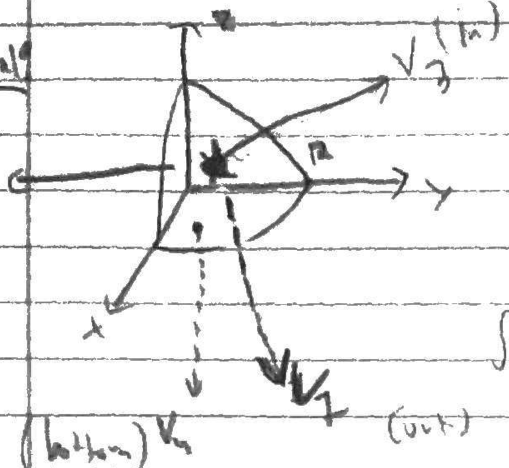
HW1

12-2-20

54) $v = r^2\cos\theta\,\vec{r} + r^2\cos\phi\,\vec{\theta} - r^2\cos\theta\sin\phi\,\vec{\phi}$

Surface Integral



a) $da = R^2\sin\theta\,d\theta\,d\phi\,\vec{r}$

$v\cdot da_1 = (R^2\cos\theta)\,R^2\sin\theta\,d\theta\,d\phi$

$\int v\cdot da = R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi$

$= R^4\left(\frac{\sin^2\theta}{2}\right)_0^{\pi/2}\left(\pi/2\right)$

$$\boxed{V_1 = \frac{\pi R^4}{4}}$$

b) V_2 = Left side, (xz-plane)

$da = -r\,dr\,d\theta\,\vec{\phi}$

$\phi = 0$, on xz-plane

$V_2\cdot da = (r^2\cos\theta\sin\phi)(r\,dr\,d\theta)$

$\boxed{V_2 = 0}$

c) V_3; $da = r\,dr\,d\theta\,\vec{\phi}$, $\phi = \pi/2$

$v\cdot da = (-r^2\cos\theta\sin\phi)(r\,dr\,d\theta)$

$= -r^3\cos\theta\,dr\,d\theta$

$= \int_0^R r^3\,dr\cdot\int_0^{\pi/2}\cos\theta\,d\theta$

$\int v_3\cdot da = (-R^4/4)(\sin\theta)\Big|_0^{\pi/2} \Rightarrow -R^4/4$

d) v_4, bottom ; $d\vec{a} = r\,dr\,d\phi\,\hat{\theta}$

$\theta = \pi/2$

$v \cdot da = (r^2 \cos\phi)(r\,dr\,d\phi)$

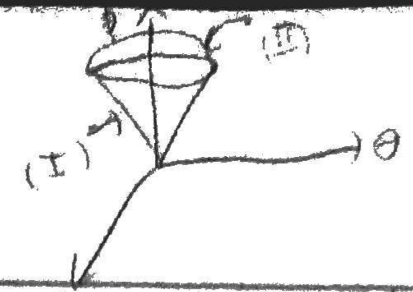
$$\int v \cdot da = \int_0^R r^3\,dr \int_0^{\pi/2} \cos\phi\,d\phi = \frac{R^4}{4}$$

$$\boxed{\int v \cdot da = \frac{\pi R^4}{4} - \frac{R^4}{4} + \frac{R^4}{4} = \frac{\pi R^4}{4}}$$

Thus $\int (\nabla \cdot v)\,d\tau = \int v \cdot da$

$$\boxed{\frac{\pi R^4}{4} = \frac{\pi R^4}{4}}$$

57)



5a) $\int v \cdot da = \int r^2 \sin\theta \,(da)$ $\qquad da = R^2 \sin\theta \, d\phi \, d\theta \, \hat{r}$

$r = R$
$\phi = 0 \to 2\pi$
$\theta = 0 \to \pi/6$

$= \int_0^{2\pi} \int_0^{\pi/6} R^2 \sin^2\theta \, d\theta \, d\phi$

PATH (I):

$= 2\pi R^4 \int_0^{\pi/6} \sin^2\theta \, d\theta \Rightarrow 2\pi R^4 \int_0^{\pi/6} \frac{1}{2}(1 - \cos 2\theta) d\theta$

$da = -\sin\theta \, d\phi \, dr \, \vec{\theta}$

$v \cdot da = 4r^3 \cos\theta \sin\theta \, d\phi \, dr$

Path (II) $= \int_0^R \int_0^{2\pi}$

$= \int_0^R \int_0^{2\pi} 4r^3 \cos\theta \sin\theta \, d\phi \, dr \Rightarrow \int_0^{2\pi} \int_0^R 4r^3 \cos\frac{\pi}{6} \cdot \frac{1}{2} \, dr \, d\phi$

$= 2\pi \sqrt{3} \int_0^R r^3 dr$

Path 1 & 2: $= 2\pi R^4 \int_0^{\pi/6} \frac{1}{2}(1 - \cos 2\theta) d\theta + 2\pi \sqrt{3} \int_0^R r^3 dr$

$= 2\pi R^4 \left(\frac{1}{2} \left(\pi/6 - \sin 2\theta/2 \Big|_0^{\pi/6} \right) + \frac{\sqrt{3}}{4} \right)$

$\boxed{= \frac{\pi R^4}{12} \left(3\sqrt{3} + 2\pi \right)}$

$$\nabla \cdot V = \frac{1}{r^2} \partial/\partial r \left(r^2 (r^2 \sin\theta) \right) + \frac{1}{r\sin\theta} \partial/\partial\theta \left[\sin\theta (4r^2 \cos\theta) \right] +$$

$$\frac{1}{r\sin\theta} \partial/\partial\phi \left(r^2 \tan\theta \right)$$

$$= 4r\sin\theta + \frac{4r}{\sin\theta}\left(\cos^2\theta - \sin^2\theta\right) = \frac{4r\cos^2\theta}{\sin\theta}$$

$$\int(\nabla \cdot V) = \int_0^{2\pi} \int_0^{\pi/6} \int_0^{R} \frac{4r\cos^2\theta}{\sin\theta} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= 4 \int_0^{2\pi} d\phi \cdot \int_0^{\pi/6} \cos^2\theta \, d\theta \cdot \int_0^{R} r^3 \, dr$$

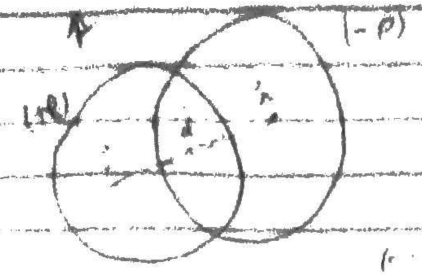
$$= \cancel{4} \cdot 2\pi \cdot \frac{R^4}{\cancel{4}} \left(\frac{\sin 2\theta}{2} \Big|_0^{\pi/6} + \pi/6 \right)$$

$$= \frac{\pi R^4}{12} \left(3\sqrt{3} + 2\pi \right)$$

Thus: $\int_V (\nabla \cdot V) \, d\tau = \oint (V \cdot \vec{n}) \, da = \frac{\pi R^4}{12}(3\sqrt{3} + 2\pi)$

HW # 2

1(a)



uniform volume charge densities

ρ:

(r-d) from (sphere 1)

From HW2: $\vec{E}_1 = \frac{\rho r}{3\varepsilon_0}\hat{r}$

$$\vec{E}_2 = -\frac{\rho(\vec{r}-\vec{d})}{3\varepsilon_0} \quad \text{(sphere 2)}$$

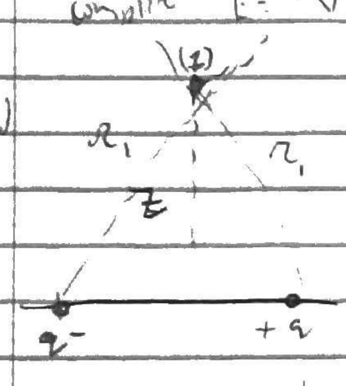
$$E_{TOT} = E_1 + E_2 = \frac{\rho}{3\varepsilon_0}(\vec{r}-\vec{r}+\vec{d}) = \boxed{\frac{\rho}{3\varepsilon_0}\vec{d}}$$

2.5) Using: $V(r) = k\sum_{i=1}^{n} q_i/r_i$ $V = k\int \lambda(r')/r \, dl'$

$V = k\int \sigma(r')/r \, da'$

compute $E = -\nabla V$

a)



$$V_{tot} = V_1 + V_2$$

$$= \frac{kq}{r_1} + \frac{kq}{r_2} = \frac{1}{4\pi\varepsilon_0}\sum \frac{q_i}{r_i}$$

$$= kq\left(1/r_1 + 1/r_2\right)$$

$$= \frac{2kq}{\sqrt{(z^2+(d/2)^2)}} = V(z)$$

$\nabla V = dV/dz\,\hat{z} + \dots$

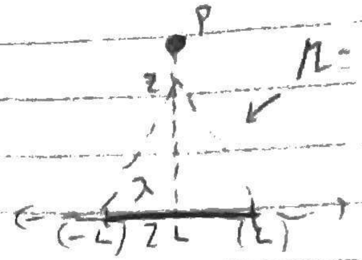
$$\vec{E} = -dV/dz\,\hat{z} = \boxed{\vec{E} = k\cdot\frac{qz}{(z^2+d^2/4)^{3/2}}}$$

$$E = \frac{-dV_{tot}}{dz}\hat{z}$$

Far-Field: $\vec{r} >> \vec{r}'$, $\|\vec{E}\| = \frac{kq_{tot}}{z^2}$

$z >> d$

(explanation ⇒ [illegible] (P))



$\mathscr{r} = \sqrt{\lambda^2 + z^2}$

2.25 b) $V = k \int \frac{\lambda(r')}{\mathscr{r}}\, dl'$

$V = \frac{1}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{\lambda\, d\lambda}{\sqrt{\lambda^2 + z^2}}$

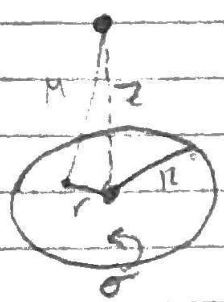
$= \frac{\lambda}{4\pi\varepsilon_0} \cdot \ln\left(\sqrt{\lambda^2 + z^2} + x\right)\Big|_{-L}^{L} \Rightarrow V_p = \frac{\lambda}{4\pi\varepsilon_0} \ln \boxed{\frac{\sqrt{L^2 + z^2} + L}{\sqrt{L^2 + z^2} - L}}$

thus, $\boxed{E = -\nabla V}$; $\vec{E} = \frac{\lambda}{4\pi\varepsilon_0} \cdot \frac{\sqrt{z^2 + L^2} - L}{\sqrt{z^2 + L^2} + L} \left[\frac{\left(\sqrt{L^2 + z^2} - L\right)\left(\frac{2z}{2\sqrt{L^2 + z^2}}\right) - \left(\sqrt{L^2 + z^2} + L\right)}{\left(\sqrt{L^2 + z^2} - L\right)^2} \right]$

$-\partial V/\partial z\, \vec{z}$

$E = \frac{-\lambda}{4\pi\varepsilon_0} \left(\frac{1}{\sqrt{L^2 + z^2} - L^2}\right)\left(\frac{-2Lz}{\sqrt{L^2 + z^2}}\right) \vec{z} \Rightarrow \boxed{E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2\lambda z}{z\sqrt{L^2 + z^2}}\, \vec{z}}$

25 c)



$V = k \int \frac{\sigma(r')}{\mathscr{r}}\, da$

$V = \frac{1}{4\pi\varepsilon_0} \int_0^R \frac{\sigma\, 2\pi r\, dr}{\sqrt{r^2 + z^2}} = \frac{\sigma}{2\varepsilon_0} \left[(r^2 + z^2)^{1/2} \right]_0^R$

$V_r = \frac{\sigma}{2\varepsilon_0} \left[(r^2 - z^2)^{1/2} - z \right]$

$E = -\nabla V \Rightarrow \frac{-\partial V}{\partial z}\, \vec{z} \Rightarrow \frac{\sigma}{2\varepsilon_0} \cdot \partial / \partial z \left[(r^2 + z)^{1/2} - z \right] \vec{z}$

$$\boxed{E_p = \frac{\sigma}{2\varepsilon_0} \left(1 - \frac{z}{(R^2 - z^2)^{1/2}} \right) \vec{z}}$$

2a) $V(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r)'}{\imath} d\tau'$, Poisson's Eqn

(Eqn. 1.102) $\frac{\nabla^2}{\imath} = -4\pi \delta^3(\imath)$

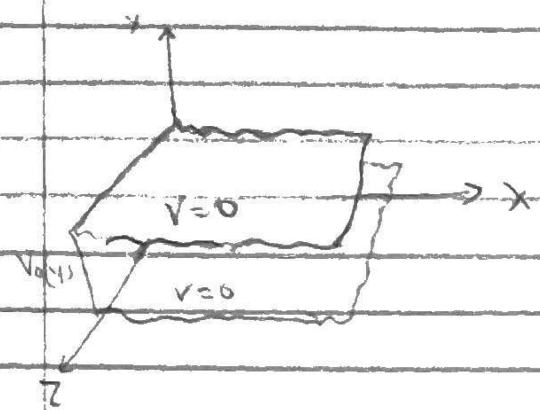
$\nabla^2 V = \nabla^2 \cdot \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')}{\imath} = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho r \cdot \nabla^2}{\imath} dV$

$$\boxed{\frac{\nabla^2}{\imath} V - 4\pi \delta(\vec{\imath}) = -\frac{\rho \vec{r}}{\varepsilon_0}}$$

HW #3

14) $\sigma(y)$ on strip at $x=0$,

- constant V_0



$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} 1/n\, e^{-n\pi x/a} \cdot \sin\left(\frac{n\pi y}{a}\right)$$

$$\sigma = -\varepsilon_0 \left[\frac{\partial V}{\partial x}\right]$$

$$\sigma(y) = -\varepsilon_0 \left[\frac{\partial V}{\partial x}\right]_{x=0}$$

$$\sigma(y) = -\varepsilon_0 \frac{\partial}{\partial x}\left\{\frac{4V_0}{\pi} \sum 1/n\, e^{-\frac{n\pi x}{a}} \cdot \sin\left(\frac{n\pi y}{a}\right)\right\}_{x=0}$$

$$\sigma(y) = -\frac{\varepsilon_0 4V_0}{\pi} \sum 1/n \left(\frac{-n\pi}{a}\right) e^{\frac{-n\pi x}{a}} \cdot \sin\left(\frac{n\pi y}{a}\right)\Bigg|_{x=0}$$

$$= \varepsilon_0 \cdot \frac{4V_0}{\pi} \cdot \frac{1}{n}\left(\frac{n\pi}{a}\right) \sum e^{-n\pi x/a} \cdot \sin\left(\frac{n\pi y}{a}\right)$$

$x=0$, this $\to 1$

$$\boxed{\sigma(y) = \frac{4\varepsilon_0 V_0}{a} \sum \sin\left(\frac{n\pi y}{a}\right)}$$