Introduction: Hatsugai-Kohmoto Model

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Wild Chicken Universality

September 23, 2025

Introduction: From Band Theory to Strong Correlation

- Band theory explains conductors/insulators based on band filling.
- **[→Mott insulator]**However, some half-filled systems remain insulating $\Rightarrow$ band theory fails.
- **[→]** Strong correlation effect.

- Doped Mott insulators show strange metal, or pesudogap phase, violating Fermi liquid theory.
- **[→]** Non-Fermi liquids (NFL): no quasiparticles, no well-defined Fermi surface. How to understand $NFL \rightarrow BCS$?

- Hubbard model is analytically intractable $\Rightarrow$ need exactly solvable toy models.
- **[→Hatsugai-Kohmoto model]** HK model provides exact solvability and captures Mott and NFL features.

HK Model: Definition and Solvability

Definition

The real-space Hamiltonian is given by:

$$\hat{H}_{\text{HK}} = -t \sum_{\langle i,j \rangle, \sigma} c^\dagger_{i\sigma} c_{j\sigma} - \mu \sum_{j,\sigma} n_{j\sigma} + \frac{U}{N_s} \sum_{j_1+j_3=j_2+j_4} c^\dagger_{j_1\uparrow} c_{j_2\uparrow} c^\dagger_{j_3\downarrow} c_{j_4\downarrow}, \tag{1}$$

- Hamiltonian:

$$\hat{H}_{\text{HK}} = \sum_k (\varepsilon_k - \mu)(n_{k\uparrow} + n_{k\downarrow}) + U n_{k\uparrow} n_{k\downarrow} \tag{2}$$

- Center-of-mass conserving interaction $\Rightarrow$ momentum decoupling.
- Each k-sector is 4-dimensional: $|0\rangle, |\uparrow\rangle, |\downarrow\rangle, |\uparrow\downarrow\rangle$.
- Exact diagonalization is possible for each k.

Ground State Structure

- For each k, 4 eigenstates:

$$\{|0\rangle,\ c_{k\uparrow}^\dagger|0\rangle,\ c_{k\downarrow}^\dagger|0\rangle,\ c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger|0\rangle\}, \tag{3}$$

and corresponding energies:

$$E_0 = 0, \quad E_\uparrow = \epsilon_k - \mu, \quad E_\downarrow = \epsilon_k - \mu, \quad E_{\uparrow\downarrow} = 2\epsilon_k - 2\mu + U.$$

  - $\varepsilon_k - \mu > 0 \Rightarrow |0\rangle$
  - $\varepsilon_k - \mu < 0 < \varepsilon_k - \mu + U \Rightarrow$ singly occupied
  - $\varepsilon_k - \mu + U < 0 \Rightarrow |\uparrow\downarrow\rangle$

- the many-body ground state takes the product form:

$$|\Psi_{\mathrm{gs}}\rangle = \bigotimes_k |\phi_k\rangle, \quad |\phi_k\rangle \in \{|0\rangle, |\uparrow\rangle, |\downarrow\rangle, |\uparrow\downarrow\rangle\}. \tag{4}$$

Phase diagram and Metal-Mott phase transition

- Consider ground state is singly-occupied, one of degenerate ground states.
- Added to the system, creating a doubly occupied site, the energy changes by:

$$\Delta E_k^+ = \epsilon_k - \mu + U$$

- A hole is doped, energy changes by:

$$\Delta E_k^- = -(\epsilon_k - \mu)$$

- $\epsilon_k \in [-W/2, W/2]$:

$$\epsilon_k^{min} = -W/2 > \mu - U, \epsilon_k^{max} = W/2 < \mu$$

$\rightarrow$,

$$\frac{\mu}{U} < 1 - \frac{W}{2U}, \quad \frac{\mu}{U} > \frac{W}{2U} \tag{5}$$

Phase Diagram and Band Filling

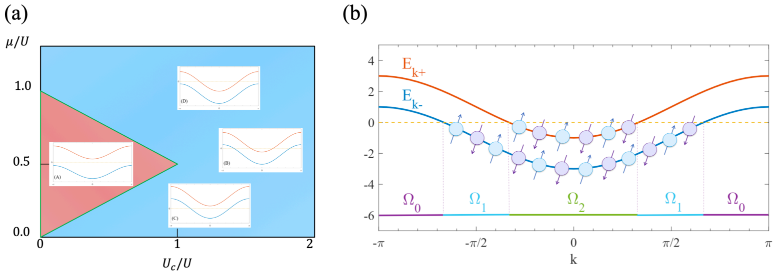


Figure: Image modified from Ref. [5]. (a) Phase diagram. (b) Schematic diagram of electron filling on the energy bands within the first Brillouin zone.

For half-filling case, **Metal-Mott phase transition occurs at** $U = W$.

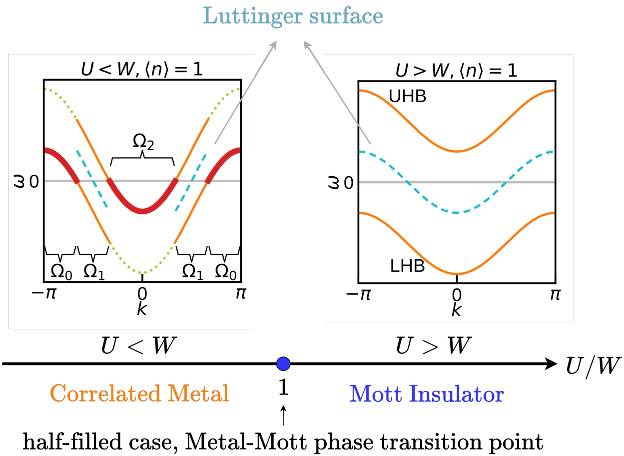


Figure: Schematic band diagrams of the Hatsugai-Kohmoto (HK) model at half-filling, modified from Ref. [2].

Violation of Luttinger Theorem in HK Model

In Fermi liquid, Luttinger theorem stating that the volume of the Fermi surface is directly proportional to the particle density.

Luttinger Theorem

$$2\int d^d k \frac{1}{(2\pi)^d}\theta(G(k,\omega=0)) = n \tag{6}$$

- Luttinger theorem: counts poles of $G(k,\omega=0)$.
- **However, counting sign change in Eq.** (6) **leads to a violation in HK model.**

Violation of Luttinger Theorem (LT)

The zero-frequency Green's function may exhibit the following two forms of sign change: poles and zeros.

We will focus on the following questions:

1. How do zeros appear?
2. Why do zeros lead to the breakdown of LT?
3. Can such a breakdown be remedied, or what improvements can be made?

1. How do zeros appear? - Zeros of $G(k, \omega = 0)$ in HK model

Green's Function of HK model

The retarded Green's function of the HK model is given by:

$$G_k(\omega) = \frac{1 - \langle n_{k\sigma} \rangle}{\omega + i0^+ - \xi_k} + \frac{\langle n_{k\sigma} \rangle}{\omega + i0^+ - (\xi_k + U)} \tag{7}$$

Consider the case of half-filling, where $\langle n_{k\sigma} \rangle = 1/2$. The Green's function can be further simplified to:

$$G^R_{k\sigma}(\omega) = \frac{1}{\omega + i0^+ - (\xi_k + \frac{U}{2}) - \frac{(\frac{U}{2})^2}{\omega + i0^+ - (\xi_k + \frac{U}{2})}} \tag{8}$$

- When $\xi_k = -U/2$, self-energy diverges, $\Rightarrow$ zero of G.
- This defines a **Luttinger surface**.

Definition of Luttinger surface (blue line): $ReG(k, \omega = 0) = 2\xi_k + U = 0$

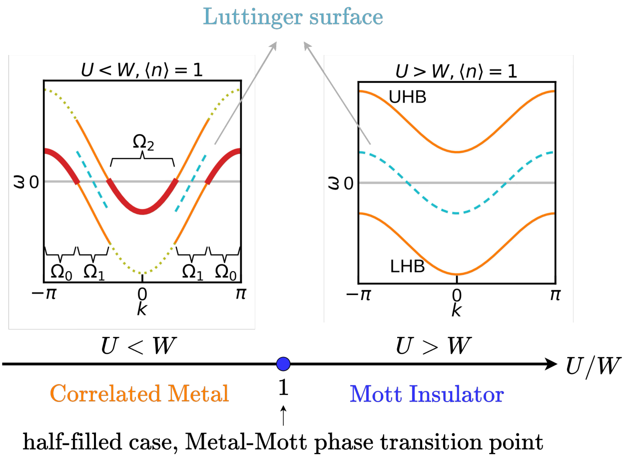


Figure: Schematic band diagrams of the Hatsugai-Kohmoto (HK) model at half-filling, modified from Ref. [2].

2. Why do zeros lead to the breakdown of LT?

Ref. [1]: The general form of the Luttinger integral is given by:

$$I_L = I_1 + I_2 \tag{9}$$

$$I_1 = \frac{i}{2\pi} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \oint_C d\omega \frac{\partial}{\partial \omega} \log(G(\mathbf{k},\omega)) = \frac{N}{2V}$$

$$I_2 = -i \int \frac{d^d\mathbf{k}}{(2\pi)^d} \oint_C \frac{d\omega}{2\pi} \left\{ G(\mathbf{k},\omega) \frac{\partial}{\partial \omega} \Sigma(\mathbf{k},\omega) \right\}$$

- $I_1 = n$, once $I_2 = 0$, then $I_L = n$, LT holds.
- But if self-energy $\Sigma(\mathbf{k},\omega)$ diverges, then $I_2 \neq 0$, LT broken.

Violation in HK model, for example, one dimension

$T \to 0$, there are two jumps in electron distribution function which constructes to two quasi-fermi surfaces:

$$\langle \hat{n}_{k\sigma} \rangle = (1 - \langle n_{k\sigma} \rangle)\, \theta_1(-\xi_k) + \langle n_{k\sigma} \rangle \theta_2(-\xi_k - U)$$

the electron density is calculated as:

$$n = 2 \left[2 \int_0^{k_{F2}} \frac{dk}{2\pi} \cdot 1 + 2 \int_{k_{F2}}^{k_{F1}} \frac{dk}{2\pi} \cdot \frac{1}{2} \right] = 2 \cdot \frac{k_{F1} + k_{F2}}{2\pi}$$

which is violated to the electron density to given by LT: $n_L = 2 \cdot \frac{2k_F}{2\pi}$.

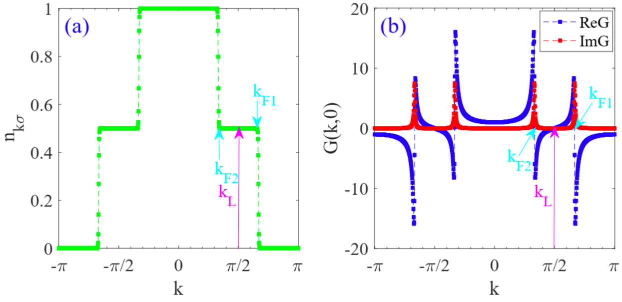


Figure: Taken from Ref. [5]. (a) Electron distribution function. (b)The corresponding real and imaginary part of Green function.

- Two jumps in electron distribution function which constructes to two quasi-fermi surfaces
- Kramers-Kronig relation:

$$\text{Re}(G^R(\omega)) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\text{Im}\, G^R(\omega')}{\omega - \omega'} d\omega' = \int_{-\infty}^{+\infty} \frac{A(\omega')}{\omega - \omega'} d\omega'$$

3. Save LT?

The relationship between PHS and LT.

While PHS can serve as a sufficient condition for Luttinger's theorem to hold, the converse does not apply. That is,

$$\text{PHS} \Rightarrow \text{LT holds} \quad \text{but} \quad \text{LT holds} \nRightarrow \text{PHS}$$

Therefore, the presence or absence of PHS should not be inferred solely from whether Luttingers theorem is satisfied.

- Proved by Seki and Yunoki [3]: particle-hole symmetry (PHS) ensures $I_2 = 0$
- However, there exist counterexamples, even when particle-hole symmetry does not exist, LT still holds.
- Waiting for careful examination ... [1]

Fermi Arc Formation

- Kun Yang et al. [4]: Modify $U \to U_k = U - 2\,T_x \cos k_x - 2\,T_y \cos k_y$.
- Green's function still solvable. Luttinger surface condition: $2\xi_k + U_k = 0$
- Luttinger surface (zeros) intersects pseudo-Fermi surfaces (poles) $\Rightarrow$ Fermi arcs.

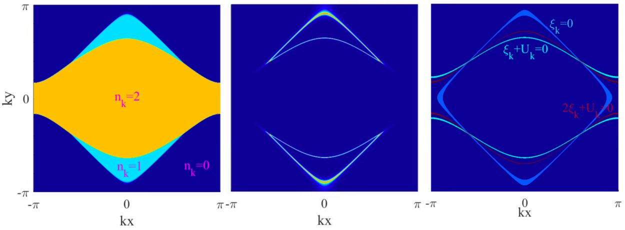
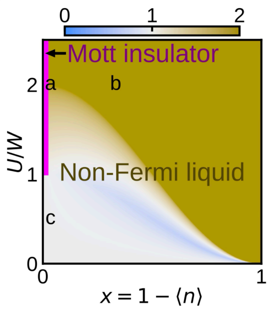


Figure: Image taken from Ref. [5]

Big region of Non-fermi Liquid based on HK model

Phillips et al. (2019)

Doped phase diagram



Doped Mott Insulator

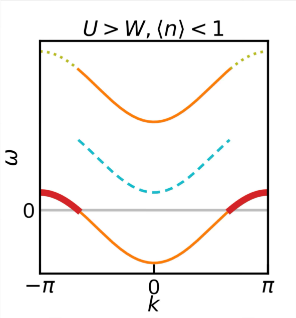


Figure: Image modified from Ref. [2]

Need unconventional **NFL** $\rightarrow$ **BCS** formation picture

s-wave Pairing in HK Model

- Add BCS-type pairing to HK: $H_{\text{pair}} = -g\Delta^\dagger\Delta$

$$H = H_{\text{HK}} - gH_p, \quad H_p = \frac{1}{L^d}\hat{\Delta}^\dagger\hat{\Delta} \tag{10}$$

  where$\hat{\Delta} = \sum_k b_k,\ b_k = c_{k\uparrow}c_{-k\downarrow}$ is the pair operator.
- Superconducting instability: for any $g > 0$, binding energy $E < 0$
- Not Bogoliubov quasiparticles, instead, composite of holons and doublons.
- Spectral weight shifts from UHB to LHB as g increases.
- Superfluid stiffness is suprressed near Mott phase.

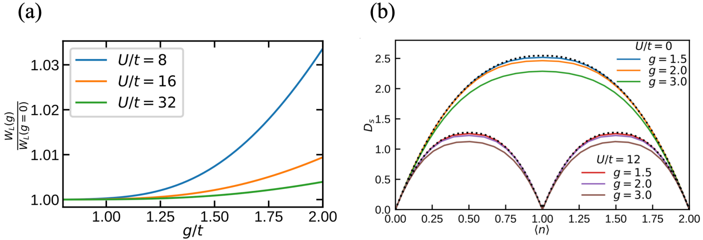


Figure: Calculated results for the HK-BCS model, modified from Ref. [2]. (a) Dynemical spectral weight transfer. (b) Superfluid stiffness.

Summary

HK model: minimal, solvable, captures Mott and NFL physics.

- HK model $\rightarrow$ Violation of Luttinger theorem
- Violates Luttinger theorem $\rightarrow$ Non-fermi liquid behavior.
- Luttinger surface $\rightarrow$ Fermi arcs.
- NFL $\rightarrow$ unconventional superconductivity.

Thanks!

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