

Solutions for Homework 5

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1 Problem 5.4

Suppose that the magnetic field in some region has the form

$$\mathbf{B} = kz\hat{\mathbf{x}} \quad (1)$$

(where k is a constant). Find the force on a square loop (side a), lying in the yz -plane and centered at the origin, if it carries a current I , flowing counterclockwise, when you look down the x axis.

Using $\mathbf{F} = I\mathbf{L} \times \mathbf{B}$, the Lorentz force for current in a magnetic field, we find

$$\mathbf{F}_{\text{net}} = Ika^2\hat{\mathbf{z}} \quad (2)$$

2 Problem 5.7

For a configuration of charges and currents confined within a volume \mathcal{V} , show that

$$\int \mathbf{J} d\tau = \frac{d\mathbf{p}}{dt} \quad (3)$$

where \mathbf{p} is the total dipole moment. [Hint: evaluate $\int_{\mathcal{V}} \nabla \cdot (x\mathbf{J}) d\tau$].

Following the hint, keeping in mind that \mathcal{V} contains all currents and charges (so none penetrate the surface \mathcal{S} enclosing \mathcal{V}):

$$\nabla \cdot (x\mathbf{J}) = x(\nabla \cdot \mathbf{J}) + \mathbf{J} \cdot (\nabla x) = -x \frac{\partial \rho}{\partial t} + J_x \quad (4)$$

$$\int_{\mathcal{V}} \nabla \cdot (x\mathbf{J}) d\tau = \int_{\mathcal{V}} \left(-x \frac{\partial \rho}{\partial t} + J_x \right) d\tau \quad (5)$$

$$\int_{\mathcal{V}} \nabla \cdot (x\mathbf{J}) d\tau = \oint_{\mathcal{S}} (x\mathbf{J}) \cdot d\mathbf{a} = 0 \quad (6)$$

$$\int_{\mathcal{V}} \left(-x \frac{\partial \rho}{\partial t} + J_x \right) d\tau = 0 \quad (7)$$

$$\int_{\mathcal{V}} x \frac{\partial \rho}{\partial t} d\tau = \int_{\mathcal{V}} J_x d\tau \quad (8)$$

$$\int_{\mathcal{V}} y \frac{\partial \rho}{\partial t} d\tau = \int_{\mathcal{V}} J_y d\tau \quad (9)$$

$$\int_{\mathcal{V}} z \frac{\partial \rho}{\partial t} d\tau = \int_{\mathcal{V}} J_z d\tau \quad (10)$$

Combine the same argument used for x with the copies of it for y and z . Multiply by the corresponding unit vector on both sides of each equation, and sum. Then, switch the order of the time-derivative with the volume integration:

$$\int_{\mathcal{V}} \mathbf{J} d\tau = \frac{d}{dt} \int_{\mathcal{V}} \mathbf{r} \rho d\tau = \frac{d\mathbf{p}}{dt} \quad (11)$$

In words, the volume integration of all current densities in a closed space is the time-derivative of the dipole moment of all charge.