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HW 4

10 $\sigma_b = P \cdot \hat{n} = \boxed{KR = \sigma_b}$

a)

$\rho_b = -\nabla \cdot P = \frac{-1}{r^2}\frac{\partial}{\partial r}(r^2 Kr) = -\frac{1}{r^2} \cdot 3Kr^2 = \boxed{-3K}$

b) $r<R$

$E = \frac{1}{3\varepsilon_0}\rho r \hat{r} = \frac{1}{3\varepsilon_0}(-3K) r\hat{r} = \boxed{\left(\frac{K}{\varepsilon_0}\right) r = E}$

$r>R$

$Q_{tot} = (KR)(4\pi R^2) + (-3K)(\frac{4}{3}\pi R^3) = 0 \therefore \boxed{E=0}$

14 $Q_{tot} = \oint_S \sigma_b \, da + \int_V \rho_b \, d\tau \qquad \sigma_b \equiv P \cdot \hat{n} \qquad \rho_b \equiv -\nabla \cdot P$

$Q_{tot} = \oint_S P \cdot da - \int_V \nabla \cdot P \, d\tau$

Due to divergence theorem $\oint_S P \cdot da = \int_V \nabla \cdot P \, d\tau$ and therefore $Q_{enclosed} = 0$

15 a) $\rho_b = -\nabla \cdot P = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{K}{r}\right) = \boxed{\frac{-K}{r^2}}$ $\qquad \sigma_b = P \cdot \hat{n}$ when $r=b$: $+P \cdot \hat{r} = K/b$

when $r=a$: $-P \cdot \hat{r} = -K/a$

$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q_{enc}}{r^2}\hat{r}$ $\qquad$ when $r<a$: $Q_{enc} = 0$ and $\therefore E=0$

when $r>b$: $Q_{enc}=0$ and $\therefore E=0$

When $a<r<b$: $Q_{enc} = \left(\frac{-K}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-K}{r'^2}\right)(4\pi r'^2)\,dr$

$= -4\pi K a - 4\pi K(r-a)$

$= \boxed{-4\pi K r}$

$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{-4\pi K r}{r^2} \cdot \hat{r} = \boxed{\frac{-K}{\varepsilon_0 r}\hat{r}}$

b) $\oint D \cdot da = Q_{enc}$

since $Q_{enc} = 0$, $D = 0$ everywhere

$D = \varepsilon_0 E + P = 0 \Rightarrow E = (-1/\varepsilon_0)P$ $\qquad$ For $r<a$ and $r>b$: $E=0$

$E = -(K/\varepsilon_0 r)\hat{r}$ when $a<r<b$

18 a) $\int D \cdot da = Q_{f_{enc}} \Rightarrow DA = \sigma A \Rightarrow \boxed{D = \sigma}$

b) $D = \varepsilon \cdot E \Rightarrow E = \frac{D}{\varepsilon}$ $\qquad E = \frac{D}{\varepsilon_1} = \frac{\sigma}{\varepsilon_1}$ for slab 1, $E = \sigma/\varepsilon_2$ for slab 2

$\varepsilon = \varepsilon_0 \varepsilon_r \therefore \varepsilon_1 = 2\varepsilon_0, \varepsilon_2 = \frac{3}{2}\varepsilon_0$

$\boxed{E_1 = \sigma/2\varepsilon_0 \quad E_2 = 2\sigma/3\varepsilon_0}$

c) $P = \varepsilon_0 \chi_e E \quad \therefore \quad P = \frac{\cancel{\varepsilon_0} \chi_e \sigma}{\cancel{\varepsilon_0} \varepsilon_r} = \frac{\chi_e \sigma}{\varepsilon_r}$

$\chi_e = \varepsilon_r - 1 \qquad P = \frac{(\varepsilon_r - 1)\sigma}{\varepsilon_r} = \left(1 - \frac{1}{\varepsilon_r}\right)\sigma \qquad P_2 = \left(1 - \frac{2}{3\varepsilon_0}\right)\sigma$

$P_1 = \left(1 - \frac{1}{2\varepsilon_0}\right)\sigma = \boxed{\frac{\sigma}{2}} = P_1 \qquad P_2 = \boxed{\frac{\sigma}{3}}$

d) $V = E_1(a) + E_2(a) = \frac{\sigma}{2\varepsilon_0} a + \frac{2\sigma}{3\varepsilon_0} a = \frac{3\sigma}{6\varepsilon_0} a + \frac{4\sigma}{6\varepsilon_0} a = \boxed{\frac{7\sigma}{6\varepsilon_0} a}$

e) $\rho_b = 0$ At the bottom of slab 1, $\sigma_b = \sigma/2 = P_1$ At bottom of slab 2, $\sigma_b = P_2 = \sigma/3$
At top of slab 1, $\sigma_b = -\sigma/2 = -P_1$ At top of slab 2, $\sigma_b = -P_2 = -\sigma/3$

f) Slab 1
above: $\sigma - \left(\frac{\sigma}{2}\right) = \frac{\sigma}{2}$
below: $\left(\frac{\sigma}{2}\right) - \left(\frac{\sigma}{3}\right) + \left(\frac{\sigma}{3}\right) - \sigma = \sigma/2$

$\therefore \boxed{E_1 = \sigma/2\varepsilon_0}$

Slab 2
Above: $\sigma - \sigma/2 + \sigma/2 - \sigma/3 = 2\sigma/3$
below: $\sigma/3 - \sigma = -2\sigma/3$ $\qquad \therefore \boxed{E_2 = \frac{2\sigma}{3\varepsilon_0}}$

$\boxed{26}$ $W = \frac{1}{2}\int \vec{D}\cdot\vec{E} = \frac{1}{2}\int_a^b \frac{Q^2}{\varepsilon(4\pi r^2)^2} d\tau + \frac{1}{2}\int_b^\infty \frac{Q^2}{\varepsilon_0 (4\pi r^2)^2} d\tau$

$= \frac{1}{2}\cdot\frac{Q^2}{16\pi^2}\left[\frac{1}{\varepsilon}\int_a^b \frac{1}{r^4}\cdot 4\pi r^2 dr + \frac{1}{\varepsilon_0}\int_b^\infty \frac{1}{r^2}\cdot 4\pi r^2 dr\right] = \frac{1}{2}\cdot\frac{Q^2}{16\pi^2}\cdot 4\pi\left[\frac{1}{\varepsilon}\int_a^b \frac{1}{r^2}dr + \frac{1}{\varepsilon_0}\int_b^\infty \frac{1}{r^2}dr\right]$

$= \frac{Q^2}{8\pi}\left[\frac{1}{\varepsilon}\left[\frac{-1}{r}\right]_a^b + \frac{1}{\varepsilon_0}\left[\frac{-1}{r}\right]_b^\infty\right] = \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{1}{1+\chi_e}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{b}\right] = \frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)}\left[\frac{1}{a} - \frac{1}{b} + \frac{1+\chi_e}{b}\right]$

$W = \boxed{\frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)}\left[\frac{1}{a} + \frac{\chi_e}{b}\right]}$

$\boxed{35}$ $\oint D\cdot da = Q_{f_{enc}}$

$D = \frac{q}{4\pi r^2}\hat{r}$ and $E = \frac{1}{\varepsilon} D = \frac{1}{\varepsilon}\left(\frac{q}{4\pi r^2}\hat{r}\right) = \boxed{\frac{q}{4\pi\varepsilon_0(1+\chi_e)}\frac{\hat{r}}{r^2} = E}$

$\varepsilon = \varepsilon_0 \varepsilon_r \Rightarrow \varepsilon = \varepsilon_0(1+\chi_e)$
$\varepsilon_r = 1 + \chi_e$

$P = \varepsilon_0 \chi_e E = \varepsilon_0 \chi_e \left(\frac{q}{4\pi\varepsilon_0(1+\chi_e)}\right)\left(\frac{\hat{r}}{r^2}\right) = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)}\frac{\hat{r}}{r^2} = P}$

$\rho_b = -\nabla\cdot P = \frac{-q\chi_e}{4\pi(1+\chi_e)}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right) = \frac{-q\chi_e}{\cancel{4\pi}(1+\chi_e)}\cdot\cancel{4\pi}\delta(x) = \boxed{\frac{-q\chi_e}{(1+\chi_e)}\delta^3(r) = \rho_b}$

$$\sigma_b = P\cdot\hat{r} = \frac{q\chi_e}{4\pi(1+\chi_e)}\cdot\frac{\hat{r}}{r^2}\cdot\hat{r} = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)r^2} = \sigma_b}$$

$$Q_{surface} = \sigma_b(4\pi r^2) = q\frac{\chi_e}{4\pi(1+\chi_e)\cancel{r^2}}\cdot\cancel{4\pi r^2} = \boxed{\frac{q\chi_e}{1+\chi_e} = Q}$$

The compensating negative would be located at the center of the surface.