Adam

(1.54) $\vec{v} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

$$\nabla\cdot\vec{v} = \frac{4r^3\cos\theta}{r^2} + \frac{r^2\cos\phi\cos\theta}{r\sin\theta} - \frac{r^2\cos\theta\cos\phi}{r\sin\theta}$$

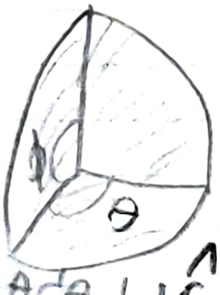
$$= 4r\cos\theta$$

$$\int_0^R\int_0^{\pi/2}\int_0^{\pi/2} 4r\cos\theta\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$\frac{\pi}{2}\int_0^R\int_0^{\pi/2} 4r^3\cos\theta\sin\theta\,dr\,d\theta = \frac{\pi}{2}\left(\frac{\sin^2\theta}{2}\right)\int_0^R 4r^3\,dr$$

$$= \frac{\pi}{2\cdot 2}\cdot R^4 = \boxed{\frac{\pi R^4}{4}}$$

$\int_A \vec{v}\,d\vec{a}$



(i) $d\vec{a} = dl_\theta\, dl_\phi\,\hat{r} = r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$ $\quad r=R$

$$\int_0^{\pi/2}\int_0^{\pi/2} r^4\cos\theta\sin\theta\,d\theta\,d\phi = \frac{\pi}{2}R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta$$

$$= \frac{R^4\pi}{4}\cos^2\theta\Big|_0^{\pi/2} = \boxed{\frac{R^4\pi}{4}}$$

(ii) $\theta = \frac{\pi}{2}$, $d\vec{a} = l\, dr\, r\, d\phi\, \hat{\theta}$, $\vec{v} \cdot d\vec{a} = r\, dr\, d\phi\, \hat{\theta}\, r^2 \cos\phi$

$$\int_0^R \int_0^{\frac{\pi}{2}} r^3 \cos\phi \, dr\, d\phi = \sin(\phi)\Big|_0^{\frac{\pi}{2}} \int_0^R r^3 = \frac{R^4}{4}$$

(iii) $\phi = 0$, $d\vec{a} = l\, dr\, r\, d\theta\, \hat{\phi}$, $\vec{v} \cdot d\vec{a} = -r^3 \cos\theta \sin\phi\, dr\, d\theta$

$\sin(0) = 0$

$$\int_0^R \int_0^{\frac{\pi}{2}} 0 = 0$$

(iv) $\phi = \frac{\pi}{2}$, $d\vec{a} = l\, dr\, r\, d\theta\, \hat{\phi}$, $\vec{v} \cdot d\vec{a} = r^3 \cos\theta \sin\phi\, dr\, d\theta$

$\sin(\frac{\pi}{2}) = 1$

$$\int_0^R \int_0^{\pi/2} r^3 \cos\theta\, dr\, d\theta = \frac{R^4}{4}$$

$$\frac{R^4}{4} - \frac{R^4}{4} + \frac{R^4 \pi}{4} = \boxed{\frac{R^4 \pi}{4}}$$

1.55 Check stokes using $\vec{v} = ay\hat{x} + bx\hat{y}$ and circle of radius R centered at origin in xy plane

$$\int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$v_x = ay$, $V_y = bx$ $V_z = 0$

$$\nabla \times \vec{v} = \hat{x}(0-0) + \hat{y}(0-0) + \hat{z}(b-a)$$

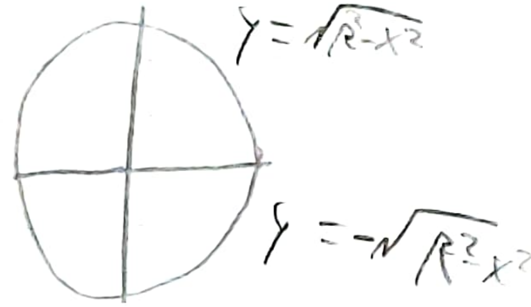
πR^2 area is known from being a circle

$$\int_S \nabla \times \vec{v} \cdot d\vec{a} = \int_0 (b-a)\, d\vec{a} = \boxed{(b-a)\pi R^2}$$

$S = \pi R^2$
$x^2 + y^2 = R^2$
$y^2 = R^2 - x^2$
$y = \sqrt{R^2 - x^2}$

$\oint_P \vec{v} \cdot d\vec{l}$

$P = \_$



(i) $d\vec{l} = dx\,\hat{x}$, $y = 2\sqrt{R^2 - x^2}$

$\vec{v} \cdot d\vec{l} = ay\,dx = a\sqrt{R^2 - x^2}\,dx$

$y = r\sin\theta$

$x = r\cos\theta$

$dx = -r\sin\theta\,d\theta$

$dy = r\cos\theta\,d\theta$

$a\,r\sin\theta(-r\sin\theta\,d\theta)$

$-\int_0^{2\pi} a r^2 \sin^2\theta\,d\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$= -ar^2 \int_0^{2\pi} \frac{1 - \cos 2\theta}{2}\,d\theta$

$= -ar^2 \left[\frac{\theta}{2} - \frac{\cos 2\theta}{4}\right|_0^{2\pi}$

$r = R$

$= -ar^2\left[\frac{0}{2} - \frac{1}{4} - \frac{2\pi}{2} + \frac{1}{4}\right] = -ar^2\pi = -aR^2\pi$

(ii) $d\vec{l} = dy\,\hat{y}$, $x = 2\sqrt{R^2 - y^2} = r\cos\theta$

$\vec{v} \cdot d\vec{l} = bx\,dy = br\cos\theta \cdot r\cos\theta\,d\theta = br^2\cos^2\theta\,d\theta$

$\int_0^{2\pi} br^2\cos^2\theta\,d\theta = br^2 \int_0^{2\pi} \cos^2\theta\,d\theta = \ldots$

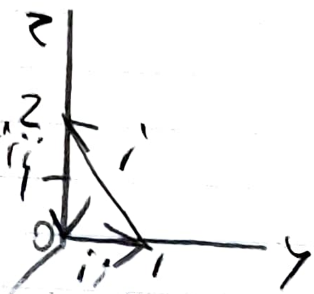
$$= br^2 \int_0^{2\pi} \frac{1+\cos 2\theta}{2} \, d\theta = br^2 \left[\frac{\theta}{2} + \frac{\cos 2\theta}{4} \right]_0^{2\pi}$$

$$= br^2 \left[\frac{2\pi}{2} + \frac{1}{4} - 0 - \frac{1}{4} \right] = br^2\pi \overset{r=R}{=} bR^2\pi$$

$$\oint_P \vec{v} \cdot d\vec{l} = bR^2\pi - aR^2\pi = \boxed{(b-a)R^2\pi}$$

1.56 $\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$

Compute line integral using figure as path. Check with stokes.



$\int \vec{v} \cdot d\vec{l}$

(i) $d\vec{l} = dz\,\hat{z} + dy\,\hat{y}$ $\quad z = -2y$

$dz = -2dy$

$$\vec{v} \cdot d\vec{l} = yz^2\,dy + (3y+z)\,dz = -\frac{z^3}{2} \cdot -\frac{dz}{2} + \left(\frac{-3z}{2} + z\right)dz$$

$$\Rightarrow \int_0^2 \frac{z^3}{4} + \frac{z}{2}\,dz = \left[\frac{z^4}{16} - \frac{z^2}{4} \right]_0^2 = \frac{16}{16} - \frac{4}{4} = 0$$

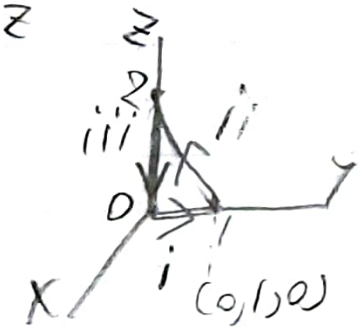
(ii) $d\vec{l} = dy\,\hat{y}$, $\quad x = 0 \quad z = 0$

$\vec{v} \cdot d\vec{l} = yz^2\,dy = 0$

(iii) $d\vec{l} = dz\,\hat{z}$, $y=0$, $x=0$

$$\int_2^0 z\,dz = \frac{z^2}{2}\Big|_2^0 = 0-2 = -2$$

(1.56) $\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$



$\int (\nabla\times\vec{v})\cdot d\vec{a} = \oint \vec{v}\cdot d\vec{l}$

$d\vec{l} = +dy\,\hat{y}$, $z=0$

$\vec{v}\cdot d\vec{l} = y\overset{0}{z^2}\,dy$, $\int \vec{v}\cdot d\vec{l} = 0$

path ii

$z(y) = 2-2y$ $\quad d\vec{l} = -dy\,\hat{y} + dz\,d\hat{z}$

$dz = -2dy$ $\quad \vec{v}\cdot d\vec{l} = (2-2y)^2 y\,dy + (2+y)(-2)dy$

$\int \vec{v}\cdot d\vec{l} = \int_1^0 4y - 4y^2 + 4y^3 - 4 - 2y\,dy$

$= \int_1^0 4y^3 - 4y^2 + 2y - 4 = y^4 - \frac{4y^3}{3} + y^2 - 4y \Big|_0^1$

$-1 + \frac{14}{3}$

(iii) $d\vec{l} = dz\,\hat{z}$ $\quad y=0$

$\int_2^0 (3y+z)\,dz = \int_2^0 z\,dz = \frac{z^2}{2}\Big|_2^0 = -2$

$\frac{14}{3} - \frac{6}{3} = \boxed{\frac{8}{3}}$

$\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$

$\int_S (\nabla \times \vec{v}) \cdot d\vec{a}$

$\nabla \times \vec{v} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)$

$+ \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$

$\hat{x}(3 - 2yz) + \hat{y}(0 - 0) + \hat{z}(0 - 0)$

$= (3 - 2yz)\hat{x}$

$z = 2 - 2y$,
$x = 0$

$\int_0^1 (3 - 2y(2 - 2y))dy$

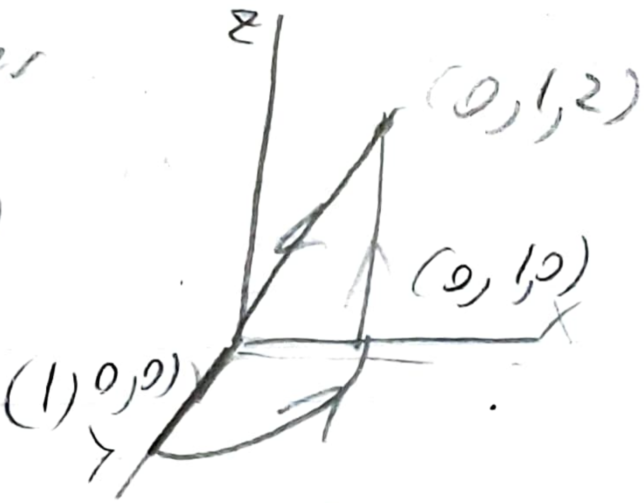
$= \int_0^1 3 - 4y + 4y^2 dy$

$= 3y - 2y^2 + \frac{4y^3}{3}\Big|_0^1$

$= 3 - 2 + \frac{4}{3} = \boxed{\frac{7}{3}}$ eh, close enough

(1.57) $\vec{V} = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}$

cart	sp	cylinder
(1,0,0)		(1,0,0)
(0,1,0)		$(1, \frac{\pi}{2}, 0)$
(0,1,2)		$(\sqrt{5}, \pi/2, 2)$



in cylinder

$r = \rho\sin\theta$ $\phi = \phi$ $z = \rho\cos\theta$

$r\sin\theta = y$

$\vec{V} =$

$r = \frac{1}{\sin\theta}$

+

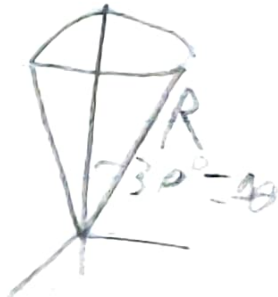
$dr =$

$\cos^2\theta - \sin^2\theta = 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$= \cos^2\theta - 1 + \cos^2\theta = 2\cos^2\theta - 1 = \cos 2\theta$

1.59 $\vec{v} = r^2 \sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r^2\tan\theta\,\hat{\phi}$

ice cream cone



$$\nabla\cdot\vec{v} = \frac{4r^3\sin\theta}{r^2} - \frac{4r^3(1-2\sin^2\theta)}{r\sin\theta}$$

$$= 4r\sin\theta + 8\sin\theta - \frac{4r}{\sin\theta}$$

$$= 12r\sin\theta - \frac{4r}{\sin\theta}$$

$$\int \nabla\cdot\vec{v}\,d\tau = \int_0^R\int_0^{\pi/6}\int_0^{2\pi} r^3(12\sin^2\theta - 4)\,dr\,d\theta\,d\phi$$

$$= 2\pi\int_0^R\int_0^{\pi/6} r^3(12\sin^2\theta - 4)\,dr\,d\theta = 2\pi\int_0^R\int_0^{\pi/6} r^3\left(12\left(\frac{1}{2} - \frac{\cos(2\theta)}{2}\right) - 4\right)dr\,d\theta$$

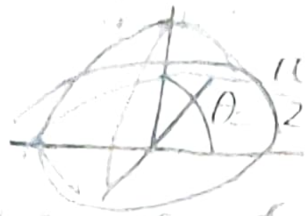
$$= 2\pi\left[\frac{12\theta}{2} - \frac{\quad}{2} - 4\theta\right]_0^{\pi/6}\int_0^R r^3\,dr$$

$$= \frac{\pi R^4}{2}\left[\frac{2\pi}{6} - \frac{3\sqrt{3}}{2}\right] = \boxed{\frac{\pi R^4}{12}\left[2\pi - 9\sqrt{3}\right]}$$

$$= \frac{\pi R^4}{12}\left[2\pi + \frac{3\sqrt{3}}{2}\right]$$

(1.62)

$\vec{a} \equiv \int_S d\vec{a}$



$\frac{\pi}{2}$ ϕ 2π R r

(a) Find vector area of hemispherical bowl

$$\vec{a} = \int_0^R \int_0^{\frac{\pi}{2}} \int_0^{2\pi} r^3 \sin\theta \, dr \, d\theta \, d\phi = 2\pi \int_0^R \int_0^{\pi/2} r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \left(-\cos\theta\right)\Big|_0^{\pi/2} \int_0^R r^2 dr = 2\pi(--1)\frac{r^3}{3}$$

$$= \boxed{\frac{2\pi r^3}{3}}$$

(b) Show $\vec{a} = \vec{0}$ for any closed surface

$$0 = \vec{a} = \oint_S d\vec{a} = \int_0^R \int_0^{\pi} \int_0^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 2\pi \int_0^R \int_0^{\pi} r^2 \sin\theta \, dr \, d\theta \Rightarrow 2\pi \left(\int_0^R \int_0^{\frac{\pi}{2}} r^3 \sin\theta \, dr \, d\theta \right.$$

$$= \left. \int_0^R \int_{\frac{\pi}{2}}^{\pi} r^3 \sin\theta \, dr \, d\theta \right) = 2\pi\left(-\cos\theta\Big|_0^{\frac{\pi}{2}} + \cos\theta\Big|_{\frac{\pi}{2}}^{\pi}\right)$$

$$\int_0^R r^2 dr = \frac{2\pi R^3}{3}\left(-(-1) + -1\right) = \frac{2\pi R^3}{3} \cdot 0 = 0$$

(c) Show $\vec{a}$ is same for all surfaces sharing boundary.

$$\int_0^R \int_0^b \int_0^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi = 2\pi \int_0^R \int_0^b r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \left(-\cos\theta \Big|_0^b \right)$$