Quiz 2

3. The Curl of $\vec{E}$-fields

• Show that $-\int_a^B \vec{E}\cdot d\vec{l} = V(\vec{b}) - V(\vec{a})$

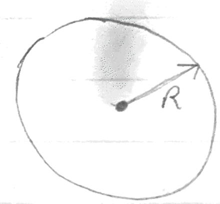
Since we know that $V(r) = -\int_0^r \vec{E}\cdot d\vec{l}$

$$V(\vec{b}) - V(\vec{a}) = -\int_0^B \vec{E}\cdot d\vec{l} + \int_0^a E\cdot d\vec{l}$$

$$= -\int_0^B \vec{E}\cdot d\vec{l} - \int_a^0 E\cdot dl = -\int_a^B \vec{E}\cdot d\vec{l}$$

$$V(\vec{r}) = -\int_\infty^r E(r')dr'$$

point charge at the origin of a closed loop



$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

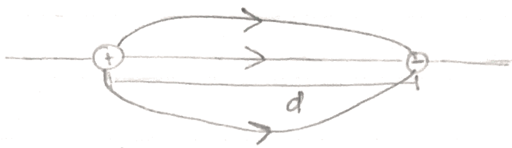
$$V(\vec{r}) = -\int_\infty^r \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^2}dr'$$

$$= \frac{+1}{4\pi\varepsilon_0}\left(\frac{q}{r'}\cancel{(-1)}\right|_\infty^r$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

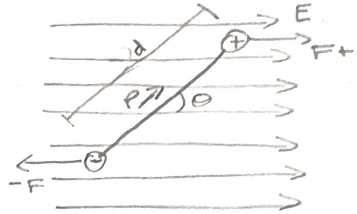
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} = \frac{kq}{r}$$

1.1 Distributions of Point Charges



$\vec{p} = q\vec{d}$ $\quad \vec{E} = E_0 \hat{x}$

Show that $\vec{\tau} = \vec{p} \times \vec{E}$



$F_+ = +qE$

$F_- = -qE$

Force components : $F_+ = +qE\sin\theta$

$F_- = -qE\sin\theta$

Torque = distance × Force

$= (d)(qE\sin\theta)$

$\tau = qdE\sin\theta$ $\quad \vec{p} = q\vec{d}$

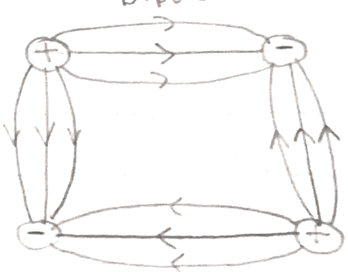
$\tau = \vec{p}\vec{E}\sin\theta$

Formula for cross products : $|\vec{A} \times \vec{B}| = AB\sin\theta$

$$\tau = \vec{p}\vec{E}\sin\theta = |\vec{p} \times \vec{E}|$$

1.2

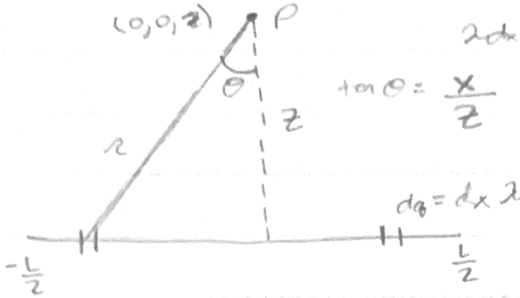
Dipole 1



Dipole 2

Since the dipole moments are symmetrical with each other the electric field vector in the center of the square is zero.

2. Continuous Charge Distributions $\qquad Q = \lambda L$



$\lambda dx = dq$

$\tan\theta = \dfrac{x}{z}$

$d\vec{E} = \dfrac{k\, dq\, \hat{\mathscr{r}}}{\mathscr{r}^2}$

$d\vec{E} = \dfrac{k\,\lambda\, dx\,(z\hat{z} - x\hat{x})}{(z^2+x^2)^1\,(z^2+x^2)^{1/2}}$

$d\vec{E} = \dfrac{k\,\lambda\, dx\,(z\hat{z} - x\hat{x})}{(z^2+x^2)^{3/2}}$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

$\vec{\mathscr{r}} = z\hat{z} - x\hat{x}$

$\mathscr{r}^2 = z^2 + x^2$

$\hat{\mathscr{r}} = \dfrac{\vec{\mathscr{r}}}{\mathscr{r}} = \dfrac{z\hat{z} - x\hat{x}}{(z^2+x^2)^{1/2}}$

$$d\vec{E} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{k\,\lambda\, dx\,(z\hat{z} - x\hat{x})}{(z^2+x^2)^{3/2}}$$

$$= k\lambda z\hat{z} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{(z^2+x^2)^{3/2}} - k\lambda\hat{x} \int_{-\frac{L}{2}}^{L/2} \frac{x\,dx}{(z^2+x^2)^{3/2}}$$

$x = z\tan(\theta)$

$dx = z\sec^2\theta\, d\theta$

$x\,dx = z^2 \tan\theta \sec^2\theta\, d\theta$

$$\vec{E} = k\lambda z\hat{z} \int_{\theta_1}^{\theta_2} \frac{z\sec^2\theta\, d\theta}{(z^2+z^2\tan^2\theta)^{3/2}} - k\lambda\hat{x} \int_{\theta_1}^{\theta_2} \frac{z^2\tan\theta\sec^2\theta\, d\theta}{(z^2+z^2\tan^2\theta)^{3/2}}$$

$$= k\lambda \frac{z^2\hat{z}}{z^3} \int_{\theta_1}^{\theta_2} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}} - k\lambda\hat{x}\frac{z^2}{z^3} \int_{\theta_1}^{\theta_2} \frac{\tan\theta\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}}$$

$$= k\lambda \frac{\hat{z}}{z} \int_{\theta_1}^{\theta_2} \frac{\sec^2\theta}{\sec^3\theta} - k\lambda\frac{\hat{x}}{z} \int_{\theta_1}^{\theta_2} \frac{\tan\theta\sec^2\theta}{\sec^3\theta}\, d\theta$$

$$= k\lambda \frac{\hat{z}}{z} \int_{\theta_1}^{\theta_2} \cos\theta\, d\theta - k\lambda\frac{\hat{x}}{z} \int_{\theta_1}^{\theta_2} \tan\theta\,/\cos\theta\; d\theta$$

$= k\lambda \frac{\hat{z}}{z} \sin\theta \Big|_{\theta_1}^{\theta_2}$

$\vec{E} = k\lambda \frac{\hat{z}}{z} (\sin\theta_2 - \sin\theta_1)$ $\quad \sin\theta_1 = \frac{-L/2}{r}$ $\quad \sin\theta_2 = \frac{L/2}{r}$

$= k\lambda \frac{\hat{z}}{z} \left(\frac{\frac{1}{2}L}{(z^2 + \frac{1}{4}L^2)^{1/2}} + \frac{(+\frac{1}{2}L)}{(z^2+\frac{1}{4}L^2)^{1/2}} \right)$

$= \frac{k\lambda \frac{\hat{z}}{z} L}{(z^2 + \frac{1}{4}L^2)^{-1/2}} = \boxed{\frac{k\lambda L \hat{z}}{z^2} \left(1 + \frac{1}{4}(L/z)^2\right)^{-1/2}}$

If $L \gg z$

$\vec{E} = \frac{k\lambda L \hat{z}}{z^2} \left(1 + \frac{1}{4}(L)^2\right)^{-1/2}$

$= \frac{k\lambda L \hat{z}}{z^2} \left(1 + \frac{L^2}{4}\right)^{-1/2}$

B) Using Gauss' Law

$\oint E \cdot da = \frac{1}{\varepsilon_0} Q_{enc}$ $\qquad Q = \lambda L$

$\oint E \cdot da = \frac{1}{\varepsilon_0}(\lambda L)$

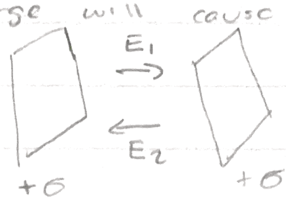
$E(a) = \frac{1}{\varepsilon_0}(\lambda L)$

$E(4\pi r^2 \cos\theta\, \hat{z}) = \frac{1}{\varepsilon_0}(\lambda L)$ $\qquad \cos\theta = \frac{z}{(z^2 + \frac{L^2}{4})^{1/2}}$

$E = \frac{\lambda L}{\varepsilon_0 4\pi r^2 z} \hat{z} \left(z^2 + \frac{L^2}{4}\right)^{-1/2}$

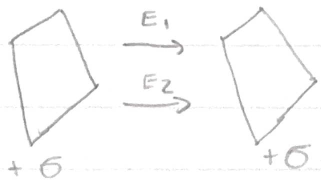
$= \frac{k\lambda L \hat{z}}{z^2} \left(1 + \frac{1}{4}(L/z)^2\right)^{-1/2}$

2. A) Electric field of a plane is $\frac{\sigma}{2\varepsilon_0}$ but two planes of positive charge will cause the total electric field to be zero



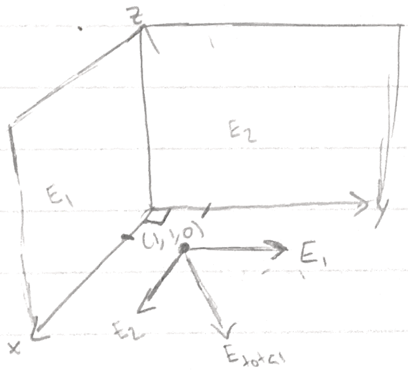
$$E_{total} = \frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = 0$$

B)



$$E_{total} = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

c)



$$\vec{E_1} = \frac{\sigma}{2\varepsilon_0}\hat{j} \qquad \vec{E_2} = \frac{\sigma}{2\varepsilon_0}\hat{i}$$

$$E_{total} = \frac{\sigma}{2\varepsilon_0}\hat{i} + \frac{\sigma}{2\varepsilon_0}\hat{j} + 0\hat{k}$$

$$E_{total} = \frac{\sigma}{2\varepsilon_0}\hat{i} + \frac{\sigma}{2\varepsilon_0}\hat{j}$$