HW #3
3, 5, 6, 13, 14, 15, 16, 19, 22, 24, 26

3 5 6 13 14 15 16 19 22 24 26
✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓

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**3.3)** $\nabla^2 V(r) = 0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V(r)}{\partial r}\right) = 0 \Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial V(r)}{\partial r}\right) = 0$$

so... $r^2 \frac{\partial V(r)}{\partial r} = A$ (constant)

$$\therefore \frac{\partial V(r)}{\partial r} = \frac{A}{r^2} \Rightarrow V(r) = \int \frac{A}{r^2}\, dr = \boxed{\frac{-A}{r} + C}$$ where $C$ = constant

$\nabla^2 V(s) = 0$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{V(s)}{\partial s}\right) = 0 \Rightarrow \frac{\partial}{\partial s}\left(s \frac{V(s)}{\partial s}\right) = 0$$

$$s \frac{V(s)}{\partial s} = A \text{ (constant)} \Rightarrow V(s) = \int \frac{A}{s}\, ds \Rightarrow \boxed{V(s) = A\ln|s| + C}$$ where $C$ = constant

**3.5)** $\nabla \cdot E_1 = \frac{\rho}{\varepsilon_0}$ $\quad \nabla \cdot E_2 = \frac{\rho}{\varepsilon_0}$ $\quad E_3 = E_2 - E_1 = \frac{\rho}{\varepsilon_0} - \frac{\rho}{\varepsilon_0} = 0$ $\quad \oint E_3 \cdot da = 0$

and $\nabla \cdot E_3 = 0$

$$\nabla \cdot (V_3 E_3) = V_3(\nabla \cdot E_3) + E_3(\nabla \cdot V_3) = 0 - E_3 \cdot E_3 = -E_3^2$$

$$\int_V \nabla \cdot (V_3 E_3)\, d\tau = \oint_S \nabla \cdot (V_3 E_3)\, da = -\int_V E_3^2\, d\tau$$

Since $\int_V E_3^2\, d\tau = 0$, it means $E_3 = 0$ and that $E_2$ and $E_1$ are equal. $\quad$ and $\int_V E_3^2\, d\tau = 0$

**3.6)** $$\int_V (V_3 \nabla^2 V_3 + \nabla V_3 \cdot \nabla V_3)\, d\tau = \oint_S (V_3 \nabla V_3)\, da$$

$$\Rightarrow -\int_V E_3^2\, d\tau = \oint_S (V_3 \nabla V_3)\, da \Rightarrow -\int_V E_3^2\, d\tau = \oint_S V_3 E_3\, da$$

so $\oint_S (V_3 E_3)\, da = -\int_V E_3^2\, d\tau$

$$\oint_S (V_3 E_3)\, da = -\int_V E_3^2\, d\tau = 0 \Rightarrow E_3 = 0,\ E_1 = E_2$$

**3.13**

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy \quad ; \text{when } n'=n$$

$$C_n = \frac{2}{a}\int V_0(y)\sin(n\pi y/a)\,dy$$

$$C_n = \frac{V_0(y)}{a}\left[\int_0^{a/2} \sin(n\pi y/a)\,dy + \int_{a/2}^{a} \sin(n\pi y/a)\,dy\right]$$

$$= \frac{2V_0(y)}{n\pi}\left[-\cos(n\pi/2)+1-\cos(n\pi/2)+(-1)^n\right]$$

$$= \frac{2V_0(y)}{n\pi}\left[1-2\cos(n\pi/2)+(-1)^n\right.$$

$n=1$ ; $=0$
$n=2$ ; $=4$
$n=3$ ; $=0$
$n=4$ ; $=0$
so $n=4k-2$

$C_n = \frac{8V_0}{n\pi}$ with $n=4k-2$ with $k=0,1,2,3,4,\ldots$

$$\boxed{V(x,y) = \frac{8V_0}{\pi}\sum_{k=0}^{\infty} \frac{e^{-(4k-2)\pi x/2}\sin((4k-2)\pi y/2)}{4k\ 2}}$$

**3.14**

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5\ldots} \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a) \qquad \sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$$

$$\sigma(y) = -\varepsilon_0 \frac{\partial}{\partial x}\left[\frac{4V_0}{\pi}\sum \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a)\right]$$

$$\sigma(y) = -\varepsilon_0 (4V_0/\pi)\sum \frac{1}{n}\left(\frac{-n\pi}{a}\right) e^{-n\pi x/a}\sin(n\pi y/a)$$

$$\sigma(y) = +\varepsilon_0\left(\frac{4V_0}{a}\right)\sum e^{-n\pi x/a}\sin(n\pi y/a)$$

at $x=0$: $\boxed{\sigma(y) = \frac{\varepsilon_0 \cdot 4 \cdot V_0}{a}\sum_{n=1,3,5,\ldots} \sin(n\pi y/a)}$

3.15) $V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$

1) $x=0, V=0$

$(A+B)(C\sin(ky)+D\cos(ky))=0$

$A=-B$

$Ae^{kx} - Ae^{-kx}$

$A(e^{kx} - e^{-kx}) \Rightarrow V(x,y) = \sinh(kx)(C\sin(ky) + D\cos(ky))$

2) $y=0, V=0$

$V(x,y) = C\sinh(kx)\sin(ky)$

3) $y=a, V=0$

$\sin(ka)=0$ $\quad k = n\pi/a$

$ka = n\pi \quad n=1,2,3,4\ldots$

$$V(x,y) = C\sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$V(b,y) = V_0(y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$\int_0^a V_0(y)\sin\left(\frac{m\pi y}{a}\right)dy = C_m \sinh\left(\frac{n\pi b}{a}\right)\left(\frac{a}{2}\right)$$

$$C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a V_0(y)\sin\left(\frac{m\pi y}{a}\right)dy$$

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

b) $$C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a V_0 \sin\left(\frac{n\pi y}{a}\right)dy$$

$$= \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\cdot V_0\left[-\cos\left(\frac{n\pi y}{a}\right)\frac{a}{n\pi}\right]_0^a$$

$\frac{2aV_0}{n\pi}$ n is odd

$\cos(\pi) = -1$

$\cos(\pi) = 1$

$$= \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\cdot\frac{V_0 a}{n\pi}(1-\cos(n\pi))$$

$$= \frac{2}{a\sin\left(\frac{n\pi b}{a}\right)}\cdot\frac{2aV_0}{n\pi} \Rightarrow V(x,y) = \boxed{\sum_{n=1,3,5,\ldots}\frac{4V_0}{\pi}\frac{\sinh\left(\frac{n\pi x}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\cdot\frac{\sin\left(\frac{n\pi y}{a}\right)}{n}}$$

3.16 $V(x,y,z) = (A\sin(kx) + B\cos(kx))(C\sin(ly) + D\cos(ly))(Ee^{\sqrt{k^2+l^2}z} + Fe^{-\sqrt{k^2+l^2}z})$

$$V(x,y,z) = A\sin(kx)\,C\sin(ly)\cdot E\underbrace{(e^{\sqrt{k^2+l^2}z} - e^{-\sqrt{k^2+l^2}z})}_{\substack{E+F=0 \\ E=-F}}$$

$$V(x,y,z) = C\sin(kx)\sin(ly)\sinh(\sqrt{k^2+l^2}\,z)$$

when $x=a$: $\sin(ka)=0$

$$ka = n\pi \qquad n=1,2,3\ldots \qquad k=\frac{n\pi}{a}$$

$\sin(la)=0$

when $y=a$: $\quad l=\frac{m\pi}{a} \qquad m=1,2,3\ldots$

$$V(x,y,z) = C_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\sqrt{n^2+m^2}\cdot\pi\cdot z}{a}\right)$$

when $z=a$

$$V(x,y,a) = V_0$$

$$V(x,y,a) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\sqrt{n^2+m^2}\cdot\pi\cdot\cancel{a}}{\cancel{a}}\right) = V_0$$

$$C_{n,m}\sinh(\sqrt{n^2+m^2}\,\pi)\left(\frac{a}{2}\right)^2 = \int_0^a\int_0^a V_0\sin\left(\frac{\pi n' x}{a}\right)\sin\left(\frac{\pi m' y}{a}\right)dx\,dy$$

$$C_{n,m}\left(\frac{a^2}{4}\right)\sinh(\sqrt{n^2+m^2}\,\pi) = V_0\left(\frac{a}{n\pi}\right)\left(\frac{a}{m\pi}\right)(1-\cos(\pi n))(1-\cos(m\pi))$$

$$C_{n,m}\left(\frac{\cancel{a^2}}{4}\right)\sinh(\sqrt{n^2+m^2}\,\pi) = \frac{4V_0\cancel{a^2}}{n\pi^2 m}$$

$$C_{n,m} = \frac{16V_0}{nm\pi^2\sinh(\sqrt{n^2+m^2}\,\pi)}$$

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n=1,3,5\ldots}\sum_{m=1,3,5\ldots}\frac{\sinh\left(\frac{\sqrt{n^2+m^2}\,z}{a}\right)}{\sinh(\sqrt{n^2+m^2}\,\pi)}\cdot\frac{\sin\left(\frac{n\pi x}{a}\right)}{n}\cdot\frac{\sin\left(\frac{m\pi y}{a}\right)}{m}}$$

3.19]   $V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l \cos\theta$

For inside: $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l \cos\theta$

For outside: $V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l \cos\theta$

$V_0 = K\cos(3\theta)$

$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$
$= aP_3\cos\theta + bP_1\cos\theta$
$= a\left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) + b\cos\theta$
$= \frac{5a}{2}\cos^3\theta + \left(\frac{-3a}{2} + b\right)\cos\theta$

$\frac{5a}{2} = 4$   $a = \frac{8}{5}$

$\frac{-3a}{2} + b = 3$

$\frac{\cancel{8}^{4}}{5}\left(\frac{-3}{\cancel{2}}\right) + b = 3$

$b = -3 + \frac{12}{5}$

$b = \frac{-3}{5}$

$V_0 = \frac{K}{5}\left(8P_3\cos\theta - 3P_1\cos\theta\right)$

$\int_0^{\pi} P_l\cos\theta \, P_l\cos\theta \sin\theta \, d\theta = \frac{2}{2l+1}$

$\sum_{l=0}^{\infty} A_l R^l P_l\cos\theta = \frac{K}{5}\left(8P_3\cos\theta - 3P_1\cos\theta\right)$

$A_1 R\left(\cancel{\frac{2}{3}}\right) = \frac{K}{5}(-3)\int_0^{\pi} P_1\cos\theta \, P_1\cos\theta \sin\theta \, d\theta$

$A_1 R = \frac{-3K}{5}\left(\cancel{\frac{2}{3}}\right) \Rightarrow \boxed{A_1 = -\frac{3K}{5R}}$

$A_3 R^3\left(\frac{2}{7}\right) = \frac{8K}{5}\int_0^{\pi} (P_3\cos\theta)^2 \sin\theta \, d\theta$

$A_3 R^3\cancel{\left(\frac{2}{7}\right)} = \frac{8K}{5}\cancel{\left(\frac{2}{7}\right)} \Rightarrow \boxed{A_3 = \frac{8K}{5R^3}}$

$$\boxed{V_{in}(r,\theta) = \frac{-3K}{5R} r P_1\cos\theta + \frac{8K}{5R^3} r^3 P_3\cos\theta}$$

$\sum_{l=0}^{\infty} A_l R^l P_l\cos\theta = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l\cos\theta$

$A_l R^l \cancel{P_l\cos\theta} = \frac{B^l}{R^{l+1}} \cancel{P_l\cos\theta}$

$A_l R^l = \frac{B_l}{R^{l+1}} \Rightarrow B_l = R^{2l+1} A_l$

$B_1 = R^{2+1} A_1$
$= R^3\left(\frac{-3K}{5R}\right) = \boxed{\frac{-3KR^2}{5}}$

$B_3 = R^7\left(\frac{8K}{5R^3}\right)$
$= \boxed{\frac{8KR^4}{5}}$

$$V_{out}(r,\theta) = \frac{B_1}{r^2} P_1 \cos\theta + \frac{B_3}{r^4} P_3 \cos\theta$$

$$\boxed{V_{out}(r,\theta) = \frac{-3KR^2}{5r^2} P_1 \cos\theta + \frac{8KR^4}{5r^4} P_3 \cos\theta}$$

$$\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} = -\frac{\sigma(\theta)}{\varepsilon_0}$$

$$\left[\frac{8KR^4(-4)}{5r^5} P_3 \cos\theta - \frac{3KR^2(-2)}{5r^3} P_1 \cos\theta\right] - \left[\frac{-3K}{5R} P_1 \cos\theta + \frac{8K}{5R^3} \cdot 3r^2 P_3 \cos\theta\right]$$

$\boxed{R=r}$ $$\left[\frac{-32K}{5r} P_3 \cos\theta + \frac{6K}{5r} P_1 \cos\theta\right] - \left[\frac{24K}{5r} P_3 \cos\theta - \frac{3K}{5r} P_1 \cos\theta\right] = \frac{-\sigma}{\varepsilon_0}$$

$$\frac{9K}{5r} P_1 \cos\theta - \frac{56K}{5r} P_3 \cos\theta = \frac{-\sigma}{\varepsilon_0} \Rightarrow \boxed{\sigma = \varepsilon_0\left[\frac{56}{5r} P_3 \cos\theta - \frac{9K}{5r} P_1 \cos\theta\right]}$$

3.22]

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l \cos\theta$$

when $r > R$

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l \cos\theta$$

$$V(r,0) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2} - r\right) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}}$$

$$\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2} - r\right) = \frac{\sigma r}{2\varepsilon_0}\left(\sqrt{1+\left(\frac{R}{r}\right)^2} - 1\right) = \frac{\sigma r}{2\varepsilon_0}\left(1 + \left(\frac{R}{r}\right)^2 \frac{1}{2} - \left(\frac{R}{r}\right)^4\left(\frac{1}{8}\right) - 1\right)$$

$$= \frac{\sigma}{2\varepsilon_0}\left(\left(\frac{R^2}{2r}\right) - \left(\frac{R^4}{8r^3}\right)\right)$$

$$B_0 = \frac{\sigma R^2}{4\varepsilon_0} \;;\; B_1 = 0$$

$$B_2 = \frac{-\sigma R^4}{16\varepsilon_0}$$

when $r < R$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell \cos\theta = \frac{\sigma}{2\varepsilon_0}\left(R\left(\sqrt{1+(r/R)^2}\right) - r\right) = \frac{\sigma}{2\varepsilon_0}\left(R\left(1+\frac{r^2}{2R^2}-\frac{r^4}{8R^4}\right)-r\right) = \frac{\sigma}{2\varepsilon_0}\left(R - r + \frac{r^2}{2R} - \frac{r^4}{8R^3}\right) = \sum_{\ell=0}^{\infty} A_\ell r^\ell$$

$$\boxed{A_0 = \frac{\sigma R}{2\varepsilon_0} \quad A_1 = \frac{-\sigma}{2\varepsilon_0} \quad A_2 = \frac{\sigma}{4\varepsilon_0 R}}$$ Northern

Southern: $$\boxed{A_0 = \frac{\sigma R}{2\varepsilon_0} \quad A_1 = \frac{\sigma}{2\varepsilon_0} \quad A_2 = \frac{\sigma}{4\varepsilon_0 R}}$$

**3.24]** $$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$\frac{s^2}{S\Phi}\left(\frac{1}{s}\Phi\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{1}{s^2}S\frac{\partial^2\Phi}{\partial\phi^2} = 0\right)\frac{s^2}{S\Phi}$$

$$\frac{s}{S}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -k^2 \qquad \Phi = A\cos k\phi + B\sin k\phi$$

$$\frac{s}{S}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) = k^2 \qquad S = s^n$$

$$V(s,\phi) = a_0 + b_0 \ln|s| + \sum_{k=1}^{\infty}\left[s^k\left(a_k\cos k\phi + b_k\sin k\phi\right) + s^{-k}\left(c_k\cos k\phi + d_k\sin k\phi\right)\right]$$

**3.26]** $\ln|s|$ and $s^{-k}$ will blow up at (inside) $s=0$ and $\ln|s|$ and $s^k$ don't work inside at $s\to\infty$.

$$\frac{\partial V_{out}}{\partial n} - \frac{\partial V_{in}}{\partial n} = \frac{-\sigma}{\varepsilon_0} \Rightarrow \sigma = -\varepsilon_0\left[\frac{\partial V_{out}}{\partial n} - \frac{\partial V_{in}}{\partial n}\right]_{s=R}$$

$$V_{in} = a_0 + \sum_{k=1}^{\infty} s^k\left(a_k\cos(k\phi) + b_k\sin(k\phi)\right)$$

$$V_{out} = a_0^* + \sum_{k=1}^{\infty} s^{-k}\left(c_k\cos(k\phi) + d_k\sin(k\phi)\right)$$

$$\sigma(\phi) = a\sin(5\phi)$$

$$\sigma(\phi) = \varepsilon_0\sum_{k=1}^{\infty}\left[\left(k s^{k-1}a_k + \frac{k}{s^{k+1}}c_k\right)\cos(k\phi) + \left(k s^{k-1}b_k + \frac{k}{s^{k+1}}d_k\right)\sin(k\phi)\right]$$

For $k=5$: $$\sigma(\phi) = \varepsilon_0\left(5s^4 b_5 + \frac{5}{s^6}d_5\right)\sin(5\phi) = a\sin(5\phi)$$

$$5s^4 b_5 + \frac{5}{s^6}d_5 = \frac{a}{\varepsilon_0}$$