A Simple Model for Antarctic Near-Surface Index of Refraction and Radio Pulse Trajectories

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Abstract

The standard model for the index of refraction in Antarctic firn is presented as a function of ice depth. The model provides a good fit to the data, but does not account for small-scale fluctuations in snow density in the upper regions of the firn. The standard model implies curved paths for radio pulses. In specific situations, Fermat's principle implies that the *shadowing effect* should occur. However, data collected in the 2011-12 season in Moore's Bay directly contradicts basic shadowing. A more complete model should include surface propagation due to ray-trapping between local snow layers.

1 A Derivation of the Density Profile

The *compressibility* χ of a simple block of material with volume $l^3 = v$ and uniform pressure p is defined as

$$\chi = -\frac{1}{v}\frac{\Delta v}{\Delta p} \tag{1}$$

Rearranging,

$$-\frac{\Delta v}{v} = \chi \Delta p \tag{2}$$

Suppose that a block comprised of snow, ice and air, known as *firn*, with volume $l^3 = v$ is compressed by a pressure p originating from a force in one direction. The density of the block is m/l^3, where m is the mass. The length of the block on the compressed side becomes $l - \epsilon = \Delta l$, the volume decreases by $\Delta v = v_f - v_i = -\epsilon l^2$, and the change in density is

$$\Delta\rho = \rho_f - \rho_i = m\left(\frac{1}{v_f} - \frac{1}{v_i}\right) = -\frac{m\Delta v}{v_f v_i} = -\rho_f \frac{\Delta v}{v} \tag{3}$$

The initial volume $v_i = v = l^3$. Substituting Eq. 2 into Eq. 3,

$$\Delta\rho = \rho_f \chi \Delta p \tag{4}$$

$$\Delta p = (\chi\rho_f)^{-1}\Delta\rho \tag{5}$$

Dividing both sides by Δl, and taking the limit that $\Delta l \to 0$ and that χ does not depend on l,

$$\frac{\Delta\rho}{\Delta l} = (\rho_f \chi)^{-1} \frac{\Delta p}{\Delta l} \tag{6}$$

$$\boxed{p' = (\chi\rho_f)^{-1} \rho'} \tag{7}$$

This compressibility result relates changes in pressure to changes in density, and will become useful for simplification in the following derivation. Let the region of firn be described by N firn blocks labelled by j, with varying density ρ_j. Let the snow surface correspond to $j = N$, $z = 0$, and the beginning of solid ice (the bottom of the firn) correspond to $j = 0$, $z = -h$. Each block has a height Δz, and a volume $v = A\Delta z$. The normal force $f = p_n A$ on block j must oppose the weight of block j, and the weight of the firn mass above block j, summed up to the surface:

$$Ap_{nj} = g\left(m_j + M\right) \tag{8}$$

$$M = \sum_{i=j+1}^{N} m_i = \sum_{i=j+1}^{N} \rho_i A dz g \tag{9}$$

$$m_j = \rho_i A dz g \tag{10}$$

Combining the Eqs. 8-10, and cancelling the common A factor,

$$p_{nj} = g \sum_{i=j}^{N} \rho_i dz \tag{11}$$

Taking the limit $dz \to 0$, $A \to \infty$ in a way that leaves $v = Adz$ constant,

$$p(z) = g \int_{z}^{0} \rho(z')dz' \tag{12}$$

Taking the derivative of both sides, applying the fundamental theorem of calculus, and rearranging

$$-g^{-1}p'(z) = \rho(z) - \rho(0) \tag{13}$$

Substituting Eq. 7 into Eq. 13 (relabelling the final density ρ_0), and rearranging

$$\rho' = -\left(g\chi\rho_0\right)\rho(z) + \left(g\chi\rho_0\right)\rho(0) \tag{14}$$

Defining two constants k_1 and k_2, Eq. 14 may be put into the following form:

$$\rho' = k_1\rho(z) + k_2 \tag{15}$$

Try the following solution, with boundary conditions $\rho(0) = \rho_s$ and $\rho(z \to -\infty) \to \rho_i$:

$$\rho(z) = A - B\exp(z/z_0) \tag{16}$$

For now, assume z_0 is a free parameter. Equation 16 solves Eq. 15 if $A = \rho_i$ and $B = \Delta\rho = \rho_i - \rho_s$, giving the final solution

$$\boxed{\rho(z) = \rho_i - \Delta\rho\exp(z/z_0)} \tag{17}$$

Physical insight on the remaining parameter z_0 may be gained by using Eq. 15 with Eq. 17 to find

$$k_1 = z_0^{-1} \tag{18}$$

From Eq. 14

$$\boxed{z_0^{-1} = g\chi\rho_0} \tag{19}$$

In conclusion, the steepness of the density profile with depth (controlled by z_0) depends directly on the compressibility of the firn.

2 Remark about the Index of Refraction Profile

For dielectric materials like snow and ice, the index of refraction is usually approximated as a linear equation of density: $n(z) \approx 1 + b\rho(z)$, and this is usually justified through expanding the Landau-Lifshitz-Looyenga equation (see below). Thus, the index versus depth of the metamorphosis from snow to ice follows a function like

$$n(z) = n_0 - n_1 e^{z/z_0} \tag{20}$$

At $z = 0$, $n(0) = n_s$ (snow), and as $|z| \gg z_0$, for $z < 0$, $n = n_{ice}$. Letting $\Delta n = n_{ice} - n_s$, the index equation becomes

$$\boxed{n(z) = n_{ice} - \Delta n e^{z/z_0}} \tag{21}$$

Notice that the compressibility of the firn (proportional to z_0^{-1}) influences also the steepness of the index of refraction profile. This is true if the index is a linear function of the density.

3 The Landau-Lifshitz-Looyenga Equation

Let the complex dielectric constants of snow and ice be ϵ_i and ϵ_s, and the dielectric constant of their mixture be $\epsilon(z)$. Further, let the two separate dielectrics each have volume fractions v_i and v_s, with volume fractions $v_i + v_s = 1$, $\rho(z) = v_i\rho_i + v_s\rho_s$. The Landau-Lifshitz-Looyenga equation gives

$$\epsilon(z) = \left(v_i\epsilon_i^{1/3} + v_s\epsilon_s^{1/3}\right)^3 \tag{22}$$

Let the real and imaginary parts of dielectric constants follow the notation $\epsilon = \epsilon' + i\epsilon''$, and defined the *loss tangent* as $\tan\delta = \epsilon''/\epsilon'$. Ice has a loss tangent of order 10^{-3} at RF frequencies, and the loss tangent of snow is smaller. To first order in $\tan\delta_i$ and $u = v_2/v_1 < 1$, with $\alpha = (\epsilon_2'/\epsilon_1')^{1/3}$, it may be shown that

$$\sqrt{\Re\epsilon(z)} = n(z) \approx v_i^{3/2}\epsilon_i'^{1/2}(1 + u\alpha) \tag{23}$$

Setting $v_2 = 0$, $v_1 = 1$ (or $u = 0$) reproduces the expected $n = \sqrt{\epsilon_i'}$ for pure ice. With $\beta = 3/2(\rho_s/\rho_i)$, $v_i^{3/2}$ is approximately

$$v_i^{3/2} \approx \left(\frac{\rho(z)}{\rho_i}\right)^{3/2} (1+u\beta)^{-1} \tag{24}$$

Combining equations, and recalling that $\epsilon_i'^{1/2} = n_{ice}$, the result is

$$\frac{n(z)}{n_{ice}} \approx \left(\frac{1+u\alpha}{1+u\beta}\right)\left(\frac{\rho(z)}{\rho_i}\right)^{3/2} \tag{25}$$

Expanding to first order about $\rho(z)/\rho_i = 1$:

$$\frac{n(z)}{n_{ice}} \approx -\frac{1}{2}\left(\frac{1+u\alpha}{1+u\beta}\right) + \frac{3}{2}\left(\frac{1+u\alpha}{1+u\beta}\right)\frac{\rho(z)}{\rho_i} \tag{26}$$

Using $n_s = \epsilon_s'^2 = 1.3$, $n_i = \epsilon_i'^2 = 1.78$, $\rho_s = 0.4$ g/cc, $\rho_i = 0.917$ g/cc, and $u = 0.1$ as an example, a linear equation for the index versus density near the firn/ice boundary layer is

$$\frac{n(z)}{n_{ice}} \approx -0.51 + 1.66\rho(z)[g/cc] \tag{27}$$

4 Fitting the Firn Model to the Data

Many analyses and derivations have been done to produce the index of refraction versus depth curve in different locations throughout Antarctica. Figure 1 contains a summary of such measurements. In the figure, the function $n(z) = A - B\exp(Cz)$ is fit to the data points. Where density data was available, the empirical conversion of $n(z) = 1.0 + 0.86\rho(z)$ has been used.

The value for A in all the fits was restricted to $n_{ice} = 1.78$, from the differential equation solution to the gravity-density problem. No restriction was placed on the value for $B = \Delta n$, but note that the results are close to $1.78 - 1.29 = 0.49$, where 1.29 is the expected value for n_s (Hanson 2013). Thus, the fits are all measuring n_s accurately. The slopes $C = z_0^{-1}$ differ across the Antarctic continent, and are statistically lower at the South Pole compared to other locations.

Table 1 summarizes the results for the fits to the points in the figure. The snow surface index of refraction is derived from the B parameter, assuming $A = n_{ice}$, and $B = \Delta n = n_{ice} - n_s$. These results may be compared to resuts from the upper 2 m ($n_s = 1.29 \pm 0.02$), obtained at the Ross Ice Shelf via multiple tecniques (Hanson 2013). The curves MB#1 and MB#2 refer to two cores drilled in Moore's Bay (Ross Ice Shelf) in 2016, and the references for the rest of the data may be found in the Table.

Note that the Schytt model quoted by (Hanson 2015) found that $q = z_0 = 35.4$ m, which is in agreement with the MB data. The Schytt model in (Hanson 2015) was fit to firn density data collected near Williams Field on the Ross Ice Shelf. For index data derived from density data, the snow/ice conversion $n(z) = 1.0 + 0.86\rho(z)$ was used (ref). The data from Ebimuna (1983) was originally quoted as pressure in kPa vs. depth, which has been converted to density via the simple formula in Shumskiy (1960):

$$z = \left(\frac{p-p_0}{\rho_i g}\right)\left\{1 - \chi\left(\frac{p+p_0}{2} - p_n\right)\right\}(1+\alpha_i\theta) + \left\{\frac{1}{\rho_0 g} - \frac{1-\chi(p_0-p_n)}{\rho_i g}\right\} p_0 \ln(p/p_0)(1+\beta_a\theta) \tag{28}$$

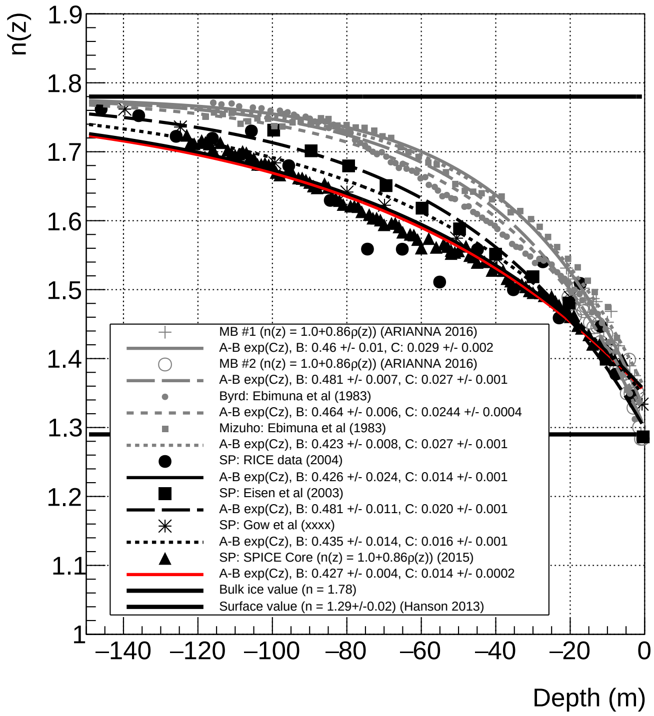

Figure 1: A summary of all the $n(z)$ data discovered for various locations in Antarctica, including Moore's Bay (MB) and the South Pole. All data points and fit lines that are black correspond to the South Pole, and the gray points and fit lines correspond to Moore's Bay, Byrd station, and Mizuho station.

The parameters in the equation of depth, z, versus pressure, p, are as follows: p_0 is the pressure at the surface, ρ_i is the density of ice (0.91670 g/cc) at a pressure $p_n = 1$ atmosphere and a temperature $\theta = 0^\circ$ C, with a volumetric compressibility of $\chi = 1.2 \times 10^{-5}$ bar^{-1}, a coefficient of linear expansion of $\alpha_i = 5.1 \times 10^{-5}$ C$^{\circ -1}$, and surface density of ρ_0. A value of 94.306 kPa is chosen for the surface pressure, corresponding to an altitude of ≈ 0.5 km. The temperature θ is the average temperature through the firn, taken to be -10 C$^\circ$ in Fig. 1 (Hanson dissertation).

5 Fermat's Principle, and Ray-Tracing

A key question for ARA/ARIANNA future designs is the expected path of a radio pulse from an Askaryan event in firn. Beginn with Fermat's principle, which states that a ray must traverse the path that minimizes the travel time. Fermat's principle is similar to the principle of least-action, in which a massive particle takes the path of least-resistance (cite Wiki Fermat's).

$$\delta S = 0 \tag{29}$$

$$\delta \int_A^B n(z)(1+\dot{y}^2)^{1/2} dx dy dz = \int_A^B L(z,\dot{y}) dx dy dz = 0 \tag{30}$$

Derivatives indicated by the dot notation are with respect to z, not time. The assumption that $x = \dot{x} = 0$ has been taken without loss of generality. Note that $\dot{y} = dy/dz$ is unit-less, and $\ddot{y}$ has

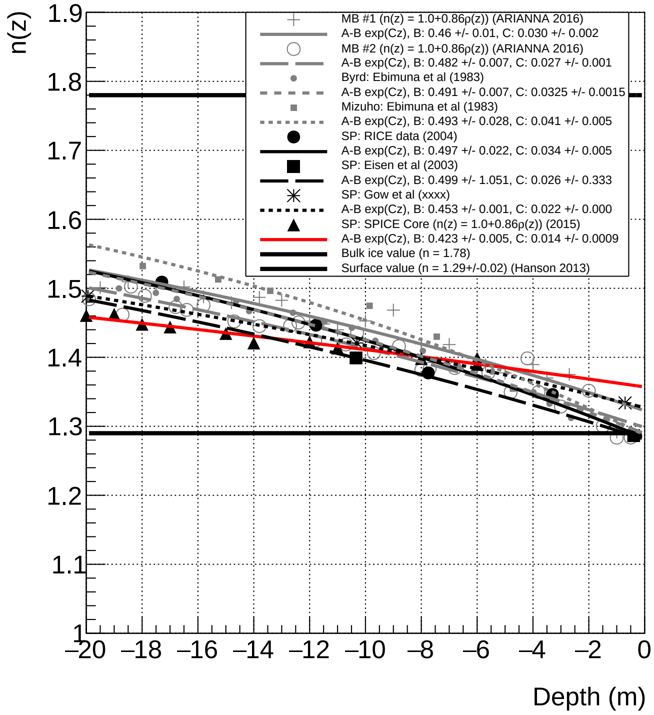

Figure 2: Figure 1, focusing on the upper few meters.

units of inverse meters. Using the Euler-Lagrange equations to minimize the variation in the path, and letting $u = \dot{y}$:

$$\frac{d}{dz}\left(\frac{\partial L}{\partial \dot{y}}\right) - \left(\frac{\partial L}{\partial y}\right) = 0 \tag{31}$$

$$\frac{d}{dz}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \tag{32}$$

$$\dot{u} = -\left(\frac{\dot{n}}{n}\right)(u^3 + u) \tag{33}$$

Note that the units are inverse meters on each side of the equation: all factors of u are unit-less, and $\dot{n}$ has units of inverse meters. Putting in the model for $n(z)$, the final equation of motion is

$$\boxed{\dot{u} = z_0^{-1}\left(\frac{\Delta n e^{z/z_0}}{n_{ice} - \Delta n e^{z/z_0}}\right)(u^3 + u)} \tag{34}$$

As a check, note the deep ice limit: $|z| \gg z_0$, $z < 0$:

$$\dot{u} = 0 \tag{35}$$

The solution to this equation of motion, after solving for z is

$$z(y) = a + by \tag{36}$$

Ref./Location	$A = n_{ice}$	B	n_s	C (m^{-1})	z_0 (m)
MB#1/Moore's Bay	1.78	0.46 ± 0.01	1.32 ± 0.01	0.029 ± 0.002	34.5 ± 2
MB#2/Moore's Bay	1.78	0.481 ± 0.007	1.299 ± 0.007	0.027 ± 0.001	37 ± 1
Ebimuna (1983)/Byrd	1.78	0.464 ± 0.006	1.316 ± 0.006	0.0244 ± 0.0004	41 ± 1
Ebimuna (1983)/Mizuho	1.78	0.423 ± 0.008	1.357 ± 0.006	0.027 ± 0.001	37 ± 1
RICE (2004)/South Pole	1.78	0.43 ± 0.02	1.35 ± 0.02	0.014 ± 0.001	71 ± 5
Eisen (2003)/South Pole	1.78	0.48 ± 0.01	1.3 ± 0.01	0.020 ± 0.001	50 ± 2.5
Gow (xxxx)/South Pole	1.78	0.435 ± 0.01	1.345 ± 0.01	0.016 ± 0.001	62.5 ± 4
SPICE (2015)/South Pole	1.78	0.427 ± 0.004	1.353 ± 0.004	0.014 ± 0.0002	71 ± 2

Table 1: The fit parameters for the curves shown in Fig. 1. The function fit to the data is $n(z) = n_{ice} - \Delta n \exp(Cz)$. The differential equation derived in the first section requires $n_{ice} = 1.78$ and $B = \Delta n = n_{ice} - n(0)$ as boundary conditions.

In other words, if the rays are far from the firn, the rays must propagate in straight lines. For the case of a shallow ray $z \to 0$, propagating initially with a horizontal velocity component satisfying $u^3 \gg u$, the main equation of motion reduces to

$$\frac{du}{dz} = \left(\frac{n_{ice} - n_s}{z_0 n_s}\right) u^3 \tag{37}$$

This is a variables-separable differential equation. Using an initial point of (y_1, z_1), a particular solution is

$$z(y) = -\frac{1}{2z_0}\left(\frac{n_{ice} - n_s}{n_s}\right)(y - y_1)^2 + z_1 \tag{38}$$

Thus, for a very shallow ray, with initial horizontal velocity, the solution dictates that the shortest travel time between two near-surface points is given by a quadratic path. As an example, use $z_0 = 37$ m and $n_s = 1.30$ to describe Moore's Bay refraction, and $z_0 = 71$ m, $n_s = 1.33$ to describe the South Pole refraction. Figure 2 compares the hypothetical ray-paths for these two cases.

The next least-restrictive approximation for the shallow depth of the ray is $\exp z/z_0 \approx 1 + z/z_0$, rather than $z \to 0$. Let $q = \Delta n / z_0$. The final solution with this limit is

$$z(y) = -\frac{1}{2}\frac{Q_1}{z_0}(y - y_1)^2 - \frac{Q_0}{Q_1} z_0 \tag{39}$$

$$Q_1 = 1 + \frac{n_{ice}}{n_s} \tag{40}$$

$$Q_0 = \frac{z_1}{z_0} + 1 + \frac{n_{ice}}{\Delta n}\left(\ln\left(\frac{n_s}{\Delta n}\right) - 2\right) \tag{41}$$

Note that, in either the limit of $z \to 0$, or $\exp z/z_0 \approx 1 + z/z_0$, the solutions are quadratic, with curvature controlled by z_0^{-1}. That is, if z_0 increases, the concavity of the ray path, and thus, the level of shadowing, decreases. It is fascinating that the same snow metamorphosis that controls the compaction from snow to ice through gravity also controls the amount of ray bending, and that this number is measurable from the density variation versus depth.

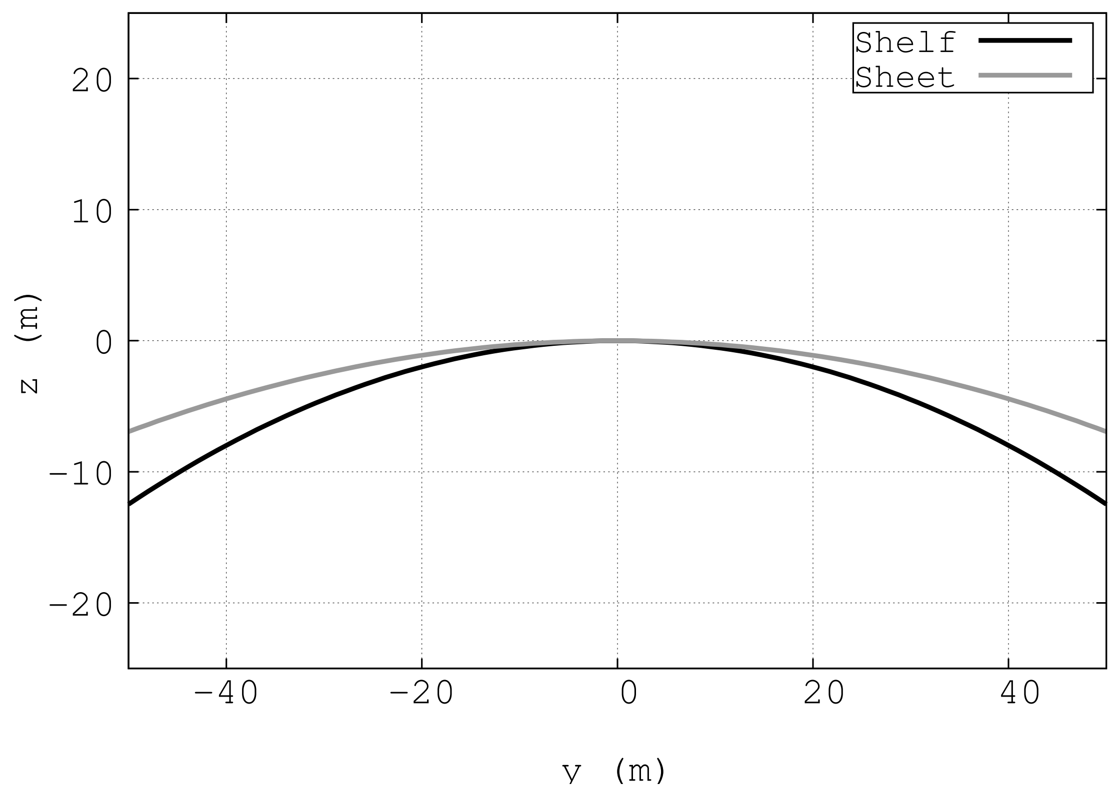

Figure 3: Examples of quadratic ray-paths in media with index of refraction profiles with the form of Eq. 21. The gray line corresponds to Eq: 21 with $z_0 = 71$ m, and the black line corresponds to Eq: 21 with $z_0 = 37$ m.

6 Horizontal and Near-Surface Propagation

Thus far, the index of refraction versus depth has been treated as a smooth function. However, perturbations from the smooth profile are introduced by variable yearly weather patterns (cite). Over-densities and under-densities can lead to local minima and maxima in the index of refraction profile. Let one such local under-density be described by a quadratic perturbation from what is otherwise an approximately constant n_0 value:

$$n(z) = n_0 - a(z - z_0)^2 \tag{42}$$
$$\dot{n} = -2a(z - z_0) \tag{43}$$
$$q = z - z_0 \tag{44}$$

Solve Eq. 33 with Eq. 42, near $q = 0$ and neglecting terms up to order q^2, the variables-separable differential equation may be solved:

$$\frac{du}{dq} \approx \left(\frac{2aq}{n_0}\right) u^3 \tag{45}$$
$$\frac{dq}{dy} \approx \pm\sqrt{-2\left(\frac{a}{n_0}\right) q^2 + C_0} \tag{46}$$

Choosing C_0 merely restricts the phase of what will become the phase of the signal. The phase of the signal is not important under the assumption that the index profile does not depend on the horizontal coordinate. In Monte Carlo codes this assumption may be relaxed. Thus,

$$\frac{dq}{dy} = \approx \pm i\sqrt{2\left(\frac{a}{n_0}\right)}q \tag{47}$$
$$\omega^2 = 2\left(\frac{a}{n_0}\right) \tag{48}$$
$$\frac{dq}{dy} = \pm i\omega q \tag{49}$$

Equation 49 admits harmonic solutions, and the spatial frequency ω is controlled by the size of the perturbation a relative to the local n_0 value:

$$q(y) = A\exp(i\omega y) + B\exp(-i\omega y) \tag{50}$$

The problem may also be worked with an over-density in the index of refraction profile, and the problem is the same up to an overall minus sign. In Eqs. 51-52, ω_- corresponds to an under-density $n(z) = n_0 - aq^2$, and ω_+ corresponds to an over-density $n(z) = n_0 + aq^2$:

$$\frac{dq}{dy} = \pm i\omega_- q \tag{51}$$
$$\frac{dq}{dy} = \mp i\omega_+ q \tag{52}$$