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Quiz 1

Vectors and Scalars

1. Prove that scalar multiplication distributes over vector addition

$$a(\vec{B}+\vec{C}) = a\vec{B} + a\vec{C}$$

$$\vec{B} = (B_x, B_y) \quad \vec{C} = (C_x, C_y)$$

$$a(\vec{B}+\vec{C}) = a[(B_x, B_y) + (C_x, C_y)]$$

$$= a[B_x + C_x, B_y + C_y]$$

$$= aB_x + aC_x, aB_y + aC_y$$

$$= (aB_x, aB_y) + (aC_x, aC_y)$$

$$= a(B_x, B_y) + a(C_x, C_y)$$

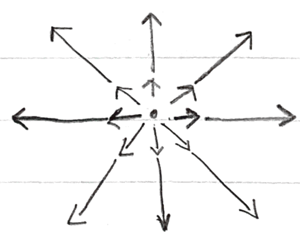
$$= a\vec{B} + a\vec{C}$$

2. $\nabla(f(x,y) + \nabla g(x,y))$

$$\nabla(f+g) = \nabla f + \nabla g \longrightarrow \nabla f(x,y) + \nabla^2 g(x,y)$$

What's wrong is that you are not able to add the gradient of $f(x,y)$ with the Laplacian of $g(x,y)$, since the gradient of $f(x,y)$ is a 1st degree partial derivative, while the Laplacian of $g(x,y)$ is a 2nd degree partial derivative. And it wouldn't make sense to add a 1st degree partial with a 2nd degree partial derivative.

3. a)



$$V_a = r = x\hat{x} + y\hat{y} + z\hat{z}$$

c) Divergence?

$$\nabla \cdot V_a = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

3B) Unit circle $r=1$ on xy plane

$x = r\cos(t)$

$y = r\sin(t)$

$z = 0$

$V_a = x\hat{x} + y\hat{y} + z\hat{z}$

$r(t) = \langle \cos(t), \sin t \rangle$

$v_a = (\cos(t), \sin(t), 0)$

$r'(t) = \langle -\sin(t), \cos(t) \rangle$

$$\int_C v \cdot d\vec{r} = \int_a^b \vec{v}(x(t), y(t)) \cdot \vec{r}'(t)\, dt$$

$$= \int_0^{2\pi} \langle \cos(t), \sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt$$

$$(\cos(t))(-\sin(t)) + (\sin(t))(\cos(t))$$

$$= \int_0^{2\pi} 0 = 0$$

This makes sense as a line integral of 0 means the path is independant. Since the curl is also zero this makes it a conservative (or "irrotational") vector, which a conservative ~~vector~~ field has the property of the line integral being path independent. This makes sense as two why the line integral was zero.

Fundamental Theorems (Stokes Theorem)

1) $\vec{v} = s^{-1}\hat{\phi}$ (s, ϕ, z)

$\int_S (\nabla \times \vec{v}) \cdot d\vec{a}$ "closed sphere"

Corollary 2: $\oint (\nabla \times v) \cdot da = 0$ for any closed surface

so...

$$\boxed{\int_S (\nabla \times \vec{v}) \cdot d\vec{a} = 0}$$

Dirac Delta Functions

$$\int_{-\infty}^{\infty} (f(x) * g(x))\, \delta(x)\, dx \qquad f(x) * g(x) = \left(\frac{f(x) - g(x)}{f(x) + g(x)}\right)$$

1a) $f(x) = \cos(x)$ $g(x) = \sin(x)$

$$\int_{-\infty}^{\infty} \left(\frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)}\right) \delta(x)\, dx = f(0)$$

$$f(0) = \left(\frac{\cos(0) - \sin(0)}{\cos(0) + \sin(0)}\right) = \frac{1-0}{1+0} = 1$$

1b) $f(x) = \cosh(x)$ $g(x) = \sinh(x)$

$$\int_{-\infty}^{\infty} \left(\frac{\cosh(x) - \sinh(x)}{\cosh(x) + \sinh(x)}\right) \delta(x)\, dx = f(0)$$

$$f(0) = \frac{\cosh(0) - \sinh(0)}{\cosh(0) + \sinh(0)} = \frac{1-0}{1+0} = 1$$

1c) $f(x) = a + ax + ax^2 + \ldots$, $g(x) = b + bx + bx^2 + \ldots$

$$\int_{-\infty}^{\infty} \left(\frac{(a + ax + ax^2 + \ldots) - (b + bx + bx^2 + \ldots)}{(a + ax + ax^2 + \ldots) + (b + bx + bx^2 + \ldots)}\right) \delta(x)\, dx = f(0)$$

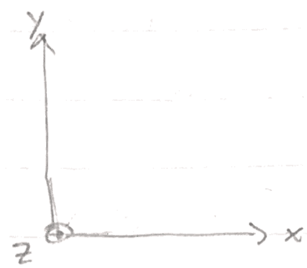
$$f(0) = \frac{(a + a(0) + a(0)^2 + \ldots) - (b + b(0) + b(0)^2 + \ldots)}{(a + a(0) + a(0)^2 + \ldots) + (b + b(0) + b(0)^2 + \ldots)}$$

$$= \frac{a-b}{a+b}$$

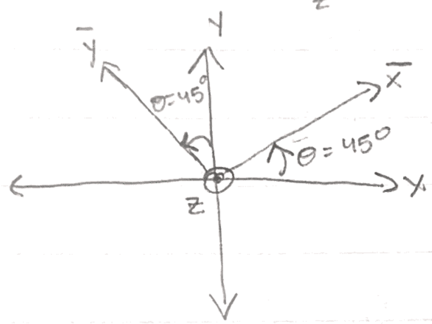
Vector Rotations

1.

A)



$\vec{v} = \hat{\imath} + \hat{\jmath}$



$$\begin{pmatrix} \bar{V}_x \\ \bar{V}_y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

B) $\bar{V}_x = V_x \cos\phi + V_y \sin\phi$

$\bar{V}_y = -V_x \sin\phi + V_y \cos\phi$

C) $\bar{V}_x^2 + \bar{V}_y^2 = V_x^2 + V_y^2$

$\bar{V}_y^2 = (-V_x \sin\phi + V_y \cos\phi)^2 = V_x^2 \sin^2\phi + V_y^2 \cos\phi - 2 V_x V_y \sin\phi \cos\phi$

$\bar{V}_x^2 = (V_x \cos\phi + V_y \sin\phi)^2 = V_x^2 \cos^2\phi + V_y^2 \sin^2\phi + 2 V_x V_y \cos\phi \sin\phi$

$\bar{V}_x^2 + \bar{V}_y^2 = (V_x^2 \sin^2\phi + V_y^2 \cos\phi - \cancel{2 V_x V_y \sin\phi \cos\phi}) + (V_x^2 \cos^2\phi + V_y^2 \sin^2\phi + \cancel{2 V_x V_y \cos\phi \sin\phi})$

$= V_x^2 \cos^2\phi + V_x^2 \sin^2\phi + V_y^2 \cos^2\phi + V_y^2 \sin^2\phi$

$= V_x^2 (\cos^2\phi + \sin^2\phi) + V_y^2 (\cos^2\phi + \sin^2\phi)$

$= V_x^2 + V_y^2$

magnitude is conserved.