1.54 $\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2 \cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r^2 \sin\phi)$

$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$

$= \frac{1}{r^2} 4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta\, r^2\cos\phi + \frac{1}{r\sin\theta}(-r^2\cos\theta\cos\phi)$

$= \frac{r\cos\theta}{\sin\theta}(4\sin\theta + \cos\phi + \cos\phi) = 4r\cos\theta$

$\int (4r\cos\theta) r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\underbrace{\int_0^R r^3\, dr}_{R^4} \underbrace{\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta}_{\frac{1}{2}} \underbrace{\int_0^{\pi/2} d\phi}_{\pi/2}$

$= R^4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{R^4\pi}{4}$

1.55 $\nabla \times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ay & bx & 0 \end{vmatrix} = 0-0\,\hat{x} - 0-0\,\hat{y} + (b-a)\hat{z}$

$x^2 + y^2 = R^2$

$dy = -(x/y)\,dx$

$v \cdot dl = \frac{1}{y}(ay^2 - bx^2)\,dx$

$y = \sqrt{R^2 - x^2}$

$\int_R^{-R} \frac{aR^2 - (a+b)x^2}{\sqrt{R^2 - x^2}}\, dx = \frac{1}{2}R^2(a-b)\sin^{-1}(x/R)\Big|_R^{-R}$

$= \frac{1}{2}R^2(a-b)\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = \frac{1}{2}\pi R^2(b-a)$

1.56 i) $x = z = 0$ $\quad v \cdot dl = (yz^2)\,dy = 0 \quad \int v \cdot dl = 0$

ii) $x = 0 \quad z = 2 - 2y \quad dz = -2dy \quad v \cdot dl = (yz^2)\,dy + (3y + z)\,dz$

$2\int_1^0 (2y^3 - 4y^2 + y - 2)\,dy = 2\left(\frac{y^4}{2} - \frac{4y^3}{3} + \frac{y^2}{2} - 2y\right)\Big|_1^0 = \frac{14}{3}$

iii) $x = y = 0 \quad dx = dy = 0 \quad v \cdot dl = (3y + z)\,dz$

$\int_2^0 z\,dz = \frac{z^2}{2}\Big|_2^0 = -2$

$\oint v \cdot dl = 0 + \frac{14}{3} - 2 = \frac{8}{3}$

$\nabla \times v \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6 & yz^2 & 3y+z \end{vmatrix} = (3 - 2yz)\hat{x} - 0-0\,\hat{y} + 0-0\,\hat{z}$

$\int_0^1 \int_0^{2-2y} (3 - 2yz)\,dy\,dz = \int_0^1 (-4y^3 + 8y^2 - 10y + 6)\,dy$

$= (-y^4 + \frac{8}{3}y^3 - 5y^2 + 6y)\Big|_0^1 = -1 + \frac{8}{3} - 5 + 6 = \frac{8}{3}$

1.57 $\quad v = (r\cos^2\theta)\,\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3r\,\hat{\phi}$

i) $\theta = \frac{\pi}{2}$ $\phi = 0$ $\quad v\cdot dl = (r\cos^2\theta)\,dr = 0 \quad \int v\cdot dl = 0$

ii) $r = 1$ $\theta = \frac{\pi}{2}$ $\quad v\cdot dl = 3r\,(r\sin\theta\, d\phi) = 3\,d\phi \quad 3\int_0^{\pi/2} d\phi = \frac{3\pi}{2}$

iii) $\theta = \theta_0$ $\phi = \frac{\pi}{2}$ $\quad v\cdot dl = (r\cos^2\theta)\,dr = \frac{4}{5} r\,dr$

$$\int_{\sqrt{5}}^{0} \frac{4}{5} r\,dr = \frac{4}{5}\frac{r^2}{2}\Big|_{\sqrt{5}}^{0} = -2$$

iiii) $\phi = \frac{\pi}{2}$ $\quad r = \frac{1}{\sin\theta}$ $\quad v\cdot dl = (r\cos^2\theta)\,dr - (r\cos\theta\sin\theta)(r\,d\theta) \quad dr = \frac{1}{\sin^2\theta}\cos\theta\,d\theta$

$$v\cdot dl = -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right)d\theta = -\frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)d\theta$$

$$= -\frac{\cos\theta}{\sin^3\theta}\,d\theta$$

$$\int_{\pi/2}^{\theta_0} \frac{\cos\theta}{\sin^3\theta}\,d\theta = \frac{1}{2\sin^2\theta}\Big|_{\pi/2}^{\theta_0} = \frac{5}{2} - \frac{1}{2} = 2$$

$$\oint v\cdot dl = 0 + \frac{3\pi}{2} - 2 + 2 = \frac{3\pi}{2}$$

$$\nabla\times v \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ r\cos^2\theta & -r\cos\theta\sin\theta & 3r \end{vmatrix} = \frac{1}{r\sin\theta}(3r\cos\theta)\,\hat{r} + -6\,\hat{\theta} + \frac{1}{r}\big(\underbrace{\cancel{-2r\cos\theta\sin\theta + 2r\cos\theta\sin\theta}}_{0}\big)\,\hat{\phi}$$

$$= 3\cot\theta\,\hat{r} - 6\,\hat{\theta}$$

$da = -r\sin\theta\, dr\, d\phi\,\hat{\theta}$ $\quad \theta = \pi/2 \quad (\nabla\times v)\cdot da = 6r\,dr\,d\phi$

$$\int_0^1 6r\,dr \int_0^{\pi/2} d\phi = 6\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{2}$$

1.59 $\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2 \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, 4r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(r^2 \tan\theta)$

$= \frac{1}{r^2}4r^3\sin\theta + \frac{1}{r\sin\theta}4r^2(\cos^2\theta - \sin^2\theta) = 4r\frac{\cos^2\theta}{\sin\theta}$

$\int 4r\frac{\cos^2\theta}{\sin\theta}(r^2\sin\theta)\,dr\,d\theta\,d\phi = \int_0^R 4r^3\,dr \int_0^{\pi/6}\cos^2\theta\,d\theta \int_0^{2\pi} d\phi$

$= 2\pi R^4\left(\frac{\pi}{12} + \frac{\sin 60°}{4}\right) = \frac{\pi R^4}{12}(2\pi + 3\sqrt{3})$

i) $r = R$ $\quad da = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$ $\quad v\cdot da = R^4\sin^2\theta\,d\theta\,d\phi$

$R^4\int_0^{\pi/6}\sin^2\theta\,d\theta\int_0^{2\pi}d\phi = 2\pi R^4\left(\frac{\pi}{12} - \frac{1}{4}\sin 60°\right) = \frac{\pi R^4}{6}\left(\pi - 3\frac{\sqrt{3}}{2}\right)$

ii) $\theta = \frac{\pi}{6}$ $\quad da = r\sin\theta\,d\phi\,dr\,\hat{\theta}$ $\quad v\cdot da = \sqrt{3}\,r^3\,dr\,d\phi$

$\sqrt{3}\int_0^R r^3\,dr\int_0^{2\pi}d\phi = \sqrt{3}\cdot\frac{R^4}{4}\cdot 2\pi = \frac{\sqrt{3}}{2}\pi R^4$

$\frac{\pi R^4}{6}\left(\pi - 3\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\pi R^4 = \frac{\pi R^4}{12}(2\pi + 3\sqrt{3})$

1.62 a) $da = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$ $\quad \int R^2\sin\theta\cos\theta\,d\theta\,d\phi\,\hat{z}$

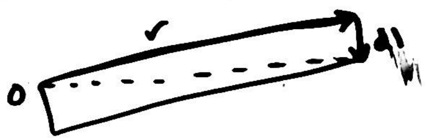
$2\pi R^2\hat{z}\int_0^{\pi/2}\cancel{\sin\theta\cos\theta\,d\theta}^{\,1/2} = \pi R^2\hat{z}$

b) $T = 1$ then $\nabla T = 0$ so $\int \nabla T\,d\tau = \oint T\,da$ is

$\int \cancelto{0}{\nabla}\,d\tau = \int da$ so $\int da = 0$

c) $a_1 \neq a_2$ $\quad \oint da = a_1 - a_2 \neq 0$

d) $\frac{1}{2}r \times dl$



$a = \frac{1}{2}\oint r \times dl$

e) $T = c\cdot r$ $\quad \nabla T = c\times(\nabla\times r) + (c\cdot\nabla)r = (c\cdot\nabla)r$

$= (c_x\partial_x + c_y\partial_y + c_z\partial_z)r = c_x\hat{x} + c_y\hat{y} + c_z\hat{z} = c$

$\oint T\,dl = \oint (c\cdot r)\,dl = -\int_s \nabla T \times da = -\int_s c\times da$

$= -c\times a = S\times a$

1.63 a) $\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot \frac{1}{r}) = \frac{1}{r^2}$

$\int v \cdot da = R \int \sin\theta d\theta d\phi = 4\pi R$

$\int (\nabla \cdot v) d\tau = \int \sin\theta \, dr \, d\theta \, d\phi = 4\pi R$

$\nabla \times (r^n \hat{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n) = \frac{1}{r^2}(n+2) r^{n+1} = (n+2) r^{n-1}$

b) $\nabla \times (r^n \hat{r}) = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r^n) - \frac{\partial}{\partial r}(r \cdot 0)\right)\hat{\theta} = 0$

$\int_V \nabla \times (r^n \hat{r}) \, dV = -\oint_S r^n \hat{r} \times dS \rightarrow$ this is zero cross product

so $\nabla \times (r^n \hat{r}) = 0$ for sphere

1.64 a) $D = -\frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 (-\tfrac{1}{2})\frac{2r}{(r^2+\epsilon^2)^{3/2}}\right) = \frac{1}{4\pi r^2}\frac{\partial}{\partial r}\left(\frac{r^3}{(r^2+\epsilon^2)^{3/2}}\right)$

$= \frac{1}{4\pi r^2}\left(\frac{3r^2}{(r^2+\epsilon^2)^{3/2}} - \frac{3}{2}\frac{r^3 2r}{(r^2+\epsilon^2)^{5/2}}\right) = \frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}}$

b) $D(0,\epsilon) = \frac{3\epsilon^2}{4\pi\epsilon^5} = \frac{3}{4\pi\epsilon^3}$

c) $D(r,0) = 0$ for $r \neq 0$ from a

d) $\int D(r,\epsilon) 4\pi r^2 dr = 3\epsilon^2 \int_0^\infty \frac{r^2}{(r^2+\epsilon^2)^{5/2}} dr$

$= 3\epsilon^2 \left(\frac{1}{3\epsilon^2}\right) = 1$