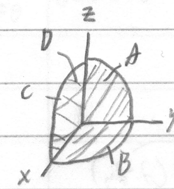
Jesse Chi-ShO.
11/2/20 PHYS

PHYS Ch. 1 HW

1.54, 55, 56, 57, 59, 62, 63, 64



1.54 $\vec{v} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$ ① ② ③

// $v_r = r^2\cos\theta,\ v_\theta = r^2\cos\phi,\ v_\phi = -r^2\cos\theta\sin\phi$

① $\frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\theta) = \frac{1}{r^2}(4r^3\cos\theta) = 4r\cos\theta$

② $\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi) = \frac{1}{r\sin\theta}(r^2\cos\theta\cos\phi) = \frac{r\cos\theta\cos\phi}{\sin\theta}$

③ $\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi) = \frac{1}{r\sin\theta}(-r^2\cos\theta\cos\phi) = -\frac{r\cos\theta\cos\phi}{\sin\theta}$ ✓

$\int_V \nabla\cdot\vec{v} = 4r\cos\theta = \int_0^{\pi/2}\int_0^{2\pi}\int_0^R 4r\cos\theta\, r^2\sin\theta\, dr\, d\theta\, d\phi \Rightarrow 4\int_0^R r^3 dr \int_0^{2\pi}\cos\theta\sin\theta\, d\theta \int_0^{\pi/2} d\phi$ ✓

$= 4\left[\frac{1}{4}r^4\right]_0^R \cdot$ // $\int_0^{2\pi}\cos\theta\sin\theta\, d\theta,\ u = \cos\theta,\ du = \sin\theta\, d\theta$ ✓

// $-\int_0^{2\pi} u\, du = -\left[\frac{u^2}{2}\right]_0^{2\pi} = -\left[\frac{\cos^2(2\pi)}{2} - \frac{\cos^2(0)}{2}\right] = \frac{+1}{2}$

$\int_V(\nabla\cdot\vec{v})\, d\tau = R^4\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi R^4}{4}}$ ✓

① // $\phi = 90°$ ② // $\theta = 0°$ ③ // $\phi = 0°$ ④ // $r = R$

$\oint_S \vec{v}\cdot d\vec{a} = \int_A \vec{v}\cdot d\vec{a} + \int_B \vec{v}\cdot d\vec{a} + \int_C \vec{v}\cdot d\vec{a} + \int_D \vec{v}\cdot d\vec{a}$ // $v_r = r^2\cos\theta,\ v_\theta = r^2\cos\phi,\ v_\phi = -r^2\cos\theta\sin\phi$

① $\iint_A \hat{\phi}\, dr\, d\theta = \int_0^{\pi/2}\int_0^R -r^2\cos\theta\sin\phi\, r\, dr\, d\theta$ // $\phi = 90 \rightarrow \sin\phi = 1 \Rightarrow \int_0^R r^3 dr \cdot \int_0^{\pi/2}\cos\theta\, d\theta$ ✓

$-\left[\frac{1}{4}R^4\right][\sin\theta]_0^{\pi/2} = \frac{-1}{4}R^4(1-0) = \frac{-1}{4}R^4$

② $\iint_B \hat{\theta}\, dr\, d\phi = \int_0^{\pi/2}\int_0^R r^2\cos\phi \cdot r\, dr\, d\phi \Rightarrow \int_0^R r^3 dr \cdot \int_0^{\pi/2}\cos\phi\, d\phi = \frac{1}{4}R^4(\sin\phi)_0^{\pi/2} = \frac{1}{4}R^4(1) = \frac{1}{4}R^4$

③ $\iint_C \hat{\phi}\, dr\, d\theta = \int_0^{\pi/2}\int_0^R -r^2\cos\theta\sin\phi \cdot r\, dr\, d\theta$ // $\phi = 0,\ \sin(0) = 0 \Rightarrow \int_0^{\pi/2}\int_0^R 0\, dr\, d\theta = 0$

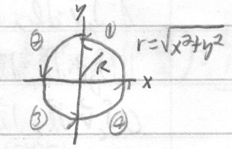
④ $\iint_D \hat{r}\, d\theta\, d\phi = \int_0^{\pi/2}\int_0^{\pi/2} r\cos\theta \cdot r^2\sin\theta\, d\theta\, d\phi \Rightarrow$ // $r = R$ // $\Rightarrow R^4\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta \cdot \int_0^{\pi/2} d\phi$

// $u = \cos\theta,\ du = \sin\theta\, d\theta \Rightarrow \int_0^{\pi/2} u\, du = \left[\frac{u^2}{2}\right]_0^{\pi/2} = \left[\frac{-\cos^2(\pi/2)}{2} + \frac{\cos^2(0)}{2}\right] = \left[0 + \frac{1}{2}\right] = \frac{1}{2}$ ✓

$R^4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi R^4}{4}}$ ✓

$\oint_S \vec{v}\cdot d\vec{a} = \frac{-1}{4}R^4 + \frac{1}{4}R^4 + 0 + \frac{\pi R^4}{4} = \boxed{\frac{\pi R^4}{4}}$

1.55 Check Stokes Theorem: $\int_S(\nabla\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$; $\vec{v} = ay\hat{x} + bx\hat{y}$



$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v_x & v_y \end{vmatrix} = \frac{\partial}{\partial x}v_y - \frac{\partial}{\partial y}v_x = \frac{\partial}{\partial x}(bx) - \frac{\partial}{\partial y}(ay) = b - a$

$\int_S(\nabla\times\vec{v})\cdot d\vec{a} = \int_0^{2\pi}\int_0^R (b-a)\cdot r\, dr\, d\theta = (b-a)\int_0^R r\, dr\int_0^{2\pi} d\theta = (b-a)\left[\frac{r^2}{2}\right]_0^R(2\pi) = (b-a)\frac{R^2}{2}(2\pi) = \boxed{\pi R^2(b-a)}$

$\oint_P \vec{v}\cdot d\vec{l} =$ // $x = r\cos\theta,\ y = r\sin\theta$, $dx = -r\sin\theta$, $dy = r\cos\theta$

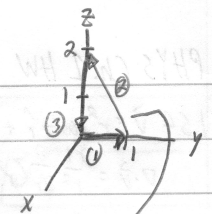
// $\vec{v}_x = a(r\sin\theta)\cdot -r\sin\theta = -a r^2\sin^2\theta$ // $\vec{v}_y = b(r\cos\theta)\cdot r\cos\theta = br^2\cos^2\theta$

$\int\vec{v}_x\, d\theta = \int_0^{2\pi} -ar^2\sin^2\theta\, d\theta = -ar^2\int_0^{2\pi}\sin^2\theta\, d\theta$ ✓ $\int\vec{v}_y\, d\theta = \int_0^{2\pi} br^2\cos^2\theta\, d\theta = br^2\int_0^{2\pi}\cos^2\theta\, d\theta = br^2\int_0^{2\pi}\frac{1}{2} + \frac{1}{2}\cos(2\theta)\, d\theta$ ✓

$-ar^2\int_0^{2\pi}\frac{1}{2} - \frac{1}{2}\cos(2\theta)\, d\theta \Rightarrow -ar^2\pi + \frac{1}{2}\left[\frac{-\sin(2\theta)}{2}\right]_0^{2\pi}$ ✓ $br^2\pi + \frac{1}{2}\left[\frac{-\sin(2\theta)}{2}\right]_0^{2\pi} \Rightarrow \vec{v}_y = br^2\pi$

$\vec{v}_x = -ar^2\pi$

$\oint_P \vec{v}\cdot d\vec{l} = \vec{v}_x + \vec{v}_y = -ar^2\pi + br^2\pi$ // $r = R \Rightarrow \boxed{(b-a)R^2\pi}$ ✓



1.56. $\int_S (\nabla\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$; $\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$

$\oint_P \vec{v}\cdot d\vec{l} = \int_{①} \vec{v}\cdot d\vec{l} + \int_{②} \vec{v}\cdot d\vec{l} + \int_{③} \vec{v}\cdot d\vec{l}$

① $d\vec{l} = +dy\,\hat{y} \Rightarrow \vec{v}\cdot d\vec{l} = yz^2\,dy \Rightarrow$ // $z=0$ // $\Rightarrow \vec{v}\cdot d\vec{l} = 0 \Rightarrow \int_{①} \vec{v}\cdot d\vec{l} = \underline{0}$

// $z = 2-2y$, $dz = -2dy$

$(2-2y)(2-2y)$

$4 = 4y - 4y + 4y^2$

$16 + \frac{32}{3} - 4 + 16$

$\frac{32}{3} - \frac{12}{3}$

② // parametrization // $d\vec{l} = dy\,\hat{y} + dz\,\hat{z} \Rightarrow \vec{v}\cdot d\vec{l} = yz^2\,dy + (3y+z)\,dz$

$\vec{v}\cdot d\vec{l} = y(2-2y)^2 dy + (3y+2-2y)(-2)\,dy \Rightarrow y(4-8y+4y^2)\,dy + (-2y-4)\,dy \Rightarrow 4y^3 - 4y^2 + 2y - 4\,dy$

$\int \vec{v}\cdot d\vec{l} = \int_1^0 4y^3 - 4y^2 + 2y - 4\,dy = \left[y^4 + \frac{4y^3}{3} + y^2 + 4y\right]_0^1 = \underline{\frac{14}{3}}$

③ $d\vec{l} = dy\,\hat{z} \Rightarrow \vec{v}\cdot d\vec{l} = (3y+z)\,dz$ // $y=0$ // $\Rightarrow \vec{v}\cdot d\vec{l} = z\,dz$

$-\int_0^2 z\,dz = -\left[\frac{z^2}{2}\right]_0^2 = \underline{-2}$

$\oint_P \vec{v}\cdot d\vec{l} = 0 + \frac{14}{3} - \frac{6}{3} = \boxed{8/3}$ ✓

$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 6x & yz^2 & 3y+z \end{vmatrix} = \hat{x}(3-2yz) - \hat{y}(0) + \hat{z}(0) \Rightarrow \nabla\times\vec{v} = (3-2yz)\,\hat{x}$

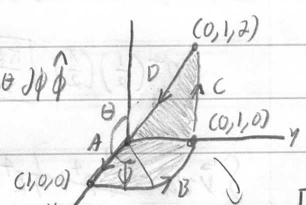
// $z = 2-2y$, $\frac{1}{2}z = 1-y$, $y = 1-\frac{1}{2}z$

$\int(\nabla\times\vec{v})\cdot d\vec{a} = \int_0^2\int_0^{1-\frac{1}{2}z} (3-2yz)\,dy\,dz \Rightarrow \int_0^2 dz\left[3y - y^2 z\right]_0^{1-\frac{1}{2}z} \Rightarrow \int_0^2 dz\left[3(1-\tfrac{1}{2}z) - z(1-\tfrac{1}{2}z)^2\right]$

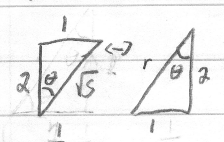
$= \int_0^2 dz\left[3 - \tfrac{3}{2}z - z(1 - z + \tfrac{1}{4}z^2)\right] = \int_0^2 dz\left[3 - \tfrac{3}{2}z - z + z^2 + \tfrac{1}{4}z^3\right] = \int_0^2 3 - \tfrac{5}{2}z - z + \tfrac{1}{4}z^3\,dz$

$= \left[3z - \frac{5}{4}z^2 + \frac{1}{3}z^3 - \frac{1}{16}z^4\right]_0^2 = \left[6 - 5 + \frac{8}{3} - 1\right] = \boxed{8/3}$ ✓

1.57. $\vec{v} = (r\cos^2\theta)\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}$. // $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$



$\oint \vec{v}\cdot d\vec{l} = \int_A \vec{v}\cdot d\vec{l} + \int_B \vec{v}\cdot d\vec{l} + \int_C \vec{v}\cdot d\vec{l} + \int_D \vec{v}\cdot d\vec{l}$



A: $\int_A \vec{v}\cdot d\vec{l}$, $d\vec{l} = dr\,\hat{r}$, $\phi = 0$, $\theta = \frac{\pi}{2} \Rightarrow \int_0^1 r\cos^2\theta\,dr = \underline{0}$

B: $\int_B \vec{v}\cdot d\vec{l}$, $d\vec{l} = r\sin\theta\,d\phi\,\hat{\theta}$, $r = 1$, $\phi = \frac{\pi}{2} \Rightarrow \int_0^{\pi/2} 3(1)^2(1)\,d\phi \Rightarrow 3\left(\frac{\pi}{2}\right) = \underline{\frac{3\pi}{2}}$

// $\sin\theta = \frac{1}{r}$

C: $\int_C \vec{v}\cdot d\vec{l}$, $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$, $\phi = \pi/2 \Rightarrow \int r\cos^2\theta\,dr - r^2\cos\theta\sin\theta\,d\theta$

// $r = \frac{1}{\sin\theta} \Rightarrow dr = \frac{-\cos\theta}{\sin^2\theta}\,d\theta$

// $\theta \in \left[\frac{\pi}{2}, \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right]$

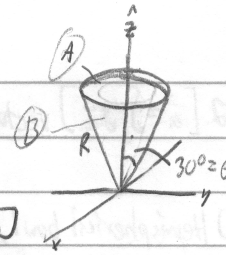
$= \int_{\pi/2}^{\sin^{-1}(1/\sqrt{5})} \left(\frac{1}{\sin\theta}\right)\cos^2\theta\left(\frac{-\cos\theta}{\sin^2\theta}\right)d\theta - \left(\frac{1}{\sin\theta}\right)^2\cos\theta\sin\theta\,d\theta \Rightarrow \int_{\pi/2}^{\sin^{-1}(1/\sqrt{5})} \frac{-\cos^3\theta}{\sin^3\theta} - \frac{\cos\theta}{\sin\theta}\,d\theta$

$= -\int_{\pi/2}^{\sin^{-1}(1/\sqrt{5})} \frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta\sin^2\theta}{\sin^3\theta}\,d\theta = -\int_{\pi/2}^{\sin^{-1}(1/\sqrt{5})} \frac{\cos^3\theta + \cos\theta\sin^2\theta}{\sin^3\theta}\,d\theta = -\int_{\pi/2}^{\sin^{-1}(1/\sqrt{5})} \frac{\cos\theta\overbrace{(\cos^2\theta + \sin^2\theta)}^{1}}{\sin^3\theta}\,d\theta$

$= -\int_{\pi/2}^{\sin^{-1}(1/\sqrt{5})} \frac{\cos\theta}{\sin^3\theta}\,d\theta \Rightarrow$ // $u = \sin\theta$, $du = \cos\theta\,d\theta$, $d\theta = \frac{du}{\cos\theta}$ // $u = \sin(\sin^{-1}(1/\sqrt{5})) = 1/\sqrt{5}$, $u = \sin(\pi/2) = 1$ $\Rightarrow -\int_1^{1/\sqrt{5}} u^{-3}\,du \Rightarrow +\left[\frac{1}{2u^2}\right]_1^{1/\sqrt{5}} \Rightarrow \left[\frac{1}{2}\frac{1}{(1/\sqrt{5})^2} - \frac{1}{2}\right] = \frac{1}{2}(5-1) = \underline{2}$

D: $\int_D \vec{v}\cdot d\vec{l}$, $d\vec{l} = dr\,\hat{r}$, $\phi = \frac{\pi}{2}$, $\theta = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) \Rightarrow \int_{\sqrt{5}}^0 r\cos^2\theta\,dr \Rightarrow \int_{\sqrt{5}}^0 r(1 - \underbrace{\sin^2\theta}_{1/5})\,dr \Rightarrow \left(1 - \frac{1}{5}\right)\left[\frac{1}{2}r^2\right]_{\sqrt{5}}^0 \Rightarrow \left(1 - \frac{1}{5}\right)\left(\frac{5}{2} - 0\right) = -\frac{4}{5}\cdot\frac{5}{2} = \underline{-2}$

So, $\oint \vec{v}\cdot d\vec{l} = 0 + \frac{3\pi}{2} + \cancel{2} \cancel{-2} = \boxed{\frac{3\pi}{2}}$



1.59, 62, 63, 64

$\theta = 30° = \pi/6$

1.59 $\vec v = r^2\sin\theta\,\hat r + 4r^2\cos\theta\,\hat\theta + r^2\tan\theta\,\hat\phi$, // $d\vec l = dr\,\hat r + r\,d\theta\,\hat\theta + r\sin\theta\,d\phi\,\hat\phi$

// $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$\int_V(\nabla\cdot v)\,d\tau = \int_S v\cdot da$ // $\theta\in[0,\pi/6]$, $\phi\in[0,2\pi]$, $r\in[0,R]$

$\nabla\cdot\vec v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot r^2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot 4r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\overset{0}{r^2\tan\theta})$

$= 4\frac{r^3}{r^2}\sin\theta + \frac{4r^2}{r\sin\theta}(\cos^2\theta - \sin^2\theta) = \cancel{4r\sin\theta} + \frac{4r\cos^2\theta}{\sin\theta} - \cancel{\frac{4r\sin^2\theta}{\sin\theta}} \Rightarrow \nabla\cdot\vec v = \frac{4r\cos^2\theta}{\sin\theta}$

$\int_V(\nabla\cdot\vec v)\,d\tau \Rightarrow \int_0^{2\pi}\int_0^{\pi/6}\int_0^R 4r^3\cos^2\theta\,dr\,d\theta\,d\phi = 4(2\pi)\int_0^{\pi/6}\cos^2\theta\,d\theta\int_0^R r^3\,dr$ // $\cos2\theta = 2\cos^2\theta - 1$, $\cos^2\theta = \frac12(\cos2\theta+1)$

$= 8\pi\int_0^{\pi/6}\frac12(\cos2\theta+1)\,d\theta\cdot\left[\frac14 r^4\right]_0^R \Rightarrow 8\pi\left(\frac12\right)\left(\frac14 R^4\right)\left\{\left[\frac12\sin(2\theta)\right]_0^{\pi/6} + [\theta]_0^{\pi/6}\right\} = R^4\pi\left\{\left(\frac12\sin\left(\frac\pi3\right) - \cancel{\frac12\sin(0)}\right) + \frac\pi6\right\}$

$= R^4\pi\left\{\frac12\cdot\frac{\sqrt3}{2} + \frac\pi6\right\} = R^4\pi\left(\frac{\sqrt3}{4} + \frac\pi6\right) = \frac{R^4\pi}{12}(3\sqrt3 + 2\pi) \Rightarrow \int_V(\nabla\cdot\vec v)\,d\tau = \frac{R^4\pi}{12}(2\pi + 3\sqrt3)$

$\oint_S\vec V\cdot da = \int_A\vec V\cdot d\vec a + \int_B\vec V\cdot d\vec a$

A: $\int_A\vec v\cdot d\vec a$, $r = R$; $d\vec a = \hat r\cdot r^2\sin\theta\,d\theta\,d\phi \Rightarrow \int_0^{2\pi}\int_0^{\pi/6} r^4\sin^2\theta\,d\theta\,d\phi = R^4(2\pi)\int_0^{\pi/6}\sin^2\theta$ // $\cos2\theta = 1 - 2\sin^2\theta$, $\sin^2\theta = \frac12(1-\cos2\theta)$

$= 2R^4\pi\int_0^{\pi/6}\frac12(1-\cos2\theta)\,d\theta \Rightarrow R^4\pi\int_0^{\pi/6}d\theta - \int_0^{\pi/6}\cos2\theta\,d\theta \Rightarrow R^4\pi\left\{\frac\pi6 - \left[\frac12\sin(2\theta)\right]_0^{\pi/6}\right\} \Rightarrow R^4\pi\left\{\frac\pi6 - \left(\frac12\sin\left(\frac\pi3\right) - \cancel{\frac12\sin(0)}\right)\right\}$

$= R^4\pi\left\{\frac\pi6 - \frac12\cdot\frac{\sqrt3}{2}\right\} \Rightarrow R^4\pi\left\{\frac\pi6 - \frac{\sqrt3}{4}\right\} \Rightarrow \frac{R^4\pi}{12}\{2\pi - 3\sqrt3\}$

B: $\int_B\vec v\cdot d\vec a$, $\theta = \frac\pi6$, $d\vec a = \hat\theta\cdot r\sin\theta\,dr\,d\phi = \hat\theta\cdot r\sin\left(\frac\pi6\right)dr\,d\phi \Rightarrow d\vec a = \hat\theta\cdot r\left(\frac12\right)dr\,d\phi \Rightarrow d\vec a = \frac12\hat\theta\,r\,dr\,d\phi$

$\int_0^{2\pi}\int_0^R\frac12 4r^2\cos\theta\cdot r\,dr\,d\phi \Rightarrow 2\cos\left(\frac\pi6\right)(2\pi)\int_0^R r^3\,dr \Rightarrow 4\pi\left(\frac{\sqrt3}{2}\right)\left[\frac14 r^4\right]_0^R \Rightarrow 2\sqrt3\pi\cdot\frac14 R^4 = \pi R^4\left(\frac{\sqrt3}{2}\right) = \frac{\pi R^4}{12}(6\sqrt3)$

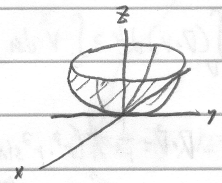
$\oint_S\vec v\cdot d\vec a = \frac{R^4\pi}{12}(2\pi - 3\sqrt3) + \frac{R^4\pi}{12}(6\sqrt3) \Rightarrow \frac{R^4\pi}{12}(2\pi - 3\sqrt3 + 6\sqrt3) \Rightarrow \frac{R^4\pi}{12}(2\pi + 3\sqrt3)$

$$\boxed{\int_V(\nabla\cdot\vec v)\,d\tau = \oint_S\vec v\cdot d\vec a = \frac{R^4\pi}{12}(2\pi + 3\sqrt3)}$$

"ordinary"

1.62 $\left[\vec{a} = \int_S d\vec{a}\right]$ <u>vector area</u>. If S is flat $\to |\vec{a}|$ is scalar area.

(a) Hemispherical bowl, radius R. // R constant // $r=R$ //



$d\vec{a} = \hat{r}\cdot R^2\sin\theta\, d\theta\, d\phi$

$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$$\vec{a} = \int_S d\vec{a} = \int_0^{2\pi}\int_0^{\pi/2} (\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z})\cdot R^2\sin\theta\, d\theta\, d\phi$$

$$= R^2\left\{\hat{x}\int_0^{2\pi}\int_0^{\pi/2}\sin^2\theta\cos\phi\, d\theta\, d\phi + \hat{y}\int_0^{2\pi}\int_0^{\pi/2}\sin^2\theta\sin\phi\, d\theta\, d\phi + \hat{z}\int_0^{2\pi}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\, d\phi\right\} \quad // \sin^2\theta = \tfrac{1}{2}(1-\cos(2\theta))$$

$$= R^2\left\{\hat{x}\int_0^{2\pi}\cos\phi\, d\phi\int_0^{\pi/2}\tfrac{1}{2}(1-\cos(2\theta))\, d\theta + \ldots\right\} \Rightarrow R^2\left\{\hat{x}\left[\sin\phi\right]_0^{2\pi}\cdot\tfrac{1}{2}\int_0^{\pi/2} 1-\cos(2\theta)\, d\theta + \ldots\right\}$$

$$= R^2\left\{\cancel{\hat{x}(0)\cdot\tfrac{1}{2}\int_0^{\pi/2} 1-\cos(2\theta)\, d\theta}^{\,0} + \hat{y}\int_0^{2\pi}\sin\phi\, d\phi\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\theta)\, d\theta + \ldots\right\} = R^2\left\{\hat{y}\left[-\cos\phi\right]_0^{2\pi}\cdot\tfrac{1}{2}\int_0^{\pi/2} 1-\cos(2\theta)\, d\theta + \ldots\right\}$$

$$= R^2\left\{\cancel{\hat{y}(0)\cdot\tfrac{1}{2}\int_0^{\pi/2} 1-\cos(2\theta)\, d\theta}^{\,0} + \hat{z}\int_0^{2\pi} d\phi\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\right\} \quad // u=\sin\theta,\ du=\cos\theta\, d\theta,\ d\theta = \tfrac{1}{\cos\theta}du$$

$$= R^2\left\{\hat{z}(2\pi)\int_0^{\pi/2} u\, du \Rightarrow 2\pi R^2\hat{z}\left[\tfrac{1}{2}u^2\right]_0^{\pi/2} \Rightarrow \pi R^2\hat{z}\left[\sin^2\theta\right]_0^{\pi/2} \Rightarrow \pi R^2\hat{z}\left(\sin^2\left(\tfrac{\pi}{2}\right)^{1} - \sin^2(0)^{0}\right)\right.$$

$$\boxed{\vec{a} = \int_S d\vec{a} = \pi R^2\hat{z}}$$

(b) $\vec{a} = \oint_S d\vec{a} = \int_V (\nabla\vec{a})\, d\tau$

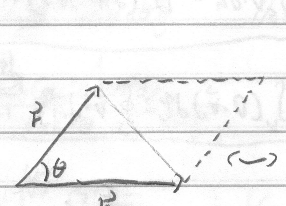
$\nabla\vec{a} = \frac{\partial}{\partial x}(\sin\theta\cos\phi) + \frac{\partial}{\partial y}(\sin\theta\sin\phi) + \frac{\partial}{\partial z}(\cos\theta) \Rightarrow \underline{\nabla\vec{a} = 0}$

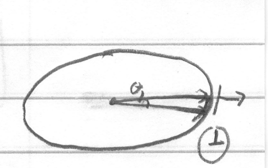
$\int_V (\nabla\vec{a})\, d\tau = 0 \Rightarrow \boxed{\vec{a} = \oint_S d\vec{a} = 0}$

(c) $\vec{a} = 0 \Rightarrow \oint(\nabla\vec{a})\cdot d\vec{a} = 0 \Rightarrow \vec{a}_1 - \vec{a}_2 = 0 \Rightarrow \boxed{\vec{a}_1 = \vec{a}_2}$

(d) $\vec{a} = \frac{1}{2}\oint \vec{r}\times d\vec{l}$.

$d\vec{a} = \frac{1}{2}(\vec{r}\times d\vec{l}) \Rightarrow \boxed{\vec{a} = \frac{1}{2}\oint \vec{r}\times d\vec{l}}$





(e) $\oint(\vec{c}\cdot\vec{r})\, d\vec{l} = -\oint_P T\, d\vec{l} = \int_S \nabla T\times d\vec{a}$. // $T = \vec{c}\cdot\vec{r}$ // $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, $\vec{c} = c_x\hat{x} + c_y\hat{y} + c_z\hat{z}$.

$\nabla T = \nabla(\vec{c}\cdot\vec{r}) = \vec{c}\times(\nabla\times\vec{r}) + (\vec{c}\cdot\nabla)\vec{r}$. // $\nabla\times\vec{r} = 0 \Rightarrow \nabla T = (\vec{c}\cdot\nabla)\vec{r} \Rightarrow \left(c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z})$

$= c_x\hat{x} + c_y\hat{y} + c_z\hat{z} = \vec{c}$. $\Rightarrow \oint(\vec{c}\cdot\vec{r})\, d\vec{l} = -\int\nabla T\times d\vec{a} = -\int\vec{c}\times d\vec{a} \Rightarrow -\vec{c}\times\int d\vec{a} \Rightarrow -\vec{c}\times\vec{a} \Rightarrow \boxed{\oint(\vec{c}\cdot\vec{r})\, d\vec{l} = \vec{a}\times\vec{c}}$

1.63

(a) $\vec{v} = \frac{\hat{r}}{r} \Rightarrow \nabla \cdot \vec{v} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{1}{r}\right) = \frac{1}{r^2}\frac{d}{dr}(r) = \frac{1}{r^2}(1) = \boxed{\frac{1}{r^2}}$

// r = R

$\oint_S \vec{v}\cdot d\vec{a} = \left(\frac{1}{R}\hat{r}\right)\left(R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}\right) = R\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = R(\cos\theta)_{\pi}^{0}(2\pi) = \boxed{4\pi R}$

$\int_V (\nabla\cdot\vec{v})\,d\tau,\ d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi \Rightarrow \int_0^R \int_0^{\pi} \int_0^{2\pi} \left(\frac{1}{\cancel{r^2}}\right)\cancel{r^2} \sin\theta\, dr\, d\theta\, d\phi =$

$= \int_0^R dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \Rightarrow R(2\pi)(\cos\theta)_{\pi}^{0} \Rightarrow \boxed{4\pi R}$ $\boxed{\int_V (\nabla\cdot\vec{v})\,d\tau = \oint_S \vec{v}\cdot d\vec{a}}$ ✓

[Divergence theorem is correct, so no discrepency w/ the integrals, so no dirac delta function.]

$\vec{v} = r^n \hat{r} \Rightarrow \nabla\cdot(r^n\hat{r}) = \frac{1}{r^2}\frac{d}{dr}(r^2\cdot r^n) = \frac{1}{r^2}\frac{d}{dr}(r^{n+2}) = \frac{1}{r^2}(n+2)r^{n+1} = (n+2)r^{-2}r^{n+1}$

$\boxed{\nabla\cdot(r^n\hat{r}) = (n+2)r^{n-1}}$ // when $n=-2$, $\nabla\cdot\left(\frac{1}{r^2}\hat{r}\right) \neq (0)r^{-3}$, but $= 4\pi\,\delta^3(\vec{r})$

(b) $\nabla\times(r^n\hat{r})$, $\vec{v} = r^n\hat{r}$

$\nabla\times(r^n\hat{r}) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\sin\theta\, v_{\phi}^{\,0} - \frac{\partial}{\partial\phi}v_{\theta}^{\,0}\right]\hat{r} = \boxed{0}$

$\int_V (\nabla\times\vec{v})\,d\tau = \int_V (0)\,d\tau = 0 = -\oint_S \vec{v}\times d\vec{a}$, $\vec{v} = r^n\hat{r}$, $d\vec{a} = r^2\sin\theta\, dr\, d\theta\, d\phi\, \hat{r}$

$\vec{v}\cdot d\vec{a} = (r^n)(r^2\sin\theta\, dr\, d\theta\, d\phi)(\hat{r}\times\hat{r})^{0} = \boxed{0}$ ✓

1.64 $\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^3(\vec{r})$, where $\vec{r}' = 0$. $\left(\nabla^2\left(\frac{1}{\mathcal{r}}\right) = -4\pi\,\delta^3(\mathcal{r})\right)$

$r = \sqrt{r^2+\epsilon^2}$ & $\epsilon \to 0$, Let $D(r,\epsilon) = \frac{-1}{4\pi}\nabla^2\frac{1}{\sqrt{r^2+\epsilon^2}}$

(a) $\nabla^2\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right) = -4\pi\,\delta^3\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right) = -4\pi\,\delta^3\left(\frac{1}{r\sqrt{1+(\epsilon/r)^2}}\right)$ // as $\epsilon\to 0 \Rightarrow -4\pi\,\delta^3\left(\frac{1}{r}\right)$

$D(r,\epsilon) = \nabla^2\left(\frac{1}{r}\right) = \frac{-1}{4\pi}\nabla^2\frac{1}{\sqrt{r^2+\epsilon^2}}$

// Laplacian, $\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\left(\frac{-1}{4\pi}\frac{1}{\sqrt{r^2+\epsilon^2}}\right)\right) \Rightarrow \frac{-1}{4\pi r^2}\frac{d}{dr}\left(r^2\cdot\left(\frac{-1}{2}\right)\left(\frac{1}{(r^2+\epsilon^2)^{3/2}}\right)(2r)\right)$

$= \frac{+1}{4\pi r^2}\frac{d}{dr}\left(\frac{r^3}{\sqrt{r^2+\epsilon^2}}\right) \Rightarrow \frac{1}{4\pi r^2}\frac{d}{dr}\left(r^3(r^2+\epsilon^2)^{-3/2}\right) = \frac{1}{4\pi r^2}\left[3r^2(r^2+\epsilon^2)^{-3/2} + r^3\left(\frac{-3}{2}\right)(r^2+\epsilon^2)^{-5/2}(2r)\right]$

$= \frac{1}{4\pi r^2}\left[\frac{3r^2}{(r^2+\epsilon^2)^{3/2}} - \frac{3r^4}{(r^2+\epsilon^2)^{5/2}}\right] \Rightarrow \frac{1}{4\pi r^2}\left[\frac{3r^2(r^2+\epsilon^2)}{(r^2+\epsilon^2)^{5/2}} - \frac{3r^4}{(r^2+\epsilon^2)^{5/2}}\right] = \frac{1}{4\pi \cancel{r^2}}\left[\frac{\cancel{3r^4} + 3\cancel{r^2}\epsilon^2 - \cancel{3r^4}}{(r^2+\epsilon^2)^{5/2}}\right]$

$= \frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}} \Rightarrow \boxed{D(r,\epsilon) = \frac{3\epsilon^2}{4\pi}(r^2+\epsilon^2)^{-5/2}}$

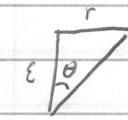
(b) $D(r,\varepsilon) = \frac{3\varepsilon^2}{4\pi}\left(r^2+\varepsilon^2\right)^{-5/2}$

$D(0,\varepsilon)$: $\frac{3\varepsilon^2}{4\pi}\left(\varepsilon^2\right)^{-5/2} = \frac{3\cancel{\varepsilon^2}}{4\varepsilon^5{}^3} = \frac{3}{4\varepsilon^3}$. $\varepsilon \to 0$: $D(0,\varepsilon) = \lim_{\varepsilon\to 0}\frac{3}{4\varepsilon^3} \simeq \boxed{\infty}$

(c) $D(r,\varepsilon) = \frac{3\varepsilon^2}{4\pi}\left(r^2+\varepsilon^2\right)^{-5/2}$

$\varepsilon \to 0,\ r \neq 0$: $D(r,\varepsilon) = \frac{3(0)^2}{4\pi}\left(r^2\right)^{-5/2} = (0)\,r^{-5} = \boxed{0}$

(d) $\int_{\text{all space}} D(r,\varepsilon)\,d\tau$, $d\tau = 4\pi r^2 dr = \int_{-\infty}^{\infty} \cancel{4\pi} r^2 \cdot \frac{3\varepsilon^2}{\cancel{4\pi}}\left(r^2+\varepsilon^2\right)^{-5/2} dr = 3\varepsilon^2 \int_{-\infty}^{\infty} \frac{r^2}{(r^2+\varepsilon^2)^{5/2}}\,dr$



// Trig. Sub: $\tan\theta = \frac{r}{\varepsilon} \Rightarrow r = \varepsilon\tan\theta$, $dr = \varepsilon\sec^2\theta$

$= 3\varepsilon^2\left[\frac{r(2r^2+3\varepsilon^2)}{6(r^2+\varepsilon^2)^{3/2}\varepsilon^2} - \frac{r}{2\cdot(r^2+\varepsilon^2)^{3/2}}\right]_{-\infty}^{\infty} \Rightarrow \left[\frac{r(2r^2+3\varepsilon^2)}{2(r^2+\varepsilon^2)^{3/2}} - \frac{3\varepsilon^2 r}{2(r^2+\varepsilon^2)^{3/2}}\right]_{-\infty}^{\infty}$

$= \left[\frac{\cancel{2}r^3+\cancel{3\varepsilon^2 r}-\cancel{3\varepsilon^2 r}}{\cancel{2}(r^2+\varepsilon^2)^{3/2}}\right]_{-\infty}^{\infty} \Rightarrow \left[\frac{r^3}{(r^2+\varepsilon^2)^{3/2}}\right]_{-\infty}^{\infty}$ // $\varepsilon \Rightarrow 0 \Rightarrow \left[\frac{\cancel{r^3}}{\cancel{r^3}}\right]_{-\infty}^{\infty} \Rightarrow \boxed{1}$