# RF Field Engineer Course: A Practical Introduction

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# Review of Week 1 Material

## Review of Week 1 Material

*A pulsed radar source consists of a 2 GHz sine wave with a PRF of 20 MHz and a duty cycle of 5%. (a) How much time passes between sine waves? (b) What are the wavelength and wavenumber of the radiation? (c) The radar signal approaches the surface of the ocean at a 45 degree angle with respect to normal. At what angle with respect to normal does the radiation reflect?*

# Week 2 Summary

## Reading: *Stimson3 ch. 1-6*

- Week 1: Units and estimation. **Key skills**: mental math, **wave concepts**
  - Electromagnetic units, estimation, and decibels
  - Waves and the wave equation
  - Reflections, refraction, and diffraction
  - Phase, amplitude, frequency, polarization
- Week 2: Basic Training in Mathematics. **Key skills**: estimate pulse bandwidth, pulse trains and uncertainty principle
  - **Complex numbers**: applications to phasors and radio waves, complex imdedance of filters and antennas
  - **Fourier series and transforms**, filters and attenuation, properties of waveforms, power spectra, and spectrograms, cross-correlation and convolution
  - **Statistics and probability**: applications to noise, signal-to-noise ratio

# Complex Numbers

*Additional useful reading: The Scientist and Engineer's Guide to Digital Signal Processing, by Steven W. Smith. California Technical Publishing (2011)*
http://www.dspguide.com

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## Why understand complex numbers?

- Radar signals have two pieces: amplitude and phase
- Complex numbers also have amplitude and phase
- There is a convenient relationship between trigonometric functions and complex numbers
- Many shared properties

1. Complex numbers 1: arithmetic and calculus
2. Complex numbers 2: the Fourier series and Fourier transform

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A complex number is an expression for which one term is proportional to $j = \sqrt{-1}$:

$$z = x + jy \tag{1}$$

To call the *complex unit* $j$ is the convention in electrical engineering, and in physics it is often called $i$.

Example of complex numbers: $(3 + 4j)$, $(x_1 + x_2 j)$. Each number has a *real* part and an *imaginary* part.

# Complex Numbers 1. Reading Stimson3 ch. 6

Operations to learn:

1. Addition
2. Subtraction
3. Real part Re and Im
4. Multiplication
5. Conjugation
6. Magnitude/Norm
7. Division

Notations to learn:

1. Cartesian
2. Polar

Addition follows the pattern of two-dimensional vectors:

$$z_1 = 3 + 4j \tag{2}$$
$$z_2 = -2 + 5j \tag{3}$$
$$z_1 + z_2 = 1 + 9j \tag{4}$$

Subtraction follows the pattern of two-dimensional vectors:

$$z_1 = 3 + 4j \tag{5}$$
$$z_2 = -2 + 5j \tag{6}$$
$$z_1 - z_2 = 5 - 1j \tag{7}$$

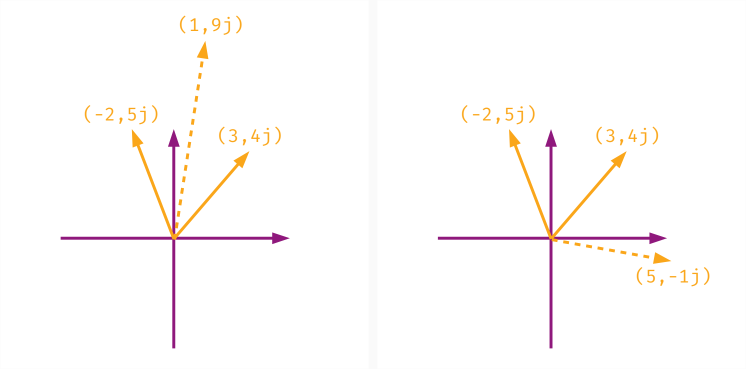


**Figure 1:** Complex addition and subtraction follows the pattern of two-dimensional vectors. (Left): Addition of $z_1$ and $z_2$. (Right): Subtraction of $z_1$ and $z_2$.

We also have the Re and Im operations:

$$z_1 = 3 + 4j \tag{8}$$

$$\mathrm{Re}\{z_1\} = 3 \tag{9}$$

$$\mathrm{Im}\{z_2\} = 4 \tag{10}$$

These are known as taking the *real*-part and the *imaginary*-part. The original complex number can be recovered by adding real and imaginary parts together:

$$z_1 = \mathrm{Re}\{z_1\} + j\,\mathrm{Im}\{z_1\} \tag{11}$$

When we add/subtract complex numbers, we combine the real parts and imaginary parts separately.

Do these operators distribute?

$$z_1 = 3 + 4j \tag{12}$$

$$z_2 = 5 + 12j \tag{13}$$

$$\mathrm{Re}\{z_1 + z_2\} = 3 + 4j + 5 + 12j = \mathrm{Re}\{8 + 16j\} \tag{14}$$

$$\mathrm{Re}\{z_1 + z_2\} = 8 = \mathrm{Re}\{z_1\} + \mathrm{Re}\{z_2\} \tag{15}$$

$$\mathrm{Im}\{z_1 + z_2\} = 16 = \mathrm{Im}\{z_1\} + \mathrm{Im}\{z_2\} \tag{16}$$

They distribute because of the associativity of addition.

*Add or subtract, then simplify:*

1. $z_1 = 7 + 7j$, $z_2 = -6 + 3j$. $z_1 + z_2 =$
2. $z_1 = 2 + 2j$, $z_2 = 3 - 3j$. $z_1 - z_2 =$
3. $z_1 = 2x + 7j$, $z_2 = 2 + 4xj$. $z_1 + z_2 =$

Let $x = -1$ and $y = 1$. *Evaluate the following expressions:*

1. $z_1 = x + yj$, $z_2 = y + xj$. $z_1 + z_2 =$
2. $z_1 = x^2 + y^2 j$, $z_2 = 2y^2 + x^2 j$. $z_1 - z_2 =$

Draw a y-axis for the imaginary part, and an x-axis for the real part, and graph the prior two exercises. Place points on the graph for $z_1$, $z_2$, and $z_1 + z_2$.

# Complex Numbers 1. Reading Stimson3 ch. 6

*Multiplication: Recall that* $j^2 = -1$.

$$z_1 = x_1 + jy_1 \tag{17}$$

$$z_2 = x_2 + jy_2 \tag{18}$$

$$z_1 \times z_2 = x_1 x_2 - y_1 y_2 + j(x_1 y_2 + x_2 y_1) \tag{19}$$

The cross-terms are straightforward, but remember the minus sign when multiplying the imaginary parts.

Examples:

1. $z_1 = 7 + 7j$, $z_2 = -6 + 3j$.
   $z_1 \times z_2 = -42 - 21 + j(21 - 42) = -63 - 21j$.
2. $z_1 = 2 + 2j$, $z_2 = 2 - 2j$. $z_1 \times z_2 = 8$. Why no imaginary part?

Another similarity with two-dimensional vectors?

$$z_1 = 4 - 1j \tag{20}$$

$$z_2 = 1 + 4j \tag{21}$$

$$z_1 \times z_2 = 8 + 15j \neq 0 \tag{22}$$

What would be the result if we were dealing with regular two-dimensional vectors? Plot Eqs. 20 and 21 with the imaginary part as the y-coordinate and the real part as the x-coordinate. Then draw a vector from the origin to the points. Do you recall the properties of the dot-product of two vectors?

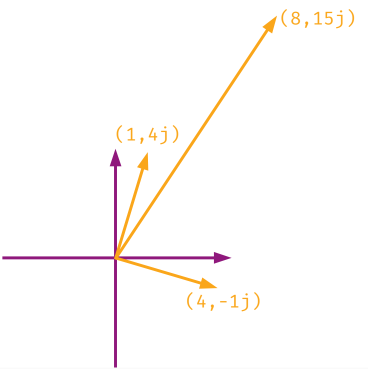


**Figure 2:** Complex multiplication resembles the *dot-product* for two-dimensional vectors, with key differences.

# Complex Numbers 1. Reading Stimson3 ch. 6

Complex conjugation: change the sign of the imaginary part.

$$z_1 = 4 - 1j \tag{23}$$

$$z_1^* = 4 + 1j \tag{24}$$

$$z_2 = = 2x + 1j \tag{25}$$

$$z_2^* = 2x - 1j \tag{26}$$

Is there a significance to $z_1 z_2^*$? What about $z_1 z_1^*$? What about $\sqrt{z_1 z_1^*}$?

What about the taking the complex conjugate of the following expression?

$$z = \frac{x_1 + jy_1}{x_2 + jy_2} \tag{27}$$

Multiply the denominator *by the complex conjugate of the denominator.*

$$z = \frac{x_1 + jy_1}{x_2 + jy_2} \times \frac{x_1 - jy_2}{x_2 - jy_2} \tag{28}$$

What happens? The real and imaginary parts *separate.* Now prove that

$$z^* = \frac{x_1 - jy_1}{x_2 - jy_2} \tag{29}$$

Let $z = (x_1 + jy_1)(x_2 + jy_2)$. Prove that

$$z^* = (x_1 - jy_1)(x_2 - jy_2) \tag{30}$$

So it seems that complex conjugation boils down to finding all the imaginary units and making them negative...*be careful about assuming this.*

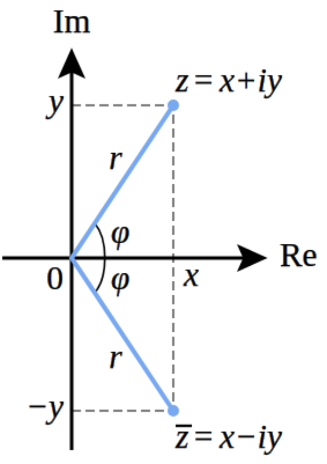


**Figure 3:** Complex conjugation flips the location of the complex number if you draw the complex number as a coordinate pair. Stay tuned for polar notation.

# Complex Numbers 1. Reading Stimson3 ch. 6

Let $z = x + jy$. Compute the following:

1. $zz^* =$
2. $\sqrt{zz^*} =$

The second item on this list has a special name: the *magnitude* or *norm* of the complex number, $|z|$.

Compute the norm of the following complex numbers:

1. $2 + 2j$
2. $3 + 4j$

Compute the norm. Do either of these simplify?

1. $\sin(x) + j\cos(x)$
2. $\cos(x) + j\sin(x)$

Division of complex numbers: remember that there are multiple steps.

$$z_1 = x_1 + jy_1 \tag{31}$$

$$z_2 = x_2 + jy_2 \tag{32}$$

$$\frac{z_2}{z_1} = \frac{x_2 + jy_2}{x_1 + jy_1} \tag{33}$$

$$\frac{z_2}{z_1} = \frac{z_2 z_1^*}{z_1 z_1^*} = \frac{z_2 z_1^*}{|z_1|^2} \tag{34}$$

$$\frac{z_2}{z_1} = \frac{\mathrm{Re}\{z_2 z_1^*\}}{|z_1|^2} + j\frac{\mathrm{Im}\{z_2 z_1^*\}}{|z_1|^2} \tag{35}$$

$$\frac{z_2}{z_1} = \frac{x_1 x_2 + y_1 y_2}{x_1^2 + y_1^2} + j\frac{x_1 y_2 - x_2 y_1}{x_1^2 + y_1^2} \tag{36}$$

Using Eq. 36, show that if $z_1 = z_2$, that $z_2/z_1 = 1$.

Evaluate the divisions below:

1. $z_1 = 1 + 4j$, $z_2 = 2 - 2j$. $z_2/z_1 =$
2. $z_1 = 1 + 1j$, $z_2 = -3 - 3j$. $z_2/z_1 =$
3. $z_1 = 1 + xj$, $z_2 = 1 - xj$, and $x = 0.5$. $z_2/z_1 =$

What if $x$ had been imaginary? Can you see a way to make the division infinity?

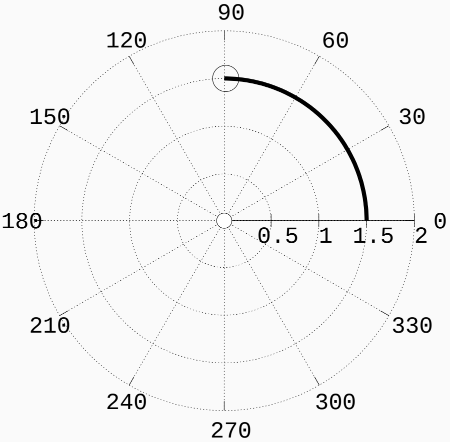


**Figure 4:** Polar coordinate systems rely on $(\rho, \phi)$ notation, rather than $(x, y)$ notation.

A useful tool for converting between polar and cartesian forms is *Euler's formula*:

$$z = |z| \cos \phi + j|z| \sin \phi \tag{37}$$

Proof of polar-notation relationship:

$$\exp(j\phi) = \sum_{i=0}^{\infty} \frac{(j\phi)^n}{n!} = \sum_{i=0}^{\infty} \frac{j^n \phi^n}{n!} \tag{38}$$

$$\exp(j\phi) = \sum_{even}^{\infty} \frac{j^n \phi^n}{n!} + \sum_{odd}^{\infty} \frac{j^n \phi^n}{n!} \tag{39}$$

$$\exp(j\phi) = \sum_{i=0}^{\infty} (-1)^n \frac{\phi^{2n}}{(2n)!} + j \sum_{i=0}^{\infty} \frac{\phi^{2n+1}}{(2n+1)!} \tag{40}$$

$$\exp(j\phi) = \cos \phi + j \sin \phi \tag{41}$$

$$|z| \exp(j\phi) = |z| \cos \phi + j|z| \sin \phi = z \tag{42}$$

# Complex Numbers 1. Reading Stimson3 ch. 6

Polar notation for complex numbers: let $z = x + jy$.

$$z = r\exp(j\phi) \tag{43}$$

$$r = |z| = \sqrt{x^2 + y^2} \tag{44}$$

$$\phi = \tan^{-1}(y/x) \tag{45}$$

Useful for multiplication and division:

$$z_1 = r_1 \exp(j\phi_1) \tag{46}$$

$$z_2 = r_2 \exp(j\phi_2) \tag{47}$$

$$z_1 \times z_2 = r_1 r_2 \exp(j(\phi_1 + \phi_2)) \tag{48}$$

$$\frac{r_2}{r_1} = \frac{r_2}{r_1} \exp(j(\phi_2 - \phi_1)) \tag{49}$$

Complex numbers in polar form are sometimes called *phasors*.

# Complex Numbers 1. Reading Stimson3 ch. 6

Convert these complex numbers from Cartesian to phasors:

1. $z_1 = 5 + 13j$
2. $z_2 = 7 - 24j$
3. $z_3 = 20 - 21j$

Divide:

1. $z_3/z_2$
2. $z_3/z_1$

Multiply:

1. $z_1 \times z_2$
2. $z_1 \times z_1^*$

How do you perform the complex conjugate of phasors? *Change the sign of the phase angle.*

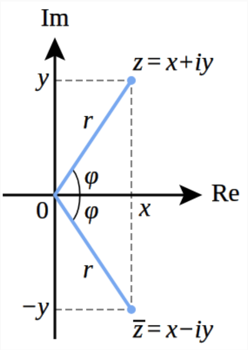


**Figure 5:** Complex conjugation flips the sign of the angle.

Notice that the procedure for finding the modulus is evident for phasors:

$$\sqrt{z_1 z_1^*} = \sqrt{r_1 r_1}\exp(j(\phi_1 - \phi_1)/2) = r_1 \tag{50}$$

Shouldn't we be saying $\pm r_1$? How does the square root function work in the complex plane?[1]

[1] Complex fractional-power functions are outside the scope of the course, as it turns out, because it requires knowledge of the topic of *branch cuts*.

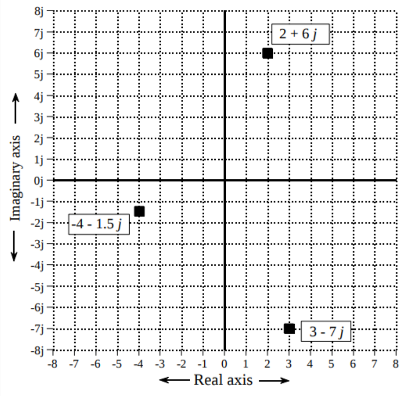


**Figure 6:** The real and imaginary axes are an *extension* of the real number line, allowing a broader representation of physical systems than just real numbers. A prime example is AC circuits.

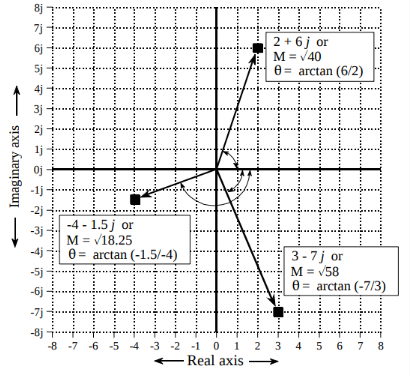


**Figure 7:** Consider the complex plane in light of trigonometry. Think of the magnitude of a complex number as the hypoteneuse of a right triangle. Then, $\mathrm{Re}\{z\} = |z| \cos\phi$, and $\mathrm{Im}\{z\} = |z| \sin\phi$, and $r = |z| = \sqrt{x^2 + y^2}$.

Recall Euler's formula. Two corollaries (prove these):

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \tag{51}$$
$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \tag{52}$$

Suppose we have now have a voltage signal

$$v_i(t) = A_i \cos(2\pi f_i t + \phi_i) \tag{53}$$

We may write

$$v_i(t) = A_i \,\mathrm{Re}\{\exp(j(2\pi f_i t + \phi_i))\} \tag{54}$$

# Complex Numbers 1. Reading Stimson3 ch. 6

What if we treat the signal as complex, but agree to take the real part at the end of our calculations?

$$v_i(t) \rightarrow A_i \exp(j(2\pi f_i t + \phi_i)) \tag{55}$$

As long as we take the real part of the right hand side, we'll have the original signal.

Now contemplate the addition of signals of the same frequency, but different amplitudes and phases. Let $x_i = 2\pi f t + \phi_i$. A signal comprised of $N$ sinusoids can be written

$$V_i(t) \rightarrow \sum_i^N a_i \exp(jx_i) \tag{56}$$

Remember that $x_i = 2\pi ft + \phi_i$. The sum of two sinusoids in the complex plane can then be written[2]

$$V(t) = a_1 \exp(jx_1) + a_2 \exp(jx_2) \tag{57}$$

1. Compute $|V|^2 = V^* V$, and $\phi_2 - \phi_1 = \pi$, $\phi_2 - \phi_1 = 0$.
2. What is $\phi_V = \tan^{-1}(\mathrm{Im}\{V\} / \mathrm{Re}\{V\})$ in each case?

Why do these results make sense? Thus, the complex numbers encapsulate the concepts of *constructive* and *destructive* interference.

[2] Notice that taking the real part distributes if the original signal is real.

# Complex Numbers 1. Reading Stimson3 ch. 6

Treating sinusoids as rotation in the complex plane is a deep subject in physics. It actually relates to some concepts in introductory physics:
`https://www.youtube.com/watch?v=jxstE6A_CYQ`

What is the effect of multiplying by a complex number as follows:

$$V_i(t) \exp(j\phi') = ? \tag{58}$$

What does this look like when graphed?

What if we continue to add terms, putting in carefully chosen phases and amplitudes? We can represent *any* periodic signal, including radio pulses and radar returns.

This is known as the **Fourier series**:

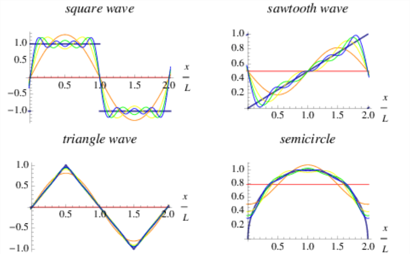


**Figure 8:** The Fourier series representing four different periodic functions.

# Examples with Octave

# Examples with Octave

Let's take the time to get octave installed on your systems:
`https://www.gnu.org/software/octave`.

If we cannot get it installed on your systems, we can always run it on the local desktops.

A good tutorial can be found here:
`https://en.wikibooks.org/wiki/Octave_Programming_Tutorial`

# Examples with Octave

Octave is a high-level *scripting* programming language. Although it is possible to write packages and compile code in octave, the most common application is executing a script that performs some analysis on digital data.

```
a = 1+1i;
b = conj(a);
a * b
```

The result of this code should be 2.0. Why? We are defining a complex number in the first line, computing the complex conjugate, and multiplying them.

Octave naturally handles vectors of numbers and matrices. Let's define a vector of complex numbers.

```
a = [1 2 3 5 7 11];
size(a)
ans = 1   6
a = a';
size(a)
```

The code in the fourth line *transposes* the vector. This means trading the rows for the columns of the vector. What begins as a $1 \times 6$ vector (one row by six columns) ends as a $6 \times 1$ vector (six rows by one column).

# Examples with Octave

Operations are as expected, but we need special notation for vectoral calculations:

```
a = 2.0;
b = 4.0;
b/a
ans = 2
b = [4.0 4.0 4.0]
b/a
ans =
2   2   2
a./b
ans =
0.5   0.5   0.5
```

Placing a dot (.) before a standard operation indicates that the operation is to be carried out in a vectoral-sense.

```
t1 = [1 2 3 4 5 6 7 8 9 10];
t2 = t1+1;
t1.*t2
ans =
2   6   12   20   30   42   56   72   90   110
```

## Examples with Octave

The colon operator (:) represents iteration in octave. Consider three cases:

```
fs = 1000.0;
t = [0.0:1/fs:10.0];
plot(t,sin(2.0 * pi * 3.0 .* t));
```

Octave should produce a plot of a sine wave with a frequency of 3.0 Hz (if the time is in seconds). How can you tell?

# Examples with Octave

Count the number of complete oscillations in 2.0 seconds. Do you see 6.0 oscillations? What is the significance of $f_s$ in the code?

```
fs = 1000.0;
t = [0.0:1/fs:10.0];
plot(t,sin(2.0 * pi * 3.0 .* t));
```

The axis command is useful for controlling the plotted region:

```
axis([0 2 -2 2]);
```

The colon operator (:) also refer to elements in a vector.

```
t(1)
ans =
0
t(2:5)
ans =
0.001   0.002   0.003   0.004
t(:)
ans =
(should be the whole vector)
t(1:end)
ans =
(should be the whole vector)
```

# Examples with Octave

Now you know how to create vectors and plot them in octave. Let's solve a few programming problems! First, let's introduce the help command.

```
help <command>
```

Type the following commands

1. help plot
2. help :

What is the purpose of the *clear* and *home* commands?

The key here is that if you are confused, the help command can boost your understanding of the correct usage. Speaking of the colon command, how do we create matrices?

## Examples with Octave

Create a matrix in octave in one of three ways:

1. Concatenate row vectors
2. Concatenate colomn vectors
3. Write it straight away

First, concatenation of row vectors:

```
a = [1:10];
b = [11:20];
A = [a; b];
display(A)
```

The semi-colon also tells the [] operators to concatenate vertically. What is the size of the matrix?

# Examples with Octave

Create a matrix in octave in one of three ways:

1. Concatenate row vectors
2. Concatenate colomn vectors
3. Write it straight away

First, concatenation of row vectors:

```
a = [1:10]';
b = [11:20]';
A = [a b];
display(A)
```

The lack of the semi-colon tells the [] operators to concatenate horizontally. What is the size of the matrix?

# Examples with Octave

Create a matrix in octave in one of three ways:

1. Concatenate row vectors
2. Concatenate colomn vectors
3. Write it straight away

First, concatenation of row vectors:

```
A = [[1 2]; [3 4];];
display(A)
```

In this case, you can actually omit the internal concatenation operators.

Can you guess how to take the transpose of the matrix?

# Examples with Octave

Create a matrix with complex numbers: First, concatenation of row vectors, and take the transpose. Do you notice something?

```
A = [[1 2i]; [3i 4];];
display(A)
```

Try using the *help* command on the transpose operator to sort out the issue. Now look up the *conj* command, and use it to take *just the transpose* of the matrix, without conjugation.

# Examples with Octave

Two additional commands that are helpful for creating square ($N \times N$) matrices:

```
ones(N,M)
zeros(N,M)
```

Create a $2 \times 2$ matrix and multiply it with a $2 \times 1$ vector. *Careful with the order; the matrix has to be on the left.*

You can imagine a matrix as an **operator** that acts on a vector. Example: two-dimensional rotation matrix (observe on board).

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{59}$$

# Examples with Octave

Code the rotation matrix (Eq. 59) for $\theta = \pi/4$, and also create three $2 \times 1$ column vectors. Multiply them and display the results. Plot them on the page and observe where they have moved.

**For those unable to perform this task at the moment, there will be a demonstration in a moment.**

- Octave command line interface (*octave-cli*)
- Octave graphical user interface (GUI)
- Variables in the workspace are persistent

# Applications to AC circuits and Radar

# Examples with Octave

We touched on complex numbers in two contexts:

- Mathematical formalism
- Programming octave

Now let's apply them to AC circuits. Consider three circuit elements: the resistor, capacitor, and inductor. They come with three symbols:

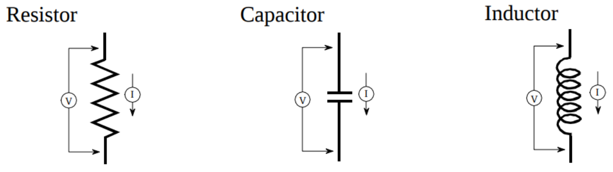


**Figure 9:** These three symbols can be combined in AC circuit diagrams to describe the flow of current and power.

# Examples with Octave

We also need the concept of Ohm's law in the AC context:

$$V(\omega) = i(\omega) Z(\omega) \tag{60}$$

The symbol $Z$ represents the *complex resistance*, or *impedance* of the circuit element. Clearly, resistance can't be complex, but we think of the phase shift introduced to the current as being carried by a complex number.

$$Z_R = R + 0i \tag{61}$$

$$Z_C = 0 + \frac{1}{j\omega C} \tag{62}$$

$$Z_L = 0 + j\omega L \tag{63}$$

To *prove* these equations, we'll need the *Fourier transform*, and some circuit analysis (future lectures). For now, think of them as just AC elements with (potentially) complex resistance.

Consider the circuit in Fig. 10. If we think of $V_{in}$ as a sinudoidal signal, and $V_{out}$ as another sinusoidal signal, we can use complex numbers to compute what will happen at a given frequency $\omega$.

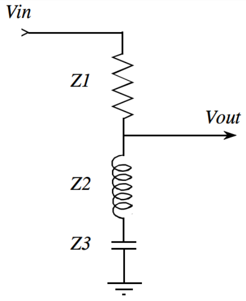


**Figure 10:** An example of an *RLC* circuit.

# Examples with Octave

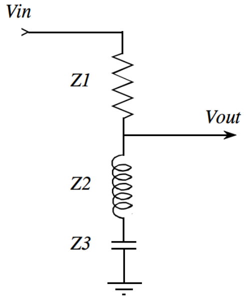


**Figure 11:** Using $V(\omega) = i(\omega)Z(\omega)$, we can solve for the ratio $V_{out}/V_{in}$.

Pretend that each AC element is some general element, and use Ohm's law to get the ratio $V_{out}/V_{in}$ (observe on board).

We should find something like this:

$$h(\omega) = \frac{Z_2 + Z_3}{Z_1 + Z_2 + Z_3} \tag{64}$$

$$\omega_{LC}^{-2} = LC \tag{65}$$

$$\tau = RC \tag{66}$$

$$k^2 = 1 - \left(\frac{\omega}{\omega_{LC}}\right)^2 \tag{67}$$

$$h(\omega) = \frac{k^4}{k^4 + (\omega\tau)^2} - j\frac{k^2\omega\tau}{k^4 + (\omega\tau)^2} \tag{68}$$

Notice that for $\omega = \omega_{LC}$, $k = 0$. What happens to $h(\omega)$ in that case? What are the limits for $\omega \ll \omega_{LC}$ and $\omega \gg \omega_{LC}$?

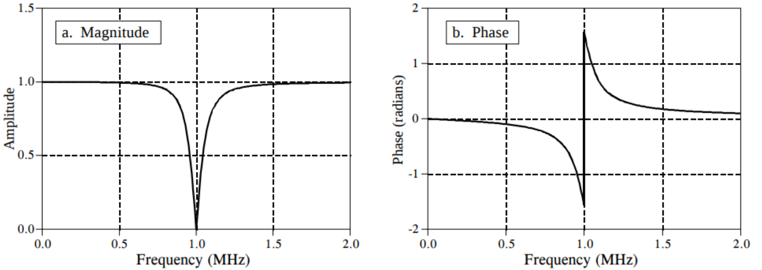


FIGURE 30-7
Notch filter frequency response. These curves are for the component values: R = 50Ω, C = 470ρF, and L = 54 µH.

**Figure 12:** The magnitude and phase of the quantity $V_{out}/V_{in}$ are plotted versus frequerncy $f = \omega/(2\pi)$ for specific values of resistance, capacitance, and inductance.

*Octave demonstration: create a script RLC.m, that reproduces Fig. 12 for given values of R, L, and C.*

- Create the script in a file entitled RLC.m
- Run the RLC script
- Plot the results
- Tweak the values of the AC elements and observe how the results change
- Understand the *RLC notch filter.*

*Application to radar:* ***Amplitude Modulation (AM)****. Using octave code, create two signals: one at a carrier frequency of 1 MHz ($f_c$) and another at 100 kHz ($f_s$).*

- Using the code, multiply the two signals and plot the result.
- Using phasors show the product of the two signals should be a third signal with frequency $f_{IF} = f_c \pm f_s$.
- Using the `randn` function, add noise to the two signals before they are multiplied.
- Using the RLC code, apply a notch filter at $f_{IF}$ *suppress noise* and *enhance signal*.

Note: often in radar, the *local oscillator* (LO) signal is mixed with radar returns of an anticipated frequency. The LO frequency is chosen to put the return at a desired IF.

# Fourier Series and Fourier Transforms

The **Fourier series** representation of a function $f(x)$ is written:

$$S(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos(nx) + B_n \sin(nx)\right) \tag{69}$$

with

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx \tag{70}$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx \tag{71}$$

Why? Imagine a radar signal is composed of several sub-signals all with different frequencies and amplitudes. There should be a way to view the *spectrum* of signals (power vs. frequency) to measure the power of each signal at each frequency. There are relationships between the time-domain and Fourier-domain version of the signal.

*Take the example of pulse trains. What is the spectrum for a sine tone at $f_c$, but modulated by a square wave with 20 percent duty cycle?*

1. The overall signal is the product of the square and the sine.
2. What is the Fourier series of a square wave?
3. The Fourier series of a sine wave is trivial.
4. Obtain the coefficients of the overall signal.
5. Is there a way to compute automatically the coefficients of any function?

Let's obtain the **Fourier series** coefficients $A_n$ and $B_n$ for a square-wave signal:

$$f(x) = 1, \quad 0 \leq x \leq \pi, \quad 0, \pi < x \leq 2\pi \tag{72}$$

(Observe on board). The result: $A_0 = 1.0$, all other $A_n = 0$, odd $B_n$ values follow $2/(n\pi)$, even $B_n = 0$ as well.

Plot a solution with the first 20 coefficients in octave. *Hint: use a for loop.*

Let's obtain the **Fourier series** coefficients $A_n$ and $B_n$ for a sine-wave signal:

$$f(x) = sin(x), \quad 0 \leq x \leq 2\pi \tag{73}$$

(Read off coefficients from the Fourier series). The result: $A_0 = 0$, $B_1 = 1$, all other $A_n = 0$ and $B_n = 0$.

**Now repeat the exercise for the product of the two signals.**

Before completing the Fourier series: Multiply the square wave and sine wave in the time-domain, and plot the result using octave.

**Demonstration.** *Note:* `signal` *package required.*

```
pkg load signal
f_s = 10.0; %Units: MHz
f_c = 1.0; %Units: MHz
f_m = 0.1; %Units: MHz
dt = 1/f_s; %Units: microseconds
t = [0:dt:300]; %Units: microseconds
m = 0.5*square(2.0*pi*f_m.*t)+0.5; %Square wave at f_m
s = sin(2.0*pi*f_c.*t); %Sine wave at f_c
d = m.*s; %Data is the product.
plot(t,d)
axis([0 300 -1.5 1.5]);
xlabel('Time (microseconds)');
ylabel('Signal (Volts)');
```

*What are the Fourier series coefficients for the sine wave times the square wave?*

$$A_n = \frac{m - m\cos(\pi m)\cos(\pi n)}{\pi(m^2 - n^2)}, \quad (m \neq n) \tag{74}$$

$$A_n = 0, \quad (m = n) \tag{75}$$

$$B_n = 0, \quad (m \neq n) \tag{76}$$

$$B_n = \frac{1}{2}, \quad (m = n) \tag{77}$$

Qualitatively, when we are not considering the carrier frequency ($m = n$), there's a function for the amplitude. When $m = n$, we just have the sine wave. The results have a 50 percent duty cycle.

The results have a 50 percent duty cycle. How can we change it to 20 percent?

```
pkg load signal
f_s = 10.0; %Units: MHz
f_c = 1.0; %Units: MHz
f_m = 0.1; %Units: MHz
dt = 1/f_s; %Units: microseconds
t = [0:dt:300]; %Units: microseconds
m = 0.5*square(2.0*pi*f_m.*t,20)+0.5; %Square wave at f_m
s = sin(2.0*pi*f_c.*t); %Sine wave at f_c
d = m.*s; %Data is the product.
plot(t,d)
axis([0 300 -1.5 1.5]);
xlabel('Time (microseconds)');
ylabel('Signal (Volts)');
```

# Fourier Series and Fourier Transforms

*Finally, is there an algorithm for the Fourier series coefficients of any signal?*

**Yes, the function is called** `fft` - the Fast Fourier Transform.

- Fourier series: represents a continuous function of time with a series of sine and cosine functions, with varying amplitudes. The amplitudes correspond to sines and cosines with different frequencies.
- Fourier transform: represent a continuous function of time as a continuous function of frequency. The function gives an amplitude for each frequency.

*Demonstration: use the* `fft` *function to compute all the Fourier series coefficients.*

First, let's split our representation of the function $f(t)$ with the Fourier series into two parts, absorb the constant into the first term, and re-scale by $1/\pi$:

$$f(t) = \frac{1}{\pi}\sum_{i=1}^{N} A_n \cos(nt) + \frac{1}{\pi}\sum_{i=1}^{N} B_n \sin(nt) \tag{78}$$

The first term is the "even" term, and the second term is the "odd" term, because of the even-ness and odd-ness of the cosines and sines.

Now our coefficient equations read (because of the missing $\pi$):

$$A_n = \int f(t)\cos(nt)dt \tag{79}$$

$$B_n = \int f(t)\sin(nt)dt \tag{80}$$

Consider the following function:

$$F(n) = A_n - jB_n = \int f(t)\cos(nt)dt - j\int f(t)\sin(nt)dt \tag{81}$$

Integrals are *linear*, in that they distribute over the same function:

$$F(n) = \int f(t)(\cos(nt) - j\sin(nt))dt = \int f(t)\exp(-jnt)dt \tag{82}$$

Now we have:

$$F(n) = \int f(t) \exp(-jnt) dt \tag{83}$$

Important question: what are the units of $t$? Don't they have to be unit-less? Otherwise we cannot take the exponential of integers times seconds. Let $t \rightarrow t/T$, where $T$ is the *period*:

$$F(n) = \frac{1}{T} \int f(t) \exp(-jnt/T) dt \tag{84}$$

Imagine a wave oscillating in the lowest *harmonic*[3]. The *frequency* of the lowest harmonic is $f = 1/T$. Suppose the integer *n selects* the harmonic (units: Hz):

$$f = \frac{n}{T} \tag{85}$$

[3] Professor: draw a picture of this.

If each integer corresponds to a phasor, we want the integer to select the *angular frequency*:

$$\omega_n = 2\pi f_n = \frac{n}{T} \tag{86}$$

This in turn causes the function $F(n)$ to be

$$F(n) = \int f(t) \exp(-j\omega_n t) dt \tag{87}$$

The lowest frequency (without $n = 0$) is $f_1 = 1/T$. Our integral is not sensitive to times longer than $T$. Also, we should make the result $F$ stable, no matter what $T$ is. Dividing by $T$ makes $F$ the average over $[-T, T]$. If $F$ is stable, we can take $T \to \infty$:

$$F(n) = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} f(t) \exp(-j\omega_n t) dt \tag{88}$$

# Fourier Series and Fourier Transforms

Finally, taking $T \to \infty$ means an infinite number of frequencies fit within $[-T, T]$. We might as well associate the frequency $n/T$ with $\omega$, since $n \in [0, \infty]$.

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t) dt} \tag{89}$$

It turns out the inverse operation looks like:

$$\boxed{F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(j\omega t) d\omega)} \tag{90}$$

Let's do some *easy* examples together. Let's obtain $F(\omega)$ for the following functions:

1. $f(t) = 1.0 \quad |t| \leq T/2$
2. $f(t) = \exp(-t/\tau)$ for $t \geq 0$, zero everywhere else.

1. $T\left(\frac{\sin(x)}{x}\right)$, where $x = \frac{\omega T}{2}$
2. $\tau/(1 + j\omega\tau)$

1. Use octave to plot each result.
2. Modify the script so that the *time-domain* signal and the *frequency-domain* signal are plotted one above the other.

To make multiple plots:

```
figure(1);
subplot(2,1,1)
plot(x,y);
subplot(2,1,2);
plot(x2,y2);
```

# Conclusion

# Reading: *Stimson3 ch. 1-6*

- Week 1: Units and estimation. **Key skills**: mental math, **wave concepts**
  - Electromagnetic units, estimation, and decibels
  - Waves and the wave equation
  - Reflections, refraction, and diffraction
  - Phase, amplitude, frequency, polarization
- Week 2: Basic Training in Mathematics. **Key skills**: estimate pulse bandwidth, pulse trains and uncertainty principle
  - **Complex numbers**: applications to phasors and radio waves, complex imdedance of filters and antennas
  - **Fourier series and transforms**, filters and attenuation, properties of waveforms, power spectra, and spectrograms, cross-correlation and convolution
  - **Statistics and probability**: applications to noise, signal-to-noise ratio