Ian Watanabe

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1. Math Bootcamp

1) a) $$(A\cdot\nabla)B = \left(A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z}\right)\hat{x} + \left(A_x \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z}\right)\hat{y}$$
$$+\left(A_x \frac{\partial B_z}{\partial x} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z}\right)\hat{z}$$

b) $$\hat{r} = \frac{\mathbf{r}}{r} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{y} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{z}$$

$$(\hat{r}\cdot\nabla)\hat{r} = \hat{x}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]$$
$$+\hat{y}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right]$$
$$+\hat{z}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$= \hat{x}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\left(\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{y}{\sqrt{x^2+y^2+z^2}}\left(\frac{-xy}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{z}{\sqrt{x^2+y^2+z^2}}\left(\frac{-xz}{(x^2+y^2+z^2)^{3/2}}\right)\right]$$
$$+\hat{y}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\left(\frac{-xy}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{y}{\sqrt{x^2+y^2+z^2}}\left(\frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{z}{\sqrt{x^2+y^2+z^2}}\left(\frac{-yz}{(x^2+y^2+z^2)^{3/2}}\right)\right]$$
$$+\hat{z}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\left(\frac{-xz}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{y}{\sqrt{x^2+y^2+z^2}}\left(\frac{-zy}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{z}{\sqrt{x^2+y^2+z^2}}\left(\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}\right)\right]$$

$$= \hat{x}\left[\frac{xy^2+xz^2-xy^2-xz^2}{(x^2+y^2+z^2)^2}\right] + \hat{y}\left[\frac{-x^2y+yx^2+yz^2-yz^2}{(x^2+y^2+z^2)^2}\right] + \hat{z}\left[\frac{-x^2z-y^2z+x^2z+y^2z}{(x^2+y^2+z^2)^2}\right]$$

$$= 0\hat{x}+0\hat{y}+0\hat{z} = \boxed{0}$$

c) $$V(r) = V_0 r^2 + V_1 \Rightarrow V(x,y,z) = V_0\left(\sqrt{x^2+y^2+z^2}\right)^2 + V_1$$

$$F = (p\cdot\nabla)E = qd\,\hat{x}\left[\hat{x}\cdot\frac{\partial}{\partial x}\left[V_0\left(x^2+y^2+z^2\right)^{1/2} + V_1\right]\right]$$

$$F = V_0 q d\left[2x\left(\frac{1}{2}\right)\frac{1}{\sqrt{x^2+y^2+z^2}}\right] = \boxed{\frac{V_0 q d x}{\sqrt{x^2+y^2+z^2}}}$$

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$$\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(r)$$

2) a) $J = \int_V e^{-r}\left(\nabla \cdot \frac{\hat{r}}{r^2}\right) d\tau = \iiint e^{-r}(4\pi\delta^3(r))\,d\tau = 4\pi(e^{-0}) = \boxed{4\pi}$

b) $\int_V f(\nabla \cdot A)\,d\tau = -\int_V A \cdot (\nabla f)\,d\tau + \oint_S fA \cdot da$

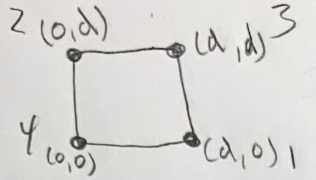
$J = \int_V e^{-r}\left(\nabla \cdot \frac{\hat{v}}{r^2}\right) d\tau = -\int_V \frac{\hat{r}}{r^2} \cdot \nabla(e^{-r})\,d\tau + \oint_S e^{-r}\frac{\hat{r}}{r^2} \cdot da$

$= -\int_V \frac{\hat{r}}{r^2}(-e^{-r}\hat{r}) \cdot 4\pi r^2 dr + \int e^{-r}\cancel{\frac{\hat{r}}{r^2} \cdot r^2} \sin\theta\, d\theta\, d\phi\, \hat{r}$

$= 4\pi\int \frac{e^{-r}}{\cancel{r^2}}\cancel{\cdot r^2}\, dr + \int e^{-r} \sin\theta\, d\theta\, d\phi = 4\pi\int_0^R e^{-r}dr + e^{-R}\int \sin\theta\, d\theta\, d\phi$

$= \left[4\pi(-e^{-r})\right]_0^R + 4\pi e^{-R} = \left[-4\pi e^{-R} + 4\pi e^{-0}\right] + 4\pi e^{-R}$

$= 4\pi e^{-0} = \boxed{4\pi}$



1) a) $E_{tot} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{d^2}\hat{r}_1 + \frac{q_2}{d^2}\hat{r}_2 + \frac{q_3}{1}\hat{r}_3\right]$

b) $\underset{(2d,0)}{E_{tot}} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{d^2}\hat{r}_1 + \frac{q_2}{\sqrt{4d^2+d^2}}\hat{r}_2 + \frac{q_3}{1}\hat{r}_3 + \frac{q_4}{4d^2}\hat{r}_4\right]$

$\underset{(0,2d)}{E_{tot}} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{\sqrt{4d^2+d^2}}\hat{r}_1 + \frac{q_2}{d^2}\hat{r}_2 + \frac{q_3}{1}\hat{r}_3 + \frac{q_4}{4d^2}\hat{r}_4\right]$

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2) $v(r) = A\frac{e^{-\lambda r}}{r}$

$E = -\nabla V = -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)\hat{r} = -A\left[\frac{r(-\lambda)e^{-\lambda r} - e^{-\lambda r}}{r^2}\right]\hat{r}$

$E = -A\left[\frac{(-r\lambda - 1)e^{-\lambda r}}{r^2}\right]\hat{r} = \boxed{\frac{A(r\lambda+1)e^{-\lambda r}}{r^2}\hat{r} = E(r)}$

$E = A(1+r\lambda)e^{-\lambda r}\frac{\hat{r}}{r^2}$

$\rho = \varepsilon_0 \nabla\cdot E = \varepsilon_0 A e^{-\lambda r}(1+r\lambda)\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) + \varepsilon_0 A\frac{\hat{r}}{r^2}\cdot\nabla\left(e^{-\lambda r}(1+r\lambda)\right)$

$\rho = \varepsilon_0 A e^{-\lambda r}(1+r\lambda)4\pi\delta^3(r) + \varepsilon_0 A\frac{\hat{r}}{r^2}\cdot\frac{\partial}{\partial r}\left[e^{-\lambda r}(1+r\lambda)\right]\hat{r}$

$= \varepsilon_0 A 4\pi\delta^3(r) + \varepsilon_0 A\frac{\hat{r}}{r^2}\left[-\lambda e^{-\lambda r}(1+r\lambda) + e^{-\lambda r}\lambda\right]\hat{r}$

$= \varepsilon_0 A 4\pi\delta^3(r) + \varepsilon_0 A\left(\frac{\hat{r}}{r^2}\right)\left[e^{-\lambda r}\lambda(\cancel{1}-\cancel{1}-r\lambda)\right]\hat{r}$

$= \varepsilon_0 A 4\pi\delta^3(r) + \varepsilon_0 A(\hat{r}/r^2)\left[-\cancel{r}\lambda^2 e^{-\lambda r}\right]\hat{r}$

$\rho = \varepsilon_0 A 4\pi\delta^3(r) - \varepsilon_0 A\frac{\lambda^2}{r}e^{-\lambda r} = \boxed{\varepsilon_0 A\left[4\pi\delta^3(r) - \frac{\lambda^2}{r}e^{-\lambda r}\right]}$

$\underline{Q} = \int\rho\, d\tau = \varepsilon_0 4\pi A\int\delta^3(r)\,d\tau - \varepsilon_0 A\lambda^2\int_0^\infty\frac{e^{-\lambda r}}{r}4\pi r^2\,dr$

$= \varepsilon_0 4\pi A - \varepsilon_0 A\lambda^2 4\pi\int_0^\infty re^{-\lambda r}\,dr$ $\quad u = r \quad du = dr \quad dv = e^{-\lambda r}dr \quad v = -\frac{1}{\lambda}e^{-\lambda r}$

$= \varepsilon_0 4\pi A - \varepsilon_0 A\lambda^2 4\pi\left[-re^{-\lambda r} + \int\frac{1}{\lambda}e^{-\lambda r}dr\right]_0^\infty$

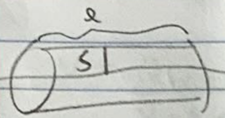
$= \varepsilon_0 4\pi A - \varepsilon_0 A\lambda^2 4\pi\left[re^{-\lambda r} - \frac{1}{\lambda^2}e^{-\lambda r}\right]_0^\infty = \varepsilon_0 4\pi A - \varepsilon_0 A\lambda^2 4\pi\left(\frac{1}{\lambda^2}\right)$

$= \varepsilon_0 A 4\pi - \varepsilon_0 A\cdot 4\pi = \boxed{0}$

$q_{in} = \lambda l$

3) $\oint E \cdot ds = q_{in}/\varepsilon_0$

$q = \lambda l$



$$E \oint ds = \frac{\lambda l}{\varepsilon_0} \Rightarrow E \cdot 2\pi s \cdot l = \frac{\lambda l}{\varepsilon_0}$$

$$\boxed{E = \frac{\lambda}{2\pi \varepsilon_0 s}}$$

b)

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1) a) $\sigma(\theta) = \varepsilon_0 \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l \cos\theta$, $A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l \cos\theta \sin\theta \, d\theta$

$C_l = \int_0^{\pi} V_0(\theta) P_l \cos\theta \sin\theta \, d\theta$

$\sigma(\theta) = \frac{\varepsilon_0}{2R} \sum_{l=0}^{\infty} (2l+1)^2 C_l P_l \cos\theta$

b) $\sigma(\theta) = \frac{\varepsilon_0}{2R} \left[C_0 P_0 \cos\theta + 9 C_1 P_1 \cos\theta + 25 C_2 P_2 \cos\theta \right]$

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2) 1. $y=0,\ V=0$

2. $x=b,\ V=0$

3. $x=-b,\ V=0$

4. $y=a;\ V=V_0$

$\hat{V}=0,\ y=0$

$\hat{V}=0,\ y=a$

$\hat{V}=-\frac{V_0 y}{a},\ b=y$

$\hat{V}=-V_0 y/a,\ -b=y$

Find $V(x,y)$

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

$$\frac{1}{X}\frac{d^2X}{dx^2}=-k^2,\quad X=Ax+B$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2}=k^2,\quad Y=Cy+D$$

$$V(x,y)=V_0\left(\frac{y}{a}\right)+\hat{V}(x,y)$$

$$\hat{V}(x,y)=\sum_{n=1}^{\infty}C_n\cosh(n\pi x/a)\sin(n\pi y/a)$$

$$\sum C_n\cosh(n\pi x/a)\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy=-\frac{V_0}{a}\int_0^a y\sin(n'\pi y/a)\,dy$$

$$\frac{a}{2}C_n\cosh\left(\frac{n\pi x}{a}\right)=-\frac{V_0}{a}\int_0^a y\sin(n'\pi y/a)\,dy$$

$$C_n=\frac{-2V_0}{a^2\cosh(n\pi b/a)}\int_0^a y\sin(n'\pi y/a)\,dy$$

$u=y$, $du=dy$, $dv=\sin\left(\frac{n'\pi y}{a}\right)dy$, $v=\left(\frac{-a}{n'\pi}\right)\cos\left(\frac{n\pi y}{a}\right)$

$$C_n=\frac{-2V_0}{a^2\cosh(n\pi b/a)}\left[-\left(\frac{ay}{n\pi}\right)\cos\left(\frac{n\pi y}{a}\right)+\int\frac{a}{n\pi}\cos\left(\frac{n\pi y}{a}\right)dy\right]_0^a$$

$$C_n=\frac{-2V_0}{a^2\cosh(n\pi b/a)}\left[\left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi y}{a}\right)-\left(\frac{ay}{n\pi}\right)\cos\left(\frac{n\pi y}{a}\right)\right]_0^a$$

$$C_n=\frac{2V_0}{a^2\cosh(n\pi b/a)}\left(\frac{a^2}{n\pi}\right)(\cos(n\pi))=\frac{2V_0}{n\pi}\cdot\frac{(-1)^n}{\cosh(n\pi b/a)}$$

$$\boxed{V(x,y)=\frac{V_0 y}{a}+\frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\cosh(n\pi b/a)}\cdot\cosh(n\pi x/a)(\sin(n\pi y/a))}$$

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3) monopole: $Q_{total} = 3q - q = 2q$

a)

$$V = \frac{Q_{tot}}{4\pi\varepsilon_0 r} = \frac{2q}{4\pi\varepsilon_0 r} = \boxed{\frac{q}{4\pi\varepsilon_0 r} = V_m}$$

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\vec{p}\cdot\hat{r}}{r^2} \qquad \vec{p} = (a\hat{z})(3q) + (0)(-q) = 3qa\hat{z}$$

$$= 3qa(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{3aq\cos\theta}{r^2}$$

$$\boxed{V(r) \approx \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{3aq\cos\theta}{r^2}\right]}$$

b) monopole: $Q_{tot} = 3q - q = 2q \quad V_{mon} = q/2\pi\varepsilon_0 r$

dipole: $\vec{p} = qa\hat{z} = qa(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$

$$V_{dipole} = \frac{(qa)\cos\theta}{4\pi\varepsilon_0 r^2} \qquad \boxed{V(r) \approx \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right]}$$

c) $Q_{tot} = 2q \quad V_{mon} = q/2\pi\varepsilon_0 r$

dipole: $\vec{p} = 3qa\hat{y} = 3qa[\sin\theta\sin\varphi\,\hat{r} + \cos\theta\sin\varphi\,\hat{\theta} + \cos\varphi\,\hat{\varphi}]$

$$V_{dip} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{3qa \cdot \sin\theta\sin\varphi}{r^2} \qquad \boxed{V(r) \approx \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\varphi}{r^2}\right]}$$