THE BALLBOT AS A DYNAMICAL SYSTEM

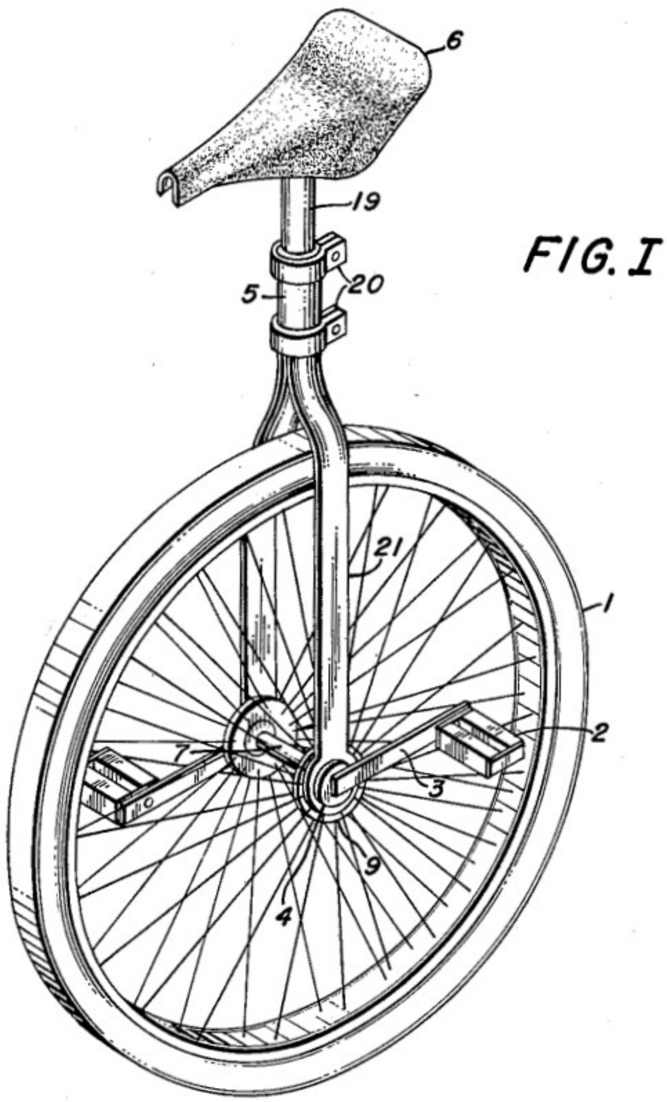

Figure 1: Patent 3,083,036 [1]

Miami University
Oxford, Ohio
2025

Samuel Skinner, Author
Dr. Van Scoy, Professor

Table of Contents

1

Introduction to the Ballbot as a Dynamical System

The ballbot is a recently patented (REF CITE PATENT) robot with a unique structure. That structure consists of a spherical base combined with a tall, narrow body. Unlike other, statically stable robots, the ballbot is inherently unstable, however the narrow and vertical body make it of significant interest for human-interactive robotics and navigation within flat environments.

Physically, the ballbot consists of two moving and interlinked parts: the ball and the body. The ball is a non-massless sphere which remains in constant contact with the ground. The body, which houses the control hardware and power supply, balances on top the ball. In practical application, there are typically three points of contact, each of which is a attached to a motor which drive the wheel and allow the robot to balance and move

From a control theory perspective, one can see that a ballbot isn't derived from any other system, but is instead, related to a 3D inverted pendulum. More specifically, the ball is modeled as a uniformly dense sphere which is attached to a uniformly dense cylinder. This system is inherently unstable under open-loop conditions, only having one unstable equilibrium when the ballbot is perfectly vertically aligned. Without active control, the ballbot will rapidly diverge from this equilibrium.

The ballbot currently has limited commercial uses, it's distinct structure, ability to maneuver narrow spaces, and ability for zero-turn rotation make it of interest for modern development in fields ranging from logistics, to security, or even personal robotics.

As will be described in subsequent chapters, we define the ballbot as having 2 perpendicular inputs representing torques on the ball, and 8 states, representing positions, angles, velocities and angular velocities of both the ball and the body. We will not be modeling the yaw rotation of the body, nor will we assume that slipping can occur between the ball and the ground. Finally, the rotation of the ball itself will not be modeled as that would transform the system into a non-holonomic system for no practical gain.

2

Modeling

Although a ballbot is a fairly complex system, it can be thought of in various ways. The most useful of which, will be as a inverted pendulum affixed atop a cart. It's states can be seen from inspection of its movement, while state equations can be derived by using lagrangian mechanics to derive a continuous nonlinear time-invariant system.

Upon inspection we can note the following states, additionally shown in 2.1.

- x positional states: x and $\dot{x}$
- y positional states: y and $\dot{y}$
- Pitch states: θ and $\dot{\theta}$
- Roll states: ϕ and $\dot{\phi}$

Additionally, we define the following constants that parametrize the simulation.

- Radius of the ball: r
- Distance between center of mass of the ball ball and the center of mass of the body: d
- Mass of the ball: m_{ball}
- Mass of the body: m_{body}
- Moment of the ball around all axes: I_{ball}
- Moment of the body around the x and y axes: I_{bodyxy}
- Moment of the body around the z axis: I_{bodyz}
- Force of gravity: g

Finally, we define the following inputs to the system.

- Ball x Torque: τ_x
- Ball y Torque: τ_y

Note, that in order to simplify the modeling, we use perpendicular torques acting on the ball, in practical application, three linearly dependent torques are actually used. These are equivalent however their relation is not explicitly defined by this document.

2.1 Definition of States

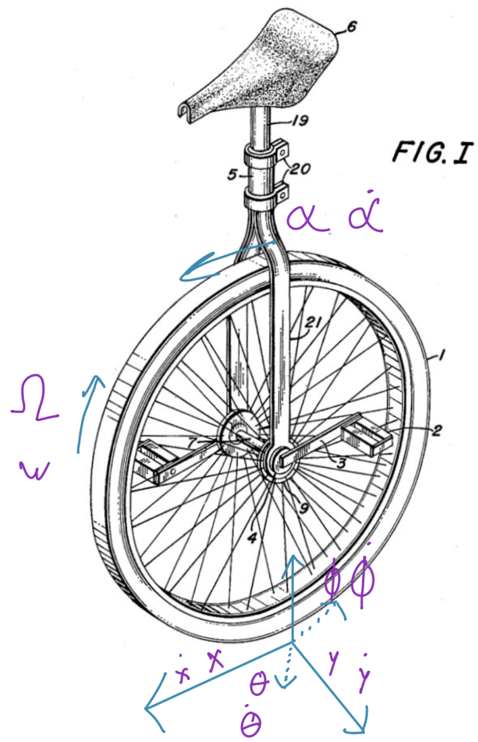

Figure 2.1: Drawing of a Unicycle with states

These states, are grouped into a set of generalized coordinates, $q = [x, y, \theta, \phi]\top$ and then combined to form the full state of the system.

$$\boldsymbol{x} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \\ \phi \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

The inputs are represented as a vector $u = \begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix}$

2.2 Derivation of State Derivatives

A general nonlinear time-invariant system is defined as $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, u)$, however for the purposes of this document, it is more helpful to think in the form

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = f(q, \dot{q})$$

Upon inspection, one can see that both sides of the equation share the state $\dot{q}$, showing that one just needs to find a function $\ddot{q} = f(q, \dot{q}, u)$ which models the dynamics of the system.

The derivation of thus function will be achieved through the use of lagrangian mechanics. Lagrangian mechanics constructs a function, called the lagrangian defined as the difference between the kinetic energies T and potential energies U of a system:

$$\mathcal{L} = T - U$$

For the ballbot system, the total kinetic energy is the sum of the translational and rotational energy of both the ball and the body. Meanwhile, the total potential energy is defined as the sum of the gravitational potential energy of the ball and the body.

By combining this, we can state the full lagrangian as: $\mathcal{L} = (E_{ball,translation} + E_{ball,angular} + E_{body,translation} + E_{body,angular}) - (E_{ball,height} + E_{body,height})$

2.2.1 Derivation of translational energies

The translational kinetic energy of any rigid body is defined by the velocity of its center of mass. While the scalar form $E = \frac{1}{2}m * v^2$ is familiar, in the context of this document, it will need to be extended into the vector case. This more general equation is written:

$$E = \frac{1}{2}m * v^\top * v$$

In our case, defining these velocities is tricky, so we rely on the fact that the time derivative of the position vector is the velocity vector. This final equation is written:

$$E = \frac{1}{2}m * \dot{p}^\top * \dot{p}$$

where p is the position vector in the frame of the object. This general formulation allows us to derive the translational energy of an object from it's position vector.

Translational energy of the ball. Starting off with the position of the ball, it can be easily seen that the position of the center of mass of the ball will be the sum of the position of it's base plus the radius of the ball upwards.

$$p_{ball} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} x \\ y \\ r \end{bmatrix} = \begin{bmatrix} x \\ y \\ r \end{bmatrix}$$

Taking the time derivative of this vector yields

$$v_{ball} = \dot{p}_{ball} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix}$$

Finally, this can be inputted into the translational energy formula to arrive at

$$E_{ball,translation} = \frac{1}{2} m * v_{ball}^{\top} * v_{ball}$$

Translational energy of the body. The position of the body's center of mass is more complex as it depends not only on the position of the ball, but also on the angles of the body (θ and ϕ) and the distance between the ball's center of mass and the body's center of mass (d)

However, the rotation of the body is defined by the rotation matrix sequence of rotating around the X axis an angle ϕ followed by a rotation around the Y axis an angle θ. These orations can be codified in the rotation matrices:

$$R_{body} = R_y(\theta) R_x(\phi) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

By using these matrices, one can easily calculate the position of the center of mass of the body as

$$p_{body} = p_{ball} + R_{body} \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}$$

Therefore, using the same method as established previously, one can calculate the velocity as the time derivative of the position vector. Do note that, because of the chain rule, and the resulting expression being dependent on θ and ϕ that the jacobian should be used in the form

$$v_{body} = \frac{dp_{body}}{dt} = \frac{\partial p_{body}}{\partial q} \frac{\partial q}{\partial t}$$

Resulting in a similar relation of

$$E_{body,translation} = \frac{1}{2} m * v_{body}^{\top} * v_{body}$$

2.2.2 Derivation of angular energies

In a similar method as in 2.2.1, one can start with the equation for the angular energy and similarly extend it to multiple. The scalar form is $E = \frac{1}{2} I \omega^2$ where I is the moment of inertia of the spinning body and ω is the angular velocity of that body. In the vector case, this similarly transforms into

$$E = \frac{1}{2} \omega^{\top} * I * \omega$$

.

Rotational energy of the ball. Since the ball never slips on the ground, it's angular velocity is directly derived from the linear velocity of the ball. For a ball of radius r, motion in the +X axis, induces a rotation around the +y axis. And, similarly, motion in the +y axis induces a rotation round the -x axis because of the right handed axes being used. Therefore,

$$\omega_{ball} = \begin{bmatrix} -\dot{y}/r \\ \dot{x}/r \\ 0 \end{bmatrix}$$

Calculating these linear accelerations is possible by splitting the linear distance function into its components via θ and then repeatedly deriving with respect to linear Additionally, on inspection we can see that angular acceleration will be linearly related to linear acceleration and the wheel's radius.

Starting with the equation for the arc length of a wheel, $s = r\Omega$ where s is the distance swept by a wheel of radius r over an angle Ω (note the intentional reuse of Ω). This equation can be derived, into the formula for the linear velocity of processing wheel $v = r\omega$ where v

is the velocity of the traveling wheel in the direction of motion and ω and $\dot{\Omega}$ are the angular velocities of the wheel. By splitting this equation into two based on components we arrive at

$$v_x = \dot{x} = r \cdot \omega \cdot \cos(\theta)$$
$$v_y = \dot{y} = r \cdot \omega \cdot \sin(\theta)$$

Finally, the derivative of this set of equations can be taken one more time find the desired formula for linear acceleration.

$$\ddot{x} = r(\dot{\omega}\cos(\theta) - \omega\sin(\theta)\dot{\theta})$$
$$\ddot{y} = r(\dot{\omega}\sin(\theta) + \omega\cos(\theta)\dot{\theta})$$

Rotation θ. Upon inspection, one notices that θ only changes based off of ϕ. In order for a unicycle to change it's θ, the rider must lean to one side and change *phi* first!

Imagine a unicycle balanced slightly over at an angle and going forward in its own local frame of reference. As it unicycle moves forward, it will rotate through θ tracing a perfect arc, dependent on the body angle ϕ. By trying some values, it can be noticed that if an angle $\phi = 0$ the radius of the circle traces goes to infinity. Meanwhile, if the angle is larger, something closer to $\phi = \pm\frac{\pi}{8}$, then theta will slowly change based on the current phi.

With this in mind, the formula for this arc is the turning radius (r_t) of a tilted. $r_t = \frac{r}{\sin(\phi)}$, the derivation of which is shown in 2.2, can be used to relate r_t, r, and ϕ. However, we now need a relation between r_t and $\dot{\theta}$. This can be derived from the formula for arc length used previously. The formula $s = r\theta$, can once again be reinterpreted as $s = r_t\theta$ where s the arc length of the turn radius that is swept by a wheel of radius r traveling= over that arc an angle θ. This means we can take the derivative of this formula to arrive at $v = r_t\dot{t}heta$. Finally, since we don't store the wheel's local forward velocity we can rewrite it in terms of angular acceleration as $\dot{\theta} = \frac{r\omega}{r_t}$. Now, by plugging in the formula derived earlier, we finally achieve $\dot{\theta} = \frac{r\omega\sin(\phi)}{r} = \omega\sin(\phi)$. Finally, by applying the product rule we can find the desired formula for $\ddot{\theta}$

$$\ddot{\theta} = \dot{\omega}\sin(\phi) + \omega\dot{\phi}\cos(\phi)$$

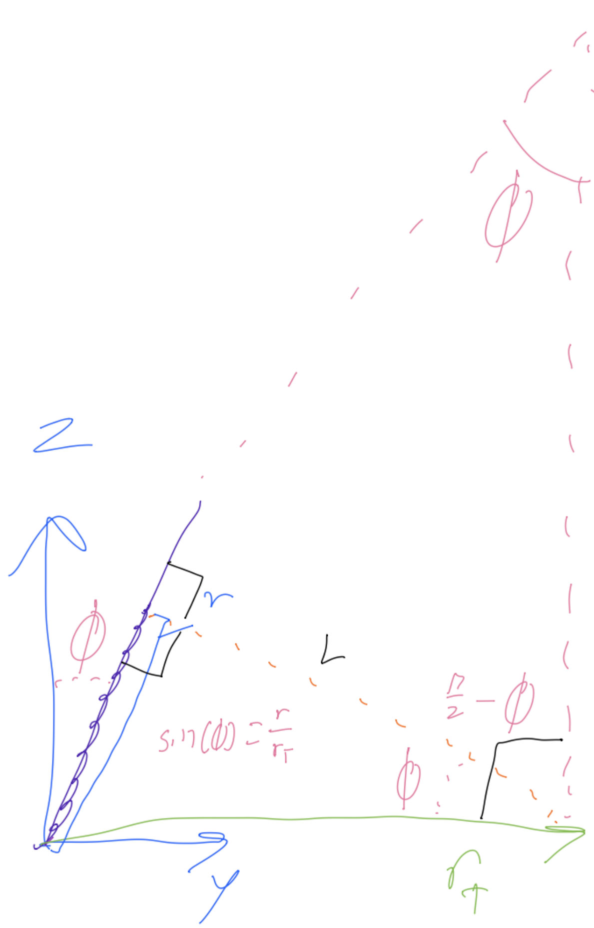

Figure 2.2: Drawing of a Unicycle with states

Rotation ϕ. Upon inspection, one notices that if perfectly balanced, a unicycle stands perfectly vertical it will stay there, however any slight perturbation will cause it to fall over. Additionally, the only perturbation that could cause this is the torque τ_2 as every other input acts such that no matter its value, there will be no change on *phi*.

From this, assuming the unicycle is not moving forward, we can create a simplified model of it. This is model in REFERENCE FIGURE. On this model, we have the force of gravity acting at both the center of mass of the wheel and on the rider. Additionally, we have a torque acting on the rider's center of mass. The gravitational forces can be split up into components along and perpendicular to the unicycle, with those along being canceled by the normal force of the unicycle. This leaves $F_{gravity,wheel}\sin(\phi)$, $F_{gravity,rider}\sin(\phi)$ and τ_2. By using newton's third law for rotation we see that the sum of torques $\sum\tau = I\alpha$ or, restated into the conventions of this document, $\sum = I\ddot{\phi}$ must be related. By combining all of these, we arrive at the formula

$$I_{pivot}\ddot{\phi} = m_w r g \sin(\phi) + m_r g(r+d)\sin(\phi) + \tau_2$$

where $I_{pivot} = m_w r + m_r(r+d)$ and is the moment corresponding to the pivot point being at the point of contact of the wheel.

This formula is accurate in the case of the unicycle when there is zero motion however once the unicycle starts to move, there are additional forces in play, the centrifugal force and the gyroscopic force.

Since we are working with torques, we need the formula for the torque caused by the centrifugal force. This can be interpreted as a torque occurring at the center of mass of the combined system of the rider and wheel perpendicular to the body of the unicycle, that is $\tau_c = d_c F_c m_c \cos(\phi)$ where d_c is the distance from the pivot to the center of mass, F_c is the centrifugal force, m_c is the total mass of the system. Additionally, note the $\cos(\phi)$ term, because the centrifugal force acts parallel to the global y axis regardless of the angle ϕ of the unicycle, it must be broken down into components that are cancelled by the unicycle's normal force and those which act as a net torque perpendicular to the axis of travel.

Starting off with d_c, it can be found by treating the unicycle as a seesaw and finding the location at which the torques balance. This formula is defined in A and resolves to $d_c = \frac{m_r r + d m_r + m_w r}{m_w + m_r}$. Next, is F_c, which additional is defined in A to be $F_c = r_w \omega \dot{\theta} m_c$. These three formulas can be combined together to form

$$\tau_c = r_w \omega \dot{\theta} m_c d_c \cos(\phi)$$

Finally, the gyroscopic torque can be calculated and, as detailed in A, its the formula is $\tau_g = I_{wz}\omega\dot{\theta}\cos(\phi)$

This can now be stated as

$$\ddot{\phi} = \frac{\tau_{gravity} + \tau_2 + \tau_c + tau_{gyroscopic}}{I_{pivot}}$$

Rotation ω and α. The omega and alpha terms are special as they are interlinked. We will address their derivations separately, resolving two separate equations, both dependent on $\dot{\omega}$ and $\ddot{\alpha}$ and then use algebraic substitution. Their derivations are explained in detail in the Appendix; however, here we will place their full definitions, where the system determinant is defined as $\Delta = [(m_w + m_r)r^2 + I_{wy}][I_{wy} + m_r d^2] - (m_r r d \cos\alpha)^2$.

The explicit formula for wheel acceleration is:

$$\dot{\omega} = \frac{(I_{wy} + m_r d^2)(m_r g d \sin\alpha) - (m_r r d \cos\alpha)(m_r g d \sin\alpha - \tau_1)}{\Delta}$$

The explicit formula for pitch acceleration is:

$$\ddot{\alpha} = \frac{[(m_w + m_r)r^2 + I_{wy}](m_r g d \sin\alpha - \tau_1) - (m_r r d \cos\alpha)(m_r g d \sin\alpha)}{\Delta}$$

Final state equation. Now that each sub-equation has been derived and the dependencies substituted, the final non-linear state update vector $\dot{x}$ can be written.

$$\dot{x} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \omega \\ \dot{\alpha} \\ r(\dot{\omega}\cos\theta - \omega\dot{\theta}\sin\theta) \\ r(\dot{\omega}\sin\theta + \omega\dot{\theta}\cos\theta) \\ \dot{\omega}\sin\phi + \omega\dot{\phi}\cos\phi \\ \frac{[m_w r + m_r(r+d)](g\sin\phi + r\omega\dot{\theta}\cos\phi) - I_{wz}\omega\dot{\theta}\cos\phi + \tau_2}{m_w r^2 + m_r(r+d)^2 + I_{wy}} \\ \frac{(m_r d^2)(\tau_1) - (m_r r d \cos\alpha)(m_r g d \sin\alpha - \tau_1 + \tau_3)}{[(m_w + m_r)r^2 + I_{wy}][m_r d^2] - (m_r r d \cos\alpha)^2} \\ \frac{[(m_w + m_r)r^2 + I_{wy}](m_r g d \sin\alpha - \tau_1 + \tau_3) - (m_r r d \cos\alpha)(\tau_1)}{[(m_w + m_r)r^2 + I_{wy}][m_r d^2] - (m_r r d \cos\alpha)^2} \end{bmatrix}$$

Some interesting things can be gleamed from this equation. Most notably the fact that τ_1 and τ_3 are highly coupled but not identical!

3

Environments

Environments in LaTeX are used to apply specific formatting to part of the document[1].

3.1 Theorems

For Theorem environments, we use the `amsthm` package. This allows us to define environments that are frequently used such as `thm` for theorem, `lem` for lemma, and so on. We can also use the `thmtools` package to create boxed or shaded theorems for emphasis. Here are some examples.

Theorem 1 (A theorem)**.** This is how we state a theorem.

Theorem 2 (Shaded)**.** For emphasis, we can put it in a shaded box.

Theorem 3 (Outlined)**.** Another way to create emphasis is with an outlined box.

Proof. We can write proofs using the `proof` environment. ∎

The `thmtools` package also provides `restatable`, which is useful if you want to state the same result more than once (say, in the introduction and later in the paper), but don't want to give it a new label and equation numbers. See the documentation for more details[2].

3.2 Lists

Create bulleted lists using the `itemize` environment. For example:

- First item

[1] `https://www.overleaf.com/learn/latex/Environments`
[2] `https://ctan.math.illinois.edu/macros/latex/contrib/thmtools/doc/thmtools-manual.pdf`

- Second item
- Third item

Numbered lists are created using the `enumerate` environment. For customization, we use the `enumitem` package with the `shortlabels` option. This allows us to write customized lists easily. For example,

`\begin{enumerate}[(i)]`
`\item first item \label{x}`
`\item second item \label{y}`
`\item third item \label{z}`
`\end{enumerate}`
	produces	(i) first item
		(ii) second item
		(iii) third item

We can refer to items using `\cref` as before. For example, the command `\cref{x,y,z}` produces Items (i) to (iii).

4

More

4.1 References and links

We use the `hyperref` package to produce a pdf with hyperlinks. We also use the `cleveref` package for facilitating references. Using the `\cref` command will automatically use the correct prefix. You can also refer to multiple things at once by using multiple arguments, or you can refer to a range using `\crefrange`. For more information, see the documentation[1].

References can be cited with the `\cite` command, which produces something like [?, ?, ?, ?]. The `cite` package ensures the citations are ordered nicely and compressed when possible.

4.2 Figures

We use the standard `figure` environment for figures. Diagrams should be placed in separate files in the `graphics/` folder and should use the `standalone` package. You can then use `includegraphics` as in Figure 4.1 to include the figure in your document. If you are using `pdflatex`, you can also use `includegraphics` to include PDF, JPEG, PNG, or other formats, but not EPS.

4.3 Tables

The `booktabs` package, which includes commands such as `\toprule`, `\midrule`, and `\bottomrule`, can be used to make nice tables. In general, never use vertical lines to separate columns. For more style tips on how to make nice tables, see[2]. Here is an example of a nice table[3].

[1] `http://mirrors.ctan.org/macros/latex/contrib/cleveref/cleveref.pdf`
[2] `https://people.inf.ethz.ch/markusp/teaching/guides/guide-tables.pdf`
[3] `https://lazyscientist.wordpress.com/2021/07/23/make-better-tables-in-latex-using-booktabs/`

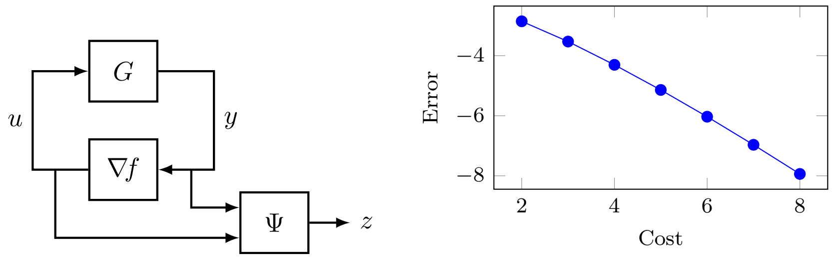

Figure 4.1: Figure captions should be long and descriptive because people actually read them, unlike the rest of the text. (Left) A block diagram make using Tikz. (Right) A plot made using Pgfplots. Note that the figures are not scaled, so the text size in the figures is consistent with the rest of the document. The source files for the figures are in the `graphics/` folder.

Table 4.1: Gravimetric analysis of silver halides in a 1.27-mL sample of sea water.

	Test Tubes				
Qty of Sample	A	B	C	D	Avg
Mass (g)	1.399	1.32	1.328	1.408	1.364
Density (g/mL)	1.10	1.04	1.05	1.109	1.07
Mass w/ Precipitate (g)	13.443	13.401	13.348	—	13.397
Mass AgCl (10^{-2} g)	9.0	9.2	8.7	—	8.9
Moles AgCl (10^{-4} mol)	6.28	6.42	6.08	—	6.50

4.4 Code

We can include snippets of code when appropriate to describe specific computer code. A sample program in Julia is shown in ??. Note that this uses the `pygmentize` package in Python, which must both be installed in order to compile. Also, the code contains unicode characters, which must be defined (see the preamble of `thesis.tex`).

A

Appendix A - Derivations

Derivation of d_c for ϕ. Looking at Figure A.1, x is the distance from the base to the center of mass, the variables in the drawing perfectly correspond to those used in the derivation of ϕ. By balancing moments, with respect to x, we arrive at the equation $(x-r)m_w = (r+d-x)m_r$ and, by solving for x, we arrive at

$$x = \frac{m_r r + d m_r + m_w r}{m_w + m_r}$$

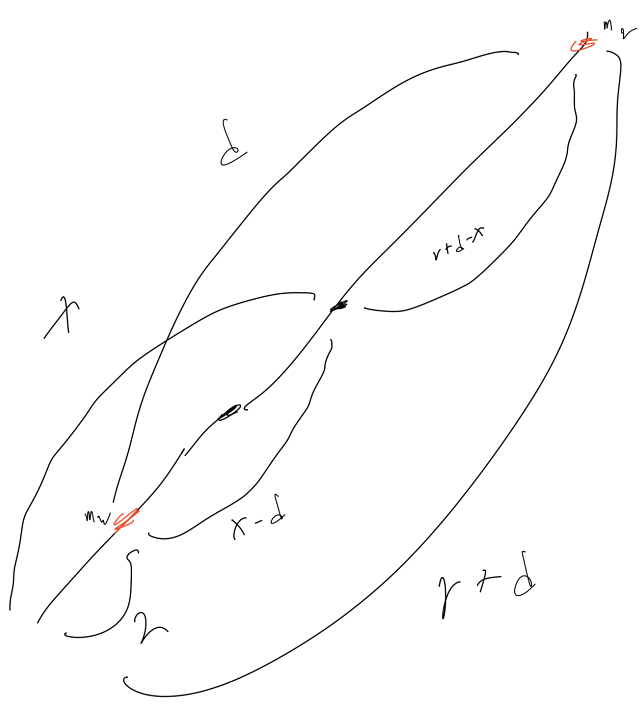

Figure A.1: Drawing of a Unicycle's center of mass points

Deivation of F_c for ϕ. As a reminder, F_c is the centrifugal force acting upon the entire unicycle at the center of mass **parallel to the global y axis**. to the frame of the unicycle. Starting off with Newton's third law, $F = ma$, we see that in order to calculate this, we need the total mass of the system, $m_c = m_w + m_r$ and the centrifugal acceleration. The formula for centrifugal acceleration is $a_c = \frac{v^2}{r}$ where v is the velocity in the direction of travel and r is the radius swept by the rotation. Thankfully, both of these correspond to formulas previously derived. $v = r_t\dot{\theta}$ restated as $r_t = \frac{v}{\dot{\theta}}$ and $v = r_w\omega$. Combing these together we arrive at $a_c = \frac{v^2}{v/\dot{\theta}} = v\dot{\theta} = r_w\omega\dot{\theta}$. Finally, this can be substituted into Newton's third law to achieve

$$F_c = r_w\omega m_c\dot{\theta}$$

Derivation of gyroscopic effect. The angular momentum of a spinning wheel is $L = I_w * \omega$ keeping with the conventions of this document, that means $L = I_{wy}\omega$. Additionally, newton's second law for rotation states that $\tau = \dot{L}$ By the small angle approximation, we can approximate $\dot{L} = L\dot{\theta}$ and with this we arrive at the formula $\tau_g = I_w\omega\dot{\theta}$. Finally, remember that this torque must act opposite as a restoring force, as it is currently written as a positive feedback loop.

$$\tau_g = -I_w\omega\dot{\theta}$$

References

[1] D. E. Cornell III. Occupant powered unicycle, U.S. Patent 3 083 036, March 26 1963.