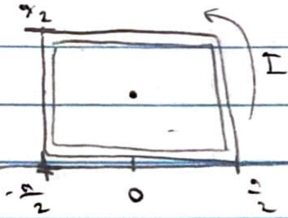
HW #5 — 5.4 5.7 5.11 5.12 5.16 5.19 5.21 5.12 Ex. 5.23 5.27 — Natasha Waldorf

5.4) Magnetic field in some region is $\vec{B} = kz\hat{x}$ where k is constant. Find the force on a square loop (side a), lying in the yz plane centered @ origin, carries current I counterclockwise from x axis.



$$F_{mag} = \int I(d\vec{l} \times \vec{B})$$
$$= I(a \times B)$$
$$= IaB$$
$$\boxed{= Iakz\hat{z}}$$

5.7) For a configuration of charges and currents confined within volume V show that $\int_V \vec{J}\, d\tau = d\vec{p}/dt$ where $\vec{p}$ is total dipole moment

$$\int_V \nabla\cdot(x\vec{J})\, d\tau = \int_V \left(-x\frac{\partial \rho}{\partial t} + J_x\right) d\tau$$

$$\nabla \cdot xJ = x(\nabla\cdot J) + J(\nabla \cdot x) = -x\frac{\partial\rho}{\partial t} + J_x$$

$$\int_V \nabla\cdot(x\vec{J})\, d\tau = \oint_S x\vec{J}\cdot d\vec{a} = 0 \quad \text{Gauss's law}$$

$$\int_V \left(-x\frac{\partial\rho}{\partial t} + J_x\right) d\tau = \oint_S x\vec{J}\cdot d\vec{a} = 0$$

$$\int_V \left(-x\frac{\partial\rho}{\partial t} + J_x\right) d\tau = 0$$

$$-\int_V x\frac{\partial\rho}{\partial t}\, d\tau + \int_V J_x\, d\tau = 0$$

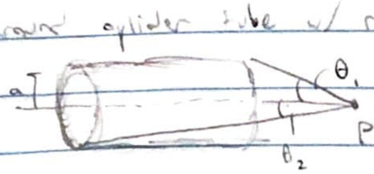
$$\int_V J_x\, d\tau = \int_V x\frac{\partial\rho}{\partial t}\, d\tau$$

same process but with $\int_V \nabla\cdot(y\vec{J})\, d\tau$ & $\int_V \nabla\cdot(z\vec{J})\, d\tau$

$$\int_V J_y\, d\tau = \int_V y\frac{\partial\rho}{\partial t}\, d\tau \qquad \int_V J_z\, d\tau = \int_V z\frac{\partial\rho}{\partial t}\, d\tau$$

$$\int_V \vec{J}\, d\tau = \int \vec{r}\frac{\partial\rho}{\partial t}\, d\tau = \frac{d}{dt}\int \vec{r}\rho\, d\tau = \frac{d\vec{p}}{dt}$$

5.11) Magnetic field at point P on axis of solenoid w/ n turns per unit length around cylinder tube w/ rad = a and current I. Field in infinite solenoid?



$$B(z) = \frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{r^2}\right) 2\pi R = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

$$B(z) = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}} \qquad I \to In\,dz \qquad z = a\cot(\theta)$$

$$dz = \frac{-a}{\sin^2\theta}d\theta$$

$$\vec{B} = \int \frac{\mu_0 In}{2}\frac{a^2}{(a^2+(a\cot(\theta))^2)}\left(\frac{-a}{\sin^2\theta}\right)d\theta$$

$$\vec{B} = -\frac{\mu_0 In}{2}\int \frac{a^3\sin^3\theta}{a^3\sin^2\theta}\, d\theta$$

$$\vec{B} = \frac{-\mu_0 In}{2}\int_{\theta_1}^{\theta_2} \sin\theta\, d\theta$$

$$\vec{B} = \frac{-\mu_0 In}{2}\left[-(\cos\theta_2 - \cos\theta_1)\right]$$

$$\boxed{\vec{B} = \frac{\mu_0 In}{2}(\cos\theta_2 - \cos\theta_1)}$$

for infinite solenoid $\theta_2 = 0°$, $\theta_1 = 180° \rightarrow \vec{B} = \frac{\mu_0 In}{2}(1-(-1))$

$\boxed{\vec{B} = \mu_0 In}$ for infinite solenoid

5.12) Calculate magnetic field at center of uniformly charged spherical shell of radius R and total charge Q spinning at constant velocity ω

$$B(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

$R \to R\sin\theta$ $\quad \sigma = \frac{Q}{4\pi R^2}$ $\quad v = \omega R = \omega R\sin\theta$

$z \to R\cos\theta$ $\quad K = \sigma v = \frac{Q}{4\pi R^2}\omega R\sin\theta = \frac{Q}{4\pi R}\omega\sin\theta$

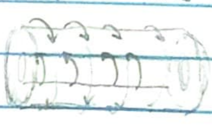
$I \to dI$ $\quad dI = KR\,d\theta = \frac{Q}{4\pi R}\omega\sin\theta R\,d\theta$

$dI = \frac{Q\omega}{4\pi}\sin\theta\,d\theta$

$$\vec{B} = \frac{\mu_0}{2}\int dI \frac{(R\sin\theta)^2}{((R\sin\theta)^2+(R\cos\theta)^2)^{3/2}}$$
$$= \frac{\mu_0}{2}\int \frac{Q\omega}{4\pi}\sin\theta\,d\theta \frac{R^2\sin^2\theta}{(R^2\sin^2\theta+R^2\cos^2\theta)^{3/2}}$$
$$= \frac{\mu_0}{2}\int \frac{Q\omega}{4\pi}\frac{R^2\sin^3\theta}{R^3}d\theta$$
$$= \frac{\mu_0 Q\omega}{8\pi R}\int_0^\pi \sin^3\theta\,d\theta$$
$$= \frac{\mu_0 Q\omega}{8\pi R}\left[-\left(\cos\theta - \frac{1}{3}\cos^3\theta\right)\right]\Big|_0^\pi$$
$$= -\frac{\mu_0 Q\omega}{8\pi R}\left[\left(-1+\frac{1}{3}\right)-\left(1-\frac{1}{3}\right)\right]$$
$$= -\frac{\mu_0 Q\omega}{8\pi R}\left(-2+\frac{1}{3}+\frac{1}{3}\right) \qquad \frac{2}{3}-\frac{6}{3}$$

$$\boxed{\vec{B} = \frac{\mu_0 Q\omega}{6\pi R}}$$

5.16) Two long coaxial solenoids each w/ current I but in opposite directions. Inner solenoid (rad a) has n_1 turns per unit length, outer (rad b) has n_2. Find $\vec{B}$ in each of three regions (i) inside inner (ii) between (iii) outside



(iii) $\boxed{\text{outside}: \vec{B} = 0}$ $\quad$ (i) $\boxed{\text{inside}: \vec{B} = \mu_0 I(n_2 - n_1)}$

(ii) $\boxed{\text{between}: \vec{B} = \mu_0 n_2 I}$ $\hat{z}$

outside field points right, inside points left $(-\hat{z})$

5.19) $I = \int_S \vec{J}\cdot d\vec{a}$ what surface?

$\int_S \vec{J}\cdot d\vec{a} = \oint \nabla\cdot\vec{J}$ Gauss

$\nabla\cdot\vec{J} = 0$ so the surface does not matter

5.21) Is Ampere's law consistent with the general rule Eq. 1.46 that divergence-of-curl is always zero? Show that Ampere's law cannot be valid outside magnetostatics. Is there any "defect" in the other 3 Maxwell equations?

1.46 : $\nabla \cdot (\nabla \times \vec{v}) = 0$

Ampere's Law: $\nabla \times \vec{B} = \mu_0 \vec{J}$ continuity $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 (\nabla \cdot \vec{J})$

$\nabla \cdot (\nabla \times \vec{B}) = -\mu_0 \frac{\partial \rho}{\partial t}$

It is not consistent with divergence of curl always being 0, only true when ρ is constant which only happens in magnetostatics.

There are no flaws like this in the other Maxwell equations

Ex. 5.12) Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{r}\, d\tau'$ s is radius s inside solenoid

$\oint \vec{A}\cdot d\vec{l} = \int \vec{B}\cdot d\vec{a}$

$\vec{A}(2\pi s) = B\pi s^2$

$2\pi s \vec{A} = \mu_0 n I \pi s^2$ $\vec{B} = \mu_0 n I$ inside solenoid

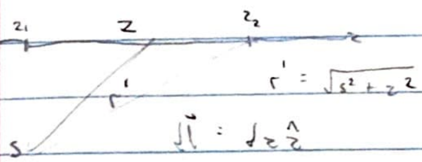
$\boxed{\vec{A} = \frac{\mu_0 n I s}{2} \text{ for } s \le R}$

$2\pi s \vec{A} = \mu_0 n I (\pi R^2)$

$\boxed{\vec{A} = \frac{\mu_0 n I R^2}{2s} \text{ for } s \ge R}$

5.22) Magnetic vector potential of finite segment of straight wire w/ current I.

$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{1}{r}\, d\vec{l}$ $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{r}\, da'$



$r' = \sqrt{s^2+z^2}$

$d\vec{l} = dz\,\hat{z}$

$\vec{A} = \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2} \frac{1}{\sqrt{s^2+z^2}}\, dz\,\hat{z}$

$\vec{A} = \frac{\mu_0 I}{4\pi} \ln\left(z + \sqrt{s^2+z^2}\right)\Big|_{z_1}^{z_2} \hat{z}$

$\vec{A} = \frac{\mu_0 I}{4\pi} \ln\left[\frac{z_2 + \sqrt{z_2^2+s^2}}{z_1 + \sqrt{z_1^2+s^2}}\right] \hat{z}$

$\vec{B} = \nabla \times \vec{A} = -\frac{\partial A_z}{\partial s}\hat{\phi}$

$= -\frac{\partial}{\partial s}\left[\frac{\mu_0 I}{4\pi} \ln\left[\frac{z_2+\sqrt{z_2^2+s^2}}{z_1+\sqrt{z_1^2+s^2}}\right]\right]\hat{\phi}$

$= \frac{-\mu_0 I}{4\pi}\left[\underbrace{\frac{s}{\sqrt{s^2+z_2^2}\,(\sqrt{s^2+z_2^2}+z_2)}}_{\text{where this } = \sin\theta_2} - \underbrace{\frac{s}{\sqrt{s^2+z_1^2}\,(\sqrt{s^2+z_1^2}+z_1)}}_{= \sin\theta_1}\right]\hat{\phi}$

& equals $\vec{B} = \frac{\mu_0 I}{4\pi s}\left[\sin\theta_2 - \sin\theta_1\right]$

5.22) Find vector potential above & below the plane surface current $\vec{K} = K\hat{x}$ over xy plane

$\vec{B} = \pm \frac{\mu_0 K}{2} \hat{y}$ $\vec{A}$ only depends on z so $\vec{A} = A(z)\hat{x}$

$$\vec{B} = \nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{A} & 0 & 0 \end{vmatrix}$$

$$\pm \frac{\mu_0 K}{2} \hat{y} = \frac{\partial}{\partial z} \vec{A} \hat{y}$$

$$\frac{\partial}{\partial z} \vec{A} = \frac{-\mu_0 K}{2}$$

$$A(z)\hat{x} = \frac{-\mu_0 K}{2}$$

$$\boxed{A = \frac{-\mu_0 K}{2} |z| \hat{x}}$$