EMHW: 4.10, 4.14, 4.15, 4.18, 4.26, 4.35

4.10) $\vec{P} = k\vec{r}$

k ↑ const.

Surface bound charge: $\sigma_b \equiv \vec{P}\cdot\hat{n}$

volume bound charge: $\rho_b \equiv -\nabla\cdot\vec{P}$

So $\sigma_b = k\vec{r}\cdot\hat{n}$ for sphere $\vec{r} \parallel \hat{n}$

$\Rightarrow \sigma_b = k|\vec{r}||\hat{n}|$

$|\vec{r}| = R$ at surface

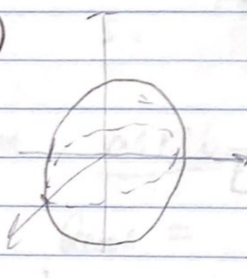
$\Rightarrow \boxed{\sigma_b = kR}$

$\rho_b = -\nabla\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr)$

$= -\frac{1}{r^2}3r^2k$

$\boxed{\rho_b = -3k}$

b.) The field inside found by Gauss' law $r<R$



$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{enc}$

$\Rightarrow EA = \frac{1}{\epsilon_0}\left(\frac{4}{3}\pi r^3\right)\rho_b$

$A = 4\pi r^2$

$E = \frac{1}{\epsilon_0}\frac{1}{3}r\rho$

this assumes $\vec{E}\parallel\vec{A}\parallel\hat{r}$

$\Rightarrow \vec{E} = \frac{1}{\epsilon_0}\frac{1}{3}r\rho\hat{r}$

$\vec{E} = \frac{1}{\cancel{3}\epsilon_0}r(-\cancel{3}k)\hat{r} = \boxed{-\frac{k}{\epsilon_0}r\hat{r} = \vec{E} \quad \text{for } r<R}$

4.10.) b.) for $r > R$

Same except now $Q_{enc} = \sigma_b(4\pi R^2) + \rho_b(\frac{4}{3}\pi R^3)$

$$= kR(4\pi R^2) - \cancel{3}k(\frac{4}{\cancel{3}}\pi R^3)$$

$$= 4\pi k R^3 - 4\pi k R^3 = 0$$

$Q_{enc} = 0 \Rightarrow \boxed{\vec{E} = 0.}$ for $r > R$.

4.14.) $\sigma_b = \vec{P}\cdot\hat{n}$

$\rho_b = -\vec{\nabla}\cdot\vec{P}$

The total charge is total surface + total volume:

$$Q_{total} = \oint_S \sigma_b da + \int_V \rho_b d\tau$$

$$= \oint_S (\vec{P}\cdot\hat{n}) da - \int_V (\vec{\nabla}\cdot\vec{P}) d\tau$$

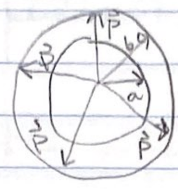
$$= \oint_S \vec{P}\cdot d\vec{a} - \int_V (\nabla\cdot\vec{P}) d\tau$$

But div. thm says $\int_V (\nabla\cdot\vec{P}) d\tau = \oint_S \vec{P}\cdot da$

So, $\boxed{Q_{total} = 0.}$

4.15.)

a.)



$$\vec{P}(\vec{r}) = \frac{k}{r}\,\hat{r}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^{\cancel{2}}\frac{k}{\cancel{r}}\right) = -\frac{k}{r^2}$$

$\sigma_{ba} = \vec{P}\cdot\hat{n} = -k/a$ ← inner surface of shell.

$\sigma_{bb} = \vec{P}\cdot\hat{n} = \frac{k}{b}$

Using Gauss' Law: at $\underline{r<a}$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{r^2}\hat{r}$$

but $Q_{enc} = 0 \Rightarrow \boxed{\vec{E} = 0}$

at $\underline{r>b}$

$Q_{enc} = 0$ since dielectric,

so $\boxed{\vec{E} = 0}$

at $\underline{a<r<b}$

$$Q_{enc} = \oint_S \sigma_{ba}\,da + \int_V \rho_b\,d\tau$$ ← volume of thin spherical shell: $4\pi r^2 dr$

$$= \oint_0^a \frac{-k}{a}\underbrace{da}_{\text{area}} + \int_a^r \frac{-k}{\cancel{r^2}}4\pi\cancel{r^2}\,dr$$

$$= -\frac{k}{\cancel{a}}(4\pi a^{\cancel{2}}) + (-k4\pi)(r-a)$$

$$= \cancel{-k4\pi a} - k4\pi r + \cancel{k4\pi a}$$

$$Q_{enc} = -k4\pi r \Rightarrow \boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{(-k4\pi r)}{r^2}\hat{r} = -\frac{k}{\epsilon_0}\frac{1}{r}\hat{r}}$$

4.15) b.) $\oint \vec{D} \cdot d\vec{a} = Q_{fenc}$

but Dielectric $\Rightarrow$ $Q_{fenc} = 0$

So $\vec{D} = 0$

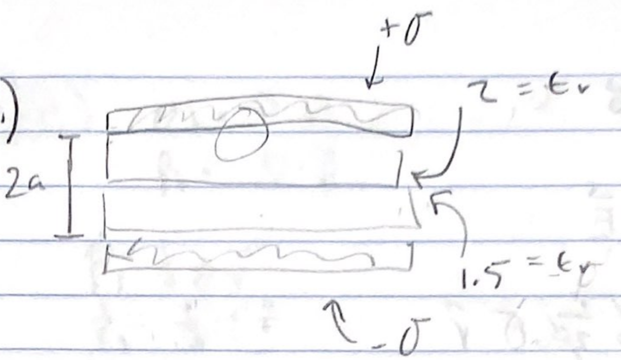
$$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{P} = \frac{k}{r} \hat{r}$$

$$0 = \epsilon_0 \vec{E} + \vec{P}$$

$$\boxed{\vec{E} = \frac{-\vec{P}}{\epsilon_0} = -\frac{k}{\epsilon_0 r} \hat{r}}$$

4.18.)



a.)

$$\int D \cdot da = Q_{fenc}$$

Pick a surface enclosing a certain amount of surface charge σ in an area A.

$$\Rightarrow \int \vec{D} \cdot d\vec{a} = D\cancel{A} = \sigma \cancel{A}$$

$$\Rightarrow D = \sigma \Rightarrow \boxed{\vec{D} = \sigma \hat{r}}$$

On the bottom $D\cancel{A} = -\sigma \cancel{A}$

$D = -\sigma$ but area normal vector is $-\hat{r}$

$$\Rightarrow \boxed{\vec{D} = \sigma \hat{r}}$$

b.) $\vec{E} = \frac{1}{\epsilon} \vec{D}$ $\qquad \epsilon = \epsilon_0 \epsilon_r$

In slab 1 $\boxed{\vec{E} = \frac{1}{2\epsilon_0} \sigma \hat{r}}$ $\qquad \epsilon = 2\epsilon_0$

$\epsilon = \frac{3}{2} \epsilon$

In slab 2 $\boxed{\vec{E} = \frac{2}{3\epsilon_0} \sigma \hat{r}}$

4.18.) c.) $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$\vec{P} = \vec{D} - \epsilon_0 \vec{E}$

$\vec{P}_1 = \sigma \hat{r} - \cancel{\epsilon_0} \frac{1}{2\cancel{\epsilon_0}} \sigma \hat{r}$

$\vec{P}_1 = \frac{1}{2} \sigma \hat{r}$

$\vec{P}_2 = \sigma \hat{r} - \cancel{\epsilon_0} \frac{2}{3\cancel{\epsilon_0}} \sigma \hat{r}$

$\vec{P}_2 = \frac{1}{3} \sigma \hat{r}$

d.) $V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l}$

$\frac{1}{2\epsilon_0} \sigma \hat{r}$ $\quad$ $\frac{3}{2\epsilon_0} \sigma \hat{r}$

$= -\left[\int_0^a \vec{E}_1 \cdot d\vec{l} + \int_a^{2a} \vec{E}_2 \cdot d\vec{l} \right]$

$= -\left[\frac{1}{2\epsilon_0} \sigma \left[\vec{l} \right]_0^a + \frac{3}{2\epsilon_0} \sigma \left[\vec{l} \right]_a^{2a} \right]$

$= -\left(\frac{1}{2\epsilon_0} \sigma a + \frac{3}{2\epsilon_0} \sigma a \right)$

$$\boxed{= -\frac{7}{6\epsilon_0} \sigma a}$$

4.18) e.) $\sigma_b = \vec{P} \cdot \hat{n}$

$\rho_b = -\nabla \cdot \vec{P}$

$\vec{P}_1 = \frac{1}{2}\sigma \hat{r}$

$\vec{P}_2 = \frac{1}{3}\sigma \hat{r}$

$\Rightarrow \rho_b = 0$ everywhere

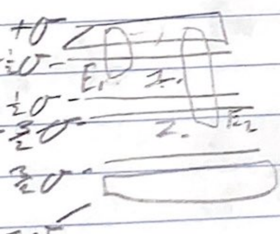
top of slab 1 : $\sigma_b = \frac{1}{2}\sigma \hat{r} \cdot -\hat{r} = -\frac{1}{2}\sigma$

bottom slab 1 : $\sigma_b = \frac{1}{2}\sigma \hat{r} \cdot \hat{r} = \frac{1}{2}\sigma$

top of slab 2 : $\sigma_b = \frac{1}{3}\sigma \hat{r} \cdot -\hat{r} = -\frac{1}{3}\sigma$

bottom slab 2 : $\sigma_b = \frac{1}{3}\sigma \hat{r} \cdot \hat{r} = \frac{1}{3}\sigma$

f.) Using Gauss' Law



$\downarrow \hat{r}$

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$EA = \frac{Q_{enc}}{\epsilon_0}$$

So, $$E_1 \cancel{A} = \frac{(\sigma - \frac{1}{2}\sigma)\cancel{A}}{\epsilon_0}$$

$$\boxed{\Rightarrow \vec{E}_1 = \frac{\sigma}{2\epsilon_0}\hat{r}}$$

$$E_2 \cancel{A} = \frac{(\sigma - \frac{1}{2}\sigma + \frac{1}{2}\sigma - \frac{1}{3}\sigma)\cancel{A}}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{E}_2 = \frac{\frac{2}{3}\sigma}{\epsilon_0}\hat{r}}$$

4.26.) $W = \frac{1}{2}\int D\cdot E\, d\tau$

$\vec{D}$: $\oint D\cdot d\vec{a} = Q_{fenc}$

$\Rightarrow DA = Q_{fenc}$

$D = \frac{Q_{fenc}}{A}$

at $r<a$

$Q_{fenc} = 0 \Rightarrow \vec{D} = 0$

at $r>a$ $\quad Q_{fenc} = Q$

$\Rightarrow D = \frac{Q}{4\pi r^2} \Rightarrow \vec{D} = \frac{Q}{4\pi r^2}\hat{r}$

$\vec{E} = \frac{\vec{D}}{\epsilon} \Rightarrow \vec{E} = \frac{Q}{\epsilon 4\pi r^2}\hat{r}$

at $a<r<b$ $\quad \epsilon = \epsilon = \epsilon_r \epsilon_0$

$$\boxed{\vec{E} = \frac{Q}{\epsilon 4\pi r^2}\hat{r}}$$

at $r>b$ $\quad \epsilon = \epsilon_0$

$$\boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}}$$

at $r<a$

$$\boxed{\vec{E} \,\&\, \vec{D} = 0}$$

4.26.) $W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau$

$$= \frac{1}{2}\int_a^\infty \left(\frac{Q}{4\pi r^2}\right)\left(\frac{Q}{\epsilon 4\pi r^2}\right) d\tau$$

$$= \frac{1}{2}\left(\frac{Q}{4\pi}\right)^2\left(\int_a^b \frac{1}{\epsilon}\frac{1}{r^4}\, d\tau + \int_b^\infty \frac{1}{\epsilon_0}\frac{1}{r^4}\, d\tau\right)$$

$d\tau \propto 4\pi r^2 dr$

$$= \frac{1}{2}\left(\frac{Q}{4\pi}\right)^2 (4\pi)\left(\frac{1}{\epsilon}\int_a^b \frac{1}{\cancel{r^4} r^2}\cancel{r^2}\, dr + \frac{1}{\epsilon_0}\int_b^\infty \frac{1}{\cancel{r^4} r^2}\cancel{r^2}\, dr\right)$$

$$= \frac{Q^2}{8\pi}\left(\frac{1}{\epsilon}\left(-\frac{1}{r}\right)\Big|_a^b + \frac{1}{\epsilon_0}\left(-\frac{1}{r}\right)\Big|_b^\infty\right)$$

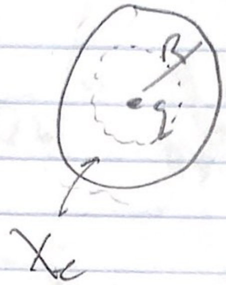
$$= \frac{Q^2}{8\pi}\left[\frac{1}{\epsilon}\left(-\frac{1}{b}+\frac{1}{a}\right) + \frac{1}{\epsilon_0}\left(\frac{1}{b}\right)\right]$$

$* \ \epsilon \equiv \epsilon_0(1+\chi_e)$

$$= \frac{Q^2}{8\pi}\left[\frac{1}{\epsilon}\left(\frac{1}{a}-\frac{1}{b}\right) + \frac{1}{\epsilon_0}\left(\frac{1}{b}\right)\right]$$

$$\boxed{W = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right) + \frac{1}{b}\right]}$$

4.35.)



$$\oint \vec{D}\cdot d\vec{a} = Q_{f\,enc}$$

$$DA = Q_{f\,enc}$$

$$\vec{D} = \frac{q}{4\pi r^2}\hat{r}$$

$$\vec{E} = \frac{1}{\epsilon}\vec{D}$$

$\epsilon = \epsilon_0(1+\chi_e)$

$$\Rightarrow \boxed{\vec{E} = \frac{q}{4\pi r^2 \epsilon}\hat{r} = \frac{q}{4\pi r^2 \epsilon_0(1+\chi_e)}\hat{r}}$$

$$\vec{P} = \epsilon_0\chi_e\vec{E}$$

$$\Rightarrow \boxed{\vec{P} = \frac{q}{4\pi r^2}\left(\frac{\chi_e}{1+\chi_e}\right)\hat{r}}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left(\frac{q}{4\pi}\left(\frac{\chi_e}{1+\chi_e}\right)\frac{\hat{r}}{r^2}\right)$$

$$= -\frac{q}{4\pi}\left(\frac{\chi_e}{1+\chi_e}\right)\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$$

$$= \frac{-q}{\cancel{4\pi}}\left(\frac{\chi_e}{1+\chi_e}\right)\cancel{4\pi}\,\delta^3(\vec{r})$$

$$\boxed{\rho_b = -q\left(\frac{\chi_e}{1+\chi_e}\right)\delta^3(\vec{r})}$$

4.35) $\sigma_b = \vec{P} \cdot \hat{n}^{\hat{r}}$

$$\sigma_b = \left. \frac{q \chi_e}{4\pi(1+\chi_e) r^2} \right|_{r=R}$$

$$\boxed{\sigma_b = \frac{q \chi_e}{4\pi(1+\chi_e) R^2}}$$

Total surface charge: $\sigma_b A = \frac{q \chi_e}{\cancel{4\pi}(1+\chi_e)\cancel{R^2}} \cancel{4\pi R^2}$

$$\boxed{= \frac{q \chi_e}{(1+\chi_e)}}$$

The negative charge is given by

$$\rho_b = -q \frac{\chi_e}{1+\chi_e} \delta^3(\vec{r})$$

which is located at $r = 0$.