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Electro Midterm

1. a) $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$\vec{A} \cdot \nabla = A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z}$$

$$(\vec{A} \cdot \nabla) B = \left(A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} \right) \hat{x}$$

$$+ \left(A_x \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z} \right) \hat{y}$$

$$+ \left(A_x \frac{\partial B_z}{\partial y} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z} \right.$$

b)

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{\dots}} \hat{x}$$

$$+ \frac{y}{\sqrt{\dots}} \hat{y}$$

$$+ \frac{z}{\sqrt{\dots}} \hat{z}$$

$$(\hat{r}\cdot\nabla)\hat{r} = \left[\frac{x}{\sqrt{\cdots}}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{\cdots}}\right) + \frac{y}{\sqrt{\cdots}}\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{\cdots}}\right) + \frac{z}{\sqrt{\cdots}}\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{\cdots}}\right)\right]\hat{x}$$

$$+ \left[\frac{x}{\sqrt{\cdots}}\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{\cdots}}\right) + \frac{y}{\sqrt{\cdots}}\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{\cdots}}\right) + \frac{z}{\sqrt{\cdots}}\frac{\partial}{\partial z}\left(\frac{y}{\sqrt{\cdots}}\right)\right]\hat{y}$$

$$+ \left[\frac{x}{\sqrt{\cdots}}\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{\cdots}}\right) + \frac{y}{\sqrt{\cdots}}\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{\cdots}}\right) + \frac{z}{\sqrt{\cdots}}\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{\cdots}}\right)\right]\hat{z}$$

For $\hat{x}$:

$$= \frac{x}{\sqrt{\cdots}}\left[\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{y}{\sqrt{\cdots}}\left[-\frac{xy}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$+ \frac{z}{\sqrt{\cdots}}\left[-\frac{xz}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \frac{\cancel{xy^2} + \cancel{xz^2}}{(x^2+y^2+z^2)^2} - \cancel{\frac{xy^2}{(x^2+y^2+z^2)^2}} - \cancel{\frac{xz^2}{(x^2+y^2+z^2)^2}}$$

$$= 0\,\hat{x}$$

For $\hat{y}$:

$$= \frac{x}{\sqrt{\cdots}}\left[-\frac{xy}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{y}{\sqrt{\cdots}}\left[\frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$+ \frac{z}{\sqrt{\cdots}}\left[-\frac{yz}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= -\cancel{\frac{yx^2}{(x^2+y^2+z^2)^2}} + \frac{\cancel{yx^2}+\cancel{yz^2}}{(x^2+y^2+z^2)^2} - \cancel{\frac{yz^2}{(x^2+y^2+z^2)^2}} = 0\hat{y}$$

For $\hat{z}$:

$$= \frac{x}{\sqrt{\cdots}}\left[-\frac{xz}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{y}{\sqrt{\cdots}}\left[-\frac{yz}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$+ \frac{z}{\sqrt{\cdots}}\left[\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= -\frac{zx^2}{(x^2+y^2+z^2)^2} - \frac{zy^2}{(x^2+y^2+z^2)^2} + \frac{zx^2+zy^2}{(x^2+y^2+z^2)^2} = 0\hat{z}$$

$$= 0\hat{x} + 0\hat{y} + 0\hat{z} = 0$$

c) $\vec{p} = q d \hat{x}$

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial V(\vec{r})}{\partial r} = \frac{1}{r^2} \left(\frac{\partial}{\partial r} \right) (r^2 \cdot V_0 r^2 + V_1)$$

$$= \frac{1}{r^2} \left(\frac{\partial}{\partial r} \right) (V_0 r^4 + V_1)$$

$$= \frac{1}{\cancel{r^2}} (V_0 \, 4 r^{\cancel{3}})$$

$$= 4 V_0 r$$

$$F = (\vec{p} \cdot \nabla) \vec{E} = \vec{p} \cdot 4 V_0 r \; \hat{x}$$

2. a)

$$J = \int_V e^{-r} \left(\nabla \cdot \frac{\hat{r}}{r^2} \right)$$

$$\nabla \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi \, \delta^3(\vec{r})$$

$$J = \int e^{-r} \left(4\pi \, \delta^3(\vec{r}) \right) = 4\pi \, \cancel{e^{-0}}^{\,1} = 4\pi$$

3. a) $P = (0, 0)$

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0}$$

$E_1 = 0$?

b) $P = (2d, 0)$

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0} = \frac{2d}{\varepsilon_0}$$

$$\vec{E} \cdot \vec{A} \equiv \frac{2d}{\varepsilon_0}$$

$$\vec{E} \, (\cancel{2}\pi x) \cancel{d} = \frac{\cancel{2d}}{\varepsilon_0}$$

? $E_2 = \dfrac{1}{\pi x \varepsilon_0} \hat{x}$ $\quad \dfrac{C/m}{m}$

c) $P = (0, 2d)$?

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0}$$

4. $V(r) = A \frac{e^{-\lambda r}}{r}$ $\vec{E}(r)$, ρ, Q

$\vec{E}(r) = -\nabla \vec{V}(r)$

$$= -\frac{\partial}{\partial r}\left(A\frac{e^{-\lambda r}}{r}\right)\hat{r}$$

$$= -\left(\frac{r\frac{\partial}{\partial r}(Ae^{-\lambda r}) - Ae^{-\lambda r}\frac{\partial}{\partial r}(r)}{r^2}\right)\hat{r}$$

$$= -\left(\frac{r(-A\lambda e^{-\lambda r}) - Ae^{-\lambda r}}{r^2}\right)\hat{r}$$

$$= \left(\frac{Ae^{-\lambda r}(1+\lambda r)}{r^2}\right)\hat{r}$$

$\nabla \cdot \vec{E}(r) = \frac{\rho}{\varepsilon_0}$ $\rho = \varepsilon_0 \nabla \cdot \vec{E}(r)$

$$= \varepsilon_0 \nabla\left(\frac{Ae^{-\lambda r}(1+\lambda r)}{r^2}\hat{r}\right)$$

$$= \varepsilon_0 A\left(e^{-\lambda r}(1+\lambda r)\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2}\cdot\nabla\left(e^{-\lambda r}(1+\lambda r)\right)\right)$$

$\nabla\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$

$e^{-\lambda r}(1+\lambda r)\delta^3(\vec{r}) = \delta^3(\vec{r})$

$$\nabla\left(e^{-\lambda r}(1+\lambda r)\right) = \hat{r}\,\frac{\partial}{\partial r}\left(e^{-\lambda r}(1+\lambda r)\right)$$
$$= \hat{r}\left(-\lambda e^{-\lambda r}(1+\lambda r) + e^{-\lambda r}\lambda\right)$$
$$= \hat{r}\left(-\lambda^2 r e^{-\lambda r}\right)$$

$$\frac{\hat{r}}{r^2}\cdot\nabla\left(e^{-\lambda r}(1+\lambda r)\right) = -\frac{\lambda^2}{r}e^{-\lambda r}$$

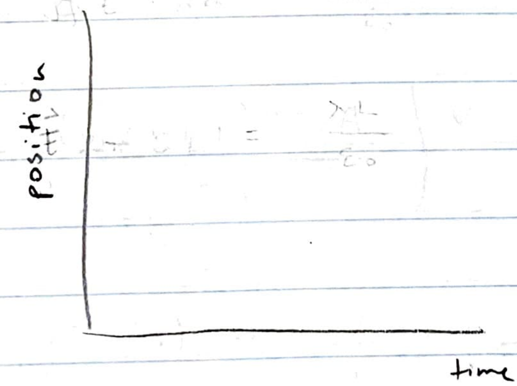
$$\rho = \varepsilon_0 A\left(4\pi\delta^3(\vec{r}) - \lambda^2/r\; e^{-\lambda r}\right)$$

$$Q = \int \rho\, d\tau = \varepsilon_0 A\left(4\pi\int \delta^3(\vec{r})\,d\tau - \lambda^2\int \frac{e^{-\lambda r}}{r}4\pi r^2\,dr\right)$$

5 a) $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$

$$\vec{E}\,2\pi s L = \frac{\lambda L}{\varepsilon_0} \qquad \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$$

b)



6. a)

$$\sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l (\cos\theta) = \frac{1}{\varepsilon_0} \sigma_0(\theta)$$

$$\sigma_0(\theta) = \varepsilon_0 \sum^{\infty} (2l+1) A_l R^{l-1} P_l (\cos\theta)$$

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$ → from book (3.69)

$$\sigma(\theta) = \frac{\varepsilon_0}{2R} \sum_{l=0}^{\infty} (2l+1)^2 C_l P_l (\cos\theta)$$

$$C_l = \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

b)

$$V_0(\theta) = P_2(\cos\theta)$$

$$l = 2$$

$$P_2(\cos\theta) = 2\cos\theta$$

$$A_2 = -2E_0$$

$$V(r,\theta) = -2E_0$$

7. $V = V_0$ when $y = a$

$V = 0$ when $y = 0$

$V = 0$ when $x = b$

$V = 0$ when $x = -b$

$V(x,y) = (A\cancel{e^{kx}} + B\cancel{e^{-kx}})(C\sin ky + D\cos ky)$

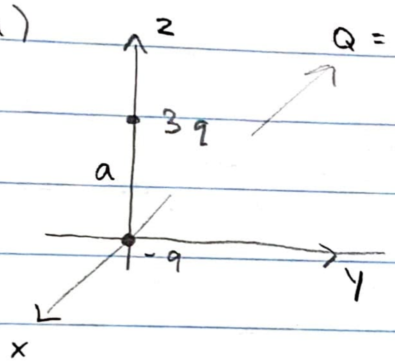
$V(x,y) = (C\sin ky + D\cos ky)$ $\quad y = a, \ V = V_0$

$V(x,a) = C\sin ka$ $\quad k = \frac{n\pi}{a}$

$V(x,y) = C\sin(n\pi/a)$

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a)$$

8. a)



$Q = 3q - 1q = 2q$ $\qquad V_{mon}(\vec{r}) = \dfrac{Q}{4\pi \varepsilon_0 r}$ → from book (3.97)

$$V_{mon} = \frac{\cancel{2}q}{2\cdot\cancel{4}\pi \varepsilon_0 r} = \frac{q}{2\pi \varepsilon_0 r}$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$$ → from book (3.99)

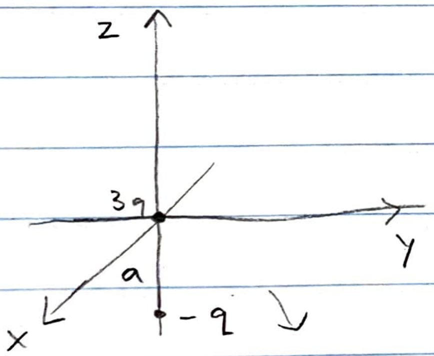
$\hat{p} = (a\hat{z})(3q) + (0)(-q) = 3qa\hat{z}$ $\qquad \hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

$\vec{p} = 3qa(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$ $\qquad \vec{p}\cdot\hat{r} = 3aq\cos\theta$

$$V_{dip}(\vec{r}) = \frac{3aq\cos\theta}{4\pi \varepsilon_0 r^2}$$

$$V(\vec{r}) = \frac{q}{2\pi \varepsilon_0 r} + \frac{3aq\cos\theta}{4\pi \varepsilon_0 r^2}$$

b)



$$V_{mon} = \frac{q}{2\pi \varepsilon_0 r}$$

$\vec{p} = (-q)a(-\hat{z}) = qa\hat{z}$

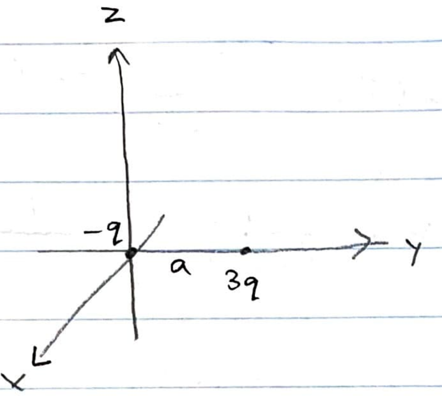
$\vec{p} = +qa(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$

$Q = 3q - 1q = 2q$

$$V_{dip} = \frac{qa\cos\theta}{4\pi \varepsilon_0 r^2}$$

$$V(\vec{r}) = \frac{q}{2\pi \varepsilon_0 r} + \frac{qa\cos\theta}{4\pi \varepsilon_0 r^2}$$

c)



$$Q = 3q - 1q = 2q$$

$$V_{mon}(\vec{r}) = \frac{q}{2\pi \varepsilon_0 r^2}$$

$$\vec{p} = 3qa\,\hat{y}$$

$$\hat{y} = \sin\theta \sin\phi \,\hat{r} + \cos\theta \sin\phi \,\hat{\theta} + \cos\phi \,\hat{\phi}$$

$$\vec{p} \cdot \vec{r} = 3qa\,\hat{y} \cdot \hat{r}$$

$$= 3qa\,(\sin\theta \sin\phi)$$

$$V_{dip}(\vec{r}) = \frac{3qa \sin\theta \sin\phi}{4\pi \varepsilon_0 r^2}$$

$$V(\vec{r}) = \frac{q}{2\pi \varepsilon_0 r^2} + \frac{3qa \sin\theta \sin\phi}{4\pi \varepsilon_0 r^2}$$