54, 55, 56, 57, 59, 62, 63, 64

Jackson

54) Check the divergence theorum for vector

$\vec{V} = r^2 \cos\theta \hat{r} + r^2 \cos\phi \hat{\theta} - r^2 \cos\theta \sin\phi \hat{\phi}$

over the over the first octant of a sphere of radius R

$div(v) = 2r\cos\theta - r^2 \sin\phi - r^2 \cos\theta \cos\phi$

$\Rightarrow 4r\cos\theta r^2 \sin\theta$

$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^R (4r\cos\theta r^2 \sin\theta)\, dr\, d\theta\, d\phi$

$\Rightarrow \iint r^4 \cos\theta \sin\theta \, d\theta\, d\phi \Rightarrow \int_0^{\pi/2} \frac{r^4}{2} d\phi \Rightarrow \frac{\pi r^4}{4}$

prove $\iiint_r \vec{v}\, ds = \oint \vec{v} \cdot d\vec{a}$, when $d\vec{a} = -r^2 \, dr\, d\phi\, \hat{z}$

the right side of the surface integral is non zero where

$d\vec{a} = r\, dr\, d\theta\, \hat{\phi}$, & $\phi = \pi/2$ $\oint \vec{v} \cdot d\vec{a} = \frac{\pi R^4}{4}$

55) Let $\vec{v} = ay\hat{x} + bx\hat{y}$, check stokes's therom

around circle path of radius R centered at origin in x-y plane

$$Curl(v) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ay & bx & 0 \end{vmatrix} \Rightarrow \hat{i}(0-0) + \hat{j}(0-0) + \hat{z}(b-a)$$

surface integral

$\int_0^r (\nabla \times \vec{v}) \cdot da = \boxed{(b-a)\pi R^2}$

was for my I got an old one

56 b)

Let $d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$

dz & $ds = 0$ so, $d\vec{l} \Rightarrow s\,d\phi\,\hat{\phi}$ $\quad s = R$

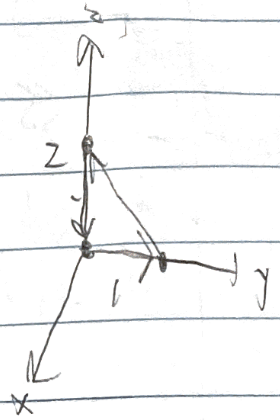
$$\oint \vec{v}\cdot d\vec{l} = \int_0^{2\pi} ay\,\hat{x}\cdot R\,d\phi\,\hat{\phi} + \int_0^{2\pi} bx\,\hat{y}\cdot R\,d\phi\,\hat{\phi}$$

$$\Rightarrow \int_0^{2\pi} -aR\sin\phi\sin\phi\,\hat{\phi}\cdot R\,d\phi\,\hat{\phi} + \int_0^{2\pi} bR\cos\phi\cos\phi\,\hat{\phi}\cdot R\,d\phi\,\hat{\phi}$$

$$\Rightarrow (b-a)\pi R^2$$

56) Compute the line integral of

$\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$ along



Check using stokes' therom.

~~Do each line integral~~

$d\vec{l} = \hat{y}\,dy$, $\vec{v}\cdot d\vec{l} = yz^2\,dy$, $z=0$ $\quad \int \vec{v}\cdot d\vec{l} = 0$

$d\vec{l} = \hat{z}\,dz$, $\vec{v}\cdot d\vec{l} = (3y+z)\,dz$, $x=y=0$

$$\int \vec{v}\cdot d\vec{l} = -\int_0^2 (3y+z)\,dz = -2$$

last path, $x=0$, $z=2-2y$, $dz = -2dy$

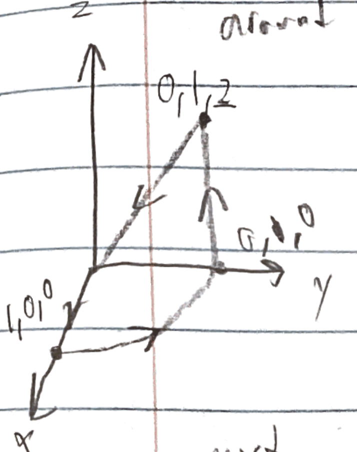
$dl = dy\,\hat{y} + dz\,\hat{z}$

$$\int \vec{v}\cdot d\vec{l} = \int_1^0 dy\,(4y^3-8y^2+2y-4) = \frac{14}{3} - 2 = 8/3$$

57) Compute the line integral of

$\vec{v} = (r\cos^2\theta)\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}$

around the path shown. Check using stokes then



- break into 4 line segments

$d\vec{l} = dr\,\hat{r}$, $\phi = 0$ & $\theta = \pi/2$

$\vec{v}\cdot d\vec{l} = r\cos^2\theta\, dr \Rightarrow 0$

next, $d\vec{l} = d\phi\,\hat{\phi}$, $\phi = 0$, $\theta = \pi/2$

$\vec{v}\cdot d\vec{l} = 3r\,d\phi$ $\quad \int_0^{3\pi/2} 3r\,d\phi = \frac{3\pi}{2}$

next,

$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$, $\vec{v}\cdot d\vec{l} = r\cos^2\theta\, dr - r^2\cos\theta\sin\theta\, d\theta$

I was lost on how to trace it w/ integral

Let $y = r\sin\theta = 1 \therefore r = \frac{1}{\sin\theta}$

then, $dr = \frac{-\cos\theta}{\sin^2\theta}\,d\theta$, $\vec{v}\cdot d\vec{l} = (-\cot^3\theta - \cot\theta)\,d\theta$

$\Rightarrow \int \vec{v}\cdot d\vec{l} = -\frac{1}{2}\frac{1}{\sin^2\theta}\Big|_{\pi/2}^{\tan^{-1}(1/2)} = 2$

last path,

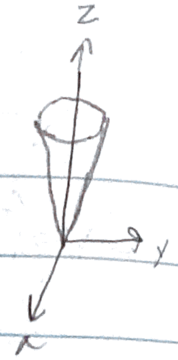
$d\vec{l} = dr\,\hat{r}$

$\int \vec{v}\cdot d\vec{l} = \int(\cos^2\theta_0\, r\,dr) \Rightarrow \cos^2\theta_0 \int_{\sqrt{5}}^{0} r\,dr = -2$

together $\frac{3\pi}{2} + 2 - 2 = \frac{3\pi}{2}$

checking: $\nabla\times\vec{v} = 3\cot\theta\,\hat{r} - 6\hat{\theta}$, $d\vec{a} = -r\,dr\,d\phi\,\hat{\theta}$

$\int \nabla\times\vec{v}\cdot d\vec{a} = \int_0^{\pi/2}\int_0^1 6r\,dr\,d\phi = \frac{3\pi}{2}$



59) Check the divergence theorem for

$\vec{v} = r^2 \sin\theta\,\hat{r} + 4r^2 \cos\theta\,\hat{\theta} + r^2 \tan\theta\,\hat{\phi}$ using the volume of the ice-cream cone shape.

$\nabla \cdot \vec{v} = 4r \cot\theta \cos\theta$, integrate over the slice

$$\int_0^R \int_0^{2\pi} \int_0^{\pi/6} (4r \cot\theta \cos\theta\, r^2 \sin\theta)\, dr\, d\theta\, d\phi$$

$$\Rightarrow 2\pi R^4 \int_0^{\pi/6} \cos^2\theta\, d\theta \Rightarrow \frac{\pi R^4}{12}(2\pi + 3\sqrt{3})$$

Must break the closed surface integral

$\vec{v} \cdot d\vec{a} = R^4 \sin^2\theta\, d\theta\, d\phi$, $\int \vec{v} \cdot d\vec{a} = 2\pi R^4 \int_0^{\pi/6} \sin^2\theta\, d\theta$

$$\Rightarrow \frac{\pi R^4}{12}(2\pi - 3\sqrt{3})$$

for the other part

$d\vec{a} = \frac{1}{2} r\, dr\, d\theta\, d\phi\, \hat{\theta}$

$$\int \vec{v} \cdot d\vec{a} = \int_0^1 \int_0^{2\pi} \sqrt{3}\, r^3\, dr\, d\phi = \frac{\pi\sqrt{3}R^4}{2}$$

together the total is $\frac{\pi R^4}{12}(2\pi + 3\sqrt{3})$

62)

Integral: $a = \int_S da$ - vector area of surface S.

If S is flat then |a| is scalar, find vector area of hemisphere of radius R.

$da = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$, there is no x & y components

the z-component of $\hat{r}$ is $\cos\theta \hat{z}$;

$$\vec{a} = 2\pi R^2 \hat{z} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta = \pi R^2 \hat{z}$$

b) $\oint_S d\vec{a} = \int_V (\nabla 1) dr = 0$ Is true because of the type of formula given in part a

So $a = 0$ for closed surface

c) Show a is same for all surfaces sharing the same boundary

$$\oint_{S_1} d\vec{a} + \oint_{S_2} d\vec{a} = \vec{a}_{total} \quad \text{but } \vec{a}_{total} = \vec{0}$$

However if the normal direction of S_1 & S_2 $\oint_{S_1} d\vec{a} - \oint_{S_2} d\vec{a} = 0$

d) Show $a = \frac{1}{2} \oint r \times dl$, where the integral is around the boundary line

let $d\vec{a} = \frac{1}{2} \vec{r} \times d\vec{l}$ since the cross product can be interpreted as the area of a parallelogram

$$\text{so } \vec{a} = \oint d\vec{a} = \oint \frac{1}{2} \vec{r} \times d\vec{l}$$

e) Show $\oint (c \cdot r) \, dl = a \times c$ for any constant vector c.

let $T = c \cdot r$

$$\oint (\vec{c} \cdot \vec{r}) d\vec{l} = \int_S \nabla(\vec{c} \cdot \vec{r}) \times d\vec{a}$$

↓

e) product rule of the gradient

$$\nabla(\vec{c}\cdot\vec{r})\,d\vec{l} = \vec{c}\times(\nabla\times\vec{r}) + (\vec{c}\cdot\nabla)\vec{r} \Rightarrow (\vec{c}\cdot\nabla)\vec{r} = \vec{c}$$

doing one last magical step, remove [illegible]

$$\oint(\vec{c}\cdot\vec{r})\,d\vec{l} = -\int_s \vec{c}\times d\vec{a} = -\vec{c}\times\vec{a} = \vec{a}\times\vec{c}$$

63) Find the divergence of $V = \frac{\hat{r}}{r}$, $\nabla\cdot(r^n\hat{r}) = (n+2)r^{n-1}$

unless $n=-2$, then it breaks $\nabla\cdot\frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$

plugging into the spherical version of the curl gives 0

64) $\nabla^2\frac{1}{r} = -4\pi\delta^3(r)$ when $r'=0$

let $D(r,\epsilon) \equiv -\frac{1}{4\pi}\nabla^2\frac{1}{\sqrt{r^2+\epsilon^2}} \Rightarrow \frac{3\epsilon^2}{4\pi}(r^2+\epsilon^2)^{-3/2}$

when $r=0$, then $\epsilon\to 0$

$$D(0,\epsilon) = \frac{3\epsilon^2}{4\pi}(\epsilon^2)^{-3/2} \to \infty$$

So as $\epsilon\to 0$, the $D\to 0$ as long as r^2 is not zero