Electromagnetic Theory: Special Presentation

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Outline

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1. The continuous limit
   - How accurate is the idea that $\Delta x \sum_i q_i \to \int dq$?
   - The spatial Fourier transform
2. The line of charges
   - Discrete, continuous
   - Far-field approximation to third order in $(1/r)^n$
   - Spatial Fourier transforms

The Continuous Limit

The Continuous Limit

The continuous limit: Why do we speak of macroscopic fields from fluid charge distributions? We know that charge is *discrete.*

The continuous limit: Why do we speak of macroscopic fluids from fluid mass distributions? We know that mass is *discrete.*

1. Consider a row of $2N+1$ point charges q along the x-axis, separated by Δx. Why $2N+1$? Place N on either side of the origin, and one at the origin. Let P be a distance z above the origin along the z-axis.

2. Consider concentric rings of point charges q along the ϕ-axis in cylindrical coordinates. Each ring is at some radius s from the origin. The separations are therefore $\Delta\phi$ and Δs, and there is one charge at the origin.

The Continuous Limit

Project goals:

1. Obtain discrete and continuous results to $\mathcal{O}(1/r)^3$
2. Calculate spatial Fourier transform of each result

The spatial Fourier transform relates position x with wavenumber $k = 2\pi/\lambda$ (inverse length units). The spatial Fourier transform of a function $f(z)$ is defined:

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikz} dz \tag{1}$$

$$f(x) = \int_{-\infty}^{\infty} \widetilde{f}(k) e^{2\pi ikz} dk \tag{2}$$

The Line of Charges

The Line of Charges

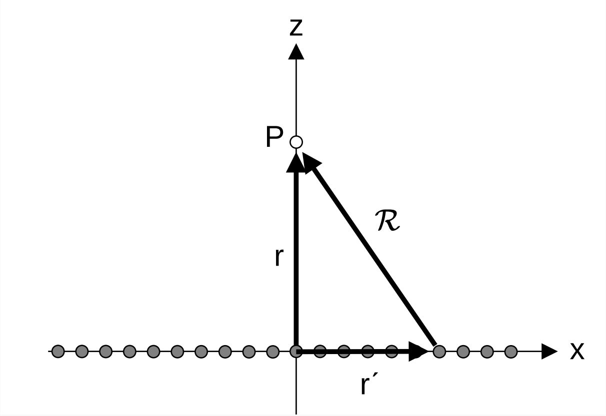

Figure 1: A row of charges extends down the x-axis and the observer position is along the z-axis.

The Line of Charges

The voltage may be treated as sum of $2N + 1$ Coulomb potentials, with the origin treated separately. Let

$$V_0 = \frac{1}{4\pi\epsilon_0}\frac{q}{z} \tag{3}$$

$$V_n = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{z^2 + (n\Delta x)^2}} \tag{4}$$

$$V(z) = V_0 + 2\sum_{n=1}^{N} V_n \tag{5}$$

Let's examine the limit that $\alpha < 1$, where $\alpha = \Delta x/z$.

The Line of Charges

The potential at z is

$$V(z) = \frac{1}{4\pi\epsilon_0}\frac{q}{z}\left(1 + 2\sum_{n=1}^{N}(1+\alpha^2 n^2)^{-1/2}\right) \tag{6}$$

$$V(z) \approx \frac{1}{4\pi\epsilon_0}\frac{q}{z}\left(1 + 2\sum_{n=1}^{N}\left(1 - \frac{1}{2}\alpha^2 n^2\right)\right) \tag{7}$$

$$V(z) \approx \frac{1}{4\pi\epsilon_0}\frac{q}{z}\left(1 + 2N - \frac{2}{2}\alpha^2\sum_{n=1}^{N} n^2\right) \tag{8}$$

$$V(z) \approx \frac{q(2N+1)}{4\pi\epsilon_0 z}\left(1 - \frac{\alpha^2}{(2N+1)}\sum_{n=1}^{N} n^2\right) \tag{9}$$

The Line of Charges

The sum of the first N squared integers is

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6} \tag{10}$$

The total charge Q is $Q = q(2N+1)$. Thus, $V(z)$ becomes

$$V(z) \approx \frac{Q}{4\pi\epsilon_0 z}\left(1 - \frac{\alpha^2}{(2N+1)}\frac{N(N+1)(2N+1)}{6}\right) \tag{11}$$

$$V(z) \approx \frac{Q}{4\pi\epsilon_0 z}\left(1 - \frac{\alpha^2 N(N+1)}{6}\right) \tag{12}$$

Suppose that $N \gg 1$, but such that $L = N\Delta x$ is constant. In that case, $N(N+1) \approx N^2$.

The Line of Charges

Notice that

$$N^2\alpha^2 = N^2\left(\frac{\Delta x}{z}\right)^2 \tag{13}$$

$$N^2\alpha^2 = \left(\frac{N\Delta x}{z}\right)^2 \tag{14}$$

$$N^2\alpha^2 = \left(\frac{L}{2z}\right)^2 = \frac{L^2}{4z^2} \tag{15}$$

Inserting Eq. 15 into Eq. 12:

$$V(z) \approx \frac{Q}{4\pi\epsilon_0 z}\left(1 - \frac{L^2}{24z^2}\right) \tag{16}$$

The Line of Charges

Let $\beta_1 = L^2/24$. The final result is

$$\boxed{V(z) \approx \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{z} - \frac{\beta_1}{z^3}\right)} \tag{17}$$

Now to compare to the continuous case. Let $dq = \lambda dx$, $\vec{r} = z\hat{z}$, and $\mathscr{r} = \sqrt{z^2 + x^2}$. Let the line of charge run from $-L/2$ to $L/2$. Let θ_1 be the angle between vertical and $\mathscr{r}$ when $x = -L/2$, and θ_2 for $x = L/2$. By the usual methods:

$$V(z) = \frac{\lambda}{4\pi\epsilon_0}\int_{\theta_1}^{\theta_2} \sec\theta d\theta \tag{18}$$

Symmetry allows:

$$V(z) = \frac{2\lambda}{4\pi\epsilon_0}\int_{0}^{\theta_2} \sec\theta d\theta \tag{19}$$

The Line of Charges

Recall that $\sec\theta_2 = \mathscr{r}/z$ and $\tan\theta_2 = L/2z$. The result of integration is

$$V(z) = \frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{\mathscr{r}}{z} + \frac{L}{2z}\right) \tag{20}$$

Using the definition $\mathscr{r} = \sqrt{z^2 + x^2}$, the logarithm can be expanded in a series:

$$V(z) \approx \frac{2\lambda}{4\pi\epsilon_0}\left(\frac{L}{2z} - \frac{\epsilon L^3}{z^3}\right) \tag{21}$$

The constant ϵ is $\epsilon = 0.0208...$ Letting $\beta_2 = 2\epsilon L^2$, and recognizing that $Q = \lambda L$, the final result is

$$\boxed{V(z) \approx \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{z} - \frac{\beta_2}{z^3}\right)} \tag{22}$$

The Line of Charges

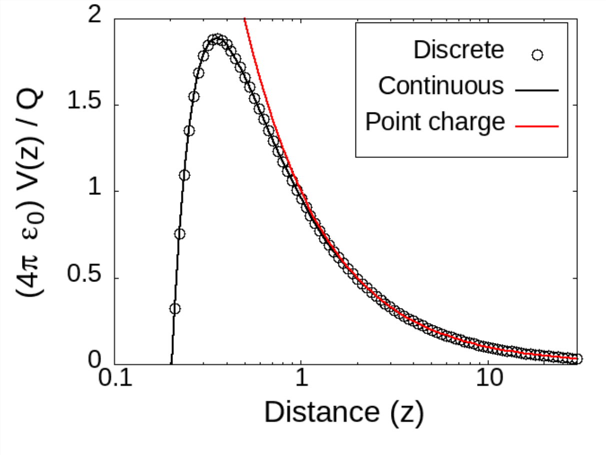

Figure 2: The potential versus z (in units of $Q/4\pi\epsilon_0$) for the discrete case (black circles), continuous linear density (black line), and a single point charge (red line).

Spatial Fourier transform

Spatial Fourier transform

What is the Fourier transform of $f(z)$?

$$f(z) = \frac{1}{z} - \frac{\beta}{z^3} \tag{23}$$

Let the Fourier transform in question be $\mathcal{F}(f(z))_k$. Relying on the linear property of Fourier transforms (indeed, all integrals), and consulting integral tables leads to the following:

$$\mathcal{F}(f(z)) = \mathcal{F}\left(\frac{1}{z}\right) - \beta\mathcal{F}\left(\frac{1}{z^3}\right) \tag{24}$$

$$\mathcal{F}\left(\frac{1}{z}\right) = -i\pi\,\mathrm{sign}(k) \tag{25}$$

$$\mathcal{F}\left(\frac{1}{z^3}\right) = -i\pi\frac{(-2i\pi k)^2}{2}\,\mathrm{sign}(k) \tag{26}$$

Spatial Fourier transform

Putting the pieces together, the magnitude of the spatial Fourier transform of $V(z)$ is

$$\boxed{|\widetilde{V}(k)| = \frac{Q}{4\pi\epsilon_0}\pi\left(1 + 2\beta_{1,2}\pi^2 k^2\right)} \tag{27}$$

The Line of Charges

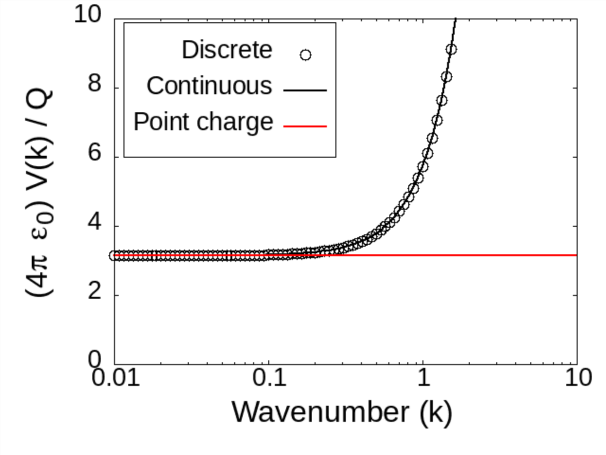

Figure 3: The spatial Fourier transform of the potential versus k (in units of $Q/4\pi\epsilon_0$) for the discrete case (black circles), continuous linear density (black line), and a single point charge (red line).

Summary

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