Jackson
Diamond

Homework 3

3.3) Find the general solution to Laplace's equation in spherical coordinates for where V depends only on r. Same for cylindrical coordinates assuming V depends on s.

In spherical $V(r) \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$

assume $r \neq 0$ & $\left(r^2 \frac{\partial V}{\partial r}\right)' = 0$

$\therefore r^2 \frac{dV}{dr} = k \rightarrow \frac{dV}{dr} = \frac{k}{r^2} \rightarrow V(r) = \frac{-k}{r} + C$

So we get $V(r) = \frac{-k}{r} + C$ for spherical coordinates, note that it has the $\frac{1}{r}$ dependence as a consequence of Laplace's equation.

For cylindrical coordinates, assume $V(r)$ depends only on s.

$V(s) = \frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{\partial V}{\partial s}\right) = 0$, assuming the same $s \neq 0$ & $\left(s \frac{\partial V}{\partial s}\right)' = 0$

it is found $s \frac{dV}{ds} = k, \Rightarrow \frac{dV}{ds} = \frac{k}{s} \Rightarrow V(s) = k\ln(s) + C$

It appears to follow $V(s) = k\ln(s) + C$ which follows the same pattern as a solution for the potential surrounding a line charge.

3.5) Prove that the field is uniquely determined when the charge density ρ is given & either V or the normal derivative $\partial V / \partial n$ is specified on each boundary surface. Do not assume the boundary are conductors, or that V is constant over any given surface.

If there are two solutions for the field in V that find ρ. Such that: $\nabla \cdot E_1 = \frac{-\rho}{\epsilon_0}$ & $\nabla \cdot E_2 = -\frac{\rho}{\epsilon_0}$

and $E_3 = E_2 - E_1$ where $\nabla \cdot E_3 = 0 \therefore \oint E_3 \cdot da = 0$

Since there is one ρ but could be many objects within the volume, and one global surface that contains all of them.

$E_3 = -\nabla V_3$ so, $\nabla \cdot (V_3 E_3) = V_3 (\nabla \cdot E_3) + E_3 (\nabla V_3)$

$\Rightarrow 0 - E_3 \cdot E_3 = -E_3^2$

$\therefore \int_V \nabla \cdot (V_3 E_3) \, d\tau \Rightarrow -\int E_3^2 \, d\tau$

Divergence ~~theorem~~

$\Rightarrow \oint V_3 E_3 \cdot da = -\int E_3^2 \, d\tau$

$\oint V_3 E_3 \cdot da = 0$ If V_1 & V_2 are specified for each surface, then V_3 will be 0 over every boundary.

It is also $= 0$ if $\partial V_1 / \partial n$ & $\partial V_2 / \partial n$ are specified.

because $\partial V_3 / \partial n = -E_3$ which is perpendicular and $= 0$.

So $\therefore \int E_3^2 \, d\tau = 0$ & $E_3 = 0$ which means $E_2 = E_1$ & the field is unique.

3.6)
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E_3

3.13)

3.6) A more elegant proof of the second uniqueness theorem uses Green's identity (Problem 1.16 c) with $T = U = V$. Supply the details.

$$\int_V (V_3 \nabla^2 V_3 + \nabla V_3 \cdot \nabla V_3) d\tau = \oint (V_3 \nabla V_3) \cdot da$$

$$\int_V (0 + \nabla V_3 \cdot \nabla V_3) d\tau = -\oint (V_3 E_3) \cdot da$$

$$-\int_V E_3^2 d\tau = -\oint (V_3 E_3) \cdot da$$

where E_3 is

$$\int_V E_3^2 d\tau = \oint V_3 E_3 \cdot da$$

3.13) Find the potential in the infinite slot of Ex 3.3 if the boundary at $x=0$ consists of two metal strips: one from $y=0$ to $y=a/2$, is held at a positive V_0, and the other, from $y=a/2$ to $y=a$, is held at a negative constant potential $-V_0$

At the end of example 3.3 they arrive at

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5..} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a)$$

So $V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$

where $C_n = \frac{2}{a}\int_0^a V_0(y) \sin(n\pi y/a) dy$

3.6 con)

The Fourier coefficient is found by applying boundary conditions

$$C_n = \frac{2V_0}{a}\left(\int_0^{a/2} \sin(n\pi y/a)\,dy - \int_{a/2}^{a} \sin(n\pi y/a)\,dy\right) = \frac{2V_0}{n\pi}\left(1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right)$$

To generalize into a sum of solutions we can find the coefficient to be zero if n is odd or a multiple of 4.

$$C_n = \frac{2V_0}{n\pi}, \quad n = (4j+2),\ j = 0,1,2\ldots$$

So the solution are $V(x,y) = \frac{8V_0}{\pi}\sum_{j=0}^{\infty} \frac{e^{(-(4j+2)\pi x/a)}\sin[(4j+2)\pi y/a]}{4j+2}$

3.14) In ex 3.3 determine the charge density $\sigma(y)$ on the strip at $x=0$, assuming it is a conductor at constant potential V_0

Since we assume its a conductor $\frac{\partial V}{\partial n} = -\sigma/\epsilon_0$

and I know $V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a)$

$$\therefore \quad \sigma(y) = -\epsilon_0 \left.\frac{\partial V}{\partial x}\right|_{x=0} = \frac{4\epsilon_0 V_0}{a}\sum_{n=0}^{\infty} \sin\left((2n+1)\frac{\pi y}{a}\right)$$

* let a be a unit of length

$\therefore$ If $y=0$ or $y=a$,

$\sigma = 0$

3.15) A rectangular pipe, running parallel to the z-axis, has three grounded metal sides, at $y=0$, $y=a$, & $x=0$. The fourth side, at $x=b$, is maintained at a specified potential $V_0(y)$. a) Develope a general formula for the potential inside the pipe. b) Find the potential capacity for the case $V_0(y) = V_0$, a constant.

From Laplace's Equation one can use separation of varibles such

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = \pm K^2 = \frac{1}{Y(y)}\frac{d^2Y}{dy^2}$$

We know the solutions for K will be exponential or sinesoidal. If you let the sinesoidal part of the solutions be for the y cordinates, a fourier series can be used that matches $V_0(y)$ of the form:

$$V(x,y) = (A\sin(Ky) + B\cos(Ky))(Ce^{Kx} + De^{-Kx})$$

$V=0$ if $y=0$ & $y=a$, $x=0$

$V=V_0(y)$ if $x=b$

let $A=0$, $B\neq 0$ & $C=-D$ while $K=\frac{n\pi}{a}$

the general solution is the sum of all specific conditions such that

$$V(x,y) = \sum_{n=0}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi x}{a}\right)$$

Using Fourier's Trick to find $C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$

If $V_0 = V_0(y)$ then $C_n = \frac{4V_0}{n\pi\sinh(n\pi)}$, $n=1,3,5,\ldots$

& the general solution is $V(x,y) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left((2n+1)\frac{\pi y}{a}\right)\sinh\left((2n+1)\frac{\pi x}{a}\right)}{(2n+1)\sinh\left(\frac{n\pi b}{a}\right)}$

3.16) A cubical box (side length a) consists of five metal plates, which are welded together & grounded. The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box.

Each side is a boundary condition meaning there are 6 which suggest sinusoidal solutions in the x & y with an exponential solution for Z.

Let: $\frac{1}{X(x)}\frac{d^2X}{dx^2} = -K^2$, $\frac{1}{Y(y)}\frac{d^2Y}{dy^2} = -L^2$, $\frac{1}{Z(z)}\frac{d^2Z}{dz^2} = (K^2+L^2)$

$\therefore$ the sum of all solutions is zero & Laplace's equation w/

The specific solutions

$$X(x) = A\cos(Kx) + B\sin(Kx)$$
$$Y(y) = C\cos(Ky) + D\sin(Ky)$$
$$Z(z) = E e^{(-z\sqrt{K^2+L^2})} + F e^{(z\sqrt{K^2+L^2})}$$

For these solutions they must follow the boundary conditions such that $A=C=0$, $B\neq 0$, $D\neq 0$, & $E+F=0$ while $K=\frac{n\pi}{a}$, & $L=\frac{m\pi}{a}$

$\therefore$ the general solution is:

$$V(x,y,z) = \sum_n \sum_m C_{n,m} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\pi z\sqrt{n^2+m^2}}{a}\right)$$

at the final boundary condition, $V_0 = \sum_n\sum_m C_{n,m} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\pi\sqrt{n^2+m^2}\right)$

Using Fourier's trick twice

$$C_{n,m}\sinh\left(\pi\sqrt{n^2+m^2}\right) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dx\,dy = \frac{16V_0}{\pi^2 nm}$$

$$\therefore V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n\ odd}\sum_{m\ odd}\left(\frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)}{nm}\frac{\sinh\left(\frac{\pi z\sqrt{n^2+m^2}}{a}\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}\right)$$

3.19) The potential at the surface of a sphere of radius R is given by $V_0 = K\cos(3\theta)$, where K is constant. Find the potential inside & outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. Assume there's no charge inside or outside the sphere.

Legendre?

Let $V_0 = \frac{K}{5}\left(8P_3(\cos\theta) - 3P_1(\cos\theta)\right)$, using Legendre Polynomials & trig formulas.

Using separation of variables the solutions for inside & outside can be found

$r \leq R$, inside: $V(r,\theta) = \sum_{L=0}^{\infty} A_L r^L P_L(\cos\theta)$ &

$r \geq R$, outside: $V(r,\theta) = \sum_{L=0}^{\infty} B_L r^{-(L+1)} P_L(\cos\theta)$

$\therefore$ the solutions must be continuous at the boundary $V_{in}(R,\theta) = V_{out}(R,\theta)$

So $A_L = \frac{2L+1}{2R^L}\int_0^{\pi} V_0(\theta) P_K(\cos\theta)\sin\theta \, d\theta$

Since there are only 2 Legendre polynomial, there are only 2 A_L values that $\neq 0$, $K=1$ & $K=3$

S.T. $A_1 = -3K/5R$ & $A_3 = 8K/5R^3$

$\therefore$ the general solution inside the shell is: $V_{in}(r,\theta) = -\frac{3K}{5R} r P_1(\cos\theta) + \frac{8K}{5R^3} r^3 P_3(\cos\theta)$

now for outside the shell, since $B_1 = A_1 R^{2L+1}$ because the potential must be continuous at $r=R$. Which means B_1 & B_3 are only nonzero

$B_1 = \frac{-3KR^2}{5}$, $B_3 = \frac{8KR^4}{5}$

$\therefore$ the general solution outside the shell is: $V_{out}(r,\theta) = -\frac{3KR^2}{5r^2} P_1(\cos\theta) + \frac{8KR^4}{5r^4} P_3(\cos\theta)$

$\therefore$ $\sigma(\theta) = \epsilon_0\left(3A_1P_1 + 7A_3R^2P_3\right)$

3.22) In Problem 2.25, you found the potential on the axis of a uniformly charged disk: $V(r,\theta)=\frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2}-r\right)$.

~~a) Use this, together with the fact that $P_L(1)=1$, to evaluate the terms in the Legendre expansion for the potential of the disk at points off the axis, assuming $r>R$.~~

~~The general solution is the sum of Legendre polys.~~

$$V(r,\theta)=\sum_{L=0}^{\infty}\frac{B_L}{r^{(L+1)}}P_L(\cos\theta),$$

$$V(r,\theta)=\sum_{L=0}^{\infty}\frac{B_L}{r^{(L+1)}}P_1(1)=\sum_{L=0}^{\infty}\frac{B_L}{r^{(L+1)}}=\frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2}-r\right)$$

~~to match the powers of R let~~ $\sqrt{r^2+R^2}=r\left(1+\frac{1}{2}\left(\frac{R}{r}\right)^2+\frac{1}{8}\left(\frac{R}{r}\right)^4\right)$

Let $B_0=\frac{\sigma R^2}{4\epsilon_0}$, $B_1=0$, & $B_2=\frac{\sigma R^4}{16\epsilon_0}$

P_0 & P_2 are the only Legendre polynomials that work

$\therefore$ the general solution is $V(r,\theta)\simeq\frac{\sigma R^2}{4\epsilon_0}\left(\frac{1}{r}-\frac{R^2}{4r^3}P_2(\cos\theta)\right)$

b) Find the potential for $r<R$ by the same method, using equ 3.66. Note you must break the interior region up into two hemispheres, above & below the disk.

3.66) $V(r,\theta)=\sum_{L=0}^{\infty}A_L r^L P_L(\cos\theta)$ with the same boundary conditions

$$V_{above}(r,\theta)\simeq\frac{\sigma}{2\epsilon_0}\left(R-rP_1(\cos\theta)+\frac{r^2}{2R}P_2(\cos\theta)\right)$$

$$V_{below}(r,\theta)\simeq\frac{\sigma}{2\epsilon_0}\left(R+rP_1(\cos\theta)+\frac{r^2}{2R}P_2(\cos\theta)\right)$$

3.24)

Solve Laplace's equation by separation of variables in cylindrical coordinates, assuming there is no dependence on z (cylindrical symmetry). Make sure to find all solutions to the radial equation. In particular, your result must accommodate the case of an infinite line charge.

Laplace's equation in this form is: $\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right)=0$

the solutions separate, to find ϕ solutions first

$$\frac{1}{\phi}\frac{d^2\Phi(\phi)}{d\phi^2}=-k^2$$, which makes sinusoidal solutions

$\Phi(\phi)=\Phi(\phi+2\pi)$ so k is an integer, other solutions for $S(s)$ of the form $\frac{s}{S(s)}\frac{d}{ds}\left(s\frac{dS(s)}{ds}\right)=k^2$

If we guess $S(s)=s^n$ of the form plugging it in gives $n=\pm k$

$S(s)=Cs^k+Ds^{-k}$, however if $k=0$ then S is constant

and there's only two solutions. $S(s)=C\ln(s)+D$

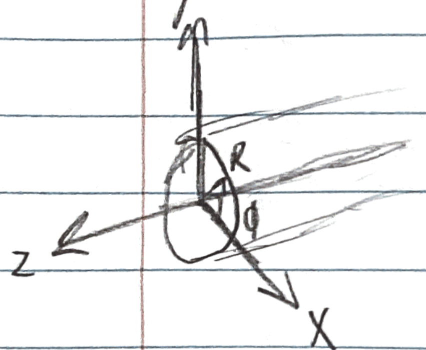
Considering $k=0$ for $\Phi(\phi)$ yields something that doesn't have periodic boundary conditions. The general way to sum it is

$$V(s,\phi)=a_0+b_0\ln(s)+\sum_{k=1}^{\infty}\left[s^k(a_k\cos(k\phi)+b_k\sin(k\phi))+s^{-k}(c_k\cos(k\phi)+d_k\sin(k\phi))\right]$$

the infinite line charge $\approx \ln(s)$

3.26)

Charge density $\sigma(\phi) = a\sin(5\phi)$, where a is a constant. This charge density is glued over the surface of an infinite cylinder of radius R. Find the potential inside & outside the cylinder. Use stuff from 3.24



Considering inside the cylinder first, getting rid of things that end at $s=0$ from the general solution gives.

$$V_{in}(s,\phi) = a_0 + \sum_{k=1}^{\infty} s^k\left(a_k\cos(k\phi) + b_k\sin(k\phi)\right)$$

and for outside you would take away things that kill the equation at $s=\infty$

$$V_{out}(s,\phi) = a_0' + \sum_{k=1}^{\infty}\frac{1}{s^k}\left(c_k\cos(k\phi) + d_k\sin(k\phi)\right)$$

let $\sigma = -\epsilon_0\left(\frac{\partial V_{out}}{\partial n} - \frac{\partial V_{in}}{\partial n}\right)\Big|_{s=R}$

$\sigma = a\sin(5\phi)$ shows that $k=5$ with all other terms vanish.

This removes a_k & c_k but b_5 & d_5 remain. To solve for them $V_{in}(R,\phi) = V_{out}(R,\phi)$.

$$V(s,\phi) = \frac{a\sin 5\phi}{10\epsilon_0}\,\frac{s^5}{R^4},\quad s \le R$$

$$V(s,\phi) = \frac{a\sin 5\phi}{10\epsilon_0}\,\frac{R^6}{s^5},\quad s \ge R$$