Introduction to GPS M-Code Signals for Onboarding of Navy Personnel

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Outline

Introduction to GPS M-Code Signals - Six Easy Pieces

1. Radio transmission equation and signal strengths
2. Signals: amplitude versus time
3. More on binary signals
4. Power spectral densities
5. Mixing signals, carrier frequencies
6. Auto-correlation functions

Synthesis: Putting the pieces together

Radio transmissions and signal strengths

1. Problem statement
2. Derivation of Friis transmission equation
3. Practical examples (interactive)
4. Application to GPS signals

Radio transmissions and signal strengths

Problem Statement

Given how far away a radio transmitter is, and the transmitter and receiver antenna characteristics, how do we predict the received signal strength?

Variables we need to understand:

- Distance between radio TX and RX: R
- *Gain* of radio TX and RX, G_t and G_r
- *Wavelength* of radio signal: λ
- Transmitted and received power: P_t and P_r

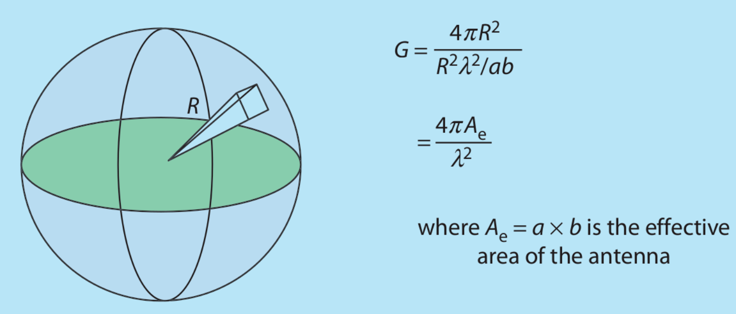


Figure 1: Adapted from Chapter 8 of *Introduction to Airborne Radar*, by Stimson, Griffiths, Baker, and Adamy. SciTech Publishing (2014).

Radio transmissions and signal strengths

How to translate the idea in the diagram into a formula for the gain:

$$s = r\theta \tag{1}$$

$$\theta_1 \approx \lambda/a \tag{2}$$

$$\theta_2 \approx \lambda/b \tag{3}$$

$$s_1 \approx R\lambda/a \tag{4}$$

$$s_2 \approx R\lambda/b \tag{5}$$

$$G = (4\pi R^2)/(s_1 s_2) = (4\pi R^2 ab)/(R^2\lambda^2) \tag{6}$$

$$A = ab \tag{7}$$

$$G = (4\pi A)/(\lambda^2) \tag{8}$$

Antenna Gain

The **gain** of a radio antenna with aperture efficiency ϵ, effective area A, radiating at a wavelength λ is

$$G = \frac{4\pi A}{\lambda^2}\epsilon \tag{9}$$

Note that the wavelength λ and frequency f are related by the *speed of light*: $f = c/\lambda$.

For radio waves in the atmosphere and space, $c \approx 0.3$ m/ns, or 0.3 km/μs.

Radio transmissions and signal strengths

Gain is typically quoted in a unit called a **dBi**: a decibel relative to isotropic sources. The decibel is a type of logarithmic unit widely used in RF fields.

$$G_{dBi} = 10 \log_{10}(G) \tag{10}$$

Interactive Question: Suppose we have a radio transmitter equipped with an antenna that operates at a wavelength of 10 cm, and an effective area of 40 cm by 40 cm. What is the gain in dBi, if the efficiency at this wavelength is 70%?

- A: 10 dBi
- B: 13 dBi
- C: 21.5 dBi
- D: 23 dBi

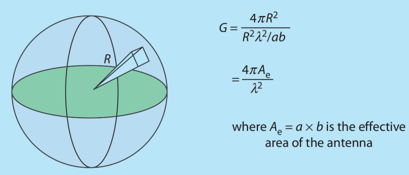
Radio transmissions and signal strengths

Now we can use the concept of *gain* to understand received power by a receiving antenna (RX) from a transmitting antenna (TX).

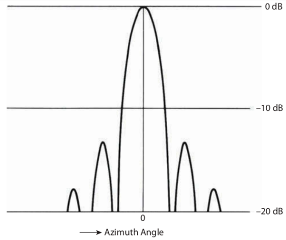
Variables we need to understand:

- Distance between radio TX and RX: R
- *Gain* of radio TX and RX, G_t and G_r
- *Wavelength* of radio signal: λ
- Transmitted and received power: P_t and P_r
- Power density: $P_t/(4\pi R^2)$

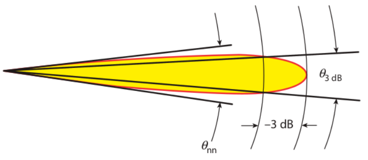
Radio transmissions and signal strengths



(a) The concept of *gain*.



(b) The concept of *radiation pattern* (dB) vs. angle.



(c) The concept of *radiation pattern*, visualized in 2D.

Figure 8-1. This three-dimensional plot shows the strength of the radiation from a pencil beam antenna.

(d) The concept of *radiation pattern*, visualized in 2D.

Now we can use the concept of *gain* to understand received power by a receiving antenna (RX) from a transmitting antenna (TX).

Suppose we have a TX transmitting P_t, received by RX with P_r. The transmitted power density will be $p = P_t/(4\pi R^2)$, augmented by G_t (Eq. 11). When p arrives at the RX, it will be collected over the A of the RX (Eq. 12). Changing variables from A to G_r (Eq. 13), we arrive at the Friis transmission formula (Eq. 14):

$$p = P_t/(4\pi R^2)G_t \tag{11}$$

$$P_r = P_t/(4\pi R^2)G_t A \tag{12}$$

$$A = \frac{G_r \lambda^2}{4\pi} \tag{13}$$

$$\frac{P_r}{P_t} = \frac{G_t G_r \lambda^2}{(4\pi R)^2} \tag{14}$$

Radio transmissions and signal strengths

Radio transmission signal strength

According to the Friis transmission formula, the ratio of received to transmitted radio power is

$$\frac{P_r}{P_t} = \frac{G_t G_r \lambda^2}{(4\pi R)^2} \tag{15}$$

Each gain factor includes non-ideal behavior due aperture efficiencies.

Interactive questions: (1) What is the RX power in dB? (2) What gain is necessary to achieve specified RX power?

Interactive question (1). For the system in the prior interactive question, imagine that the TX and RX are similar antennas separated by $R = 900$ m. We have $\lambda = 0.1$ m, $G_t = 13$ dBi, and $G_r = 13$ dBi. What is P_r/P_t in dB? This is known as *path loss.*

- A: -25 dB
- B: -75 dB
- C: -125 dB
- D: -175 dB

Interactive question (2). For the system in the prior interactive question, imagine that the design needs to change such that the overall path loss is now -60 dB. We cannot change R or λ, so all we can do is boost the gain. What should the new gain be, in dB, for the antennas?

- A: 5 dB
- B: 10 dB
- C: 20 dB
- D: 60 dB

Application to GPS signals: large path losses. In Fig. 3, three reference TX send signals to one RX. Knowing the signal strength of each signal constrains (via a system of equations) the location of the RX. The trouble is the distances represented by L1, L2, and L3.

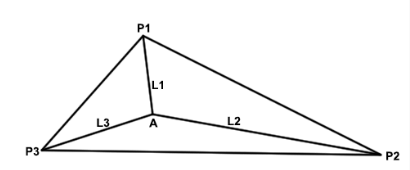


Figure 3: Trilateration with several GPS satellite TX and one RX.

A typical distance from a GPS RX to a satellite is 20,000 km.

Radio transmissions and signal strengths

Interactive question (3). What is the path loss of a TX/RX system (according to the Friis transmission equation) if $R = 20,000$ km? Assume $\lambda = 1$ m and $G = 10$ dB.

- A: -148 dB
- B: -158 dB
- C: -168 dB
- D: -204 dB

Satellite signals introduce path losses that must be counteracted if the signal is to be used for navigation.

Signals: amplitude versus time

Signals: amplitude versus time

1. Basic anatomy of a signal - part 1: units of time and amplitude
2. Example 1 with GNU Octave: plotting signals
3. **Special topic: sinusoids and complex signals**
4. Square pulses and binary sequences
5. **Special topic: counting in binary**
6. Sampling theorem
7. Example 2 with GNU Octave: aliasing, part 1

Signals: amplitude versus time

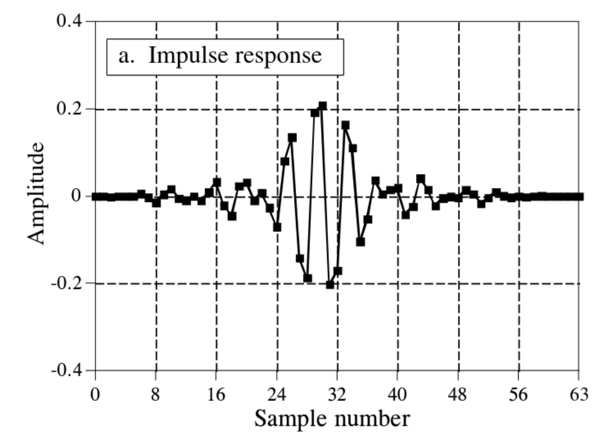


Figure 4: An example of a sampled, digitized signal representing the basic signal produced by an RF circuit when that circuit receives a pulse.

Signals: amplitude versus time

Time-domain signal properties:

1. Units of the y-axis: (A) amplitude from radio receivers as a voltage. For example, μV. (B) digitized samples, as in [0 : 2048], with the numbers corresponding to specific voltages.
2. Units of the x-axis: (A) specific times. For example, μs. (B) sampled times, such as [0 : 63], with samples related by Δt.

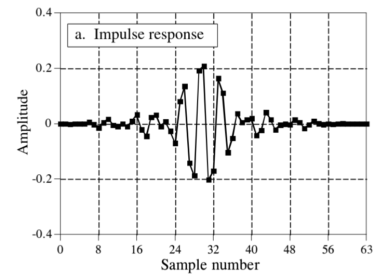


Figure 5: Amplitude versus sample number.

Signals: amplitude versus time

GNU Octave code example: “amplitude_time.m”

```
t_samples = 0:255;
sampling_frequency = 200.0; %Hz
delta_t = 1/sampling_frequency; %seconds
t = t_samples*delta_t;
frequency = 2.0; %Hz
signal = cos(2.0*pi*frequency.*t);
plot(t,signal,'linewidth',3,'color','black');
axis([0 max(t) -2 2]);
xlabel('Time (s)');
ylabel('Amplitude (V)');
set(gca(),'fontname','Calibri','fontsize',20);
grid on;
print('-dpdf','amp_time_2.pdf');
```

Signals: amplitude versus time

GNU Octave code example: “amplitude_time.m”

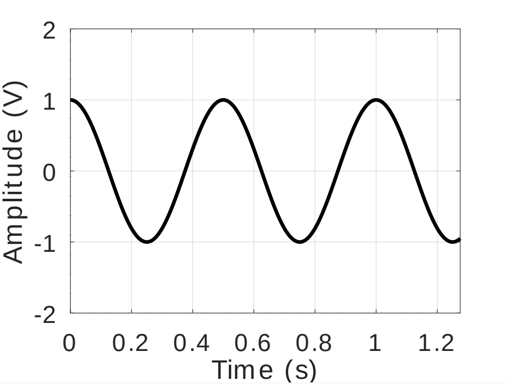


Figure 6: The figure created by the GNU Octave example.

Signals: amplitude versus time

GNU Octave exercise: Make the following changes, separately, to understand the impact on the signal:

```
t_samples = 0:511;
signal = 2*cos(2.0*pi*frequency.*t);
frequency = 4.0; %Hz
```

Other important quantities:

- *Power*: square each amplitude sample, and sum each value.
- *Average value*: sum the amplitude samples and divide by N.

```
P = sum(signal.^2); %Power
average_value = mean(signal); %Average
```

Other important quantities:

- *Power*: square each amplitude sample, and sum each value. What happens to the power when you calculate it with $f = 2$ Hz or $f = 4$ Hz?
- *Average value*: sum the amplitude samples and divide by N. What should be the average of a sinusoid, when the signal includes a whole number of periods?

Signals: amplitude versus time

Complex Signals: A tool for understanding signals

Let $j = \sqrt{-1}$, f be the frequency of a sinusoidal signal, and t be the time. Euler's theorem states that

$$\exp(2\pi jft) = \cos(2\pi ft) + j\sin(2\pi ft) \tag{16}$$

The *real* part of the signal is $\cos(2\pi ft)$, and the *imaginary* part is $\sin(2\pi ft)$.

Also note, exponentials and complex numbers have some useful properties:

- $\exp(x)\exp(y) = \exp(x+y)$
- $(x+jy) + (a+jb) = (x+a) + j(y+b)$

Treating real signals as complex

$$A\cos(2\pi ft) = \Re\{A\exp(2\pi jft)\} \tag{17}$$

By "taking the real part," we recover the cosine. "Taking the imaginary part" of a complex exponential gives the sine portion:

$$A\sin(2\pi ft) = \Im\{A\exp(2\pi jft)\} \tag{18}$$

Treating signals as exponentials allows us to multiply them easily, to understand product signal behavior.

Interactive Question (4). What happens when you multiply two signals with frequencies f_1 and f_2? (Start complex, then take the real part at the end).

Interactive Question (5). What happens when you multiply two signals with the same frequency f, but with different phases?

$$s_1(t) = a\cos(2\pi ft + \phi_1) \tag{19}$$

$$s_2(t) = a\cos(2\pi ft + \phi_2) \tag{20}$$

$$T(t) = s_1(t)s_2(t) \tag{21}$$

(Treat them as complex, then take the real part at the end).

Final GNU Octave exercise: Make the following change to the amplitude versus time code to understand the impact on the signal:

```
t_samples = 0:255;
sampling_frequency = 10.0; %Hz
delta_t = 1/sampling_frequency; %seconds
t = t_samples*delta_t;
frequency = 2.0; %Hz
```

How does this distort the signal?

- *Frequency greater than sampling frequency divided by 2*
- *Frequency less than sampling frequency divided by 2*

More on binary signals

More on binary signals

1. Binary numbers: digital signals from analog signals
2. Practical examples (interactive)
3. Chips and spreading symbols, chip rate, and data rate
4. Example 3 with GNU Octave: plotting a pseudo-random code sequence
5. Binary offset carriers

More on binary signals

First of all, what is an *analog* circuit?

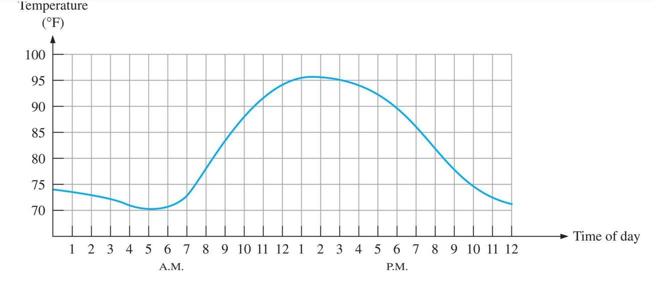


Figure 7: An example of an analog signal from a temperature sensor, converted from voltage.

More on binary signals

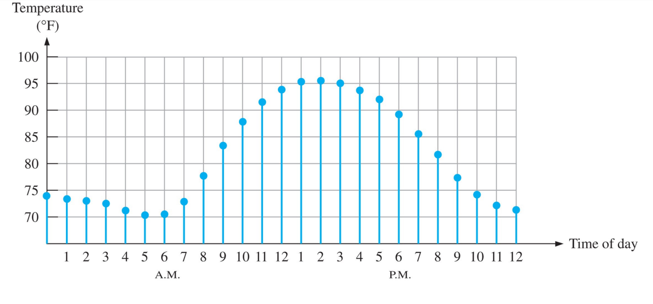


Figure 8: An example of that same signal, digitized and sampled.

Digital data forms the basis of computation:

- Noise issues, lossless transmission
- Constructed from *digits* … 1 and 0

More on binary signals

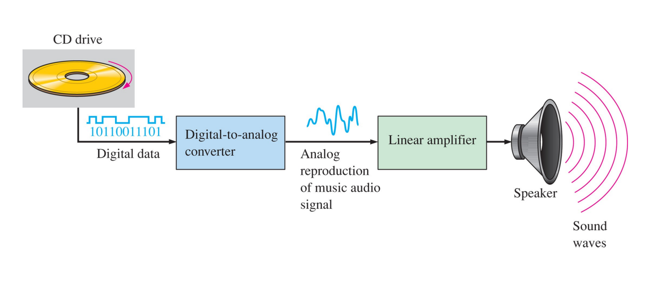


Figure 9: An example of a digital signal converted from binary to analog voltage signal.

More on binary signals

How do we build up digital data from analog signals?

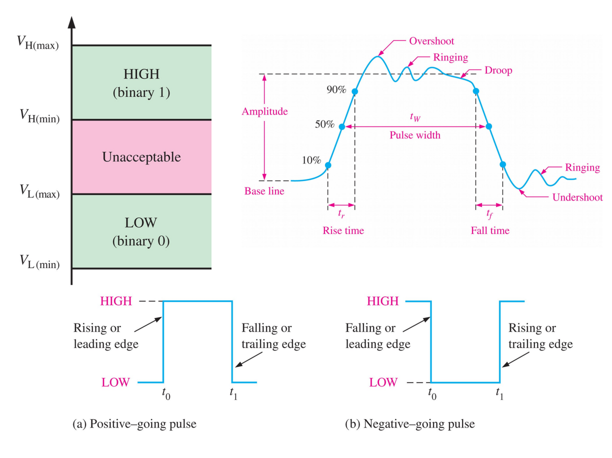


Figure 10: Logical "1" and "0."

More on binary signals

Terminology for digital signals:

1. **Frequency**, f and **period**, T: Signals per second, time between signals ($f = 1/T$).
2. **Pulse width**, t_{W}: time duration a pulse is HIGH.
3. **Duty cycle**: $t_{\mathrm{W}}/T \times 100\%$

More on binary signals

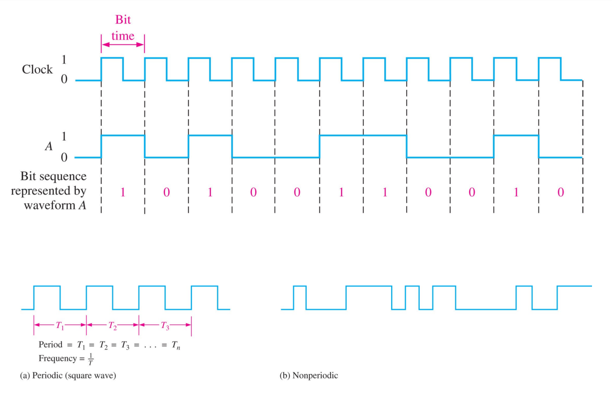


Figure 11: A clock signal is an example of a digital bitstream: alternating 1 and 0. It has a period and a frequency. Data can be *periodic* or *non-periodic*. (Professor: do some examples here).

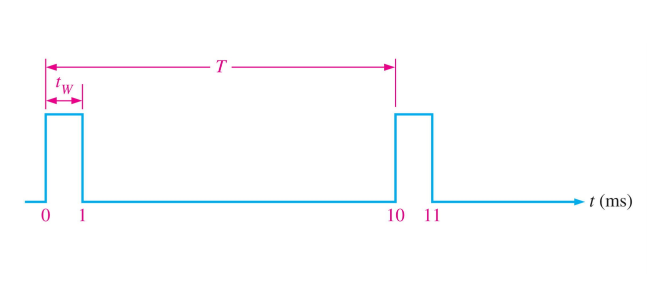


Figure 12: A periodic pulse demonstrating the concept of duty cycle. (Professor: do an example here and vary duty cycle).

What is the duty cycle in Fig. 12? What is the frequency?

More on binary signals

A script to produce a binary signal (install the “signal” package):

```
pkg load signal
t_samples = 0:255;
sampling_frequency = 200.0; %Hz
delta_t = 1/sampling_frequency; %seconds
t = t_samples*delta_t;
frequency = 2.0; %Hz
signal = square(2.0*pi*frequency.*t);
plot(t,signal,'linewidth',3,'color','black');
axis([0 max(t) -2 2]);
xlabel('Time (s)');
ylabel('Amplitude (V)');
set(gca(),'fontname','Calibri','fontsize',20);
grid on;
print('-dpdf','amp_time_3.pdf');
```

More on binary signals

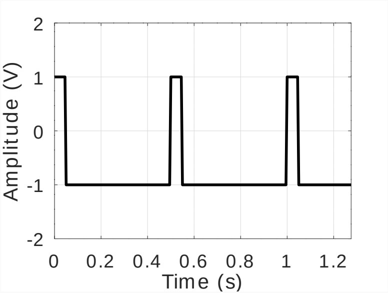


Figure 13: A periodic pulse demonstrating the concept of duty cycle. (Professor: do an example here and vary duty cycle). What is the frequency, period, and duty cycle of the signal?

More on binary signals

Make the following change to the code:

```
...
frequency = 2.0; %Hz
signal = square(2.0*pi*frequency.*t,10);
plot(t,signal,'linewidth',3,'color','black');
...
print('-dpdf','amp_time_4.pdf');
```

Interactive Question (6). If a binary signal repeats with period 1 μs, but has a pulse with of 0.2μs, what is the duty cycle?

- A: 10 percent
- B: 20 percent
- C: 200 percent
- D: 2 percent

More on binary signals

Interactive Question (7). Draw in your notes a binary signal representing the following bit sequence: 11001101. Bonus: How could we make our code represent this signal?

```
t_samples = 0:255;
sampling_frequency = 200.0; %Hz
delta_t = 1/sampling_frequency; %seconds
t = t_samples*delta_t;
period = length(t_samples)/8;
seq = [1 1 0 0 1 1 0 1];
signal = []
for i=seq
        signal = [signal ones(1,period)*i];
endfor
plot(t,signal,'linewidth',3,'color','black');
axis([0 max(t) -2 2]);
xlabel('Time (s)');
ylabel('Amplitude (V)');
set(gca(),'fontname','Calibri','fontsize',20);
grid on;
print('-dpdf','amp_time_4.pdf');
```

Encoding our signals. What if we don't want someone to intercept our data and use it? Consider the following signal:

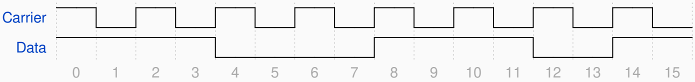


Figure 14: (Top) A binary carrier signal, and (Bottom) a binary sequence representing some data: 11001101, at the same rate as the clock signal.

Notice: The pulse width of the "Carrier" signal in Fig. 14 is one half the minimum pulse width of the "Data" signal. For one period of the carrier, we have one data bit. The fastest *data rate* is equal to the carrier frequency.

More on binary signals

Encoding our signals. What if we don't want someone to intercept our data and use it? Consider the following signal:

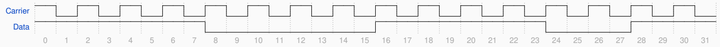


Figure 15: (Top) A binary carrier signal, and (Bottom) a binary sequence representing the same data as Fig. 14, 11001101, at one half the rate as in Fig. 14.

Notice: The pulse width of the "Carrier" signal in Fig. 14 is one fourth the minimum pulse width of the "Data" signal. For two periods of the carrier, we have one data bit. The fastest *data rate* is equal half of the carrier frequency.

More on binary signals

Now multiply the carrier and the data signals. The result is a *pseudo-random code*:

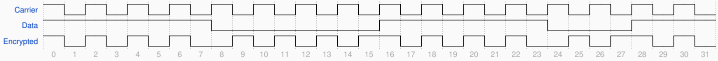


Figure 16: (Top) A binary carrier signal, (Middle) The data stream. (Bottom) the product of the carrier and the data stream.

Notice: The encrypted data stream now appears as random data at the carrier frequency. If the data stream is already encrypted through binary techniques, the signal at the bottom of Fig. 16 would be very difficult to intercept and use.

More on binary signals

The spreading symbol: By taking the data bit and multiplying it by 1010, the data is said to be *spread* by a *spreading symbol* 1010.

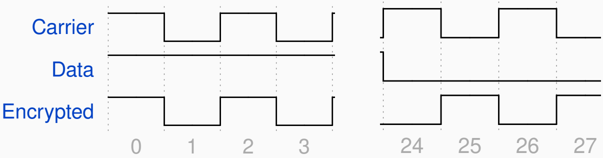


Figure 17: (Left) The first data bit. (Right) The second to last data bit.

Interactive Question (8). Suppose we have a binary carrier with a period of 0.1 μs, and a data stream of 1011 0010 to transmit. (a) Write out the encrypted data sequence by multiplying the carrier with the data stream. Assume the data rate is half the frequency of the carrier. (b) How long (in time) is the encrypted data?

Some vocabulary:

1. **Chip**: the unit of the data in the encrypted result.
2. **Symbol**: the pieces of data made from multiple chips.
3. **Chip rate**: the rate of the encrypted signal.
4. **Symbol rate**: the rate of the data stream.
5. **Spreading factor**: the ratio of the chip rate to the data rate.

Power spectral densities

1. Basic anatomy of a signal - part 2: units of frequency and power
2. The Fourier transform and FFT algorithm
3. Example 4 with GNU Octave: aliasing, part 2
4. Units: dB, dBw and dBm, and dB/Hz
5. Practical examples (interactive)

Mixing signals, carrier frequencies

Mixing signals, carrier frequencies

1. Mathematics of mixing signals:
    - Trigonometric identities
    - Complex signals
2. Block diagram of mixing
3. Example 5 with GNU Octave: moving a signal to a carrier frequency

Auto-correlation functions

1. Mixing a signal with itself: auto-correlation function (ACF)
2. Example 6 with GNU Octave: auto-correlation of a square pulse
3. Applications to GPS signal timing