Question - Calculate the Optimal Weights which Minimizes Variance

Given $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_m}\}$ where $x_i \in \mathbb{R}^n$ find $w \in \mathbb{R}^m$ which minimized the Variance of y given by $y = w_1 \boldsymbol{x_1} + w_2 \boldsymbol{x_2} + \cdots + w_m \boldsymbol{x_m}$ where $\forall i \; w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$.

Remark. *The question is given at Question 44984132 on StackOverflow.*

Answer - Calculate the Optimal Weights which Minimizes Variance

The above can be written as following:

$$
\begin{aligned}
\arg \min_{w} \quad & \frac{1}{2} \left\| X w - \frac{1}{m} e^T X w e \right\|_2^2 \\
\text{subject to} \quad & w \succeq 0 \\
& e^T w = 1
\end{aligned}
$$

Where X is composed by $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_m}\}$ as its columns, $w = [w_1, w_2, \dots, w_m]^T$ and $e = [1, 1, \dots, 1]^T$, $e \in \mathbb{R}^m$.

The above is a Convex Problem where the solution is limited to the Unit Simplex.

A method to solve is using Projected Sub Gradient Method. The idea is to apply a Sub Gradient step and project the result onto the Unit Simplex.

In order to so one have to calculate the following:

- The Sub Gradient (Gradient in the case above as the function is smooth) of the Objective Function.
- The Projection onto the unit simplex.

The Gradient of the Objective Function

$$\begin{aligned}
\frac{\partial}{\partial w} f\left(w\right) &= \frac{\partial}{\partial w} \left(\frac{1}{2} \left\| X w - \frac{1}{m} e^{T} X w e \right\|_{2}^{2} \right) \\
&= \frac{\partial}{\partial w} \left(X w - \frac{1}{m} e^{T} X w e \right) \frac{\partial}{\partial \left(X w - \frac{1}{m} e^{T} X w e \right)} f\left(w\right) \\
&= \left(X^{T} - \frac{1}{m} \frac{\partial}{\partial w} \left(e^{T} X w e \right) \right) \left(X w - \frac{1}{m} e^{T} X w e \right) \\
&= \left(X^{T} - \frac{1}{m} X^{T} e e^{T} \right) \left(X w - \frac{1}{m} e^{T} X w e \right)
\end{aligned}$$

The Projection onto the Unit Simplex

There are 2 options to apply this:

- Solving the Projection Minimization as done in MathExchange Answer 2338491.
- Iteratively projecting onto the Non Negative Half Space and the set of vectors which their sum is 1.

Remark. *The answer (With MATLAB code) is given at Answer 195787 on StackOverflow.*