Donny Diaz

Hw Chapter 1 p 54, 55, 56, 57, 59, 62, 63, 64

1.54) check the divergence thm for the func

$V = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

$\int(\nabla\cdot v)\,d\tau = \oint v\cdot da$

we know

$\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

$\nabla\cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r^2\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial(-r^2\cos\theta\sin\phi)}{\partial\phi}$

$\nabla\cdot v = 4r\cos\theta + r\cot\phi\cos\theta - r\cos\phi\cot\theta = 4r\cos\theta$

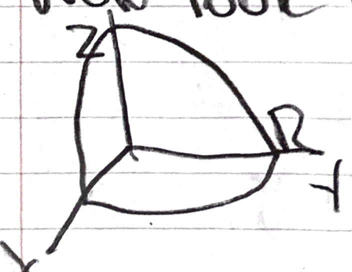
$\int(\nabla\cdot v)\,d\tau = \int_0^R\int_0^{\pi/2}\int_0^{\pi/2}(4r\cos\theta)\,\underline{r^2\sin\theta\,dr\,d\theta\,d\phi}$

$= 4\int_0^R r^3\,dr\cdot\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\cdot\int_0^{\pi/2}d\phi$

$u=\sin\theta$, $du=\cos\theta$

$= 4\left(\frac{R^4}{4}\right)\cdot\left(\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}\right)\cdot\frac{\pi}{2} = R^4\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \boxed{\frac{\pi R^4}{4}}$

Now look at $\oint v\cdot da$, ~~there are~~ 4 different parts



1) $da = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$ front surface (spherical)

$v\cdot da = (R^2\cos\theta)R^2\sin\theta\,d\theta\,d\phi$

$R^4\int_0^{\pi/2}\int_0^{\pi/2}\cos\theta\sin\theta\,d\phi\,d\theta = R^4\int_0^{\pi/2}\cos\theta\sin\theta\cdot\left(\frac{\pi}{2}\right)$

$u=\sin\theta$, $du=\cos\theta$

$= \frac{\pi R^4}{2}\left(\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}\right) = \frac{R^4\pi}{4}$

1.54 cont) left surface $da = -r\,dr\,d\theta\,\hat{\phi}$ we know $\phi = 0$ for xz plane

$v \cdot da = (-r^2 \cos\theta \sin\phi)(-r\,dr\,d\theta) =$

$= r^3 \cos\theta \sin(0)\,dr\,d\theta = 0$

so $\int v \cdot da = 0$

3) for the right side: $da = r\,dr\,d\theta\,\hat{\phi}$

$\phi = \frac{\pi}{2}$ for yz plane

$v \cdot da = (-r^2 \cos\theta \sin\phi)(r\,dr\,d\theta)$

$= (-r^2 \cos\theta \sin\frac{\pi}{2})(r\,dr\,d\theta) = -r^3 \cos\theta\,dr\,d\theta$

$\int v \cdot da = \int_0^R \int_0^{\pi/2} -r^3 \cos\theta\,d\theta\,dr = \int_0^R -r^3 (\sin\theta\Big|_0^{\pi/2})\,dr$

$= \int_0^R -r^3\,dr = -\frac{r^4}{4}\Big|_0^R = -\frac{R^4}{4}$

4) Now for the bottom part

$da = r\,dr\,d\phi\,\hat{\theta}$ $\theta = \pi/2$

$v \cdot da = (r^2 \cos\phi)(r\,dr\,d\phi) = r^3 \cos\phi\,dr\,d\phi$

$\int v \cdot da = \int_0^R \int_0^{\pi/2} r^3 \cos\phi\,d\phi\,dr$

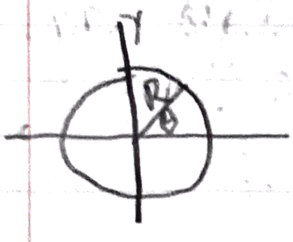
$= \int_0^R r^3 \cdot (\sin\phi\Big|_0^{\pi/2})\,dr = \int_0^R r^3\,dr = \frac{R^4}{4}$

Then total $\int v \cdot da = \frac{R^4 \pi}{4} + 0 + -\frac{R^4}{4} + \frac{R^4}{4} = \boxed{\frac{\pi R^4}{4}}$ ✓

So divergence thm is applied

55) Check stokes thm using func $v = ay\hat{x} + bx\hat{y}$ (a and b are constants) and circular path of radius R, centered at origin in the XY plane

Stokes Thm states $\int_S (\nabla \times v)\cdot da = \oint_P v\cdot dl$



$x = R\cos\theta \quad y = R\sin\theta$

$$\nabla \times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ay & bx & 0 \end{vmatrix}$$

$$\nabla\times v = \left(0 - \frac{\partial bx}{\partial z}\right)\hat{x} - \left(0 - \frac{\partial ay}{\partial z}\right)\hat{y} + \left(\frac{\partial bx}{\partial x} - \frac{\partial ay}{\partial y}\right)\hat{z}$$

$$= 0\hat{x} - 0\hat{y} + (b-a)\hat{z}$$

da for Area vector $= \pi R^2 \hat{z}$

So $\int(\nabla\times v)\cdot dA = \int (b-a)\hat{z}\cdot \pi R^2\hat{z} = \underline{\pi R^2(b-a)}$

Now for $\oint_P v\cdot dl$

$dl = dx\hat{x} + dy\hat{y}$, $\quad dx = -R\sin\theta d\theta \quad dy = R\cos\theta d\theta$

$dl = -R\sin\theta\hat{x} + R\cos\theta\hat{y}$

$$\int v\cdot dl = \int (ay\hat{x} + bx\hat{y})\cdot(-R\sin\theta\,\hat{x}\,d\theta + R\cos\theta\,\hat{y}\,d\theta)$$

$$= -\int_0^{2\pi} aR^2\sin^2\theta\, d\theta + \int_0^{2\pi} bR^2\cos^2\theta\, d\theta$$

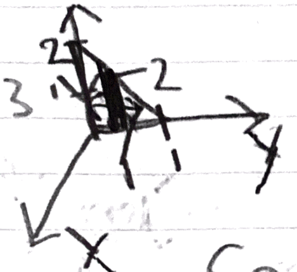
$$= -\frac{aR^2}{2}\int_0^{2\pi}(1-\cos2\theta)\,d\theta + \frac{bR^2}{2}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta$$

$$= -\frac{aR^2}{2}\left(\theta + \frac{\sin2\theta}{2}\Big|_0^{2\pi}\right) + \frac{bR^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\Big|_0^{2\pi}\right.$$

$= -\frac{aR^2}{\cancel{2}}(\cancel{2}\pi) + \frac{bR^2}{\cancel{2}}(\cancel{2}\pi) = \underline{\pi R^2(b-a)}$

So stokes thm applies.

5b) Compute line integral of $v = 6\hat{i} + yz^2\hat{j} + (3y+z)\hat{k}$ along triangular path. Check answer using stokes thm



Stokes thm: $\int_S(\nabla \times \vec{v})\cdot da = \oint \vec{v}\cdot dI$ ✓

along $\underline{\text{path 1}}$, $x=0$ and $z=0$. Thus $dx = dz = 0$ and $dI = dy\hat{j}$

So $\int_0^1 \vec{v}\cdot dI = \int_0^1 yz^2\,dy$ and $z=0$

So $\int \vec{v}\cdot dI = 0$

for path 2, eqn of the line is $z - z_1 = m(y - y_1)$

$m = \frac{z_2 - z_1}{y_2 - y_1} = \frac{2-0}{0-1} = -2$

then $z - 0 = -2(y-1) = 2(1-y)$ $\quad z = 2(1-y)$, $dz = -2dy$

$dI = dy\hat{j} + dz\hat{k}$

$\int \vec{v}\cdot dI = \int yz^2\,dy + (3y+z)\,dz$

$= \int y\overset{(2-2y)^2}{(2-2y)^2}\,dy + (\overset{y+2}{3y + (2-2y)})\,{-2dy}$

$\quad y(4 - 8y + 4y^2)$

$= \int_1^0 (4y - 8y^2 + 4y^3)\,dy - (2y+4)\,dy$

$= 2y^2 - \frac{8y^3}{3} + y^4 - y^2 - 4y \Big|_1^0 = \left(y^2 + y^4 - \frac{8y}{3} - 4y\right)\Big|_1^0$

$0 - \left(1 + 1 - \frac{8}{3} - 4\right) = +2 + \frac{8}{3}$

for path 3, $x=0$ and $y=0$, thus $dx = dy = 0$ and $dI = dz\hat{k}$

$2 < z < 0$

limit of z is 2 to 0 $\int \vec{V}\cdot d\vec{l} = \int_2^0 (3y+2z)\,dz$

$= \int_2^0 (3(0)+2z)\,dz = \int_2^0 2z\,dz = \frac{z^2}{2}\Big|_2^0 = 0 - 2 = -2$

line integral for whole path is $\int_1 \vec{V}\cdot d\vec{l} + \int_2 \vec{V}\cdot d\vec{l} + \int_3 \vec{V}\cdot d\vec{l}$

So we get $0 + 2 + \frac{8}{3} - 2 = \boxed{\frac{8}{3}}$

~~Now we check stokes thm~~

First solve $\nabla\times\vec{V} = \nabla\times(6\hat{i} + yz^2\hat{j} + (3y+2z)\hat{k})$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6 & yz^2 & 3y+2z \end{vmatrix} = \left(\frac{\partial(3y+2z)}{\partial y} - \frac{\partial(yz^2)}{\partial z}\right)\hat{i} - \left(\frac{\partial(3y+2z)}{\partial x} - \frac{\partial 6}{\partial z}\right)\hat{j}$

$+ \left(\frac{\partial(yz^2)}{\partial x} - \frac{\partial(6)}{\partial y}\right)\hat{k} = (3-2yz)\hat{i} - (0-0)\hat{j} + (0-0)\hat{k}$

$= (3-2yz)\hat{i}$

$\int_S (\nabla\times\vec{V})\cdot da \qquad da = dz\,dy\,\hat{i}$

$= \int_0^1 \int_0^{2(1-y)} (3-2yz)\,dz\,dy$ type I

$= \int_0^1 \left(3z - yz^2 \Big|_0^{2(1-y)}\right) dy = \int_0^1 (3(\overset{2-2y}{2(1-y)})) - y(2(1-y))^2\,dy$

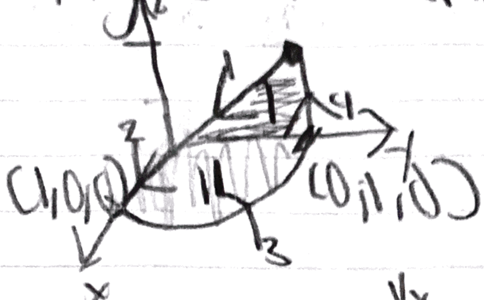
$= \int_0^1 (6-6y) - 4y(1-y)^2)\,dy$

$(1-y)(1-y) = (1+2y+y^2)$

$= \int_0^1 (6-6y) - (4y - 8y^2 + 4y^3) = \int_0^1 6 - 10y + 8y^2 - 4y^3\,dy$

$= 6y - 5y^2 + 8\frac{y^3}{3} - y^4 \Big|_0^1 = (6 - 5 + \frac{8}{3} - 1) = \boxed{\frac{8}{3}}$ ✓

8) Compute line integral of $\mathbf{v} = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}$ around path shown in fig 1.50 Check your answer using Stokes thm.



Stokes thm
$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint \mathbf{v} \cdot d\mathbf{l}$

$$\nabla \times \mathbf{v} = \nabla \times (\overset{v_r}{r\cos^2\theta}\,\hat{r} \overset{v_\theta}{-(r\cos\theta\sin\theta)}\hat{\theta} + \overset{v_\phi}{3r}\hat{\phi})$$

$$= \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\, v_\phi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial(r v_\phi)}{\partial r}\right)\hat{\theta}$$

$$+ \frac{1}{r}\left(\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right)\hat{\phi}$$

$$= \frac{1}{\cancel{r}\sin\theta}(3\cancel{r}\cos\theta)\hat{r} + \frac{1}{r}(0 - 6r)\hat{\theta} + \frac{1}{r}(2r\cos\theta\sin\theta - 2r\sin\theta\cos\theta)\hat{\phi}$$

$$= 3\cot\theta\,\hat{r} - 6\hat{\theta} + 0\hat{\phi}$$

There are two areas, the triangular face and the bottom face.

For triangular: $d\mathbf{a} = -r\,dr\,d\theta\,\hat{\phi}$

For bottom: $d\mathbf{a} = -r\sin\theta\,dr\,d\phi\,\hat{\theta}$

$$\underbrace{\int(\nabla\times\mathbf{v})\cdot d\mathbf{a}}_{\text{triangular}} + \underbrace{\int(\nabla\times\mathbf{v})\cdot d\mathbf{a}}_{\text{bottom}}$$

$$0 \qquad = \int_0^{\pi/2}\int_0^1 6r\,dr\,d\phi$$

$$= \int_0^{\pi/2} d\phi \cdot \int_0^1 6r\,dr = \frac{\pi}{2}\cdot\left(\frac{6}{2}\right) = \boxed{\frac{3\pi}{2}}$$

Now we do $\oint v \cdot dI$

For path 1

$\phi = \frac{\pi}{2}$, $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ and $\sqrt{5} < r < 0$

$v \cdot dI = \left(r\cos^2\left(\tan^{-1}\left(\frac{1}{2}\right)\right)\right)dr$

$= r(\cos^2(26.57))dr = .8\,r\,dr$

$\int_{\sqrt{5}}^{0} .8\,r\,dr = .8\left(\frac{r^2}{2}\Big|_{\sqrt{5}}^{0}\right) = .8\left(0 - \frac{5}{2}\right) = -2$

For path 2

$\theta = \pi/2$, $\phi = 0$ and r goes from 0 to 1 $0 < r < 1$

$v \cdot dI = ((r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\hat{\theta} + 3r\hat{\phi}) \cdot (dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi})$

$= (r\cos^2\theta)dr - (r^2\cos\theta\sin\theta)d\theta + (3r^2\sin\theta)\,d\phi$

$= r\cos^2\left(\frac{\pi}{2}\right)dr - 0 + 3r^2\,d\phi = 0 - 0 + 3r^2(0) = \underline{0}$

For path 3

$r = 1$, $\theta = \frac{\pi}{2}$ and ϕ goes from $0 \to \pi/2$ $0 < \phi < \frac{\pi}{2}$

$v \cdot dI = \left((1)\cos^2\left(\frac{\pi}{2}\right)\right)dr - \left(1\cos\frac{\pi}{2}\sin\frac{\pi}{2}\right)d\theta + \left(3(1)^2\sin\frac{\pi}{2}\right)d\phi$

$= 3\,d\phi$

$\int v \cdot dI = \int_0^{\pi/2} 3\,d\phi = \frac{3\pi}{2}$

For path 4

$r = \frac{1}{\sin\theta}$ and $dr = -\frac{1}{\sin^2\theta}\cos\theta\,d\theta$ and $\phi = \frac{\pi}{2}$

θ changes from $\frac{\pi}{2}$ to $\tan^{-1}\left(\frac{1}{2}\right)$

$V \cdot dl = (r\cos^2\theta)dr - (r^2\cos\theta\sin\theta)d\theta + 0$

$= \left(\frac{1}{\sin\theta}\cos^2\theta\right)\left(\frac{-\cos\theta}{\sin^2\theta}d\theta\right) - \left(\left(\frac{1}{\sin\theta}\right)^2\cos\theta\sin\theta\right)d\theta$

$= -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta}\right)d\theta = -\left(\frac{\cos^3\theta + \cos\theta\sin^2\theta}{\sin^3\theta}\right)d\theta$

$= -\frac{\cos\theta}{\sin\theta}\left(\frac{\overbrace{\cos^2\theta + \sin^2\theta}^{1}}{\sin^2\theta}\right)d\theta = -\frac{\cos\theta}{\sin^3\theta}d\theta$

$\int_{\pi/2}^{\tan^{-1}(1/2)} -\frac{\cos\theta}{\sin^3\theta}d\theta$ $\qquad u = \sin\theta$, $du = \cos\theta\, d\theta$

$= -\int_{\pi/2}^{\tan^{-1}(1/2)} \frac{du}{u^3} = +\frac{1}{2\sin^2\theta}\Big|_{\pi/2}^{\tan^{-1}(1/2)}$

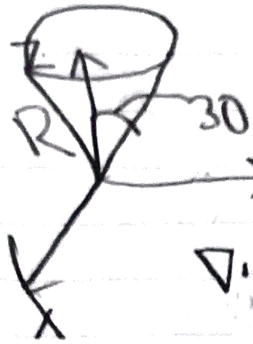
$\frac{u^{-3+1}}{-2}$

$= \frac{1}{2\sin^2(\tan^{-1}(1/2))} - \frac{1}{2} = \frac{1}{2(1/5)} - \frac{1}{2} = 2$

So the total line integral is $\oint V \cdot dl = \cancel{-2} + 0 + \frac{3\pi}{2} + \cancel{2}$

$= \boxed{\frac{3\pi}{2}}$

59) Check the divergence thm for the func $V = r^2\sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r^2\tan\theta\,\hat{\phi}$ using the volume of the cone in fig 1.52



divergence thm

$\int(\nabla\cdot v)\,d\tau = \oint v\cdot da$

$\nabla\cdot v = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

$= \frac{1}{r^2}(4r^3\sin\theta) + \frac{4r^2}{r\sin\theta}(-\sin^2\theta + \cos^2\theta) + 0$

$= \frac{\sin}{\sin}\frac{4r\sin\theta}{1} + \frac{4r}{\sin\theta}(\cos 2\theta) = \frac{4r}{\sin\theta}(\sin^2\theta + \cos 2\theta)$

$= \frac{4r}{\sin\theta}(\sin^2\theta + \cos^2\theta - \sin^2\theta) = \frac{4r}{\sin\theta}\cos^2\theta$

r: 0 to R ϕ: 0 to 2π θ = 0 to $\pi/6$ (b/c θ is at 30°)

$\int_0^{2\pi}\int_0^{\pi/6}\int_0^R \left(4r\frac{\cos^2\theta}{\cancel{\sin\theta}}\right) r^2 \cancel{\sin\theta}\, dr\, d\theta\, d\phi$

$= \overset{2\pi}{\int_0^{2\pi} d\phi} \cdot \int_0^{\pi/6}\cos^2\theta\, d\theta \cdot \overset{R^4}{\int_0^R 4r^3\, dr}$

use symbolab ↓

$\int_0^{\pi/6}\cos^2\theta\, d\theta = \int_0^{\pi/6}\frac{1+\cos 2\theta}{2} = \frac{1}{2}\int_0^{\pi/6}dx + \frac{1}{2}\int_0^{\pi/6}\cos 2\theta\, d\theta$ $u = 2\theta$ $\frac{du}{2} = \frac{\theta}{2}$

$\frac{1}{2}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right) = \frac{\pi}{12} + \frac{\sqrt{3}}{8}\times 3 = \frac{2\pi + 3\sqrt{3}}{24}$

$= \cancel{2}\pi R^4\left(\frac{2\pi + 3\sqrt{3}}{\cancel{24}_{12}}\right) = \frac{\pi R^4}{12}(2\pi + \sqrt{3})$

To find area, look at the ice cream and the cone

For cone $\theta = \pi/6$, $0 \le \phi \le 2\pi$, $0 \le r \le R$

$da = r\sin\theta \, d\phi \, dr \, \hat{\theta}$

$v \cdot da = 4r^2\cos\theta \, (r\sin\theta \, d\phi \, dr) = 4r^3\cos\theta\sin\theta \, d\phi \, dr$

$\int v \cdot da = \int_0^{2\pi}\int_0^R 4r^3\cos\theta\sin\theta \, dr \, d\phi = \int_0^{2\pi}\cos\theta\sin\theta \, d\phi \cdot \int_0^R 4r^3$

$= R^4 \cdot (2\pi\cos\theta\sin\theta)$, $\theta = \pi/6$

$= R^4 \cdot 2\pi\cos\frac{\pi}{6}\sin\frac{\pi}{6} = \cancel{2}\pi R^4\left(\frac{\sqrt{3}}{\cancel{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}\pi R^4}{2}$

For ice cream cone

$r = R$, $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi/6$

So $da = R^2\sin\theta \, d\phi \, d\theta \, \hat{r}$

$v \cdot da = (R^2\sin\theta)(R^2\sin\theta \, d\theta \, d\phi) = R^4\sin^2\theta \, d\theta \, d\phi$

$\int v \cdot da = R^4\int_0^{2\pi}\int_0^{\pi/6}\sin^2\theta \, d\theta \, d\phi = 2\pi R^4\int_0^{\pi/6}\sin^2\theta \, d\theta$

$= 2\pi R^4\left(\frac{1}{2}\left(\theta - \frac{1}{2}\sin(2x)\Big|_0^{\pi/6}\right)\right) = 2\pi R^4\left(\frac{1}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)\right)$

$2\pi R^4\left(\frac{2\pi - 3\sqrt{3}}{24}\right) = \frac{\pi R^4}{12}(2\pi - 3\sqrt{3})$

total

$\int v \cdot da = \frac{\sqrt{3}\pi R^4}{2} + \frac{\pi R^4}{12}(2\pi - 3\sqrt{3}) =$

$= \frac{6\sqrt{3}\pi R^4 + \pi R^4(2\pi - 3\sqrt{3})}{12} = \frac{\pi R^4}{12}(6\sqrt{3} + (2\pi - 3\sqrt{3})$

$= \boxed{\frac{\pi R^4}{12}(2\pi + 3\sqrt{3})}$ divergence thm does apply

62) $a \equiv \int_S da$ is called vector area of surface S. If S happens to be flat, then $|a|$ is ordinary (scalar) area

a) Find vector area of a hemispherical bowl of radius R

$da = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$\hat{x}$ and $\hat{y}$ will be 0, $\hat{z}$ or $\hat{r}$ will be $\cos\theta \, \hat{z}$

$$a = \int_S da = \int_0^{2\pi}\int_0^{\pi/2} R^2 \sin\theta \cos\theta \, d\theta \, d\phi \, \hat{z}$$

$$= 2\pi R^2 \hat{z} \int_0^{\pi/2} \sin\theta\cos\theta \, d\theta = 2\pi R^2 \left(\frac{\sin^2\theta}{2} \Big|_0^{\pi/2} \right) \hat{z}$$

$$= \pi R^2 \hat{z}$$

b) Show that $a = 0$ for any closed surface

$T = 1$ for any closed surface

$$\int_V (\nabla T) \, d\tau = \oint_S T \, da$$

$$\int_V \nabla(1) \, d\tau = \oint da = 0$$

c) Show that a is the same for all surfaces sharing the same boundary

let $a_1 \neq a_2$, if we put them together to make a closed surface then

$$\oint da = a_1 - a_2 \neq 0$$

d) Show that $a = \frac{1}{2}\oint r \times dI$

When dividing the conical surface up into infitesimal triangular surfaces. Area of 1 triangle' $da = \frac{1}{2}(b \times h)$, $r = b$ $dI = h$
Since we add up all the triangles
we will get that $a = \int da = \oint \frac{1}{2}(r \times dl) = \frac{1}{2}\oint (r \times dI)$

e) Show that $\oint (c \cdot r)\, dI = a \times c$ for any constant vector c.
Hint: let $T = c \cdot r$ in prob 1.61e
from prob 1.61] $\int_S \nabla T \times da = -\oint_P T\, dI$

let $T = c \cdot r$, then $\nabla T = \nabla(c \cdot r) = c \times (\nabla \times r) + (c \cdot \nabla) r$
$\nabla \times r = 0$ and $c \cdot \nabla = \left(c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z}\right)$

So $(c \cdot \nabla) r = \left(c_x \frac{\partial}{\partial x} + c_y \frac{\partial}{\partial y} + c_z \frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$
$= c_x \hat{i} + c_y \hat{j} + c_z \hat{k} = c$

$\int T\, dI = \oint (c \cdot r)\, dI = \int_S \nabla T \times da$

$\oint (c \cdot r)\, dI = -\int_S c \times da = -c \times \int_S da = -c \times a = a \times c$

So $\oint (c \cdot r)\, dI = a \times c$

63) Find the divergence of the func $v = \frac{\hat{r}}{r}$
First compute directly as in Eq 1.84. Test result using divergence thm as in Eq 1.85. Is there a delta func at the origin? What is the general formula for the divergence of $r^n \hat{r}$?

Eq 1.84 is $\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$

1.85 $\oint v \cdot da = \int \left(\frac{1}{R^2}\hat{r}\right) \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$

$= \left(\int_0^{\pi} \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi$

We know $v_r = \frac{1}{r}$, $v_\theta = 0$, $v_\phi = 0$

Then $\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^{\cancel{2}} \frac{1}{\cancel{r}}\right)\right) = \boxed{\frac{1}{r^2}}$ divergence

Now use divergence Thm

$\nabla \cdot v = \frac{1}{r^2}$ and $d\tau = 4\pi r^2 dr$

$\int (\nabla \cdot v) d\tau = \int \left(\frac{1}{r^2}\right) 4\pi r^2 dr = \int_0^R 4\pi \, dr = \underline{4\pi R}$

Now look at $\oint v \cdot da$, $r = R$, $v = \frac{1}{R}\hat{r}$, and $da = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$\int v \cdot da = \int \left(\frac{1}{\cancel{R}}\right)(R^{\cancel{2}} \sin\theta \, d\theta \, d\phi) = R\left(\int_0^{\pi} \sin\phi\right)\left(\int_0^{2\pi} d\phi\right)$

$= R\left(-\cos\theta \Big|_0^{\pi}\right)(2\pi) = 2\pi R(+(+1) + +(1)) = \underline{4\pi R}$

divergence thm applys.

There is no delta func at orgin b/c surface integral is dependent of R.

$v = r^n \hat{r}$ so $v_r = r^n$ $v_\theta = 0$, $v_\phi = 0$

$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2(r^n)) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^{2+n}) = \frac{1}{r^2}(n+2) r^{n+1}$

divergence when $n \neq -2$

$= \boxed{(n+2)\, r^{n-1}}$ ~~for $n \neq -2$~~

b) Find curl of $r^n \hat{r}$. Test conclusion using prob 1.61b

Prob 1.61B $\int_V (\nabla \times v)\, d\tau = -\oint_S v \times da$

$$\nabla \times v = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial(r v_\phi)}{\partial r}\right)\hat{\theta}$$

$$+ \frac{1}{r}\left(\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right)\hat{\phi}$$

$v_r = r^n$
$v_\theta = 0$
$v_\phi = 0$

$$\nabla \times v = \frac{1}{r\sin\theta}(0)\hat{r} + \frac{1}{r}(0)\hat{\theta} + \frac{1}{r}(0)\hat{\phi} = \vec{0}$$

So $\int_V 0\, d\tau = 0 = -\oint_S v \times da$

$v \times da = 0$ b/c da is in the $\hat{r}$ direction as $\vec{v}$.

64) $\nabla^2(1/r) = -4\pi\delta^3(\vec{r})$, replace r w $\sqrt{r^2+\varepsilon^2}$, watch what happens as $\varepsilon \to 0$. Let

$$D(r,\varepsilon) = -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\varepsilon^2}}$$

to demonstrate that this goes to $\delta^3(\vec{r})$ as $\varepsilon \to 0$.

a) Show that $D(r,\varepsilon) = (3\varepsilon^2/4\pi)(r^2+\varepsilon^2)^{-5/2}$

First find $\nabla^2 \frac{1}{\sqrt{r^2+\varepsilon^2}}$ using spherical coords for the laplacian.

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{1}{\sqrt{r^2+\varepsilon^2}}\right)\right) + 0 + 0$$

use symbolab

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{-r}{(r^2+\varepsilon^2)^{3/2}}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-r^3}{(r^2+\varepsilon^2)^{3/2}}\right)$$

$$= \frac{1}{r^2}\left(\frac{-3r^2\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}}\right) = \frac{-3\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}}$$

Now we bring in $-\frac{1}{4\pi}$

$$D(r,\varepsilon) = \left(\frac{3\varepsilon^2}{4\pi}\right)(r^2+\varepsilon^2)^{-5/2}$$

b) check that $D(0,\varepsilon) \to \infty$ as $\varepsilon \to 0$

$$\lim D(0,\varepsilon) = \frac{3\varepsilon^2}{4\pi(\varepsilon^2)^{5/2}} = \frac{3\cancel{\varepsilon^2}}{4\pi\varepsilon^{\cancel{5}3}} = \frac{3}{4\pi\varepsilon^3}$$

This will go to ∞ as ε approaches 0 since the denominator will get smaller and smaller thus making $D(0,\varepsilon)$ bigger.

c) check that $D(r,\varepsilon) \to 0$ as $\varepsilon \to 0$, for all $r \neq 0$

Since $r \neq 0$ then $(r^2+\varepsilon^2)$ will get bigger and bigger as ε approaches and when the denominator gets bigger then $D(r,\varepsilon)$ will start to approach 0.

d) Check that the integral of $D(r,\varepsilon)$ over all space is 1

$$\text{so } \int_0^\infty D(r,\varepsilon)\, 4\pi r^2\, dr = \int_0^\infty \frac{3\varepsilon^2}{\cancel{4\pi}(r^2+\varepsilon^2)^{5/2}} \cancel{4\pi} r^2\, dr$$

$$= 3\varepsilon^2 \int_0^\infty \frac{r^2}{(r^2+\varepsilon^2)^{5/2}}\, dr$$ use integral calculator online $$= 3\varepsilon^2\left(\frac{1}{3\varepsilon^2}\right) = 1$$