3.3, 3.5, 3.6, 3.13,
3.14, 3.15, 3.16,
3.19, 3.22, 3.24, 3.26

Erandi Macias

Electro hw 3 :

3.3 -

Spherical:

$$\nabla^2 V(r) = 0$$

$$\nabla^2 V(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V(r)}{\partial r}\right) + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V(r)}{\partial\theta}\right)}^{0}$$

$$+ \cancel{\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V(r)}{\partial\phi^2}\right)}^{0} = 0$$

$$\nabla^2 V(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V(r)}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \qquad \text{slope} = 0$$

$$r^2\frac{\partial V}{\partial r} = c \qquad c = \text{some constant}$$

$$\partial r \cdot \frac{\partial V}{\partial r} = \frac{c}{r^2}\cdot\partial r \qquad V(r) = \int\frac{c}{r^2}\partial r$$

$$= -\frac{c}{r} + k$$

cylindrical:

$$\nabla^2 V(s) = 0$$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V(s)}{\partial s}\right) = 0$$

$$s\frac{\partial V}{\partial s} = c \qquad \frac{\partial V}{\partial s} = \frac{c}{s}$$

$$V(s) = \int \frac{c}{s}\,ds = c\ln(s) + k$$

* 3.5 -

$$\vec{\nabla}\vec{E}_1 = \frac{1}{\varepsilon_0}\rho \qquad \vec{\nabla}\vec{E}_2 = \frac{1}{\varepsilon_0}\rho$$

Same as proof of second uniqueness

$$\oint \vec{E}_1 \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_i \qquad \oint \vec{E}_2 \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_i$$

$$\oint \vec{E}_1 \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{tot} \qquad \oint \vec{E}_2 \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{tot}$$

$$\vec{E}_3 = \vec{E}_1 - \vec{E}_2$$

$$\vec{\nabla}\vec{E}_3 = 0$$

$$\oint \vec{E}_3 \cdot d\vec{a} = 0 \qquad \oint V_3 \vec{E}_3 \cdot d\vec{a} = -\int_V (E_3)^2\,d\tau$$

$$\int (E_3)^2 = 0 \qquad \vec{E}_2 = \vec{E}_1$$

* 3.6 -

$V = T = V_3$

$$\int_V [V_3 \nabla^2 V_3 + \vec{\nabla} V_3 \cdot \vec{\nabla} V_3] \, d\tau = \oint_S V_3 \vec{\nabla} V_3 \cdot d\vec{a}$$

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\varepsilon_0} + \frac{\rho}{\varepsilon_0} = 0$$

$$\vec{\nabla} V_3 = -\vec{E}_3 \qquad \int_V E_3^2 \, d\tau = -\oint_S V_3 \vec{E}_3 \, d\vec{a}$$

$= 0$

* 3.13 -

Ex. 3.3

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$$

$x \to \infty \Rightarrow V \to 0$

$$V(x,y) = e^{-k}(C\sin(ky) + D\cos(ky))$$

$y = 0 \Rightarrow V = 0$

$$V(x,y) = Ce^{-kx}\sin(ky)$$

$y = a \Rightarrow V = 0$

$C\sin(ka) = 0 \qquad ka = n\pi \qquad n = 1, 2, 3 \ldots$

$$k = \frac{n\pi}{a}$$

$$V(x,y) = Ce^{-\frac{n\pi y}{a}}\sin\left(\frac{n\pi y}{a}\right)$$

Fourier trick Series

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) V_0$$

$$C_n = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{n\pi y}{a}\right) dy$$

$$C_n = \frac{2V_0}{a}\left(\int_0^{a/2} \sin(n\pi y/a)\, dy - \int_{a/2}^{a} \sin\left(\frac{n\pi y}{a}\right) dy\right)$$

$$= \frac{2V_0}{n\pi}\left(1+(-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right)$$

* 3.14 -

$$V(x,y) = \frac{4}{\pi} V_0 \sum_n \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n} \qquad \hat{n} = \hat{x}$$

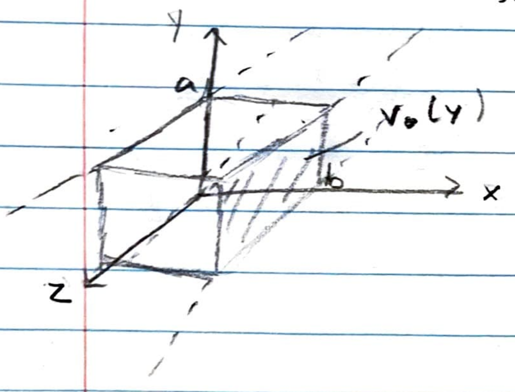
$$\sigma(y) = -\varepsilon_0 \frac{\partial}{\partial x}\left[\frac{4}{\pi} V_0 \sum_n \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)\right]_{x=0}$$

$$= +\varepsilon_0 \frac{4}{\pi} V_0 \sum \frac{1}{\cancel{n}}\left(\cancel{+}\frac{\cancel{n}\cancel{\pi}}{a} \cancel{e^{-\frac{n\pi x}{a}}}^{1}\right) \sin\left(\frac{n\pi y}{a}\right)\Big|_{x=0}$$

$$\sigma(y) = \frac{4\varepsilon_0 V_0}{a} \sum_n \sin\left(\frac{n\pi y}{a}\right)$$

* 3.15 -

same as 3.3



$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$$

exp ≠ 0

$x = 0 \Rightarrow V = 0$

$y = 0,\ a = 0 \Rightarrow V = 0$

$x = b \Rightarrow V = V_0(y)$

$$V(x,y) = AC\left(e^{n\pi x/a} - e^{-n\pi x/a}\right)\sin(n\pi y/a)$$

$$= (2AC)\sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$V(x,y) = \sum_{n=}^{\infty} C_n \sinh(n\pi x/a) \sin(n\pi y/a)$$

$$V_0(y) = \sum C_n \sinh(n\pi b/a) \sin(n\pi y/a)$$

$$C_n \sinh(n\pi b/a) = 2/a \int_0^a V_0(y) \sin(n\pi y/a)\, dy$$

$$C_n = \frac{2}{a \sinh(n\pi b/a)} \int_0^a V_0(y) \sin(n\pi y/a)\, dy$$

* 3.16 -

Same as ex. 3.5

(i) $V=0 \Rightarrow x=0$

(ii) $V=0 \Rightarrow x=a$

(iii) $V=0 \Rightarrow y=0$

(iv) $V=0 \Rightarrow y=a$

(v) $V=0 \Rightarrow z=0$

(vi) $V = V_0(x,y,z)$ when $z=a$?

$X(x) = A\sin(kx) + B\cos(kx)$

$Y(y) = C\sin(ly) + D\cos(ly)$

$Z(z) = Ee^{\sqrt{k^2+l}} + Fe^{-\sqrt{k^2+l^2}}$

$$V(x,y,z) = (A\sin(kx) + B\cancel{\cos(kx)}^{0})(C\sin(ly) + D\cancel{\cos(ly)}^{0})(Ee^{\sqrt{k^2+l}} + Fe^{-\sqrt{k^2+l^2}})$$

$$V(x,y,z) = 2E\sinh(\pi\sqrt{n^2+m^2}\,(z/a))$$

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh(\pi\sqrt{n^2+m^2}\,(z/a))$$

$$C_{n,m}\sinh(\pi\sqrt{n^2+m^2}) = \left(\frac{2}{a}\right)^2 V_0\int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dx\,dy$$

$$= \frac{16V_0}{\pi^2 nm}$$

$$V(x,y,x) = \frac{16V_0}{\pi^2}\sum\sum \frac{\sin(n\pi x/a)\sin(m\pi y/a)}{nm}\frac{\sin(\pi\sqrt{n^2+m^2}(z/a))}{\sinh(\pi\sqrt{n^2+m^2})}$$

3.19-

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Inside : From ex. 3.9

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad (r \leq R)$$

Outside :

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \quad (r \geq R)$$

$$V_0 = k\cos 3\theta$$

$\cos(3\theta)$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(2\theta+\theta) = \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)$$

$$= [2\cos^2\theta - 1]\cos\theta - 2\sin\theta\cos\theta\sin\theta$$

$$= 2\cos^3\theta - \cos\theta - 2\cos\theta(\sin^2\theta)$$

$$= 2\cos^3\theta - \cos\theta - 2\cos\theta[1-\cos^2\theta]$$

$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$

$$= 4\cos^3\theta - 3\cos\theta$$

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$

$$= a P_3(\cos\theta) + b P_1(\cos\theta)$$

$$= a\left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) + b\cos\theta$$

$$= \frac{5a}{2}\cos^3\theta + \left(-\frac{3a}{2} + b\right)\cos\theta$$

$\frac{5a}{2} = 4 \qquad a = 8/5$

$-\frac{3a}{2} + b = -3$

$-\frac{3}{\cancel{2}}\left(\frac{\cancel{8}^{4}}{5}\right) + b = -3$

$b = \frac{12}{5} - 3 = -3/5$

$V(\theta) = k\cos 3\theta$

$= \frac{k}{5}\left(8P_3(\cos\theta) - 3P_1(\cos\theta)\right)$

$\int_0^\pi P_\ell(\cos\theta)\, P_\ell(\cos\theta)\sin\theta\, d\theta = \frac{2}{2\ell+1}$

$\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \frac{k}{5}\left(8P_3(\cos\theta) - 3P_1(\cos\theta)\right)$

$A_1 R\left(\frac{2}{3}\right) = \frac{k}{5}(-3)\int_0^\pi P_1\cos\theta\, P_1(\cos\theta)(\sin\theta)\, d\theta$

$= -\frac{3k}{5}\left(\frac{2}{3}\right)$

$A_1 = -\frac{3k}{5R}$

$$\cancel{\left(\frac{2}{7}\right)} R^3 A_3 = \frac{8k}{5} \int_0^{\pi} \left(P_3(\cos\theta)\right)^2 \sin\theta \, d\theta \quad \cancel{2/7}$$

$$A_3 = \frac{8k}{5R^3} \qquad A_1 = -\frac{3k}{5R}$$

$$V_{in}(r,\theta) = -\frac{3k}{5R} r P_1(\cos\theta) + \frac{8k}{5R^3} r^3 P_3(\cos\theta)$$

$$\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}}$$

$$B_\ell = R^{2\ell+1} A_\ell$$

$$B_1 = R^3 A_1 = R^3\left(-\frac{3k}{5R}\right) = -\frac{3kR^2}{5}$$

$$B_3 = R^7 \frac{8k}{5R^3} = \frac{8kR^4}{5}$$

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$= \frac{B_1}{r^2} P_1(\cos\theta) + \frac{B_3}{r^4} P_3(\cos\theta)$$

$$= -\frac{3kR^2}{5r^2} P_1(\cos\theta) + \frac{8kR^4}{5r^4} P_3(\cos\theta)$$

$$= \frac{k}{5}\left(8\left(\frac{R}{r}\right)^4 P_3(\cos\theta) - 3\left(\frac{R}{r}\right)^2 P_1(\cos\theta)\right)$$

$$\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} = -\frac{\sigma(\theta)}{\varepsilon_0}$$

$$\frac{k}{5}\left(8R^4(-4r^{-5})P_3(\cos\theta) - 3R^2(-2r^{-3})P_1(\cos\theta)\right)$$

$$-\frac{k}{5}\left(\frac{8(3r^2)}{R^3}P_3(\cos\theta) - \frac{3}{R}P_1(\cos\theta)\right) = -\frac{\sigma(\theta)}{\varepsilon_0}$$

$$\frac{k}{5}\left(-\frac{32}{R}P_3(\cos\theta) - \frac{6}{R}P_1(\cos\theta) - \frac{24}{R}P_3(\cos\theta)\right.$$

$$\left.+ 3/R(\cos\theta)\right)$$

$$\varepsilon_0\frac{k}{5}\left(\frac{56}{R}P_3(\cos\theta) - \frac{9}{R}P_1(\cos\theta)\right) = \sigma(\theta)$$

3.22 -

a) $r > R$

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(\cancelto{0}{A_l} r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$= \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$$

$$V(r,0) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2} - r\right) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}$$

$$\sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2} - r\right)$$

$$= \frac{\sigma r}{2\varepsilon_0}\left(\sqrt{1+\left(\frac{R}{r}\right)^2} - 1\right)$$

b) $r < R$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$V(r,0) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+r^2} - r\right) = \sum_{\ell=0}^{\infty} A_\ell r^\ell$$

$$= \frac{\sigma}{2\varepsilon_0} \sqrt{1+\left(\frac{R}{r}\right)^2} + 1$$

* 3.24 -

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \cancel{\frac{\partial^2 V}{\partial z^2}}^{\,0}$$

$$V(s,\phi) = S(s)\,\Phi(\phi)$$

$$\frac{1}{s}\Phi\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{s^2}S\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = C_1 \qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = C_2$$

$$C_1 + C_2 = 0$$

??.?

3.26 -

Inside : $V(s,\phi) = a_0 + \sum_{k=1}^{\infty} s^{k} (a_k \cos k\phi + b_k \sin k\phi)$

outside : $V(s,\phi) = \bar{a}_0 + \sum_{k=1}^{\infty} s^{-k} (c_k \cos k\phi + d_k \sin k\phi)$

$$\sigma = -\varepsilon_0 \left(\frac{\partial V_{out}}{\partial s} - \frac{\partial V_{in}}{\partial s} \right) \Big|_{s=R}$$

$s = R$

$$a \sin 5\phi = -\varepsilon_0 \sum_{k=1}^{\infty} \left[-\frac{k}{R^{k+1}} \left(\cancelto{0}{c_k} \cos k\phi + d_k \cancelto{0}{\sin k\phi} \right) - kR^{k-1} \left(a_k \cancelto{0}{\cos k\phi} + b_k \cancelto{0}{\sin k\phi} \right) \right]$$

$$V(s,\phi) = \frac{a \sin 5\phi}{10 \varepsilon_0} \frac{s^5}{R^4} \qquad s \le R$$

$$V(s,\phi) = \frac{a \sin 5\phi}{10 \varepsilon_0} \frac{R^6}{s^5} \quad , \quad s \ge R$$