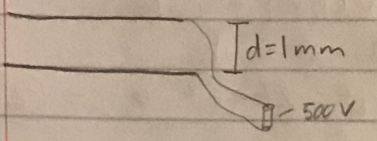
HW Set 4 # 4.1, 4.7, Ex. 4.2 (for inside and outside of the sphere), 4.10, 4.15, 4.18   Bonus: 4.6

4.1 H atom: Bohr radius $= \frac{1}{2}$ Angstrom



fraction: $d/_{\text{Bohr radius}} = 1\,mm/_{\frac{1}{2}\text{ Angstrom}}$

$$= \frac{1\,mm}{5\times10^{-8}\,mm} = \boxed{2\times10^{7}}$$

$$\vec{p} = \alpha\vec{E} \rightarrow p = \alpha E, \quad \alpha_H = 1.48\times10^{-31}m^3, \quad p = qd$$

$$E = p/\alpha = qd/\alpha$$

$$E = \frac{q(1mm)}{1.48\times10^{-31}\,m^3} = \frac{q(.001\,\cancel{m})}{1.48\times10^{-31}\,m^{\cancel{3}\,2}} = 6.76\times10^{27}(q)\,m^{-2}$$

$$\vec{E} = 6.76\times10^{27}(q)\,\vec{d}\,(m^{-2})$$

$$|\vec{V}| = |-\nabla\cdot\vec{E}| \rightarrow V = \nabla\cdot E$$

$$\boxed{V \approx 6.76\times10^{27}q\,(m^{-2})}$$

4.7 $W = \int\vec{F}\cdot d\vec{r} = \int\vec{\tau}\cdot d\hat{\phi}$ $\vec{\tau} = \vec{p}\times\vec{E}$

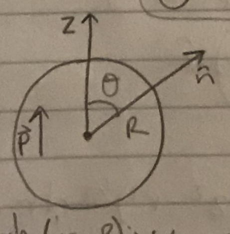
$-W = U$

$$W = \int(\vec{p}\times\vec{E})\cdot d\hat{\phi} = \int_{\theta_1}^{\theta_2}(\vec{p}\times\vec{E})\cdot\hat{\phi}\,d\theta$$

$$= \int_{\theta_1}^{\theta_2}(\hat{\phi}\times\vec{p})\cdot\vec{E}\,d\theta = \vec{p}\cdot\vec{E}$$

$$\boxed{U = -(\vec{p}\cdot\vec{E})}$$

Ex 4.2



find electric field inside & outside sphere

$\rho_b = 0 \qquad \sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta$

$\vec{E} = -\nabla V$

inside $(r<R)$: $V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{P\cos\theta}{\cancel{r^2}}\right)\frac{\cancel{4}}{3}\cancel{\pi} r^{\cancel{3}} = \frac{P\cos\theta\cdot r}{3\varepsilon_0}, \quad r\cos\theta = z$

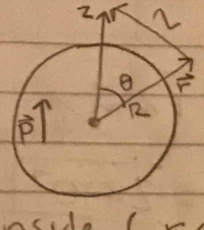
$$\vec{E} = -\nabla\left(\frac{Pz}{3\varepsilon_0}\right) = -\frac{P}{3\varepsilon_0}\hat{z}$$

outside $(r>R)$: $V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{p\cdot\cos\theta}{4\pi\varepsilon_0\cdot r^2}$

$$\vec{E} = -\nabla\left(\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\right) = -\left(\frac{-2p\cos\theta}{4\pi\varepsilon_0 r} + \frac{p(-\sin\theta)}{4\pi\varepsilon_0 r^2}\right)$$

$$\vec{E} = \frac{2kp\cos\theta}{r} + \frac{kp\sin\theta}{r^2}$$

4.10



$\vec{P} = k\vec{r}$ $\qquad \vec{E} = -\nabla \cdot V$

a) $\rho_b = -\nabla \cdot P = -\nabla(k\vec{r}) = -k$

$\sigma_b = \vec{P} \cdot \hat{n} = k\vec{r} \cdot \vec{r} = k$

$z^2 = r^2 + \mathfrak{r}^2 \rightarrow \mathfrak{r}^2 = z^2 - r^2 \rightarrow \mathfrak{r} = (z^2 - r^2)^{1/2}$

b) inside $(r < R)$:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{k}{(z^2-r^2)^{1/2}}(4\pi r^2) + \frac{(-k)}{(z^2-r^2)^{1/2}}\left(\frac{4}{3}\pi r^3\right)\right), \quad z = r\cos\theta$$

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{k(4\pi r^2)}{(r^2\cos^2\theta - r^2)^{1/2}}\right) - \frac{1}{4\pi\varepsilon_0}\left(\frac{k(\frac{4}{3}\pi r^3)}{(r^2\cos^2\theta - r^2)^{1/2}}\right)$$

$$V = \frac{k r^{\cancel{2}}}{\varepsilon_0(\cancel{r^2}(\cos^2\theta - 1))^{1/2}} - \frac{k r^{\cancel{3}2}}{3\varepsilon_0(\cancel{r^2}(\cos^2\theta - 1))^{1/2}}$$

$$= \frac{kr}{\varepsilon_0(-\sin^2\theta)^{1/2}} - \frac{kr^2}{\varepsilon_0(-\sin^2\theta)^{1/2}} = \frac{kr}{\varepsilon_0(i\sin\theta)} - \frac{kr^2}{\varepsilon_0(i\sin\theta)}$$

$$E = -\nabla\left(\frac{kr}{\varepsilon_0(i\sin\theta)} - \frac{kr^2}{\varepsilon_0(i\sin\theta)}\right) = -\left[\left(\frac{k}{i\varepsilon_0\sin\theta} - \frac{2kr}{i\varepsilon_0\sin\theta}\right) + \left(\frac{kr}{i\varepsilon_0}(-\csc\theta\cot\theta) + \frac{kr^2}{i\varepsilon_0}(+\csc\theta\cot\theta)\right)\right]$$

$$= -\left[\left(\frac{k(1-2r)}{i\varepsilon_0\sin\theta}\right) + \frac{kr\cancel{\csc\theta}\cot\theta}{i\varepsilon_0}(-1+r)\cdot\frac{\cancel{\sin\theta}}{\sin\theta}\right]$$

$$\vec{E} = -\left(\frac{k(1-2R) + kR(R-1)\cot\theta}{i\varepsilon_0\sin\theta}\right)\hat{r}$$

r to ∞ $\qquad$ outside $(r > R)$:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\int_r^\infty k(z^2-r^2)^{-1/2}\,dr + \int_0^{2\pi}\int_0^{\pi}\int_r^\infty -k(z^2-r^2)^{-1/2}\,dr\,d\theta\,d\phi\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(k\left[\frac{-1}{\cancel{2}r}\frac{(z^2-r^2)^{-\frac{1}{2}+1}}{\cancel{-\frac{1}{2}+1}}\right]_r^\infty - k(2\pi^2)\left[\frac{-1}{\cancel{2}r}\frac{(z^2-r^2)^{-\frac{1}{2}+1}}{\cancel{-\frac{1}{2}+1}}\right]_r^\infty\right)$$

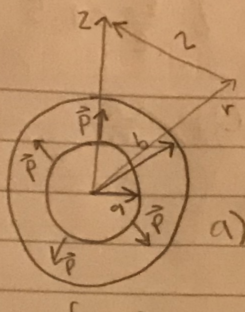
$$= \frac{1}{4\pi\varepsilon_0}\left(k\left[\frac{-(z^2-\cancel{(\infty)})^{1/2}}{\cancel{\infty}} + \left(+\frac{(z^2-r^2)^{1/2}}{r}\right)\right] - 2\pi^2 k\left(+\frac{(z^2-r^2)^{1/2}}{r}\right)\right)$$

$$= \frac{k}{4\pi\varepsilon_0}\left(\frac{(z^2-r^2)^{1/2}}{r}\right) - \frac{\pi k}{2\varepsilon_0}\left(\frac{(z^2-r^2)^{1/2}}{r}\right) = \left(\frac{k}{4\pi\varepsilon_0} - \frac{\pi k}{2\varepsilon_0}\right)\left(\frac{(z^2-r^2)^{1/2}}{r}\right)$$

$$\vec{E} = \left(\frac{k}{4\pi\varepsilon_0} - \frac{\pi k}{2\varepsilon_0}\right)\left(\frac{(z^2-r^2)^{1/2}(1) + r\left(\frac{1}{\cancel{2}}(z^2-r^2)^{-1/2}\cdot\cancel{2}r\right)}{r^2}\right)\hat{r}$$

$$\vec{E} = \left(\frac{k}{4\pi\varepsilon_0} - \frac{\pi k}{2\varepsilon_0}\right)\left(\frac{\sqrt{z^2-r^2} + r^2(z^2-r^2)^{-1/2}}{r^2}\right)\hat{r}$$

4.15



$$ɿ^2 = z^2 - r^2 \rightarrow ɿ = (z^2 - r^2)^{1/2}$$

$$\vec{P}(\vec{r}) = (k/r)\,\hat{r}$$

a) $\sigma_b = \vec{P} \cdot \hat{n} = k/r$

$$\rho_b = -\nabla \cdot \vec{P} = -\nabla(k/r) = -(-k/r^2) = k/r^2$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \oint_S \frac{\sigma_b}{ɿ} da' + \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_b}{ɿ} d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0} \oint_S \frac{(k/r)}{(z^2-r^2)^{1/2}} da + \frac{1}{4\pi\varepsilon_0} \int_V \frac{(k/r^2)}{(z^2-r^2)^{1/2}} d\tau$$

$$= \frac{k}{4\pi\varepsilon_0} \left[\int_0^{\pi}\int_{r_0}^{r_1} \frac{1}{r\,(z^2-r^2)^{1/2}} dr\,d\theta + \int_0^{2\pi}\int_0^{\pi}\int_{r_0}^{r_1} \frac{1}{r^2(z^2-r^2)^{1/2}} dr\,d\theta\,d\phi \right.$$

$$= \frac{k}{4\pi\varepsilon_0} \left[\int_0^{\pi}\int_{r_0}^{r_1} \frac{r}{r^2(z^2-r^2)^{1/2}} dr\,d\theta + \int_0^{2\pi}\int_0^{\pi}\int_{r_0}^{r_1} \frac{r}{r^3(z^2-r^2)^{1/2}} dr\,d\theta\,d\phi \right]$$

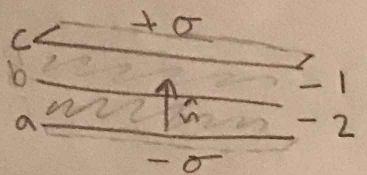
$u = z^2 - r^2 \rightarrow r^2 = z^2 - u$ $\qquad$ $u = z^2 - r^2 \rightarrow r^2 = z^2 - u$

$du = -2r$ $\qquad$ $du = -2r$ $\qquad$ $r^3 = (z^2-u)^{3/2}$

$r = -\frac{1}{2} du$ $\qquad$ $r = -\frac{1}{2} du$

$$= \frac{-\cancel{\pi} k}{8\cancel{4\pi}\varepsilon_0} \int \frac{(-\frac{1}{2} du)}{(z^2-u)(u)^{1/2}} + \frac{-\cancel{2}\pi^{\cancel{2}} k}{4\cancel{2}\cancel{4\pi}\varepsilon_0} \int \frac{-\frac{1}{2} du}{(z^2-u)^{3/2}(u)^{1/2}}$$

$$= \frac{-k}{8\varepsilon_0} \int \frac{du}{z^2 u^{1/2} - u^{3/2}} + \frac{-\pi k}{4\varepsilon_0} \int \frac{du}{(z^2-u)^{3/2}(u)^{1/2}}$$



thickness = a
total distance = 2a

1: $\frac{\varepsilon_0}{\varepsilon} = 2$

2: $\frac{\varepsilon_0}{\varepsilon} = 1.5$

4.18 a) $\oint \vec{D}\cdot d\vec{a} = Q_f \rightarrow \vec{D}\cdot\vec{A} = Q_f \rightarrow \vec{D} = Q_f/\vec{A}$

$\boxed{\vec{D}_1 = +\sigma/a^2}$, $\boxed{\vec{D}_2 = -\sigma/a^2}$

b) $\vec{D} = \varepsilon\vec{E} \rightarrow \vec{E} = \vec{D}/\varepsilon$

$\boxed{\vec{E}_1 = +\sigma/\varepsilon a^2}$, $\boxed{\vec{E}_2 = -\sigma/\varepsilon a^2}$

c) $\vec{D} = \varepsilon_0\vec{E} + \vec{P} \rightarrow \vec{P} = \vec{D} - \varepsilon_0\vec{E}$

$$\vec{P}_1 = \frac{+\sigma}{a^2} - \varepsilon_0\left(\frac{+\sigma}{\varepsilon a^2}\right) = \frac{\sigma - 2\sigma}{a^2} = \boxed{\frac{-\sigma}{a^2}}$$

$$\vec{P}_2 = \frac{-\sigma}{a^2} - \varepsilon_0\left(\frac{-\sigma}{\varepsilon a^2}\right) = \frac{-\sigma + 1.5\sigma}{a^2} = \boxed{\frac{\sigma}{2a^2}}$$

d) $V(a) - V(b) = -\int_a^b \vec{E}\cdot d\vec{l}$, $\Delta V = \int_a^b \vec{E}\cdot d\vec{l} + \int_b^c \vec{E}\cdot d\vec{l}$

$$\Delta V = \vec{E}_2\, l + \vec{E}_1\, l = \frac{-\sigma}{\varepsilon a^{\cancel{2}}}(\cancel{a}) + \frac{+\sigma}{\varepsilon a^{\cancel{2}}}(\cancel{a})$$

$\boxed{\Delta V = 0}$

e) $\rho_f = -\nabla\cdot\vec{P}$, $\sigma_b = \vec{P}\cdot\hat{n}$

$\rho_{f_1} = -\nabla\left(\frac{-\sigma}{a^2}\right) = \boxed{\sigma/a^2}$ $\quad \rho_{f_2} = -\nabla(\sigma/2a^2) = \boxed{-\sigma/2a^2}$

$\sigma_{b_1} = (-\sigma/a^2)\cdot\hat{n} = \boxed{-\sigma/a^2}$ $\quad \sigma_{b_2} = (\sigma/2a^2)\cdot\hat{n} = \boxed{\sigma/2a^2}$

f) $$V = k\left(\oint_S \frac{\sigma_b}{2}\,da' + \int_V \frac{\rho_b}{2}\,d\tau'\right)$$

$$\vec{E}_1 = -\nabla V_1 = -\nabla k\left(\oint_S \frac{(-\sigma/a^2)}{2}\,da_1 + \int_V \frac{(\sigma/a^2)}{2}\,d\tau_2\right)$$

$$\vec{E}_2 = -\nabla V_2 = -\nabla k\left(\oint_S \frac{(\sigma/2a^2)}{2}\,da_2 + \int_V \frac{(-\sigma/2a^2)}{2}\,d\tau_2\right)$$

Bonus: 4.6 $\vec{N} = \vec{p}\times\vec{E}$, $\quad \vec{p} = p_x\hat{x} + p_y\hat{y} + p_z\hat{z}$, $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$

$$\vec{N} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ p_x & p_y & p_z \\ E_x & E_y & E_z \end{vmatrix} = (p_yE_z - p_zE_y)\hat{x} - (p_xE_z - p_zE_x)\hat{y} + (p_xE_y - p_yE_x)\hat{z}$$

$\vec{p}$ will come to rest parallel to the plane