Test seminar #2:
- Lecture 7, 8, 9, 10
- Seminar 4, 5

Lecture 07

The dynamical system associated to an autonomous differential equation in $\mathbb{R}^n$

(1) $\dot{x} = f(x)$, ~~f~~ $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \in \mathbb{N}^*$ $f \in C^1$

- the unknown : $t \in \mathbb{R} \longmapsto x(t) \in \mathbb{R}^n$

$\dot{x} \rightsquigarrow x'$ the notation ~~for~~ used by Isaac Newton for the derivative w.r.t. time

ex:

$\dot{x} = tx$ is non-autonomous (linear) (has "t")

$\dot{x} = x$ is autonomous (linear) (has no "t")

$\dot{x} = 1 - x^2$ autonomous, nonlinear

The existence and uniqueness theorem for IVP

We assume that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $\eta \in \mathbb{R}^n$ and consider the IVP

$$(2) \begin{cases} \dot{x} = f(x) \\ x(0) = \eta \end{cases}$$

We have that the IVP (2) has a unique sol, denoted by $\varphi(\cdot, \eta)$ whose maximal interval of definition is denoted by $J_\eta = (\alpha_\eta, \beta_\eta) \subset$

when $\varphi(\cdot, \eta)$ is bounded in the future (bounded when on $[0, \beta_\eta)$) we have $\beta_\eta = +\infty$

when $\varphi(\cdot, \eta)$ is bounded in the past (on $(\alpha_\eta, 0]$) we have $\alpha_\eta = -\infty$; when $\varphi(\cdot, \eta)$ is bounded on J_η then $J_\eta = \mathbb{R}$

Terminology

η - the initial state $\in \mathbb{R}^n$ of the system

$\varphi(t, \eta) \in \mathbb{R}^n$ - the state of the system at time t when η is the initial state

$\mathbb{R}^n$ - the state space of the dynamical system

Def:

$\eta^\circ \in \mathbb{R}^n$ is said to be equilibrium (stationary) state (point) of (1) when $\varphi(t, \eta^\circ) = \eta^\circ$, $\forall t \in \mathbb{R}$

Rk:

- η° is an equil. point of (1) $\Leftrightarrow$ the IVP $\begin{cases} \dot{x} = f(\ldots \\ x(0) = \ldots \end{cases}$

has the unique sol a constant function , $\varphi(t,\eta^*)=\eta^*$, $\forall t\in\mathbb{R}$

$\Leftrightarrow f(\eta^*)=0$

T

- assume that $\exists \lim_{t\to\infty} \varphi(t,\eta)=\eta^* \in\mathbb{R}^n$

- then η^* is an equil. point of (1)

Proof:

$\boxed{n=1}$

$$\dot{\varphi}(t,\eta)=f(\varphi(t,\eta))$$

Hyp, f is cont $\Rightarrow \exists \lim_{t\to\infty} f(\varphi(t,\eta))=f(\eta^*)$ $\Big\}\Rightarrow$

$\Rightarrow \exists \lim_{t\to\infty} \dot{\varphi}(t,\eta)=f(\eta^*)$

Lemma : $f\in C^1$ $\exists \lim_{t\to\infty} f(t)=\eta^*$ and th $\exists \lim_{t\to\infty} \dot{f}(t) \Rightarrow$

$\Rightarrow$ the limit is 0 $(\lim_{t\to\infty} f'(t)=0)$

We apply the mean value theor. on $[m,m+1]$, $m\in\mathbb{N}^*$

$\exists\ a_m\in(m,m+1)$ s.t.

$$f(m+1)-f(m)=f'(a_m)$$

$\lim_{m\to\infty} f(m)=\eta^*$

$\lim_{m\to\infty} f(m+1)=\eta^*$ $\Big\} \Rightarrow \lim_{m\to\infty} \dot{f}(a_m)=0$

$\exists \lim_{t\to\infty} \dot{f}(t)$ $\Big\}\Rightarrow$

$\lim_{m\to\infty} f'(a_m)=0$

$\Rightarrow \lim_{t \to \infty} f(t) = 0$

Def:

Let $\eta^* \in \mathbb{R}$ equilb. point of (1)

- we say that η^* is an attractor of (1) if $\exists$ a neighborhood V_* of η^* s.t. $\lim_{t \to \infty} \varphi(t, \eta) = \eta^*$, $\forall \eta \in V_*$

- when η^* is an attractor, we ~~define~~ ~~obtain~~ define its ~~base~~ in basin of attraction $A_{\eta^*} = \{ \eta \in \mathbb{R}^n / \lim_{t \to \infty} \varphi(t, \eta) = \eta^* \}$

when $A_{\eta^*} = \mathbb{R}^n$ we say that η^* is a global attractor

- when we replace " $t \to \infty$ " to " $t \to -\infty$ " we can give the similar notions replacing "attractor" with "repeller"

Def :

- the function $(t, \eta) \longmapsto \varphi(t, \eta)$ is said to be the <u>flow</u> of (1)

Def :

Let $\eta \in \mathbb{R}^n$

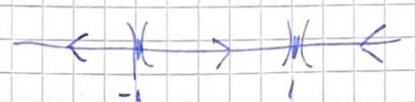
- the orbit (trajectory) of η is $\gamma_\eta = \{ \varphi(t, \eta) \,/\, t \in J_\eta \}$

Rk:

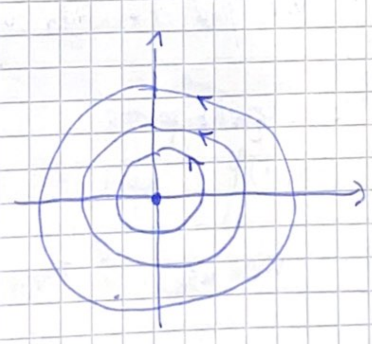
- γ_η is the image of $\varphi(\cdot, \eta)$

- the <u>phase portrait</u> of (1) is the representation of some "significant" orbits , together with an arrow on each side that indicates the future

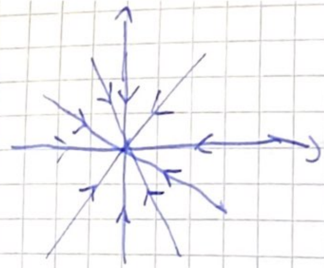
ex: $\dot{x} = 1 - x^2$



$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$$



$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$$



General procedure to represent the phase portrait of (1) when $\boxed{n=1}$ (scalar dyn sys)

<u>Lemma</u>

Let $f: \mathbb{R} \to \mathbb{R}$ be a C^1 function

- any nonconstant sol of $\dot{x} = f(x)$ is a strictly monotone function

<u>Step 1</u>: We find the equilib. points; we solve the eq $f(x) = 0$

<u>Step 2</u>: We find the sign of f on the intervals delimit[ed] by the equilib. points

<u>Step 3</u>: We use: "The orbits of $\dot{x} = f(x)$ are the ones corresponding to the equilib. points and the open intervals delimited by the ...

- we represent on $\mathbb{R}$ the orbits and an arrow on each orbit according to the rules:

  - if $f > 0$ on the orbit, then the arrow points to the right
  - if $f < 0$ ——————————— " ——————— to the left

ex:

- the orbits of $\dot{x} = 1 - x^2$ are $(-\infty; -1)$, $\{-1\}$, $(-1, 1)$, $\{1\}$, $(1, +\infty)$

How to read the ~~fase~~ phase portrait?

$\dot{x} = 1 - x^2$

- Deduce that $\varphi(\cdot, 0)$ is bounded, strictly increasing, defined on $\mathbb{R}$, $\lim_{t \to \infty} \varphi(t, 0) = 1$, $\lim_{t \to -\infty} (t, 0) = -1$

Explanation:

$0 \in (-1, 1)$, which is ~~iem~~ an orbit $\Rightarrow$

$\Rightarrow \gamma_0 = (-1, 1) \Rightarrow$ the image $\varphi(\cdot, 0)$ is $(-1, 1)$

$\Rightarrow \varphi(\cdot, 0)$ is bounded $\overset{\textcircled{T}_1}{\Rightarrow} I_0 = \mathbb{R}$

Phase port. $\Rightarrow \varphi(\cdot, 0)$ is strictly increasing

$\lim_{t \to \infty} \varphi(t, 0) = 1$

$\lim_{t \to -\infty} \varphi(t, 0) = -1$

PP $\Rightarrow$ $\eta^* = 1$ is an attractor and $A_1 = (-1, +\infty)$

$\eta^* = -1$ is a repeller and $B_{-1} = (-\infty; 1)$

The linearization method

$f: \mathbb{R} \to \mathbb{R}$, C^1 function

Let η^* be an equilib. point of $\dot{x} = f(x)$

If $f'(\eta^*) < 0 \Rightarrow \eta^*$ is an attractor

If $f'(\eta^*) > 0 \Rightarrow \eta^*$ is a repeller

Rk:

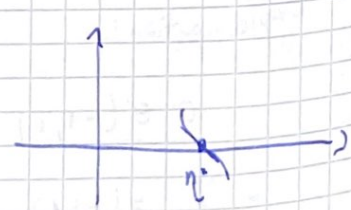
When $f'(\eta^*) = 0 \Rightarrow$ the linearization method fails

$\dot{x} = -x^3$, $\dot{x} = x^3$, $\dot{x} = x^2$, $\dot{x} = 0$ (a repres. the PP and discuss the stability of $\eta^* = 0$)

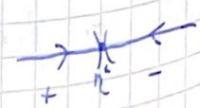
Proof (intuitive)

$f'(\eta^*) < 0$

$f(\eta^*) = 0$



x	η^*
f	$+ \ 0 \ -$



① Problem (24) (from the list)

$k > 0$ parameter ; $\dot{x} = -k(x-21)$

is the model of Newton for cooling processes, here

$x(t)$ being the temperature of a cup of tea at the moment "t"

k - constant that depends on the enviroment

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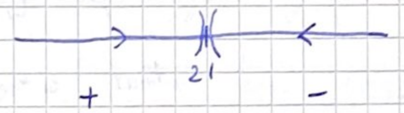
a) Find the flow and represent the phase portr.

b) initial temp $45^\circ C \longrightarrow 37^\circ C$ after 10 minutes

Find ? ? $\to 37^\circ C$ after 20 minutes

$t=0$, $t=1$ is 1 min



$\varphi(t, \eta)$

$\eta = ?$ s.t. $\varphi(20, \eta) = 37$

$\varphi(10, 45) = 37$

(an eq. in "k")

$\eta \in \mathbb{R}$ $\begin{cases} \dot{x} = -k(x-21) \\ x(0) = \eta \end{cases}$ $\varphi(t, \eta)$ the sol of this IVP

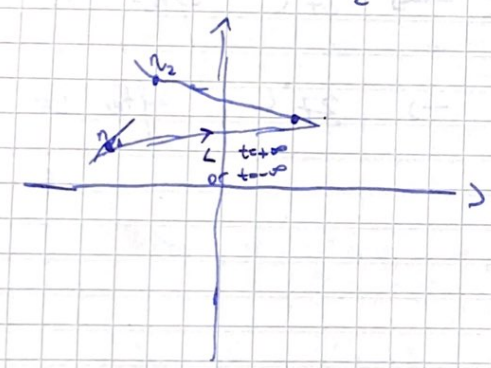
Lecture 08

Planar dynamical system

(1) $\dot{x} = f(x)$ where $f \in C^1$ $(\mathbb{R}^2 \to \mathbb{R}^2)$

the unknown $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^2$ is the state space

L7: state space, eq point, orbit, phase portrait



Rk:

1) for any $\eta \in \mathbb{R}^2$, $\exists$ an orbit γ_η s.t. $\eta \in \gamma_\eta$

2) for $\eta_1, \eta_2 \in \mathbb{R}^2$, $\eta_1 \neq \eta_2$ we have that either $\gamma_{\eta_1} \cap \gamma_{\eta_2} = \emptyset$ or $\gamma_{\eta_1} = \gamma_{\eta_2}$

3) an orbit γ_η ends only at infinity or near an attractor or repeller.

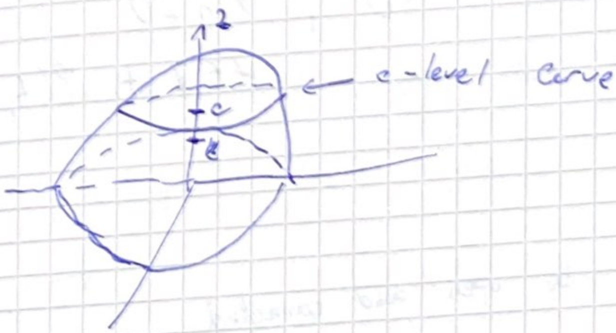
Def:

Let $U \subset \mathbb{R}^2$ and $H : U \to \mathbb{R}$ continuous function,

$c \in \mathbb{R}$

- the c-level curve of H is Γ_c

$\Gamma_c = \{ x \in U \mid H(x) = c \}$



Def: $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = x^2 + y^2$

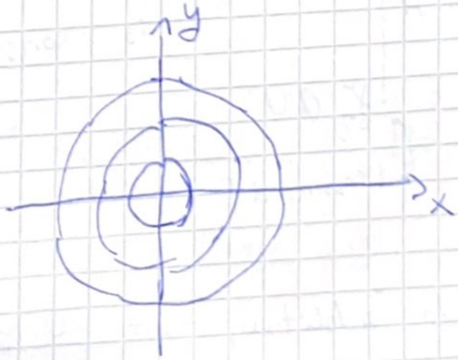
Represent in $\mathbb{R}^2$ the level curves of H

$c \in \mathbb{R}$ $\quad x^2 + y^2 = c$

$c > 0 \Rightarrow$ circle centered in the origin with radius $\sqrt{c}$

$c = 0 \Rightarrow (0,0)$

$c < 0 \Rightarrow \emptyset$



Def:
Let $U \subset \mathbb{R}^2$ be open and connected

and $H : U \to \mathbb{R}$ C^1 function

- we say H is a first integral of (1) in U

if ~~(…)~~

(i) H is not locally constant

(ii) $H(\varphi(t, \eta)) = H(\eta)$, $\forall \eta \in U$

$\forall t$ s.t. $\varphi(t, \eta) \in U$

Def:
Let $U \subset \mathbb{R}^2$ be open and connected

- we say that U is an invariant set for (1) if

$\forall \eta \in U$ we have $\gamma_\eta \in U$

Rk:

~~(ii) $\Leftrightarrow$ $\forall \eta \in U$ we γ~~

(ii) $\Leftrightarrow$ $H\Big|_{\gamma_\eta \cap U}$ is constant

(restricted)

Assume, in addition, that U is an invariant set of

$\Rightarrow \forall \eta \in U$, $\gamma_\eta \subseteq U$

(ii) $\Leftrightarrow$ $\gamma_\eta \subseteq \Gamma_{H(\eta)}$ (the orbits of (1) are contained the level curve of a first integral)

ex: $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$

Check that (0,0) the only eq. point, which is neither an attractor nor a repeller

Check, using the def, that $H: \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = x^2 + y^2$ is a global first integral

Def:
- a first integral in $\mathbb{R}^2$ is ~~set~~ said to be a global first integral

$$f(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$$

eq point : $f(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x = y = 0$

Indeed (0,0) is the only eq. point

- we need the expression of the flow

Let $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$, the IVP $\begin{cases} \dot{x} = -y \\ \dot{y} = x \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$

we use reduction to a

second order dif eq in x

$y = -\dot{x}$

$\left.\begin{aligned} \ddot{x} &= -\dot{y} \\ \dot{y} &= x \end{aligned}\right\} \Rightarrow \ddot{x} = -x \Rightarrow \ddot{x} + x = 0$

$r^2 + 1 = 0$

$r^2 = -1 \Rightarrow r_{1,2} = \pm i \rightarrow \begin{matrix} \cos t \\ \sin t \end{matrix}$

$$\begin{cases} x = c_1 \cos t + c_2 \sin t \\ y = c_1 \sin t - c_2 \cos t \end{cases}$$

$x(0) = c_1 = \eta_1$

$y(0) = -c_2 = \eta_2$

$\Rightarrow \varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 \cos t - \eta_2 \sin t \\ \eta_1 \sin t + \eta_2 \cos t \end{pmatrix}$

$\forall t \in \mathbb{R} \quad \forall \eta \in \mathbb{R}^2$

Assume by contradiction that $(0,0)$ is an attractor

Then η close to $(0,0)$ $\quad 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\lim_{t \to \infty} \varphi(t, \eta) = 0$ False (we have no lim for sin/cos)

same for repeller

$H \in C^1(\mathbb{R}^2)$ is not locally constant

$H(\varphi(t,\eta)) = (\eta_1 \cos t - \eta_2 \sin t)^2 + (\eta_1 \sin t + \eta_2 \cos t)^2$

(we have to prove that in final we have no t)

$H(\varphi(t,\eta)) = \eta_1^2 + \eta_2^2 = H(\eta)$, $\forall t \in \mathbb{R}$

ex:

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$$

Check that $(0,0)$ only eq point which is a global attractor

Check that $H : \mathbb{R} \times (0; +\infty) \to \mathbb{R}$

$H(x,y) = x \dfrac{x}{y}$ is a first integral

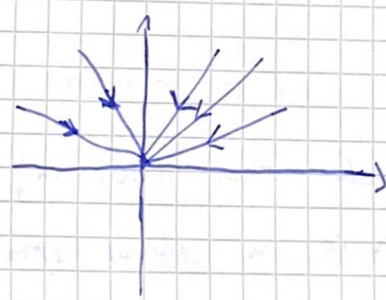
Repres. the ph. port.

$y > 0 \Rightarrow \dot{x} < 0 \Rightarrow$ the arrow must point to the left compute the flow

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases} \qquad \varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^{-t} \\ \eta_2 e^{-t} \end{pmatrix}$$

$\lim_{t \to \infty} \varphi(t, \eta) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \Rightarrow$ the eq point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a glob attractor

$\forall \eta \in \mathbb{R}^2$



$\frac{x}{y} = c \Rightarrow x = cy$ (lines with different slopes)

Note that $H_2 : \mathbb{R} \times (-\infty; 0) \to \mathbb{R}$

$H_2(x, y) = \frac{x}{y}$ is also a first integral

Let $\eta = \begin{pmatrix} \eta_1 \\ 0 \end{pmatrix} \in \mathbb{R}^2 \Rightarrow$

$\Rightarrow \varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^{-t} \\ 0 \end{pmatrix}$

$\gamma_\eta = \left\{ \begin{pmatrix} \eta_1 e^{-t} \\ 0 \end{pmatrix} . t \in \mathbb{R} \right\}$

Proposition (another method to check that a given function is a first integral)

Let $U \subset \mathbb{R}^2$ open and connected, $H : U \to \mathbb{R}$, C^1 and is not locally constant

$$H \text{ is a first integral of } \begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases} \iff$$

$$\iff f_1(x,y) \cdot \frac{\partial H}{\partial x}(x,y) + f_2(x,y) \cdot \frac{\partial H}{\partial y}(x,y) = 0 \quad (2)$$

$\forall (x,y) \in U$

Proof

$$H \text{ is a first integral} \iff H(\varphi(t,\eta)) = H(\eta) \iff$$

$$\iff \frac{d}{dt} H(\varphi(t,\eta)) = 0$$

$$H(\varphi_1(t,\eta), \varphi_2(t,\eta))$$

$$\iff \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \quad \cancel{H(\varphi_1(t,\eta))}$$

$$\frac{\partial H}{\partial x}(\varphi(t,\eta)) \cdot \underbrace{\dot{\varphi}_1(t,\eta)}_{f_1(\varphi(t,\eta))} + \frac{\partial H}{\partial y}(\varphi(t,\eta)) \cdot \underbrace{\dot{\varphi}_2(t,\eta)}_{f_2(\varphi(t,\eta))} = 0$$

$\cancel{\frac{\partial H}{\partial x}(\varphi(}$

$\varphi(t,\eta)$ can be an arbitrary point in U

$\cdot$ = derivative w.r.t. t

A procedure to find a first integral of $\begin{cases} \dot{x} = f_1 \\ \dot{y} = f_2 \end{cases}$

Step 1

$$\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)} \quad (3)$$

Step 2

- we integrate (3) and write the general sol

$H(x,y) = c \quad , \; c \in \mathbb{R}$

Step 3

- find a domain U for H and check that it is the first integral using (2)

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$$

$$\frac{dy}{dx} = \frac{x}{-y}$$

(sep. of variables)

$$y\,dy = -x\,dx \quad \Big| \int$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c \quad \Big| \cdot 2$$

$$y^2 + x^2 = 2c$$

$$H(x,y) = y^2 + x^2 \quad , \quad H : \mathbb{R}^2 \to \mathbb{R}$$

check

" $-y \frac{\partial H}{\partial x} + x \frac{\partial H}{\partial y} = 0$ " , $\forall (x,y) \in \mathbb{R}^2$

$-y \cdot 2x + x \cdot (2y) = 0$

$-2xy + 2xy = 0$ True

Lecture 09

Stability of equilibria of planar system

① $\dot{x} = f(x)$

$\eta^* \in \mathbb{R}^2$ be an equil. point for of (1) (eg $f(\eta^*) = 0_2$)

($\forall \eta \in \mathbb{R}^2$ $t \mapsto \varphi(t, \eta)$ the unique sol of IVP

$\begin{cases} \dot{x} = f(x) \\ x(0) = \eta \end{cases}$)

Def:

1) η^* is an attractor of (1) when $\exists V \in V(\eta^*)$

st. $\forall \eta \in V$

$\lim_{t \to \infty} \varphi(t, \eta) = \eta^*$

2) η^* is a repeller —— $\lim_{t \to -\infty} \varphi(t, \eta) = \eta^*$

3) η^* stable when $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall \eta \in$

with $\| \eta - \eta^* \|_{\mathbb{R}^2} < \delta$ we have

$$\| \varphi(t, \eta) - \eta^* \|_{\mathbb{R}^2} < \varepsilon \quad , \forall t \in [0; +\infty)$$

4) η^* is unstable when it is not stable

1. Stability of linear plane systems
2. The linearization method to study the stability of an equilibrium point of a nonlinear system
3. Examples

1.

(2) $\dot{x} = Ax$, $A \in M_2(\mathbb{R})$ $\det A \neq 0$

Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the eigenvalues of A

Rk:

$A\eta = 0$ $\Leftrightarrow$ to find eq. points

1) $\det A = \lambda_1 \cdot \lambda_2$

2) $\det A \neq 0 \Leftrightarrow \eta^* = 0_2 \in \mathbb{R}^2$ is the unique eq. point of

3) $\det A \neq 0 \Leftrightarrow \lambda_1 \neq 0 \wedge \lambda_2 \neq 0$