = 2\rho$

$\vec{E} = \frac{k\,2\rho}{d^2}\hat{d} = \frac{1}{\cancel{4}\pi\varepsilon_0}\frac{\cancel{2}\rho}{d^2}\hat{d} = \boxed{\frac{\rho\,\hat{d}}{2\pi\varepsilon_0 d^2}}$

2.25 Eq 27: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$ $\qquad$ Eq 30: $V = \frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(\vec{r}')}{\mathscr{r}}da'$

$= k\int(\lambda(\vec{r}')/\mathscr{r})\,dl'$

a) $V = \frac{1}{4\pi\varepsilon_0}\left(\frac{+q}{\mathscr{r}} + \frac{+q}{\mathscr{r}}\right) = \frac{\cancel{2}q}{\cancel{4}\pi\varepsilon_0(\mathscr{r})} = \frac{q}{2\pi\varepsilon_0(\mathscr{r})}$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = (d/2)\hat{x} - (z)\hat{z}$

$\mathscr{r} = (d^2/4) + z^2$

$V = \frac{q}{2\pi\varepsilon_0(d^2/4 + z^2)} = \boxed{\frac{q}{(\pi\varepsilon_0 d^2/2) + (2\pi\varepsilon_0 z^2)}}$

$E = -\nabla V = -\frac{d}{dz}\,q\left((\pi\varepsilon_0\frac{d^2}{2}) + (2\pi\varepsilon_0 z^2)\right)^{-1}$

$= -q\left((\pi\varepsilon_0\frac{d^2}{2} + 2(2\pi\varepsilon_0 z)\right)^{-1}$

$= \boxed{-q\cdot\left((\pi\varepsilon_0(d^2/2) + (4\pi\varepsilon_0 z)\right)^{-1}}$

- if Figure 34a was changed to $-q$, then the potential would equal 0
- it would suggest that the field is constant

b) $\lambda(\vec{r}') = \lambda(2L) = 2\lambda L$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = z\hat{z} - x\hat{x} \rightarrow \mathscr{r} = (z^2 + x^2)^{1/2}$, $dl' = dx$

$V = \frac{1}{\cancel{4}_2\pi\varepsilon_0}\int_{-L}^{L}\frac{\cancel{2}\lambda L\,dx}{(z^2+x^2)^{1/2}} = \frac{\lambda L}{2\pi\varepsilon_0}\int_{-L}^{L}\frac{dx}{(z^2+x^2)^{1/2}}$

c) $\sigma(\vec{r}\,') = \sigma \pi R^2$, $\mathscr{r} = (z^2 + R^2)^{1/2}$, $da' = dR$

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma \pi R^2 (dR)}{(z^2+R^2)^{1/2}} = \frac{\sigma \cancel{\pi}}{4\cancel{\pi}\varepsilon_0} \int \frac{R^2\, dR}{(z^2+R^2)^{1/2}}$$

$U = z^2 + R^2$

$R^2 = U - z^2$

$R = (U - z^2)^{1/2}$

$dR = \frac{1}{2}(U-z^2)^{-\frac{1}{2}} dU$

$$= \frac{\sigma}{4\varepsilon_0} \int \frac{(U-z^2)' \frac{1}{2}(U-z^2)^{-1/2}\, dU}{U}$$

$$= \frac{\sigma}{8\varepsilon_0} \int \frac{\sqrt{U-z^2}}{U}\, dU = \frac{\sigma}{8\varepsilon_0} \int \frac{\sqrt{U(1-\frac{z^2}{U})}}{U}\, dU$$

2.29 Eq 29: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\, d\tau'$

$\nabla V = k \frac{\rho}{\mathscr{r}} \longrightarrow \nabla^2 V = 0$ ✓