Electromagnetc Theory: PHYS330

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Summary

Week 5 Summary

1. Current density and continuity equation
2. The divergence and curl of $\vec{B}$-fields
3. The magnetic vector potential, $\vec{B} = \nabla \times \vec{A}$
   - Vector calculus theorems
   - Boundary conditions
   - Multipole expansion
4. Magnetic fields in matter
   - Magnetization
   - Field of a magnetized object
   - The auxiliary field, $\vec{H}$
   - Linear magnetic media

Current density and continuity equation

Current density and continuity equation

Let the *current density* $\vec{J}$ be defined by

$$\vec{J} = \rho \vec{v} \tag{1}$$

Units: current per unit area (other definitions available for different geometries). So it's reasonable to obtain the whole scalar current by integrating:

$$I = \int_{\mathcal{S}} \vec{J} \cdot d\vec{a} \tag{2}$$

If we want to account for the charge leaving a volume $\mathcal{V}$ through a closed surface $\mathcal{S}$ is

$$\oint_{\mathcal{S}} \vec{J} \cdot d\vec{a} = \int_{\mathcal{V}} (\nabla \cdot \vec{J}) d\tau \tag{3}$$

$$\int_{\mathcal{V}} (\nabla \cdot \vec{J}) d\tau = -\frac{d}{dt} \int_{\mathcal{V}} \rho d\tau = -\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau \tag{4}$$

Current density and continuity equation

This is true for *any* volume, so the integrands must be equal:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \tag{5}$$

This is called the continuity equation, and it also arises in quantum mechanics. If $\partial \rho / \partial t = 0$, then we have a **steady current.**

Suppose we have a current density $\vec{J}(\vec{r}) = I_0(t)\hat{r}/r^2$, with $I_0(t) = \delta(t - t_0)$. Find $\rho(t)$, the charge density as a function of time in the region containing $\vec{J}$. (Breakout rooms).

The Divergence of B-fields

The Divergence of B-fields

The Biot-Savart law states that

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau' \tag{6}$$

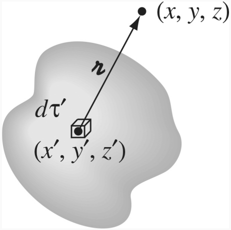

Figure 1: Definitions of coordinates in variables for derivation of divergence of B-fields. The gray region represents charges and current densities.

The Divergence of B-fields

Take the divergence of the Biot-Savart law, but then use a product rule for the integrand.

$$\nabla \cdot \vec{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\vec{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau' \tag{7}$$

$$\nabla \cdot \left(\vec{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\nabla \times \vec{J}) - \vec{J} \cdot \left(\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \tag{8}$$

- $\nabla \times \vec{J} = 0$, because this is like taking $df(x)/dx'$.
- We showed in Chapter 1 that $\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 0$. Is this visually obvious?

Thus,

$$\boxed{\nabla \cdot \vec{B} = 0} \tag{9}$$

The Divergence of B-fields

From warmup exercises, we know that we can therefore write

$$\vec{B} = \nabla \times \vec{A} \tag{10}$$

(Breakout rooms): create three divergence-less vector fields. One in Cartesian coordinates, one in cylindrical coodinates, and one in spherical. Exclude trivial cases like $\vec{B} = 0$.

The Curl of $\vec{B}$-fields

The Curl of $\vec{B}$-fields

Because $\vec{B}$-fields have no divergence, we can write

$$\vec{B} = \nabla \times \vec{A} \tag{11}$$

Because the curl of the gradient of a scalar function is zero, we can choose[1]

$$\nabla \cdot \vec{A} = 0 \tag{12}$$

Since $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$,

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}} \tag{13}$$

[1] We can always find a scalar function whose gradient we are free to add to $\vec{A}$ that makes the divergence go away.

The Curl of $\vec{B}$-fields

Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

- First, what is $\vec{B}$, from Ampère's Law?
- Why can we *not* just do this business, as with Poisson's equations for $V(\vec{r'})$?

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'})}{\mathcal{r}} d\tau' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\mathcal{r}} d\vec{l'} \tag{14}$$

- Notice that

$$\oint \vec{A} \cdot d\vec{l} = \int (\nabla \times \vec{A}) \cdot d\vec{a} = \int \vec{B} \cdot d\vec{a} = \Phi_B \tag{15}$$

  by Stoke's Theorem.
- Obtain $\oint \vec{A} \cdot d\vec{l}$ Ampèrian loop of radius s, and $\vec{B}$ from Ampère's Law ...

Boundary Conditions

Boundary Conditions

What boundary conditions exist for $\vec{B}$ and $\vec{A}$ at surface currents?

$\vec{B}$-fields

1. Review of a surface current, $\vec{B}$-field of a uniform surface current
2. Apply divergence theorem for $\vec{B}_{\perp}$
3. Apply Ampère's Law for $\vec{B}_{||}$

$\vec{A}$-fields

1. Divergence
2. $\oint \vec{A} \cdot d\vec{l}$

Boundary Conditions

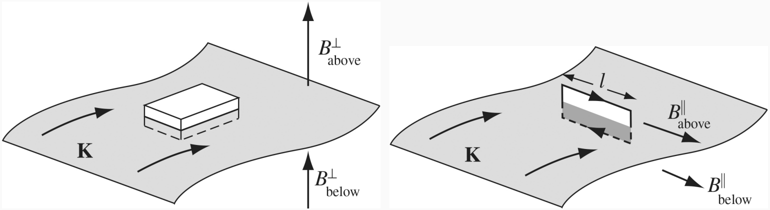

Figure 2: (Left) Perpendicular B-field condition (Right) Parallel B-field condition.

Multipole Expansion for Vector Potential

Multipole Expansion for Vector Potential

It's still true that the generator function for the Legendre polynomials is $1/\mathscr{r}$:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n} P_n(\cos\alpha) \tag{16}$$

(Remember that α is the angle between r and r'). Therefore for any current loop:

$$\vec{A} = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\mathscr{r}} d\vec{l} \tag{17}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^{n} P_n(\cos\alpha) d\vec{l} \tag{18}$$

Multipole Expansion for Vector Potential

Use Eq. 18 to find the $n = 0$ and the $n = 1$ terms.

1. Can you explain the result for the $n = 0$ term on physical grounds?
2. Show that the second term is

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha \, d\vec{l'} \tag{19}$$

3. Convince yourself that $\hat{r} \cdot \vec{r'} = r' \cos\alpha$.
4. Now we're going on a trip down memory lane...

Multipole Expansion for Vector Potential

Recall from the Ch. 1 homework that

$$\oint (\vec{c} \cdot \vec{r'}) d\vec{l'} = \vec{a} \times \vec{c} \tag{20}$$

where $\vec{a}$ is the "area vector."

$$\vec{a} = \int_{\mathcal{S}} d\vec{a'} \tag{21}$$

The vector field $\vec{c}$ is a constant one. Let $\vec{c} = \hat{r}$ to find

$$\oint (\hat{r} \cdot \vec{r'}) d\vec{l'} = \vec{a} \times \hat{r} \tag{22}$$

Multipole Expansion for Vector Potential

Putting it all together for the $n = 1$ term:

$$\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\left(I \int_{\mathcal{S}} d\vec{a'} \right) \times \hat{r}}{r^2} \tag{23}$$

Define the vector $\vec{m}$ as

$$\vec{m} = I \int_{\mathcal{S}} d\vec{a'} \tag{24}$$

So that

$$\boxed{\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}} \tag{25}$$

Multipole Expansion for Vector Potential

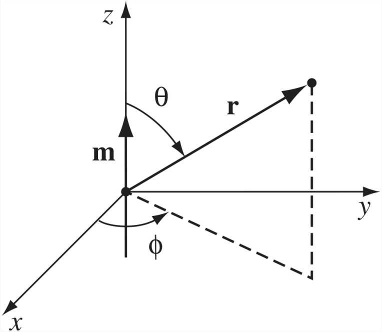

Figure 3: Choose this geometry for the magnetic dipole.

1. Evaluate the dipole term for the vector potential with this geometry
2. Compute the curl

Conclusion

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