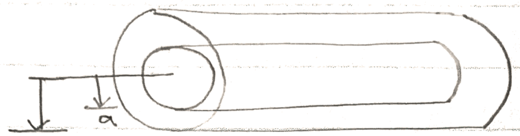
Homework 2.43, 2.50, 3.1, 3.3, 3.13, 3.14, 3.15

2.43)




Capacitance per unit length ??

$q_{inside} = +$ ($a$)  $q_{outer} = -$ ($b$)

$$V(+) - V(-) = -\int_{(-1)}^{(+)} E \cdot dl \qquad E = \frac{q}{2\pi s l \epsilon_0}\hat{s}$$

$$V(a) - V(b) = -\frac{q}{2\pi l \epsilon_0}\int_b^a \frac{1}{s}\,ds$$

$$V = (V_+) - (V_-) = V(a) - V(b) = \frac{-q}{2\pi l \epsilon_0}\left(\ln(s)\Big|_b^a\right)$$

$$V = \frac{-q}{2\pi l \epsilon_0}\left(\ln(a) - \ln(b)\right) = \frac{q}{2\pi l \epsilon_0}\left(\ln(b) - \ln(a)\right)$$

$$V = \frac{-q}{2\pi l \epsilon_0}\ln\left(\frac{b}{a}\right)$$

$$V = \frac{q}{C} \Rightarrow C = \frac{q}{V} = \frac{2\pi l \epsilon_0}{\ln\left(\frac{b}{a}\right)}$$

$$\boxed{\frac{C}{l} = \frac{2\pi \epsilon_0}{\ln\left(\frac{b}{a}\right)}}$$

2.50) $V(r) = A\dfrac{e^{-\lambda r}}{r}$ electric field $E(r)$?? Charge density $\rho(r)$?? total charge $Q$??

Electric field $E(r)$

$$E = -\nabla V$$
$$= -\left(\frac{\partial}{\partial x}(V) + \frac{\partial}{\partial y}(V) + \frac{\partial}{\partial r}(V)\right)$$
$$= -\left(0 + 0 + \frac{\partial}{\partial r}\left(A\frac{e^{-\lambda r}}{r}\right)\hat{r}\right)$$
$$= -A\left(\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)\hat{r}\right)$$
$$= -A\left(\frac{re^{-\lambda r}(-\lambda) - e^{-\lambda r}}{r^2}\right)\hat{r}$$
$$= -A\left(\frac{-r\lambda e^{-\lambda r} - e^{\lambda r}}{r^2}\right)\hat{r}$$

$$\boxed{E = Ae^{-\lambda r}\left(\frac{r\lambda + 1}{r^2}\right)\hat{r}}$$

Charge Density $\rho(r)$

$$\nabla \cdot E = \frac{1}{\varepsilon_0}\rho(r)$$

$$\rho(r) = (\nabla \cdot E)\varepsilon_0$$
$$\rho(r) = (\varepsilon_0)\,\nabla \cdot \left(Ae^{-\lambda r}\left(\frac{r\lambda + 1}{r^2}\right)\hat{r}\right)$$
$$\rho(r) = \varepsilon_0\left(Ae^{-\lambda r}(r\lambda+1)\left(\nabla \cdot \frac{\hat{r}}{r^2}\right) + \left(\frac{\hat{r}}{r^2}\right)\nabla\left(Ae^{-\lambda r}(r\lambda+1)\right)\right)$$
$$\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(r)$$

$$\rho(r) = \varepsilon_0\left(Ae^{-\lambda r}(r\lambda+1)(4\pi\delta^3(r)) + \left(\frac{\hat{r}}{r^2}\right)\nabla\left(Ae^{-\lambda r}(r\lambda+1)\right)\right)$$
$$\left(e^{-\lambda r}(r\lambda+1)\right)(4\pi\delta^3(r)) = 4\pi\delta^3(r)$$
$$\rho(r) = \varepsilon_0\left(A4\pi\delta^3(r) + \left(\frac{\hat{r}}{r^2}\right)\nabla\left(Ae^{-\lambda r}(r\lambda+1)\right)\right)$$
$$\nabla\left(Ae^{-\lambda r}(r\lambda+1)\right) = \left(0 + 0 + \frac{\partial}{\partial r}\left(Ae^{-\lambda r}(r\lambda+1)\right)\hat{r}\right)$$
$$= A\left(\frac{\partial}{\partial r}\left(e^{-\lambda r}(r\lambda+1)\right)\hat{r}\right)$$
$$= A\left(\frac{\partial}{\partial r}\left(e^{-\lambda r}\right)(r\lambda+1) + \frac{\partial}{\partial r}(r\lambda+1)\left(e^{-\lambda r}\right)\right)$$

3.1) $V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\varepsilon_0 R}$

average potential over a spherical surface of radius $R$

$z < R$


$V = \frac{q}{4\pi\varepsilon_0 r}$ (potential at a point)

$= \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{\sqrt{z^2+R^2-2Rz\cos\theta}}$

$r^2 = z^2+R^2-2Rz\cos\theta$

$r = \sqrt{z^2+R^2-2Rz\cos\theta}$

average potential

$$V_{avg} = \frac{1}{4\pi R^2}\int V \cdot da$$

$$= \frac{1}{4\pi R^2}\int\left(\frac{q}{4\pi\varepsilon_0}\right)\left(\frac{1}{\sqrt{z^2+R^2-2Rz\cos\theta}}\right)\left(R^2\sin\theta\, d\theta\, d\phi\right)$$

$$= \left(\frac{1}{4\pi R^2}\right)\left(\frac{q}{4\pi\varepsilon_0}\right)\int\frac{1}{\sqrt{z^2+R^2-2Rz\cos\theta}}R^2\sin\theta\, d\theta\, d\phi$$

$z < R \rightarrow \sqrt{z^2+R^2-2Rz\cos\theta} \approx \sqrt{R^2} = R$

$$= \left(\frac{1}{4\pi \cancel{R^2}}\right)\left(\frac{q}{4\pi\varepsilon_0}\right)\int\frac{1}{R}\cancel{R^2}\sin\theta\, d\theta\, d\phi$$

$\int_0^{\pi}\int_0^{2\pi}\sin\theta\, d\theta\, d\phi$

$(2)(2\pi) = 4\pi$

$$= \left(\frac{1}{\cancel{4\pi}}\right)\left(\frac{q}{4\pi\varepsilon_0}\right)\left(\frac{\cancel{4\pi}}{R}\right)$$

$$= \frac{q}{4\pi\varepsilon_0 R} \rightarrow \boxed{V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\varepsilon_0 R}}$$

3.3) Laplace eq. in spherical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} V\right) + \cancel{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} V\right)} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\phi} V\right)}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} V\right) = 0$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} V\right) = 0$$

$$\partial V = \frac{1}{r^2}\left(\underbrace{r^2 \frac{\partial V}{\partial r}}_{\text{constant}}\right)\partial r$$

$$\int \partial V = \int \frac{1}{r^2}(C_0)\,\partial r$$

$$\boxed{V = -\frac{1}{r}(C_0) + C_1}$$

cylindrical coordinates

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \cancel{\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}} + \cancel{\frac{\partial^2 V}{\partial z^2}}$$

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$

$$\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$

$$\partial V = \frac{1}{s}\left(\underbrace{s\frac{\partial V}{\partial s}}_{\text{constant}}\right)\partial s$$

$$\int \partial V = \int \frac{1}{s}(C_0)\,ds$$

$$\boxed{V = C_0 \ln(s) + C}$$

3.13) $V(0,y) = \begin{cases} V_0, & 0 < y < a/2 \\ -V_0, & a/2 < y < a \end{cases}$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\frac{n\pi y}{a}$$

$$V(x,y) = \sum_{n=2,6,\dots} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)$$

$$= \frac{8V_0}{\pi} \sum_{n=2,6,\dots} \frac{e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)}{n}$$

$$C_n = \frac{2}{a}\int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \frac{2}{a}\left(\int_0^{a/2} V_0 \sin\frac{n\pi y}{a}\,dy - \int_{a/2}^{a} V_0 \sin\frac{n\pi y}{a}\,dy\right)$$

$$= \left(\frac{2}{\cancel{a}}\right)(V_0)\left(\frac{\cancel{a}}{n\pi}\right)\left(\left(-\cos\frac{\pi n y}{a}\Big|_0^{a/2}\right) + \left(\cos\frac{n\pi y}{a}\Big|_{a/2}^{a}\right)\right)$$

$$= \frac{2V_0}{n\pi}\left(-\cos\left(\frac{n\pi}{2}\right) + \cos(0) + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$C_n = \frac{2V_0}{n\pi}\left(1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right)$$

for any odd $n$, $C_n = 0$

$n=2$ $\quad C_2 = \frac{\cancel{2}V_0}{\cancel{2}\pi}\left(1 + 1 - 2\cos\left(\frac{2\pi}{2}\right)\right) = \frac{4V_0}{\pi}$ or $\frac{8V_0}{2\pi}$

$n=4$ $\quad C_4 = \frac{2V_0}{4\pi}\left(1 + 1 \overbrace{-2\cos\left(\frac{4\pi}{2}\right)}^{-2}\right) = 0$

$n=6$ $\quad C_6 = \frac{2V_0}{6\pi}\left(1 + 1 - 2\cos\left(\frac{6\pi}{2}\right)\right) = \frac{4V_0}{3\pi}$ or $\frac{8V_0}{6\pi}$

$n=8$ $\quad C_8 = \frac{2V_0}{8\pi}\left(1 + 1 \overbrace{-2\cos\left(\frac{8\pi}{2}\right)}^{-2}\right) = 0$

when $n = 4, 8, \dots$ $C_n = 0$ $\qquad$ when $n = 2, 6, \dots$

$$C_n = \frac{8V_0}{n\pi}$$

3.14) Charge density on the strip at $x=0$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} e^{\frac{-n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$$\sigma = -\varepsilon_0 \left(\frac{\partial V}{\partial x}\right)$$

$$\sigma = -\varepsilon_0 \frac{\partial}{\partial x}\left(\frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} e^{\frac{-n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)\right)$$

$$= -\varepsilon_0 \left(\frac{4V_0}{\pi}\right) \frac{\partial}{\partial x}\left(\sum_{n=1,3,5,\ldots} \frac{1}{n} e^{\frac{-n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)\right)$$

$$= \cancel{-}\varepsilon_0 \left(\frac{4V_0}{\cancel{\pi}}\right) \sum_{n=1,3,5,\ldots} \frac{1}{\cancel{n}}\left(\cancel{-}\frac{\cancel{n\pi}}{a}\right) e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$$= \frac{4V_0\varepsilon_0}{a} \sum_{n=1,3,5,\ldots} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

when $x=0$ $$\sigma(0) = \frac{4V_0\varepsilon_0}{a} \sum_{n=1,3,5,\ldots} \sin\left(\frac{n\pi y}{a}\right)$$