Numerical Energy Flux Calculation

Robert Lahmann[*], Erlangen Centre for Astroparticle Physics (ECAP)

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Radio emission from a source is simulated by tracing rays that leave the emitter at polar angles α_{tx} with separation of $\Delta\alpha_{\text{tx}}$. In azimuth, rotational symmetry is implied.

Assuming a three-dimensional isotropic power radiation, each ray can be associated with a fraction of the total radiated power P_0 of

$$P_{\text{ray}} = P_0 \frac{\Delta\alpha_{\text{tx}} \cos\alpha_{\text{tx}} \Delta\psi}{4\pi}$$

where $\Delta\psi$ is the implied azimuthal spacing of the rays in the three-dimensional extension.

The energy flux, i.e. energy per time and area or power per area, is expected to scale as r^{-2} with the distance r for isotropic radiation from a point source or as r^{-1} for radiation from an "infinite line source", where in this case r is the shortest distance of a given point from the line source.

Hence we define an energy flux received by a test antenna at a certain position. By determining the energy flux from "counting rays that hit the antenna" and using different positions for the test antennas, the r-dependence for a given pattern of rays travelling through the medium can be estimated.

The energy flux Φ through a given area A can then be estimated numerically as

$$\Phi = \frac{\sum_{\text{rays passing A}} P_{\text{ray}}}{A}$$

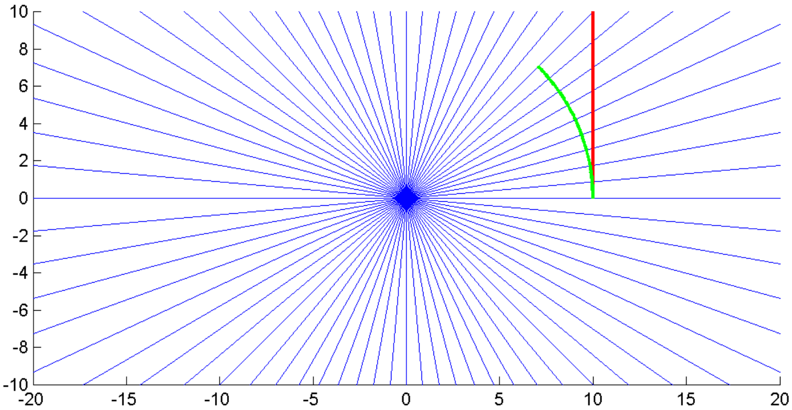

Figure 1: Power flow radiating from a point source in two dimensions

*robert.lahmann@fau.de

Please note that the expression contains no term $\cos\alpha_{\text{A}}$ to account for projection effects of a ray passing the area A at an angle α_{A} w.r.t. the normal on the area as it is required for analytical calculations. To understand this, we look at the power irradiating a given area, i.e. we look at the numerator of our numerical definition of the flux Φ. The situation is demonstrated in two dimensions in Fig. 1.

Clearly, the red and green lines are passed by the same number of rays, because the two lines have the same opening angle w.r.t. the point source. For the numerical calculation of the power by "ray counting" hence the power of each ray is counted irrespective of the angle at which it intersects the line. For an integration along the lines (i.e. the surface in three dimensions) the angle between the direction of the power flow and the area would need to be taken into account and would lead to the same power irradiating the green and red area.

Obviously the flux through the red area (we now pretend the model is in three dimensions) is smaller than that through the green area since the former area is bigger and the power is the same. The green area would show the perfect r^{-2}-scaling expected for a point source in three dimensions. The red area has no exactly defined distance to the source, r varies with the z-position. In three dimension, the red area is part of a cylinder surface, the green one part of a sphere. For the cylinder:

$$A_{\text{cyl}} = \ell \cdot \rho\Delta\psi$$

where ℓ is the extension of the area in z-direction and ρ the radial distance in the xy-plane of the cylinder surface from the point source.

The area on the sphere can be calculated:

$$\begin{aligned} dA_{\text{sph}} &= r^2 \cdot \cos\beta d\beta\Delta\psi \\ A_{\text{sph}} &= r^2 \cdot [\sin\beta]_{\beta_1}^{\beta_2}\Delta\psi \\ &= r^2 \cdot [\sin\beta_2 - \sin\beta_1]\Delta\psi \end{aligned}$$

Here β_1 is the angle of the lower edge of the antenna w.r.t. the source ($\beta_1 = 0°$ in the example in Fig. 1) and β_2 the upper edge ($\beta_2 = 45°$ in Fig. 1) . And for small $\Delta\beta := \beta_2 - \beta_1$ as expected:

$$\begin{aligned} A_{\text{sph}} &= r^2 \cdot [\sin(\beta_1 + \Delta\beta) - \sin\beta_1]\Delta\psi \\ &= r^2 \cdot [\sin\beta_1\cos(\Delta\beta) + \cos\beta_1\sin(\Delta\beta) - \sin\beta_1]\Delta\psi \\ &\approx r^2 \cdot [\sin\beta_1 + \cos\beta_1\Delta\beta - \sin\beta_1]\Delta\psi \\ &= r^2 \cdot \cos\beta_1\Delta\beta\Delta\psi \end{aligned}$$

Note further that $r\cos\beta_1 = \rho$ and that for small $\Delta\beta$ the arc length $\Delta\beta \cdot r$ approaches ℓ from the expression for A_{cyl} above.

Finally we get two expressions for the flux, depending on the area that is looked at:

$$\begin{aligned} \Phi_{\text{cyl}} &= \frac{P_0}{4\pi}\frac{\cancel{\Delta\psi}\sum_{\text{rays passing A}}(\Delta\alpha_{\text{tx}}\cos\alpha_{\text{tx}})}{\ell \cdot \rho\cancel{\Delta\psi}} \\ \Phi_{\text{sph}} &= \frac{P_0}{4\pi}\frac{\cancel{\Delta\psi}\sum_{\text{rays passing A}}(\Delta\alpha_{\text{tx}}\cos\alpha_{\text{tx}})}{r^2 \cdot [\sin\beta_2 - \sin\beta_1]\cancel{\Delta\psi}} \end{aligned}$$

Note that the power associated to a ray P_{ray} could in principle also be derived from an integration, yielding a difference of sin-values instead of the term $\Delta\alpha_{\text{tx}}\cos\alpha_{\text{tx}}$. The increase of precision for reasonable (i.e. small) values of $\Delta\alpha_{\text{tx}}$ is marginal, however, and the chosen formulation of P_{ray} probably makes it easier to change the emission characteristics to something different than isotropic.