# Electromagnetc Theory: PHYS330

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November 9, 2020

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# Summary

# Week 2 Summary

1. Homework discussions
   - Proofs! Glorious proofs.
   - Exercises with *checking* fundamental theorems
2. Electrostatics and Coulomb forces
   - Charge distributions, superposition, and the Coulomb force
   - A note about the *far-field*
   - Setting up integrals, taking limits, checking units
   - The divergence of electric fields
   - The curl of electric fields
3. Electric Potential
   - Definitions, fundamental theorem for gradients
   - Reference points
   - Laplace equation ...
4. Work, energy, and conductors

# Homework

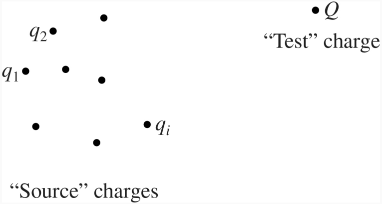
## Homework, Week 2

Unlike last week, these exercises come from *within* the chapter. Ideally, you should look at all of the problems within the chapter as you study.

- Exercise 2.5
- Exercise 2.6
- Exercise 2.9
- Exercise 2.12
- Exercsie 2.16
- Exercise 2.18
- Exercise 2.25
- Exercise 2.29

# Charge distributions, Superposition, and the Coulomb Force

# Charge distributions, Superposition, and the Coulomb Force



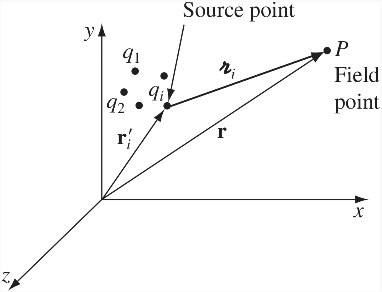




**Figure 1:** The basic problem of electrostatics. Note the definition of the separation vector, and the vectors to the field point and to all the source charges.

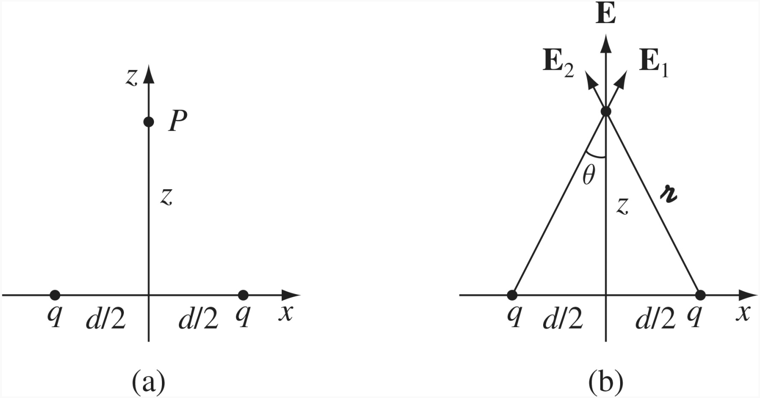



**Figure 2:** Begin with a dipole, and then a *physical* dipole.

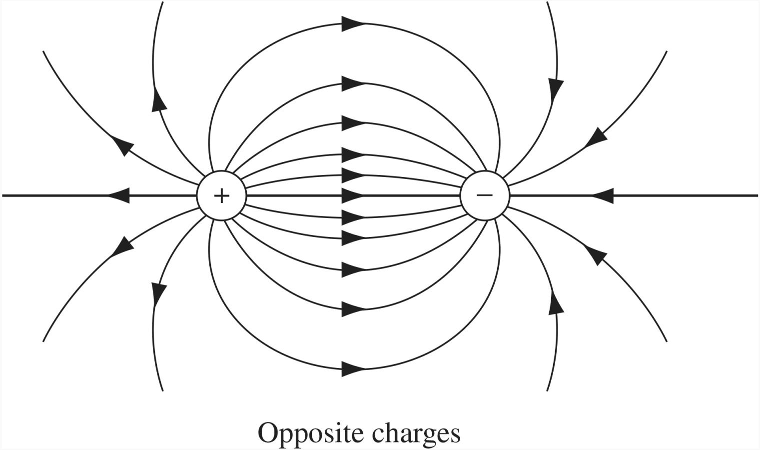



**Figure 3:** Field of a *physical* dipole.

# Charge distributions, Superposition, and the Coulomb Force

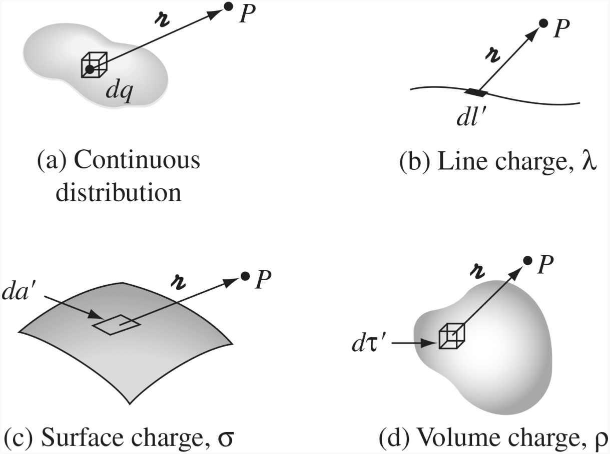



**Figure 4:** The continuous limit implies a variety of symmetries and geometries over which we integrate, rather than sum.

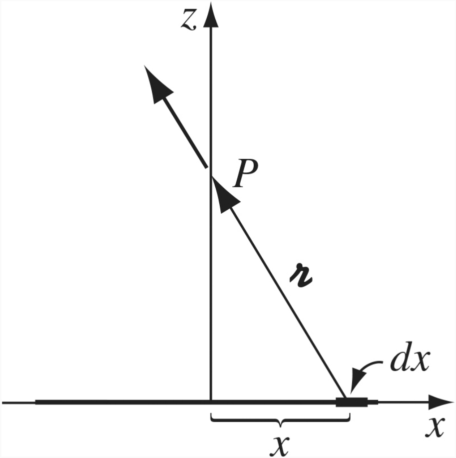



**Figure 5:** A coninuous line density of charge. Integration yields the electric field.

Useful calculations:

1. Continuous line charge, length $L$.
2. Continuous plane of charge, radius $R$.
3. Loop of charge, radius $R$, a distance $z$ above the center.

Why are these interesting? One example is that these shapes are used as *antennas*. Give some alternating current at the right voltage and impedance to a shape of metal, then you've got your antenna that radiates a certain way.

**Professor do these examples.**[1]

[1] Remember from PHYS180? Remember? Yeah...good times.

# A Note about the Far-Field

# The Far-Field

One way to express the **far-field** approximation (compare to Fraunhofer and Fresnel limits in diffraction):

$$
\begin{aligned}
\vec{r} &= \vec{r}' + \vec{\mathscr{r}} && (1)\\
\vec{\mathscr{r}} &= \vec{r} - \vec{r}' && (2)\\
\mathscr{r} &= \sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + r'^2} && (3)\\
\mathscr{r} &= r\sqrt{1 - 2\vec{r}\cdot\vec{r}' r^{-2} + r'^2 r^{-2}} && (4)\\
\mathscr{r} &\approx r\sqrt{1 - 2\vec{r}\cdot\vec{r}' r^{-2}} && (5)\\
\mathscr{r} &\approx r\left(1 - \vec{r}\cdot\vec{r}' r^{-2}\right) && (6)\\
\mathscr{r} &\approx r\left(1 - \hat{r}\cdot\vec{r}' r^{-1}\right) && (7)\\
\mathscr{r} &\approx r - \hat{r}\cdot\vec{r}' && (8)
\end{aligned}
$$

# The Far-Field

**Repeat the charged loop calculation**, but replace $\mathscr{r} \approx r - \hat{r} \cdot \vec{r}'$ at the outset. What happens?

# The Divergence of $\vec{E}$-fields

# The Divergence of $\vec{E}$-fields

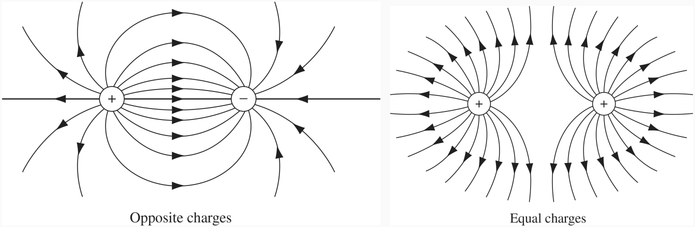



**Figure 6:** Field lines are an important theoretical concept.

# The Divergence of $\vec{E}$-fields

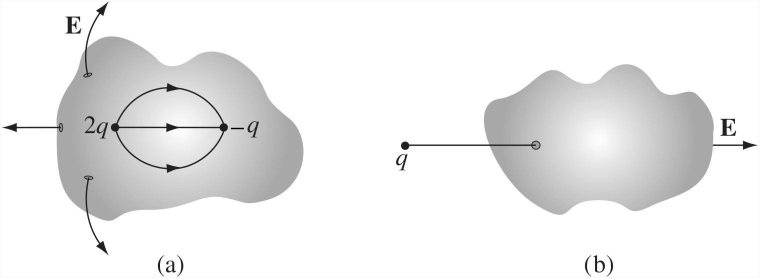



**Figure 7:** The concept of a closed Gaussian surface.

# The Divergence of $\vec{E}$-fields

$$\boxed{\oint \vec{E}_i \cdot d\vec{a} = \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\frac{q_i\hat{r}}{r^2}\cdot r^2\sin\theta d\theta d\phi\hat{r} = \frac{q_i}{\epsilon_0}} \tag{9}$$

$$\vec{E} = \sum_{i=1}^{n}\vec{E}_i \tag{10}$$

$$\oint \vec{E}\cdot d\vec{a} = \sum_{i=1}^{n}\left(\oint \vec{E}_i\cdot d\vec{a}\right) \tag{11}$$

$$\oint \vec{E}\cdot d\vec{a} = \sum_{i=1}^{n}\left(\frac{q_i}{\epsilon_0}\right) \tag{12}$$

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{tot}}{\epsilon_0} \tag{13}$$

Gauss' Law: the total flux is proportional to the contained charge.

## The Divergence of $\vec{E}$-fields

The divergence theorem:

$$\oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \int_{\mathcal{V}} (\nabla \cdot \vec{E}) d\tau \tag{14}$$

Remark that the total charge is the integral over the 3D charge density:

$$\frac{Q_{tot}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho d\tau \tag{15}$$

This implies

$$\oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \int_{\mathcal{V}} (\nabla \cdot \vec{E}) d\tau = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho d\tau \tag{16}$$

Looking at the last two expressions:

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}} \tag{17}$$

# The Divergence of $\vec{E}$-fields

*Differential form* of Gauss' Law:

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}} \tag{18}$$

Consider a different argument:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho(\vec{r}\,') d\tau' \tag{19}$$

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left( \frac{\hat{\hat{\mathscr{r}}}}{\mathscr{r}^2} \right) \rho(\vec{r}\,') d\tau' \tag{20}$$

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi\delta^3(\mathscr{r})\rho(\vec{r}\,') d\tau' \tag{21}$$

$$\nabla \cdot \vec{E} = \frac{4\pi}{4\pi\epsilon_0} \int \delta^3(\vec{r} - \vec{r}\,')\rho(\vec{r}\,') d\tau' \tag{22}$$

$$\nabla \cdot \vec{E} = \rho(\vec{r})/\epsilon_0 \tag{23}$$

# The Divergence of $\vec{E}$-fields

(Refresh with delta-functions): Use this charge distribution and Eq. 19 to find the $\vec{E}$-field.

$$\rho(\vec{r}') = q\delta^3(\vec{r}' - x\hat{x}) - q\delta^3(\vec{r}' + x\hat{x}) \tag{24}$$

# The Divergence of $\vec{E}$-fields

**Symmetry in the Application of Gauss' Law**:

If the E-field and the area element are *always* orthogonal,

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{tot}}{\epsilon_0} \tag{25}$$

$$|\vec{E}| A = \frac{Q_{tot}}{\epsilon_0} \tag{26}$$

$$\vec{E} = \frac{Q_{tot}}{\epsilon_0 A} \hat{n} \tag{27}$$

This trick can be used even when the charge distribution is not uniform, but *does exhibit* symmetry.

# The Divergence of $\vec{E}$-fields

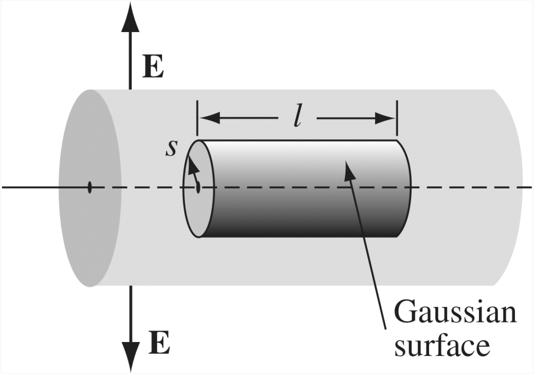



**Figure 8:** Use cylindrical symmetry to apply Gauss' Law. The charge distribution function is $\rho(s) = ks$. Obtain the field (a) *inside* the object, then (b) outside the object.

# The Divergence of $\vec{E}$-fields

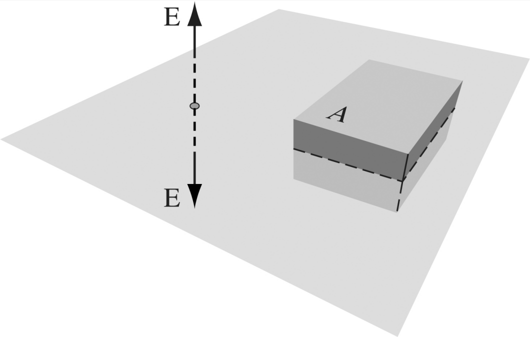



**Figure 9:** Use Cartesian symmetry to apply Gauss' Law. The charge distribution function is $\rho(x, y) = +\sigma$. Obtain the field above (or below) the charged plane.

# The Divergence of $\vec{E}$-fields

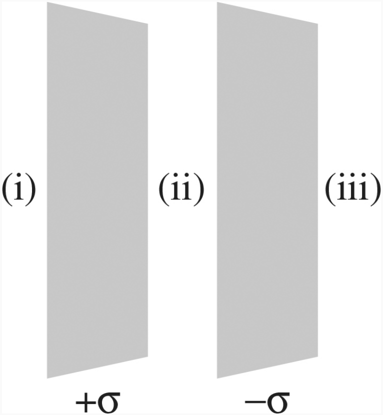



**Figure 10:** Combinations of "Gaussian charged objects." What about the field between two line charges?

# The Curl of $\vec{E}$-fields

## The Curl of $\vec{E}$-fields

The $\vec{E}$-field of a point charge at the origin ($\vec{r}' = 0$), and the line element in spherical coordinates:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{28}$$

$$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi} \tag{29}$$

$$\vec{E}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr \tag{30}$$

$$\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \tag{31}$$

$$\oint \vec{E}\cdot d\vec{l} = 0 \quad (\vec{a} = \vec{b}) \tag{32}$$

$$\nabla \times \vec{E} = 0 \tag{33}$$

Any combination of point charges will also lead to zero curl. Why? Superposition.

# The Curl of $\vec{E}$-fields

Define a function, then, that encapsulates *path-independence*:

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E} \cdot d\vec{l} \tag{34}$$

- $\mathcal{O}$ is a reference point, naturally taken to be $\infty$ (far from origin)
- $V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$
- $0 = V(\vec{a}) - V(\vec{a}) = -\oint \vec{E} \cdot d\vec{l} = 0$

Fundamental theorem for gradients:

$$V(\vec{b}) - V(\vec{a}) = \int_{\vec{a}}^{\vec{b}} \nabla V \cdot d\vec{l} = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l} \tag{35}$$

$$\vec{E} = -\nabla V \tag{36}$$

# The Curl of $\vec{E}$-fields

Path independence:

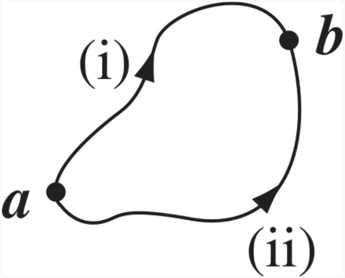



**Figure 11:** If the line integral was not path-independent, then path i minus path ii would not be zero. Path i and ii form a closed line integral.

# The Curl of $\vec{E}$-fields

Two more ideas:

$$\vec{E} = -\nabla V \tag{37}$$

$$\nabla \cdot \vec{E} = -\nabla^2 V \tag{38}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{39}$$

Equation 39 is known as the *Poisson Equation*. If $\rho = 0$ in some region:

$$\boxed{\nabla^2 V = 0} \tag{40}$$

Equation 40 is known as the Laplacian of the potential or Laplace's Equation. Solving it is the subject of Ch. 3.

# The Curl of $\vec{E}$-fields

Show that the potential of a point charge $q$ at the origin is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{41}$$

Indeed, collections of point charges lead to

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i} \tag{42}$$

A collection of *many* point charges smoothed into a continuous distribution leads to

$$V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho' d\tau'}{\mathscr{r}} \tag{43}$$

# The Curl of $\vec{E}$-fields

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho' d\tau'}{\mathscr{r}} \tag{44}$$

(a) Derive the potential due to a line charge of length $L$, directly above the center of the line. (b) Take the gradient in cylindrical coordinates to obtain $\vec{E}$.

# Conclusion

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    - Proofs! Glorious proofs.
    - Exercises with *checking* fundamental theorems
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    - Charge distributions, superposition, and the Coulomb force
    - A note about the *far-field*
    - Setting up integrals, taking limits, checking units
    - The divergence of electric fields
    - The curl of electric fields
3. Electric Potential
    - Definitions, fundamental theorem for gradients
    - Reference points
    - Laplace equation ...
4. Work, energy, and conductors