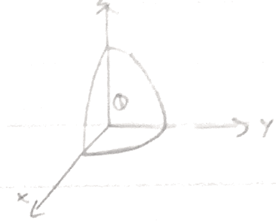
Ch 1 54, 55, 56, 57, 59, 62, 63, 64

54. $v = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

Check for divergence $\frac{1}{8}$ of a sphere



make sure both sides are equal

$$\int (\nabla\cdot v)\,d\tau = \int v\cdot da$$

$$(\nabla\cdot v) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(v_\phi)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2(r^2\cos\theta)) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta(r^2\cos\phi)) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$$

$$= \frac{1}{\cancel{r^2}}(4r^{\cancel{3}}\cos\theta) + \cancel{\frac{1}{r\sin\theta}(r^2\cos\theta\cos\phi)} - \cancel{\frac{1}{r\sin\theta}(r^2\cos\theta\cos\phi)}$$

$$= 4r\cos\theta$$

$$= \int_0^R\int_0^{\pi/2}\int_0^{\pi/2}(4r\cos\theta)\,r^2 dr\sin\theta\, d\theta\, d\phi$$

$$= \int_0^R\int_0^{\pi/2}\int_0^{\pi/2} 4r^3\cos\theta\sin\theta\, dr\, d\theta\, d\phi$$

split into 3 parts → $\int_0^{\pi/2} 1\, d\phi = \frac{\pi}{2}$ $\int_0^R 4r^3 dr = \cancel{4}\frac{r^4}{\cancel{4}}\Big|_0^R = R^4$ $\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$= \int_0^{\pi/2} u\, du = \frac{u^2}{2} = \frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = \frac{1}{2}$

$$= \left(\frac{\pi}{2}\right)(R^4)\left(\frac{1}{2}\right)$$

$$\int(\nabla\cdot v)\,d\tau = \frac{\pi R^4}{4}$$

$\int v\cdot da$

$da = R^2\sin\theta\, d\theta\, d\phi\,\hat{r}$ $r = R$

$v\cdot da = (R^2\cos\theta)R^2\sin\theta\, d\theta\, d\phi = R^4\cos\theta\sin\theta\, d\theta\, d\phi$

$$\int v\cdot da = \int_0^{\pi/2}\int^{\pi/2} R^4\cos\theta\sin\theta\, d\theta\, d\phi \rightarrow R^4\int_0^{\pi/2} 1\, d\phi \cdot \left(\frac{\pi R^4}{2}\right)$$

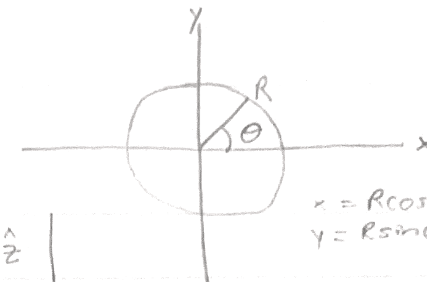
$\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta = \frac{1}{2}$

$$\left(\frac{\pi R^4}{2}\right)\left(\frac{1}{2}\right) = \frac{\pi R^4}{4}$$

$$\boxed{\int(\nabla\cdot v)\,d\tau = \int v\cdot da = \frac{\pi R^4}{4}}$$

55. $V = ay\,\hat{x} + bx\,\hat{y}$ Stoke's theorem

$$\int_S (\nabla \times V)\cdot dA = \oint V \cdot dI$$



$x = R\cos\theta$
$y = R\sin\theta$

$$\nabla \times V = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ay & bx & 0 \end{vmatrix}$$

$$= 0 + 0 + \hat{z}\left(\frac{\partial}{\partial x}(bx) - \frac{\partial}{\partial y}(ay)\right) = (b-a)\,\hat{z}$$

$da = \pi R^2\,\hat{z}$

$$\int_S (\nabla \times V)\cdot dA = \int_S (b-a)\hat{z} \cdot (\pi R^2 \hat{z}) = \pi R^2 (b-a)$$

$\oint V \cdot dI =$ $\quad V = ay\,\hat{x} + bx\,\hat{y}$

$dI = dx\,\hat{x} + dy\,\hat{y}$ $\quad x = R\cos\theta \quad dx = -R\sin\theta\, d\theta \quad y = R\cos\theta\, d\theta \quad dy = R\cos\theta\, d\theta$

$dI = (-R\sin\theta\, d\theta)\,\hat{x} + (R\cos\theta\, d\theta)\,\hat{y}$

$$\oint (ay\hat{x} + bx\hat{y}) \cdot \left[(-R\sin\theta\, d\theta)\hat{x} + (R\cos\theta\, d\theta)\hat{y}\right]$$

$$= \oint \left[-aR^2\sin^2\theta\, d\theta + bR^2\cos^2\theta\, d\theta\right]$$

Identities: $\sin^2 x = \dfrac{1-\cos(2x)}{2}$

$\cos^2 x = \dfrac{1+\cos(2x)}{2}$

$$= -\int_0^{2\pi} aR^2 \sin^2\theta\, d\theta + \int_0^{2\pi} bR^2\cos^2\theta\, d\theta$$

$$= -\frac{aR^2}{2}\int_0^{2\pi} 1 - \cos 2\theta\, d\theta + \frac{bR^2}{2}\int_0^{2\pi} 1 + \cos(2\theta)\, d\theta$$

$$= -\frac{aR^2}{2}\left(\theta - \tfrac{1}{2}\sin(2\theta)\Big|_0^{2\pi}\right) + \frac{bR^2}{2}\left(\theta + \tfrac{1}{2}\sin(2\theta)\Big|_0^{2\pi}\right)$$

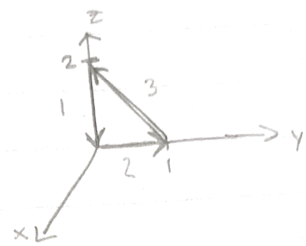
$$= -\frac{aR^2}{\cancel{2}}(\cancel{2}\pi) + \frac{bR^2}{\cancel{2}}(\cancel{2}\pi)$$

$$= -a\pi R^2 + b\pi R^2 = (b-a)\pi R^2$$

$$\boxed{\int (\nabla \times V)\cdot da = \oint V \cdot dI = (b-a)\pi R^2}$$

56. $v = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$

$\int_S (\nabla \times v) \cdot da = \oint v \cdot dl$



$$\nabla \times V = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6 & yz^2 & 3y+z \end{vmatrix} = \left(\frac{\partial}{\partial y}(3y+z) - \frac{\partial}{\partial z}(yz^2)\right)\hat{x} - 0 + 0 = (3-2yz)\hat{x}$$

$$\int_S (\nabla \times v) \cdot da = \int_S (3-2yz)\hat{x} \cdot (dy\,dz)\hat{x} = \int_S (3-2yz)\,dy\,dz$$

$z = ??$ $z = 2(1-y)$

$$= \int_0^{z=2(1-y)} \int_0^y (3-2yz)\,dy\,dz = \int_0^y \left(3z - yz^2 \Big|_0^{2(1-y)}\right) dy$$

$(2-2y)(2-2y) = 4-4y-4y+4y^2$
$= 4-8y+4y^2$

$$= \int_0^y 3(2(1-y)) - y(2(1-y))^2\,dy = \int_0^y 6-6y-4y+8y^2-4y^3\,dy$$

$$= \int_0^y 6-10y+8y^2-4y^3\,dy$$

y is only from 0 to 1 $\quad = \int_0^1 6-10y+8y^2-4y^3\,dy$

$$= \left(6y-5y^2+\frac{8}{3}y^3-y^4 \Big|_0^1\right) = 6-5+\frac{8}{3}-1 = \frac{8}{3}$$

$\oint v \cdot dl$

① Since $x=0$ and $y=0$ $\quad$ z is between 0 to 2

$dl = dz\,\hat{z}$ $\quad \int_2^0 (3y+z)\,dz = \int_2^0 z\,dz = \frac{z^2}{2}\Big|_2^0 = -2$

② Since $x=0$ and $z=0$ $\quad$ y is between 0 to 1

$dl = dy\,\hat{y}$ $\quad \int_0^1 yz^2\,dy = \int_0^1 0\,dy = 0$

(0,0,2)

③ (0,1,0)

$z - z_1 = m(y - y_1)$

$2 - 0 = m(0-1) \quad 2 = m(-1) \quad m = -2$

$z - 0 = -2(y-1) \Rightarrow z = 2(1-y)$ ☆ use this on integral for $\int_s (\nabla \times v) \cdot da$

y is from 0 to 1

$dz = -2dy$

$$\int v \cdot dI = \int (y z^2 dy) + (3y + z\,dz)$$

$$= \int_1^0 4y(1-y)^2 dy - (3y + 2(1-y))2dy$$

$$= \int_1^0 4y(1-2y+y^2)dy \; (-6y - 4 + 4y)dy$$

$$= \int_1^0 4y - 8y^2 + 4y^3 - 6y - 4 + 4y \; dy$$

$$= \int_1^0 2y - 4 - 8y^2 + 4y^3 \; dy$$

$$= \left(y^2 - 4y - \frac{8}{3}y^3 + y^4 \right|_1^0$$

$$= -1 + 4 + \frac{8}{3} - 1$$

$$= 2 + \frac{8}{3} = \frac{14}{3}$$

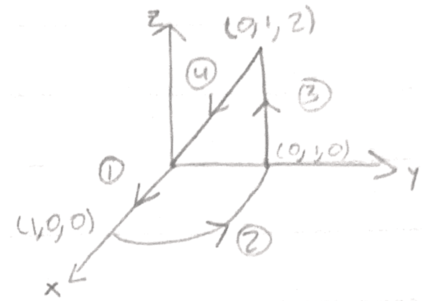
$$\int v \cdot dI = -2 + 0 + \frac{14}{3} = -\frac{6}{3} + 0 + \frac{14}{3}$$

$$= \frac{8}{3}$$

$$\boxed{\int_s (\nabla \times v) \cdot da = \oint v \cdot dI = \frac{8}{3}}$$

57. $v = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}$

$\oint v \cdot dI$



$\theta = \frac{\pi}{2} \quad \phi = 0 \quad r$ from 0 to 1

1 $v \cdot dI = [(r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}] \cdot [dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}]$

$= (r\cos^2\theta)\,dr - (r^2\cos\theta\sin\theta)\,d\theta + (3r^2\sin\theta)\,d\phi$

$= (r\cancel{\cos^2(\tfrac{\pi}{2})})\,dr - (r^2\cancel{\cos(\tfrac{\pi}{2})}\sin(\tfrac{\pi}{2}))\,d\theta + (3r^2\cancel{\sin(\tfrac{\pi}{2})})\,d\phi = 0$

1 $\int v \cdot dI = 0$

2 $v \cdot dI = (r\cos^2\theta)\,dr - (r^2\cos\theta\sin\theta)\,d\theta + (3r^2\sin\theta)\,d\phi$

$r = 1 \quad \theta = \frac{\pi}{2} \quad \phi$ from 0 to $\frac{\pi}{2}$

$= (\cos^2\cancel{(\tfrac{\pi}{2})})\,dr - (\cos\cancel{(\tfrac{\pi}{2})}\sin(\tfrac{\pi}{2}))\,d\theta + (3\sin\cancel{(\tfrac{\pi}{2})})\,d\phi = 3\,d\phi$

$= \int_0^{\pi/2} 3\,d\phi = 3\phi\Big|_0^{\pi/2} = \frac{3\pi}{2}$ 2

3 $v \cdot dI = (r\cos^2\theta)\,dr - (r^2\cos\theta\sin\theta)\,d\theta$

$r = \frac{1}{\sin\theta} \quad dr = -\frac{1}{\sin^2\theta}\cos\theta\,d\theta \quad \phi = \frac{\pi}{2}$

$\theta = \frac{\pi}{2} \quad \theta' = \tan^{-1}(\frac{1}{2}) = 0.46364$

$v \cdot dI = \left(\frac{1}{\sin\theta} \cdot \cos^2\theta\right)\left(-\frac{1}{\sin^2\theta} \cdot \cos\theta\,d\theta\right) - \left(\frac{1}{\sin^2\theta}\cos\theta\sin\theta\right)d\theta$

$= \left(\frac{\cos^2\theta}{\sin\theta}\right)\left(\frac{-\cos\theta}{\sin^2\theta}\,d\theta\right) - \left(\frac{\cos\theta}{\sin\theta}\,d\theta\right)$

$= \left(\frac{-\cos^3\theta}{\sin^3\theta} - \frac{\cos\theta}{\sin\theta}\right)d\theta = \left(\frac{-\cos^3\theta - \cos\theta\sin^2\theta}{\sin^3\theta}\right)d\theta$

$= -\frac{\cos\theta}{\sin\theta}\left(\frac{\overbrace{\cos^2\theta + \sin^2\theta}^{1}}{\sin^2\theta}\right)d\theta = -\frac{\cos\theta}{\sin^3\theta}\,d\theta$

$= \int_{\pi/2}^{0.46364} -\frac{\cos\theta}{\sin^3\theta}\,d\theta \quad \begin{matrix} u = \sin\theta \\ du = \cos\theta\,d\theta \end{matrix} = \int_{\frac{\pi}{2}}^{0.46364} -\frac{1}{u^3}\,du = +\frac{1}{2}u^{-2} = \frac{1}{2\sin^2\theta}\Bigg|_{\frac{\pi}{2}}^{0.46364}$

$= \frac{1}{2\sin^2(0.46364)} - \frac{1}{2\sin^2(\frac{\pi}{2})} = 2$ 3

(11) $v \cdot dI = (r\cos^2\theta)\,dr - (r^2\cos\theta\sin\theta)\,d\theta + (3r^2\sin\theta)\,d\phi$

$\phi = \frac{\pi}{2} \quad \theta = \tan^{-1}\left(\frac{1}{2}\right) \quad r = \sqrt{0^2+1^2+2^2} = \sqrt{5} \quad 0 \text{ to } \sqrt{5}$

$v \cdot dI = r\cos^2\left(\tan^{-1}\left(\frac{1}{2}\right)\right) dr = 0.8r\,dr$

$= \int_{\sqrt{5}}^{0} 0.8r\,dr = \frac{0.8}{2} r^2 \Big|_{\sqrt{5}}^{0} = -2$

$\oint v \cdot dI = 0 + \frac{3\pi}{2} \cancel{+2} \cancel{-2} = \frac{3\pi}{2}$

$\int (\nabla \times V) \cdot da \qquad V = (r\cos^2\theta)\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}$

$$\nabla \times V = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ r\cos^2\theta & -r\cos\theta\sin\theta & 3r \end{vmatrix}$$

$$= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta(3r)) - \cancel{\frac{\partial}{\partial\phi}(r\cos\theta\sin\theta)}\right)\hat{r}$$

$$+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\cos^2\theta) - \frac{\partial}{\partial r}(r(3r))\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r(r\cos\theta\sin\theta)) - \frac{\partial}{\partial\theta}(r\cos^2\theta)\right)\hat{\phi}$$

$$= \frac{1}{\cancel{r}\sin\theta} 3\cancel{r}\cos\theta\,\hat{r} + \frac{1}{\cancel{r}}(-6\cancel{r})\,\hat{\theta} + 0$$

$$= 3\cot\theta\,\hat{r} - 6\hat{\theta}$$

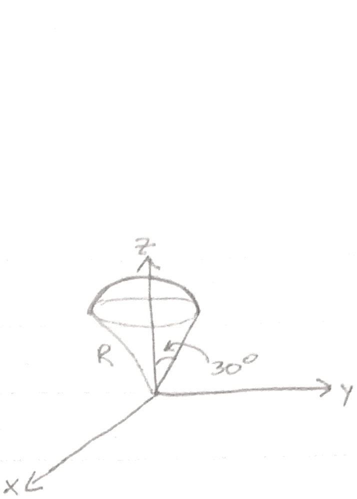
$\int (\nabla \times v) \cdot da \qquad da = -r\,dr\,d\phi\,\hat{\theta}$

$$\int_0^1 \int_0^{\pi/2} 6r\,dr\,d\phi = \int_0^{\pi/2} \left(3r^2 \Big|_0^1\right) d\phi$$

$$= \int_0^{\pi/2} 3\,d\phi = \frac{3\pi}{2}$$

$$\boxed{\int (\nabla \times v) \cdot da = \oint v \cdot dI = \frac{3\pi}{2}}$$

59. $v = \overset{v_r}{r^2\sin\theta}\,\hat{r} + \overset{v_\theta}{4r^2\cos\theta}\,\hat{\theta} + \overset{v_\phi}{r^2\tan\theta}\,\hat{\phi}$



Divergence: $\int (\nabla\cdot v)\,d\tau = \oint v\cdot da$

$$\nabla\cdot V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(v_\phi)$$

$$= \frac{1}{\cancel{r^2}}(4\cancel{r^3}\sin\theta) + \frac{1}{\cancel{r}\sin\theta}(4r^{\cancel{2}}\cos(2\theta)) + \frac{1}{r\sin\theta}(0)$$

$$= \frac{4r\sin^2\theta}{\sin\theta} + \frac{4r\cos(2\theta)}{\sin\theta} = \frac{4r}{\sin\theta}(\cancel{\sin^2\theta} + \cos^2\theta - \cancel{\sin^2\theta})$$

$$= 4r\frac{\cos^2\theta}{\sin\theta} \qquad \int\left(4r\frac{\cos^2\theta}{\sin\theta}\right)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^R\int_0^{\pi/6}\int_0^{2\pi} 4r^3\cos^2\theta\,dr\,d\theta\,d\phi$$

$$\int_0^R 4r^3 = r^4\Big|_0^R = R^4 \qquad \int_0^{2\pi} 1\,d\phi = 2\pi$$

$$= \int_0^{\pi/6}\cos^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/6} 1 + \cos(2\theta)\,d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/6}$$

$$= \frac{1}{2}\left(\frac{\pi}{6} + \frac{1}{2}\sin(\pi/3)\right)$$

$$= \frac{\pi}{12} + \frac{1}{4}\sin(\pi/3)$$

$$\int(\nabla\cdot v)\,d\tau = (R^4)(2\pi)\left(\frac{\pi}{12} + \frac{3}{12}\sin(\pi/3)\right) = \left(\frac{\pi R^4}{12}\right)(2\pi + 3\sin(\pi/3))$$

$$= \left(\frac{\pi R^4}{12}\right)(2\pi + 3\sqrt{3})$$

$\oint v\cdot da$ $\quad da = R^2\sin\theta\,d\phi\,d\theta\,\hat{r}$ $\quad r = R$ $\quad \phi = 0$ to 2π $\quad \theta: 0$ to $\frac{\pi}{6}$

$$v\cdot da = (R^2\sin\theta)R^2\sin\theta\,d\theta\,d\phi$$

$$\int v\cdot da = \int_0^{\pi/6}\int_0^{2\pi} R^4\sin^2\theta\,d\theta\,d\phi \qquad \int_0^{2\pi} R^4\,d\phi = 2\pi R^4 \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

$$= \int_0^{\pi/6}\sin^2\theta\,d\theta = \int_0^{\pi/6}\frac{1}{2} - \frac{\cos(2\theta)}{2}\,d\theta$$

$$\left(\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right)\Big|_0^{\pi/6} = \frac{\pi}{12} - \frac{1}{4}\sin(\pi/3) = \frac{\pi}{12} - \frac{3}{12}\sin(\pi/3)$$

$$(2\pi R^4)\left(\frac{\pi}{12} - \frac{3}{12}\sin(\pi/3)\right) = \left(\frac{\pi R^4}{12}\right)(2\pi - 3\sqrt{3})$$

$\int v \cdot da^{\text{cone}}$ $da = r\sin\theta \, d\phi \, dr \, \hat{\theta}$ $\theta = \frac{\pi}{6}$ $\phi = 0 \text{ to } 2\pi$ $r: 0 \text{ to } R.$

$$v \cdot da = (4r^2\cos\theta)(r\sin\theta \, d\phi \, dr) = 4r^3\cos\theta\sin\theta \, d\phi \, dr$$

$$\int v \cdot da = \int_0^R \int_0^{2\pi} 4r^3 \cos\theta \sin\theta \, d\phi \, dr$$

$$= R^4 \int_0^{2\pi} \cos\theta \sin\theta \, d\phi$$

$$= R^4 (\cancel{2}\pi) \underbrace{\cos(\pi/6)}_{\frac{\sqrt{3}}{2}} \underbrace{\sin(\pi/6)}_{\frac{1}{\cancel{2}}} = \frac{\sqrt{3}\pi R^4}{2}$$

$$\int v \cdot da = \left(\frac{\pi R^4}{12}\right)(2\pi - 3\sqrt{3}) + \frac{\sqrt{3}\pi R^4}{2}$$

$$= \left(\frac{\pi R^4}{12}\right)(2\pi - 3\sqrt{3}) + 6\sqrt{3}\left(\frac{\pi R^4}{12}\right)$$

$$= \left(\frac{\pi R^4}{12}\right)(2\pi - 3\sqrt{3} + 6\sqrt{3})$$

$$= \frac{\pi R^4}{12}(2\pi + 3\sqrt{3})$$

$$\boxed{\int (\nabla \cdot v) \, d\tau = \oint v \cdot da = \frac{\pi R^4}{12}(2\pi + 3\sqrt{3})}$$

62. $a = \int_S da$

a) hemispherical bowl vector area = ??

$$a = \int_S R^2 \sin\theta \cos\theta \, d\theta \, d\phi$$

$$a = \int_0^{2\pi} \int_0^{\pi/2} R^2 \sin\theta \cos\theta \, d\theta \, d\phi$$

$$= 2\pi R^2 \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \quad u = \sin\theta \quad du = \cos\theta \, d\theta$$

$$= 2\pi R^2 \int_0^{\pi/2} u \, du$$

$$= 2\pi R^2 \left(\frac{\sin^2\theta}{2} \right|_0^{\pi/2} = (\cancel{\tfrac{1}{2}})(\cancel{2}\pi R^2)$$

$$\boxed{a = \pi R^2}$$

B) $a = 0$ for any closed surface

Stoke's Theorem : $\oint (\nabla \times v) \cdot da = 0$ for any closed surface

$$\oint_S da = 0 \text{ for any closed surface}$$

$$a = 0$$

c) If $a_1 \neq a_2$ then $\oint da = 0$, for a closed area

$$\oint da = a_1 - a_2 \neq 0$$

D) $a = \frac{1}{2} \oint r \times dI$

$$da = \frac{1}{2} (r \times dI)$$

$$a = \int_S da = \frac{1}{2} \oint r \times dI$$

e) $\oint (c \cdot r) \, dI = a \times c$ Let $T = c \cdot r$

$$\int_S \nabla T \cdot da = - \oint_P T \, dI$$

$T = c \cdot r$

$$\nabla T = \nabla (c \cdot r) = c \times (\overset{0}{\nabla \cancel{\times} r}) + (c \cdot \nabla) r$$

$$(c \cdot \nabla) r = \left(\frac{\partial}{\partial x} c_x + \frac{\partial}{\partial y} c_y + \frac{\partial}{\partial z} c_z \right) (x \hat{x} + y \hat{y} + z \hat{z})$$

$$= c_x \hat{x} + c_y \hat{y} + c_z \hat{z} = C$$

$$\int T \, dl = \oint (c \cdot r) \, dl$$

$$\oint_P T \, dl = \int_S \nabla T \times da$$

$$\oint (c \cdot r) \, dl = - \int_S c \times da$$

$$= - c \times \cancel{\int_S} da = - c \times a$$

$$= \int (c \cdot r) \, dl = - c \times a$$

$$= a \times c$$

63. a) $v = \frac{\hat{r}}{r}$ $\quad v = v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}$

$v_r = \frac{1}{r} \quad v_\theta = 0 \quad v_\phi = 0$

$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^{\cancel{2}} \cdot \frac{1}{\cancel{r}}\right)$

$\boxed{\nabla \cdot v = \frac{1}{r^2} \text{ divergence using 1.84}}$

$\int (\nabla \cdot v)\, d\tau \qquad \nabla \cdot V = \frac{1}{r^2} \qquad d\tau = 4\pi r^2 dr$

$= \int \left(\frac{1}{r^2}\right) 4\pi r^2\, dr$

$= 4\pi \int_0^R dr = 4\pi\left(1\,|_0^R\right) = 4\pi R$

$\int v \cdot da \qquad da = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r} \qquad v = \frac{1}{r}\hat{r} \quad r = R$

$= \int \left(\frac{1}{\cancel{R}}\right)\hat{r} \cdot (R^{\cancel{2}} \sin\theta\, d\theta\, d\phi\, \hat{r})$

$= \int R\sin\theta\, d\theta\, d\phi$

$= \int_0^\pi \int_0^{2\pi} R\sin\theta\, d\theta\, d\phi$

$= 2\pi R \int_0^\pi \sin\theta\, d\theta = 2\pi R\left(-\cos\theta\,\right|_0^\pi = 2\pi R(2) = 4\pi R$

$\boxed{\int (\nabla \cdot v)\, d\tau = \int v \cdot da = 4\pi R \text{ using 1.85}}$

There is no delta function at the origin as the integral was dependent of R.

General formula for divergence ?? $\quad v = r^n \hat{r} \qquad v = v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}$

$v_r = r^n \quad v_\theta = 0 \quad v_\phi = 0$

$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)$

$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n)$

$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^{2+n}) = \frac{1}{r^2}(n+2)r^{n+2-1} = (n+2)r^{n+1-2}$

$= \boxed{(n+2)r^{n-1}}$

general formula

63 B) curl? $r^n \hat{r}$ $\quad V = r^n \hat{r}$ $\quad V = V_r \hat{r} + V_\theta \hat{\theta} + V_\phi \hat{\phi}$ $\quad V_r = r^n \quad V_\theta = 0 \quad V_\phi = 0$

$$\nabla \times V = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ r^n & 0 & 0 \end{vmatrix} = 0 - 0 + 0 = 0$$

$\boxed{\text{curl} \quad \nabla \times V = 0}$

64. a) $D(r, \varepsilon) = -\frac{1}{4\pi} \nabla^2 \frac{1}{\sqrt{r^2 + \varepsilon^2}}$

$$\nabla^2 \frac{1}{\sqrt{r^2+\varepsilon^2}} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \frac{1}{\sqrt{r^2+\varepsilon^2}}\right) + 0 + 0 \qquad (r^2+\varepsilon^2)^{-1/2}$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \frac{-r^3}{(r^2+\varepsilon^2)^{3/2}}$$

$$= \frac{1}{r^2} \frac{\left(-3r^2(r^2+\varepsilon^2)^{3/2} - (-r^3)(3r(r^2+\varepsilon^2)^{1/2})\right)}{(r^2+\varepsilon^2)^3}$$

$$= \frac{-3\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}} \qquad D(r,\varepsilon) = \frac{3\varepsilon^2}{4\pi\,(r^2+\varepsilon^2)^{5/2}}$$

B) $D(0, \varepsilon) \to \infty$ as $\varepsilon \to 0$

The denominator will get bigger as $\varepsilon \to 0$ which will inturn increase $D(0, \varepsilon) \to \infty$.

c) $D(0, \varepsilon) \to 0$ $\quad \varepsilon \to 0$ $\quad r \neq 0$

If $r \neq 0$

D) $\int_0^\infty D(r,\varepsilon)\, 4\pi r^2 dr = 3\varepsilon^2 \int_0^\infty \frac{1}{(r^2+\varepsilon^2)^{5/2}} dr =$