Solutions for Homework 2

Dr. Jordan Hanson - Whittier College Dept. of Physics and Astronomy

February 26, 2022

1 Problem 2.5

Find the electric field a distance z above the center of a circular loop of radius r that carries a uniform line charge λ.

Start by filling in the pieces of the Coulomb effect:

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{dq'}{\mathscr{r}^2}\hat{\mathscr{r}} \tag{1}$$

- $\boldsymbol{\mathscr{r}} = z\hat{z} - R\hat{s}$
- $\mathscr{r} = \sqrt{z^2+R^2} = z\sqrt{1+(R/z)^2} = z\sqrt{1+\epsilon^2}$. (If $\epsilon \to 0$, this represents the far-field).
- $\mathscr{r}^2 = z^2 + R^2$
- $\hat{\mathscr{r}} = (\hat{z} - \epsilon\hat{s})/(1+\epsilon^2)^{1/2}$
- $dq' = \lambda R d\phi'$, because cylindrical coordinates are the correct choice here.
- Note that $\epsilon = R/z$, so if $z = 0$ then $\epsilon \to \infty$, and $\epsilon \to 0$ if $z \gg R$.
- $Q = 2\pi R\lambda$, the total charge.

Integrate to add up the $d\mathbf{E}$ to find $\mathbf{E}$:

$$\mathbf{E} = \frac{\lambda R}{4\pi\epsilon_0 z^2(1+\epsilon^2)^{3/2}}\int_0^{2\pi} d\phi'(\hat{z} - \epsilon\hat{s}) \tag{2}$$

By symmetry, the $\hat{s}$ term evaluates to zero. The result is

$$\mathbf{E} = \frac{2\pi\lambda R\hat{z}}{4\pi\epsilon_0 z^2(1+\epsilon^2)^{3/2}} = \frac{Q\hat{z}}{4\pi\epsilon_0 z^2(1+\epsilon^2)^{3/2}} \tag{3}$$

Checks: $\mathbf{E} = 0$ if $z = 0$ because $\epsilon \to \infty$. Also, we find the far-field effect if $z \ll R$, because $\epsilon \to 0$. The units also check out.

2 Problem 2.6

Find the electric field a distance z above a the center of a flat circular disc of radius R that carries a uniform surface charge σ. What does your formula give in the limit $R \to \infty$? Also check the case $z \gg R$.

Start by filling in the pieces of the Coulomb effect:

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{dq'}{\mathscr{r}^2}\hat{\mathscr{r}} \tag{4}$$

- $\boldsymbol{\mathscr{r}} = z\hat{z} - s\hat{s}$. As in Problem 2.5, the $\hat{s}$-component will vanish upon integration.
- $\mathscr{r}^2 = z^2 + s^2$
- $\hat{\mathscr{r}} = (z\hat{z} - s\hat{s})/(z^2+s^2)^{1/2}$
- $dq' = \sigma da' = s ds d\phi$, because cylindrical coordinates work best here.
- $Q = \pi R^2\sigma$ is the total charge.
- Let $z\tan\theta = s$, so that $ds = z\sec^2\theta d\theta$, and $\theta_0 = \tan^{-1}(R/z)$

Integrate to add up the $d\mathbf{E}$ to find $\mathbf{E}$:

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} z\hat{z} \int_0^R \frac{sds}{(s^2+z^2)^{3/2}} \tag{5}$$
$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \cos\theta|_0^{\theta_0}\,\hat{z} = \frac{\sigma}{2\epsilon_0}\left(1-\cos\theta_0\right)\hat{z} \tag{6}$$

We know what is $\tan\theta = R/z$, but what is $\cos\theta_0$? (*Hint: draw the triangle and then find the hypoteneuse*). The result is

$$\mathbf{E} = \frac{\sigma\hat{z}}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) \tag{7}$$

The units check automatically because of the units of σ (Coulombs per meter squared), and the ϵ_0 in the denominator. If $R \to \infty$, then the second term vanishes and the field is

$$\mathbf{E} \to \frac{\sigma\hat{z}}{2\epsilon_0} \tag{8}$$

This form of the field is the boundary condition near a charged surface. To check the limit that $z \gg R$, first factor a z:

$$\mathbf{E} = \frac{\sigma\hat{z}}{2\epsilon_0} z\left(z^{-1} - (z^2+R^2)^{-1/2}\right) = \frac{\sigma\hat{z}}{2\epsilon_0} z\left(z^{-1} - z^{-1}(1+(R/z)^2)^{-1/2}\right) \tag{9}$$

Now replace $(1+(R/z)^2)^{-1/2} \approx (1-(1/2)(R/z)^2)$, and notice the $1/z$ terms cancel:

$$\mathbf{E} = \frac{\sigma\hat{z}}{2\epsilon_0} z\left(\frac{1}{2z}\left(\frac{R}{z}\right)^2\right) \tag{10}$$

Multiply top and bottom by π, and recall that $Q = \pi R^2$ to find the point-charge field:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q\hat{z}}{z^2} \tag{11}$$