

Solutions for Homework 4

Dr. Jordan Hanson - Whittier College Dept. of Physics and Astronomy

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1 Problem 4.10

A sphere of radius R carries a polarization

$$\mathbf{P}(\mathbf{r}) = k\mathbf{r} \quad (1)$$

In Eq. 1, k is a constant and \mathbf{r} is the vector from the center.

- (a) Calculate the bound charges σ_b and ρ_b .
- (b) Find the field inside and outside the sphere.

(a) The surface bound charge is at radius R , so $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = kR$. The volumetric bound charge is $\rho_b = -\nabla \cdot \mathbf{P} = -3k$. Note the total charge should add up to zero: $(4\pi R^2)\sigma_b + (4/3)\pi R^3\rho_b = 4\pi R^3k - 4\pi R^3k = 0$. (b) The field of any constant volumetric charge density $-3k$ should be calculable via Gauss' law. We find, after integrating over a Gaussian surface of radius $r < R$:

$$\oint \mathbf{E} \cdot d\mathbf{a} = \mathbf{E} \cdot \mathbf{A} = \frac{1}{\epsilon_0} \rho = \frac{-4k\pi r^3}{\epsilon_0} \quad (2)$$

$$\mathbf{E} = -\frac{3kr}{\epsilon_0} \hat{\mathbf{r}} = -\mathbf{P}/\epsilon_0 \quad (3)$$

(b) Note that, because the net charge is zero, the field outside the sphere is zero.

2 Problem 4.14

When you polarize a neutral dielectric, the charge moves a bit, but the total remains zero. This fact should be reflected in the bound charges σ_b and ρ_b . Prove from Eqs. 4.11 and 4.12 that the total bound charge vanishes.

First, let's integrate the total volumetric bound charge:

$$-q = \int_V \rho_b d\tau = - \int_V \nabla \cdot \mathbf{P} d\tau = - \oint \mathbf{P} \cdot \hat{\mathbf{n}} da = - \oint \sigma_b da \quad (4)$$

Next, the total surface bound charge is

$$q = \oint_S \sigma_b da \quad (5)$$

Now we see that $Q = -q + q = 0$.

3 Problem 4.15

A thick spherical shell (inner radius a , outer radius b) is made of dielectric material with a frozen-in polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r} \hat{\mathbf{r}} \quad (6)$$

In Eq. 6, k is a constant and r is the distance from the center. There is no free charge in the problem. Find the electric field in all three regions by two different methods:

- (a) Locate all the bound charge, and use Gauss' Law to calculate the field it produces.
- (b) Use Eq. 4.23 to find \mathbf{D} , and then get \mathbf{E} from Eq. 4.21. [Notice the second method is faster, and it avoids any explicit reference to the bound charges].

For (a), locating all the bound charge:

- $r = a$:

$$\sigma_a = \mathbf{P} \cdot (-\hat{\mathbf{r}}) = -\left(\frac{k}{a}\right) \hat{\mathbf{r}} \quad (7)$$

- $a < r < b$:

$$\rho = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{k}{r} \right) = -\frac{k}{r^2} \quad (8)$$

- $r = b$:

$$\sigma_b = \frac{k}{b} \quad (9)$$

- What is the total charge for this neutral object?

$$Q = 4\pi(b-a) - 4\pi \int_a^b \rho d\tau = 4\pi(b-a) - 4\pi(b-a) = 0 \quad (10)$$

The neutral object has $Q = 0$. Using Gauss' Law, $\mathbf{E} = 0$ for $r < a$, $\mathbf{E} = -\mathbf{P}/\epsilon_0$, and for $r > b$, $\mathbf{E} = 0$ because total charge is 0.

For (b), we find that since there is no free charge, $\mathbf{D} = 0$ and by the definition of \mathbf{D} , $\mathbf{E} = -\mathbf{P}/\epsilon_0$. Thus, we achieve the same result by making reference to only the free charge.

4 Problem 4.18

The space between the plates of a parallel plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness a , so the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate $-\sigma$.

- (a) Using Gauss' law for \mathbf{D} , we find that $\mathbf{D} = -\sigma\hat{\mathbf{z}}$ throughout the system.
- (b) Using $\mathbf{D} = \epsilon\mathbf{E}$, we find slab 1 has $\mathbf{E} = -(\sigma/2\epsilon_0)\hat{\mathbf{z}}$ and slab 2 has $\mathbf{E} = -(2\sigma/3\epsilon_0)\hat{\mathbf{z}}$.
- (c) Using the definition of \mathbf{D} , and the results from (b), we find $\mathbf{P} = -(\sigma/2)\hat{\mathbf{z}}$ for slab 1. For slab 2 we find $\mathbf{P} = -(\sigma/3)\hat{\mathbf{z}}$.
- (d) We know that $\Delta V = E\delta z = E_1a + E_2a = (E_1 + E_2)a = (7\sigma a)/(6\epsilon_0)$.
- (e) There is no ρ_b because the polarization densities are constant. For the bound surface charges, there is $-\sigma/2$ at the top of slab 1, and $\sigma/2$ at the bottom of slab 1 from medium 1. There is $-\sigma/3$ at the top of slab 2 from medium 2, and $\sigma/3$ at the bottom of slab 2 from medium 2. At the interface, if we add surface charges, we find $\sigma/6$.
- (f) To find the same fields as above, we must combine total charge from free and bound charges at the top and bottom surfaces.

5 Problem 4.26

A spherical conductor, of radius a , carries a charge Q . It is surrounded by linear dielectric material of susceptibility χ_e , out to radius b . Find the energy of this configuration.

The goal is to find \mathbf{E} and \mathbf{D} , so that we can apply

$$W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d\tau \quad (11)$$

The results are:

- Displacement inside the conductor ($r < a$): 0
- Displacement outside the conductor ($r > a$): $Q/(4\pi r^2)\hat{\mathbf{r}}$
- Field inside the conductor: 0
- Field outside the conductor but inside the dielectric ($a < r < b$): $Q/(4\pi\epsilon r^2)\hat{\mathbf{r}}$.
- Field outside the dielectric ($r > b$): $Q/(4\pi\epsilon_0 r^2)\hat{\mathbf{r}}$.

Inserting all these into into Eq. 11, and integrating:

$$W = \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)} \left(\frac{1}{a} + \frac{\chi_e}{b} \right) \quad (12)$$

6 Proble 4.35

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