Solutions for Homework 4

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1 Problem 4.10

A sphere of radius R carries a polarization

$$\mathbf{P}(\mathbf{r}) = k\mathbf{r} \tag{1}$$

In Eq. 1, k is a constant and and $\mathbf{r}$ *is the vector from the center.*

- *(a) Calculate the bound charges* σ_b *and* ρ_b*.*
- *(b) Find the field inside and outside the sphere.*

(a) The surface bound charge is at radius R, so $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = kR$. The volumetric bound charge is $\rho_b = -\nabla\cdot\mathbf{P} = -3k$. Note the total charge should add up to zero: $(4\pi R^2)\sigma_b + (4/3)\pi R^3\rho_b = 4\pi R^3 k - 4\pi R^3 k = 0$. (b) The field of any constant volumetric charge density $-3k$ should be calculable via Gauss' law. We find, after integrating over a Gaussian surface of radius $r < R$:

$$\oint \mathbf{E}\cdot d\mathbf{a} = \mathbf{E}\cdot\mathbf{A} = \frac{1}{\epsilon_0}\rho = \frac{-4k\pi r^3}{\epsilon_0} \tag{2}$$

$$\mathbf{E} = -\frac{3kr}{\epsilon_0}\hat{\mathbf{r}} = -\mathbf{P}/\epsilon_0 \tag{3}$$

(b) Note that, because the net charge is zero, the field outside the sphere is zero.

2 Problem 4.14

When you polarize a neutral dielectric, the charge moves a bit, but the total remains zero. This fact should be reflected in the bound charges σ_b *and* ρ_b*. Prove from Eqs. 4.11 and 4.12 that the toal bound charge vanishes.*

First, let's integrate the total volumetric bound charge:

$$-q = \int_{\mathcal{V}} \rho_b d\tau = -\int_{\mathcal{V}} \nabla\cdot\mathbf{P} d\tau = -\oint \mathbf{P}\cdot\hat{\mathbf{n}} da = -\oint \sigma_b da \tag{4}$$

Next, the total surface bound charge is

$$q = \oint_S \sigma_b da \tag{5}$$

Now we see that $Q = -q + q = 0$.

3 Problem 4.15

A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a frozen-in polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}} \tag{6}$$

In Eq. 6, k is a constant and r is the distance from the center. There is no free charge in the problem. Find the electric field in all three regions by two different methods:

- (a) Locate all the bound charge, and use Gauss' Law to calculate the field it produces.
- (b) Use Eq. 4.23 to find $\mathbf{D}$, and then get $\mathbf{E}$ from Eq. 4.21. [Notice the second method is faster, and it avoids any explicit reference to the bound charges].

For (a), locating all the bound charge:

- $r = a$:

$$\sigma_a = \mathbf{P} \cdot (-\hat{\mathbf{r}}) = -\left(\frac{k}{a}\right)\hat{\mathbf{r}} \tag{7}$$

- $a < r < b$:

$$\rho = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right) = -\frac{k}{r^2} \tag{8}$$

- $r = b$:

$$\sigma_b = \frac{k}{b} \tag{9}$$

- What is the total charge for this neutral object?

$$Q = 4\pi(b-a) - 4\pi\int_a^b \rho d\tau = 4\pi(b-a) - 4\pi(b-a) = 0 \tag{10}$$

  The neutral object has $Q = 0$. Using Gauss' Law, $\mathbf{E} = 0$ for $r < a$, $\mathbf{E} = -\mathbf{P}/\epsilon_0$, and for $r > b$, $\mathbf{E} = 0$ because total charge is 0.

For (b), we find that since there is no free charge, $\mathbf{D} = 0$ and by the definition of $\mathbf{D}$, $\mathbf{E} = -\mathbf{P}/\epsilon_0$. Thus, we achieve the same result by making reference to only the free charge.

4 Problem 4.18

The space between the plates of a parallel plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness a, so the total distance between the plates is $2a$*. Slab 1 has a dielectric constant of* 2*, and slab 2 has a dielectric constant of* 1.5*. The free charge density on the top plate is* σ *and on the bottom plate* $-\sigma$*.*

- (a) Using Gauss' law for $\mathbf{D}$, we find that $\mathbf{D} = -\sigma\hat{\mathbf{z}}$ throughout the system.
- (b) Using $\mathbf{D} = \epsilon\mathbf{E}$, we find slab 1 has $\mathbf{E} = -(\sigma/2\epsilon_0)\hat{\mathbf{z}}$ and slab 2 has $\mathbf{E} = -(2\sigma/3\epsilon_0)\hat{\mathbf{z}}$.
- (c) Using the definition of $\mathbf{D}$, and the results from (b), we find $\mathbf{P} = -(\sigma/2)\hat{\mathbf{z}}$ for slab 1. For slab 2 we find $\mathbf{P} = -(\sigma/3)\hat{\mathbf{z}}$.
- (d) We know that $\Delta V = E\delta z = E_1 a + E_2 a = (E_1 + E_2)a = (7\sigma a)/(6\epsilon_0)$.
- (e) There is no ρ_b because the polarization densities are constant. For the bound surface charges, there is $-\sigma/2$ at the top of slab 1, and $\sigma/2$ at the bottom of slab 1 from medium 1. There is $-\sigma/3$ at the top of slab 2 from medium 2, and $\sigma/3$ at the bottom of slab 2 from medium 2. At the interface, if we add surface charges, we find $\sigma/6$.
- (f) To find the same fields as above, we must combine total charge from free and bound charges at the top and bottom surfaces.

5 Problem 4.26

A spherical conductor, of radius a, carries a charge Q. It is surrounded by linear dielectric material of susceptibility χ_e*, out to radius b. Find the energy of this configuration.*

The goal is to find $\mathbf{E}$ and $\mathbf{D}$, so that we can apply

$$W = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E}\, d\tau \tag{11}$$

The results are:

- Displacement inside the conductor ($r < a$): 0
- Displacement outside the conductor ($r > a$): $Q/(4\pi r^2)\hat{\mathbf{r}}$
- Field inside the conductor: 0
- Field outside the conductor but inside the dielectric ($a < r < b$): $Q/(4\pi\epsilon r^2)\hat{\mathbf{r}}$.
- Field outside the dielectric ($r > b$): $Q/(4\pi\epsilon_0 r^2)\hat{\mathbf{r}}$.

Inserting all these into into Eq. 11, and integrating:

$$W = \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a} + \frac{\chi_e}{b}\right) \tag{12}$$

6 Problem 4.35

A point charge q is in embedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Find the electric field, the polarization, and the bound charge densities, σ_b and ρ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?

- Use Gauss' Law with electric displacement: $\mathbf{D} = q/(4\pi r^2)\hat{\mathbf{r}}$.
- The electric field is then $\mathbf{E} = \mathbf{D}/\epsilon$, or

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0(1+\chi_e)r^2}\hat{\mathbf{r}} \tag{13}$$

- For polarization, note that linear dielectrics have $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$, so

$$\mathbf{P} = \left(\frac{\chi_e}{1+\chi_e}\right)\mathbf{D} \;=\; \left(\frac{\chi_e}{1+\chi_e}\right)\left(\frac{q}{4\pi r^2}\right)\hat{\mathbf{r}} \tag{14}$$

- The surface and volume bound charge densities are:

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{r}} = \left(\frac{\chi_e}{1+\chi_e}\right)D \tag{15}$$

$$\rho_b = -\nabla\cdot\mathbf{P} = -q\left(\frac{\chi_e}{1+\chi_e}\right)\delta^3(\mathbf{r}) \tag{16}$$

- The total surface bound charge is $Q = q\left(\frac{\chi_e}{1+\chi_e}\right)$, but the negative compensating charge is at the origin:

$$Q = \int \rho_b d\tau = -q\left(\frac{\chi_e}{1+\chi_e}\right) \tag{17}$$