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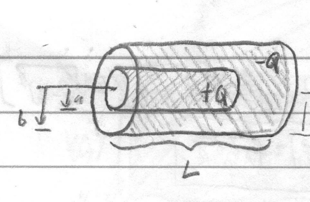
11/16/20 PHYS

HW3

2.43, 50
3.1, 3, 13, 14, 15



$\boxed{2.43}$ Capacitance per unit length. Assume +Q on inner cylinder, -Q outer.

$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}$, $da = 2\pi s L \Rightarrow E\cdot 2\pi s L = \frac{1}{\epsilon_0} Q \Rightarrow E = \frac{1}{2\pi\epsilon_0}\frac{Q}{s}$

$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{Q}{s}\hat{s}$

2 potential, potential difference: $V(b)-V(a) = \int_a^b \vec{E}\cdot d\vec{l}$ // $d\vec{l} = ?$ → cylindrical coords. $\Rightarrow d\vec{l} = ds\,\hat{s} + \dots$

$V(b)-V(a) = \int_a^b \frac{1}{2\pi\epsilon_0}\frac{Q}{s}\,ds \Rightarrow \frac{Q}{2\pi\epsilon_0}\int_a^b \frac{1}{s}\,ds \Rightarrow \frac{1}{2\pi\epsilon_0} Q\left[\ln(s)\right]_a^b$

$= \frac{1}{2\pi\epsilon_0} Q\left[\ln(b)-\ln(a)\right] \Rightarrow V_{tot} = V(b)-V(a) = \frac{1}{2\pi\epsilon_0} Q \ln(b/a)$

$C = Q/V \Rightarrow \frac{\cancel{Q}}{\frac{1}{2\pi\epsilon_0}\cancel{Q}\ln(b/a)} = \frac{2\pi\epsilon_0}{\ln(b/a)} \Rightarrow \boxed{C = \frac{2\pi\epsilon_0}{\ln(b/a)}}$

$\boxed{2.50}$ Potential $V(\vec{r}) = A\frac{e^{-\lambda r}}{r}$, constants A & λ.

$E(\vec{r}) = -\nabla V = -\frac{\partial}{\partial r}\left(A\frac{e^{-\lambda r}}{r}\right) \Rightarrow -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right) = -A\left(\frac{r(-\lambda)e^{-\lambda r} - e^{-\lambda r}(1)}{r^2}\right) = +A\left(\frac{+\lambda r e^{-\lambda r} + e^{-\lambda r}}{r^2}\right)$

$= \frac{A}{r^2}\left(e^{-\lambda r} + \lambda r e^{-\lambda r}\right) \Rightarrow \frac{A}{r^2}e^{-\lambda r}(1+\lambda r) \Rightarrow \boxed{\vec{E}(\vec{r}) = \frac{A}{r^2}e^{-\lambda r}(1+\lambda r)\,\hat{r}}$

$\nabla^2 V = -\rho/\epsilon_0 \Rightarrow \rho = -\epsilon_0\cdot\nabla^2 V = -\epsilon_0\cdot\nabla\cdot(\nabla V) \Rightarrow +\epsilon_0\,\nabla\cdot\vec{E} \Rightarrow \rho = \epsilon_0(\nabla\cdot\vec{E})$

$\rho = \epsilon_0\,\nabla\cdot\left(A\frac{\hat{r}}{r^2}e^{-\lambda r}(1+\lambda r)\right) \Rightarrow \epsilon_0 A\left[e^{-\lambda r}(1+\lambda r)\cdot\nabla\cdot\frac{\hat{r}}{r^2} + \frac{\hat{r}}{r^2}\cdot\nabla\left(e^{-\lambda r}(1+\lambda r)\right)\right]$ // $\nabla\cdot\frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$

$= \epsilon_0 A\left[e^{-\lambda r}(1+\lambda r)\cdot 4\pi\delta^3(\vec{r}) + \frac{\hat{r}}{r^2}\left(e^{-\lambda r}(\lambda) + (-\lambda)e^{-\lambda r}(1+\lambda r)\right)\right]$ // $f(x)\delta(x) = f(0)\delta(x)$; $f(r) = e^{-\lambda r}(1+\lambda r^2)$

$= \epsilon_0 A\left[(1)\cdot 4\pi\delta^3(\vec{r}) + \frac{\hat{r}}{r^2}\left(\cancel{\lambda e^{-\lambda r}} - \cancel{\lambda e^{-\lambda r}} - \lambda^2 r e^{-\lambda r}\right)\right] \Rightarrow \epsilon_0 A\left[4\pi\delta^3(\vec{r}) + \frac{\hat{r}}{r}\left(-\lambda^2 e^{-\lambda r}\right)\right]$

$\boxed{\rho = \epsilon_0 A\left[4\pi\delta^3(\vec{r}) - \frac{1}{r}\lambda^2 e^{-\lambda r}\right]}$

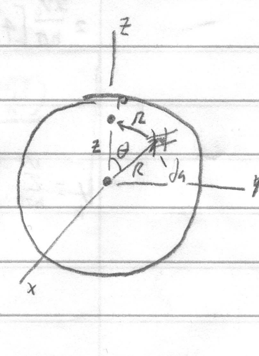
$Q = \int\rho\,d\tau = \int \epsilon_0 A\left(4\pi\delta^3(\vec{r}) - \frac{1}{r}\lambda^2 e^{-\lambda r}\right)dr = \epsilon_0 A\left(4\pi\overbrace{\int\delta^3(\vec{r})d\tau}^{1} - \lambda^2\int\frac{1}{r}e^{-\lambda r}\,d\tau\right)$ // $d\tau = 4\pi r^2 dr$

$= \epsilon_0 A\left(4\pi - \lambda^2 4\pi\int r e^{-\lambda r}dr\right)$ // Int. by parts: $u = r$, $dv = e^{-\lambda r}dr$ // $\int u\,dv = uv - \int v\,du$

$= \epsilon_0 A\left(4\pi - \lambda^2 4\pi\left[\frac{-r}{\lambda}e^{-\lambda r} + \frac{1}{\lambda}\int e^{-\lambda r}dr\right]\right)$ $du = dr$, $v = \int e^{-\lambda r}dr = \frac{-1}{\lambda}e^{-\lambda r}$

$= \epsilon_0 A\left(4\pi - \lambda^2 4\pi\left[\frac{-r}{\lambda}e^{-\lambda r} + \frac{1}{\lambda}\left(\frac{-1}{\lambda}e^{-\lambda r}\right)\right]\right) \Rightarrow \epsilon_0 A\left(4\pi + \lambda^2 4\pi\left[-r\lambda e^{-\lambda r} - \frac{1}{\lambda^2}e^{-\lambda r}\right]_{r=0}^{r=\infty}\right) \Rightarrow \epsilon_0 A\left(4\pi - \lambda^2 4\pi\left(\frac{1}{\lambda^2}\right)\right)$

$Q = \epsilon_0 A(\cancel{4\pi} - \cancel{4\pi}) \Rightarrow \boxed{Q = 0}$



$\boxed{3.1}$ $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$, where $r^2 = z^2 + R^2 - 2zR\cos\theta$. But, $z < R$. // $r^2 = z^2 + R^2 - 2zR\cos\theta$

$V(\vec{r}) = \frac{1}{4\pi R^2}\oint_{sphere} V\,da \Rightarrow V_{ave} = \frac{1}{4\pi R^2}\left(\frac{q}{4\pi\epsilon_0}\right)\int(z^2+R^2-2zR\cos\theta)^{-1/2}R^2\sin\theta\,d\theta\,d\phi$

$V_{ave} = kq\left(\frac{1}{4\pi R^2}\right)R^2\int\frac{1}{(z^2+R^2-2zR\cos\theta)^{1/2}}\sin\theta\,d\theta\,d\phi \Rightarrow kq\left(\frac{1}{4\pi}\right)\int_0^{2\pi}d\phi\int_0^{\pi}\frac{\sin\theta\,d\theta}{(z^2+R^2-2zR\cos\theta)^{1/2}}$

// $u = -2zR\cos\theta$, $du = 2zR\sin\theta\,d\theta$, $d\theta = \frac{du}{2zR\sin\theta}$

$= kq\left(\frac{1}{2}\right)\int_0^{\pi}\frac{1}{2zR}\cdot\frac{1}{(z^2+R^2-u)^{1/2}}du \Rightarrow kq\left(\frac{1}{4zR}\right)\int_0^{\pi}\frac{d\theta}{(z^2+R^2-u)^{1/2}} \Rightarrow kq\left(\frac{1}{4zR}\right)\left[2(z^2+R^2-u)^{1/2}\right]_0^{\pi}$

$= kq\left(\frac{1}{2zR}\right)\left[\sqrt{z^2+R^2-2zR\cos\theta}\right]_0^{\pi} \Rightarrow kq\left(\frac{1}{2zR}\right)\left[\sqrt{z^2+R^2+2zR} - \sqrt{z^2+R^2-2zR}\right]$

$\underline{V_{ave}} = k_q\left(\frac{1}{2zR}\right)\left[\sqrt{(z+R)^2} - \sqrt{(R-z)^2}\right] \Rightarrow k_q\left(\frac{1}{2zR}\right)\left[(z+R) + (R+z)\right] \Rightarrow k_q\left(\frac{1}{2zR}\right)(2z)$

$\boxed{V_{ave}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}}$

For collection of charges, V_{ave} = net V produced @ center. So, V_{center} of external charges & V_{enc} of enclosed charges.

So, $V_{center} = \frac{1}{4\pi\varepsilon_0}\frac{q}{z}$ & $V_{enc} = \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{R}$

$$V_{ave} = \sum V \Rightarrow V_{ave} = \frac{1}{4\pi\varepsilon_0}\frac{q}{z} + \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{R} \Rightarrow \boxed{V_{ave} = V_{center} + \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{R}}$$

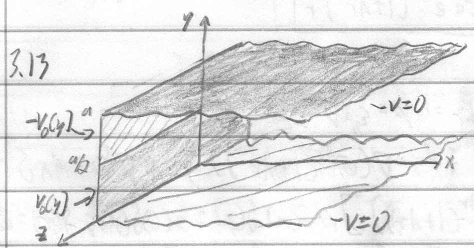
3.3 $\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \cdot \frac{dV}{dr}\right) = 0 \Rightarrow \frac{d}{dr}\left(r^2 \cdot \frac{dV}{dr}\right) = 0 \Rightarrow \left(r^2 \cdot \frac{dV}{dr}\right) + c = 0 \Rightarrow r^2 \cdot \frac{dV}{dr} = c \Rightarrow \frac{dV}{dr} = \frac{c}{r^2} \Rightarrow \int \frac{dV}{dr} = \int \frac{c}{r^2}$

$\boxed{V = -\frac{c_1}{r} + c_2}$ //constants c_1 & c_2

$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0 \Rightarrow \frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0 \Rightarrow \left(s \cdot \frac{\partial V}{\partial s}\right) + c = 0 \Rightarrow s \cdot \frac{\partial V}{\partial s} = c \Rightarrow \frac{\partial V}{\partial s} = \frac{c}{s} \Rightarrow \int \frac{\partial V}{\partial s} = \int \frac{c}{s}$

$V = \ln(c_1 s) + c_2 \Rightarrow \boxed{V = c_1 \ln(s) + c_2}$ //constants c_1 & c_2.

3.13



$V(x,y) = \sum_{n=1}^{\infty} C_n e^{\frac{-n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$, w/ $C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$.

Assume $C_n = \begin{cases} +V_0(y), & 0<y<a/2 \\ -V_0(y), & a/2<y<a \end{cases}$

$C_n = \frac{2}{a}\int_0^{a/2} V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy + \int_{a/2}^{a} -V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$

$C_n = \frac{2}{a}V_0\left[\int_0^{a/2}\sin\left(\frac{n\pi y}{a}\right)dy - \int_{a/2}^{a}\sin\left(\frac{n\pi y}{a}\right)dy\right]$ // $u = \frac{n\pi y}{a}$, $du = \frac{n\pi}{a}dy$, $dy = \frac{a}{n\pi}du$

$= \frac{2V_0}{a}\left[\int_0^{a/2}\sin(u)\frac{a}{n\pi}du - \int_{a/2}^{a}\sin(u)\frac{a}{n\pi}dy\right] = \frac{2V_0}{a}\cdot\frac{a}{n\pi}\left[\left(-\cos\left(\frac{n\pi y}{a}\right)\right)_{y=0}^{y=a/2} + \left(+\cos\left(\frac{n\pi y}{a}\right)\right)_{y=a/2}^{y=a}\right]$

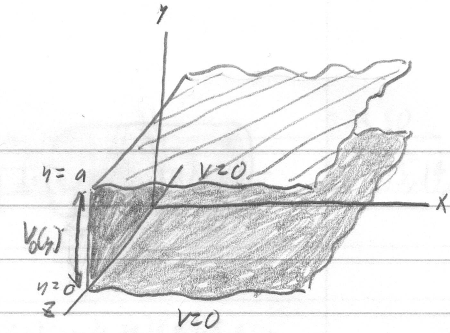
$= \frac{2V_0}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + 1 + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right] = \frac{2V_0}{n\pi}\left[1 + \cos(n\pi) - 2\cos\left(\frac{n\pi}{2}\right)\right]$ // n=1, 1-1+0=0 ; n=3, 1-1+0=0 ; n=5, 1-1+0=0

// n odd, n=2, 6, 10, ... ⇐ // n=2, 1+1+2=4 ; n=4, 1+1-2=0 ; n=6, 1+1+2=4

$= \frac{2V_0}{n\pi}[4] \Rightarrow C_n = \begin{cases} \frac{8V_0}{n\pi}, & n=2,6,10,\ldots \\ 0, & \end{cases}$

$V_n = \sum_{n=1}^{\infty}\frac{8V_0}{n\pi}e^{\frac{-n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right) \Rightarrow \boxed{V_n = \frac{8V_0}{\pi}\sum_{n=2,6,10,\ldots}^{\infty}\frac{e^{\frac{-n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right)}{n}}$

3.14 $\sigma(y) = ?$ (From Ex 3.3) $V(x,y) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_0}{n\pi} e^{\frac{-n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$ ✓



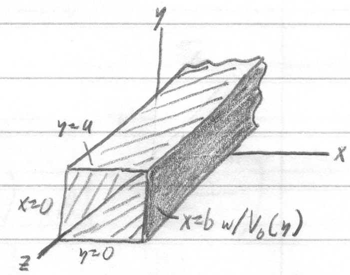
$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} e^{\frac{-n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$, $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$ ✓ // $\sigma(y) = -\epsilon_0 \frac{\partial V}{\partial x}$

$\sigma(y) = \frac{-4V_0\epsilon_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \cdot \sin\left(\frac{n\pi y}{a}\right) \frac{\partial}{\partial x}\left(e^{\frac{-n\pi x}{a}}\right) \Rightarrow \frac{+4V_0\epsilon_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \cdot \sin\left(\frac{n\pi y}{a}\right)\left(+\frac{n\pi}{a}\right) \cancel{e^{\frac{-n\pi x}{a}}}^{0}$ // $x=0$ ✓

$$\boxed{\sigma(y) = \frac{4V_0\epsilon_0}{a} \sum_{n=1,3,5,\ldots}^{\infty} \sin\left(\frac{n\pi y}{a}\right)}$$

3.15

(a) $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$.



Boundary Conditions: (i) $V=0$ when $y=0$
(ii) $V=0$ when $y=a$
(iii) $V=0$ when $x=0$
(iv) $V=V_0$ when $x=b$

So, $V(x,y) = X(x)Y(y)$. ✓

$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$ ✓

$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$ ✓

$f(x) + g(y) = 0$ ✓

$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = C_1$, $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = C_2$, $C_1 + C_2 = 0 \Rightarrow \frac{\partial^2 X}{\partial x^2} = k^2 X$, $\frac{\partial^2 Y}{\partial y^2} = -k^2 Y$. $\Rightarrow X(x) = Ae^{kx} + Be^{-kx}$, $Y(y) = C\sin ky + D\cos ky$ ✓

$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$ // (i) $\to y=0, D=0$; (iii) $\to x=0, A+B=0, A=-B$ ✓

$= (A(e^{kx} - e^{-kx}))\cdot C\sin ky$ // $\sinh(kx) = \frac{e^{kx} - e^{-kx}}{2} \Rightarrow 2A\sinh(kx)\,C\sin(ky)$ // (ii) $\to y=a$, $k = \frac{n\pi}{a}$ ✓

$\underline{V(x,y) = (2AC)\sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}$ ✓

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}$$

(b) $V_0(y) = V_0$ (a constant)? // $x=b$ ✓

//even/odd function?

$V(b,y) = V_0 = \sum_{n=1}^{\infty} \underbrace{C_n \sinh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi y}{a}\right) \to C_n\sinh\left(\frac{n\pi b}{a}\right)\begin{cases} n \text{ odd } (1,3,5,\ldots), & \frac{4V_0}{n\pi} \\ n \text{ even } (2,4,6,\ldots), & 0 \end{cases}$ ✓ So, n is odd!

$C_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{4V_0}{n\pi} \Rightarrow C_n = \frac{4V_0}{n\pi\sinh(n\pi b/a)}$ ✓

$V(x,y) = \sum_{n=1}^{\infty} \frac{4V_0}{n\pi} \cdot \frac{1}{\sinh\left(\frac{n\pi b}{a}\right)} \cdot \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \Rightarrow \boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{n=1}^{\infty} \frac{\sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}}$