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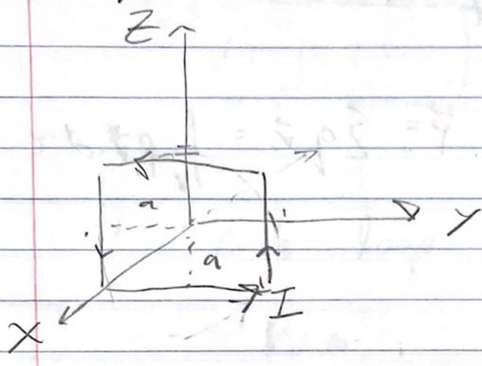
EM HW 5: 5.4, 5.7, 5.11, 5.12, 5.16, 5.19, 5.21, Ex 5.12, 5.23, 5.27

5.4.) $\vec{B} = kz\hat{x}$



$$\vec{F}_{mag} = \int I(d\vec{l} \times \vec{B})$$

The left and right side cancel due to symmetry.

On top $z = a/2 \Rightarrow \vec{B} = k\left(\frac{a}{2}\right)\hat{x}$

So, $\vec{F} = I\int(-d\hat{y}) \times \left(k\frac{a}{2}\hat{x}\right)$

$\vec{F} = -Iak\left(\frac{a}{2}\right)\hat{z}$

On bottom $z = -a/2 \Rightarrow \vec{B} = -k\left(\frac{a}{2}\right)\hat{x}$

$\vec{F} = I\int(d\hat{y}) \times \left(-k\frac{a}{2}\hat{x}\right)$

$\vec{F} = Iak\left(\frac{a}{2}\right)\hat{z}$

So $\boxed{\vec{F}_{total} = Ia^2k\hat{z}}$

5.7.) Show $\int_V \vec{J}\, d\tau = d\vec{p}/dt$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \qquad \vec{p} = \sum q\vec{r} = \int_V \rho \vec{r}\, d\tau$$

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\int_V \rho \vec{r}\, d\tau$$

$$= \int_V \frac{d\rho}{dt}\vec{r}\, d\tau$$

$$= -\int_V (\nabla \cdot \vec{J})\vec{r}\, d\tau$$

$$\nabla \cdot x\vec{J} = x(\nabla \cdot \vec{J}) + \vec{J}\cdot(\nabla x)$$

$$\hookrightarrow \vec{J}\cdot\hat{x} = J_x$$

So $\quad -x(\nabla \cdot \vec{J}) = J_x - \nabla \cdot x\vec{J}$

Plugging into integral

$$= \int_V J_x\, d\tau - \int_V (\nabla \cdot x\vec{J})\, d\tau$$

$$\hookrightarrow = \int_S x\vec{J}\cdot d\vec{a} = 0$$

since charges within volume.

$$= \int_V J_x\, d\tau$$

doing the same for y and z

$$-\int (\nabla \cdot \vec{J})\vec{r}\, d\tau = \boxed{\int \vec{J}\, d\tau = \frac{d\vec{p}}{dt}}$$

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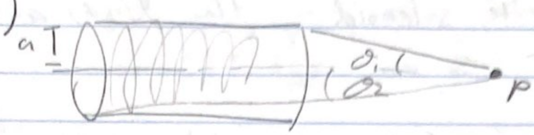
5.11.)



For each loop $dB(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}$

$R = a$

For n loops $dB = \frac{\mu_0 n I}{2} \frac{a^2}{(a^2+z^2)^{3/2}}$

adding up all contributions over interval

$$B = \frac{\mu_0 n I}{2} \int \frac{a^2}{(a^2+z^2)^{3/2}} dz$$

now $\tan\theta = \frac{a}{z} \Rightarrow z = \frac{a}{\tan\theta} = a\cot\theta$

$\Rightarrow dz = -a\csc^2\theta = \frac{-a}{\sin^2\theta}$

$$\frac{a^2}{(a^2+a^2\cot^2\theta)^{3/2}} = \left(\frac{\cancel{a^2}}{a^{\cancel{3}}}\right) \frac{1}{(1+\cot^2\theta)^{3/2}} \frac{(\sin^2)^{3/2}}{(\sin^2)^{3/2}}$$

$$= \frac{1}{a} \frac{\sin^3\theta}{(\sin^2+\cos^2)^{3/2}} = \frac{\sin^3\theta}{a}$$

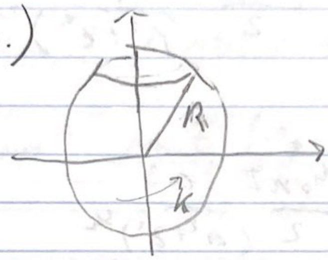
$$\Rightarrow B = \frac{\mu_0 n I}{2} \int_{\theta_1}^{\theta_2} -\frac{\sin^{\cancel{3}}\theta}{\cancel{a}} \frac{\cancel{a}}{\sin^{\cancel{2}}\theta} d\theta$$

$$= \frac{\mu_0 n I}{2} - \int_{\theta_1}^{\theta_2} \sin\theta\, d\theta = \boxed{\frac{\mu_0 n I}{2}(\cos\theta_1 - \cos\theta_2)}$$

5.11.) for an infinite solenoid the limits are 0 and π

$$\Rightarrow \quad B = \frac{\mu_0 n I}{2}\left(\cos(0) - \cos(\pi)\right) = \underline{\underline{\mu_0 n I}}$$

5.12.)



$$dB = \frac{\mu_0 \, dI}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$$

$$R = R\sin\theta$$
$$z = R\cos\theta$$

$$\Rightarrow dB = \frac{\mu_0 \, dI}{2} \frac{(R\sin\theta)^2}{(R^2\sin^2\theta + R^2\cos^2\theta)^{3/2}} = \frac{\mu_0 dI}{2} \frac{\cancel{R^2}\sin^2\theta}{R^{\cancel{3}\,1}}$$

$$dB = \frac{\mu_0}{2} \frac{\sin^2\theta}{R} \, dI$$

In this case $dI = K R \, d\theta$, $K = \sigma v$

$$v = \omega R \sin\theta$$

$$Q = 4\pi R^2 \sigma \Rightarrow \sigma = \frac{Q}{4\pi R^2}$$

$$\Rightarrow dI = \frac{Q}{4\pi \cancel{R^2}} \omega R \sin\theta R \, d\theta$$

$$= \frac{\omega}{4\pi} Q \sin\theta \, d\theta$$

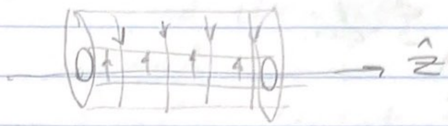
$$B = \frac{\mu_0}{2R} \frac{Q\omega}{4\pi} \int_0^{\pi} \sin^2\theta \sin\theta \, d\theta = \frac{\mu_0 Q \omega}{8\pi R}\left(\frac{4}{3}\right)$$

$$B = \frac{\mu_0 Q \omega}{6\pi R}$$

Say spinning in $\hat{z}$

$$\boxed{\vec{B} = \frac{\mu_0 Q \omega}{6\pi R} \hat{z}}$$

5.16.)



i.) inside both

$\vec{B}_{outer} = +\mu_0 n_2 I \; \hat{z}$

$\vec{B}_{inner} = -\mu_0 n_1 I \hat{z}$

$\boxed{\vec{B}_{total} = \mu_0 I (n_2 - n_1) \hat{z}}$

ii.) inbetween

$\vec{B}_{inner} = 0$

$\vec{B}_{outer} = +\mu_0 n_2 I \hat{z}$

$\therefore \boxed{\vec{B}_{total} = \mu_0 n_2 I \hat{z}}$

iii.) The field is zero outside solenoids $\Rightarrow$

$\vec{B}_{total} = 0.$

5.19.) $I_{enc} = \int_S \vec{J} \cdot d\vec{a}$

$\nabla \cdot \vec{J} = 0$

For solenoidal fields, the surface does not matter

5.21) $\nabla \times \vec{B} = \mu_0 \vec{J}$

Now the continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 (\nabla \cdot \vec{J}) = -\mu_0 \frac{\partial \rho}{\partial t}$$

but $\frac{\partial \rho}{\partial t}$ = zero since steady currents.

So only valid if ρ is constant!

Ex 5.12) Find the vector potential of an infinite solenoid with n turns per unit length, radius R, current I.

Note $\oint \vec{A}\cdot d\vec{l} = \int (\nabla\times\vec{A})\cdot d\vec{a} = \int \vec{B}\cdot d\vec{a} = \Phi$

flux of $\vec{B}$ through loop

So, $\oint \vec{A}\cdot d\vec{l} = \int \vec{B}\cdot d\vec{a}$

For an amperian loop of radius s within solenoid

$$A(2\pi s) = \underbrace{B}_{=\mu_0 n I \text{ inside solenoid}}(\pi s^2)$$

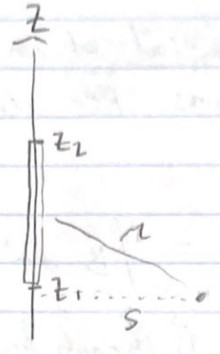
$$\Rightarrow \boxed{A = \frac{\mu n I s}{2}\hat{\phi} \quad \text{for } s \le R}$$

For a loop outside...

$$\int \vec{B}\cdot d\vec{a} = \mu_0 n I(\pi R^2) \text{ since field only goes out to } R$$

$$\text{Thus } \boxed{A = \frac{\mu_0 n I}{2}\frac{R^2}{s}\hat{\phi} \quad \text{for } s \ge R}$$

5.23.)



$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{\mathscr{r}}\,dl'$$

$$\vec{I} = I\hat{z}$$

$$\mathscr{r} = \sqrt{z^2+s^2}$$

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} I \int_{z_1}^{z_2} \frac{1}{\sqrt{z^2+s^2}}\,dz\,\hat{z}$$

integral looked up

$$= \frac{\mu_0}{4\pi} I \left[\ln\left(z+\sqrt{z^2+s^2}\right)\right]_{z_1}^{z_2}\hat{z}$$

$$= \frac{\mu_0}{4\pi} I \left(\ln\left(z_2+\sqrt{z_2^2+s^2}\right) - \ln\left(z_1+\sqrt{z_1^2+s^2}\right)\right)\hat{z}$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} I \ln\left(\frac{z_2+\sqrt{z_2^2+s^2}}{z_1+\sqrt{z_1^2+s^2}}\right)\hat{z}}$$

$$B = \nabla\times\vec{A} = -\frac{\partial A}{\partial s}\hat{\phi}$$ ← only one that exists!

$$= -\frac{\mu_0 I}{4\pi}\frac{\partial}{\partial s}\left(\ln\left(z_2+\sqrt{z_2^2+s^2}\right) - \ln\left(z_1+\sqrt{z_1^2+s^2}\right)\right)$$

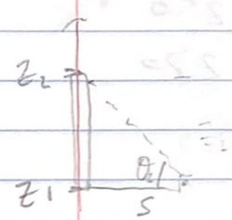
$$= \frac{-\mu_0 I}{4\pi}\left[\frac{1}{z_2+\sqrt{z_2^2+s^2}}\,\frac{1}{\cancel{2}\sqrt{z_2^2+s^2}}(\cancel{2}s) - \frac{1}{z_1+\sqrt{z_1^2+s^2}}\,\frac{1}{\cancel{2}\sqrt{z_1^2+s^2}}(\cancel{2}s)\right]$$

$$= \frac{-\mu_0 I}{4\pi}\left[\frac{z_2-\sqrt{z_2^2+s^2}}{\cancel{z_2^2}-\cancel{z_2^2}+s^2}\,\frac{s}{\sqrt{z_2^2+s^2}} - \frac{z_1-\sqrt{z_1^2+s^2}}{\cancel{z_1^2}-\cancel{z_1^2}+s^2}\,\frac{s}{\sqrt{z_1^2+s^2}}\right]$$

5.23.) $$= \frac{-\mu_0 I}{4\pi}\left[\frac{\cancel{s}z_2 - \cancel{s}\sqrt{z_2^2+s^2}}{s^{\cancel{2}}\sqrt{z_2^2+s^2}} - \frac{\cancel{s}z_1 - \cancel{s}\sqrt{z_1^2+s^2}}{s^{\cancel{2}}\sqrt{z_1^2+s^2}}\right]$$

$$= \frac{\cancel{-}\mu_0 I}{4\pi}\left(\frac{\cancel{-}1}{s}\right)\left[\frac{z_2}{\sqrt{z_2^2+s^2}} + \cancel{1} - \frac{z_1}{\sqrt{z_1^2+s^2}} - \cancel{1}\right]$$

$$= \frac{\mu_0 I}{4\pi}\frac{1}{s}\left[\underbrace{\frac{z_2}{\sqrt{z_2^2+s^2}}}_{\sin(\theta_2)} - \underbrace{\frac{z_1}{\sqrt{z_1^2+s^2}}}_{\sin(\theta_1)}\right]$$



$$= \boxed{\frac{\mu_0 I}{4\pi}\frac{1}{s}\left(\sin\theta_2 - \sin\theta_1\right)}$$

Eq 5.37 ✓

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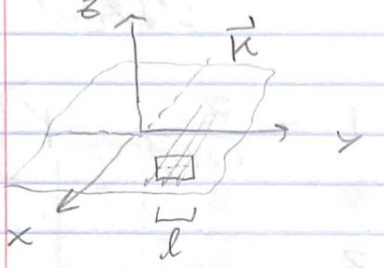
5.27.) Find vector potential $\vec{A}$.



$$\vec{K} = K\hat{x}$$

From ex 5.8

$$\vec{B} = \begin{cases} +(\mu_0/2)K\hat{y} & \text{for } z<0 \\ -(\mu_0/2)K\hat{y} & \text{for } z>0 \end{cases}$$

For surface currents

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{r}\, da'$$

Thus $\vec{A}$ is parallel to $\vec{K}$.

Since sheet of current is infinite in xy, A can only possibly depend on z.

Thus $\vec{A} = A(z)\hat{x}$

$$\Rightarrow \vec{B} = \nabla \times \vec{A} = \begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A(z) & 0 & 0 \end{vmatrix} = +\frac{\partial A(z)}{\partial z}\hat{y} = \frac{\mu_0 K}{2}\hat{y}$$

all constants, so

$$\boxed{\vec{A} = -\frac{\mu_0 K}{2} z\,\hat{x}}$$