Ch. 1 HW

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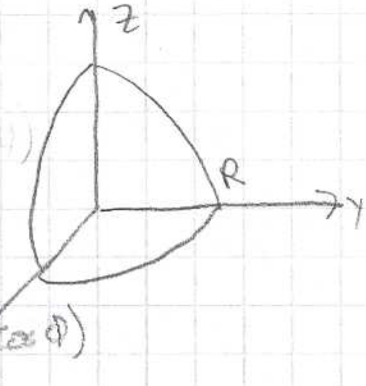
PHYS 330



1.54) $v = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

$$\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, r^2\cos\phi\right)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\phi\right)$$

$$= \frac{1}{r^2}4r^3\cos + \frac{1}{r\sin\theta}\cos\theta\, r^2\cos\phi + \frac{1}{r\sin\theta}\left(-r^2\cos\theta\cos\phi\right)$$

$$= 4r\cos\theta$$

$$\int(\nabla\cdot v)\,d\tau = \int_0^R\int_0^{\pi/2}\int_0^{\pi/2}(4r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 4\int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi$$

$$= 4\left(\tfrac{1}{4}r^4\Big|_0^R\right)\left(\tfrac{1}{2}\sin^2(\theta)\Big|_0^{\pi/2}\right)\left(\pi/2\right) = 4\left(\frac{R^4}{4}\right)\left(\frac{\sin^2(\pi/2)}{2}\right)\left(\pi/2\right)$$

$$= \boxed{\frac{\pi R^4}{4}}$$

1.55) $v = ay\hat{x} + bx\hat{y}$

$$\nabla \times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & bx & 0 \end{vmatrix} = \hat{z}(b-a)$$

$$\int_S (\nabla \times v) \cdot da = (b-a)\pi R^2$$

$$v \cdot dl = (ay\hat{x} + bx\hat{y}) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z})$$

$$= ay\,dx + bx\,dy \qquad x^2 + y^2 = R^2 \Rightarrow 2x\,dx + 2y\,dy = 0.$$

$$dy = -(x/y)\,dx$$

$$v \cdot dl = ay\,dx + bx(-x/y)\,dx = \frac{1}{y}(ay^2 - bx^2)\,dx$$

$$\int v \cdot dl = \int_R^{-R} \frac{aR^2 - (a+b)x^2}{\sqrt{R^2 - x^2}}\,dx \qquad \text{semicircle} = y = \sqrt{R^2 - x^2}$$

$$= \frac{1}{2}R^2(a-b)\sin^{-1}\left(\frac{x}{R}\right)\Big|_R^{-R} = \frac{1}{2}R^2(a-b)(\sin^{-1}(-1) - \sin^{-1}(1))$$

$$= \frac{1}{2}R^2(a-b)\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = \boxed{\frac{1}{2}\pi R^2(b-a)}$$

1.56.) $\int (\nabla \times \vec{v}) \cdot d\vec{a} = \oint \vec{v} \cdot dl$

$\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$

path 1

$d\vec{l} = +dy\hat{y}, \quad z=0$

$\vec{v} \cdot d\vec{l} = y z\, dy, \quad \int \vec{v} \cdot d\vec{l} = \emptyset$

path 2

$z(y) = 2 - 2y \qquad d\hat{l} = dy\hat{y} + dz\hat{z}$

$dz = -2dy \qquad \vec{v} \cdot d\vec{l} = yz^2 dy + (3y+z)dz$

$\vec{v} \cdot d\vec{l} = y(2-2y)^2 dy + (3y+2-2y)(-2)dy$

$\int_1^0 dy(\ldots) = 14/3 \qquad 14/3 - 6/3 = \boxed{8/3}$ ✓

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ 6 & yz^2 & 3y+z \end{vmatrix} = \hat{x}(3-2yz) - \hat{y}(0-0) + \hat{z}(0-0)$$

$\nabla \times \vec{v} = (3-2yz)\hat{x}$

$\int \nabla \times \vec{v} \cdot d\vec{a} = \int_0^2 \int_0^{1-\frac{1}{2}z} (3-2yz)\, dy\, dz$

$z = 2-2y$

$\frac{1}{2}z = 1-y$

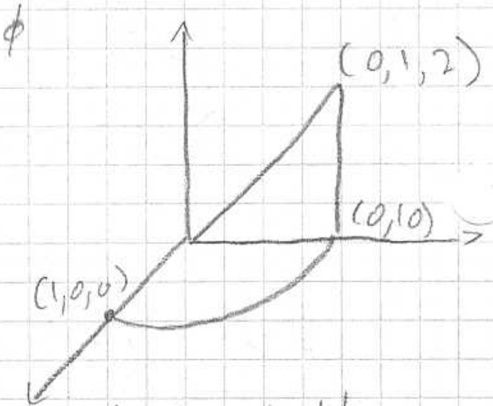
$y = 1 - \frac{1}{2}z$

$= \int_0^2 dz \left\{ 3y - zy^2 \right\}_0^{1-\frac{1}{2}z}$

$= \int_0^2 dz \left\{ 3(1-\tfrac{1}{2}z) - z(1-\tfrac{1}{2}z)^2 \right\}$

$= \boxed{\frac{8}{3}}$ ✓

1.57) $v = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\phi$



I. $\theta = \frac{\pi}{2}$ $\phi = 0,$ $r: 0 \to 1$

$$v \cdot dl = [(r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\phi] \cdot [dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}]$$

$$= (r\cos^2\theta)dr - (r^2\cos\theta\sin\theta)d\theta + (3r^2\sin\theta)d\phi$$

$$= (r\cos^2(\pi/2))dr - (r^2\cos(\tfrac{\pi}{2})\sin(\pi/2))d\theta + 0 = \emptyset$$

$$\int_0^1 v \cdot dl = \underline{\emptyset}$$

II $r = 1,$ $\theta = \frac{\pi}{2}$ $\phi: 0 \to \pi/2$

$$v \cdot dl = ((1)\cos^2(\tfrac{\pi}{2}))dr - ((1)^2\cos(\tfrac{\pi}{2})\sin(\pi/2))d\theta + (3r^2\sin(\pi/2))d\phi$$

$\underline{= 3d\phi}$ $\int_0^{\pi/2} 3d\phi = 3\phi\Big|_0^{\pi/2} = \underline{\frac{3\pi}{2}}$

III $r = \frac{1}{\sin\theta}$ $dr = -\frac{1}{\sin^2\theta}\cos\theta d\theta$ $\theta = \frac{\pi}{2} \to \theta_0 = \tan^{-1}(\frac{1}{2})$

$\phi = \frac{\pi}{2}$

$$v \cdot dl = (r\cos^2\theta)(dr) - (r\cos\theta\sin\theta)(rd\theta)$$

$$= \frac{\cos^2\theta}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\right)d\theta - \frac{\cos\theta\sin\theta}{\sin^2\theta}d\theta$$

$$= -\frac{\cos\theta}{\sin^3\theta}d\theta \to \int v \cdot dl = -\int_{\pi/2}^{\theta_0}\frac{\cos\theta}{\sin^3\theta}d\theta$$

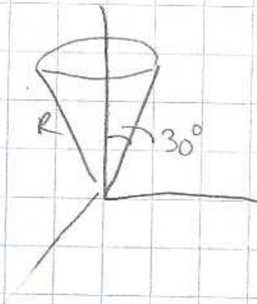
$$= \frac{1}{2\sin^2\theta}\Big|_{\pi/2}^{\theta_0} = \underline{2}$$

IV $\theta = \theta_0$ $\phi = \frac{\pi}{2}$ $r: \sqrt{5} \to 0$ $v \cdot dl = (r\cos^2\theta)(dr) = \frac{4}{5}rdr$

$$\int v \cdot dl = \frac{5}{4}\int_{\sqrt{5}}^{0} r\,dr = \frac{4}{5}\frac{r^2}{2}\Big|_{\sqrt{5}}^{0} = -2$$

$$\oint v \cdot dl = 0 + \frac{3\pi}{2} + 2 - 2 = \boxed{\frac{3\pi}{2}}$$

1.59a) $v = r^2\sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r^2\tan\theta\,\hat{\phi}$



$$\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, 4r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\tan\theta)$$

$$= \frac{1}{r^2}4r^3\sin\theta + \frac{1}{r\sin\theta}4r^2(\cos^2\theta - \sin^2\theta)$$

$$= 4r\frac{\cos^2\theta}{\sin\theta} \qquad \int_0^R 4r^3dr\int_0^{\pi/6}\cos^2\theta\, d\theta\int_0^{2\pi}d\phi$$

$$= R^4\, 2\pi\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{\pi/6} = \boxed{\frac{\pi R^4}{12}(2\pi + 3\sqrt{3})}$$

1.62) $a = \int_S da$

a.) $da = R^2\sin\theta\, d\phi\,\hat{r}$ $\qquad a = \int R^2\sin\theta\cos\theta\, d\theta\, d\phi\,\hat{z}$

$$= 2\pi R^2\hat{z}\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = 2\pi R^2\hat{z}\,\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}$$

$\underline{a = \pi R^2\hat{z}}$

b.) $T = 1$ $\qquad \int_\tau(\nabla T)d\tau = \oint_S T\,da$ $\qquad \nabla T = 0$ $\qquad \oint_S da = 0$

$\Downarrow$

$\underline{a = 0}$

c.) $a_1 \neq a_2$ $\qquad \oint da = a_1 - a_2 \neq 0$

d.) $a = \frac{1}{2}\oint r\times dl$ $\qquad da = \frac{1}{2}(r\times dl)$ area of parallelogram

$$a = \oint da = \underline{\frac{1}{2}\oint r\times dl}$$

e.) $\oint(c\cdot r)dl = a\times c$

$\int\nabla T\cdot da = \oint T\,dl$ $\qquad T = c\cdot r$ $\qquad \nabla T = \nabla(c\cdot r)$

$c = \left(c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z}\right)$ $\qquad (c\cdot\nabla)r = c_x\hat{x} + c_y\hat{y} + c_z\hat{z} = c$

$$\oint T\,dl = \oint(c\cdot r)dl = -\int(\nabla T)\times da = -\int c\times da = -c\times\int da$$

$\boxed{= c\times a}$

1.63) $v = \frac{\hat{r}}{r}$ 1.84

$$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

1.85 $\oint v \cdot da = \int \left(\frac{1}{R^2}\hat{r}\right) \cdot \left(R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right)$

$$= \int \left(\frac{1}{R^2}\hat{r}\right) \cdot \left(R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right)$$

$$= \left(\int_0^{\pi} \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = \boxed{4\pi}$$

$v = \frac{\hat{r}}{r}$ $\quad v = v_r \hat{r} + v_\theta \hat{\theta} + v_\phi \hat{\phi}$

$v_r = \frac{1}{r}$ $\quad v_\theta = 0$ $\quad v_\phi = 0$

$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)$ $\quad \nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}\right) = \boxed{\frac{1}{r^2}}$

$$\int (\nabla \cdot v) \, d\tau = \int \left(\frac{1}{r^2}\right) 4\pi r^2 \, dr = 4\pi \int_0^R dr = 4\pi R$$

$$\int v \cdot da = \int \left(\frac{1}{R}\hat{r}\right) \cdot \left(R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right) = R \int \sin\theta \, d\theta \, d\phi$$

$$= R\left(-\cos\theta\Big|_0^{\pi}\right)\left(\phi\right)_0^{2\pi} = \boxed{4\pi R}$$

$\int (\nabla \cdot v) \, d\tau = \int v \cdot da$

$v = r^n \hat{r}$ $\quad v = v_r \hat{r} + v_\theta \theta + v_\phi \hat{\phi}$

$v_r = r^n$ $\quad v_\theta = 0$ $\quad v_\phi = 0$ $\quad \nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)$

$$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+2}) = \frac{1}{r^2}(n+2) r^{n+1}$$

$$\boxed{\nabla \cdot (r^n \hat{r}) = (n+2) r^{n-1} \quad n \neq -2}$$

b.) $\nabla \times v = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta \, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right)\hat{\theta}$

$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right)\hat{\phi}$ $\quad v_r = r^n \quad v_\theta = 0 \quad v_\phi = 0$

$v = r^n \hat{r}$ $\quad \nabla \times v = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta(0)) - \frac{\partial (0)}{\partial \phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial (r^n)}{\partial \phi} - \frac{\partial}{\partial r}(r 0)\right)\hat{\theta}$

$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r(0)) - \frac{\partial (r^n)}{\partial \theta}\right)\phi = \underline{0}$

1.64) $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(r)$ Eq. 1.102 $r'=0$ $\sqrt{r^2+\epsilon^2}$

$$D(r,\epsilon) \equiv -\frac{1}{4\pi}\nabla^2\frac{1}{\sqrt{r^2+\epsilon^2}}$$

$$D = -\frac{1}{4\pi}\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(-\frac{1}{2}\right)\frac{2r}{(r^2+\epsilon^2)^{3/2}}\right] = \frac{1}{4\pi r^2}\frac{d}{dr}\left[\frac{r^3}{(r^2+\epsilon^2)^{3/2}}\right]$$

$$= \frac{1}{4\pi r^2}\left[\frac{3r^2}{(r^2+\epsilon^2)^{3/2}} - \frac{3}{2}\frac{r^3\,2r}{(r^2+\epsilon^2)^{5/3}}\right] = \frac{1}{4\pi r^2}\frac{3r^2}{(r^2+\epsilon^2)^{5/2}}(r^2+\epsilon^2-r^2)$$

$$= \boxed{\frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}}} \checkmark$$

b) $D(0,\epsilon) = \dfrac{3\epsilon^2}{4\pi\epsilon^5} = \dfrac{3}{4\pi\epsilon^3}$ $\epsilon \to 0$ $\epsilon \to \infty$ ✓

c) $D(r,0) = 0$ $r \neq 0$ $\dfrac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}} = 0$ if $\epsilon = 0$ ✓
$r \neq 0$

d) $\int D(r,\epsilon)\,4\pi r^2\,dr = 3\epsilon^2\int_0^\infty \dfrac{r^2}{(r^2+\epsilon^2)^{5/2}}\,dr = 3\epsilon^2\left(\dfrac{1}{3\epsilon^2}\right) = \underline{1}$ ✓