Def: (the type of a linear system):

- we say that eq point O_2 of (2) is a :

- ~~NODE~~ **NODE** when either both ~~eigen~~ $\lambda_1 \leq \lambda_2 < 0$ or $0 < \lambda_1 \leq \lambda_2$

~~- we say that~~

- **SADDLE** : $\lambda_1 < 0 < \lambda_2$
- **CENTER** : $\lambda_{1,2} = \pm i\beta$, $\beta \in \mathbb{R}^*$
- **FOCUS** : $\lambda_{1,2} = \alpha \pm i\beta$, $\alpha, \beta \in \mathbb{R}^*$

(T)

1) If $Re(\lambda_1) < 0 \wedge Re(\lambda_2) < 0 \Rightarrow \eta^* = O_2$ of (2) is a global attractor

2) If $Re(\lambda_1) > 0 \wedge Re(\lambda_2) > 0 \Rightarrow \eta^* = O_2$ of (2) is a global repeller

3) A center is stable ; all the sol. of (2) with a center is a periodic function

4) A saddle is unstable

② The linearization method to study the stability an eq. point of nonlin. sys.

(1) $\dot{x} = f(x)$

~~f(x)=~~

$f(x)$ $\quad\quad$ η^*

$$f(x) = \underbrace{f(\eta^*)}_{=0} + \underbrace{f'(\eta^*)}_{\text{Jacobian matrix}}(x - \eta^*)$$

- for a function $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ its Jacobian matrix in $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad$ $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

$$Jf(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{pmatrix}$$

$f(x) \to Jf(\eta^*) \cdot (x - \eta^*)$

We consider

(3) $\dot{x} = Jf(\eta^*) \cdot x$

Def:

1) the linear sys (3) is called the linearization of ~~the~~ (1) around the equilib point η^*

2) assume that the det $Jf(\eta^*) \neq 0$

when the eq. point O_2 of (3) is a NODE / SADDLE / CENTER / FOCUS we say that eq. point η^* of (1) a LINEAR NODE / LINEAR SADDLE / LINEAR CENTER / LINEAR FOCUS

3) Let $\lambda_1, \lambda_2 \in \mathbb{C}$ eigenvalues of $Jf(\eta^*)$ we say that η^* is a hyperbolic eq. point of (1) when $Re(\lambda_1) \neq 0$ and $Re(\lambda_2) \neq 0$

Rk: - if det $Jf(\eta^*) = 0 \Rightarrow \eta^*$ is not hyperbolic

- assume that det $Jf(\eta^*) \neq 0$

- if η^* is a linear center $\Rightarrow \eta^*$ is not hyperbolic; in the rest of the cases it is hyperbolic

(T) Let η^* be a hyperbolic eq. point of the nonlinear system (1)

- if η^* is a linear attracting node or focus $\Rightarrow \eta^*$ is an attractor for the (1)

- if η^* is a linear reppelling node or focus $\Rightarrow \eta^*$ is a repeller for the (1)

- if η^* is a linear saddle $\Rightarrow$ η^* is an unstable eq. p

~~③~~ Rk:

If we have to study the stability of the eq. p
a second order scalar

$$\ddot{x} = f(x, \dot{x})$$

we have to write the equivalent planar sys of
unknowns

$$x \wedge y = \dot{x}$$

This system is

$$\begin{cases} \dot{x} = y \\ \dot{y} = f(x, y) \end{cases}$$

③

1. The idealized pendulum equation

$$\ddot{\theta} + \nu\dot{\theta} + \sin\theta = 0$$

second order non linear d.e.

$\nu > 0$ damping constant

$\nu = 0$ ideal case

$$\ddot{\theta} + \sin\theta = 0$$

$x = \theta$

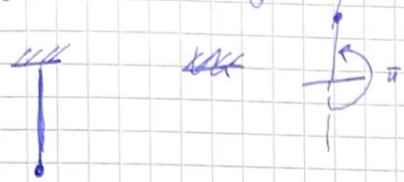
$y = \dot{\theta}$

$$f(x,y) = \begin{pmatrix} y \\ -\sin x \end{pmatrix}$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x \end{cases}$$

non lin. planar sys.

Study the stability of the eq.



look for eq. : $f(x,y)=0 \Leftrightarrow \begin{cases} y=0 \\ -\sin x=0 \end{cases} \Rightarrow \begin{cases} x=k\pi \\ y=0 \end{cases}, k\in\mathbb{Z}$

$(k\pi, 0)$, $k\in\mathbb{Z}$

Physically we have 2 eq points $\eta_1^*=(0,0)$

$\eta_2^*=(\pi,0)$

Since our system is nonlinear we apply the lin method

$$Jf=\begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix} \qquad \eta_2^*=(\pi,0)$$

$$Jf(\pi,0)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}=0 \Rightarrow \lambda^2-1=0 \Rightarrow \lambda_{1,2}=\pm 1$$

$\Rightarrow \eta_2^*$ is a linear saddle, hyperbolic $\overset{\text{T}}{\Rightarrow}$ η_2^* is unstable

$\eta_1^* = (0,0)$

$$Jf(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda_1 & 1 \\ -1 & -\lambda_2 \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0$$

$$\lambda_{1,2} = \pm i$$

$\Rightarrow \eta_1^*$ is a linear center and it is not hyperbolic

thus linearization method fails.

(f)

- if η^* is a linear center there exists a first integral of (1) well defined in a neighborhood η^* then η^* is a stable eq. point of (1)

$$\frac{dy}{dx} = \frac{-\sin x}{y} \Rightarrow y\,dy = -\sin x\,dx \quad \Big| \int$$

$$\frac{y^2}{2} = \cos x + C \quad , \ C \in \mathbb{R}$$

$$H(x,y) = \frac{1}{2}y^2 - \cos x$$

$H : \mathbb{R}^2 \to \mathbb{R}$, C^1

check $\dfrac{\partial H}{\partial x} \cdot y + \dfrac{\partial H}{\partial y} \cdot (-\sin x) = 0$, $\forall (x,y) \in \mathbb{R}^2$

True $\sin x\, y - \sin x\, y = 0$

$\Rightarrow$ H is a global first integral
(0,0) lin. center
$\Big\}$ (T) $\Rightarrow$ η^* is stable

Lecture 10

Planar dynamical system (cont)

1. Phase portraits for linear planar systems
2. Phase portraits for planar systems using polar coords

1.

(1) $\dot{X} = AX$, $A \in M_2(\mathbb{R})$ $X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

$\lambda_1, \lambda_2 \in \mathbb{C}$ the 2 eigenval. of A $\det A \neq 0$

Recall from L9

$\det A \neq 0 \Leftrightarrow$ the only equilibrium of (1) is $\eta^* = 0 \in \mathbb{R}^2$

$\det A \neq 0 \Leftrightarrow \lambda_1 \neq 0 \wedge \lambda_2 \neq 0$

Def + (T) from L9

if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \leq \lambda_2 < 0 \Rightarrow \eta^* = 0_2$ is a NODE, (type)

(stability $\rightarrow$) global attractor ; $\nexists$ global first integral

if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \geq \lambda_2 > 0 \Rightarrow \eta^* = 0_2$ is a NODE,

global repeller ; $\nexists$ global first integral if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < 0 < \lambda_2 \Rightarrow \eta^* = 0_2$ is a SADDLE

unstable ; $\exists$ first global integral

if $\lambda_{1,2} = \pm i\beta$, with $\beta \in \mathbb{R}^*$ $\Rightarrow \eta^* = 0_2$ is a CENTER

stable ; $\exists$ first global integral

if $\lambda_{1,2} = \alpha \pm i\beta$ with $\alpha, \beta \in \mathbb{R}^*$ $\Rightarrow \eta^* = 0_2$ is a FOCUS

$\nexists$ global first integral and

$$\begin{cases} \text{global attracter when } \alpha < 0 \\ \text{global repeller when } \alpha > 0 \end{cases}$$

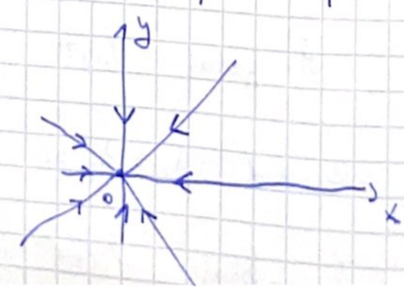
Typical phase portrait of a NODE

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$$

$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ diagonal matrix $\Rightarrow \lambda_1 = -1$, $\lambda_2 = -1$

$\Rightarrow$ the equilb. $\eta^* = 0_2$ is a NODE , global attracter

From L8 we have the phase portrait



Typical phase portrait of a SADDLE

$$\begin{cases} \dot{x} = -x \\ \dot{y} = y \end{cases} \qquad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = 1 \end{matrix} \Rightarrow$$

$\Rightarrow$ the equil. $\eta^* = 0_2$ is a SADDLE, unstable

Find the flow

The flow : $\cancel{\varphi(x)}$ $\varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^{-t} \\ \eta_2 e^{t} \end{pmatrix}$

~~the~~ A global first integral $H : \mathbb{R}^2 \to \mathbb{R}$

$$H(x,y) = xy$$

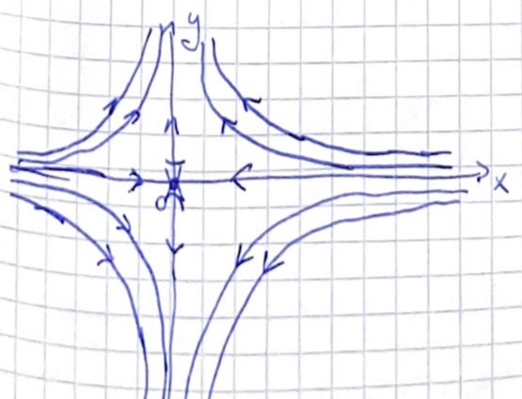
The phase portrait : the level curves of H $\quad xy = c$, $c \in \mathbb{R}$

$xy = 1$

$y = \frac{1}{x}$ $\qquad f : \mathbb{R}^* \to \mathbb{R}$, $f(x) = \frac{1}{x}$

$$f'(x) = -\frac{1}{x^2} < 0$$

x	$-\infty$		0		$+\infty$
$\frac{1}{x}$	0	$\searrow$	$-\infty \,\vert\, +\infty$	$\searrow$	0



$xy = -1 \Rightarrow y = -\frac{1}{x}$

$xy = 0 \Rightarrow x = 0 \lor y = 0$

Typical phase portrait of a CENTER

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases} \qquad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \det(A - \lambda I_2) = 0$$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

Indeed $\eta^* = 0_2$ is a CENTER. STABLE, it is a gl

from L8:

$H : \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = x^2 + y^2$

the phase portrait is

$$\begin{cases} \dot{x} = -y \\ \dot{y} = 4x \end{cases}$$

Typical phase portrait of a FOCUS

$$\begin{cases} \dot{x} = x - y \\ \dot{y} = x + y \end{cases} \qquad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2} = 1 \pm i$$

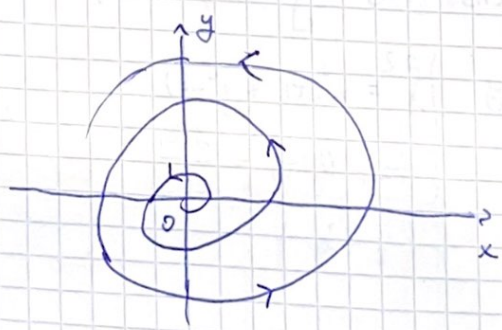
$$(Re(\lambda_{1,2}) = 1 > 0)$$

$\eta^* = 0_2$ is a FOCUS, global repeller

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

The flow is: $\varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^t \cos t - \eta_2 e^t \sin t \\ \eta_1 e^t \sin t + \eta_2 e^t \cos t \end{pmatrix}$

Particular case: $\eta_2 = 0$



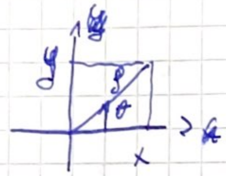
$$\varphi(t, \eta_1, 0) = \begin{pmatrix} \eta_1 e^t \cos t \\ \eta_1 e^t \sin t \end{pmatrix}$$

param eq: $\begin{cases} x = \eta_1 e^t \cos t \\ y = \eta_1 e^t \sin t \end{cases}$

$$\begin{cases} \rho^2 = x^2+y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

$\rho^2 = x^2+y^2 = \eta_1^2 e^{2t} \Rightarrow \rho(t) = |\eta_1| e$

$\tan\theta = \tan t \Rightarrow \theta(t) = t$



global = it doesn't from where we start (desen)

2. Phase portraits using polar coords

$$(1) \quad \begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$$

- to transform (1) to polar coords means to consider new unknowns as $(\rho(t), \theta(t))$ instead of $(x(t), y(t))$ related by

$$\begin{cases} \rho^2 = x^2(t) + y^2(t) \\ \tan\theta(t) = \dfrac{y(t)}{x(t)} \end{cases} \quad (2)$$

Step 1

- take the derivative w.r.t. t in (2)

$$(3) \quad \begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\dot{y}\cdot x - y\dot{x}}{x^2} \end{cases}$$

Step 2

- replace in (3) $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$

Then replace

$$x = \rho \cos\theta$$
$$y = \rho \sin\theta$$

Come back to $\begin{cases} \dot{x} = x - y \\ \dot{y} = x + y \end{cases}$. We want to transform it to polar coords.

$$\begin{cases} \rho\dot{\rho} = x\cdot(x-y) + y(\cancel{y}x+y) \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{x(x+y) - y(x-y)}{x^2} \end{cases} \Rightarrow \begin{cases} \rho\dot{\rho} = x^2 + y^2 \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{x^2+y^2}{x^2} \end{cases}$$

$$\Rightarrow \begin{cases} \rho\dot{\rho} = \rho^2 \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\rho^2}{\rho^2\cos^2\theta} \end{cases} \Rightarrow \begin{cases} \dot{\rho} = \rho \\ \dot{\theta} = 1 \end{cases} \Rightarrow \begin{cases} \rho(t) = c_1 e^t \\ \theta(t) = t + c_2 \end{cases}$$

$\dot{\rho} > 0 \Rightarrow \rho$ increasing $\Rightarrow$ depart from O_2

$\dot{\rho} < 0 \Rightarrow \rho$ decreasing $\Rightarrow$ approach O_2

$\dot{\rho} = 0 \Rightarrow \rho$ constant along an orbit $\Rightarrow$ the orbit is a circle centered in O_2

$\dot{\theta} > 0 \Rightarrow \theta \nearrow \Rightarrow$ ~~counter clo~~ counterclockwise rotation

$\dot{\theta} < 0 \Rightarrow \theta \swarrow \Rightarrow$ clockwise rotation

$\dot{\theta} = 0 \Rightarrow \theta$ is constant $\Rightarrow$ the orbit lie on a line through O_2

ex:

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2) \end{cases}$$

a) Check that $\varphi(t, 1, 0) = (\cos t, \sin t)$

b) ~~Poto~~ Pass to polar coords and represent the phase

c) Read the p.p. and specify the stability of the equil $(0,0)$. There is an attractor?

a)

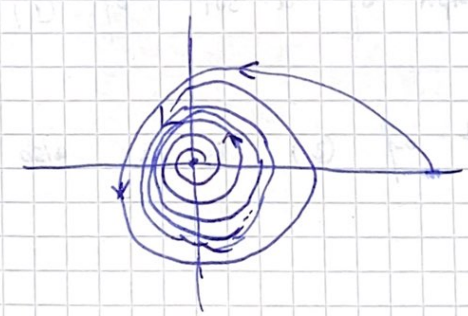
Recall that, by the def, $\varphi(t,1,0)$ is the sol of the

$$\text{IVP} \quad (*) \begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = x + y(1-x^2-y^2) \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

- we have to replace $x = \cos t$, $y = \sin t$ in $(*)$ and show that we obtain valid relations. Ok

b)

$$\begin{cases} \dot{\rho} = \rho(1-\rho^2) \\ \dot{\theta} = 1 \end{cases}$$



$x = \cos t$

$y = \sin t$

$(0,0)$ eq point

$\dot{\theta} = 1 > 0 \Rightarrow \theta \nearrow$

$\dot{\rho} = \rho(1-\rho^2)$

$= \rho \underbrace{(1+\rho)}_{>0}(1-\rho)$

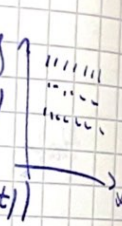
Lecture 11

I The direction field associated to a d.e

II ~~the~~ Short introduction to ~~no~~ numerical methods

I

(1) $y'(x) = f(x, y(x))$ (2) $\begin{cases} \dot{x}(t) = g_1(x(t), y(t)) \\ \dot{y}(t) = g_2(x(t), y(t)) \end{cases}$



Assumptions: $f \in C^1(\mathbb{R}^2)$, $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$

Recall: we have existence and uniqueness for IVP

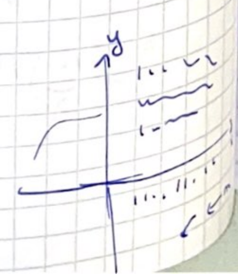
The graph of a sol of (1) is a plane curve ($\mathbb{R}^2$)

An orbit of (2) is a also a plane curve

Def:

1. The direction field associated to eq.(1) de is a collection of vectors

To a point $(x_0, y_0) \in \mathbb{R}^2$ we associate a vector of slope $m = f(x_0, y_0)$



2. The direction field associated to sys (2) is a collection of vectors

To a point $(x_0, y_0) \in \mathbb{R}^2$ we associate a vector of slope $m = \dfrac{g_2(x_0,y_0)}{g_1(x_0,y_0)}$

ex:

$$(3) \quad y' = 1 - \frac{x}{y^2}$$

$$(4) \quad \begin{cases} \dot{x} = y^2 \\ \dot{y} = -x + y^2 \end{cases}$$

Note that (3) and (4) have the same dir. field

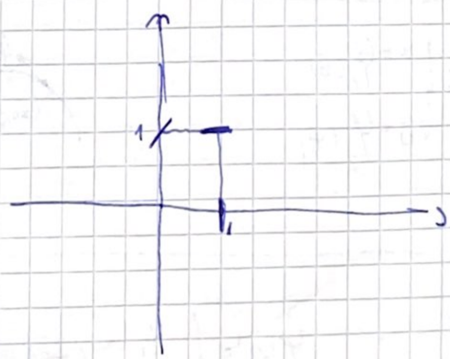
Draw the vectors corresponding to the points $(1,1)$, $(1,0)$, $(0,1)$

$$\frac{g_2(x,y)}{g_1(x,y)} = \frac{-x+y^2}{y^2} = 1 - \frac{x}{y^2} = f(x,y) \Rightarrow$$ (3) and (4) have the same direction field

$(1,1) \Rightarrow m_1 = 0$

$(1,0) \Rightarrow m_2 = \infty$

$(0,1) \Rightarrow m_3 = 1$



$0 \Rightarrow \parallel O_x$

$1 \Rightarrow \parallel$ first bisec

$\infty \Rightarrow \parallel O_y$

$g(x,y) = (g_1(x,y), g_2(x,y))$

this vector has the slope $\dfrac{g_2}{g_1}$

Property

Let $(x_0, y_0) \in \mathbb{R}^2$

I Denote by $\varphi : J \to \mathbb{R}$ the sol of the IVP $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$

Then $\varphi'(x_0) = f(x_0, y_0)$, i.e the vector of the dir. field associated to (x_0, y_0) is tangent to the sol curve that passes through the point

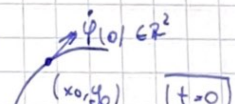
II Denote by $\varphi : J \to \mathbb{R}^2$ the sol of the IVP $\begin{cases} \dot{x} = g_1(x,y) \\ \dot{y} = g_2(x,y) \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$

~~Then~~

Then ~~is~~ the tangent vector to the orbit of φ in (x_0,y_0) is $\dot{\varphi}(0)$

---

orbit $\begin{cases} x = \varphi_1(t) \\ y = \varphi_2(t) \end{cases}$



$$\dot{\varphi}(0) = \begin{pmatrix} g_1(x_0, y_0) \\ g_2(x_0, y_0) \end{pmatrix} \quad \text{whose slope is} \quad \frac{g_2(x_0,y_0)}{g_1(x_0,y_0)}$$

(In ~~oth~~ other words, the dir field ~~tangent vector~~ is tangent to the orbits of sys (2))

Def:

Let $m \in \mathbb{R} \cup \{\mp\infty\}$

The m- isocline of (1) is $\{(x,y) \in \mathbb{R}^2 / m = f(x,y)\}$

The m-isocline of (2) is $\{(x,y) \in \mathbb{R}^2 / m = \frac{g_2(x,y)}{g_1(x,y)}\}$

ex:

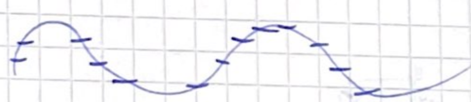
$$(5) \begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases} \qquad (6) \; y' = -\frac{x}{y}$$

a) Note that they have the same dir field

b) Find and represent the isoclines

c) Find the shape of the orbits of (5)

0 -isocline



we take $y = mx$ (for dir field) vectors

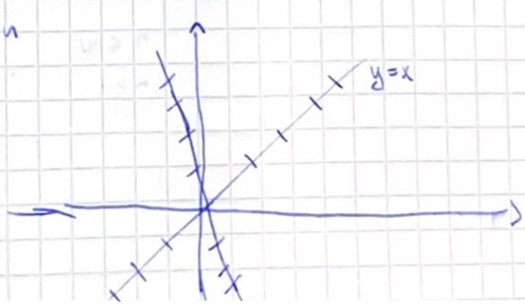
a)

$$\frac{g_2(x,y)}{g_1(x,y)} = -\frac{x}{y} = f(x,y)$$

b)

$m \in \mathbb{R} \cup \{\infty\}$ the m -isocline has the equation

$m = -\frac{x}{y} \Rightarrow y = -\frac{1}{m}x$ this is a line through the origin



$m = -1 \Rightarrow y = x$

This is the -1 -isocline

Note that $m \cdot \left(-\frac{1}{m}\right) = -1 \Rightarrow$ the dir field is $\perp$ to any isocl[ine]

So any orbit is orthogonal to any line that passes [through]

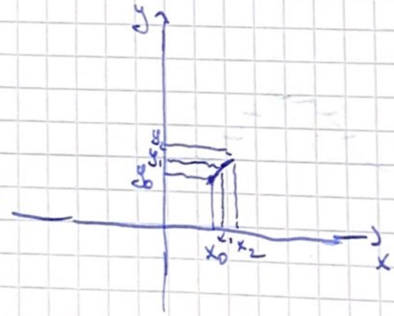
the origin

Thus, any orbit is a circle centered in the origin

II Short introduction to numerical methods

$$(1)\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \qquad f \in C^1(\mathbb{R}^2)$$

has a unique sol $\varphi : J \to \mathbb{R}$



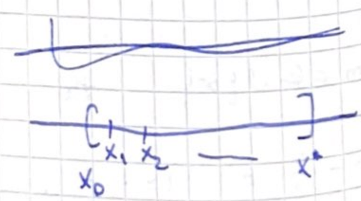
$m_0 = f(x_0, y_0)$

$m_1 = f(x_1, y_1)$

$\vdots$

The Euler's numerical method

$[x_0, x^*] \subset J$



take [...]
of [...]
$x_0 < x_1$ [...]

$n \in \mathbb{N}$
$n \geq 1$ - number of steps

$y - y_0 = m_0 (x - x_0)$

$y_1 = y_0 + (x_1 - x_0) \cdot f(x_0, y_0)$

$\vdots$

$y_{k+1} = y_k + (x_{k+1} - x_k) \cdot f(x_k, y_k) \quad , \quad k = \overline{0, n-1}$

$\underline{\underline{y_i}}$ is an "approximation" of the exact value $\varphi(x_i)$

the result of a numerical method

Usually, $x_{k+1} - x_k = h \quad , \forall \, k = \overline{0, n-1}$ $\quad h > 0$ is called the step size

$$\Rightarrow \begin{cases} y_{k+1} = y_k + h \cdot f(x_k, y_k) \quad , \quad k = \overline{0, n-1} \\ x_{k+1} = x_k + h \end{cases}$$

ex:

$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

$x_0 = 0$, $y_0 = 1$

We know that the unique sol is

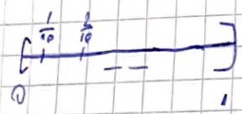
$\varphi(x) = e^x$

We fix the interval $[0; 1]$

We take the stepsize $h_n = \frac{1}{n} \quad , \quad n \geq 1$ fixe

We compute y_n, an approx. of $\varphi(1) = e$

We will prove that $\lim_{n \to \infty} y_n = e$



$n = 10$

Note that if we take h_n the number of steps to cover the interval $[0,1]$ is n

Write the Euler's numerical formula

$$\begin{cases} x_{k+1} = x_k + \frac{1}{n} \\ y_{k+1} = y_k + \frac{1}{n} y_k \\ x_0 = 0 \\ y_0 = 1 \end{cases}$$

$f(x,y) = y$

$$\Rightarrow \begin{cases} x_k = \frac{k}{n} \\ y_{k+1} = \left(\frac{1}{n} + 1\right) y_k \end{cases}$$

$$\begin{cases} x_k = \frac{k}{n} \\ y_k = \left(1 + \frac{1}{n}\right)^k \end{cases}, \quad k = \overline{0,n-1}$$

$$y_n = \left(1 + \frac{1}{n}\right)^n \approx e, \quad \forall n \in \mathbb{N}^*$$