Cedric Evans

1

HW 3

3) $$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \Rightarrow$$

$$r^2 \frac{dV}{dr} = \text{const.} \Rightarrow$$

$$\frac{\text{const.}}{r^2} = \frac{dV}{dr} \Rightarrow$$

$$\cancel{\int \frac{c}{r^2} = \int \frac{dV}{dr}}$$

$$\frac{c}{r^2}\,dr = dV \Rightarrow$$

$$\int dV = \int \frac{c}{r^2}\,dr \Rightarrow$$

$$V = c \int \frac{1}{r^2}\,dr \Rightarrow$$

$$V = c\left(-\frac{1}{r}\right) + \underline{k}$$

constant from integration

$$V = -\frac{c}{r} + k$$

$$\nabla^2 V = \frac{1}{s}\frac{d}{ds}\left(s \frac{dV}{ds}\right) = 0 \Rightarrow$$

$$s \frac{dV}{ds} = c \Rightarrow$$

$$\frac{c}{s} = \frac{dV}{ds}$$

$$\int dV = \int \frac{c}{s}\,ds \Rightarrow$$

$$V = c \ln s + k$$

5) ~~First follows 2nd uniqueness theorem~~ $\Rightarrow$

$$\cancel{\oint_S V_3 E_3 \cdot da = -\int_V (E_3)^2\, d\tau}$$

5) $\nabla \cdot E_1 = \frac{1}{\varepsilon_0}\rho$, $\nabla \cdot E_2 = \frac{1}{\varepsilon_0}\rho$ $\Rightarrow$

$\oint E_1 \cdot da = \frac{1}{\varepsilon_0} Q_i$, $\oint E_2 \cdot da = \frac{1}{\varepsilon_0} Q_i$ $\Rightarrow$

$E_3 = E_1 - E_2$, $\Rightarrow$ $\nabla E_3 = 0$ $\Rightarrow$

$\oint E_3 \cdot da = 0$ $\Rightarrow$

$\oint_S V_3 E_3 \cdot da = -\int_V (E_3)^2 d\tau$ $\Rightarrow$

if $V_3 = 0$ then...

$0 = \int_V E_3^2 d\tau$

~~also~~ by this then $E_1 = E_3$

6) $U = T = V_3$

$\oint_V [V_3 \nabla^2 V_3 + \nabla V_3 \cdot \nabla V_3] d\tau = \oint_S V_3 \nabla V_3 \cdot da$ $\Rightarrow$

$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2$ $\Rightarrow$

$-\rho/\varepsilon_0 + \rho/\varepsilon_0 = 0$,

$\nabla V_3 = -E_3$ $\Rightarrow$

$\int_V E_3^2 d\tau = -\oint_S V_3 E_3 \cdot da$

13) $V(x,y) = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy \Rightarrow$

if $V_0(y) = \begin{cases} V_0, & \text{for } 0<y<\frac{a}{2} \\ -V_0, & \text{for } a/2<y<a \end{cases} \Rightarrow$

$V(x,y) = \frac{2}{a}\int_0^a (V_0 - V_0)(y)\sin\left(\frac{n\pi y}{a}\right)dy \Rightarrow$

$V(x,y) = \frac{2}{a}\int_0^{a/2}\sin\left(\frac{n\pi y}{a}\right)dy - \frac{2}{a}\int_0^{a/2}\sin\left(\frac{n\pi y}{a}\right)dy \Rightarrow$

(integration) $\Rightarrow$

$\frac{2V_0}{a}\left\{\frac{-\cos(n\pi y/a)}{(n\pi/a)}\Big|_0^{a/2} + \frac{\cos(n\pi y/a)}{(n\pi/a)}\Big|_{a/2}^{0}\right\} \Rightarrow$

$\frac{2V_0}{n\pi}\left\{-\cos\left(\frac{n\pi y}{a}\right)\Big|_0^{a/2} + \cos\left(\frac{n\pi y}{a}\right)\Big|_{a/2}^{0}\right\} \Rightarrow$

$\frac{2V_0}{n\pi}\left\{-\left(\cos\frac{n\pi a}{\cancel{(a2)}} - \cos(0)\right) + \left(\cos(0) - \cos\left(\frac{n\pi a}{2}\right)\right)\right\}$

$\Rightarrow$

$\frac{2V_0}{n\pi}\left\{1+(-1)^n - (1+(-1)^n)\right\}$

14) $V(x,y) = \frac{4V_0}{\pi} \sum \frac{1}{n} e^{(-n\pi x/a)} \sin(n\pi y/a)$,

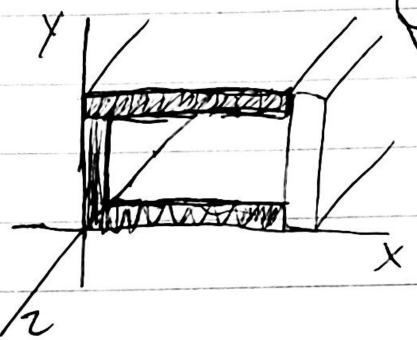
$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n} \Rightarrow$

$\sigma(y) = -\varepsilon_0 \frac{\partial}{\partial x}\left\{ \frac{4V_0}{\pi} \sum \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a) \right\} \Rightarrow$

$-\varepsilon_0 \frac{4V_0}{\cancel{\pi}} \sum \frac{1}{\cancel{n}}\left(-\frac{\cancel{n\pi}}{a}\right) e^{-n\pi y/a} \sin(n\pi y/a) \Rightarrow$

$\sigma(y) = \frac{4\varepsilon_0 V_0}{a} \sum \sin(n\pi y/a)$

15) a) $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ $\left\{ \begin{array}{l} V(x,0)=0 \\ V(x,a)=0 \\ V(0,y)=0 \\ V(b,y) = V_0(y) \end{array} \right\}$ similar to that warm up in class



$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky) \Rightarrow$

$D = 0$, $B = -A$, $\cancel{ka}\ \pi^{ka} \Rightarrow$

$V(x,y) = AC(e^{n\pi x/a} - e^{-n\pi x/a}) \sin(n\pi y/a) =$

$\underbrace{(2AC)}_{\text{Const.}} \sinh(n\pi x/a) \sin(n\pi y/a) \Rightarrow$

$V(x,y) = C_n \sinh(n\pi x/a) \sin(n\pi y/a) \Rightarrow$

$V_0 = C_n \sinh(n\pi x/a) \sin(n\pi y/a) \Rightarrow$

Fourier's Trick $\Rightarrow$

$C_n \sinh(n\pi b/a) = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a)\, dy \Rightarrow$

$$C_n = \frac{2}{a\sinh(n\pi b/a)} \int_0^a V_0(y)\sin(n\pi y/a)\,dy$$

b) $$C_n = \frac{2}{a\sinh(n\pi b/a)} V_0 \int_0^a \sin(n\pi y/a)\,dy \Rightarrow$$

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)dy = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2a}{n\pi} & \text{if } n \text{ is odd} \end{cases} \Rightarrow$$

$$V_{(x,y)} = \cancel{4V_0}\ \frac{2V_0}{\cancel{a}\sinh(n\pi b/a)} \cdot \frac{2\cancel{a}}{n\pi} = \frac{4V_0}{n\pi\sinh(n\pi b/a)} \Rightarrow$$

$$V(x,y) = \frac{4V_0}{n\pi} \sum \frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{\sinh(n\pi b/a)}$$

1b) $$\begin{cases} V=0, & x=0 \\ V=0 & x=a \\ V=0, & y=0 \\ V=0, & y=a \\ V=0, & z=0 \\ V=V_0, & z=a \end{cases}$$

$X_{(x)} = A\sin(kx) + B\cos(kx)$,

$Y_{(y)} = C\sin(ly) + D\cos(ly)$,

$Z_{(z)} = Ee^{\sqrt{k^2+l^2}} + Ge^{-\sqrt{k^2+l^2}}$

$\Rightarrow$

$k = \frac{n\pi}{a}$

$l = m\pi/a$

$E = -G \Rightarrow$

$$Z(z) = 2E\sinh\left(\pi\sqrt{n^2+m^2}(z/a)\right) \Rightarrow$$

$$V(x,y,z) = C_{n,m}\sin(n\pi x/a)\sin(m\pi y/a)\sinh\left(\pi\sqrt{n^2+m^2}\,z/a\right)$$

$$V_0 = \left[C_{n,m}\sinh\left(\pi\sqrt{n^2+m^2}\right)\right]\sin(n\pi x/a)\sin(m\pi y/a) \Rightarrow$$

$$C_{n,m}\sinh\left(\pi\sqrt{n^2+m^2}\right) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a\int_0^a \sin(n\pi x/a)\sin(m\pi y/a)\,dx\,dy \Rightarrow$$

(0 ~~odd~~ if n,m = ~~odd~~ even)

$$\cancel{V\sin\left(\frac{n\pi x}{a}\right)}$$

$$V(x,y,z) = \frac{4V_0}{a^2}\sum_n\sum_m \frac{2}{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\frac{1}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}$$

19) $V_0(\theta) = k\cos(3\theta) = k[4\cos^3\theta - 3\cos\theta] = k[\alpha P_3(\cos\theta) + \beta P_1\cos\theta]$
$\Rightarrow$

$4\cos^3\theta - 3\cos\theta = \alpha[\frac{1}{2}(5\cos^3\theta - 3\cos\theta] + \beta\cos\theta = \frac{5\alpha}{2}\cos^3\theta +$
$(\beta - \frac{3}{2}\alpha)\cos\theta$

$4 = \frac{5\alpha}{2} \Rightarrow 8 = 5\alpha = \alpha = \frac{8}{5}$

$-3 = \beta - \frac{3}{2}\alpha \Rightarrow -3 + \frac{3}{2}\alpha = \beta$

$\beta = -3 + \frac{3}{2}\left(\frac{8}{5}\right) \Rightarrow$

$\beta = -3 + \frac{24}{10} \Rightarrow \beta = -3 + \frac{12}{5} = -\frac{3}{5} \Rightarrow$

$V_0\theta = k[\frac{8}{5}P_3\cos\theta + \left(\frac{-3}{5}\right)P_1\cos\theta]$

$$V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l \; P_l(\cos\theta), & r \le R \\ \sum_{l=0}^{\infty} (B_l/r^{l+1}) P_l(\cos\theta), & r \ge R \end{cases} \Rightarrow$$

$$A_l = \frac{(2l+1)}{2R^l}\int_0^{\pi} V_0(\theta) P_l\cos\theta \sin\theta \, d\theta \Rightarrow$$

$$A_l = \frac{(2l+1)}{2R^l}\int_0^{\pi} k\left[\frac{8}{5}P_3\cos\theta + \frac{-3}{5}P_1\cos\theta\right] P_l\cos\theta\sin\theta\, d\theta \Rightarrow$$

$$\frac{k}{5}\left(\frac{2l+1}{2R^l}\right)\left\{8\int P_3\cos\theta P_l\cos\theta\sin\theta\, d\theta - 3\int P_1\cos\theta P_l\cos\theta\sin\theta\, d\theta\right\} \Rightarrow$$

$$\frac{k}{5}\left(\frac{2l+1}{2R^l}\right)\left\{8\frac{2}{(2l+1)}\delta_{l3} - 3\frac{2}{(2l+1)}\delta_{l1}\right\} \Rightarrow$$

$$\frac{k}{5}\frac{1}{R^l}\left\{8\delta_{l3} - 3\delta_{l1}\right\} \Rightarrow$$

$$V(r,\theta) = -\frac{3k}{5R} r P_1(\cos\theta) + \frac{8k}{5R^3} r^3 P_3(\cos\theta) \Rightarrow$$

$$\frac{k}{5}\left[8\left(\frac{r}{R}\right)^3 P_3\cos\theta - 3\left(\frac{r}{R}\right)P_1\cos\theta\right]$$

$$B_l = A_l R^{2l+1} \Rightarrow \begin{cases} 8kR^4/5 \\ -3kR^2/5 \end{cases} \Rightarrow$$

$$V(r,\theta) = \frac{-3kR^2}{5}\frac{1}{r^2}P_1(\cos\theta) + \frac{8kR^4}{5}\frac{1}{r^4}P_3(\cos\theta) \Rightarrow$$

$$\sigma(\theta) = \varepsilon_0 \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l \cos\theta \Rightarrow \varepsilon_0 \left[3A_1 P_1 + 7A_3 R^2 P_3\right] \Rightarrow$$

$$\varepsilon_0\left[3\left(-\frac{3k}{5R}\right)P_1 + 7\left(\frac{8k}{5R^3}\right)R^2 P_3\right] \Rightarrow$$

$$\frac{\varepsilon_0 k}{5R}\left[-9P_1\cos\theta + 56P_3\cos\theta\right]$$

22) a) $V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} \underbrace{P_l(\cos\theta)}_{P_l(1)} \quad (r>R) \Rightarrow$

$$\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} = \frac{\sigma}{2\varepsilon_0}\left[\underbrace{\sqrt{r^2+R^2}}_{r>R} - r\right]$$

$\Rightarrow r\sqrt{1+(R/r)^2} \Rightarrow$

$$r\left[1+\frac{1}{2}(R/r)^2 - \frac{1}{8}(R/r)^4\right] \Rightarrow$$

sum term

$$\sum \frac{B_l}{r^{l+1}} = \frac{\sigma}{2\varepsilon_0} r\left[1 + \frac{1}{2}(R/r)^2 - \frac{1}{8}(R/r)^4 + \ldots - 1\right] = \frac{\sigma}{2\varepsilon_0}\underbrace{\left(\frac{R^2}{2r} - \frac{R^4}{8r^3}\right)}_{\text{sum term}}$$

if $l \to \infty$ & $l_{initial} = 0$ then...

$$\frac{B_0}{r^{0+1}} \Rightarrow B_0 = \frac{\sigma R^2}{4\varepsilon_0},$$

$$V(r,\theta) = \frac{\sigma R^2}{4\varepsilon_0}\underbrace{\left[\frac{1}{r} - \frac{R^2}{4r^3}P_2(\cos\theta) + \ldots\right]}_{\text{sum term}}$$

b) For $r<R \Rightarrow 0 \le \theta \le \pi/2$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \Rightarrow$$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\varepsilon_0}\left[\underbrace{\sqrt{r^2+R^2}-r}_{\text{similar to part a)}}\right] \Rightarrow$$

$$R\sqrt{1+(r/R)^2} = R\left[1+\frac{1}{2}(r/R)^2-\frac{1}{8}(r/R)^4\right] \Rightarrow$$

$$\sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\varepsilon_0}\left[R+\frac{1}{2}\frac{r^2}{R}-\frac{1}{8}\frac{r^4}{R^3}+\dots+-r\right] \Rightarrow$$

$$A_0 = \frac{\sigma}{2\varepsilon_0} R \Rightarrow$$

$$V(r,\theta) = \frac{\sigma}{2\varepsilon_0}\left[R - r P_1 \cos\theta + \frac{1}{2R} r^2 P_2(\cos\theta)\right]$$

24) $$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0 \Rightarrow$$

let $V(s,\phi) = S(s)\,\Phi(\phi) \Rightarrow$

$$\frac{1}{s}\Phi\frac{d}{ds}\left(s\frac{dS}{ds}\right)+\frac{1}{s^2}S\frac{\partial^2\Phi}{\partial\phi^2} = 0 \Rightarrow$$

$$\frac{s^2}{S\Phi}\left[\frac{1}{s}\cancel{\Phi}\frac{d}{ds}\left(s\frac{dS}{ds}\right)+\frac{1}{s^2}\cancel{S}\frac{d^2\Phi}{d\phi^2}\right] = [0]\cdot\frac{s^2}{S\Phi} \Rightarrow$$

$$\underbrace{\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)}_{S = \text{const.}}+\underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{\Phi = \text{const}} = 0$$

$(\text{Const})_1 + (\text{Const})_2 = 0 \Rightarrow$

$\text{Const}_1 = -(\text{Const}_2) \Rightarrow$

if $C_2 = -C_2$ then $C_2 = -k^2$ for $\frac{d^2\Phi}{d\phi^2} = -k^2\Phi \Rightarrow$

$A\cos k\phi + B\sin k\phi$

if C_2 is $(-)$ the $C_1 = (+)k \Rightarrow$

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = k^2 S$$

(got stuck here)