Solutions for Homework 2

Dr. Jordan Hanson - Whittier College Dept. of Physics and Astronomy

February 26, 2022

1 Problem 2.5

Find the electric field a distance z above the center of a circular loop of radius r that carries a uniform line charge λ.

Start by filling in the pieces of the Coulomb effect:

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{dq'}{\mathscr{r}^{\,2}}\hat{\mathscr{r}} \tag{1}$$

- $\boldsymbol{\mathscr{r}} = z\hat{z} - R\hat{s}$
- $\mathscr{r} = \sqrt{z^2+R^2} = z\sqrt{1+(R/z)^2} = z\sqrt{1+\epsilon^2}$. (If $\epsilon \to 0$, this represents the far-field).
- $\mathscr{r}^{\,2} = z^2 + R^2$
- $\hat{\mathscr{r}} = (\hat{z} - \epsilon\hat{s})/(1+\epsilon^2)^{1/2}$
- $dq' = \lambda R d\phi'$, because cylindrical coordinates are the correct choice here.
- Note that $\epsilon = R/z$, so if $z = 0$ then $\epsilon \to \infty$, and $\epsilon \to 0$ if $z \gg R$.
- $Q = 2\pi R\lambda$, the total charge.

Integrate to add up the $d\mathbf{E}$ to find $\mathbf{E}$:

$$\mathbf{E} = \frac{\lambda R}{4\pi\epsilon_0 z^2(1+\epsilon^2)^{3/2}}\int_0^{2\pi} d\phi'(\hat{z} - \epsilon\hat{s}) \tag{2}$$

By symmetry, the $\hat{s}$ term evaluates to zero. The result is

$$\mathbf{E} = \frac{2\pi\lambda R\hat{z}}{4\pi\epsilon_0 z^2(1+\epsilon^2)^{3/2}} = \frac{Q\hat{z}}{4\pi\epsilon_0 z^2(1+\epsilon^2)^{3/2}} \tag{3}$$

Checks: $\mathbf{E} = 0$ if $z = 0$ because $\epsilon \to \infty$. Also, we find the far-field effect if $z \ll R$, because $\epsilon \to 0$. The units also check out.

2 Problem 2.6

Find the electric field a distance z above a the center of a flat circular disc of radius R that carries a uniform surface charge σ. What does your formula give in the limit $R \to \infty$? Also check the case $z \gg R$.

Start by filling in the pieces of the Coulomb effect:

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{dq'}{\mathscr{r}^{\,2}}\hat{\mathscr{r}} \tag{4}$$

- $\boldsymbol{\mathscr{r}} = z\hat{z} - s\hat{s}$. As in Problem 2.5, the $\hat{s}$-component will vanish upon integration.
- $\mathscr{r}^{\,2} = z^2 + s^2$
- $\hat{\mathscr{r}} = (z\hat{z} - s\hat{s})/(z^2+s^2)^{1/2}$
- $dq' = \sigma da' = s ds d\phi$, because cylindrical coordinates work best here.
- $Q = \pi R^2\sigma$ is the total charge.
- Let $z\tan\theta = s$, so that $ds = z\sec^2\theta d\theta$, and $\theta_0 = \tan^{-1}(R/z)$

Integrate to add up the $d\mathbf{E}$ to find $\mathbf{E}$:

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} z\hat{z} \int_0^R \frac{sds}{(s^2+z^2)^{3/2}} \tag{5}$$

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \cos\theta|_0^{\theta_0}\, \hat{z} = \frac{\sigma}{2\epsilon_0} \left(1-\cos\theta_0\right) \hat{z} \tag{6}$$

We know what is $\tan\theta = R/z$, but what is $\cos\theta_0$? (*Hint: draw the triangle and then find the hypoteneuse*). The result is

$$\mathbf{E} = \frac{\sigma\hat{z}}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) \tag{7}$$

The units check automatically because of the units of σ (Coulombs per meter squared), and the ϵ_0 in the denominator. If $R \to \infty$, then the second term vanishes and the field is

$$\mathbf{E} \to \frac{\sigma\hat{z}}{2\epsilon_0} \tag{8}$$

This form of the field is the boundary condition near a charged surface. To check the limit that $z \gg R$, first factor a z:

$$\mathbf{E} = \frac{\sigma\hat{z}}{2\epsilon_0} z\left(z^{-1} - (z^2+R^2)^{-1/2}\right) = \frac{\sigma\hat{z}}{2\epsilon_0} z\left(z^{-1} - z^{-1}(1+(R/z)^2)^{-1/2}\right) \tag{9}$$

Now replace $(1+(R/z)^2)^{-1/2} \approx (1-(1/2)(R/z)^2)$, and notice the $1/z$ terms cancel:

$$\mathbf{E} = \frac{\sigma\hat{z}}{2\epsilon_0} z\left(\frac{1}{2z}\left(\frac{R}{z}\right)^2\right) \tag{10}$$

Multiply top and bottom by π, and recall that $Q = \pi R^2$ to find the point-charge field:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q\hat{z}}{z^2} \tag{11}$$

3 Problem 2.9

Suppose the electric field in some region is found to be $\mathbf{E} = kr^3\hat{r}$, in spherical coordinates (k is some constant). (a) Find the charge densiy ρ. (b) Find the total charge contained in a sphere of radius R, centered at the origin. (Do it two ways).

(a) This is a straightforward application of Gauss' Law in differential form, noting that only the $\hat{r}$ term matters by symmetry:

$$\nabla\cdot\mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mathbf{E}\cdot\hat{r}\right) + ... = 5kr^2 \tag{12}$$

Thus the charge distribution is

$$\rho(r) = 5k\epsilon_0 r^2 \tag{13}$$

(b) The total charge Q may be found by (i) direct integration of Eq. 13, or (ii) by using Gauss' Law. First, (i):

$$Q = \int \rho(r)d\tau = \int_0^R\int_0^\pi\int_0^{2\pi} 5k\epsilon_0 r^2 r^2 \sin\theta dr d\theta d\phi = (4\pi)(5k\epsilon_0)\int_0^R r^4 dr = 4\pi k\epsilon_0 R^5 \tag{14}$$

Now, method (ii):

$$Q = \epsilon_0 \oint \mathbf{E}\cdot d\mathbf{a} = \epsilon_0 \int_0^{2\pi}\int_0^\pi kR^3\hat{r}\cdot\hat{r}R^2 d\theta d\phi = 4\pi k\epsilon_0 R^5 \tag{15}$$

4 Problem 2.12

Use Gauss' Law to find the electric field inside a uniformly charged solid sphere (charge density ρ).

Note that

- $d\mathbf{a} \propto \hat{r}$
- $\mathbf{E} \propto \hat{r}$ and only varies with r, by symmetry.
- Along a spherical Gaussian surface at radius r, $\mathbf{E}$ is constant.
- Thus, $\oint \mathbf{E} \cdot d\mathbf{a} = \mathbf{E} \cdot \mathbf{A}$, where $\mathbf{A}$ is the surface area of the Gaussian surface oriented in the $\hat{r}$ direction.

We now have

$$\mathbf{E} \cdot \mathbf{A} = \frac{1}{\epsilon_0}\rho \int d\tau' = \frac{\rho}{\epsilon_0}(4\pi) \int_0^r r'^2 dr' \tag{16}$$

The final result is

$$\mathbf{E}(\mathbf{r}) = \frac{\rho \mathbf{r}}{3\epsilon_0} = \frac{\rho r \hat{r}}{3\epsilon_0} \tag{17}$$