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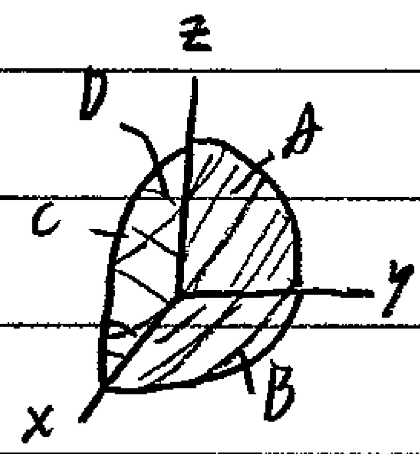
11/2/20 PHYS

PHYS Ch. 1 HW

1.54, 55, 56, 57, 59, 62, 63, 64



1.54 $\vec{v} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

$\nabla\cdot\vec{v} = \overset{①}{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)} + \overset{②}{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta)} + \overset{③}{\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}$

// $v_r = r^2\cos\theta,\ v_\theta = r^2\cos\phi,\ v_\phi = -r^2\cos\theta\sin\phi$

① $\frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\theta) = \frac{1}{r^2}(4r^3\cos\theta) = 4r\cos\theta$

② $\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi) = \frac{1}{r\sin\theta}(r^2\cos\theta\cos\phi) = \frac{r\cos\theta\cos\phi}{\sin\theta}$

③ $\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi) = \frac{1}{r\sin\theta}(-r^2\cos\theta\cos\phi) = -\frac{r\cos\theta\cos\phi}{\sin\theta}$ //

$\int_V \nabla\cdot\vec{v} = 4r\cos\theta = \int_0^R\int_0^{2\pi}\int_0^{\pi/2} 4r\cos\theta\, r^2\sin\theta\, dr\,d\theta\,d\phi \Rightarrow 4\int_0^R r^3 dr \int_0^{2\pi}\cos\theta\sin\theta\,d\theta \int_0^{\pi/2} d\phi$ //

$= 4\left[\frac{1}{4}r^4\right]_0^R \cdot$ // $\int_0^{2\pi}\cos\theta\sin\theta\,d\theta,\ u=\cos\theta,\ du = \sin\theta\,d\theta$ //

// $-\int_0^{2\pi} u\,du = -\left[\frac{u^2}{2}\right]_0^{2\pi} = -\left[\frac{\cos^2(2\pi)}{2} - \frac{\cos^2(0)}{2}\right] = \frac{+1}{2}$

$\int(\nabla\cdot\vec{v})\,d\tau = R^4\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi R^4}{4}}$ ✓

① // $\phi = 90°$ ② // $\theta = 0°$ ③ // $\phi = 0°$ ④ // $r = R$

$\int_S \vec{v}\cdot d\vec{a} = \int_A \vec{v}\cdot d\vec{a} + \int_B \vec{v}\cdot d\vec{a} + \int_C \vec{v}\cdot d\vec{a} + \int_D \vec{v}\cdot d\vec{a}$ // $v_r = r^2\cos\theta,\ v_\theta = r^2\cos\phi,\ v_\phi = -r^2\cos\theta\sin\phi$

① $\int\int_A \hat{\phi}\,dr\,d\theta = \int_0^R\int_0^{\pi/2} -r^2\cos\theta\sin\phi\, r\,dr\,d\theta$ // $\phi = 90 \to \sin\phi = 1 \Rightarrow \int_0^R r^3 dr\cdot\int_0^{\pi/2}\cos\theta\,d\theta$ //

$-\left[\frac{1}{4}R^4\right]\left[\sin\theta\right]_0^{\pi/2} = \frac{-1}{4}R^4(1-0) = \frac{-1}{4}R^4$

② $\int\int_B \hat{\theta}\,dr\,d\phi = \int_0^R\int_0^{\pi/2} r^2\cos\phi\cdot r\,dr\,d\phi \Rightarrow \int_0^R r^3 dr\cdot\int_0^{\pi/2}\cos\phi\,d\theta = \frac{1}{4}R^4(\sin\phi)_0^{\pi/2} = \frac{1}{4}R^4(1) = \frac{1}{4}R^4$

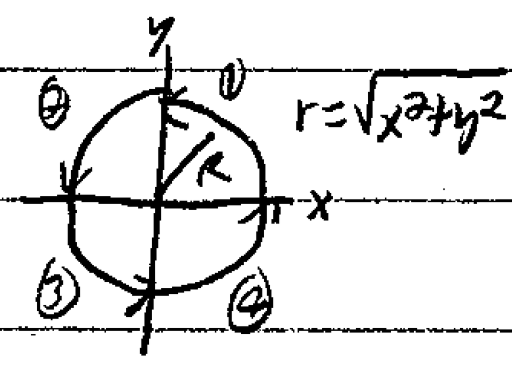
③ $\int\int_C \hat{\phi}\,dr\,d\theta = \int_0^R\int_0^{\pi/2} -r^2\cos\theta\sin\phi\cdot r\,dr\,d\theta$ // $\phi = 0,\ \sin(\phi) = 0 \Rightarrow \int_0^R\int_0^{\pi/2} 0\,dr\,d\theta = 0$

④ $\int\int_D \hat{r}\,d\theta\,d\phi = \int_0^{\pi/2}\int_0^{\pi/2} r^2\cos\theta\cdot r^2\sin\theta\,d\theta\,d\phi \Rightarrow$ // $r = R$ // $\Rightarrow R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2} d\phi$

// $u = \cos\theta,\ du = \sin\theta\,d\theta \Rightarrow \int_0^{\pi/2} u\,du = \left[\frac{u^2}{2}\right]_0^{\pi/2} = \left[\frac{-\cos^2(\pi/2)}{2} + \frac{\cos^2(0)}{2}\right] = \left[0 + \frac{1}{2}\right] = \frac{1}{2}$ //

$R^4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi R^4}{4}}$ ✓ //

$\int_S \vec{v}\cdot d\vec{a} = \cancel{\frac{-1}{4}R^4} + \cancel{\frac{1}{4}R^4} + 0 + \frac{\pi R^4}{4} = \boxed{\frac{\pi R^4}{4}}$



1.55 Check Stokes Theorem: $\int_S(\nabla\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$; $\vec{v} = ay\,\hat{x} + bx\,\hat{y}$

$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v_x & v_y \end{vmatrix} = \frac{\partial}{\partial x}v_y - \frac{\partial}{\partial y}v_x = \frac{\partial}{\partial x}(bx) - \frac{\partial}{\partial y}(ay) = b - a$

$\int_S(\nabla\times\vec{v})\cdot d\vec{a} = \int_0^R\int_0^{2\pi}(b-a)\cdot r\,dr\,d\theta = (b-a)\int_0^R r\,dr\int_0^{2\pi} d\theta = (b-a)\left[\frac{r^2}{2}\right]_0^R(2\pi) = (b-a)\frac{R^2}{2}(2\pi) = \boxed{\pi R^2(b-a)}$

$\oint_P \vec{v}\cdot d\vec{l} =$ // $x = r\cos\theta,\ y = r\sin\theta,\ dx = -r\sin\theta,\ dy = r\cos\theta$

// $\vec{v}_x = a(r\sin\theta)\cdot -r\sin\theta = -a r^2\sin^2\theta$ // $\vec{v}_y = b(r\cos\theta)\cdot r\cos\theta = br^2\cos^2\theta$

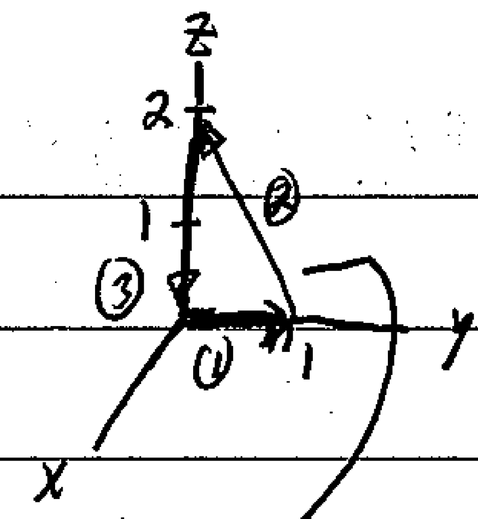
$\int\vec{v}_x\,d\theta = \int_0^{2\pi} -ar^2\sin^2\theta\,d\theta = -ar^2\int_0^{2\pi}\sin^2\theta\,d\theta$ // $\int\vec{v}_y\,d\theta = \int_0^{2\pi} br^2\cos^2\theta\,d\theta = br^2\int_0^{2\pi}\cos^2\theta\,d\theta = br^2\int_0^{2\pi}\frac{1}{2} + \frac{1}{2}\cos(2\theta)\,d\theta$ //

$-ar^2\int_0^{2\pi}\frac{1}{2} - \frac{1}{2}\cos(2\theta)\,d\theta \Rightarrow -ar^2\pi + \frac{1}{2}\left[\frac{-\sin(2\theta)}{2}\right]_0^{2\pi}$ $\cancel{0}$ // $br^2\pi + \frac{1}{2}\left[\frac{-\sin(2\theta)}{2}\right]_0^{2\pi}$ $0 \Rightarrow \vec{v}_y = br^2\pi$

$\vec{v}_x = -ar^2\pi$

$\oint_P \vec{v}\cdot d\vec{l} = \vec{v}_x + \vec{v}_y = -ar^2\pi + br^2\pi$ // $r = R \Rightarrow \boxed{(b-a)R^2\pi}$ ✓

1.56. $\int_S (\nabla\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$; $\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$



$\oint_P \vec{v}\cdot d\vec{l} = \underset{①}{\int \vec{v}\cdot d\vec{l}} + \underset{②}{\int \vec{v}\cdot d\vec{l}} + \underset{③}{\int \vec{v}\cdot d\vec{l}}$

① $d\vec{l} = +dy\,\hat{y} \Rightarrow \vec{v}\cdot d\vec{l} = yz^2\,dy \Rightarrow // z=0 // \Rightarrow \vec{v}\cdot d\vec{l} = 0 \Rightarrow \int \vec{v}\cdot d\vec{l} = 0$

$(2-2y)(2-2y)$

$4 - 4y - 4y + 4y^2$

② // parametrerization // $d\vec{l} = dy\,\hat{y} + dz\,\hat{z} \Rightarrow \vec{v}\cdot d\vec{l} = yz^2\,dy + (3y+z)\,dz$ $// z = 2-2y,\ dz = -2dy$

$\vec{v}\cdot d\vec{l} = y(2-2y)^2 dy + (3y+2-2y)(-2)dy \Rightarrow y(4-8y+4y^2)dy + (-2y-4)dy \Rightarrow 4y^3 - 8y^2 + 2y - 4\,dy$

$16 + \frac{32}{3} - 4 + 16$

$\int \vec{v}\cdot d\vec{l} = \int_1^0 4y^3 - 8y^2 + 2y - 4\,dy = \left[y^4 + \frac{8y^3}{3} + y^2 + 4y\right]_0^1 = \frac{14}{3}$

$\frac{32}{3} - \frac{12}{3}$

③ $d\vec{l} = dz\,\hat{z} \Rightarrow \vec{v}\cdot d\vec{l} = (3y+z)\,dz \quad // y=0 // \Rightarrow \vec{v}\cdot d\vec{l} = z\,dz$

$-\int_0^2 z\,dz = -\left[\frac{z^2}{2}\right]_0^2 = -2$

$\oint_P \vec{v}\cdot d\vec{l} = 0 + \frac{14}{3} - \frac{6}{3} = \boxed{8/3}$ ✓

$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ 6x & yz^2 & 3y+z \end{vmatrix} = \hat{x}(3-2yz) - \hat{y}(0) + \hat{z}(0) \Rightarrow \nabla\times\vec{v} = (3-2yz)\hat{x}$

$// z = 2-2y,\ \frac{1}{2}z = 1-y,\ y = 1-\frac{1}{2}z$

$\int(\nabla\times\vec{v})\cdot d\vec{a} = \int_0^2\int_0^{1-\frac{1}{2}z}(3-2yz)\,dy\,dz \Rightarrow \int_0^2 dz\left[3y - y^2 z\right]_0^{1-\frac{1}{2}z} \Rightarrow \int_0^2 dz\left[3(1-\tfrac{1}{2}z) - z(1-\tfrac{1}{2}z)^2\right]$

$= \int_0^2 dz\left[3 - \frac{3}{2}z - z(1 - z + \frac{1}{4}z^2)\right] = \int_0^2 dz\left[3 - \frac{3}{2}z - z + z^2 - \frac{1}{4}z^3\right] = \int_0^2 3 - \frac{5}{2}z + z^2 - \frac{1}{4}z^3\,dz$

$= \left[3z - \frac{5}{4}z^2 + \frac{1}{3}z^3 - \frac{1}{16}z^4\right]_0^2 = \left[6 - 5 + \frac{8}{3} - 1\right] = \boxed{8/3}$ ✓

1.57