Electromagnetc Theory: PHYS330

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Summary

Week 4 Summary

1. Atoms, polarizations, and dipole moments
2. $\vec{P}$, dipole per unit volume, and bound charges
3. $\vec{D}$, the electric displacement
4. Linear dielectrics

Dipole Moment, and The Electric Field of a Dipole

From the multipole expansion, the monopole and dipole terms are (asynchronous video content):

$$V(r,\theta)_{n=0} = \frac{1}{4\pi\epsilon_0}\frac{\int_{all}\rho(\vec{r'})d\tau'}{r} \tag{1}$$

$$V(r,\theta)_{n=1} = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\int_{all}\vec{r'}\rho(\vec{r'})d\tau'}{r^2} \tag{2}$$

$$\vec{p} = \int_{all}\vec{r'}\rho(\vec{r'})d\tau' \tag{3}$$

Valid for any charge distribution, useful for getting far-field E-fields:

$$\boxed{\vec{p} = \int_{all} \vec{r'} \rho(\vec{r'}) d\tau'} \tag{4}$$

1. Discrete charge distribution example: asynch. video content
2. Continuous: “similar” to calculating the moment of inertia of a mass distribution (only it’s linear vector, not r'^2).

Dipole Moment, and The Electric Field of a Dipole

For a dipole, with no other charges ($\vec{p} = q\vec{d}$):

$$V_{n=1}(r, \theta) = \frac{\hat{r} \cdot \vec{p}}{4\pi\epsilon_0 r^2} \tag{5}$$

1. Using Eq. 5, calculate the electric field of a dipole in spherical coordinates.
2. Are there any regions for which the E-field is zero?
3. What happens when you set $\partial E/\partial\theta = 0$?
4. Recall that the energy density of a field is $u_E = \frac{1}{2}\epsilon_0 \vec{E} \cdot \vec{E}$. For a given radius R, is the energy density higher at $\theta = 0$ or $\theta = \pi/2$?

Breakout rooms.

Atoms, polarizations, and dipole moments

Atoms, polarizations, and dipole moments

Suppose an external field $\vec{E}$ induces a dipole moment $\vec{p}$ in an atomic charge distribution:

$$\boxed{\vec{p} = \alpha \vec{E}} \tag{6}$$

This statement is empirical, but it's true for "ordinary" field strengths: field isn't strong enough to ionize the atom.

What is the electric field a distance d from the center of a uniformly charged sphere of radius a? *[Hint: use Gauss' law, and assume ρ is constant in spherical coordinates].*

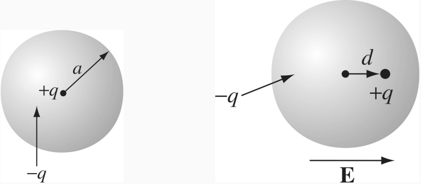

Figure 1: A simple atom-like charge distribution in (left) no external field, and (right) a uniform external field.

Atoms, polarizations, and dipole moments

Result:

$$E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3} \tag{7}$$

But then assume that $p = qd$, so

$$\alpha = 4\pi\epsilon_0 a^3 \tag{8}$$

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.67	0.396	24.1	1.64	43.4	59.4

Figure 2: Do you understand the units of this table? The numbers are quoted as $\alpha/4\pi\epsilon_0$, *in units of* 10^{-30} m^3. *What would they be in namometers cubed?*

The trouble is that we cannot easily measure the volume of an atom (realm of quantum mechanics).

Molecules can also have a *permanent* dipole moment: polar molecules.

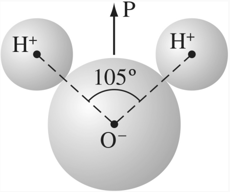

Figure 3: How would you calculate the dipole moment here?

Show that the torque on such a molecule in an external field is $\vec{\tau} = \vec{p} \times \vec{E}$ (Professor example).

Atoms, polarizations, and dipole moments

If you have (approximately) aligned polar molecules with an external field $\vec{E} = E_0\hat{x}$, and then *reverse* the direction of the field, in what direction is the torque? Assume the dipole moments are in the xy-plane.

- A: $-\hat{z}$
- B: $\hat{y}$
- C: $\hat{z}$
- D: The torque is zero

Atoms, polarizations, and dipole moments

In summary, there are two reasons there could be dipole moments within a material:

1. The atoms are *stretched* and you get an $\alpha = 4\pi\epsilon_0 a^3$
2. The atoms or molecules are *rotated* and you get a dipole moment $\vec{p}$ per atom/molecule.

Macroscopically, it is easier to demonstrate the polar molecule effect: `https://youtu.be/riMrg_kO__w`

$\vec{P}$, dipole per unit volume, and bound charges

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We need to understand the field of a polarized material.
Suppose we introduce the *dipole moment per unit volume*, $\vec{P}$.

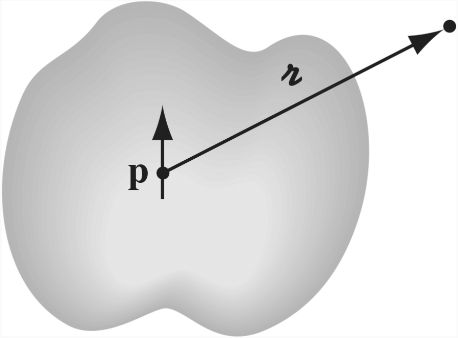

Figure 4: The definition of the dipole moment per unit volume, and geometry.

$\vec{P}$, dipole per unit volume, and bound charges

This implies that

$$\vec{p} = \vec{P} d\tau' \tag{9}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \tag{10}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\vec{P}(\vec{r'}) \cdot \hat{\boldsymbol{\mathscr{r}}}\, d\tau'}{\mathscr{r}^2} \tag{11}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \vec{P}(\vec{r'}) \cdot \nabla \left(\frac{1}{\mathscr{r}} \right) d\tau' \tag{12}$$

$$\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla(f) \tag{13}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \left\{ \int_{\mathcal{V}} \nabla' \cdot \left(\frac{\vec{P}}{\mathscr{r}} \right) d\tau' - \int_{\mathcal{V}} \frac{1}{\mathscr{r}} \left(\nabla' \cdot \vec{P} \right) d\tau' \right\} \tag{14}$$

$\vec{P}$, dipole per unit volume, and bound charges

$$V(r) = \frac{1}{4\pi\epsilon_0}\left\{\int_{\mathcal{V}} \nabla' \cdot \left(\frac{\vec{P}}{\mathscr{r}}\right) d\tau' - \int_{\mathcal{V}} \frac{1}{\mathscr{r}} \left(\nabla' \cdot \vec{P}\right) d\tau'\right\} \tag{15}$$

$$V(r) = \frac{1}{4\pi\epsilon_0}\left\{\oint_{\mathcal{S}} \left(\frac{\vec{P}}{\mathscr{r}}\right) \cdot d\vec{a}' - \int_{\mathcal{V}} \frac{1}{\mathscr{r}} \left(\nabla' \cdot \vec{P}\right) d\tau'\right\} \tag{16}$$

$$d\vec{a}' = da' \hat{n} \tag{17}$$

$$\sigma_b = \vec{P} \cdot \hat{n} \tag{18}$$

$$\rho_b = -\nabla \cdot \vec{P} \tag{19}$$

$$V(r) = \frac{1}{4\pi\epsilon_0}\left\{\oint_{\mathcal{S}} \frac{\sigma_b}{\mathscr{r}} da' - \int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}} d\tau'\right\} \tag{20}$$

The appearance of *bound charge*.

$\vec{P}$, dipole per unit volume, and bound charges

Suppose we have a sphere with uniform polarization in the z-direction (and it is constant). The ρ_b is zero because

- A: There is no bound charge inside a sphere.
- B: The divergence of a constant is zero.
- C: By symmetry.
- D: Otherwise the integral over ρ_b would diverge.

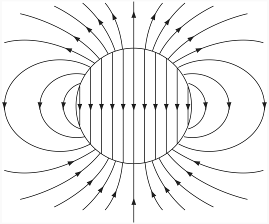

Figure 5: The uniformly polarized sphere. $\sigma_b = \vec{P} \cdot \hat{n} = P\cos\theta$.

$\vec{P}$, dipole per unit volume, and bound charges

Think for a moment: in your own words, why do the field lines point in *opposite* directions just inside and just outside the surface of the sphere?

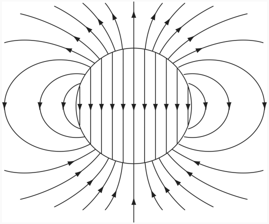

Figure 6: The uniformly polarized sphere. $\sigma_b = \vec{P} \cdot \hat{n} = P \cos\theta$.

$\vec{P}$, dipole per unit volume, and bound charges

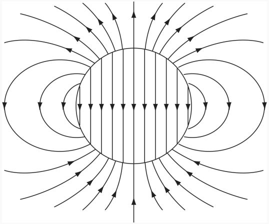

Figure 7: The uniformly polarized sphere. $\sigma_b = \vec{P} \cdot \hat{n} = P \cos\theta$.

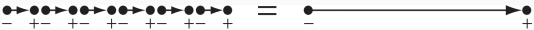

Figure 8: Whenever you think of bound charge density versus surface charge density, think of this picture.

$\vec{P}$, dipole per unit volume, and bound charges

Conceptual question: Given Eq. 20, what is the potential due a disk of surface bound charge σ_b at a point slightly above the surface? *[Hint: if it helps, think of $\sigma_b = \vec{P_0} \cdot \hat{z}$, where P_0 is a constant.]*

$\vec{D}$, the electric displacement

Conclusion

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