Definition 1 (Complex of R-modules 1.10)**.** A **complex** of R-Modules is a sequence of modules F_i and maps $F_i to F_{i-1}$ such that the compositions $F_{i+1} \to F_i \to F_{i-1}$ are all zero.

The **homology** of this complex at F_i is the module

$$\ker(F_i \to F_{i-1}) \operatorname{im}(F_{i+1} \to F_i)$$

A **free resolution** of an R-module M
is a complex

$$\mathcal{F} : \ldots \to F_n \overset{\to}{\phi_n} \ldots \to F_1 \overset{\to}{\phi_1} F_0$$

of free R-Modules such that $\operatorname{coker} \phi_1 = M$ and $\mathcal{F}$ is exact (sometimes we add " $\to 0$ " to the right of $\mathcal{F}$ and then insist that $\mathcal{F}$ be exact except at F_0). We shall sometimes abuse this notation and say that an exact sequence

$$\mathcal{F} : \ldots \to F_n \overset{\to}{\phi_n} \ldots \to F_1 \overset{\to}{\phi_1} F_0 \to M \to 0$$

is a resolution of M. The image of the map ϕ_i is called the ith syzygy module of M. A resolution $\mathcal{F}$ is a **graded free resolution** if R is a graded ring, the F_i are graded free modules, and the maps are homogeneous maps of degree 0. Of course only graded modules can have graded free resolutions. If for some $n < \inf$ we have $F_{n+1} = 0$, but $F_i \neq 0 \forall 0 \leq i \leq n$, then we shall say that $\mathcal{F}$ is a **finite resolution of length** n.

Theorem 1 (Hilberts syzygy theorem1.13)**.** *If $R = l[x_1, \ldots, x_r]$, then every finitely generated graded R-Module has a finite graded free resolution of lenght $\leq r$.*

Theorem 2 (Buchberger's Criterion 15.8)**.** *The elements $g_1, \ldots, g_t$ form a Gröbner basis $\iff$ $h_{ij} = 0$ forall i and j*

Notation 1 ()**.** In the following we let F be a free module with basis and let M be a submodule of F generated by monomials $m_1, \ldots, m_t$. Let

$$\phi : \oplus_{j=1}^{t} S\epsilon_j \to F; \phi(\epsilon_j) = m_j$$

be a homomorphism from a free module whose image is M. For each pair of indices i, j such that m_i and m_j involve the same basis element of F, we define

$$m_{ij} := m_i / \operatorname{GCD}(m_i, m_j),$$

and we define σ_{ij} to be the element of $\ker \phi$ given by

$$\sigma_{ij} := m_{ji}\epsilon_i - m_{ij}\epsilon_j.$$

Lemma 3 ([1][15.1)**.**] With notation as above , $\ker \phi$ is generated by the σ_{ij}.

Definition 2 (initial term 325)**.** If $>$ is a monomial order, then for any $f \in F$ we define the **initial term of f**, written $\mathbf{in}_>(f)$ to be the greatest term term of f with respect to the order $>$, and if M is a submodule of F we define $in_>(M)$ to be the monomial submodule generated by the elements $in_<(f)$ forall $f \in M$. When there is no danger of confusion we will simply write **in** in place of $\mathbf{in}_>$.

Definition 3 (Groebnerbasis 328)**.** A **Gröbner basis** with respect to an order $>$ on a free module with basis F is a set of elements $g_1, \ldots, g_t \in F$ such that if M is the submodule of F generated by $g_1, \ldots, g_t$, then $in_>(g_1), \ldots, in_>(g_t)$ generate $in_>(M)$. We then say, that $g_1, \ldots, g_t$ is a **Gröbner basis for** M

Proposition-Definition 4 (15.6)**.** *Let F be a free S-module with basis and monomial order $>$. If $f, g_1, \ldots, g_t \in F$ then there is an expression*

$$f = \sum f_i g_i + f' with f' \in F, f_i \in S_i,$$

where none of the monomials of f' is in $(in(g_1), \ldots, in(g_t))$ and

$$in\,(f) \geq in\,(f_i g_i)$$

*for every i. Any such f' is called a **remainder** of f with respect to $g_1, \ldots, d_t$, and an expression $f = \sum f_i g_i + f'$ satisfying the condition of the proposition is called a **standard expression** for f in terms of g_i.*

Algorithmus 1 (Division Algorithm 328)**.** Lef F be a free S-module with basis and a fixed monomial order. If $f, g_1, \ldots, g_t \in F$, then we may produce a standard expression

$$f = \sum m_u g_{s_u} + f'$$

for f with respect to $g_1, \ldots, g_t$ by defining the indices s_u and the terms m_u inductively. Having chosen $s_1, \ldots, s_p$ and $m_1, \ldots, m_p$, if

$$f'_p := f - \sum_{u=1}^{p} m_u g_{s_u} \neq 0$$

and m is the maximal term of f'_p that is divisible by some $in\,(g_i)$, then we choose

$$s_{p+1} = i, m_{p+1} = m/in\,(g_i)$$

This process terminates when either $f'_p = 0$or no $in\,(g_i)$ divides a monomial of f'_p; the remainder f' is then the last f'_p produced.

Algorithmus 2 (Buchberger's Algorithm 333)**.** In the situation of Theorem 15.8, suppose that M is a submodule of F, and let $g_1, \ldots, g_t \in M$ be a set of generators of M . Compute the remainders h_{ij}. If all of the $h_{ij} = 0$, then the g_i form a Gröbner basis for M. If some $h_{ij} \neq 0$,then replace $g_1, \ldots, g_t$ with $g_1, \ldots, g_t, h_{ij}$, and repeat the process. As the submodule generated by the initial forms of $g_1, \ldots, g_t, h_{ij}$ is strictly larger than that generated by the initial forms of $g_1, \ldots, g_t$, this process must terminate after finitely many steps. The upperbound

$$b = ((r+1)(d+1)+1)^{2^{(s+1)}(r+1)}$$

, where

$r =$ number of variables$d =$ maximum degree of the polynomials g_i , and $s =$ the degree of the Hilbert polynomial (this

Definition 4 (334)**.**

$$\tau_{ij} := m_{ji}\epsilon_i - m_{ij}\epsilon_j - \sum_u f_u{}^{(ij)}\epsilon_u$$

, for $i < j$ such that $in\,(g_i)$ and $in\,(g_j)$ involve the same basis element of F.

Theorem 5 (Schreyer [1][15.10]**.** *] With the notation as above, suppose that $g_1, \ldots, g_t$ is a Gröbner basis. Let $>$ be the monomial order on $\oplus_{j=1}^{t} S\epsilon_j$ defined by taking $m\epsilon_u > n\epsilon_v \iff$*

$$\in (mg_u) > \in (ng_v) \, \textit{with respect to the given order on} F$$

or

$$\in (mg_u) = \in (ng_v) \, (\textit{up to a scalar}) \, \textit{but} u < v.$$

The τ_{ij} generate the syzygies on the g_i. In fact, the τ_{ij} are a Gröbner basis for the syzygies with respect to the order $>$, and $\in (\tau_{ij}) = m_{ji}\epsilon_i$.

Beweis. We show first that the initial term of τ_{ij} is $m_{ji}\epsilon_i$. We have

$$m_{ji} \in (g_i) = m_{ij} \in (g_j) \,,$$

and these terms are by hypothesis greater than any that appear in the $f_u^{(ij)} g_u$. Thus, $\in (\tau_{ij})$ is either $m_{ji}\epsilon_i$ or $-m_{ij}\epsilon_j$ by the first part of the definition of $>$, and since $i < j$ we have $m_{ji}\epsilon_i > m_{ij}\epsilon_j$.
Now we show that the τ_{ij} form a Gröbner basis. Let $\tau = \sum f_v \epsilon_v$ be any syzygy. We must show that $\in (\tau)$ is divisable by one of the $\in (\tau_{ij})$; that is, $\in (\tau)$ is a multiple of some $m_{ji}\epsilon_i$ with $i < j$. For each index v, set $n_v\epsilon_v = \in (f_v\epsilon_v)$. Since these terms cannot cancel with each other, we have $\in (\sum f_v\epsilon_v) = n_i\epsilon_i$ for some i. Let $\sigma = \sum' n_v\epsilon_v$ be the sum over all indices v for which $n_v \in (g_v) = n_i \in (g_i)$ up to a scalar; all indices v in this sum must be $\geq i$ because we assume that $n_i\epsilon_i$ is the initial term of τ.
Thus, σ is a syzygy on the $\in (g_v)$ with $v \geq i$. By 3 , all such syzygies are generate by the σ_{uv} for $u, v \geq i$, and the ones in which ϵ_i, appears are the σ_{ij} for $j > i$. It follows that the coefficient n_i is in the ideal generated by the m_{ji} for $j > i$, and we are done. □

Literatur

[1] David Eisenbud. *Commutative Algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, 1995.