EM: HW #3 3.3, 3.5, 3.6, 3.13, 3.14, 3.15, 3.16, 3.19, 3.22, 3.24, 3.26

3.3.) $\nabla^2 V = 0$

in spherical coordinates for $V(r)$

$$\nabla^2 V(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$

$$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$

$$\Rightarrow r^2 \frac{dV}{dr} = \text{const.} = c$$

$$\frac{dV}{dr} = \frac{c}{r^2} \Rightarrow \boxed{V = -\frac{c}{r} + k}$$

in cylindrical for $V(s)$

$$\nabla^2 V(s) = \frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0$$

$$\frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0$$

$$\Rightarrow s\frac{dV}{ds} = \text{const.} = c$$

$$\frac{dV}{ds} = \frac{c}{s} \Rightarrow \boxed{V = c\ln(s) + k}$$

3.5.) Suppose there are two electric fields that encapsulate charge density ρ and V is given on each boundary or $\frac{\partial V}{\partial n}$.
$\hat{n}$ at surface

Then gauss' law in diff. form states

$$\nabla \cdot E_1 = \frac{1}{\varepsilon_0}\rho \qquad \nabla \cdot E_2 = \frac{1}{\varepsilon_0}\rho$$

Thus for a surface enclosing total charge

$$\oint \bar{E}_1 \cdot da = \frac{1}{\varepsilon_0} Q_{tot} \qquad \oint E_2 \cdot da = \frac{1}{\varepsilon_0} Q_{tot}$$

Now consider $E_3 = E_2 - E_1$.

$$\nabla \cdot E_3 = \nabla \cdot (E_2 - E_1) = \nabla \cdot E_2 - \nabla \cdot E_1 = \frac{1}{\varepsilon_0}\rho - \frac{1}{\varepsilon_0}\rho$$

$$\Rightarrow \nabla \cdot E_3 = 0$$

and $\oint \bar{E}_3 \cdot da = \frac{1}{\varepsilon_0}\rho\,(\text{volume}) = 0$ also.

3.6.) Green's identity : $\cancel{\oint_V [T\nabla^2 U - U\nabla^2 T] d\tau = \oint_S [T\nabla U - U\nabla T]\cdot da}$

$$\int_V [T\nabla^2 U + (\nabla T)\cdot(\nabla U)]\, d\tau = \oint_S (T\nabla U)\cdot da$$

If $T = U = V_3$, $V_3 = V_2 - V_1$

$$\int_V [V_3 \nabla^2 V_3 + (\nabla V_3)\cdot(\nabla V_3)]\, d\tau = \oint_S V_3 \nabla V_3 \cdot da$$

Since V_2, V_1 both satisfy Poisson's equation

$$\nabla^2 V_3 = \nabla^2 (V_2 - V_1) = \nabla^2 V_2 - \nabla^2 V_1 = \frac{-\rho}{\epsilon_0} - \frac{-\rho}{\epsilon_0} = 0$$

$$\Rightarrow \nabla^2 V_3 = 0$$

Now $$\int_V (\nabla V_3)^2\, d\tau = \oint_S V_3 \nabla V_3 \cdot da$$

$$\nabla V_3 = -E_3$$

$$\int_V (E_3)^2\, d\tau = -\oint_S V_3 E_3 \cdot da$$

as in the proof of 2nd unique thm. , if surfaces equipotentials $\Rightarrow V_3$ const.

$$-\oint_S V_3 E_3 \cdot da = -V_3 \oint_S E_3 \cdot da$$

but since $E_3 = E_2 - E_1$ $(V_3 = V_2 - V_1)$

$$\oint E_3 \cdot da = 0$$

3b.) cont.

$$\Rightarrow \int_V E_3^2 \, d\tau = 0$$

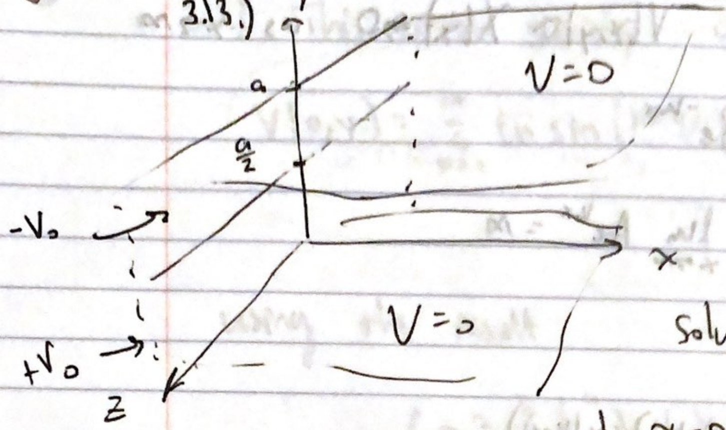
Only satisfied when $E_3 = 0$ since $E_3^2 \geq 0$.

Thus $E_3 = E_2 - E_1 = 0$ or

$$E_2 = E_1$$

and the electric field in the region satisfying poissons equation is unique. ☒

3.13.)



$V(x,y) \Rightarrow \nabla^2 V(x,y) = 0$ inside the slot.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Solutions must satisfy

1. $x=0,\ 0 \le y \le a/2 \rightarrow V = V_0$
2. $x=0,\ \frac{a}{2} \le y \le a \rightarrow V = -V_0$
3. $y = a \rightarrow V = 0$
4. $y = 0 \rightarrow V = 0$
5. $V \rightarrow 0$ as $x \rightarrow \infty$.

Propose solution $V(x,y) = X(x)\,Y(y)$

Thus, $$Y \frac{d^2X}{dx^2} + X \frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

This implies each term constant thus

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1, \quad \frac{d^2X}{dx^2} = C_1 X$$

call $C_1 = k^2$

$$\frac{d^2X}{dx^2} = k^2 X \Rightarrow X = Ae^{kx} + Be^{-kx}$$

for Y, $$\frac{1}{Y}\frac{d^2Y}{dy^2} = C_2 \quad \frac{d^2Y}{dy^2} = C_2 Y$$

call $C_2 = -k^2$

$$\frac{d^2Y}{dy^2} = -k^2 Y \Rightarrow Y = C\sin(ky) + D\cos(ky)$$

3.13.) Now we must have $V(x,y) \propto X(x) \to 0$ as $x \to \infty$

$$X(x) = Ae^{kx} + Be^{-kx}$$

$$\lim_{x\to\infty} Be^{-kx} = 0 \qquad \lim_{x\to\infty} Ae^{kx} = \infty$$

So $A = 0$.

$$V(x,y) = Be^{-kx}\left(\left(\sin(ky) + D\right)\cos(ky)\right)$$

rewriting constants

$$V(x,y) = e^{-kx}\left(C\sin(ky) + D\cos(ky)\right)$$

cond. 4 requires $D = 0$

$$V(x,y) = e^{-kx}\left(\left(C\sin(ky)\right)\right)$$

cond. 3. requires $\sin(ka) = 0$

$$\Rightarrow K = \frac{n\pi}{a}$$

$$V(x,y) = C_n e^{-kx}\sin\left(\frac{n\pi}{a}y\right)$$

Now we have a lin. comb. of these solutions

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi}{a}y\right)$$

3.13.) This must satisfy condition at $x=0$.

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y) = \begin{cases} +V_0 & 0 < y < a/2 \\ -V_0 & a/2 < y < a \end{cases}$$

using the result

$$C_n = \frac{2}{a}\int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

yields

$$C_n = \frac{2}{a}\left[\int_0^{a/2} V_0 \sin\left(\frac{n\pi y}{a}\right) dy - \int_{a/2}^{a} V_0 \sin\left(\frac{n\pi y}{a}\right) dy \right]$$

$$= \frac{2V_0}{a}\left[\left(\frac{-a}{n\pi}\right) \cos\left(\frac{n\pi}{a} y\right)\Big|_0^{a/2} + \left(\frac{a}{n\pi}\right) \cos\left(\frac{n\pi}{a} y\right)\Big|_{a/2}^{a} \right]$$

$$= \frac{2V_0}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + \cos(0) + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right) \right]$$

$$= \frac{2V_0}{n\pi}\left[1 + (-1)^n - 2\cos\left(\frac{n}{2}\pi\right) \right]$$

$n=1 \quad 1-1-2\cos\left(\frac{\pi}{2}\right) = 0$

$n=2 \quad 1+1-2\cos(\pi) = \cancel{0}\ 4$

$n=3 \quad 1-1-2\cos\left(\frac{3\pi}{2}\right) = 0$

$n=4 \quad 1+1-2\cos(2\pi) = 0$

only not zero for evens not divisible by 4

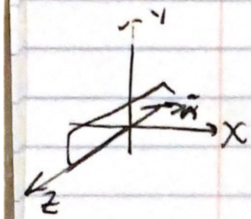
$$\Rightarrow C_n = \begin{cases} \dfrac{8V_0}{n\pi} & n = 2, 6, 10, 14, \ldots \\ 0 & \text{else} \end{cases}$$

3.13.)

$$\Rightarrow V(x,y) = \frac{8V_0}{\pi} \sum_{n=2,6,10,14,\ldots}^{\infty} \frac{1}{n} e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a}y\right)$$

3.14) The potential inside the slot is given by

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a)$$



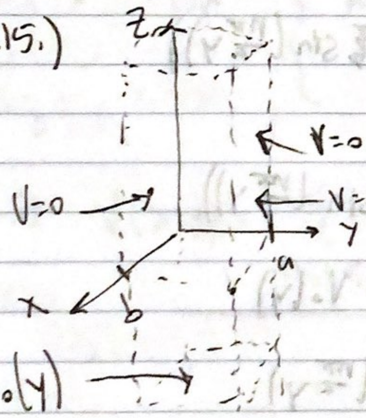
$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$ where $\hat{n}$ is normal to surface
↳ in x direction

$$\Rightarrow \sigma = -\epsilon_0 \frac{\partial V}{\partial x} = -\epsilon_0 \frac{4V_0}{\pi} \frac{\partial}{\partial x}\left(\sum \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin n\pi y/a\right)$$

$$= \cancel{-}\epsilon_0 \frac{4V_0}{\pi} \sum \frac{1}{n}\left(+\frac{n\pi}{a}\right) e^{-\frac{n\pi}{a}x} \sin(n\pi y/a)$$

$$\sigma = \frac{4V_0\epsilon_0}{a} \sum_{n=1,3,5,\dots} e^{-\frac{n\pi}{a}x} \sin(n\pi y/a)$$

3.15.)



$\nabla^2 V = 0$ z coordinate doesn't matter

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

w/ conditions
1. $V(x,0) = 0$
2. $V(x,a) = 0$
3. $V(0,y) = 0$
4. $V(b,y) = V_0(y)$

we know this using sep. of var.

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

This time let Y be sinusoidal to satisfy $=0$ at $y=0, y=a$.

3.15)

$$\frac{\partial^2 Y}{\partial y} = -k^2 Y$$

$$\Rightarrow Y = C\sin(ky) + D\cos(ky)$$

$$\frac{\partial^2 X}{\partial x} = k^2 X \Rightarrow X = Ae^{kx} + Be^{-kx}$$

$$\Rightarrow V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$$

$\cos(0) = 1 \Rightarrow D = 0.$ using 1.

$\sin(ka) = 0 \Rightarrow ka = n\pi \Rightarrow k = \frac{n\pi}{a}$ $n \in \mathbb{Z}^+$ 2.

$A + B = 0 \Rightarrow A = -B \Rightarrow B = -A$ using 3.

$$\Rightarrow V(x,y) = (Ae^{kx} - Ae^{-kx})(C\sin(\tfrac{n\pi}{a}y))$$

$$= \underbrace{AC}_{C_n}(e^{kx} - e^{-kx})(\sin(\tfrac{n\pi}{a}y))$$

$$V(x,y) = \sum_{n=1}^{\infty} C_n (e^{kx} - e^{-kx})(\sin(\tfrac{n\pi}{a}y))$$

Must satisfy 4. @ $V(b,y) = V_0(y)$

$$V_0(y) = \sum_{n=1}^{\infty} \underbrace{C_n(e^{kb} - e^{-kb})}_{\text{const.}} \sin(\tfrac{n\pi}{a}y)$$

Fourier's trick

$$C_n(e^{kb} - e^{-kb}) = \frac{2}{a}\int_0^a V_0(y)\sin(\tfrac{n\pi}{a}y)\,dy$$

$$= V_0(y)\left(\frac{-a}{n\pi}\right)\frac{2}{a}\left[\cos(\tfrac{n\pi}{a}y)\right]_0^a$$

3.19.) * actually for $V_0(y) \neq$ const. the furthest we can go is

$$V(x,y) = \sum_{n=1}^{\infty} \left(\left(\frac{2}{a}\right)\left(\frac{1}{e^{kb}-e^{-kb}}\right)\int_0^a V_0(y)\sin\left(\frac{n\pi}{a}y\right)dy \right)\left(e^{kx}-e^{-kx}\right)\left(\sin\frac{n\pi}{a}y\right)$$

a.)

Since $V_0(y)$ inside integral w/ resp. to y.

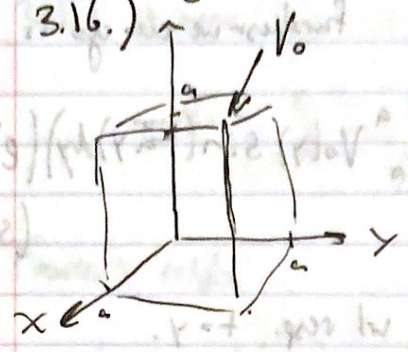
If $V_0(y) = V_0 =$ const.

$$C_n = -\frac{2V_0}{e^{kb}-e^{-kb}}\left(\frac{1}{n\pi}\right)\underbrace{\left(\cos(n\pi) - \cos(0)\right)}_{\substack{=0 \text{ if } n \text{ even} \\ = -2 \text{ if } n \text{ odd}}}$$

$$\Rightarrow C_n = \begin{cases} \frac{4V_0}{n\pi}\left(e^{kb}-e^{-kb}\right)^{-1} & n \text{ odd} \\ 0 & \text{else} \end{cases}$$

b.) $$\Rightarrow V(x,y) = \sum_{n=1,3,5,\dots} \frac{4V_0}{n\pi}\left(e^{kb}-e^{-kb}\right)^{-1}\left(e^{kx}-e^{-kx}\right)\left(\sin\left(\frac{n\pi}{a}y\right)\right)$$

3.16.)



all else $V=0$.

1. $V(x,y,a) = V_0$
2. $V(a,x,z) = 0$
3. $V(x,a,z) = 0$
4. $V(0,x,z) = 0$
5. $V(x,0,z) = 0$
6. $V(x,y,0) = 0$

$\nabla^2 V = 0$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

$$C_1 + C_2 + C_3 = 0$$

want $C_3 > 0$, $C_2 < 0$, $C_1 < 0$

let $C_1 = -k^2$, $C_2 = -l^2$, $C_3 = k^2 + l^2$

$$\frac{\partial^2 X}{\partial x^2} = -k^2 X \Rightarrow X = A\sin(kx) + B\cos(kx)$$

$$\Rightarrow Y = C\sin(ly) + D\cos(ly)$$

$$\frac{\partial^2 Z}{\partial z^2} = (k^2+l^2) Z \Rightarrow Z = Ee^{\sqrt{k^2+l^2}\,z} + Fe^{-\sqrt{k^2+l^2}\,z}$$

4. $X(0) = 0 \Rightarrow B = 0$
5. $Y(0) = 0 \Rightarrow D = 0$
6. $Z(0) = 0 \Rightarrow E + F = 0 \Rightarrow F = -E \Rightarrow Z = E\left(e^{\sqrt{k^2+l^2}\,z} - e^{-\sqrt{k^2+l^2}\,z}\right)$ $\hookrightarrow \sinh(\ )$
2. $X(a) = 0 \Rightarrow A\sin(ka) = 0 \Rightarrow k = \frac{n\pi}{a}$
3. $Y(a) = 0 \Rightarrow C\sin(la) = 0 \Rightarrow l = \frac{m\pi}{a}$

$$\Rightarrow V(x,y,z) = XYZ = A\sin\left(\frac{n\pi}{a}x\right) C\sin\left(\frac{m\pi}{a}y\right) E\sinh\left(\sqrt{k^2+l^2}\,z\right)$$

$$V(x,y,z) = \sum_m \sum_n C_{n,m} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)\sinh\left(\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2}\,z\right)$$

3.16) at $z=a$ ...

$$V_0 = \sum_m \sum_n \underbrace{C_{m,n} \sinh\left(\pi \sqrt{\frac{n^2+m^2}{a^2}}\, a\right)}_{\text{const.}} \sin\left(\frac{n\pi}{a}x\right)\left(\sin\left(\frac{m\pi}{a}y\right)\right)$$

Fourier's trick ...

$$C_{m,n} \sinh(\dots) = \left(\frac{2}{a}\right)^2 \int_0^a\int_0^a V_0 \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right) dx\,dy$$

$$= \frac{4}{\cancel{a^2}} V_0 \frac{\cancel{a^2}}{\pi^2 mn}\left[\cos\left(\frac{n\pi}{a}\right)\right]_0^a \left[\cos\left(\frac{m\pi}{a}\right)\right]_0^a$$

$$= \frac{4V_0}{\pi^2 mn} \underbrace{\left[\cos(n\pi) - \cos(0)\right]}_{\substack{=0 \text{ n even} \\ =-2 \text{ n odd}}} \underbrace{\left[\cos(m\pi) - \cos(0)\right]}_{\substack{=0 \text{ n even} \\ =-2 \text{ n odd}}}$$

So,

$$C_{m,n}\sinh(\dots) = \begin{cases} \dfrac{16V_0}{\pi^2 mn} & m,n \text{ odd} \\ 0 & \text{else} \end{cases}$$

$\Rightarrow$

$$V(x,y,z) = \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{16V_0}{\pi^2}\left(\frac{1}{mn}\right)\left(\sinh\left(\pi\sqrt{n^2+m^2}\right)\right)^{-1} \left(\sin\left(\frac{n\pi}{a}x\right)\right)\left(\sin\left(\frac{m\pi}{a}y\right)\right) \left(\sinh\left(\pi\sqrt{\frac{n^2+m^2}{a^2}}\, z\right)\right)$$

3.19.) sphere at radius R $\qquad V_0 = k\cos 3\theta$.

In spherical coor. w/ azimuthal sym.

$$V(r,\theta) = R\,\Theta$$

$$R = Ar^l + \frac{B}{r^{l+1}} \qquad \Theta = P_l \cos\theta$$

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \qquad P_1 = x$$

$$P_3(x) = \frac{5x^3-3x}{2}$$

Inside sphere: $B=0$ (since $\frac{B}{r}$ @ $r=0 \to \infty$).

$$\Rightarrow V(r,\theta) = \sum_l A r^l P_l(\cos\theta)$$

to apply boundary condition need $V_0 = k\cos(3\theta)$ in terms of $\cos\theta$.
google yields $V_0 = k[4\cos^3\theta - 3\cos\theta]$.
get in terms of P_n

$$V_0 = \cancel{k}[A P_3(\cos\theta) + B P_1(\cos\theta)] = \cancel{k}[4\cos^3\theta - 3\cos\theta]$$

$$= A\left(\tfrac{1}{2}\,5x^3 - 3x\right) + Bx = 4x^3 - 3x$$

$$= \tfrac{5}{2}Ax^3 + \left(B - \tfrac{3}{2}A\right)x = 4x^3 - 3x$$

$$\Rightarrow 4 = \tfrac{5}{2}A \qquad -3 = B - \tfrac{3}{2}A$$

$$A = \tfrac{8}{5} \qquad B = -3 + \tfrac{3}{\cancel{2}}\left(\tfrac{\cancel{8}^{4}}{5}\right) = -3 + \tfrac{12}{5}$$

$$B = -3/5$$

$$\Rightarrow V_0 = k\left[\tfrac{8}{5}P_3(\cos\theta) + \left(-\tfrac{3}{5}\right)P_1(\cos\theta)\right]$$

3.19) so, at $r=R$ $V_0(\theta) = \sum_l^{\infty} A_l R^l P_l(\cos\theta)$

$\Rightarrow$ using Fourier's trick (and orthogonality of P_l's)

$$A_l R^l \frac{2}{2l+1} = \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

$$= \frac{2l+1}{2R^l} \left[k \int_0^{\pi} \frac{8}{5} \underbrace{P_3 P_l \sin\theta d\theta}_{\begin{cases} \frac{2}{2l+1} & l=3 \\ 0 & \text{else} \end{cases}} - k \int \frac{3}{5} \underbrace{P_1 P_l \sin\theta d\theta}_{\begin{cases} \frac{2}{2l+1} & l=1 \\ 0 & \text{else} \end{cases}} \right]$$

$$= \frac{\cancel{2l+1}}{\cancel{2}R^l} (k) \left[\underbrace{\frac{8}{5}\left(\frac{\cancel{2}}{\cancel{2l+1}}\right)}_{l=3} - \underbrace{\frac{3}{5}\left(\frac{\cancel{2}}{\cancel{2l+1}}\right)}_{l=1} \right]$$

$$A_l = \begin{cases} \frac{8k}{5R^3} & l=3 \\ \frac{-3k}{5R} & l=1 \\ 0 & \text{else} \end{cases}$$

$\Rightarrow$ $V(r,\theta) = \sum_l A_l r^l P_l(\cos\theta)$ becomes

$$\boxed{V(r,\theta) = \frac{-3k}{5R} r P_1(\cos\theta) + \frac{8k}{5R^3} r(\cos\theta)}$$

3.14) Outside sphere: $A_l = 0$ (since $A_l(\infty) \Rightarrow \infty$)

$$\Rightarrow \quad V(r,\theta) = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Since the potential must be continuous at R,

$$\sum_l A_l r^l P_l(\cos\theta) = \sum \frac{B_l}{r^{l+1}} P_l \cos\theta$$

$$\Rightarrow \quad B_l = A_l r^{2l+1}$$

$$\Rightarrow \quad B_l = \begin{cases} \frac{8k}{5\cancel{R^3}} R^{\cancel{7}4} & l=3 \\ \frac{-3k}{5\cancel{R}} R^{\cancel{3}2} & l=1 \\ 0 & \text{else} \end{cases}$$

$\frac{R^5}{r^2}$

Thus $$V(r,\theta) = \frac{-3kR^2}{5}\frac{1}{r^2}P_1(\cos\theta) + \frac{8kR^4}{5}\frac{1}{r^4}P_3(\cos\theta)$$

Eq 3.81 says $$\sigma_0(\theta) = \epsilon_0 \sum_l (2l+1) A_l R^{l-1} P_l \cos\theta$$

$$= \epsilon_0\left(3\left(\frac{-3k}{5R^{\cancel{3}1}}\right)P_1(\cos\theta) + 7\left(\frac{8k}{5R^{\cancel{3}1}}\right)\cancel{R^2}\, P_3(\cos\theta)\right)$$

$$\sigma_0 = \frac{k\epsilon_0}{5R}\left(-9P_1(\cos\theta) + 56\, P_3(\cos\theta)\right)$$

3.22.) ~~$V(r,0) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2} - R\right)$~~

$$V(r,0) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2} - r\right)$$

$r > R \Rightarrow A_l = 0.$

$$\Rightarrow V(r,\theta) = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$V(r,0) = \sum_l \frac{B_l}{r^{l+1}} P_l(1) \qquad * P_l(1) = 1$$

$$= \sum_l \frac{B_l}{r^{l+1}} = \frac{\sigma}{2\epsilon_0}\left(\underbrace{\sqrt{r^2+R^2}}_{} - r\right)$$

$$\hookrightarrow = \sqrt{\frac{r^4}{r^2} + \frac{R^2 r^2}{r^2}} = r\sqrt{1+\frac{R^2}{r^2}}$$

taylor expansion abt. $x=0$ of $\sqrt{1+\frac{R^2}{r^2}} = 1 + \frac{1}{2}\frac{R^2}{r^2} - \frac{1}{8}\left(\frac{R^2}{r^2}\right)^2 + \ldots$

$$\Rightarrow \sum_l \frac{B_l}{r^{l+1}} = \frac{\sigma r}{2\epsilon_0}\left(1 + \frac{1}{2}\frac{R^2}{r^2} - \frac{1}{8}\frac{R^4}{r^4} + \ldots - 1\right)$$

$$= \cancel{\frac{\sigma r}{2\epsilon_0}} + \frac{\sigma R^2}{4\epsilon_0 r} - \frac{1\sigma R^4}{16\epsilon_0 r^3} + \ldots - \cancel{\frac{\sigma r}{2\epsilon_0}}$$

$l=0$: $\frac{B_0}{r^1} = \frac{\sigma R^2}{4\epsilon_0 r}$

$l=1$: $\frac{B_1}{r^2} = 0$

$l=2$: $\frac{B_2}{r^3} = -\frac{\sigma R^4}{16\epsilon_0 r^3}$

$\vdots$

$$\Rightarrow B_0 = \frac{\sigma R^2}{4\epsilon_0}, \quad B_2 = \frac{-\sigma R^4}{16\epsilon_0}, \ldots$$

cont. for even l.

3.22.) So,

$$V(r,\theta) = \sum_{l} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$V(r,\theta) = \frac{\sigma R^2}{4\epsilon_0}\left[\frac{1P_0}{r} - \frac{R^2}{4r^3}P_2(\cos\theta) + \dots\right]$$

For $r < R$: $\Rightarrow B_l = 0$

So, $V(r,\theta) = \sum_{l} A_l r^l P_l(\cos\theta)$

$* P_l(1) = 1$

For top hemisphere:

$$V(r,\theta) = \sum_{l} A_l r^l P_l(1) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2} - r\right)$$

$$\Rightarrow \sum A_l r^l = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2} - r\right)$$

$\hookrightarrow R\boxed{\sqrt{1+\left(\frac{r}{R}\right)^2}}$

$\hookrightarrow \approx 1 + \frac{1}{2}\frac{r^2}{R^2} - \frac{1}{8}\left(\frac{r^2}{R^2}\right)^2$

~~Same trick yields~~

$$\cancel{\sum A_l r^l = \frac{\sigma r}{2\epsilon_0}(}$$

$$\sum A_l r^l = \frac{\sigma}{2\epsilon_0}\left(R + \frac{1}{2}\frac{r^2}{R} - \frac{1}{8}\frac{r^4}{R^3} - r\right)$$

$l=0$: $A_0 = \frac{\sigma R}{2\epsilon_0}$

$l=1$: $A_1 r = \frac{-\sigma}{2\epsilon_0} r$

$l=2$: $A_2 r^2 = \frac{\sigma r^2}{4\epsilon_0 R}$

$$\Rightarrow A_0 = \frac{\sigma}{2\epsilon_0}R, \quad A_1 = \frac{-\sigma}{2\epsilon_0}, \quad A_2 = \frac{\sigma}{4\epsilon_0 R}$$

3.22.) So,

$$V(r,\theta) = \sum_{l} A_l r^l P_l(\cos\theta)$$

$$= \frac{\sigma}{2\epsilon_0}\left[R P_0(\cos\theta) - r P_1(\cos\theta) + \frac{1}{2R} r^2 P_2(\cos\theta) - \dots \right]$$

In lower hemisphere on axis $\theta = \pi$.

$\Rightarrow \quad P_l(\cos(\pi)) = P_l(-1) = (-1)^l$ ← from google

$$V(r,\pi) = \sum_{l} (-1)^l A_l r^l = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2} - r\right)$$

same as before upto:

$l=0: \quad A_0 = (-1)^0 \frac{\sigma}{2\epsilon_0} R$

$l=1: \quad (-1)^1 A_1 r = \frac{-\sigma}{2\epsilon_0} r$

$l=2: \quad (-1)^2 A_2 r^2 = \frac{\sigma r^2}{4\epsilon_0 R}$

$\Rightarrow A_0 = \frac{\sigma}{2\epsilon_0} R,\ A_1 = \frac{\sigma}{2\epsilon_0},$

$A_2 = \frac{\sigma}{4\epsilon_0 R}$

$$\Rightarrow V(r,\theta) = \frac{\sigma}{2\epsilon_0}\left[R P_0(\cos\theta) + r P_1(\cos\theta) + \frac{1}{2R} r^2 P_2(\cos\theta) + \dots \right]$$

3.24.) $\nabla^2 V = 0 = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \cancel{\frac{\partial^2 V}{\partial z^2}}$ cylindrical symmetry

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Look for solutions $V(s,\phi) = S(s)\Phi(\phi) \ldots$

$$\Rightarrow \frac{1}{s}\Phi\frac{\partial}{\partial s} s\frac{\partial S}{\partial s} + \frac{1}{s^2} S \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$s\Phi\frac{\partial}{\partial s} s\frac{\partial S}{\partial s} + S\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\frac{s}{S}\frac{\partial}{\partial s} s\frac{\partial S}{\partial s} + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

two const. terms.

$$\frac{s}{S}\frac{\partial}{\partial s} s\frac{\partial S}{\partial s} = C_1 \quad \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = C_2$$

$$C_1 + C_2 = 0.$$

Now V periodic in $\phi \Rightarrow \Phi$ periodic.

So let $C_2 = -k^2$

$$\Rightarrow \frac{\partial^2 \Phi}{\partial \phi^2} = -k^2\Phi$$

$$\Rightarrow \Phi = A\sin(k\phi) + B\cos(k\phi)$$

So $C_1 = k^2$

$$\Rightarrow s\frac{\partial}{\partial s} s\frac{\partial S}{\partial s} = k^2 S$$

Consider $S = s^n$

$$s\frac{d}{ds}\left(s(n)s^{n-1}\right) = s\,n(n)s^{n-1} = n^2 s^n = k^2 s^n$$

$$\Rightarrow n = \pm k$$

3.24.) So,

$$S = C s^{k} + D s^{-k}.$$

What if $k=0$?

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = \text{const.}$$

$$\frac{d}{ds} s \frac{dS}{ds} = \text{const.}$$

$$\Rightarrow s\frac{dS}{ds} = \text{const} = C$$

$$\frac{dS}{ds} = \frac{C}{s} \Rightarrow S = C\ln(s) + D$$

also:

$$\Phi(k=0) = 0 + B = B$$

Then for $k=0$:

$$V = B(C\ln(s) + D) = C_0\ln(s) + D_0$$

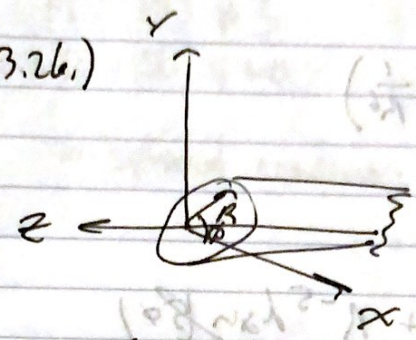
otherwise ($k > 0$):

$$V(s,\phi) = (A\sin(k\phi) + B\cos(k\phi))(C s^{k} + D s^{-k})$$

Summing over k's:

$$V(s,\phi) = C_0\ln(s) + D_0 + \sum_{k=1}^{\infty} (A\sin(k\phi) + B\cos(k\phi))(C s^{k} + D s^{-k})$$

3.26.)



$$\sigma(\phi) = a\sin(5\phi)$$

$$V(s,\phi) = C_0\ln(s) + D_0 + \sum_{k=1}^{\infty}\left(A\sin(k\phi) + B\cos(k\phi)\right)(Cs^{k} + Ds^{-k})$$

Inside: ignore s^{-k} since $\frac{1}{s^k} \to \infty$ at $s=0$.
ignore $\ln(s)$ since $\ln(0) \to -\infty$

So, $V(s,\phi) = D_0 + \sum_{k=1}^{\infty}\left(A\sin(k\phi) + B\cos(k\phi)\right)Cs^{k}$

renaming consts...

$$V(s,\phi) = a_0 + \sum_{k=1}^{\infty} \cancel{B_k} s^{k}\left(b\sin(k\phi) + c\cos(k\phi)\right)$$

outside: ignore s^{k} since at $s=\infty \to \infty$
ignore $\ln$ since $\ln(\infty) = \infty$

So, $V(s,\phi) = a_0 + \sum_{k=1}^{\infty} s^{-k}\left(d\sin(k\phi) + e\cos(k\phi)\right)$

now using

$$\sigma = -\epsilon_0\left(\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n}\right)\Bigg|_{s=R} \quad *n=s$$

$$\sigma = -\epsilon_0\left(\sum_{k=1}^{\infty} -kR^{k-1}\left(b\sin(k\phi) + c\cos(k\phi)\right) \mp \sum_{k=1}^{\infty} +kR^{-k-1}\left(d\sin(k\phi) + e\cos(k\phi)\right)\right)$$

$= a\sin(5\phi)$, so $k=5$, $c=e=0$

$$a\sin\cancel{(5\phi)} = -\epsilon_0\left(-5R^{4}\, b\sin(5\phi) + (-5)R^{-6}\, d\,\cancel{\sin(5\phi)}\right)$$

$$a = +\epsilon_0\left[+5R^{4}b + \frac{5d}{R^{6}}\right]$$

3.26.) $a = 5\epsilon_0 \left(bR^4 + d\frac{1}{R^6} \right)$ 6

enforcing cont. of V:

$$\cancel{a} + R^5 b\sin(\cancel{5\phi}) = \cancel{c} + R^{-5} d\sin(\cancel{5\phi})$$

$$b = \frac{d}{R^{10}} \iff d = R^{10} b$$

So, $a = 5\epsilon_0 (bR^4 + bR^4)$

$$a = 10\epsilon_0 R^4 b$$

$$b = \frac{a}{10\epsilon_0 R^4} \quad \text{inside}$$

$$d = \frac{aR^6}{10\epsilon_0} \quad \text{outside} \qquad *k=5$$

So, $$\boxed{V(s,\phi) = \begin{cases} \dfrac{a}{10\epsilon_0 R^4} s^5 \sin(5\phi) & \underline{\text{inside}} \\ \dfrac{aR^6}{10\epsilon_0} \dfrac{1}{s^5} \sin(5\phi) & \underline{\text{outside}} \end{cases}}$$