3.3 3.5 3.6
3.13 3.14 3.15 3.16
3.19 3.22 3.24 3.26

Matthew Bueven

EMT HW #3

3.3

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$ in spherical coords dependent on r

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \Rightarrow r^2 \frac{dV}{dr} = k$$

$$r^2 \frac{dV}{dr} = k \Rightarrow \frac{dV}{dr} = \frac{k}{r^2} \Rightarrow \boxed{V(r) = -\frac{k}{r} + C}$$

where C is constant & potential V(r) depends on r only.

$$\nabla^2 V = \frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0$$ in cylindrical coords. dependent on s

$$s\frac{dV}{ds} = k \Rightarrow \frac{dV}{ds} = \frac{k}{s}$$

$$\Rightarrow dV = \frac{k}{s}\,ds$$ integral of poly is a log

$$\Rightarrow \boxed{V(s) = k\ln(s) + C}$$

3.5) Two fields obeying Gauss's Law

$\nabla^2 E_1 = -\rho / \varepsilon_0 \qquad \nabla^2 E_2 = -\rho/\varepsilon_0$

Difference between fields $= E_1 - E_2 = E_3$

$\nabla \cdot E_3 = (-\rho/\varepsilon_0) - (-\rho/\varepsilon_0) = \underline{0}$

Green's Id

$$\int_V [T\nabla^2 U) + \nabla U \cdot \nabla T] \, d\tau = \oint (T \nabla U) \, ds$$

$E_3 =$ pot diff

$T = U = E_3 \qquad E_3 = -\nabla V_3$

$$\int_V [(E_3 \nabla^2 V_3) + (\nabla V_3 \cdot \nabla V_3] \, d\tau = \oint E_3 (\nabla V_3) \, ds$$

$\nabla^2 V_3 = 0$

$$\int_V [(E_3 (0) + (\nabla V_3)^2] \, d\tau = -\oint E_3 V_3 \, ds$$

$$\int_V (\nabla V_3)^2 d\tau = \int_V (E_3)^2 d\tau$$

$$0 = \int_V (E_3)^2 d\tau \Rightarrow \int_V (E_3)^2 d\tau = 0$$

So $E_3 = 0$ & $E_2 = E_1$

3.6

LOST

3.13) Using Ex 3.3

Running parallel to xz plane => $V(x,y) = X(x)\,Y(y)$

Boundary Cond:

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)$$

$$\Rightarrow C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

$0<y<a = 0<y<a/2$ & $a/2<y<a$

$$V(0,y) = \begin{cases} +V_0 & 0<y<a/2 \\ -V_0 & a/2<y<a \end{cases}$$

$$C_n = \frac{2}{a}\left[\int_0^{a/2} V_0 \sin\frac{n\pi y}{a}\,dy - \int_{a/2}^{a} V_0 \sin\frac{n\pi y}{a}\,dy\right]$$

$$= \frac{2V_0}{n\pi}\left\{1+(-1)^n - 2\cos\frac{n\pi}{2}\right\}$$

$C_n = 0$ when $n =$ odd so $C_n = 0$

C_n when $n=4$

$$C_n = \frac{2V_0}{n\pi}\left\{1+(-1)^4 - 2\cos\frac{4\pi}{2}\right\} = \frac{2V_0}{n\pi}\left\{1+1-2\right\} = 0$$

$n = 2, 6, 10 \ldots$ then $C_n = \frac{8V_0}{n\pi}$

$$C_n = \frac{4V_0}{n\pi}, \quad n = (4j+2), \quad j = 0, 1, 2$$

$$V(x,y) = \sum_{j=0}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$\Rightarrow$ Sub $\quad n = 4j+2$

$$\boxed{V(x,y) = \frac{8V_0}{\pi} \sum_{j=0}^{\infty} \frac{e^{-(4j+2)\pi x/a} \sin((4j+2)\pi y/a)}{4j+2}}$$

3.14 Conductor

$$\frac{dV}{dn} = \frac{-\sigma}{\varepsilon_0} \Rightarrow \sigma = -\varepsilon_0 \frac{dV}{dn}$$

$$\sigma(y) = -\varepsilon_0 \left[\frac{dV}{dx}\right]_{x=0}$$

$V(x,y)$ in infinite slot

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5} \frac{1}{n} e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)$$

Substitute.

$$\sigma(y) = -\varepsilon_0 \frac{d}{dx}\left\{\frac{4V_0}{\pi} \sum \frac{1}{n} e^{\frac{-n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)\right\}\Big|_{x=0}$$

$$= -\varepsilon_0 \frac{4V_0}{\pi} \sum \frac{1}{n}\left(-\frac{n\pi}{a}\right) e^{\frac{-n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)\Big|_{x=0}$$

$$= \varepsilon_0 \frac{4V_0}{\cancel{\pi}} \frac{1}{\cancel{n}}\left(\frac{n\pi}{a}\right) \sum e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)\Big|_{x=0}$$

$\Rightarrow$

$$\boxed{\sigma(y) = \frac{4\varepsilon_0 V_0}{a} \sum_{n=1,3,5,\ldots} \sin\left(\frac{n\pi y}{a}\right)}$$

2.15) a) General potential formula

Lapl x Ey $= \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0$

Boundary Condi $V(x,0) = 0$

$V(x,a) = 0$

$V(0,y) = 0$

$V(b,y) = V_0(y)$

$V(x,y) = (Ae^{kx} + Be^{-kx})(\sin ky + D\cos ky)$

$V(x,0) = 0 \rightarrow (Ae^{kx} + Be^{-kx})D = 0$

$0 = (A+B)(\sinh_y = A = B$

$V(x,y) = A((e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}}) \sin\frac{n\pi y}{a} = 2A(\sinh(\frac{n\pi x}{a})\sin(\frac{n\pi y}{a})$

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$V_0(y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

* Fouriers: $C_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$

$$C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

b) $V_0(y) = V_0$ then $C_n = \frac{4V_0}{n\pi\sinh(n\pi)}$

$$\Rightarrow V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5}\frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}$$

3.16 3 diff eq describe separable solutions

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = -k^2$$

$$\frac{1}{Y(y)}\frac{d^2Y}{dy^2} = -l^2$$

$$\frac{1}{Z(z)}\frac{d^2Z}{dz^2} = (k^2+l^2)$$

$\Rightarrow X(x) = A\sin(kx) + B\cos(kx)$

$Y(y) = C\sin(lx) + D\sin(ly)$

$Z(z) = Ee^{(-z\sqrt{k^2+l^2})} + Fe^{(z\sqrt{k^2+l^2})}$

$A = C = 0,\ B \neq 0,\ D \neq 0,\ k = n\pi/a,\ l = m\pi/a$

& $E + F = 0$

Put together $= V(x,y,z) = \sum_n \sum_m C_{n,m} \sin\left(\frac{n\pi x}{a}\right)$

$\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\pi\sqrt{\frac{n^2+m^2}{a}}\right)$

Evaluate $C_{n,m}$ with bound cond.

$$V_0 = \sum_n \sum_m \left[C_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\pi\sqrt{n^2+m^2}\right)\right]$$

Fourier:

$$C_{nm}\sinh\left(\pi\sqrt{n^2+m^2}\right) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dx\,dy$$

$$= \frac{16V_0}{\pi^2 nm}$$

So

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_n\sum_m \left(\frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)}{n\cdot m}\right)\left(\frac{\sinh\left(\pi\frac{\sqrt{n^2+m^2}}{a}\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}\right)$$

3.19

3.27) General solution:

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \quad \text{for } r > R$$

Polynomials gen sols

$$V(r,0) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2} - r\right)$$

Using Eq 1: $V(r,0) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

$$= \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(1) \text{ so } \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r^2+R^2} - r\right]$$

$$\sqrt{r^2+R^2} = r\left(1+\frac{R^2}{r^2}\right)^{1/2} = r\left[1+\frac{1}{2}\left(\frac{R^2}{r^2}\right) - \frac{1}{8}\left(\frac{R}{r}\right)^4\right)$$

$B_0 = \sigma R^2/4\varepsilon_0 \quad B_1 = 0 \quad B_2 = \sigma R^4/16\varepsilon_0$

$$V(r,\theta) = \frac{\sigma R^2}{4\varepsilon_0 r} - \frac{\sigma R^4}{16\varepsilon_0 r^3} P_2(\cos\theta) + \ldots$$

$$\approx \frac{\sigma R^2}{4\varepsilon_0}\left(\frac{1}{r} - \frac{R^2}{4r^3} P_2(\cos\theta)\right)$$

b) varies from 0 to $\rho/2$ for $r < R$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \Rightarrow \sum_{l=0}^{\infty} A_l r^0$$

$$\sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r^2+R^2} - r\right]$$

$$\sqrt{r^2+R^2} = R\sqrt{1+\frac{r^2}{R^2}}$$

$$\Rightarrow \sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\varepsilon_0}\left[R + \frac{r^2}{2R} - \frac{r^4}{8R^3} + \dots - r\right]$$

$A_0 = \sigma R/2\varepsilon_0 \qquad A_1 = -\sigma/2\varepsilon_0 \qquad A_2 = \sigma/4\varepsilon_0 R$

$$V(r,\theta) = \frac{\sigma}{2\varepsilon_0}\left[R - r\cos\theta \; \frac{r^2}{2R} P_2(\cos\theta)\right]$$

Southern: $V(r,\pi) = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r^2+R^2} - r\right]$

$$\sum_{l=0}^{\infty} (-1)^l A_l r^l = \frac{\sigma}{2\varepsilon_0}\left[R + \frac{r^2}{2R} - \frac{r^4}{8R^3} + \dots - r\right]$$

So

$$V(r,\theta) = \frac{\sigma}{2\varepsilon_0}\left(R + rP_1(\cos\theta) + \frac{r^2}{2R}P_2(\cos\theta)\right)$$

3.24) Laplace eq with no z dependence

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right)+\frac{1}{s^2}\left(\frac{d^2V}{d\phi^2}\right)=0$$

Sep $V(s,\phi)=S(s)\,\Phi(\phi)$

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{d}{ds}\left(S(s)\Phi(\phi)\right)\right)+\frac{1}{s^2}\frac{d^2}{d\phi^2}\left(S(s)\Phi(\phi)\right)=0$$

$$\frac{1}{s}\Phi\frac{d}{ds}\left(s\frac{dS}{ds}\right)+\frac{1}{s^2}S\frac{d^2\Phi}{d\phi^2}=0$$

$$c_1=\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right), \qquad c_2=\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

$c_1+c_2=0$ set $c_2=-k^2$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-k^2 \Rightarrow \Phi=A\cos k\phi+B\sin k\phi$$

$$c_1=k^2 \Rightarrow s\frac{d}{ds}\left(s\frac{dS}{ds}\right)=k^2S$$

$$\Rightarrow s\frac{d}{ds}\left(s\,n s^{n-1}\right)=k^2s^n \Rightarrow n^2s^n=k^2s^n$$

$$n=\pm k \Rightarrow s^2\frac{d^2S}{ds^2}+s\frac{d^2S}{ds^2}-k^2S=0$$

$$=S(s)=Cs^k+Ds^{-k}$$

Put $k=0$ in diff equation

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right)=k^2S \Rightarrow =0 \Rightarrow s\frac{dS}{ds}=E$$

$$\Rightarrow \frac{dS}{ds}=\frac{E}{s}$$

$S = C\ln \ln(s) + D$

D & C are constants too $k=0$ in diffr

$$\frac{1}{\phi} \frac{d^2\Phi}{d\phi^2} = 0 \Rightarrow \frac{d\Phi}{d\phi} = A$$

$$d\Phi = A\,d\phi \Rightarrow \Phi = A\phi + B$$

We set

$$V(s,\phi) = \sum_{k=1}^{\infty} \begin{bmatrix} s^k(a_k\cos k\phi + b_k\sin k\phi) + \\ s^{-k}(c_k\cos k\phi + d_k\sin k\phi) \end{bmatrix} + \begin{matrix}(C\ln(s) + D) \\ + (A\phi + B)\end{matrix}$$

Neglect $(A\phi + B)$

$$V(s,\phi) = \sum_{k=1}^{\infty} \begin{bmatrix} s^k(a_k\cos k\phi + b_k\sin k\phi) \\ + s^{-k}(c_k\cos k\phi + d_k\sin k\phi) \end{bmatrix} + a_0 + b_0\ln(s)$$

b_0 & a_0 are constants so pull into the front.

$$\boxed{V(s,\phi) = a_0 + b_0\ln(s) + \sum_{k=1}^{\infty}\left[s^k(a_k\cos(k\phi) + b_k\sin(k\phi)) + s^{-k}(c_k\cos(k\phi) + d_k\sin(k\phi))\right]}$$

3.26) Inside:

s^{-k} s^{-k} blow up at $s=0$

$$V(s,\phi) = a_0 + \sum_{k=1}^{\infty} s^k (a_k \cos k\phi + b_k \sin k\phi)$$

Outside: $$V(s,\phi) = a_0' + \sum_{k=1}^{\infty} \frac{1}{s^k} (c_k \cos k\phi + d_k \sin k\phi)$$

$\ln(s)$ & s^k blow up at $s \to \infty$

$$\sigma = -\varepsilon_0 \left(\frac{dV_{out}}{dn} - \frac{dV_{in}}{dn} \right)\Big|_{s=R}$$

$$\Rightarrow a \sin 5\phi = -\varepsilon_0 \sum_{k=1}^{\infty} \Big\{ -\frac{k}{R^{k+1}} (c_k \cos k\phi + d_k \sin k\phi) -$$
$$k R^{k-1} (a_k \cos k\phi + b_k \sin k\phi) \Big\}$$

a_k & $c_k = 0$ b_k & $d_k = 0$ except $k=5$, $a = 5\varepsilon_0 \left(\frac{1}{R^6} d_5 + R^4 b_5 \right)$

$\Rightarrow a = 5\varepsilon_0 (R^4 b_5 + R^4 b_5) \Rightarrow 10\varepsilon_0 R^4 b_5$

$$b_5 = \frac{a}{10\varepsilon_0 R^4}, \quad d_5 = \frac{a R^6}{10\varepsilon_0}$$

$$\boxed{V(s,\phi) = \frac{a \sin 5\phi}{10\varepsilon_0} \frac{s^5}{R^4} \text{ for } s \leq R \qquad V(s,\phi) = \frac{a \sin 5\phi}{10\varepsilon_0} \frac{R^6}{s^5} \text{ for } s \geq R}$$