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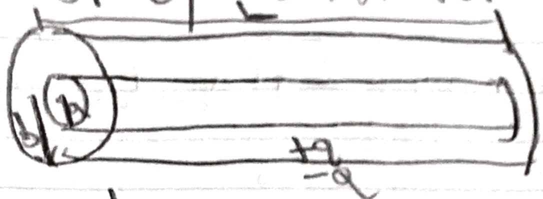
11/16

Homework 3 p 2.43, 2.50, 3.1, 3.3, 3.13–15

2.43) Find capacitance per unit length of two coaxial metal cylindrical tubes, of radii a and b



$E = \cancel{\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}} = \frac{1}{2\pi s\epsilon_0}\hat{s}$ for $a<s<b$

$V = V(a) - V(b) = -\int_b^a \left(\frac{q}{2\pi s\epsilon_0}\right) ds$

$V = \frac{-q}{2\pi\epsilon_0 L}\int_b^a \frac{ds}{s} = \frac{-q}{2\pi\epsilon_0 L}\ln s \Big|_b^a$

$= \frac{-q}{2\pi\epsilon_0 L}(\ln(a) - \ln(b)) = \frac{q}{2\pi\epsilon_0 L}\left(\ln\left(\frac{b}{a}\right)\right)$

$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L \cancel{q}}{\cancel{q}\ln\left(\frac{b}{a}\right)}$ or $\frac{C}{L} = \boxed{\frac{2\pi\epsilon_0}{\ln(b/a)}}$

2.50) electric potential of some configuration is given by expression $V(r) = A\frac{e^{-\lambda r}}{r}$, A and λ constants.

Find $E(r)$, $\rho(r)$ and total charge Q

$E = -\nabla V = -\frac{\partial}{\partial r}\left(A\frac{e^{-\lambda r}}{r}\right) = -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)$ symbolab

$= -A\left(\frac{-\lambda r e^{-\lambda r} - e^{-\lambda r}}{r^2}\right)\hat{r} = \boxed{A e^{-\lambda r}(\lambda r + 1)\frac{\hat{r}}{r^2}}$

$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \quad \rho = (\nabla \cdot E)\varepsilon_0$

$\rho = \varepsilon_0 \left(Ae^{-\lambda r}(1+r\lambda)(\nabla \cdot \frac{\hat{r}}{r^2}) + (\frac{\hat{r}}{r^2}) \cdot \nabla (Ae^{-\lambda r}(1+r\lambda)) \right)$

we know $\nabla \cdot (\frac{\hat{r}}{r^2}) = 4\pi \delta^3(\vec{r})$

$\rho = \varepsilon_0 (Ae^{-\lambda r}(1+r\lambda)(4\pi \delta^3(\vec{r})) + (\frac{\hat{r}}{r^2}) \cdot \nabla (Ae^{-\lambda r}(1+r\lambda)))$

$[e^{-\lambda r}(1+r\lambda)](4\pi \delta^3(r)) = 4\pi \delta^3(r)$

$\rho = \varepsilon_0 (A 4\pi \delta^3(r) + (\frac{\hat{r}}{r^2}) \cdot \nabla (Ae^{-\lambda r}(1+r\lambda)))$

$\nabla (Ae^{-\lambda r}(1+r\lambda)) = \hat{r} \frac{\partial}{\partial r}(Ae^{-\lambda r}(1+r\lambda))$

$= \hat{r}(-r\lambda^2 e^{-\lambda r})A$

$\frac{\hat{r}}{r^2} \cdot [-r\lambda^2 e^{-\lambda r}]A = -\frac{\lambda^2 e^{-\lambda r}}{r} A$

so $\rho = \boxed{\varepsilon_0 [A 4\pi \delta^3(r) - \frac{\lambda^2 A e^{-\lambda r}}{r}]}$

$Q = \int \rho \, d\tau = \int_0^\infty A\varepsilon_0 (4\pi \delta^3(r) - \frac{\lambda^2 e^{-\lambda r}}{r}) \, d\tau$

$= 4\pi\varepsilon_0 A \int (\delta^3(r)) d\tau - A\varepsilon_0 \lambda^2 \int \frac{e^{-\lambda r}}{r} d\tau \qquad d\tau = 4\pi r^2 dr$

$= 4\pi\varepsilon_0 A(1) - A\varepsilon_0 \lambda^2 \int \frac{e^{-\lambda r}}{\cancel{r}} 4\pi r^{\cancel{2}} dr$

$= 4\pi\varepsilon_0 A - 4\pi\varepsilon_0 A \lambda^2 \int r e^{-\lambda r} dr$

$u = r \qquad v = \frac{-e^{-\lambda r}}{\lambda}$

$du = 1 \qquad dv = e^{-\lambda r}$

$u = -kr$
$du = -k$

$kA - kAk^2\left(-\frac{re^{-kr}}{k} + \frac{1}{k}\int e^{-kr}\,du\right)$

$= kA - kAk^2\left(-\frac{re^{-kr}}{k} - \frac{e^{-kr}}{k^2}\right)\Big|_0^{\infty} = kA - \cancel{k}A\cancel{k^2}\left(\frac{1}{\cancel{k^2}}\right)$

$\boxed{Q = kA - kA = 0}$

3.1) find avg potential over spherical surface of radius R due to a pt charge q located inside. Show that in general

$V_{avg} = V_{center} + \frac{Q_{enc}}{4\pi\varepsilon_0 R}$ where V_{center} is potential at the center due all the external charge, Q_{enc} is total charge

$z<R \quad V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ $\qquad r^2 = z^2 + R^2 - 2Rz\cos\theta$
$r = (z^2 + R^2 - 2Rz\cos\theta)^{1/2}$

$V = \frac{kq}{(z^2+R^2-Rz\cos\theta)^{1/2}}$ $\qquad V_{avg} = \frac{1}{4\pi R^2}\oint V\,da$

$V_{ave} = \frac{1}{4\pi R^2}kq\int (z^2+R^2-2Rz\cos\theta)^{-1/2}R^2\sin\theta\,d\theta\,d\phi$

for $z<R$ $\quad (z^2+R^2-2Rz\cos\theta)^{-1/2} \approx (R^2)^{-1/2} = R^{-1}$

So: $V_{avg} = \frac{kq\cancel{R^2}}{4\pi\cancel{R^2}}\int\frac{1}{R}\sin\theta\,d\theta\,d\phi = \boxed{\frac{q}{4\pi\varepsilon_0 R}}$

Value of potential V at pt r is equal to avg value of potential over any spherical surface centered on
$V_{center} = V_{avg}$

If more than 1 charge kept inside sphere potential is $\frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{R}$, Potential due to external charge is V_{center}

$$\boxed{\text{So } V_{avg} = V_{center} + \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{R}}$$

33) Find general soln to laplace eqn in spherical coords, where V depends only on r. Do same for cylindrical coords assuming V depends on s.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0 \text{ since } v \text{ depends on } r$$

$$r^2 \frac{\partial V}{\partial r} = C_0 \text{ then } \partial V = \frac{C_0}{r^2}\partial r$$

$$V = -\frac{C_0}{r} + C_1$$

for cylindrical coords

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$

$$\text{So } s\frac{\partial V}{\partial s} = C_0 \quad \int \partial V = \int \frac{C_0}{s}\partial s$$

$$\boxed{V = C_0 \ln|s| + C_1}$$

3.13) Find potential in infinite slot of Ex 3.3 if boundary at $X=0$ consist of 2 metal strips; one from $0<y<a/2$ is held at constant potential V_0 and the other $a/2<y<a$ at $-V_0$

from ex 3.3, $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$, $V(x,y)=X(x)Y(y)$

$$V(x,y)=\sum_{i=1}^{\infty} C_n e^{-n\pi x/a}\sin\left(\frac{n\pi y}{a}\right)$$

we know $C_n=\frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$

boundary condition for $x=0$, $V(0,y)$

$$V(0,y)=\begin{cases} V_0, & 0<y<a/2 \\ -V_0, & a/2<y<a \end{cases}$$

$$C_n=\frac{2}{a}\left(\int_0^{a/2} V_0 \sin\frac{n\pi y}{a}\,dy-\int_{a/2}^{a} V_0 \sin\frac{n\pi y}{a}\,dy\right)$$

$u=\frac{n\pi y}{a}$ $du=\frac{n\pi}{a}$

$$=\frac{2}{a}\left(V_0\frac{a}{n\pi}\left(-\cos\frac{n\pi y}{a}\Big|_0^{a/2}\right)+V_0\frac{a}{n\pi}\left(\cos\frac{n\pi y}{a}\Big|_{a/2}^{a}\right)\right)$$

$$=\frac{2}{\cancel{a}}\frac{V_0\cancel{a}}{n\pi}\left(-\cos\frac{n\pi}{2}+\cos 0+\left(\underbrace{\cos n\pi}_{-1^n}-\cos\frac{n\pi}{2}\right)\right)$$

$$C_n=\frac{2V_0}{n\pi}\left(1+(-1)^n-2\cos\frac{n\pi}{2}\right)$$

$C_n=0$ for all odd n #'s

and for $n=2$

$$C_n=\frac{\cancel{2}V_0}{\cancel{2}\pi}\left(1+1-2(-1)\right)=\frac{4V_0}{\pi}$$

$n=4 \quad C_n = \frac{V_0}{2\pi}\left(1+1-2\overset{1}{\cos 2\pi}\right)=0$

So $C_n=0$ for $n=4,8,12\ldots$

So for $n=2,6,10\ldots$ $C_n=\frac{8V_0}{n\pi}$

then $\boxed{V(x,y)=\frac{8V_0}{\pi}\sum_{n=2,6\ldots}^{\infty}\frac{e^{-n\pi x/a}\sin(n\pi x/a)}{n}}$

3.14) For ex 3.3, determine charge density $\sigma(y)$ on strip at $x=0$, assuming it is a conductor of constant potential V_0

We know $V(x,y)=\frac{4V_0}{\pi}\sum_{n=1,3,5}\frac{e^{-n\pi x/a}\sin(n\pi y/a)}{n}$

$\sigma(y)=-\varepsilon_0\frac{\partial V}{\partial x} = -\varepsilon_0\frac{4V_0}{\pi}\frac{\partial}{\partial x}\left(\sum\frac{e^{-n\pi x/a}\sin(n\pi y/a)}{n}\right)$

$=-\frac{4\varepsilon_0 V_0}{\pi}\left(\sum\frac{\left(-\frac{n\pi}{a}\right)e^{-n\pi x/a}\sin(n\pi y/a)}{n}\right)$

$\sigma(y)$ at $x=0$ $= \cancel{+}\frac{4\varepsilon_0 V_0}{\cancel{\pi}}\left(\frac{+\cancel{n}\cancel{\pi}}{a}\right)\sum_{n=1,3,\ldots}\frac{\sin(n\pi y/a)}{\cancel{n}}$

$=\boxed{\frac{4\varepsilon_0 V_0}{a}\sum_{n=1,3,\ldots}\sin\left(\frac{n\pi y}{a}\right)}$

3.15) rectangular pipe, running parallel to z-axis (from $-\infty$ to $+\infty$), has 3 grounded metal sides, at $y=0$, a, and $x=0$. Fourth side, at $x=b$ is maintained at specified potential $V_0(y)$

a) develop general formula for potential inside pipe

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Boundary conditions

1) $V=0$ when $y=0$ $V(x,0)$
2) $V=0$ when $y=a$ $V(x,a)$
3) $V=0$ when $x=0$ $V(0,y)$
4) $V=V_0$ when $x=b$ $V(b,y)$

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

apply B.C 1 $0 = (Ae^{kx} + Be^{-kx})D$ $\underline{D=0}$

apply B.C 3 $0 = (A+B)C\sin ky$ $A=-B$, $B=-A$

apply B.C 2 $0 = (Ae^{kx} + -Ae^{-kx})(C\sin ka)$

$A(e^{kx} - e^{-kx})(C\sin ka)$

$0 = 2AC\sinh(kx)(\sin ka)$

$c=0$ when $ka = \frac{n\pi y}{a}$

$$V(x,y) = 2AC\sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

General soln $V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$

apply B.C 4 $V_0(y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$

Then use fourier analysis

$$C_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

$$C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

b) Find potential explicitly for the case $V_0(y) = V_0$ (constant)

$V_0(y) = V_0$

so $C_n = \dfrac{2V_0}{a\sinh(n\pi b/a)} \int_0^a \sin\left(\dfrac{n\pi y}{a}\right) dy$

$u = \dfrac{n\pi y}{a} \quad du = \dfrac{n\pi}{a}$

$$C_n = \frac{2V_0}{\cancel{a}\sinh\left(\frac{n\pi b}{a}\right)} \frac{\cancel{a}}{n\pi}\left(-\cos\frac{n\pi y}{a}\right)\Big|_0^a$$

$$= \frac{2V_0}{n\pi\sinh\left(\frac{n\pi b}{a}\right)}\left(-\cos n\pi + 1\right)$$

$1 - \cos n\pi = 0$ when n is even

$1 - \cos n\pi = 2$ when n is odd

$$C_n = \frac{4V_0}{n\pi \sinh(n\pi b/a)}$$

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,\ldots} \frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}}$$