

# DESIGN OF PROGRAMMING LANGUAGES

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Fall 2015  
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## **IMPORTANT CALCULATION MODELS** **3**

**1. FUNCTIONAL MODEL** **3**

**2. LOGIC MODEL** **3**

**3. IMPERATIVE MODEL** **4**

**SUMMARY** **4**

## **SYNTAX** **5**

**GRAMMAR** **5**

TYPES OF GRAMMARS 6

WREN LANGUAGE 8

AMBIGUITY 8

**STATIC SEMANTIC** **9**

CONTEXT CONSTRAINTS 9

SEMANTIC ERRORS 10

ABSTRACT SYNTAX 10

ATTRIBUTE GRAMMARS 12

## **SEMANTICS** **13**

**1. OPERATIONAL SEMANTICS** **13**

**2. DENOTATIONAL SEMANTICS** **15**

A WHILE LANGUAGE 16

**NONSTANDARD SEMANTICS** **18**

ABSTRACT INTERPRETATION 18

## **IMPERATIVE AND DECLARATIVE** **19**

**FUNCTIONAL LANGUAGE** **19**

**DECLARATIVE LANGUAGE TEST** **19**

**THE WORLD OF EXPRESSIONS AND THE WORLD OF STATEMENTS** **19**

CHURCH-ROSEN PROPERTY (CONFLUENCE) 20

REFERENTIAL TRANSPARENCY 20

## **LAMBDA-CALCULUS** **21**

UNTYPED  $\lambda$ -CALCULUS: 21

SCOPE 21

OPERATIONAL SEMANTICS 21

**EVALUATE STRATEGIES** **22**

1. FULL BETA-REDUCTION 22

2. NORMAL ORDER 22

3. CALL BY NAME (NON-STRICT OR LAZY) 22

4. CALL BY VALUE (STRICT) 22

**PROGRAMMING IN  $\lambda$**  **23**

CHURCH BOOLEAN 23

CHURCH NUMERALS 24

<b>RECURSION</b>	<b>25</b>
FIXED POINT	25
CALL-BY-NAME Y COMBINATOR (OR FIXED POINT COMBINATOR)	25
CALL-BY-VALUE Z COMBINATOR	25
<b>SOME PROGRAMMING LANGUAGES</b>	<b>26</b>
<b>LISP</b>	<b>26</b>
HISTORICAL LISP STRUCTURE	26

## Session 1

## Definitions

- A **computational model** is collection of values and operations.
- A **computation** is the application of a sequence of operations to values to yield another value.
- A **programme** is a specification of a computation.
- A **programming language** is a notation for writing programmes.

# IMPORTANT CALCULATION MODELS

## 1. Functional Model

**Values:** functions

**Operations:** function operations

**Example:**

$Sd(xs) = \text{sqrt}(v)$

Where

$n = \text{length}(xs)$

$v = \text{fold}(\text{plus}, \text{map}(\text{sqr}, xs)) / n - \text{sqr}(\text{fold}(\text{plus}, xs) / n)$

\*Pure functional lang: Haskell

\*functional langs: ML – Lisp ...

\*This language is "Declarative".

\*Functional and Logical languages are Declarative.

## 2. Logic Model

**Values:** facts, definitions of relations

**Operations:** logical inferences

**Example:**

1. Human(Socrates)
2. Human(Penelope)
3. Mortal(x) if Human(x)

- |  |   |
|--|---|
| 4. $\neg \text{mortal}(y)$   | Assumption                                |
| 5. $x=y$   | (3),(4) and unification and Modus Tollens |
| 6. $\neg \text{human}(y)$  | (3),(4) and unification and Modus Tollens |
| 7. $y=\text{Socrates}$   | (1),(5),(6) and unification               |
| 8. $y=\text{Penelope}$   | (1),(5),(6) and unification               |
| 9. Contradiction( $\neg \text{human}(\text{Socrates})$ and $\text{human}(\text{Socrates})$ ) |   |

\*"Prolog" is a logical language or framework.

### 3. Imperative Model

**Values:** states

**Operations:** state transitions

Example:

constant pi = 3.14

input (radius)

circumference := 2 \* pi \* radius

output(circumference)

\*languages: Pascal - C - C++ - Java – Assembly

### Summary

COMPUTATIONAL MODEL	VALUE	OPERATION	EXAMPLE
IMPERATIVE	state	State transition	C
FUNCTIONAL	functional	Function application	Haskell
LOGIC	Facts and relations	rules	Prolog

## Session 2

# SYNTAX

A language: **Syntax, Semantics, Pragmatics.**

\*The amount of expressiveness of a language.

\*Chomsky, The linguist said the following

## Grammar

A grammar  $(\Sigma, N, P, S)$  consists of four parts;

- 1-  $\Sigma$  : terminal symbols or alphabet
- 2-  $N$  : nonterminal symbols or syntactic alphabet
- 3-  $P$  : productions or rules
- 4-  $S$  : the start symbol

### BNF (Backus-Naur Form):

`<declaration> ::= var <variable list> : <type>;`

\*BNF is called a metalanguage, because it defines languages.

#### Example

`var x,y : int ;`

### Definition

**Vocabulary:** terminals and nonterminals.

A production  $\alpha ::= \beta$

\* $\alpha$  must have at least one nonterminal.

## Types of Grammars

### Type 0 (Unrestricted grammars):

At least one nonterminal occurs on the left side of a rule.

#### Example

$\mathbf{a}\langle\text{thing}\rangle\mathbf{b} ::= \mathbf{b}\langle\text{another thing}\rangle$

### Type 1 (Context-sensitive grammars):

The right side contains no fewer symbols than the left.

#### Example

$\langle\text{thing}\rangle\mathbf{b} ::= \mathbf{b}\langle\text{thing}\rangle$

rules would be like this:

$\alpha \langle B \rangle \gamma ::= \alpha \beta \gamma$

### Type 2 (Context-free grammars):

The left side is a single nonterminal.

#### Example

$\langle A \rangle ::= \beta$

rules would be like this:

$\langle \text{expression} \rangle ::= \langle \text{expression} \rangle \mathbf{a} \langle \text{term} \rangle$

\*BNF is a rule for specifying Type 2 languages and programming languages are defined by it.

### Type 3 (Regular grammars):

The left side is a single nonterminal.

#### Example

$\langle A \rangle ::= \beta$

rules would be like this:

$\langle A \rangle ::= \mathbf{a}$

or

$\langle A \rangle ::= \mathbf{a} \langle B \rangle$

\*These languages would be accepted by Finite automata.

#### Example

A grammar for binary numbers

$\langle \text{binary number} \rangle ::= \mathbf{0}$

$\langle \text{binary number} \rangle ::= \mathbf{1}$

$\langle \text{binary number} \rangle ::= \mathbf{0} \langle \text{binary number} \rangle$

$\langle \text{binary number} \rangle ::= \mathbf{1} \langle \text{binary number} \rangle$

or

$\langle \text{binary number} \rangle ::= \mathbf{0} | \mathbf{1} | \mathbf{0} \langle \text{binary number} \rangle | \mathbf{1} \langle \text{binary number} \rangle$

### Example

A grammar for a natural language

```

<sentence> ::= <noun phrase><verb phrase> .
<noun phrase> ::= <determiner><noun> | <determiner><noun><prepositional phrase>
<verb phrase> ::= <verb> | <verb><noun phrase> | <verb><noun phrase><prepositional phrase>
<prepositional phrase> ::= <preposition><noun phrase>
<noun> ::= boy | girl | cat | telescope
<determiner> ::= a | the
<verb> ::= say | go | shop | saw
<preposition> ::= by | with

```

\*The latter language is Ambiguous.

### Example

A context-sensitive grammar

```

<sentence> ::= a b c | a <thing> b c
<thing> b ::= b <thing>
<thing> c ::= <thing> b c c
a <other> ::= a a | a a <thing>
b <other> ::= <other> b

```

The language would be like this:

$\{a^n b^n c^n \mid n \in \mathbb{Z}^+\}$

### Definition

A grammar is **Ambiguous** if some phrases in the language generated by the grammar has two or more distinct derivation trees.



## Session 3

## Wren Language

Syntaxes are either:

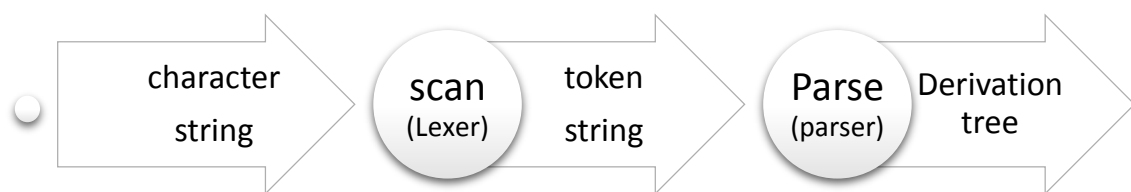
**Lexical syntax**

→ Lexical Analysis (scanning)

or

**Phrase-Structure syntax**

→ Syntactic Analysis (Parsing)



## Ambiguity

Having more than one derivation tree for a statement in a language.

## Example

if exp1 then if exp2 then cmd1 else cmd2

## Session 4

## static semantic

## Example

```

Program illegal is
    var a : boolean;
begin
    a := 5
end

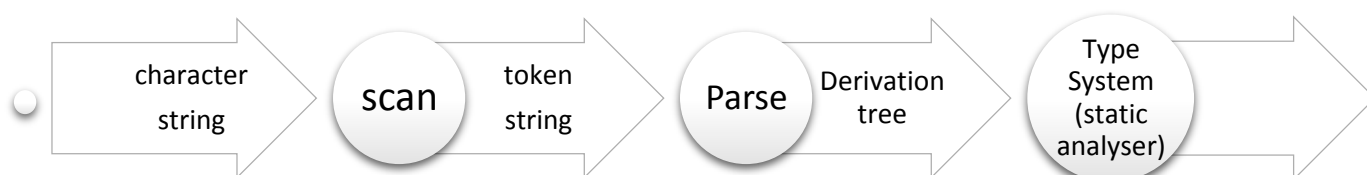
```

There are two ideas about this problem:

It's a **syntactic** problem.

It's a **static semantic** problem.

so we are going to need a **static analyser**:



Static semantics cannot be analysed by context free machines.

We do not use a Type 1 (context sensitive) machine because it is much harder to have a context sensitive compiler.

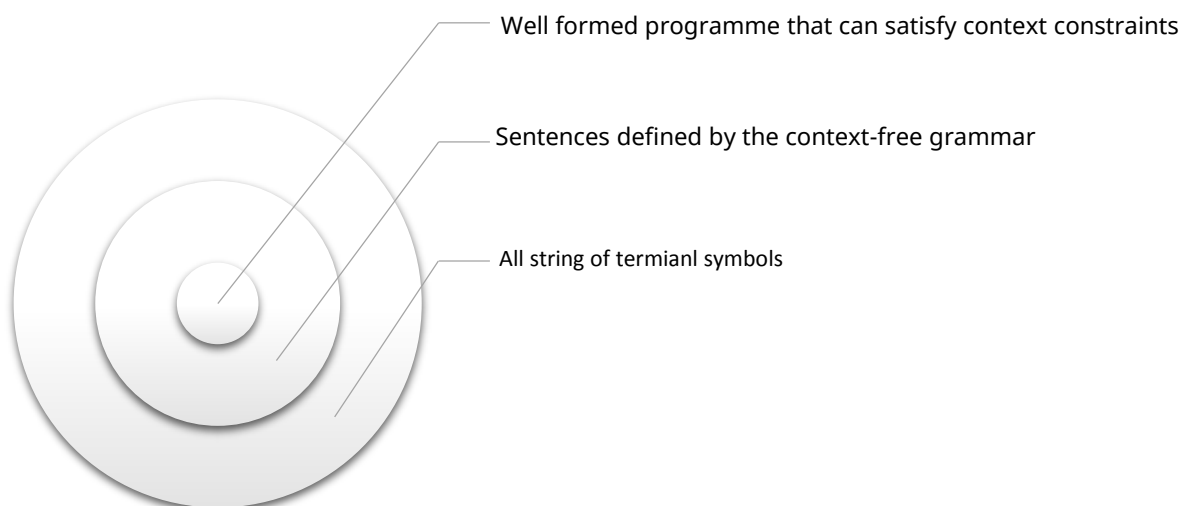
## Context constraints

1. All identifiers that appear in a block must be declared in that block
2. No identifier may be declared more than once in a block.
3. An identifier occur in a read command must be an integer variable.

.

.

.



## Semantic Errors

### Example

1. An attempt is made to divide by zero.
2. A variable that has not been initialised has been accessed.
3. A read command is executed when the input file is empty.
4. type mismatch.

## Abstract Syntax

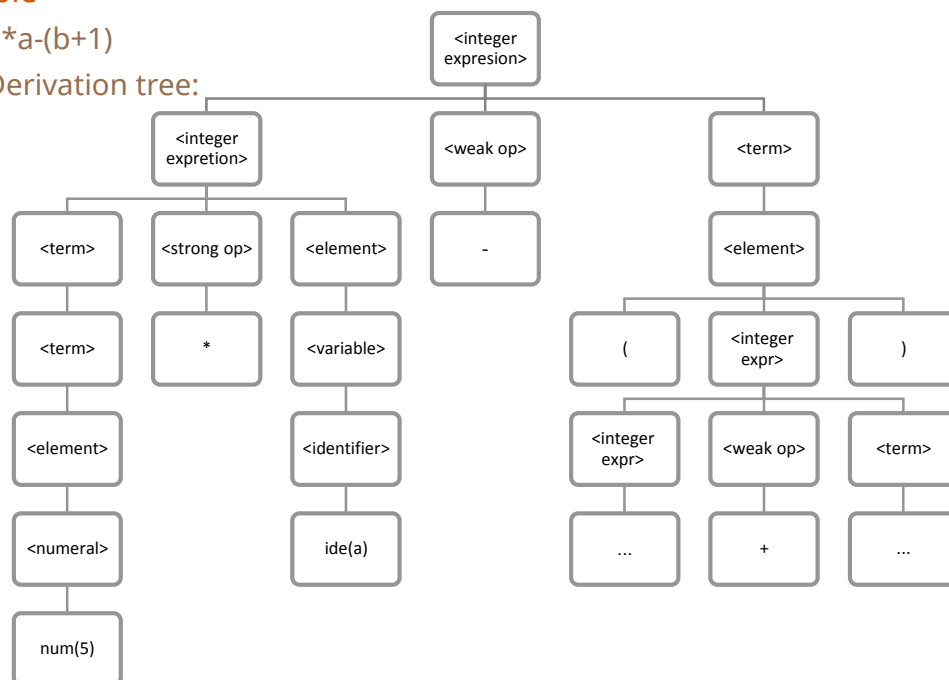
Is a way of fixing the redundancy in a concrete syntax.

\***Concrete** is the opposite of Abstract.

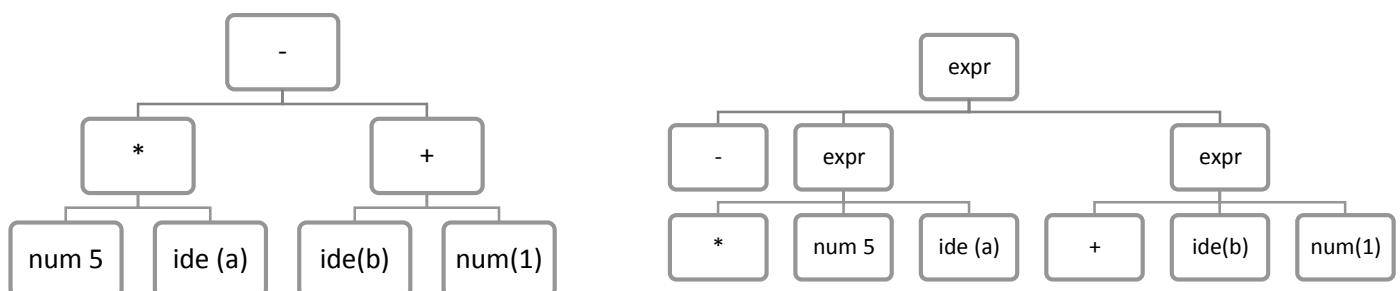
### Example

5\*a-(b+1)

Derivation tree:



## AST : Abstract Syntax Tree



<expression>  $\rightarrow$  ...  $\rightarrow$     operations  
   numerals  
   identifiers  
   boolean constants

## Syntactic Categories: Expression , Numeral , Identifier

Now we can define an abstract syntax by removing **unit rules** (rules without terminals) unless they have a basic component.



## Session 6

## SEMANTICS

- 1- Operational Semantics
- 2- Denotational Semantics (معنا شناسی دلالتی) → John Michel's book (The Bible of Denotational Semantics :)
- 3- Axiomatic Semantics

## 1.Operational Semantics

## Example

$t ::= \text{true} \mid \text{false} \mid \text{if } t \text{ then } t \text{ else } t \mid 0 \mid \text{succ } t \mid \text{pred } t \mid \text{iszero } t$

Inductive Definitions (judgment of **t term**):

$$\frac{}{\text{true term}} \quad (\text{Axiom})$$

$$\frac{}{\text{false term}} \quad (\text{Axiom})$$

$$\frac{t_1 \text{ term } t_2 \text{ term } t_3 \text{ term}}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \text{ term}} \quad (\text{proper machine})$$

$v ::= \text{true} \mid \text{false} \quad \text{values}$

Evaluation: (Small-Step)

$$\frac{}{\text{if true then } t_2 \text{ else } t_3 \rightarrow t_2}$$

$$\frac{}{\text{if false then } t_2 \text{ else } t_3 \rightarrow t_3}$$

$$\frac{\text{if } t_1 \rightarrow t_1'}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \rightarrow \text{if } t_1' \text{ then } t_2 \text{ else } t_3}$$

## Theorem - Determinacy of one-step evaluation

if  $t \rightarrow t'$  and  $t \rightarrow t''$ , then  $t' = t''$

## Definition

A term **t** is in **normal form** if no evaluation rule applies to it.

## Theorem

In our language (the above language) If **t** is normal form, then **t** is a value.

## Definition

The **Multistep evaluation** relation  $\rightarrow^*$  is the reflexive and transitive closure of  $\rightarrow$ .

if  $t \rightarrow t'$ , then  $t \rightarrow^* t'$

$t \rightarrow^* t$

if  $t \rightarrow^* t'$  and  $t' \rightarrow^* t''$ , then  $t \rightarrow^* t''$

### Theorem

For every term  $t$ , there is some normal form  $t'$  such that  $t \rightarrow^* t'$ .

$v ::= \text{true} \mid \text{false} \mid \text{nv}$

$\text{nv} ::= 0 \mid \text{succ } \text{nv}$

$$\frac{t_1 \rightarrow t_1'}{\text{succ } t_1 \rightarrow \text{succ } t_1'}$$

$$\frac{}{\text{pred } 0 \rightarrow 0}$$

$$\frac{}{\text{pred}(\text{succ } \text{nv}_1) \rightarrow \text{nv}_1}$$

$$\frac{t_1 \rightarrow t_1'}{\text{pred } t_1 \rightarrow \text{pred } t_1'}$$

$$\frac{}{\text{iszero } 0 \rightarrow \text{true}}$$

$$\frac{}{\text{iszero } (\text{succ } \text{nv}_1) \rightarrow \text{false}}$$

$$\frac{t_1 \rightarrow t_1'}{\text{iszero } t_1 \rightarrow \text{iszero } t_1'}$$

### Definition

A closed term is **stuck** if it is normal form but not a value.

\*further studies: middle weight Java and its Operational semantics.

## Session 7

## 2. Denotational Semantics

Christopher Strachey and Dana Scott had introduced it back in 1960s.

Denotational Semantics consists of :

**Object language (programme)**  
and **Meta language (mathematics).**

### Example

Object language:  $x := 0; y := 0; \text{ while } x \leq z \text{ do } (y := y + x; x := x + 1)$

Meta language:  $F(z) = 1 + 2 + 3 + \dots + z$

**Compositionality** is a feature of this kind of semantics meaning that if elements of two phrases are the same then the phrases are the same.

### Example

$B \equiv B', P \equiv P', Q \equiv Q' \rightarrow \text{if } B \text{ then } P \text{ else } Q \equiv \text{if } B' \text{ then } P' \text{ else } Q'$

### Example

Denotational semantics for binary numbers:

$e ::= n \mid e + e \mid e - e$

$n ::= b \mid nb$

$b ::= 0 \mid 1$

\*[[e]] = The parse tree of e

$E[[e]]$  is the meaning of e

$E[[0]] = 0$

0 is from the meta language (the mathematical language).

$E[[1]] = 1$

$E[[nb]] = E[[n]] * 2 + E[[b]]$

$E[[e_1 + e_2]] = E[[e_1]] + E[[e_2]]$

Some arithmetic expressions:

$e ::= v \mid n \mid e + e \mid e - e$

$n ::= d \mid nd$

$d ::= 0 \mid 1 \mid \dots \mid 9$

$v ::= x \mid y \mid z \mid \dots$

A **programme** is a function from states to states.  $P: \text{states} \rightarrow \text{states}$

A **State** is a function from variables to values.  $S: \text{Variables} \rightarrow \text{values}$



Now we define  $E[[e]](S)$

$$E[[x]](S) = S(x)$$

$$E[[0]](S) = 0$$

...

$$E[[9]](S) = 9$$

$$E[[nd]](S) = E[[n]](S) * 10 + E[[d]](S)$$

$$E[[e_1 + e_2]](S) = E[[e_1]](S) + E[[e_2]](S)$$

$$E : \text{Parse tree} \rightarrow (\text{states} \rightarrow \mathbb{N})$$

$$E[[e]](S)$$

$$E : \text{parse tree} \rightarrow \mathbb{N}^{\text{states}} \quad \text{parse tree} \rightarrow (\text{states} \rightarrow \mathbb{N})$$

$$E[[e]](S)$$

$$C : \text{Parse tree} \rightarrow (\text{state} \rightarrow \text{state})$$

\*both E and C give us a function from states in return.

## A While Language

$P ::= x := e \mid P; P \mid \text{if } e \text{ then } P \text{ else } P \mid \text{while } e \text{ do } P$

State = Variables  $\rightarrow$  Values

Command = States  $\rightarrow$  States

$\text{modify}(s, x, a) = \lambda v \in \text{variables} [\text{if } v = x \text{ then } a \text{ else } s(v)]$  \* $s(v)$  is the value of  $v$  in the state  $s$

$C[[P]](s)$  is supposed to be a meaning function

$$C[[x := e]](s) = \text{modify}(s, x, E[[e]](s))$$

$$C[[P_1; P_2]](s) = C[[P_2]](C[[P_1]](s))$$

$$C[[\text{if } e \text{ then } P_1 \text{ else } P_2]](s) = \text{if } E[[e]](s) \text{ then } C[[P_1]](s) \text{ else } C[[P_2]](s)$$

$$C[[\text{while } e \text{ do } P]](s) = \text{if not } E[[e]](s) \text{ then } s \text{ else } C[[\text{while } e \text{ do } P]](C[[P]](s))$$

## Session 8

## Example

if  $x > y$  then  $x := y$  else  $y := x$

$s_0(x) = 1$  ,  $s_0(y) = 2$

$s_1(x) = 1$  ,  $s_1(y) = 1$

$$\begin{aligned} C[[\text{if } x > y \text{ then } x := y \text{ else } y := x]](s_0) &= \text{if } E[[x > y]](s_0) \text{ then } C[[x := y]](s_0) \text{ else } C[[y := x]](s_0) \\ &= C[[y := x]](s_0) \\ &= \text{modify}(s_0, y, E[[x]](s_0)) = \text{modify}(s_0, y, 1) = s_1 \end{aligned}$$

## Example

$P = x := 0; y := 0; \text{while } x \leq z \text{ do } (y := y + x; x := x + 1)$

$s_0(z) = 2$

$s_1 = \text{modify}(s_0, x, 0)$

$s_2 = \text{modify}(s_1, y, 0)$

$$\begin{aligned} C[[P]](s_0) &= C[[\text{while } x \leq z \text{ do } (y := y + x; x := x + 1)]](s_2) \\ &= \text{if not } E[[x \leq z]](s_2) \text{ then } s_2 \text{ else } C[[\text{while } x \leq z \text{ do } (y := y + x; x := x + 1)]](C[[y := y + x; x := x + 1]](s_2)) \\ &= C[[\text{while } x \leq z \text{ do } (y := y + x; x := x + 1)]](s_3) \\ &= \dots \\ &= s' \quad \text{which } s'(x) = 3, s'(y) = 3, s'(z) = 2 \end{aligned}$$

$C$  is a **partial function** meaning that it is undefined for some programmes.

## Example

$C[[\text{while } x = x \text{ do } x := x]](s) = ?$

or

$C[[\text{while } x = y \text{ do } x := y]](s) = s$	if $s(x) \neq s(y)$
undefined	otherwise

## Nonstandard Semantics

They are used in **programme analysis** (Data flow analysis, ... , Abstract interpretation).

### Abstract interpretation

#### Example

To check programmes to make sure that every variable is initialised before it is used.

\*Methods of programme analysis can only be **conservative** so they wouldn't have a false positive which means they're sound, because of the halting problem being unsolvable.

error error state

variables  $\rightarrow \{ \text{init}, \text{uninit} \}$

states =  $\{ \{ \text{error} \} \cup \{ \text{variables} \rightarrow \{ \text{init}, \text{uninit} \} \}$

$C[[P]](s)$

$E[[e]](s) = \text{err}$  if  $e$  contains any variable  $y$  with  $s(y) = \text{uninit}$

$E[[e]](s) = \text{OK}$  otherwise

for example

$C[[x:=e]](s) = \text{if } E[[e]](s) = \text{OK} \text{ then } \text{modify}(s, x, \text{init}) \text{ else error}$

$C[[P1;P2]](s) = \text{if } C[[P1]](s) = \text{error} \text{ then error else } C[[P2]](C[[P1]](s))$

$s_1 * s_2 = \lambda v \in \text{Variables} \text{ if } s_1(v) = s_2(v) = \text{init} \text{ then init else uninit}$

$C[[\text{if } e \text{ then } P1 \text{ else } P2]](s) = \text{if } E[[e]](s) = \text{err} \text{ or } C[[P1]](s) = \text{error} \text{ or } C[[P2]](s) = \text{error} \\ \text{then error else } C[[P1]](s) * C[[P2]](s)$

$C[[\text{if } 0=1 \text{ then } x:=0 \text{ else } x:=1; y:=2]](s_0) = \text{modify}(s_0, x, \text{init})$

## Session 9

Michel's book chapter 4

# IMPERATIVE AND DECLARATIVE

There are four kinds of sentences in natural languages:

- Imperative
- Declarative
- Interrogative
- Exclamatory

Programming languages are one of the first two.

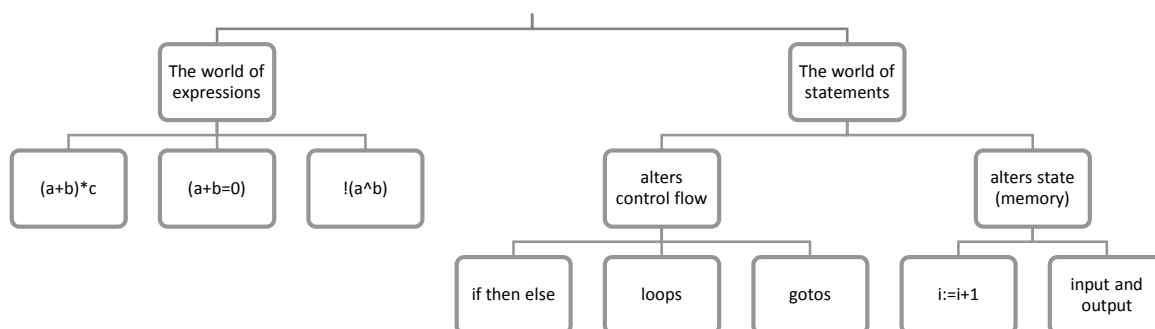
## Functional language

A programming language in which most computation is done by evaluation of expressions that contain functions. Like *Lisp*, *Haskell* and *ML* languages.

## Declarative Language Test

Within the scope of specific declaration of  $x_1, \dots, x_n$ , all occurrences of an expression  $e$  containing only variables  $x_1, \dots, x_n$  have the same value.

## The World Of Expressions and The World Of Statements



### Example

$z := (z * a * y + b) * (z * a * y + c)$

$\Rightarrow t := z * a * y$

$z := (t + b) * (t + c)$

this is OK in the world of expressions

$y := (z * a * y + b); z := (z * a * y + c);$

$\Rightarrow t := z * a * y$

$y := t + b \quad z := t + c$

but this is not OK in the world of statements

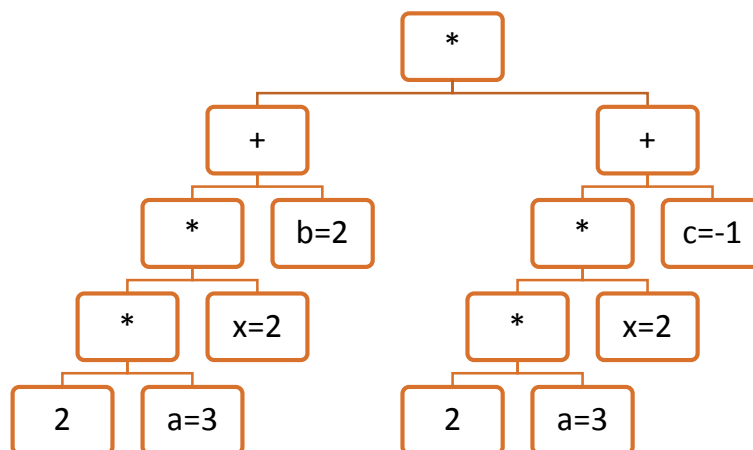
## Church-Rosen Property (Confluence)

No matter what the order of decoration, as long as we obey the structure of the tree, we will always get the same.

\*a language with this property is called **confluent**.

\* It doesn't matter which leaf we start from.

### Example



$(2ax+b)(2ax+c)$

$a=3, b=2, c=-1, x=2$

In this tree, to decorate (calculate) every node, we need to decorate its children first.

### Example

**a+2\*F(b)**

*function F(x: integer) L integer*

*begin*

*F:=x\*x;*

*end*

this id pure functional

*function F(x: integer) L integer*

*begin*

*a:=a+1;*

*F:=x\*x;*

*end*

but this is not pure functional

## Referential Transparency

### Example

"I saw Walter get into his car."

"I saw Walter get into his Ferrari."

This sentence is referentially transparent.

"He was called William Rufus because of his red beard."

"He was called William IV because of his red beard."

This sentence is **not** referentially transparent.

## Session 10

# LAMBDA-CALCULUS

(Pierce's book – season 5)

**Core calculus** is the basic language which other languages have been built on it.

**Lambda-calculus ( $\lambda$ -calculus)** is the core of functional languages. (Introduced by Alonso Church)

**Pi-calculus ( $\pi$ -calculus)** is the core of concurrent languages.

**Object-calculus** is the core of object-oriented languages.

Untyped  $\lambda$ -calculus:

$t ::= x \mid \lambda x.t \mid tt$

rules:

- Application associates to left.

$s \ u \ t = ((s \ u) \ t)$

- The bodies of abstractions are taken to extend as far to the right as possible.

$\lambda x. \lambda y. xyx = \lambda x. (\lambda y. (xy)x)$

## scope

binding: an element can be bound or free and free elements are variables.

for every  $x, x > y$ . "y" is a variable and "for every" is a binder

in  $\lambda$ -calculus  $\lambda$  is the **binder**.

## Example

in  $\lambda z. \lambda x. \lambda y. x(yz)$  there is no variable because all of the elements have a binder.  
this term is closed.

Closed terms are also called combinatory.

The most famous closed term is the identity function:  $\text{id} = \lambda x. x$

## Operational Semantics

$(\lambda x. t)t' \rightarrow [x \mapsto t']t$  (beta-reduction)

or

$[t'/x] t$

## Example

$(\lambda x. x)y \rightarrow y$

$(\lambda x. x(\lambda x. x))(ur) \rightarrow (ur)(\lambda x. x)$

redex = reducible expression :  $(\lambda x. t)t'$

## Evaluate Strategies

Consider this term:

$$(\lambda x.x)((\lambda x.x)(\lambda z.(\lambda x.x)z))$$

or  $\text{id} (\text{id} (\lambda z.\text{id } z))$

### 1. Full beta-reduction

$$\begin{aligned} \text{id} (\text{id} (\lambda z.\text{id } z)) &\rightarrow \text{id} (\text{id} (\lambda z.z)) \\ &\rightarrow \text{id} (\lambda z.z) \\ &\rightarrow \lambda z.z \\ &\not\rightarrow \end{aligned}$$

### 2. Normal order

Starting from the outmost.

$$\begin{aligned} \text{id} (\text{id} (\lambda z.\text{id } z)) &\rightarrow \text{id} (\lambda z.\text{id } z) \\ &\rightarrow \lambda z.\text{id } z \\ &\rightarrow \lambda z.z \\ &\not\rightarrow \end{aligned}$$

### 3. Call by name (non-strict or lazy)

Starting from the outmost.

No reductions inside abstractions.

$$\begin{aligned} \text{id} (\text{id} (\lambda z.\text{id } z)) &\rightarrow \text{id} (\lambda z.\text{id } z) \\ &\rightarrow \lambda z.\text{id } z \\ &\not\rightarrow \end{aligned}$$

### 4. Call by value (strict)

Starting from the outmost.

No reductions inside abstractions.

Reduction can only be applied when the argument is a value (a  $\lambda$  abstraction).

$$\begin{aligned} \text{id} (\text{id} (\lambda z.\text{id } z)) &\rightarrow \text{id} (\lambda z.\text{id } z) \\ &\rightarrow \lambda z.\text{id } z \\ &\not\rightarrow \end{aligned}$$

## Session 11

Programming in  $\lambda$ 

$$+ : R^*R \rightarrow R$$

$$+(2,3)=5$$
**Currying:**

$$+ : R \rightarrow R^R$$

a function that gives back another function

$$(+2)(3)=5$$

+2 is a function that gets 3 as an argument.

## Church Boolean

$$\text{tru} = \lambda t. \lambda f. t \quad \text{tru } v \text{ } w = v$$

$$\text{fls} = \lambda t. \lambda f. f \quad \text{fls } v \text{ } w = w$$
we need to have a *test* like this:
$$\begin{array}{ll} \text{test } b \text{ } v \text{ } w = & v \quad b \text{ is tru} \\ & w \quad b \text{ is fls} \end{array}$$
so the definition of *test* would be like:
$$\text{test} = \lambda l. \lambda m. \lambda n. l m n$$
now if we give true *v w* to *test*:
$$\begin{aligned} \text{test true } v \text{ } w &\rightarrow (\lambda m. \lambda n. \text{tru } m \text{ } n) \text{ } v \text{ } w \\ &\rightarrow (\lambda n. \text{tru } v \text{ } n) \text{ } w \\ &\rightarrow (\text{tru } v \text{ } w) \\ &\rightarrow (\lambda t. \lambda f. t) \text{ } v \text{ } w \\ &\rightarrow (\lambda f. v) \text{ } w \\ &\rightarrow v \end{aligned}$$

$$\text{and} = \lambda b. \lambda c. b \text{ } c \text{ } \text{fls}$$

$$\text{and tru tru} = \text{tru tru fls} = \text{tru}$$

$$\text{or} = \lambda b. \lambda c. b \text{ } \text{tru } c$$

$$\text{or fls tru} = \text{fls tru tru} = \text{tru}$$

$$\text{neg} = \lambda b. b \text{ } \text{fls } \text{tru}$$

$$\text{pair} = \lambda f. \lambda s. \lambda b. b \text{ } f \text{ } s$$

$$\text{fst} = \lambda p. p \text{ } \text{tru}$$

$$\text{scd} = \lambda p. p \text{ } \text{fls}$$

$$\text{fst}(\text{pair } v \text{ } w) = \text{fst}(\lambda b. b \text{ } v \text{ } w) = (\lambda p. p \text{ } \text{tru}) (\lambda b. b \text{ } v \text{ } w) = \text{tru } v \text{ } w = v$$



## Church Numerals

$$C_0 = \lambda s. \lambda z. z$$
$$C_1 = \lambda s. \lambda z. sz$$
$$C_2 = \lambda s. \lambda z. s(sz)$$
$$C_3 = \lambda s. \lambda z. s(s(sz))$$
$$scc = \lambda n. \lambda s. \lambda z. s(ns z) \text{ successor}$$
$$scc\ C_n = \lambda s. \lambda z. s(s(s... (sz))) = C_{n+1}$$
$$plus = \lambda m. \lambda n. \lambda s. \lambda z. ms(ns z)$$
$$times = \lambda m. \lambda n. m(plus\ n)\ C_0$$
$$iszero = \lambda m. m\ (\lambda x. fls)\ tru$$
$$zz = pair\ C_0\ C_0$$
$$ss = \lambda p. pair\ (snd\ p)\ (plus\ C_1\ (snd\ p))$$
$$prd = \lambda m. fst\ (m\ ss\ zz) \text{ predecessor}$$

## Session 12

## Recursion

## Fixed Point

$$f: A \rightarrow A$$

Its Fixed point is  $x \in A : f(x) = x$

\*if  $x$  is a fixed point in  $f$  then,  $f(f(f(\dots f(x)\dots)) = x$

Factorial is:

$$f(0) = 1$$

$$f(n+1) = (n+1) * f(n)$$

or

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \times f(n-1) & \text{otherwise} \end{cases}$$

or

$$f : n \mapsto \begin{cases} 1 & \text{if } n = 0 \\ n \times f(n-1) & \text{otherwise} \end{cases}$$

Let's define the functional  $F(f) = f'$ :

$$f' : n \mapsto \begin{cases} 1 & \text{if } n = 0 \\ n \times f(n-1) & \text{otherwise} \end{cases}$$

The only fixed point of  $F$  is the factorial function.

Call-by-name  $y$  combinator (or fixed point combinator)

Now we need a combinator  $\text{fix} : F \rightarrow \text{the fixed point of } F$

$$y = \lambda h. (\lambda x. h(xx)) (\lambda x. h(xx)) \quad \text{this is introduced by Church}$$

## Example

$$yF = (\lambda x. F(xx)) (\lambda x. F(xx)) = F (\lambda x. F(xx)) (\lambda x. F(xx)) = F (yF)$$

so  $yF$  is the fixed point of  $F$

Now for the factorial:

$$\text{fct} = \lambda f. \lambda n. \text{if } n=0 \text{ then } 1 \text{ else times } n \text{ f}(n-1)$$

$$\text{factorial} = y \text{ fct} \quad \text{meaning the fixed point of fct}$$

## Example

$$\begin{aligned} y \text{ fct } 2 &= \text{fct } (y \text{ fct}) 2 \\ &= (\lambda f. \lambda n. \text{if } n=0 \text{ then } 1 \text{ else times } n \text{ f}(n-1)) (y \text{ fct}) 2 \\ &= (\lambda n. \text{if } n=0 \text{ then } 1 \text{ else } n * (y \text{ fct}) (n-1)) 2 \\ &= \text{if } 2=0 \text{ then } 1 \text{ else } 2 * (y \text{ fct}) (2-1) \\ &= 2 * (y \text{ fct}) (1) \\ &= \dots \end{aligned}$$

Call-by-value  $Z$  combinator

$$Z = \lambda h. (\lambda x. h (\lambda y. xxy)) (\lambda x. h (\lambda y. xxy)) \quad \text{this is introduced by Gordon Plotkin}$$

which  $ZF = F(ZF)$  (to prove this we need to accept  $\lambda x. mx = m$  which is called Eta-equivalence.)

## Session 13

# SOME PROGRAMMING LANGUAGES

## Lisp

Abbreviation of "List Processor"

Developed in MIT at late 50s (by John McCarthy's team)

Motivating application: for symbolic computations and exploratory programming.

### Example

$$\begin{aligned} \text{Integ } x^2 dx &> x^3/+C \\ 2x^2+x^3 &> x^2 (2+x) \end{aligned}$$

Some products:

- emacs
- gtk

Some developments and branches:

- Maclisp (MIT 1960s)
- Scheme (MIT 1970s)
- Common Lisp

Lisp project:

- Motivating application
- Abstract machine (IBM 704)

\*concrete  $\leftrightarrow$  abstract : concrete programme are less portable and abstract ones are less efficient.

- Theoretical foundation

An Article to read:

Recursive functions of symbolic expressions and their computation by machine.

CACM, 3(4), 184-195 (1960)

## Historical Lisp structure

### Prefix

$(+ 1 2 3 4) \rightarrow 1+2+3+4$

### Atom

$\langle \text{atom} \rangle ::= \langle \text{symbol} \rangle \mid \langle \text{number} \rangle$

$\langle \text{smb} \rangle ::= \langle \text{char} \rangle \mid \langle \text{smb} \rangle \langle \text{char} \rangle \mid \langle \text{smb} \rangle \langle \text{digit} \rangle$

$\langle \text{num} \rangle ::= \langle \text{digit} \rangle \mid \langle \text{num} \rangle \langle \text{digit} \rangle$

### S-expressions and Lists

dotted pair : a.a

$\langle \text{sexp} \rangle ::= \langle \text{atom} \rangle \mid (\langle \text{sexp} \rangle . \langle \text{sexp} \rangle)$

## Functions and special forms

cons, car , cdr , eq , atom  
 cond, lambda, define, quote, eval  
 +, -, \*

Until the above section Lisp is pure functional  
 here are some functions that make Lisp impure.

rplaca, rplacd, set, setq

\* T true  
 nil false

### Examples

(quote cons) → Makes an atom "cons"  
 (cons a b) → A pair containing the values of **a** and **b**  
 (cons (quote a)(quote b)) → A pair containing the atom "a" and "b"  
 '(+ 1 2) or (quote (+ 1 2)) → the list (+ 1 2)  
 (+ 1 2) → 3

### Examples

A function to find something in a list

```
(define find (lambda(x y)
  (cond ((equal y nil) nil)
        ((equal x (car y)) x)
        (true (find x (cdr y)))))
))
```

now to use it we can say:

```
(find 'apple '(pear peach apple banana fig))
```