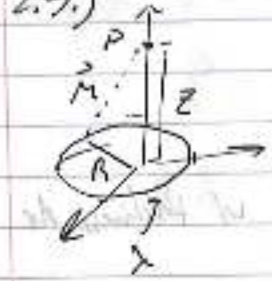
Andrew Hanscholder

E&M HW2:

2.5, 2.6, 2.9, 2.12, 2.16, 2.18, 2.25, 2.29

2.5.)



$$\vec{E} = \int d\vec{E} = \int \frac{K\,dq'}{\mathscr{r}^2}\hat{\mathscr{r}}$$

$$\vec{r} = z\hat{z} \qquad dq' = \lambda dl' = \lambda dl_\phi = \lambda R d\phi$$

$$\vec{r}' = R\hat{s}$$

$$\vec{\mathscr{r}} = z\hat{z} - R\hat{s}$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{|\mathscr{r}|} = \frac{z\hat{z} - R\hat{s}}{\sqrt{z^2+R^2}}$$

$$d\vec{E} = \frac{K\lambda R\,d\phi}{(z^2+R^2)^{3/2}}(z\hat{z} - R\hat{s})$$

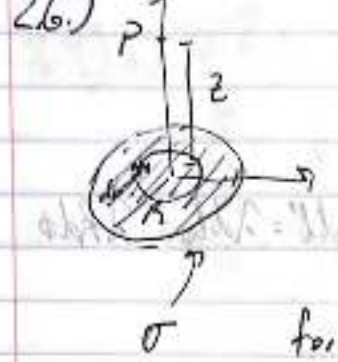
$$\vec{E} = \int d\vec{E} = \frac{K\lambda R}{(z^2+R^2)^{3/2}}\left[\int_0^{2\pi} z\hat{z}\,d\phi - \underbrace{\int_0^{2\pi} R\hat{s}\,d\phi}_{\emptyset}\right]$$

$\emptyset$ ← zero due to circular symmetry in radial direction.

$$= \frac{K\lambda R}{(z^2+R^2)^{3/2}}(z\hat{z}\,2\pi)$$

$$\boxed{\vec{E} = \frac{K(\lambda R 2\pi)z}{(z^2+R^2)^{3/2}}\hat{z} = \frac{KQz}{(z^2+R^2)^{3/2}}\hat{z}}$$

2.6.)



$$\vec{E} = \int d\vec{E} = \int \frac{K\,dq'}{r^2}\hat{r}$$

Consider concentric rings w/ thickness dr

for a ring w/ radius r

$$\vec{E}_r = \frac{KQz}{(z^2+R^2)^{3/2}}\hat{z} \qquad *Q = \sigma dr(2\pi r)$$

adding all rings $\vec{E}_{total} = \int_0^R \vec{E}_r$

$$= K2\pi z\hat{z}\int_0^R \frac{r}{(z^2+R^2)^{3/2}}dr$$

let $u=(z^2+R^2) \Rightarrow du = 2r\,dr$

$dr = \frac{1}{2r}du$

$$= K2\pi z\hat{z}\int_0^R \frac{1}{2}\frac{1}{u^{3/2}}du$$

$$= K2\pi z\hat{z}\left[\frac{1}{\cancel{2}}\cdot -\cancel{2}\,u^{-\frac{1}{2}}\right]_0^R$$

$$= K2\pi z\hat{z}\left[-(z^2+R^2)^{-\frac{1}{2}}\right]_0^R$$

$$= K2\pi z\hat{z}\left(\frac{-1}{\sqrt{z^2+R^2}} + \frac{1}{\sqrt{z^2}}\right)$$

$$\boxed{\vec{E}_{total} = K2\pi z\left(\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right)\hat{z}}$$

2.9) a) $\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\rho$

$\vec{E} = kr^3\hat{r}$

$\nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(kr^5)$

$= \frac{1}{r^2}5kr^4$

$\nabla\cdot\vec{E} = 5kr^2 = \frac{1}{\epsilon_0}\rho$

$\Rightarrow \boxed{\rho = 5\epsilon_0 k r^2}$

b) $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

$Q_{enc} = \epsilon_0\oint\vec{E}\cdot d\vec{a}$

$\cancel{= \epsilon_0 \int kr^3\hat{r}\cdot 4\pi r^2\hat{r}}$

$\cancel{= \epsilon_0 4\pi k r^5}$

for a sphere $\vec{E}\cdot d\vec{a} = E\,da$ since vectors parallel

$Q_{enc} = \epsilon_0\oint E\,da = \epsilon_0 E\oint da = \epsilon_0 EA$

$E = kr^3$

$A = 4\pi r^2$

$\boxed{Q_{enc} = \epsilon_0 k 4\pi r^5}$

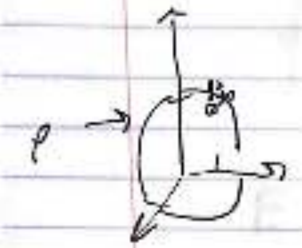
or ~~Qenc = ρ(volume)~~ $Q_{enc} = \int\rho\,d\tau$

$= \int_0^r (5\epsilon_0 k r^2)(4\pi r^2\,dr)$

$= 20\pi\epsilon_0 k\int_0^r r^4\,dr$

$\boxed{Q_{enc} = 4\pi\epsilon_0 k r^5}$

2.12.) $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$



$\vec{E} \cdot d\vec{a} = |\vec{E}||d\hat{a}| \cos\theta = E da$

$\oint E da = E \oint da = EA, \quad A = 4\pi r^2$

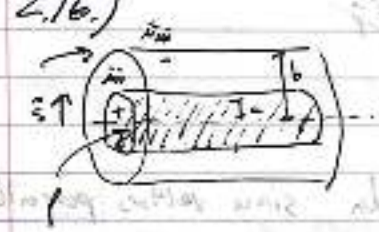
So, $E = \frac{Q_{enc}}{4\pi r^2 \epsilon_0} = \frac{\rho \text{ (volume)}}{4\pi\epsilon_0 r^2}$

$= \frac{\rho \frac{4}{3}\pi r^3}{4\pi \epsilon_0 r^2} = \frac{\rho r}{3\epsilon_0}$

this field points radially, so

$\boxed{\vec{E} = \frac{\rho r}{3\epsilon_0} \hat{r}}$

2.16.)



i) $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

$\oint \vec{E} d\vec{a} = EA \qquad A = 2\pi s l$

$E = \frac{Q_{enc}}{2\pi\epsilon_0 s l} \qquad Q_{enc} = \rho \pi s^2 l$ (same s, l)

$E = \frac{\rho s}{2\epsilon_0}$ points in $\hat{s}$ direction...

$\boxed{\vec{E} = \frac{\rho s}{2\epsilon_0} \hat{s}}$

2/23

2.16.) ii.) pick a cylinder enclosing the inner one...

$b > s > a$

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$Q_{enc} = \rho \frac{1}{2}\pi a^2 l$

$A = 2\pi s^2 l$

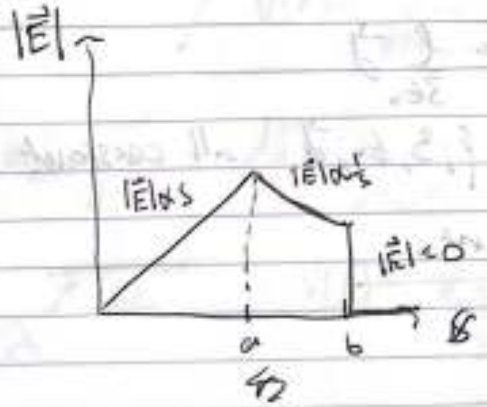
$$E = \frac{\rho 2\pi a^2 l}{2\pi \epsilon_0 s^2 l} = \frac{\rho a^2}{2\epsilon_0 s^2}$$

points radially... in $\hat{s}$

$$\boxed{\vec{E} = \frac{\rho a^2}{2\epsilon_0 s}\hat{s}}$$

iii.) outside both $Q_{enc} = 0$

So, $\oint \vec{E} \cdot d\vec{a} = 0 \Rightarrow \boxed{\vec{E} = 0}$



2.18.)

From 2.12: $\vec{E} = \frac{\rho r}{3\epsilon_0}\hat{r}$

at each point

$$\vec{E}_{total} = \vec{E}_+ + \vec{E}_-$$

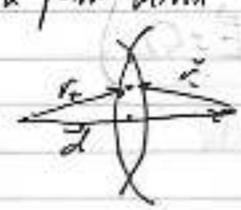
$$= \frac{\rho}{3\epsilon_0}\left(r_+\hat{r}_+ - r_-\hat{r}_-\right)$$

$r_+\hat{r}_+ = \vec{r}_+$

$r_-\hat{r}_- = \vec{r}_-$

$$\vec{E}_{total} = \frac{\rho}{3\epsilon_0}\left(\vec{r}_+ - \vec{r}_-\right)$$

at a point within intersection



$$\vec{d} = \vec{r}_+ - \vec{r}_-$$

$$\Rightarrow \vec{E}_{total} = \frac{\rho}{3\epsilon_0}\vec{d}$$

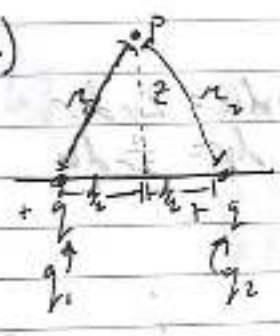
$\rho, 3, \epsilon_0, \vec{d}$ all constant

$\Rightarrow \vec{E}$ constant

2.25.) 2.27 $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1} \frac{q_i}{r_i}$

2.26 $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r} dq$

a.)



$r_1 = \sqrt{z^2 + (\frac{d}{2})^2} = r_2$

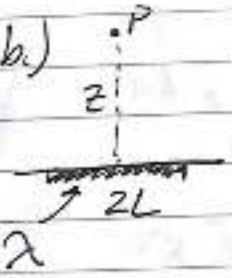
$\sum_{i=1} \frac{q_i}{r_i} = \frac{q_1}{r_1} + \frac{q_2}{r_2}$

$= \frac{q_1}{\sqrt{z^2+(\frac{d}{2})^2}} + \frac{q_2}{\sqrt{z^2+(\frac{d}{2})^2}}$

b/c $q_1 = q_2$

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{2q}{\sqrt{z^2+(\frac{d}{2})^2}}}$$

b.)



$dl' = dl_x = dx$

$r = \sqrt{z^2 + x^2}$

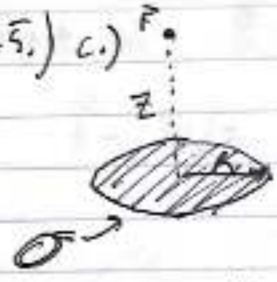
$V(\vec{r}) = k \int \frac{\lambda}{r} dl' = k\lambda \int \frac{1}{\sqrt{z^2+x^2}} dx$

$= k\lambda \left[\ln \left| x + \sqrt{z^2+x^2} \right| \right]_{-L}^{L}$

integral looked up

$$\boxed{V(\vec{r}) = k\lambda \ln \left| \frac{L + \sqrt{z^2+L^2}}{-L + \sqrt{z^2+L^2}} \right|}$$

2.25.) c.)



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\eta}\, da'$$

for a disk $\int da = \int_0^R 2\pi r\, dr$

$\eta = \sqrt{z^2+r^2}$

$$V(\vec{r}) = k\sigma \int_0^R \frac{2\pi r\, dr}{\sqrt{z^2+r^2}}$$

$u = z^2+r^2 \Rightarrow du = 2r\,dr$

$dr = \frac{1}{2r}\,du$

$$= k\sigma\pi \int_0^R u^{-\frac{1}{2}}\, du$$

$$= \frac{2k\sigma\pi}{\cancel{2}\,1}\left[u^{\frac{1}{2}}\right]_0^R = \frac{k\sigma\pi}{2}\left[(z^2+r^2)^{\frac{1}{2}}\right]_0^R$$

$$= \frac{2k\sigma\pi}{2}\left[(z^2+R^2)^{\frac{1}{2}} - z\right]$$

$$= \frac{2\sigma\pi}{2\cdot 4\pi\epsilon_0}\left[(z^2+R^2)^{\frac{1}{2}} - z\right]$$

$$\boxed{V(\vec{r}) = \frac{\sigma}{2\epsilon_0}\left[(z^2+R^2)^{\frac{1}{2}} - z\right]}$$

2.25.) cont. a) $\vec{E} = -\nabla V = -\left(\frac{\partial V_x}{\partial x}\hat{x} + \frac{\partial V_y}{\partial y}\hat{y} + \frac{\partial V_z}{\partial z}\hat{z}\right)$

$$\vec{E} = -\nabla V = \frac{1}{4\pi\epsilon_0}\left(\frac{-1}{\cancel{2}}\right)\frac{2q}{\left(\sqrt{z^2+(\frac{d}{2})^2}\right)^3}(\cancel{2}z)\,\hat{z}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left(\sqrt{z^2+(\frac{d}{2})^2}\right)^3}\,\hat{z}$$

b.) $\vec{E} = -\nabla V = \cancel{k\lambda\left(\frac{-L+\sqrt{z^2+L^2}}{L+\sqrt{z^2+L^2}}\right)\left(\frac{1}{2}\frac{2z}{\sqrt{z^2+L^2}}\right)\left(\frac{1}{2}\frac{2z}{(z^2+L^2)}\right)}$

$$= \frac{-\partial}{\partial z}\left[k\lambda\left(\ln|L+\sqrt{z^2+L^2}| - \ln|-L+\sqrt{z^2+L^2}|\right)\right]\hat{z}$$

$$= -k\lambda\left[\frac{1}{L+\sqrt{z^2+L^2}}\,\frac{1}{\cancel{2}}\frac{\cancel{2}z}{\sqrt{z^2+L^2}} - \left(\frac{1}{-L+\sqrt{z^2+L^2}}\,\frac{1}{\cancel{2}}\frac{\cancel{2}z}{\sqrt{z^2+L^2}}\right)\right]\hat{z}$$

$$= -k\lambda\frac{z}{\sqrt{z^2+L^2}}\left[\frac{1}{L+\sqrt{z^2+L^2}} - \frac{1}{-L+\sqrt{z^2+L^2}}\right]\hat{z}$$

$$= -k\lambda\frac{z}{\sqrt{z^2+L^2}}\left[\frac{-L+\cancel{\sqrt{z^2+L^2}} - L - \cancel{\sqrt{z^2+L^2}}}{z^2+\cancel{L^2}-\cancel{L^2}}\right]\hat{z}$$

$$= k\lambda\frac{\cancel{z}}{\sqrt{z^2+L^2}}\left[\frac{2L}{z^{\cancel{2}}}\right]\hat{z}$$

$$\boxed{\vec{E} = \frac{2L\lambda}{4\pi\epsilon_0}\frac{1}{z\sqrt{z^2+L^2}}\,\hat{z}}$$

2.28) c) $\vec{E} = -\nabla V$

$$= -\frac{\partial}{\partial z}\left[\frac{\sigma}{2\epsilon_0}\left[(z^2+R^2)^{\frac{1}{2}} - z\right]\right]\hat{z}$$

$$= -\frac{\sigma}{2\epsilon_0}\left[\frac{\cancel{2}z}{\cancel{2}\sqrt{z^2+R^2}} - 1\right]\hat{z}$$

$$\boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]\hat{z}}$$

2.29.) 2.29 $V(\vec{r}) = k\int \frac{\rho(r')}{\mathscr{r}}\,d\tau'$

$$\nabla^2 V = k\,\nabla^2\int\frac{\rho(r')\,d\tau'}{\mathscr{r}}$$

$$= k\int \rho(r')\,\nabla^2\frac{1}{\mathscr{r}}\,d\tau'$$ *from eq 1.102 $\nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\mathscr{r})$

$$= k\int \rho(r')\,(-4\pi\delta^3(\mathscr{r}))\,d\tau'$$ * $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$$= k(-4\pi)\int \rho(\vec{r}')\,\delta^3(\vec{r} - \vec{r}')\,d\tau'$$

$$= \frac{-4\pi}{4\pi\epsilon_0}\,\rho(\vec{r})$$

$$\boxed{\nabla^2 V = -\frac{\rho(\vec{r})}{\epsilon_0}}$$