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Electro hw #1 1.54, 1.55, 1.56, 1.57, 1.59, 1.62, 1.63, 1.64

1.54-

$$\vec{v} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$$

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\cos\theta) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$$

$$= \frac{1}{\cancel{r^2}}(4r^{\cancel{3}}\cos\theta) + \cancel{\frac{1}{r\sin\theta}(r^2\cos\theta\cos\phi)}$$

$$+ \cancel{\frac{1}{r\sin\theta}(-r^2\cos\theta\cos\phi)}$$

$$= 4r\cos\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} r^2\cos\theta\, r^2\sin\theta\, d\theta\, d\phi = R^4\,\pi/2\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta$$

$$= R^4\,\pi/2\;\tfrac{1}{2} = R^4\,\pi/4$$

$$R^4\,\pi/4 = R^4\pi/4 + \underbrace{S_2 + S_3 + S_4}_{0}$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{R}(1+2+3)(R^2\sin\theta\, d\theta\, d\phi)$$

spherical ?

* 1.55 -

$$\vec{v} = ay\,\hat{x} + bx\,\hat{y}$$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ ay & bx & 0 \end{vmatrix} = \hat{x}(0-0) + \hat{y}(0-0) + \hat{z}\left(\frac{\partial}{\partial x}(bx) - \frac{\partial}{\partial y}(ay)\right)$$

$$= \hat{z}(b-a)$$

$$\int \vec{v} \cdot d\vec{l} = \int \vec{z}(b-a) \cdot \hat{z}\,(r\,dr\,d\phi)$$

$$= \int_0^{2\pi} \int_0^{R} \hat{z}(b-a) \cdot \hat{z}\,(r\,dr\,d\phi)$$

$$= (b-a)\left(\int_0^{R} r\,dr\right)\left(\int_0^{2\pi} d\phi\right)$$

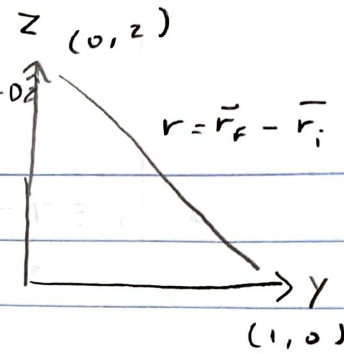
$$= (b-a)\left(\frac{R^2}{\cancel{2}}\right)(\cancel{2}\pi) = (b-a)R^2\pi$$

$$= \pi R^2 (b-a)$$

Like ex. 1.7?

$0\hat{x} + 2\hat{z} - \hat{y} + 0\hat{z}$

$\vec{r} = 2\hat{z} - \hat{y}$



1.56 -

$$\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$$

$x = z = 0 \quad dx = dz = 0 \quad y: 0 \to 1 \quad \vec{v}\cdot d\vec{l} = (yz^2)dy = 0$

$\int \vec{v}\cdot d\vec{l} = 0$

$x = 0 \quad z = 2-2y \quad dz = -2dy \quad y: 1 \to 0$

$\vec{v}\cdot d\vec{l} = (yz^2)dy + (3y+z)dz = y(2-2y)^2 dy - (3y+2-2y)2dy$

$d\vec{l} = dz\hat{z} + dy\hat{y}$

$$\int v \cdot d\vec{l} = 2\int_1^0 (2y^3 - 4y^2 + y - 2)\,dy$$

$$= \left[2\left(\frac{y^4}{2} - \frac{4y^3}{3} + \frac{y^2}{2} - 2y\right)\right]_1^0$$

$$= \frac{14}{3}$$

$x = y = 0 \quad dx = dy = 0 \quad z: 2 \to 0 \quad \vec{v}\cdot d\vec{l} = (3y+z)dz$

$= z\,dz$

$$\int \vec{v}\cdot d\vec{l} = \int_2^0 z\,dz = \left[\frac{z^2}{2}\right]_2^0 = -2$$

$$\int \vec{v}\cdot d\vec{l} = 0 + \frac{14}{3} - 2 = 8/3$$

1.56 cont. -

Stokes thm $\oint \vec{v} \cdot d\vec{l}$

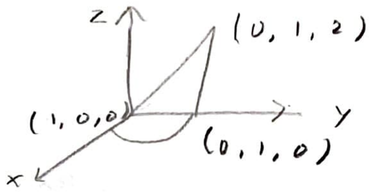
$$\nabla \times \vec{v} = \frac{\partial}{\partial y}(3y+z) - \frac{\partial}{\partial z}(yz^2) = 3 - 2yz$$

$$\int \nabla \times \vec{v} \cdot d\vec{a} = \iint (3-2yz)\, dy\, dz = \int_0^1 \left[\int_0^{2-2y} (3-2yz)\, dz \right] dy$$

$$= \int_0^1 \left[3(2-2y) - 2y\, \tfrac{1}{2}(2-2y)^2 \right] dy$$

$$= \int_0^1 (-4y^3 + 8y^2 - 10y + 6)\, dy$$

$$= \left[(-y^4 + \tfrac{8}{3}y^3 - 5y^2 + 6y) \right]_0^1 = 8/3$$



* 1.57 -

$$\vec{v} = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}$$

$$d\vec{l} = dr\,\hat{r} + d\theta\,\hat{\theta} + d\phi\,\hat{\phi}$$

(1, 0, 0)
(0, 1, 0)
(0, 1, 2)

At origin: ①

$$\vec{v}\cdot d\vec{l} = (r\cos^2\theta)\,dr = 0$$

$$\int \vec{v}\cdot d\vec{l} = 0$$

② $r = 1$, $\theta = \pi/2$ $\phi : 0 \to \pi/2$

$$\vec{v}\cdot d\vec{l} = (3r)(r\sin\theta\, d\phi) = 3\,d\phi$$

$$\int \vec{v}\cdot d\vec{l} = \int_0^{\pi/2} 3\,d\phi$$

$$= 3\int_0^{\pi/2} d\phi = 3\pi/2$$

③ $\phi = \pi/2$ $r\sin\theta = y = 1$ $dr = \frac{-1}{\sin^2\theta}\cos\theta\, d\theta$

$$r = \frac{1}{\sin\theta}$$

$\theta : \pi/2 \to \pi/4$

$$\vec{v}\cdot d\vec{l} = (r\cos^2\theta)\,dr - (r\cos\theta\sin\theta)\,r\,d\theta$$

$$= \frac{\cos^2\theta}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\right)d\theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}\,d\theta$$

1.57 - cont.

$$= -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$= -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)d\theta$$

$$= -\frac{\cos\theta}{\sin^3\theta}\,d\theta$$

$$\int \vec{v}\cdot d\vec{l} = \int_{\pi/2}^{\pi/4} -\frac{\cos\theta}{\sin^3\theta}\,d\theta = -\frac{1}{2\sin^2\theta}\Bigg|_{\pi/2}^{\pi/4}$$

$$= \frac{1}{2(1/2)} - \frac{1}{2(1)} = 1 - 1/2 = 1/2$$

(4) $\theta = \pi/4$ $\quad \phi = \pi/2$ $\quad r: \sqrt{2} \to 0$ $\quad \vec{v}\cdot d\vec{l} = (r\cos^2\theta)\,dr$

$$= 1/2\, r\, dr$$

$$\int \vec{v}\cdot d\vec{l} = \int_{\sqrt{2}}^{0} 1/2\, r\,dr = 1/2 \int_{\sqrt{2}}^{0} r\,dr$$

$$= \frac{1}{2}\left[\frac{r^2}{2}\right]_{\sqrt{2}}^{0} = -\frac{1}{4}(2) = -1/2$$

$$\oint \vec{v}\cdot d\vec{l} = 0 + \frac{3\pi}{2} + \frac{1}{2} - \frac{1}{2} = \frac{3\pi}{2}$$

stoke's theorem $\int (\nabla \times \vec{v}) \, d\vec{a}$

$$\vec{\nabla} \times \vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta \, 3r) - \frac{\partial}{\partial\phi}(-r\sin\theta\cos\theta)\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\cos^2\theta) - \frac{\partial}{\partial r}(r\,3r)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(-r\,r\cos\theta\sin\theta) - \frac{\partial}{\partial\theta}(r\cos^2\theta)\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}\left[3r\cos\theta\right]\hat{r} + \frac{1}{r}\left[-6r\right]\hat{\theta} + \frac{1}{r}\left[-2r\cos\theta\sin\theta + 2r\cos\theta\sin\theta\right]\hat{\phi}$$

$$= 3\cot\theta\,\hat{r} - 6\hat{\theta}$$

$$\int (\vec{\nabla} \times \vec{v}) \, d\vec{a} = \int_0^1 6r\,dr \int_0^{\pi/2} d\phi$$

$$= 6\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = 3\pi/2$$

1.59 -

$$\vec{v} = r^2 \sin\theta \hat{r} + 4r^2 \cos\theta \hat{\theta} + r^2 \tan\theta \hat{\phi}$$

$$\nabla \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^2 \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \, 4r^2 \cos\theta)$$

$$+ \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} (r^2 \tan\theta)$$

$$= \frac{1}{r^2} 4r^3 \sin\theta + \frac{1}{r \sin} 4r^2 (\cos^2\theta - \sin^2\theta)$$

$$= \frac{4r}{\sin\theta} (\cancel{\sin^2\theta} + \cos^2\theta - \cancel{\sin^2\theta})$$

$$= 4r \frac{\cos^2\theta}{\sin\theta}$$

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^R \left(4r \frac{\cos^2\theta}{\cancel{\sin\theta}}\right) (r^2 \cancel{\sin\theta} \, d\phi \, d\theta \, dr)$$

$$= \int_0^R 4r^3 \, dr \int_0^{\pi/2} \cos^2\theta \, d\theta \int_0^{2\pi} d\phi = R^4 (\pi/4)(2\pi) \ ?$$

$$\cancel{4} \left[\frac{r^4}{\cancel{4}} \right]_0^R$$

$$\int_0^{\pi/2} \frac{1 + \cos(2\theta)}{2} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} 1 + \cos(2\theta) \, d\theta$$

$$= \frac{1}{2} \left(\int_0^{\pi/2} 1 \, d\theta + \int_0^{\pi/2} \cos(2\theta) \, d\theta \right)$$

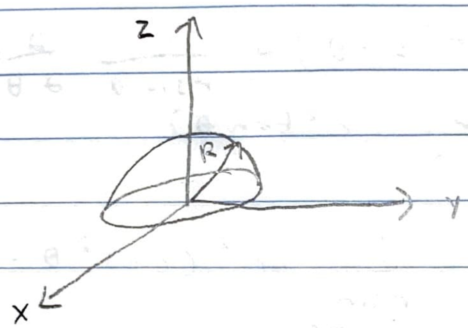
$$\int_0^{\pi/2} 1 \, d\theta = \pi/2 \qquad \int_0^{\pi/2} \cos(2\theta) \, d\theta = 0$$

$$\frac{1}{2} \left[\frac{\pi}{2} + 0 \right] = \pi/4$$

$\ast$ 1.62 -

$$\vec{a} = \int_S d\vec{a} \qquad \hat{z}$$

a)



$$d\vec{a} = R^2 \sin\theta \, d\theta \, d\phi$$
$$da_z = R^2 \sin\theta \cos\theta \, d\theta \, d\phi$$

$$\vec{a} = \int d\vec{a} = \int R^2 \sin\theta \, d\theta \, d\phi \, \hat{z}$$

$$= \int_0^{\pi/2} \int_0^{2\pi} R^2 \sin\theta \, d\theta \, d\phi \, \hat{z}$$

$$= 2\pi R^2 \hat{z} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

$$= 2\pi R^2 \hat{z} \left[\frac{\sin^2\theta}{2} \right]_0^{\pi/2}$$

$$= \pi R^2 \hat{z}$$

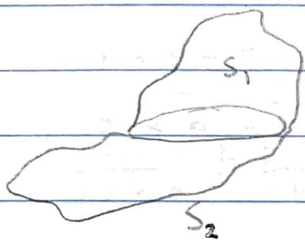
b) $\vec{a} = \vec{0}$ 1.61 (a)

$$\int (\nabla T) \, d\tau = \oint T \, d\vec{a} \qquad T = 1$$

$$\left(\hat{x} \frac{d}{dx} + \dots \right) 1$$

$$\oint d\vec{a} = 0$$

c)

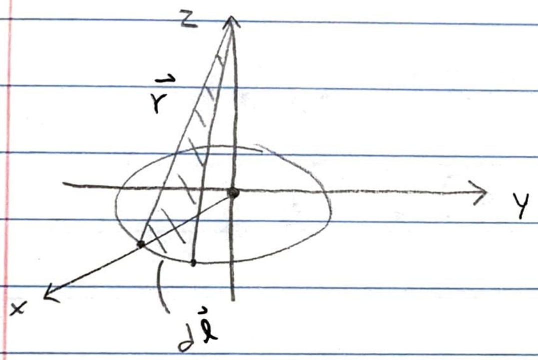


$\vec{a}_{tot} = 0$ for closed surfaces

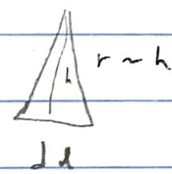
$\vec{a}_1 + \vec{a}_2$

d)

$$\vec{a} = \frac{1}{2}\oint \vec{r} \times d\vec{l}$$



$\frac{1}{2} bh$



$\frac{1}{2} r\,dl$

$$d\vec{a} = \frac{1}{2}(r \times d\vec{l})$$

$r \times d\vec{l}$ is the area of the parallelogram and the direction is perpendicular to its surface

e)

$$\oint (\vec{c}\cdot\vec{r})\,d\vec{l} = \vec{a} \times \vec{c}$$

$T = \vec{c}\cdot\vec{r}$ / 1.61e

$$\vec{\nabla} T = \vec{\nabla}(\vec{c}\cdot\vec{r}) = \vec{c} \times (\overset{0}{\cancel{\vec{\nabla}\times\vec{r}}}) + (\vec{c}\cdot\vec{\nabla})\vec{r}$$

$$\oint T\,d\vec{l} = \oint (\vec{c}\cdot\vec{r})\,d\vec{l} = -\int (\vec{\nabla} T) \times d\vec{a}$$

$$A \times B = - B \times A$$

$$= - \int \vec{c} \times d\vec{a} = - \vec{c} \times \int d\vec{a} = - c \times a$$

$$= a \times c$$

1.63-

a) $$V = \frac{\hat{r}}{r}$$

$$\nabla \cdot V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r) = \frac{1}{r^2}$$

$$\oint \vec{v} \cdot d\vec{a} = \int \left(\frac{1}{R}\hat{r}\right) R^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$$

$$= R \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 4\pi R$$

$$\nabla \times (r^n \hat{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+2})$$

$$= \frac{1}{r^2}(n+2) r^{n+1}$$

$$= (n+2) r^{n-1}$$

* b)

$$\vec{\nabla} \times \vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(v_\phi \sin\theta - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

1.63 - cont

$\vec{v} = r^n \hat{r}$

$$\nabla \times \vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(0(\sin\theta)) - \frac{\partial}{\partial\phi}(0)\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r^n) - \frac{\partial}{\partial r}(r(0))\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r(0)) - \frac{\partial}{\partial\theta}(r^n)\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}[0 - 0]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}(0) - 0\right]\hat{\theta}$$

$$+ \frac{1}{r}[0 - 0]\hat{\phi} = 0$$

Laplacian

D4MF

* 1.64-

$$D(r, \epsilon) = -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\epsilon^2}}$$

a)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)$$

$$\frac{1}{\sqrt{r^2-\epsilon^2}}$$

$$= -\frac{1}{4\pi}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right)\right]\right.$$

$$+ \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\,\cancel{\frac{\partial}{\partial\theta}\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right)}\right]^{0}$$

$$\left.+ \frac{1}{r^2\sin^2\theta}\cancel{\frac{\partial^2}{\partial\phi^2}\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right)}^{0}\right)$$

$$= -\frac{1}{4\pi r^2}\frac{d}{dr}\left(r^2\left[-\frac{r}{(r^2+\epsilon^2)^{3/2}}\right]\right)$$

$$= -\frac{1}{4\pi r^2}\frac{d}{dr}\left(r^2\left[-\frac{r}{(r^2+\epsilon^2)^{3/2}}\right]\right)$$

$$= -\frac{1}{4\pi r^2}\frac{d}{dr}\left(-\frac{r^3}{(r^2+\epsilon^2)^{3/2}}\right)$$ → using wolfram

$$= -\frac{1}{4\pi r^2}\left(-\frac{3\epsilon^2 r^2}{(r^2+\epsilon^2)^{5/2}}\right)$$

$$= \frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}}$$

b) $r \to 0$

$$D(0, \varepsilon) = \frac{3\varepsilon^2}{4\pi\varepsilon^5} = \frac{3}{4\pi\varepsilon^3} \to \infty$$

c) $D(r, 0) = 0$ for $r \neq 0$ as shown in part a

d)

$$\int D(r, \varepsilon)\, d\tau = 3 \int_0^{\pi/2} \tan^2\theta \cos^2\theta \, d\theta$$

using wolfram

$$= 3\left[\frac{1}{3}\right] = 1$$